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ERASMUS SCHOOL OF ECONOMICS

Bachelor Thesis

Parametric Portfolio Policies: Multi-period Policies Based on Power Utility

Author:
Bas van der Noll

Supervisor:
B.F. Diris PhD.

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Abstract

This research develops a method which constructs portfolio weights that maximize a power utility of an investor, while parametrizing these portfolio weights. This parametrizing is done with the use of mechanically managed portfolios. In the first place 'conditional' portfolios are used. These portfolios invest wealth in an asset proportional to the value of a state variable. Secondly, the concept of 'timing' portfolios is used. These portfolios invest wealth in an asset in one period and in the risk-free rate in all other periods. Since this method does not account for compounding, also a method that accounts for compounding is developed. To check whether the methods works well, they are compared to the nowadays 'traditional' method, which depends on an underlying econometric VAR-model. The methods that are developed in this research show good in-sample performance and robustness compared to the 'traditional' method, whereas out-of-sample they are outperformed by the 'traditional' method. The advantages of parametrizing the portfolio weights are more profound. However, the performance, especially out-of-sample, is dominated by the 'traditional' method.

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1 Introduction

As an investor, you constantly seek for the optimal trade off between risk and return. This trade off naturally depends on how a portfolio, in which you have invested, is constructed. As a portfolio exists of a fixed number of assets with a certain weight attached to each of the assets, constructing a portfolio which gives an optimal return-risk trade off, boils down to choosing the weights the right way.
Nowadays, research on the portfolio choice problem is becoming quite popular for different reasons. First, choosing the weights of the assets in a portfolio is of great importance for actually managing a portfolio, in the sense that the success of the investment heavily depends on the result the portfolio generates. Secondly, recent research reveals that returns on different assets may be predictable, saying that state variables, like dividend-price ratio, contain information about future returns. This enables an investor to anticipate on upcoming returns and in this way reduces its risk while maximizing the return. Finally, current complex problems may be solved faster because of the technological improvements of computers. As the returns on assets change tick-by-tick, processing historical information faster enables the investor to act more quickly on recent developments in the market.

Over the last decades, several different techniques for constructing portfolio weights have been developed. Most popular and nowadays 'traditional' called technique depends on an econometric model which models the behaviour of assets returns using state variables and has the power of predicting returns. Using the implications of this model, it is possible to construct optimal portfolio weights. However, in order to let this econometric model work properly, we need to make some (statistical) assumptions on the conditional distribution of returns. Recent development on this method has been achieved by Jurek & Viceira(2010). They provided a recursive, analytical solution for this portfolio choice problem, whereas until then portfolio weights were obtained by using simulation.
In stead of focussing on the returns of the assets, Brandt & Santa-Clara(2006) argue that if state variables contain information about asset returns and asset returns imply certain weights, it is also sensible to assume that state variables 'directly' contain information about the optimal weights. By parametrizing the weights we do not have to make any statistical assumption on the returns, as we do with the specification of the econometric model described earlier. This should make the model more robust for misspecification. Using the method of parametrizing, the paper describes how multi-period investment strategies can be constructed. Results show that the method produces a good trade off between risk and return.
Another research, conducted by Brandt, Santa-Clara & Valkanov(2009), provides a similar method of parametrizing portfolio weights like Brandt & Santa-Clara(2006), only now applying this method to a large number of stocks and only for strategies one period ahead. The method produced robust performance, in and out of sample. They also compared their method to the 'traditional' method and concluded that compared to their own approach, this 'traditional' method produces "notoriously noisy and unstable results".
Just a few weeks ago, Van Der Noll et al.(2011) provided a supplementary research, by thoroughly comparing the 'traditional' approach with the method as developed by Brandt & Santa-Clara(2006) for one and multi-period strategies. The problem they faced was that, initially, the method of Brandt & Santa-Clara is based on a different utility function than the 'traditional' method described by Jurek & Viceira(2010). Brandt & Santa-Clara provide an iterative method, developed by Brandt et. al(2005) for optimizing along different utility functions. Only Van Der Noll et al.(2011) show that this iterative method, under certain circumstances, fails to produce useful weights and results. The combination of an increasing horizon and a larger number of state variables caused the the iterative method to fail, in the sense that it did not converge to a useful solution. So, it should be good to develop a new method, in order to compare the technique of Brandt & Santa-Clara to the method as defined by Jurek & Viceira.

That is were this research fits in. The goal of this paper is to investigate whether it is possi-

ble to apply the (multi-period) technique of Brandt & Santa-Clara to the utility function provided by Jurek & Viceira. Hence, I will explain why the utility function used by Jurek & Viceria is so interesting.
The utility function used by Brandt & Santa-Clara(2006) considers shocks of the returns. This method only rewards with high utility if you have high returns and low variance. In other words, when the volatility of the returns increase, the utility drops, because it punishes high variance. In practice, this means that large shocks upwards are punished almost just as hard as large shocks downwards. An utility function which only considers the mean and variance of returns is called a quadratic utility. Main drawback is thus that upward and downward shocks are treated almost equally.
Lets take a look at the utility function used by Jurek & Viceira. This form of utility, called power utility, also considers higher order moments than only the first (mean) and second moment (variance). This results in the fact that only sharp drops in returns are heavily punished and large increases are rewarded with a slight increase of utility. It is easy to see, that this form of utility is much more realistic compared to quadratic utility. In practice, investors do not mind upwards shocks of returns, but they want to avoid sharp downward shocks.
Campbell & Viceira (2002) did research on the difference between the quadratic and power utility. It seems that utility functions incorporate the behaviour of the coefficients of absolute and relative risk aversion. To understand how utility functions work, I will discuss the results of Campbell & Viceira(2002) briefly.
Absolute risk aversion is the willingness to pay an absolute amount of wealth to avoid the gamble (*risk*) of a certain absolute size. It is generally assumed that absolute risk aversion decreases, or at least not increases, in the wealth you currently poses. It means that for example a billionaire will be relatively unconcerned with an absolute risk that might worry a poor person and is willing to pay less to avoid that risk. Formally, the coefficient of absolute risk aversion is defined as the ratio between the second and first derivative of the utility function in a certain point of wealth.
Relative risk aversion is the willingness to pay a fraction of your current wealth to avoid the gamble (*risk*) of a given size relative to your current wealth. A plausible assumption to make, is that this coefficient of relative risk aversion is independent of the amount of wealth you posses and is thus constant. In other words, people with different amounts of wealth, make the same decision if the relative risk, compared to their wealth, is the same. The coefficient of relative risk aversion is defined as the product between the current wealth and the coefficient of absolute risk aversion in that point of wealth.
Campbell & Viceira(2002) show that the quadratic utility has coefficients of absolute and relative risk that both increase in wealth. Inspecting the properties of the power utility, they found that the absolute risk aversion decreases in wealth and the relative risk aversion is constant in wealth. This is exactly what we are looking for because it is realistic and therefore considering power utility is relevant.

The motivation for performing this research is now clear. Applying the multi-period technique developed by Brandt & Santa-Clara(2006) to a power utility. Jurek & Viceira(2010) found an analytical solution for a power utility problem, due to the fact that they assumed the returns to be distributed normally. Because the power of the technique of Brandt & Santa-Clara(2006) is that they do not make any statistical assumptions, an analytical solution for our problem is still not available, just as Brandt(2010) argued. To find a solution for the problem, we will use, just as Brandt, Santa-Clara & Valkanov(2009), numerical optimization.
In order to find out how good the method provided by Brandt & Santa-Clara(2006) works for multiple-periods strategies applied to power utility, I will compare its results to the results produced by the method of Jurek & Viceira(2010).

The structure of the paper will be as follows. I first describe the method I will use in the the section 'Methods'. After that, I will discuss on which data I will apply the proposed technique in the section 'Data'. Now that is is clear what method is used and to what data I will apply it, the results will be elaborated on in the section called 'Results'. Finally, I conclude this paper with

‘Conclusion’ and ‘Recommendations and Further Research’.

2 Methods

In this section I will elaborate on the method used by Brandt & Santa-Clara(2006), called the BSC-method from now on, and make this method applicable to power utility. The BSC-method considers an investor that has a certain amount of wealth to invest in a portfolio. Let me take a look at how the investor wants to allocates his money optimally by satisfying the following optimization over his future wealth:

$$\max E_t \left[\frac{1}{1-\gamma} (W_{t+1})^{1-\gamma} \right], \tag{1}$$

where W_{t+1} represents the wealth at the end of period $t+1$ and γ represent the risk aversion. The function between brackets is called the power utility. Hence, we maximize the expected utility of the investor. Let R_t^f be the gross risk-free rate at time t and r_{t+1}^p is equal to the excess portfolio return from time t to $t+1$ (Note the difference in dating: the risk-free rate for the period t to $t+1$ is determined at time t, whereas the returns for assets in the same time frame are determined at $t+1$. Also remark that I will use capitals for gross returns and small characters for excess returns throughout the rest of the paper). Given this form of notation and assuming the initial wealth to be 1, we have

$$W_{t+1} = (R_t^f + r_{t+1}^p). \tag{2}$$

Filling (2) into (1) we get

$$\max_{x_t} E_t \left[\frac{1}{1-\gamma} (R_t^f + x_t^\top r_{t+1})^{1-\gamma} \right], \tag{3}$$

where x_t represent the weights chosen at time t and r_{t+1} is the vector with excess returns on the included assets.
Now that the incentives of the investor are clear, let me continue to explain how the weights in the portfolio are constructed using the BSC-method method. The key insight of the BSC-method is parametrizing the portfolio weights and therefore usage of ‘conditional’ and ‘timing’ portfolios. Using these two concepts, this method augments the assets space with hypothetical assets.

2.1 Parametrizing portfolio weights and ‘conditional’ portfolios

As we have seen in (3), we want to maximize the expected utility for every time t by choosing x_t right. As mentioned before, the BSC-method assumes that state variables contain information about weights and thus are the weights predictable using this state variables. In other words, we can assume that the portfolio weights are linear in a vector of K state variables z_t (where the first state variable is generally a constant) and therefore parametrize the weights as:

$$x_t = \theta z_t \tag{4}$$

where θ is a $N \times K$ matrix (N is the number of risky assets). Now our maximization problem becomes:

$$\max_{\theta} E_t \left[\frac{1}{1-\gamma} (R_t^f + (\theta z_t)^\top r_{t+1})^{1-\gamma} \right] \tag{5}$$

From linear algebra I use that

$$(\theta z_t)^\top r_{t+1} = z_t^\top \theta^\top r_{t+1} = vec(\theta)(z_t \otimes r_{t+1}) \tag{6}$$

where vec(θ) piles up the columns of the matrix θ and $\otimes$ is the Kronecker product of two matrices. Now we can write:

$$\tilde{x} = vec(\theta) \tag{7}$$

$$\tilde{r}_{t+1} = z_t \otimes r_{t+1} \tag{8}$$

Our problem now has the following form:

$$\max_{\tilde{x}} E_t \left[\frac{1}{1-\gamma} (R_t^f + \tilde{x}^\top \tilde{r}_{t+1})^{1-\gamma} \right] \tag{9}$$

Because the same $\tilde{x}$ maximizes the conditional expected utility over all times t, it therefore maximizes the unconditional expected utility:

$$\max_{\tilde{x}} E \left[\frac{1}{1-\gamma} (R_t^f + \tilde{x}^\top \tilde{r}_{t+1})^{1-\gamma} \right] \tag{10}$$

which is the same formulation as the problem of finding portfolio weights $\tilde{x}$ for the expanded asset space with returns $\tilde{r}_{t+1}$. The augmented set of assets can be interpreted as managed portfolios, each of which invest in the basis assets an amount that is proportional to the value of the state variable. These terms are called 'conditional' portfolios. In section 2.4 we will see how the augmented asset space looks like after applying the concept of 'conditional' portfolios.

However, now we can only apply an one-period ahead strategy. In order to construct H-period ahead policies, I use the concept of 'timing' portfolios.

2.2 'Timing' portfolios

So far, we only considered one-period policies. In order to create H-period policies, we use the concept of 'timing' portfolios. The main idea behind 'timing' portfolios is that we construct a portfolio for every period, that invests in the risky assets in that period and in the risk-free rate in all the other periods. Taking for example a two-period strategy. The objective function is the following:

$$\max E_t \left[\frac{1}{1-\gamma} (r^p_{t\to t+2} + R^f_{t\to t+2})^{1-\gamma} \right] \tag{11}$$

where $r^p_{t\to t+2} + R^f_{t\to t+2}$ represents the sum of two-period excess return and the two-period return on the risk-free rate ($R_t^f R_{t+1}^f$). This can be written as:

$$\begin{aligned} r^p_{t\to t+2} + R^f_{t\to t+2} &= \left(R_t^f + x_t^\top r_{t+1}\right)\left(R_{t+1}^f + x_t^\top r_{t+2}\right) \\ &= x_t^\top \left(R_{t+1}^f r_{t+1}\right) + x_{t+1}^\top \left(R_t^f r_{t+2}\right) + R_t^f R_{t+1}^f + \left(x_t^\top r_{t+1}\right)\left(x_{t+1}^\top r_{t+2}\right) \end{aligned} \tag{12}$$

The first line in the expression refers to the two-period gross portfolio return. The second line is the first line rewritten and is decomposed into four terms terms. The first term can be interpreted as a mechanically managed portfolio that invests the wealth in the risky asset in the first period and in the risk-free rate in the second period. The second term represents similar portfolio, only now investing the wealth in the first period in the risk-free rate and in the second period in the risky asset. This concept of investing wealth in the risky asset the one period and investing the wealth in the risk-free rate all the other periods is what we call 'timing' portfolios. The third term is the multi-period gross return on the risk-free rate.
The last term represents the compounding term. If we compare the magnitude of this term to the first two terms, its magnitude will be like a 100th percent per year, because $\left(x_t^\top r_{t+1}\right)\left(x_{t+1}^\top r_{t+2}\right)$ is a multiplication of two excess returns. Both these excess returns have magnitude of a certain percent per year. Because its magnitude is not that large, we assume this term to be negligible.

So, a two-period 'timing' portfolios has a structure like $\tilde{r}_{t\to t+2} = \left[R_{t+1}^f r_{r+1}, R_t^f r_{r+2}\right]$. Generalizing this to a H-period 'timing' portfolio, we use

$$\tilde{r}_{t\to t+H} = \left\{ \prod_{\substack{i=0 \\ i\neq j}}^{H-1} R_{t+i}^f r_{t+j+1} \right\}_{j=0}^{H-1}, \tag{13}$$

where each term represents a portfolio that invests in the risky assets at time $t+j$ and in the risk-free rate at all other times $t+i$, with $i \neq j$. Brandt & Santa-Clara(2006) use non-overlapping periods, as estimating with overlapping periods comes with the usual statistical problems. Estimating with overlapping periods causes the error terms to be subjected to autocorrelation, resulting the standard errors to be faulty. Although these statistics are quite interesting, I will not consider them and therefore I will be able to use overlapping periods for estimation. This becomes quite handy when considering long horizons. Using non-overlapping periods, it is clear to see that the number of independent observations dramatically decreases with an increasing horizon. As generally known, if we have less observations to estimate, the precision of estimation decreases. Since I will not consider the statistics of the estimation, I will be able to use all the information available and increase the precision of the estimates as far as possible.

In order to create complete multi-period strategies combined with 'conditional' portfolios, I will replace r_{t+j+1} by $r_{t+j+1} \otimes z_{t+j}$. In this way we incorporate the concept of 'conditional' portfolios.

There are different kinds of portfolios to construct. When you predefine the the assets you will invest in and you do not change the weights of these assets during the investment period, we call this portfolio 'static'. When you do change the weights during the investment period, because new information has come available, we call the portfolio 'dynamic'. Using the concepts of 'conditional' and 'timing' portfolios, we simulate a dynamic portfolio choice problem in a static way. In a static way, because we predefine all the assets and choose the weights accordingly. During the investment period we do not change the weights in $\tilde{x}$. The actual portfolio weights do change because they are dependent on the value of the state variables.

2.3 Optimization

In this section I will explain how I come to a final solution of the problem. Because this portfolio choice problem with power utility does not have an exact, analytical solution, I will use numerical optimization. The problem will be defined as in (10). Because directly optimizing this utility function comes with problems for the optimization routines in MATLAB, I will use a Taylor expansion of this function around the expected wealth (and therefore the expected value of gross portfolio return). Hence, the k^{th}-order Taylor approximation of the utility function is defined as

$$\begin{aligned} E\left[U\left(W_{t\to t+H}\right)\right] \approx\ & U\left(E\left[W_{t\to t+H}\right]\right) \\ & + \frac{1}{2!}U''\left(E\left[W_{t\to t+H}\right]\right)E\left[\left(W_{t\to t+H}-E\left[W_{t\to t+H}\right]\right)^2\right] \\ & + \frac{1}{3!}U'''\left(E\left[W_{t\to t+H}\right]\right)E\left[\left(W_{t\to t+H}-E\left[W_{t\to t+H}\right]\right)^3\right] \\ & + \ldots \\ & + \frac{1}{k!}U^{K}\left(E\left[W_{t\to t+H}\right]\right)E\left[\left(W_{t\to t+H}-E\left[W_{t\to t+H}\right]\right)^k\right] \end{aligned} \tag{14}$$

This optimization problem has two important parameters to choose. The first parameter to fixate is the point where we will wrap the expansion around, the expected value of final wealth $E\left[W_{t\to t+H}\right]$. When difference between the final solution and this 'wrap-around'-point increases, the quality of approximation of final utility decreases. If the final solution is close to this 'wrap-around'-point , the approximation of final utility will be close to the actual utility the investor gets at the terminal consumption date. Hence, I develop a strategy to approximate the final solution as close as possible. In order to do so, I will first develop a $(H-1)$-period policy with a 'simple', arbitrary start value, for example the $(H-1)$-period average growth of the risk-free rate. The final wealth that is generated is the 'wrap-around'-point for the actual H-period problem. Because we first optimize a $(H-1)$-period problem before optimizing the H-period problem, we call this dynamic consistent.

The second parameter to choose is the order of the Taylor approximation. In general, the higher

the order of approximation, the more accurate the approximation is. However, if the order of expansion increases, so does the complexity of computations. If the chosen order is too high, I miss the goal of 'simplifying' the problem to make the MATLAB routines applicable. On the other hand, if the chosen order of expansion is too low, the final solution will not be accurate and the solution to the problem will be sub-optimal. Although we could apply dynamic consistence for the 'wrap-around'-point, here we do not have that possibility.
Brandt, Santa-Clara & Valkanov(2009) show that optimizing a k^{th}-order Taylor expansion, boils down to optimizing the first k moments of the distribution of the portfolio return. Hence, the first four moments (mean, variance, skewness and kurtosis) are important. However, considering only the first four moment, does not make the Taylor expansion accurate enough. I have tried several different orders of approximation and the sixth order seemed to be the highest order for which the MATLAB routines did not get too complex, while the results seem to be accurate.

2.4 Example of application of 'conditional' and 'timing' portfolios

Because this approach is relatively new, it is good to present some illustrative examples of the steps I take in order to augment the asset space. Consider an asset space with excess returns of a stock index and a bond index with (for simplicity) only 6 observations.

$$\begin{bmatrix} r_1^s & r_1^b \\ r_2^s & r_2^b \\ \vdots & \vdots \\ r_6^s & r_6^b \end{bmatrix}. \tag{15}$$

Now the initial asset space is clear, I will continue with defining the structure of the state variable. Suppose I adopt two state variables: a constant (which is always included) and for example the dividend-price-ratio. The matrix of the state variables looks like:

$$\begin{bmatrix} 1 & z_0^1 \\ 1 & z_1^1 \\ \vdots & \vdots \\ 1 & z_5^1 \end{bmatrix}, \tag{16}$$

where the dating reflects again the fact that z is known at the beginning of the return period. When applying an one-period strategy, the conditional asset space is constructed as in equation (8)

$$\begin{bmatrix} r_1^s & r_1^b & z_0^1 r_1^s & z_0^1 r_1^b \\ r_2^s & r_2^b & z_1^1 r_2^s & z_1^1 r_2^b \\ \vdots & \vdots & \vdots & \vdots \\ r_6^s & r_6^b & z_5^1 r_6^s & z_5^1 r_6^b \end{bmatrix}. \tag{17}$$

The resulting asset space now contains two regular assets and two 'conditional' portfolios. The interpretation of the 'conditional' portfolios is as follows: the amount invested in an asset is proportional to the value of the state variable. Hence, it is clear to see that we use the information of the state variable to forecast weights. After estimating $\tilde{x}$, the weights for the stock index are computed as $x_t^s = \tilde{x}_1 + \tilde{x}_3 z_t$. Similarly for the weights of the bond index it holds that: $x_t^b = \tilde{x}_2 + \tilde{x}_4 z_t$. These are the solutions for the one-period strategy. If we consider multi-period strategies, we need to apply the concept of 'timing' portfolios. The 'timing' portfolios are constructed using equation (13). Applying this concept to our small 2-period policy example, the 'timing' portfolios look like

$$\begin{bmatrix} r_1^s R_1^f & R_0^f r_2^s & r_1^b R_1^f & R_0^f r_2^b \\ r_2^s R_2^f & R_1^f r_3^s & r_2^b R_2^f & R_1^f r_3^b \\ \vdots & \vdots & \vdots & \vdots \\ r_5^s R_5^f & R_4^f r_6^s & r_5^b R_5^f & R_4^f r_6^b \end{bmatrix}. \tag{18}$$

From this augmented assets space it is clear to see how this concept of 'timing' portfolio works. Every row of the matrix above represents a policy for a two-period strategy, for every asset. The corresponding weights have the following interpretation: "stock in period 1", "stock in period 2", "bond in period 1" and "bond in period 2". In order to complete the augmenting process, with 'conditional' and 'timing' portfolios, we use (13) only replace r_{t+j+1} by $r_{t+j+1} \otimes z_{t+j}$. The final augmented assets space has then the following structure:

$$\tilde{r} = \begin{bmatrix} r_1^s R_1^f & R_0^f r_2^s & r_1^b R_1^f & R_0^f r_2^b & z_0^1 r_1^s R_1^f & R_0^f z_1^1 r_2^s & z_0^1 r_1^b R_1^f & R_0^f z_1^1 r_2^b \\ r_2^s R_2^f & R_1^f r_3^s & r_2^b R_2^f & R_1^f r_3^b & z_1^1 r_2^s R_2^f & R_1^f z_2^1 r_3^s & z_1^1 r_2^b R_2^f & R_1^f z_2^1 r_3^b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_5^s R_5^f & R_4^f r_6^s & r_5^b R_5^f & R_4^f r_6^b & z_4^1 r_5^s R_5^f & R_4^f z_5^1 r_6^s & z_4^1 r_5^b R_5^f & R_4^f z_5^1 r_6^b \end{bmatrix}. \tag{19}$$

The interpretation of this last assets space is like (e.g.: $z_0^1 r_1^s R_1^f$): "Stock in period 1, conditional of the level of z". The final portfolio return is calculated as:

$$r^p_{t\to t+2} + R^f_{t\to t+2} = \tilde{x}^\top \tilde{r}_{t\to t+2} + R_t^f R_{t+1}^f \tag{20}$$

This portfolio return will be used to plug into the utility function and to calculate the final utility of the investor.

Because the usage of 'conditional' and 'timing' portfolios, equation (4) yields that x_t has the following structure:

$$x_t = \begin{bmatrix} x^s_{t,t} & x^s_{t,t+1} & \dots & x^s_{t,t+H-1} & x^b_{t,t} & x^s_{t,t+1} & \dots & x^b_{t,t+H-1} \end{bmatrix}. \tag{21}$$

The notation and interpretation of this 'weight' space is as follows. For example $x^s_{t,t+1}$ yields the weight for the stock for the second period, and that second period is at time t. For convenience, I write:

$$x_{t,t} = \begin{bmatrix} x^s_{t,t} \\ x^b_{t,t} \end{bmatrix} \tag{22}$$

Now the process of augmenting the asset space is clear, I will continue discussing a refinement of this problem.

2.5 Refinement

As discussed in section 2.2, using 'timing' portfolios, I discard the presences of compounding terms. In this section I will propose a extension in order to account for these compounding terms.

From the previous section I know that one can parametrize the portfolio weights using 'conditional' and 'timing' portfolios. In order to account for the compounding terms I will calculate the actual portfolio weights in every iteration of the optimization, using that (4) which yields the 'weight' space (21). Using the notation of (22), I plug these weights into

$$\begin{aligned} r^p_{t\to t+H} + & R^f_{t\to t+H} = \\ & \left(R_t^f + x_{t,t}^\top r_{t+1}\right)\left(R_{t+1}^f + x_{t+1,t+1}^\top r_{t+2}\right)\dots\left(R_{t+H-1}^f + x_{t+H-1,t+H-1}^\top r_{t+H}\right), \end{aligned} \tag{23}$$

you get the H-period return, which accounts for compounding. Now we can optimize over θ. Because the usage of (4) and the subsequent structure as in (22), it is clear to see that this multi-period return depends on θ.

My expectation is that the difference between the model accounting for compounding and the model that does not account for compounding increases when the investment horizon becomes larger. Because when there are more investment periods, you miss more compounding terms and therefore the economic loss will be larger.

2.6 Limitations and Assumptions

Although the new proposed method and its refinement have their advantages, they do have their disadvantages. I will discuss some practical issues, their implications and solutions.

When a state variable predicts a large return for assets in the next period, the method will enlarge the weight for assets for the next period. This causes a problem when the return turns out to be quite low or even worse: negative. Suppose the portfolio return in that period becomes less than -100%, which implicates negative wealth for that investment period. The power utility produces therefore a positive utility. This is a serious problem, since we maximize the utility and normally the utility does not become larger than zero. The start of the problem lies in the fact that the portfolio return became less than -100%.
A common used solution for this problem is as follows: suppose when investing in a portfolio, you also buy an option, that when the portfolio return turns out to be less than -100%, you execute the right to sell the stock at such a price that your final wealth becomes 0,001 (assuming the wealth at the beginning of the period was 1). Remark that this option should be priced and this price should be taken in consideration. To avoid that this portfolio choice problem becomes too complex and comprehensive, I assume this option to be free. Although this assumption is not realistic, it is necessary in order to complete this research. The choice for 0,001 instead of 0,01 or 0,0001 or even an other value is quite arbitrary. However, it is clear to see that choosing 0 as payout of the option, will result in a utility of $-\infty$. Obviously this causes great inconveniences for calculation and interpretation of the results. In Appendix Tables A.1-A.3, I made an in-sample comparison between the cases of 0,01 and 0,001. This comparison shows that the choosing a different value of the pay-out, does not change the relative performances heavily in most of the cases.

3 Data

In this section I will elaborate on the data I will use in this research. To do so, I will refer to earlier research to motivate why to use the chosen data. The data is measured quarter-annually and the time-frame of the data is from the 1^{st} quarter of 1926 until the 4^{th} quarter of 2008. This gives 332 observations. First I will discuss the available asset before discussing the state variables I will use in this research.

For all the used data, the time series are presented in Appendix Figures B.1-B.6.

3.1 Assets

In this research I will use the same data as Goyal & Welch(2008) used in their research. Main difference is that I will not use the continuously compounded returns, but 'simple' returns, like Brandt & Santa-Clara(2006). Notation is as follows: capitals represent gross returns, whereas small characters represent excess returns.

Risk-free rate (R^f): This time series represents the gross returns on the risk-free rate.
Excess stock return (r^s): I use the $S\&P$ index returns from the center for Research in Security Press (CRSP) month end values. Like Brandt & Santa-Clara(2006), I will use just the excess returns and not the continuously compounded ones. The excess return is calculated as $r^s = \frac{R^s}{R^f} - 1$, where R^s represents the gross return on the stock index.
Excess bond return (r^b): I use long-term government bond returns to model the excess bond return. The excess bond returns are calculated as $r^b = \frac{R^b}{R^f} - 1$, where R^b is the gross return on the long-term government bond.

3.2 State variables

Although Van Der Noll et al.(2011) introduced 15 different state variables, I will use significantly less predictors. More importantly, the research is not about the variable selection. Using previous research, I will use the following state variables:

Dividend-Price Ratio (*DP*): The dividend-price ratio is the log of the ratio of the dividends and price. The price is the value of the S&P 500 index and the dividend is the 12 month moving sum of the dividends paid on the S&P 500 index. Calculation of this ratio is like $ln(\frac{D}{P})$, where D represents the dividend and P the price. Motivation to choose this state variable is that in the past dozens of researches (for example Campbell and Shiller (1988 & 1989), Fama and French (1988) and Cochrane (1997)) showed that the dividend-price ratio contains information about future excess stock return.

Treasury Bill rate (*TBL*):The Treasury Bill rate is the interest on a three-month Treasury bill. Note that this return is annualized and is calculated as $ln(1+TBL)$. Again, Campbell(1987) and Fama & French(1989) did research on this predictor.

Term spread (*TMS*): The term spread is difference between the log of the long term yield on government bonds and the log of Treasury-bills. It is modelled as $ln(1 + LTY) - ln(1 + TBL)$, where LTY represent the long term yield on government bonds and TBL is equal to the Treasury-bill rate. For example Campbell(1987) preformed research on the usage of this variable to predict stock returns

Now I will continue this paper by discussing the results.

4 Results

In this section I will elaborate on the results. I will use different measures in order to compare the three different models: the 'standard model' (abbreviated as *stand.*) as discussed in section 'Methods'; the 'standard model accounting for compounding' (abbreviated as *stand. comp.* or *standard compounding*), discussed in subsection 'Refinement'; and 'traditional model' (abbreviated as *trad.*). To be clear about the differences in the methods, I will give a brief description of how the 'traditional' methods works.

The basis of the 'traditional' method is an econometric model. The so-called VAR(1)-model (Vector AutoRegressive model with 1 lag) models the behaviour of the continuously compounded asset returns. In this case, the VAR-model is specified as:

$$y_{t+1} = \Phi_0 + \Phi_1 y_t + v_{t+1}. \tag{24}$$

where Φ_0 is a $(1 + N + K) \times 1$ vector of intercepts. Here, N denotes the number of risky assets and K denotes the amount of included state variables. Φ_1 is a $(1 + N + K) \times (1 + N + K)$ square matrix of slope coefficients and v_{t+1} is a $(1 + N + K) \times 1$ vector with error terms which are assumed to be homoskedastic and normally distributed:

$$v_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_v). \tag{25}$$

The vector x_{t+1} is structured as

$$y_{t+1} = \begin{bmatrix} ln(R_t^f) \\ \bar{r}_{t+1} \\ z_{t+1} \end{bmatrix}, \tag{26}$$

with

$$\bar{r}_{t+1} = \begin{bmatrix} ln(1 + r_{t+1}^s) \\ ln(1 + r_{t+1}^b) \end{bmatrix}. \tag{27}$$

The Σ_v matrix is a covariance matrix containing the covariances between all the elements in y_{t+1}. The final solution of the methods has the following structure:

$$x_{t+H-\tau}^{(\tau)} = A_0^{(\tau)} + A_1^{(\tau)} y_{t+H-\tau}, \tag{28}$$

where τ represents the number of periods remaining until the final consumption date. The precise construction of the A-matrices is described meticulously in Van Der Noll et al.(2011) and Jurek & Viceira(2010). The most important characteristic of the method is that the construction of the A-matrices is done in an analytical and recursive way. Therefore, the weights are constructed recursive as well.

Now that all three models are clear, let me continue by explaining how I will compare the results.

In order to compare the three different methods, I will vary different parameters of the models. First I look at different investment horizons. Arbitrarily, I choose the investment horizon H to be 4 (1 year), 20 (5 years) and 40 (10 years) (note that the data is quarter-annually measured). Next I will vary on the coefficient of relative risk aversion, γ. Again arbitrarily, I choose γ to be 2, 5 and 10.
Subsequently I will change the set of included state variables. The usage of these state variables will be as follows. I will use five different state variable combinations. Note that all the different combinations always adopt a constant as a state variable. The first one does not include any state variables and therefore only a constant. The next three sets of state variables each adopt one of the three state variables (referred with the abbreviation mentioned in section 'Data') and the final set of state variables includes all three state variables (referred to as '*DP, TBL & DP*').
Finally, I will also check the difference in the results when the crisis during the 1930's is excluded from the sample. Hence, I choose the start date not to be 1926Q1, but 1945Q1, 19 years later. It is possible that the crisis period influences the model estimates heavily and therefore change the weights and returns of the portfolio. This second sample, which excludes the volatile 30's, will be referred to as the 'sub-sample'.
In total, I will evaluate 90 different model set-ups across three methods. This is necessary to make a thorough comparison.

Now it is clear how the models are set up, I will continue explaining how I will measure the relative performance of the three methods. First, I will take a look at the weights. This is where portfolio construction is all about and therefore it is crucial to elaborate on them.
Next measure of comparison will be the achieved utility. Note that I skip the evaluation of the returns, purely because we are interested in the value of the utility. Hence, we optimize over the expected utility and therefore it is the most essential comparison measure.
Just as Van Der Noll et al.(2011), I will evaluate the out-of-sample (OOS) performance. This measure is important, because in practice we use out-of-sample predictions to construct a portfolio. When out-of-sample performance is quite good, the method has the power of constructing a good performing portfolio in reality.
Finally I will elaborate on the robustness of the methods. Brandt & Santa-Clara(2006) argue that their method of parametrizing the weights is a robust method, because you do not have to make any statistical assumptions on the conditional distribution of the returns.

I start with the evaluation of the weights that are produced by the three methods.

4.1 Weight evaluation

In this section I will elaborate on the weights produced by the three methods. I do this, because the weights the methods attach the the different assets, form the basis of the portfolio that is constructed. In the end, the weights determine returns and the returns determine the performance of the portfolio. To capture characteristics of the weight I will elaborate on 3 scenarios. The mean weights for all periods for the stock index for these three scenarios are graphically represented below in Figures 1-3. From these figures I can make several observations.

Figure 1: Mean weights as a percentage of wealth for the stock index when $\gamma = 10$, an investment horizon of 4 periods and *DP,TBL & TMS* included, using 1926Q1-2008Q4

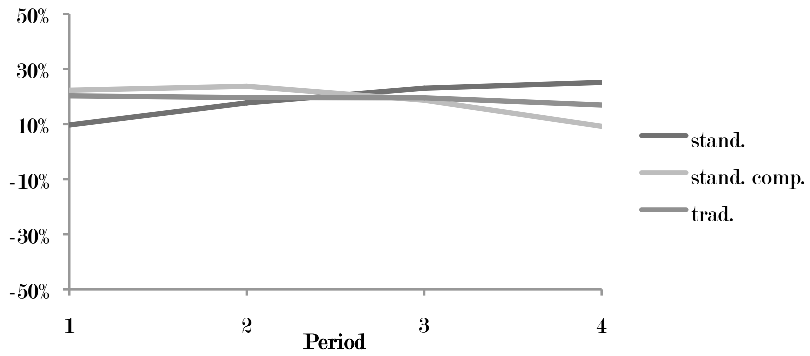


Figure 2: Mean weights as a percentage of wealth for the stock index when $\gamma = 5$, an investment horizon of 20 periods and *DP,TBL & TMS* included, using 1926Q1-2008Q4

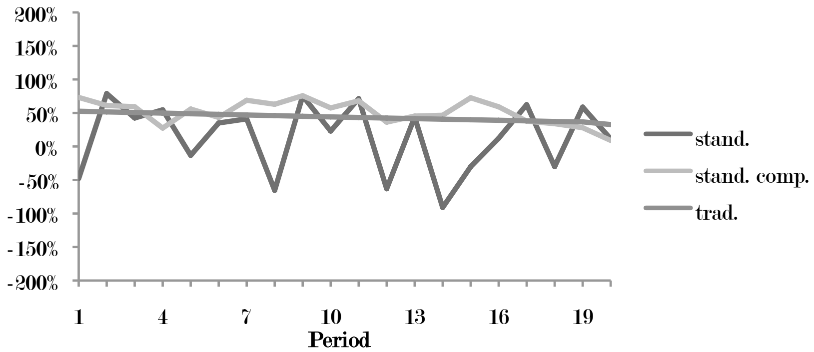
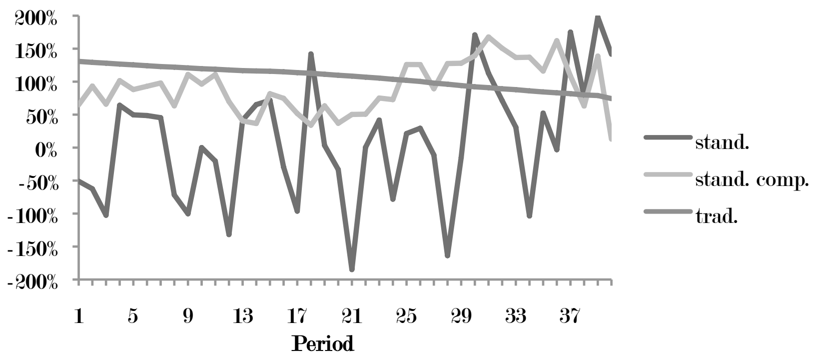


Figure 3: Mean weights as a percentage of wealth for the stock index when $\gamma = 2$, an investment horizon of 40 periods and *DP,TBL & TMS* included, using 1926Q1-2008Q4



The first observation is that the weight for both the 'standard' methods seem to be much more volatile, at least for 20- and 40-period strategies. Note that there is a slight difference in volatility between the 'standard' methods, where the 'standard compounding' method produces the less volatile weights of the two. A possible explanation for the fact that the 'traditional' method produces less volatile weights, is that the weights produced by the 'standard' methods have a direct, linear relationship with the state variables. Therefore, shocks in state variables are directly observable in the weights. The relationship between the weights and the state variables when using the 'traditional' method is not that clear, because of the usage of an econometric model an its implications. Rapidly changing weights could indicate however that the weights adapt quickly to new market information or shocks. In this way, new information is quickly adopted and can be used to forecast weights more accurately. Therefore, high volatility in weights is not a bad characteristic.

Secondly, the weights tend to get more extreme when the remaining horizon lengthens. Especially the methods that account for compounding, the 'traditional' and 'standard compounding' method, produce weights that incorporate this characteristic. Other researchers (Jurek & Viceira (2010), Van Der Noll et al.(2011)) also discovered this characteristic. Both argued that more extreme weights when the remaining horizon is longer, are driven by intertemporal hedging motives. This basically means that when the remaining horizon lengthens, you are able to hedge your risks. When the remaining horizon decreases, you have less periods available to hedge your risks and therefore the weights in assets (and with that, the risk you take) decreases.
Interesting to see is that the weights produced by the 'traditional' method *monotonically* decrease, where the weights produced by the two 'standard' methods decrease, but not *monotonically.* A possible explanation for this difference, is that the weights of the 'traditional' method depend on each other. In order to calculate the weights in the last period, you need the weights of the period before the last period. This recursion originates from the solution of the 'traditional' method, since this is an exact and recursive solution. Moreover, the weights produced by the both the 'standard' methods do not have that clear dependency.
Let w^f be the weight of the risk-free rate. We can use that

$$w^f = 1 - w^b - w^s, \tag{29}$$

as w^b is the weight of the bond index and w^s the weight for the stock index. Therefore, we would expect increasing weights for the bond index and risk-free rate as the remaining horizons decreases. This is again the incorporation of the intertemporal hedging motives. When getting closer to the terminal consumption date, you would prefer to invest less risky assets, like bonds and the risk-free rate. Mean weight series for the same parameter combinations as before are shown in the Appendinx Figures B.7-B.12. Hence, we see that in some cases the weights increase when the remaining horizon decreases. Only for the 4-period investment horizon combined with a relative risk aversion of 10, we see that the weight for the risk-free rate decreases. For all other cases, it is clear to see that the weights increase as the remaining horizon decreases, or at least not decreases.

Now that the characteristics of the weights are revealed, let me take a look what impact these weights have on the performance.

4.2 Utility evaluation

In this section I will evaluate the relative performance of the models. The evaluation of performance is done by comparing the achieved utilities. Hence, this measure is one of the important ones, since all three methods are designed to optimize over the utility at the terminal consumption date. However, just comparing utilities is difficult, because the size of the differences do not give a good interpretation. Therefore, I will use the certainty equivalent of utility (CE_U) as a measure. The certainty equivalent can be interpreted as the risk-free return an investor needs in order to obtain the same final utility. Formally, the certainty equivalent is defined as:

$$U(CE_U) = E\left[U(x)\right]. \tag{30}$$

To calculate the certainty equivalent, we simply replace the expected utility with the sample average of the utility. Applying this concept of measurement to power utility, we find that

$$\frac{1}{1-\gamma}\left(CE_U\right)^{1-\gamma} = \frac{1}{T}\sum_{t=0}^{T-H}\left(\frac{1}{1-\gamma}(W_{t+H})^{1-\gamma}\right). \tag{31}$$

4.2.1 In-sample

Throughout this section, in-sample results will be evaluated using the results presented in the Appendix Tables A.4-A.9. I start with indicating which combination of state variables performs best. This is done to keep the comparison convenient and understandable. It is done by ranking which state variable combination performs best for every coefficient of relative risk aversion γ and for every investment horizon H. The Tables 1-3 show the ranking for $\gamma = 5$ for all the different investment horizons. The tables for the other values of γ and the other sample period can be found in Appendix Tables A.10-A.15. The ranking is done based on the certainty equivalents produced when including the different combinations of state variables. When including a state variable combination that results in the highest certainty equivalent, this combination is classified as *'Best'*. When a state variable combination produces the lowest certainty equivalent, it is classified as *'Worst'*.

Table 1: Ranking table for $\gamma = 5$ and H=4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand.	DP, TBL & TMS	TMS	TBL	DP	None
stand. comp.	DP, TBL & TMS	TMS	TBL	DP	None
trad.	TMS	None	TBL	DP	DP, TBL & TMS

Table 2: Ranking table for $\gamma = 5$ and H=20

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand.	DP, TBL & TMS	DP	TMS	None	TBL
stand. comp.	DP, TBL & TMS	TBL	TMS	DP	None
trad.	TMS	None	TBL	DP, TBL & TMS	DP

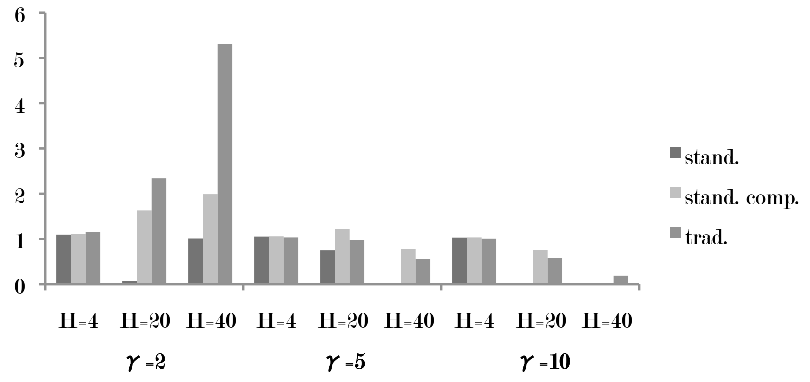
Table 3: Ranking table for $\gamma = 5$ and H=40

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand.	DP, TBL & TMS	DP	TMS	None	TBL
stand. comp.	DP, TBL & TMS	TBL	TMS	DP	None
trad.	TMS	None	TBL	DP, TBL & TMS	DP

From this ranking table we see that the state variable combinations *TMS* and *DP, TBL & TMS* dominate the performances, in the sense that when including this state variables, the portfolio performs best. *DP, TBL & TMS* and *TMS* perform as best state variable combinations 7 times and 11 times out of 27 cases respectively (3 γ's, 3 horizons and 3 methods). Looking at the sub-sample which excludes the 30's crisis, we see that these same state variable combinations dominate the other state variable combinations 12 (*DP, TBL & TMS*) and 6 times (*TMS*). Therefore, from now on I will only consider these state variable combinations.

First we take a look at the performance across different values for γ and H when *TMS* is included. The results are represented graphically below in Figure 4.

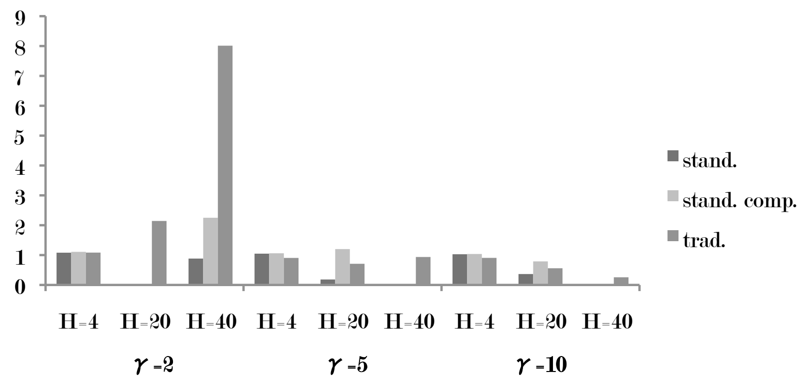
Figure 4: CE's when *TMS* is included for the 1926Q1-2008Q4



From this figure I can draw important conclusions. The drop in certainty equivalents for the 'traditional' method is relatively large compared to the drop of certainty equivalents of both the 'standard' methods when γ increases. This holds for all the different investment horizons. Although the 'traditional' method outperforms both the 'standard' methods for low values of γ, it is defeated for higher values of relative risk aversion. This is an indication that both the 'standard' methods are more suitable for investors who preferably invest with less risk involved.

Taking a look at the results when *DP, TBL & TMS* are included in the methods. The results are graphically represented in Figure 5.

Figure 5: CE's when *DP, TBL & TMS* are included for the 1926Q1-2008Q4



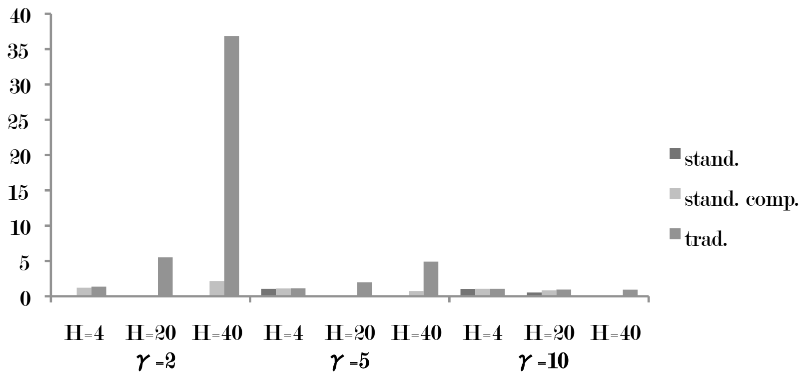
Basically, we can draw the same conclusions here. However, remark that for H=20 (combined with $\gamma = 2$) & H=40 (when γ is 5 or 10), the 'standard' method fails to produce sensible results. Its compounding counterpart fails only for H=40 when $\gamma = 5$ or $\gamma = 10$. Reasons for these results will be discussed later on.

Inspecting the performance of the other model set-ups, which include *DP*, or *TBL* or no state variables at all, we can draw basically the same conclusions as before. Data supporting this conclusion can be found in Appendix Tables A.4-A.6.

However, taking a look at the sub-sample, I conclude that the 'traditional' method outperforms both 'standard' methods 39 out of 45 different model set ups, whereas for the total sample, this number is only 20 out of 45. Looking more closely at these results, we notice that the difference in performance is due to the fact that the 'traditional' method produces extraordinary good results.

More importantly, both the 'standard' methods do not seem to decrease in performance. To illustrate the difference in performance, one should compare Figure 5 to Figure 6 which is shown below.

Figure 6: CE's when *DP, TBL & TMS* are included for the 1945Q1-2008Q4



Especially with a 40-period investment horizon and $\gamma = 2$, the 'traditional' method seems to work brilliantly. This holds for all the different combinations of state variables.

Another observation one can make Figures 4-6 and Appendix Tables A.4-A.9 is that the 'standard' method rarely outperforms it counterpart that accounts for compounding. In 45 different scenarios the 'standard compounding' method works better 40 times for the total sample. Considering the sub-sample, you will see that the 'standard' methods only works better 8 times out of 45 scenarios. Remarkable: the difference in performance increases when the investment horizon increases. This is exactly the expectation I pronounced in the section 'Refinement'. Because when the investment horizon increases you miss a larger number of compounding terms and therefore will experience more economic loss. Note that the 5 scenarios in the complete sample where the 'standard' methods produces a better certainty equivalent are exceptions, because in those cases the optimization failed to found a suitable solution for both methods. The same holds for the sub-sample.

This brings us directly to the next topic: optimization problems. I have experienced that, for some scenarios (H=20 & $\gamma = 2$ or H=40 & $\gamma = 5$ or 10), the optimization used while constructing portfolio weights according the 'standard' method did not converged to a useful solution, in the sense that the final utilities became very negative. The same problem can be determined for the 'standard compounding' method, only here this holds only for an investment horizon of 40 periods and a γ equal to 5 or 10.
That the optimization becomes tough when the investment horizon increases is not surprising. The number of hypothetical assets grows with the number of investment periods and so does the number of variables I optimize over. For example, a 40-period strategy with 2 assets and 3 states variables involves an optimization over 320 individual variables. As we only have 332 observation, we can say that there is roughly 1 observation per variable available. Hence, failure of the optimization is understandable.
Understanding why γ plays a role in the failure of the optimization, has a more technical background. Increasing the γ results in an utility function with higher order of non-linearity. In general, when the order of non-linearity increases, the process of optimization gets tougher.
Take for example $\gamma = 10$ with the 'standard' method which does not account for compounding. Having a final wealth of 0,001 (as discussed in section 'Limitations') or even smaller, this results in a utility of order $-1 \cdot 10^{26}$. Having a 40-period horizon and 1 state variable, you still need to optimize 160 variables using 332 observation. It is clear to see that the optimization does not have

the power to change the precision of the estimation of the variables with roughly 2 observations per variable and it can therefore hardly improve the negative return, given the high order of non-linearity.
The reason that the 'standard compounding' methods works slightly better, in the sense that it sometimes produces sensible results where the 'standard' method fails to do so, is that using compounding, the estimation of final wealth is much more accurate and the chance of have such a low returns as 0,001 (which used to be a negative return), is very small. Hence, the issue of very small returns is for the 'standard compounding' method much less of a problem, although it still is in some cases.

Next, I will evaluate efficiency of the methods, relative to each other.

4.2.2 Out-of-sample

In this section I will compare the out-of-sample (OOS) performance of the three methods. The OOS performance is measured as follows. I start with an initial in-sample period ($t_1 - t_{IS}$). From this period I estimate the coefficients of all three methods. Using these estimates, I am able to construct the weights at the end of time t_{IS} and use these to generate a portfolio return r^p_{I+1}. The next step is to estimate the coefficients from the period t_1 until t_{IS+1} and construct the weights for the portfolio returns r^p_{IS+2}. Hence, I use an expanding windows for estimating the model coefficients.

Because this method is quite time-consuming, I restrict the variations of the model parameters. Therefore, I choose γ to be only 5 and 10, discarding $\gamma = 2$. This is done because for $\gamma = 2$, I expect the 'traditonal' method to prevail in all cases, as it did on the in-sample performance. From section 4.2, it has become clear that the state variable combinations *TMS* and *DP, TBL & TMS* generate the best results. This is an indication that these state variables have the power of predicting asset returns. Therefore, I will only adopt these combinations of state variables when evaluating the OOS-performance of the three methods. As I have also seen in the IS-evaluation, the 40-period horizon does not seem to work for all scenarios, due to insufficient data. When considering OOS performance, the available data to estimate coefficients becomes even less and therefore I do not expect the 40-period policy produce sensible OOS results. Therefore I will only consider 4 and 20-period policies.
Also a conclusion from section 4.2 is that the results of the 'traditional' heavily change when in- or excluding the 30's crisis. For that reason I will consider two different sample periods. The first includes the 30's crisis by adopting 1926Q1-1965Q4 as in-sample and 1966Q1-1989Q4 as out-of-sample. It is easy to see that I start with a 40-year IS period and a 24-year OOS period. For the second sample, I use the same amount of available data, because both 'standard' methods seem to depend on the amount of data available. Accordingly, I choose a sample without the 30's crisis, namely 1945Q1-1984Q4 as the IS period and 1985Q1-2008Q4 as OOS period.

The results are shown in Tables 4 and 5. To check whether these results are of real value, I first check the in-sample results. This is done, because the samples differ from the samples used in section 4.2.1. Comparing the IS results from the tables below with the results from Appendix Tables A.4-A.9. Let me first compare the results for the sample with the 30's crisis included. Where I concluded previous section that the 'standard' method outperformed the 'traditional' method for the shortest horizon, here we cannot make that observation. It seems that for this shorter sample, both the method increased their performances, only the 'traditional' method improved even more. Also the 'standard compounding' method improved it performance on the shorter sample. Only where this method first outperformed the 'traditional' method on all scenarios I consider in this section, it now is outperformed on all these same scenarios. Excluding the 30's crisis, I see that all methods increased their performance on most of the scenarios. Only now the mutual differences in performances do not change dramatically. Moreover, the differences in performances between the 'standard' and 'traditional' method decreased in most of the cases. The same holds for the

Table 4: UC's and relative changes IS and OOS for 1926Q1-1989Q4

When a percentage is made bold, it means that it is 'better', in the sense that is higher than its in- or out-of-sample counterpart

			UC						% Δ *i.r.t trad.*		% Δ *i.r.t trad.*		% Δ *i.r.t stand.*	
			IS			OOS			IS	OOS	IS	OOS	IS	OOS
H	γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand.*	*stand.comp*	*stand.comp*	*stand.comp*	*stand.comp*
4	5	TMS	1,07	1,08	1,15	0,60	0,74	1,07	-6,47%	-43,43%	-6,09%	-30,99%	0,41%	22,00%
4	10	TMS	1,06	1,06	1,10	0,91	0,92	1,06	-3,17%	-14,31%	-3,16%	-13,40%	0,01%	1,07%
20	5	TMS	1,07	1,33	2,05	0,92	0,00	1,60	-47,85%	-42,32%	-35,40%	-99,89%	23,89%	-99,81%
20	10	TMS	0,62	1,12	1,55	0,79	0,70	1,35	-59,98%	-41,68%	-28,16%	-48,09%	79,51%	-10,99%
4	5	DP, TBL & TMS	1,10	1,10	1,15	0,71	0,85	0,93	-4,92%	-23,32%	-4,64%	-8,61%	0,29%	19,18%
4	10	DP, TBL & TMS	1,07	1,07	1,10	0,89	0,91	0,99	-2,07%	-9,40%	-2,05%	-7,71%	0,02%	1,87%
20	5	DP, TBL & TMS	0,99	1,49	2,28	0,33	0,00	1,08	-56,46%	-69,12%	-34,80%	-99,79%	49,74%	-99,31%
20	10	DP, TBL & TMS	0,49	1,19	1,61	0,95	0,75	1,12	-69,90%	-14,87%	-26,09%	-33,07%	145,54%	-21,38%

Table 5: UC's and relative changes IS and OOS for 1945Q1-2008Q4

When a percentage is made bold, it means that it is 'better', in the sense that is higher than its in- or out-of-sample counterpart

			UC						% Δ *i.r.t trad.*		% Δ *i.r.t trad.*		% Δ *i.r.t stand.*	
			IS			OOS			IS	OOS	IS	OOS	IS	OOS
H	γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand.*	*stand.comp*	*stand.comp*	*stand.comp*	*stand.comp*
4	5	TMS	1,11	1,09	1,10	1,05	0,82	0,91	-1,62%	-22,35%	-0,48%	-13,68%	1,15%	11,18%
20	5	TMS	1,82	1,22	1,53	1,46	0,53	0,80	-32,99%	-63,84%	-15,54%	-45,39%	26,05%	51,04%
4	10	TMS	1,07	1,06	1,07	1,04	0,97	0,99	-0,34%	-6,04%	0,29%	-4,25%	0,64%	1,91%
20	10	TMS	1,39	1,01	1,25	1,26	1,04	1,02	-27,67%	-17,61%	-10,06%	-18,86%	24,34%	-1,51%
4	5	DP, TBL & TMS	1,11	1,09	1,12	1,02	0,93	0,98	-1,92%	-8,34%	0,69%	-3,08%	2,66%	5,73%
20	5	DP, TBL & TMS	1,79	0,01	0,00	0,75	0,00	0,00	-99,39%	-99,71%	-99,99%	-99,98%	-98,67%	-93,45%
4	10	DP, TBL & TMS	1,07	1,07	1,08	1,02	0,99	1,00	-0,28%	-3,60%	1,01%	-2,43%	1,30%	1,22%
20	10	DP, TBL & TMS	1,39	0,90	1,40	0,92	0,37	0,81	-35,27%	-59,65%	0,57%	-12,58%	55,38%	116,66%

relative performances between the ‘standard compounding’ and ‘traditional’ method.

Now that for the different sub-sample the changes in performance is discussed, I continue with the out-of-sample performance. In all the considered cases, the ‘traditional’ method outperforms both the ‘standard’ as well as the ‘standard compounding’ method. Remarkable to see is that the ‘traditional’ method produces in almost all cases a certainty equivalent above 1, whereas both the ‘standard’ methods fail to do so, in all cases.

This result, that the ‘traditional’ method performs better OOS, is not surprising, considering the in-sample performance for this period. However, in the previous section, the ‘standard’ methods performed, relative to the ‘traditional’ method, better. The changes in relative performance is due to the fact that the ‘traditional’ method increased drastically.

Because out-of-sample performance is the only performance measure that counts in reality, the ‘traditional’ method is preferred. It is supreme to both the ‘standard’ methods, in all considered cases.

4.3 Robustness evaluation

In this section I will discuss the robustness of the three methods. For robustness are different definitions available. Hence, I use that a method is robust if the results it produces are consistent when varying the different parameters (investment horizon, coefficient of relative risk aversion, included state variables and samples). Observations I make and conclusions I will draw below, are all based on previous sections and their tables and figures.

I start with fixating the investment horizon H. When doing this, note that the results across the different states variables combinations reflect more or less the same characteristics. Robustness across different state variable combinations will be elaborated on later. Hence, I will check how the results change over the different values of γ.
As mentioned in the previous section, the ‘traditional’ method fluctuates the most when changing the value of relative risk aversion. This can clearly be concluded from Figures 4 & 5. Hence, you could say that the results are not consistent over values of γ and therefore both the ‘standard’ models seem to be more robust when considering different γ’s.

When fixating γ and start evaluating over the different investment horizons, we again see that the ‘traditional’ methods seems to fluctuate more in its performance than its ‘standard’ and ‘standard compounding’ counterparts. However, both the ‘standard’ methods, do have problems when the horizon increases combined with high values of relative risk aversion, due to optimization limitations. When considering all horizons, remark that both the ‘standard compounding’ and ‘traditional’ method benefit from larger investment horizons for $\gamma = 2$, however, the ‘traditional’ methods benefits most. The ‘standard’ method however produces only relatively good results for the smallest investment horizon. For $\gamma = 5$, the story is quite different. Both the ‘standard’ and ‘traditional’ method decline in certainty equivalents, whereas the ‘standard compounding’ method produces up-and-down performances (low for H=4, high for H=20 and low again for H=40). For $\gamma = 10$, I observe that all CE’s decline when the investment horizon increases, although the ‘traditional’ methods tends to deteriorate faster in performance. Concluding I would say that the ‘standard compounding’ method is the most robust method over the different investment horizons.

Now I will discuss the impact of using different state variable combinations. I will do this like I did in the previous paragraph by fixating γ and the investment horizons. Starting again with $\gamma = 2$, it is clear to see that the performance of the ‘traditional’ method changes more than the performance of the ‘standard compounding’ method. Considering $\gamma = 5$, the results of the ‘standard compounding’ method are even more consistent over the different state variable combinations. Only for H=40 the ‘standard compounding’ method seems to change almost just as heavily as the 'traditional' method. Again, note that the 'standard' method does not produce useful results for the largest investment horizon. When looking at the performance for 10 as a value for the relative risk aversion, both the 'standard' methods fail to produce usable results. Considering only the first two investment horizons, I observe that for the smallest horizon both the 'standard' methods beat the 'traditional' methods in terms of consistency across the different state variables. For the 20-period investment period, the 'standard' method again fails, whereas its compounding counterpart and the 'traditional' methods are equally as consistent for different predictors included. Hence, the 'standard compounding' method seems to be the most robust method concerning different state variables.

The last part of this robustness evaluation is the impact of different samples. Excluding the 30's crisis form the sample resulted in extraordinary results for the 'traditional' method. Both the 'standard' and 'standard compounding' method generate more or less same results compared to the full sample. This indicates that 'traditional' method heavily depends on the distribution, or at least the volatility, of the returns. This is exactly what you would expect, reminding that the 'traditional' method relies on an econometric model which assumes the returns to be distributed normally. The fact that the performance of both the 'standard' methods are independent the distribution of the returns, makes the methods quite robust. This is exactly the expectation Brandt & Santa-Clara(2006) pronounced in their research.

Finally, I conclude that the 'standard compounding' method is the most robust method I evaluate in this research. Because the 'standard' method fails in 15 out of 45 scenarios, I find it the less robust method. The 'traditional' method is not quite robust, in the sense that it does not perform as consistent as the 'standard compounding' method.

5 Conclusion

This research extends the method developed by Brandt & Santa-Clara(2006). The method I have introduced uses the same parametrization as them, has a different goal and has therefore different techniques to solve the portfolio choice problem.

Goal of this research is to examine if the method of Brandt & Santa-Clara (2006) is applicable to a power utility, whereas Brandt & Santa-Clara only used quadratic utility. This is relevant, because power utility seems to reflect the true utility functions of investors. However, where there is a simple analytical solutions for the quadratic utility available, power utility cannot be analytical optimized, unless you make some strong assumptions on the conditional return distribution of the assets. That is why I used numerical optimization to solve this portfolio choice problem.

This method is all about the mechanically managed portfolios. More specific, by using the concepts of 'conditional' and 'timing' portfolios, the initial asset space is augmented with hypothetical assets. 'Conditional' portfolios invest in the asset proportional the the value of state variables. 'Timing' portfolios invest in the assets in one single period and in the risk-free rate all other periods. Drawback of this method is that it discards compounding it choosing the optimal weights. Hence, I have developed an extension to the original method, which accounts for this compounding effect.

To evaluate how well the methods (the method I described and its extension) work, I compare them to the 'traditional' method. This method is being used widely in the industry of portfolio management. Weights constructed by this method depend on an underlying econometric model. Main drawback of this method is exactly this econometric model. In order to rely on this model, strong statistical assumptions on the conditional distribution of the asset returns need to be made. Therefore, this model is likely to mismodel the returns when they are not distributed as assumed and will produce more or less useless weights. The model is thus vulnerable for misspecification.

Recent development on this method was achieved by Jurek& Viceira(2010). They developed an analytical, recursive solution for the power utility portfolio choice problem. The great power of the method I developed is the fact that this method does not need to make any statistical assumptions on the conditional distribution of asset returns. It should therefore be more robust for misspecification. This robustness has already been confirmed by Van Der Noll et al.(2011) for small horizons (maximum 1 year).

I first discussed the weights produced by the three methods. Remarkable was the volatility of the weights produces by the newly proposed 'standard' methods. A sensible explanation is that this volatility directly follows from the linear relationship between the weights and the included state variables. Another remarkable characteristic of the weights for all methods, is that the weights for the asset index seem to get more extreme as the remaining investment horizon increases. This is exactly what Van Der Noll et al. and Jurek & Viceira found. They argued that this is a result of the presence of intertemporal hedging motives. This means that risks you take can be hedged between periods. When the remaining horizon decreases, you have less possibilities to hedge you position, so the weights, and with that the risk, get less extreme.

Next I evaluated the utility produces by the methods, as these methods are designed to produce optimal utilities, in the sense that they maximize the expected utility. First I evaluated the in-sample performance. Results show that *TMS* and *DP, TBL & TMS* were the best state variable combinations, in the sense that they generated the highest certainty equivalents. I have seen that for the lowest value of relative risk aversion, the 'traditional' method performs best. For the highest value of γ the 'standard compounding' method performs best. However, the combination of high values of relative risk aversion and large investment horizons causes the optimization to fail, in the sense that it does not generate useful results.
This failure partly has to do with the number of variables we optimize over. When including all predictors and estimating a 40-period horizon, the optimization considers 320 variables whereas there are at maximum only 332 observations available; roughly 1 observation per variable. No wonder the optimization experiences problems.
On the other hand has the optimization difficulties with the order of non-linearity. The order of non-linearity increases when the value of γ increases. In general, it holds that if the non-linearity increases, the optimization gets tougher.
Secondly, the performance of the 'traditional' method heavily depends on the which time frame of returns is used. Because when excluding the 30's crisis, this method produces extraordinary results, whereas both the 'standard' methods generate more or less similar results. This is an indication, that the assumption of a normal distribution of the returns may hold when the 30's crisis is excluded.

Secondly I considered out-of-sample performance. Since this performance measure is the most important one, because it simulates reality, in the sense that we are making future predictions. Although Van Der Noll et al.(2011) found that the 'standard' method outperformed the 'traditional' method in most of the cases, I conclude contrary in this research. In all considered cases, the out-of-sample performance of the 'traditional' is better than both the 'standard' methods. Main differences in parameters between this research and Van Der Noll et al.(2011) is the investment horizon and the included set of state variables.

Finally, I elaborated on the robustness of the methods. Although the 'traditional' method works better is some cases, the robustness is far form ideal. Starting with the fluctuating results over the different γ's, investment horizons, state variable combinations and sample periods. The 'standard' method may be a little more robust, it does not produce sensible result 1 in 3 times on average. The most robust method of all three is the 'standard compounding' method. It produces consistent results when varying the different parameters.

Finally, I conclude that neither of the three methods outperforms both the other methods. However, the ‘traditional’ method generates good results, only these results heavily depends on the parameters you choose. On the other hand, the ‘standard compounding’ method produces good results, but the performance decreases when looking at the out-of-sample performances.

6 Recommendations and Further Research

The goal is this research was to find a portfolio weights constructing method that is applicable to power utility while parametrizing the portfolio weights. Although this research was quite thorough, there are some aspects of this research that may be improved.

The first improvement of the results may be found in including the excess assets returns as state variables. It is generally known that historical returns may contain information about future returns. In this research I chose not include the excess asset returns as state variables. When including all 4 available state variables (including a constant) and 2 excess assets returns as state variables, the dimensionality of the optimization increases dramatically, making the optimization too complex and the ratio of observations per variable too low. Hence, research on possible state variables one can include in general, is a good follow-up on this research.

Another possible improvement of this research and its results can be achieved by using ‘shrinkage’. When using ‘shrinkage’, it is possible to improve the estimates and may get more accurate results. Since the optimization considers 160 or 320 variables in some cases, improvement of precision must be an achievable target.
Another possible adjustment is the restriction of extreme weights. The weights are directly dependent on the state variables, large shocks in these predictors cause the weights to get more extreme. When restricting these weights to avoid extreme position, it is possible to reduce the risk significantly. Since especially downward risk is heavily punished by the power utility, the method of parametrizing the weights may benefit from this risk-reducing approach.
It may be a good step to apply a different optimization algorithm. Although MATLAB has numerous optimization algorithms, a search to more efficient and less time consuming algorithm, should improve the results. Also tweaking the optimization characteristics (start value, number of iterations and so on) may improve the results.

The best possible improvement, is the development of a exact solution for this portfolio choice problem. With some assumptions it is already possible (as seen in Jurek & Viceira(2010)), but key to this method is *not* to make any assumptions. Since it took literally decades to find an exact solution for the ‘traditional’ method, I guess it will take a lot of time and effort to find an exact solution for the ‘standard’ method.
In the mean time, a good improvement can be found in dimensionality reduction. For example, when considering en 40-period problem, with 2 assets and 4 state variables (a constant and 3 time-varying variables), we optimize over 320 variables. During this research, I only had 332 observations available. Roughly 1 observation per variable. When applying dimensionality reduction, one should be able to increase the ratio of observations per variable.

Although, since the original paper of Brandt & Santa-Clara in 2006 the method of parametrizing portfolio weights has developed on multiple aspects (different utility function, other data), there is enough to improve on. Especially with the ‘fresh’ crisis of ‘07 until now, where assumptions do not hold in all cases, it good to expand the research on this field. Being independent of assumptions, especially in times of uncertainty, is a great good.

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A Appendix: Tables

Table A.1: UC's for all scenarios for 1926Q1-2008Q4, when the pay-out of the option is 0,01 for a 4-period investment horizon.
Differences are the differences with the results if the actual used pay-out value is 0,001.

γ	State Variable	*stand.*	Δ	*stand. comp.*	Δ	*trad.*	Δ
4	No state variables	1,07	0,00	1,08	0,00	1,09	0,00
4	DP	1,07	0,00	1,08	0,00	1,05	0,00
4	TBL	1,08	0,00	1,09	0,00	1,09	0,00
4	TMS	1,10	0,00	1,11	0,00	1,16	0,00
4	DP TBL TMS	1,08	0,00	1,11	0,00	1,08	0,00
4	No state variables	1,05	0,00	1,05	0,00	1,03	0,00
4	DP	1,04	0,00	1,05	0,00	0,94	0,00
4	TBL	1,05	0,00	1,05	0,00	1,03	0,00
4	TMS	1,05	0,00	1,06	0,00	1,03	0,00
4	DP TBL TMS	1,05	0,00	1,06	0,00	0,90	0,00
4	No state variables	1,03	0,00	1,03	0,00	1,01	0,00
4	DP	1,02	0,00	1,03	0,00	0,94	0,00
4	TBL	1,03	0,00	1,03	0,00	1,01	0,00
4	TMS	1,03	0,00	1,03	0,00	1,01	0,00
4	DP TBL TMS	1,03	0,00	1,04	0,00	0,91	0,00

Table A.2: UC's for all scenarios for 1926Q1-2008Q4, when the pay-out of the option is 0,01 for a 20-period investment horizon.
Differences are the differences with the results if the actual used pay-out value is 0,001.

γ	State Variable	*stand.*	Δ	*stand. comp.*	Δ	*trad.*	Δ
20	No state variables	1,28	0,00	1,47	0,00	1,72	0,00
20	DP	0,74	0,00	1,34	0,00	1,67	0,00
20	TBL	0,71	0,00	1,55	0,00	1,76	0,00
20	TMS	0,48	0,40	1,63	0,00	2,34	0,00
20	DP TBL TMS	0,04	0,04	0,00	0,00	2,14	0,00
20	No state variables	0,69	0,00	1,14	0,00	0,96	0,00
20	DP	0,74	0,00	1,13	0,00	0,74	0,00
20	TBL	0,82	0,00	1,21	0,00	0,95	0,00
20	TMS	0,75	0,00	1,22	0,00	0,98	0,00
20	DP TBL TMS	0,18	0,00	1,20	0,00	0,71	0,00
20	No state variables	0,02	0,01	0,68	0,00	0,57	0,00
20	DP	0,31	0,00	0,72	0,00	0,56	0,00
20	TBL	0,02	0,02	0,76	0,00	0,58	0,00
20	TMS	0,02	0,01	0,76	0,00	0,58	0,00
20	DP TBL TMS	0,37	0,00	0,79	0,00	0,56	0,00

Table A.3: UC's for all scenarios for 1926Q1-2008Q4, when the pay-out of the option is 0,01 for a 40-period investment horizon.

Differences are the differences with the results if the actual used pay-out value is 0,001.

γ	State Variable	*stand.*	Δ	*stand. comp.*	Δ	*trad.*	Δ
40	No state variables	0,62	0,50	2,14	0,00	2,92	0,00
40	DP	0,59	0,00	2,12	0,00	3,94	0,00
40	TBL	0,48	0,35	2,38	-0,03	3,18	0,00
40	TMS	1,01	0,00	1,99	0,00	5,31	0,00
40	DP TBL TMS	0,88	0,00	2,26	0,01	8,01	0,00
40	No state variables	0,03	0,02	0,60	0,00	0,50	0,00
40	DP	0,00	0,00	0,00	0,00	0,75	0,00
40	TBL	0,03	0,02	0,00	0,00	0,54	0,00
40	TMS	0,00	0,00	0,78	0,00	0,56	0,00
40	DP TBL TMS	0,00	0,00	0,00	0,00	0,94	0,00
40	No state variables	0,00	0,00	0,00	0,00	0,18	0,00
40	DP	0,00	0,00	0,00	0,00	0,23	0,00
40	TBL	0,01	0,01	0,29	-0,01	0,19	0,00
40	TMS	0,00	0,00	0,02	0,01	0,19	0,00
40	DP TBL TMS	0,00	0,00	0,00	0,00	0,26	0,00

Table A.4: Mean Wealth, mean Utility and Certainty Equivalents for 1926Q1-2008Q4 with a 4-period investment horizon

		Mean Wealth			Mean Utility			Certainty Equivalent			% Δ *i.r.t trad.*		%Δ *i.r.t. stand.*
γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*stand. comp*
2	No State Variables	1,13	1,13	1,22	-0,93	-0,92	-0,92	1,07	1,08	1,09	-1,53%	-0,78%	0,76%
2	DP	1,13	1,13	1,33	-0,93	-0,92	-0,95	1,07	1,08	1,05	2,41%	3,47%	1,03%
2	TBL	1,14	1,15	1,22	-0,93	-0,92	-0,91	1,08	1,09	1,09	-1,21%	-0,12%	1,11%
2	TMS	1,17	1,18	1,41	-0,91	-0,90	-0,86	1,10	1,11	1,16	-5,35%	-4,40%	1,00%
2	DP, TBL & TMS	1,19	1,19	1,58	-0,93	-0,90	-0,92	1,08	1,11	1,08	-0,29%	2,28%	2,58%
5	No State Variables	1,08	1,08	1,09	-0,21	-0,21	-0,22	1,05	1,05	1,03	1,53%	1,90%	0,36%
5	DP	1,08	1,08	1,11	-0,21	-0,21	-0,31	1,04	1,05	0,94	10,36%	11,15%	0,71%
5	TBL	1,08	1,08	1,10	-0,21	-0,20	-0,22	1,05	1,05	1,03	1,92%	2,48%	0,55%
5	TMS	1,10	1,10	1,15	-0,20	-0,20	-0,22	1,05	1,06	1,03	1,89%	2,35%	0,44%
5	DP, TBL & TMS	1,10	1,10	1,17	-0,21	-0,20	-0,37	1,05	1,06	0,90	15,84%	17,21%	1,19%
10	No State Variables	1,06	1,06	1,07	-0,09	-0,09	-0,10	1,03	1,03	1,01	1,58%	1,81%	0,23%
10	DP	1,06	1,06	1,07	-0,09	-0,09	-0,19	1,02	1,03	0,94	8,85%	9,46%	0,55%
10	TBL	1,06	1,06	1,07	-0,09	-0,08	-0,10	1,03	1,03	1,01	1,94%	2,35%	0,40%
10	TMS	1,07	1,07	1,09	-0,08	-0,08	-0,10	1,03	1,03	1,01	2,30%	2,65%	0,35%
10	DP, TBL & TMS	1,07	1,07	1,10	-0,09	-0,08	-0,27	1,03	1,04	0,91	13,47%	14,41%	0,83%

Table A.5: Mean Wealth, mean Utility and Certainty Equivalents for 1926Q1-2008Q4 with a 20-period investment horizon

		Mean Wealth			Mean Utility			Certainty Equivalent			% Δ *i.r.t trad.*		%Δ *i.r.t. stand.*
γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*stand. comp*
2	No State Variables	1,73	1,89	2,43	-0,78	-0,68	-0,58	1,28	1,47	1,72	-25,65%	-14,37%	15,17%
2	DP	1,44	1,97	3,13	-1,35	-0,75	-0,60	0,74	1,34	1,67	-55,67%	-20,14%	80,15%
2	TBL	2,05	1,98	2,46	-1,40	-0,65	-0,57	0,71	1,55	1,76	-59,41%	-11,97%	116,89%
2	TMS	2,21	2,15	5,74	-13,64	-0,61	-0,43	0,07	1,63	2,34	-96,87%	-30,30%	2124,59%
2	DP, TBL & TMS	0,37	0,21	9,03	-147,42	-971,19	-0,47	0,01	0,00	2,14	-99,68%	-99,95%	-84,82%
5	No State Variables	1,27	1,60	1,60	-1,13	-0,15	-0,30	0,69	1,14	0,96	-28,52%	18,37%	65,56%
5	DP	1,35	1,61	1,72	-0,83	-0,15	-0,84	0,74	1,13	0,74	0,17%	52,80%	52,55%
5	TBL	1,46	1,62	1,61	-0,55	-0,12	-0,30	0,82	1,21	0,95	-13,75%	27,16%	47,43%
5	TMS	1,39	1,72	2,11	-0,79	-0,11	-0,27	0,75	1,22	0,98	-23,33%	24,72%	62,66%
5	DP, TBL & TMS	1,41	1,68	2,45	-222,10	-0,12	-0,99	0,18	1,20	0,71	-74,15%	69,43%	555,45%
10	No State Variables	1,12	1,46	1,42	$-1{,}78 \cdot 10^{24}$	-3,69	-16,56	0,00	0,68	0,57	-99,72%	18,16%	42707,68%
10	DP	1,16	1,47	1,46	-4159,97	-2,01	-19,69	0,31	0,72	0,56	-44,83%	28,85%	133,56%
10	TBL	1,17	1,48	1,43	$-7{,}12 \cdot 10^{23}$	-1,31	-13,86	0,00	0,76	0,58	-99,70%	29,98%	43284,45%
10	TMS	1,09	1,51	1,61	$-1{,}42 \cdot 10^{24}$	-1,33	-14,16	0,00	0,76	0,58	-99,72%	30,10%	4686,41%
10	DP, TBL & TMS	1,24	1,53	1,74	-957,77	-0,94	-20,98	0,37	0,79	0,56	-34,59%	41,26%	115,97%

Table A.6: Mean Wealth, mean Utility and Certainty Equivalents for 1926Q1-2008Q4 with a 40-period investment horizon

			Mean Wealth			Mean Utility			Certainty Equivalent		% Δ *i.r.t trad.*		%Δ *i.r.t. stand.*
γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*stand. comp*
2	No State Variables	2,41	3,51	6,77	-7,77	-0,47	-0,34	0,13	2,14	2,92	-95,59%	-26,61%	1563,44%
2	DP	1,68	3,64	10,19	-1,70	-0,47	-0,25	0,59	2,12	3,94	-85,03%	-46,32%	258,66%
2	TBL	3,06	3,92	6,38	-8,27	-0,41	-0,31	0,12	2,42	3,18	-96,20%	-24,06%	198,76%
2	TMS	2,21	3,40	44,64	-0,99	-0,50	-0,19	1,01	1,99	5,31	-80,94%	-62,54%	96,55%
2	DP, TBL & TMS	2,15	3,63	118,09	-1,13	-0,44	-0,12	0,88	2,25	8,01	-88,98%	-71,92%	154,69%
5	No State Variables	3,04	2,45	2,90	$-5{,}14{\cdot}10^{9}$	-1,99	-4,11	0,00	0,60	0,50	-99,47%	19,90%	22439,38%
5	DP	0,62	0,01	3,45	$-1{,}26{\cdot}10^{12}$	$-1{,}56{\cdot}10^{22}$	-0,78	0,00	0,00	0,75	-99,91%	-100,00%	-99,70%
5	TBL	3,04	0,53	2,94	$-4{,}28{\cdot}10^{9}$	$-4{,}11{\cdot}10^{13}$	-2,92	0,00	0,00	0,54	-99,49%	-99,95%	-89,89%
5	TMS	2,78	2,80	5,28	-6,93·1010	-0,69	-2,52	0,00	0,78	0,56	-99,75%	38,09%	56133,49%
5	DP, TBL & TMS	0,31	0,13	8,26	$-4{,}74{\cdot}10^{24}$	$-4{,}51{\cdot}10^{22}$	-0,33	0,00	0,00	0,94	-100,00%	-100,00%	220,16%
10	No State Variables	5,72	2,80	2,24	$-3{,}05{\cdot}10^{24}$	$-1{,}76{\cdot}10^{25}$	$-5{,}39{\cdot}10^{5}$	0,00	0,00	0,18	-99,18%	-99,32%	-17,69%
10	DP	0,04	0,08	2,41	$-1{,}36{\cdot}10^{54}$	$-9{,}29{\cdot}10^{54}$	$-7{,}47{\cdot}10^{4}$	0,00	0,00	0,23	-100,00%	-100,00%	-19,19%
10	TBL	5,83	2,99	2,32	$-1{,}90{\cdot}10^{24}$	-5625,93	$-4{,}08{\cdot}10^{5}$	0,00	0,30	0,19	-99,16%	60,95%	18999,27%
10	TMS	5,60	3,63	2,93	$-9{,}15{\cdot}10^{24}$	$-1{,}52{\cdot}10^{24}$	$-3{,}50{\cdot}10^{5}$	0,00	0,00	0,19	-99,30%	-99,15%	-22,06%
10	DP, TBL & TMS	0,05	0,08	3,83	$-4{,}37{\cdot}10^{65}$	$-1{,}49{\cdot}10^{55}$	$-2{,}33{\cdot}10^{4}$	0,00	0,00	0,26	-100,00%	-100,00%	1355,30%

Table A.7: Mean Wealth, mean Utility and Certainty Equivalents for 1945Q1-2008Q4 with a 4-period investment horizon

			Mean Wealth			Mean Utility			Certainty Equivalent		% Δ *i.r.t trad.*		%Δ *i.r.t. stand.*
γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*stand. comp*
2	No State Variables	1,16	1,16	1,33	-0,92	-0,91	-0,84	1,09	1,10	1,19	-8,19%	-7,15%	1,14%
2	DP	1,25	1,23	1,50	-0,91	-0,88	-0,81	1,10	1,13	1,24	-11,61%	-8,55%	3,46%
2	TBL	1,26	1,22	1,34	-0,90	-0,89	-0,83	1,11	1,13	1,20	-7,87%	-5,87%	2,17%
2	TMS	1,29	1,25	1,56	-4,95	-0,87	-0,78	0,20	1,15	1,29	-84,32%	-10,97%	467,75%
2	DP, TBL & TMS	1,56	1,40	1,83	-16,77	-0,82	-0,73	0,06	1,22	1,37	-95,63%	-10,82%	1941,93%
5	No State Variables	1,10	1,09	1,14	-0,20	-0,20	-0,18	1,06	1,06	1,08	-2,27%	-1,70%	0,58%
5	DP	1,13	1,12	1,18	-0,20	-0,19	-0,17	1,06	1,08	1,10	-3,41%	-1,70%	1,77%
5	TBL	1,13	1,12	1,14	-0,19	-0,19	-0,18	1,06	1,07	1,09	-1,95%	-1,06%	0,90%
5	TMS	1,15	1,14	1,20	-0,19	-0,18	-0,17	1,07	1,08	1,11	-3,12%	-2,23%	0,92%
5	DP, TBL & TMS	1,23	1,20	1,26	-0,20	-0,16	-0,16	1,06	1,11	1,12	-5,65%	-1,21%	4,70%
10	No State Variables	1,07	1,07	1,09	-0,08	-0,08	-0,07	1,04	1,04	1,05	-0,90%	-0,57%	0,33%
10	DP	1,09	1,09	1,11	-0,08	-0,07	-0,07	1,04	1,05	1,05	-1,33%	-0,45%	0,88%
10	TBL	1,08	1,08	1,09	-0,08	-0,07	-0,07	1,04	1,05	1,05	-0,59%	-0,09%	0,50%
10	TMS	1,10	1,10	1,12	-0,07	-0,07	-0,07	1,05	1,05	1,06	-0,90%	-0,59%	0,31%
10	DP, TBL & TMS	1,13	1,12	1,14	-0,07	-0,06	-0,06	1,05	1,06	1,06	-1,59%	0,14%	1,76%

Table A.8: Mean Wealth, mean Utility and Certainty Equivalents for 1945Q1-2008Q4 with a 20-period investment horizon

			Mean Wealth			Mean Utility			Certainty Equivalent		% Δ *i.r.t trad.*		%Δ *i.r.t. stand.*
γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*stand. comp*
2	No State Variables	2,30	2,27	3,99	-0,69	-0,59	-0,38	1,44	1,69	2,63	-45%	-36%	17%
2	DP	3,33	2,49	9,25	-13,56	-0,52	-0,29	0,07	1,91	3,43	-98%	-44%	2487%
2	TBL	3,25	2,49	4,22	-34,83	-0,53	-0,36	0,03	1,87	2,77	-99%	-32%	6422%
2	TMS	2,57	2,52	11,00	-38,76	-0,51	-0,25	0,03	1,95	3,99	-99%	-51%	7452%
2	DP, TBL & TMS	0,39	0,17	28,93	-116,28	-920,09	-0,18	0,01	0,00	5,51	-100%	-100%	-87%
5	No State Variables	1,48	1,75	1,97	-0,28	-0,08	-0,08	0,97	1,33	1,34	-28%	-1%	37%
5	DP	1,75	1,85	2,59	-0,13	-0,05	-0,03	1,18	1,48	1,68	-29%	-12%	25%
5	TBL	1,76	1,83	2,00	-0,19	-0,06	-0,06	1,07	1,46	1,42	-25%	2%	36%
5	TMS	1,64	1,91	2,74	-0,34	-0,06	-0,06	0,93	1,44	1,46	-36%	-1%	55%
5	DP, TBL & TMS	0,43	0,32	3,99	$-7{,}96\cdot10^{6}$	$-1{,}79\cdot10^{14}$	-0,02	0,01	0,00	1,97	-99%	-100%	-99%
10	No State Variables	1,50	1,53	1,59	-10,70	-2,56	-7,21	0,60	0,71	0,63	-4%	12%	17%
10	DP	1,29	1,56	1,82	-23,72	-0,76	-0,34	0,55	0,81	0,88	-38%	-9%	47%
10	TBL	1,30	1,59	1,61	-283,65	-0,29	-4,50	0,42	0,90	0,66	-37%	35%	115%
10	TMS	1,45	1,60	1,86	-40,06	-1,26	-6,47	0,52	0,76	0,64	-18%	20%	47%
10	DP, TBL & TMS	1,28	1,62	2,24	-29,87	-0,54	-0,17	0,54	0,84	0,95	-44%	-12%	56%

Table A.9: Mean Wealth, mean Utility and Certainty Equivalents for 1945Q1-2008Q4 with a 40-period investment horizon

		Mean Wealth			Mean Utility			Certainty Equivalent			% Δ *i.r.t trad.*		%Δ *i.r.t. stand.*
γ	State Variables	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*trad.*	*stand.*	*stand. comp.*	*stand. comp*
2	No State Variables	2,92	3,86	18,92	-19,46	-0,40	-0,15	0,05	2,47	6,62	-99,22%	-62,65%	4713,62%
2	DP	0,03	0,12	187,68	$-3{,}45\cdot10^{5}$	$-1{,}45\cdot10^{5}$	-0,09	0,00	0,00	11,59	-100,00%	-100,00%	137,36%
2	TBL	6,77	14,79	20,36	-1,91	-313,18	-0,14	0,52	0,00	7,28	-92,81%	-99,96%	-99,39%
2	TMS	8,70	13,69	166,48	-1,08	-241,18	-0,07	0,93	0,00	14,14	-93,43%	-99,97%	-99,55%
2	DP, TBL & TMS	0,61	3,84	1276,79	-1729,20	-0,46	-0,03	0,00	2,16	36,84	-100,00%	-94,13%	373936,41%
5	No State Variables	3,63	0,02	4,35	-38,41	$-5{,}35\cdot10^{17}$	-0,41	0,28	0,00	0,89	-67,95%	-100,00%	-99,99%
5	DP	0,60	2,79	10,86	$-2{,}06\cdot10^{16}$	-0,44	0,00	0,00	0,87	3,03	-100,00%	-71,37%	1470110,54%
5	TBL	3,37	1,68	4,44	-19,19	$-8{,}22\cdot10^{9}$	-0,18	0,34	0,00	1,09	-69,06%	-99,78%	-99,30%
5	TMS	3,07	2,88	8,91	-108,47	-0,22	-0,28	0,22	1,03	0,97	-77,48%	5,98%	370,69%
5	DP, TBL & TMS	0,42	2,50	26,37	$-2{,}78\cdot10^{19}$	-0,79	0,00	0,00	0,75	4,91	-100,00%	-84,73%	7695490,82%
10	No State Variables	6,38	3,42	2,79	$-2{,}10\cdot10^{7}$	$-6{,}70\cdot10^{4}$	$-4{,}29\cdot10^{4}$	0,12	0,23	0,24	-49,74%	-4,83%	89,34%
10	DP	0,05	0,53	4,39	$-2{,}66\cdot10^{51}$	$-3{,}00\cdot10^{35}$	-2,98	0,00	0,00	0,69	-100,00%	-99,99%	5816,08%
10	TBL	6,89	2,46	2,85	$-7{,}73\cdot10^{10}$	$-4{,}12\cdot10^{24}$	$-2{,}04\cdot10^{4}$	0,05	0,00	0,26	-81,42%	-99,45%	-97,02%
10	TMS	6,68	1,90	3,87	$-1{,}84\cdot10^{7}$	$-1{,}37\cdot10^{29}$	$-3{,}00\cdot10^{24}$	0,12	0,00	0,25	-50,98%	-99,82%	-99,63%
10	DP, TBL & TMS	0,06	1,05	7,00	$-1{,}64\cdot10^{71}$	$-5{,}40\cdot10^{25}$	-0,20	0,00	0,00	0,94	-100,00%	-99,88%	11312806,41%

Table A.10: Ranking table for $\gamma = 2$ and H=4 using 1926Q1-2008Q4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand	TMS	TBL	DP, TBL & TMS	No State Variables	DP
stand. comp.	DP, TBL & TMS	TMS	TBL	DP	No State Variables
trad.	TMS	TBL	No State Variables	DP, TBL & TMS	DP

Table A.11: Ranking table for $\gamma = 2$ and H=20 using 1926Q1-2008Q4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand	No State Variables	DP	TBL	TMS	DP, TBL & TMS
stand. comp.	TMS	TBL	No State Variables	DP	DP, TBL & TMS
trad.	TMS	DP, TBL & TMS	TBL	No State Variables	DP

Table A.12: Ranking table for $\gamma = 2$ and H=40 using 1926Q1-2008Q4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand	TMS	DP, TBL & TMS	DP	No State Variables	TBL
stand. comp.	TBL	DP, TBL & TMS	No State Variables	DP	TMS
trad.	DP, TBL & TMS	TMS	DP	TBL	No State Variables

Table A.13: Ranking table for $\gamma = 10$ and H=4 using 1926Q1-2008Q4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand.	TMS	TBL	DP, TBL & TMS	No State Variables	DP
stand. comp.	DP, TBL & TMS	TMS	TBL	DP	No State Variables
trad.	No State Variables	TBL	TMS	DP	DP, TBL & TMS

Table A.14: Ranking table for $\gamma = 10$ and H=20 using 1926Q1-2008Q4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand.	DP, TBL & TMS	DP	TBL	TMS	No State Variables
stand. Comp.	DP, TBL & TMS	TBL	TMS	DP	No State Variables
trad.	TBL	TMS	No State Variables	DP	DP, TBL & TMS

Table A.15: Ranking Table for $\gamma = 10$ and H=40 using 1926Q1-2008Q4

	Best	2^{nd} *best*	3^{rd} *best*	4^{th} *best*	*Worst*
stand.	TBL	No State Variables	TMS	DP	DP, TBL & TMS
stand. Comp.	TBL	TMS	No State Variables	DP	DP, TBL & TMS
trad.	DP, TBL & TMS	DP	TMS	TBL	No State Variables

B Appendix: Figures

Figure B.1: The gross risk-free rate for 1926Q1-2008Q4

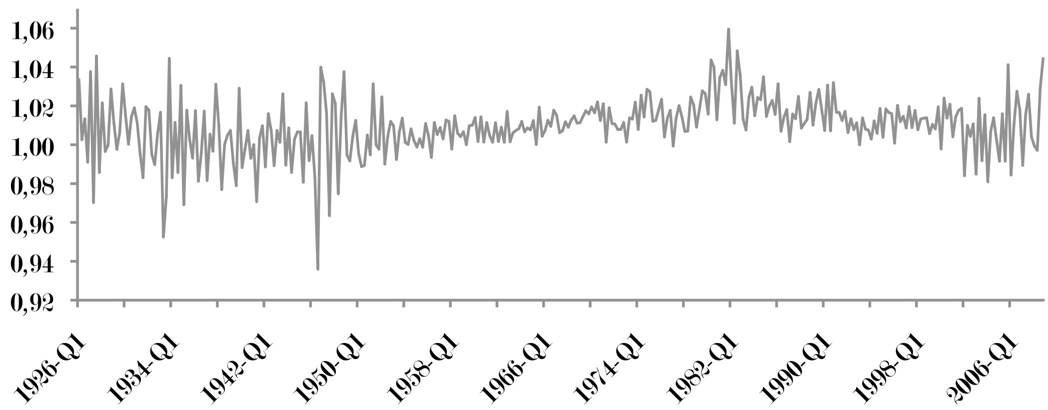

Figure B.2: Excess stock returns for 1926Q1-2008Q4

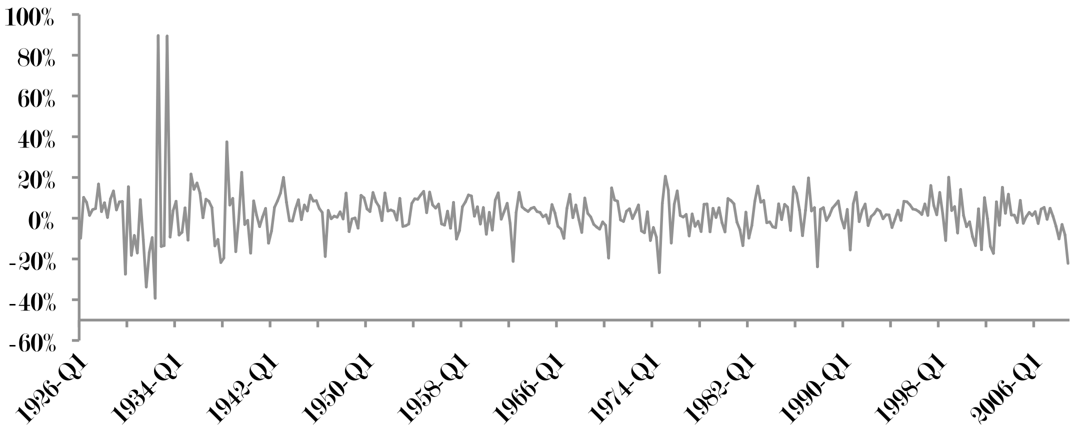

Figure B.3: Excess bond returns for 1926Q1-2008Q4

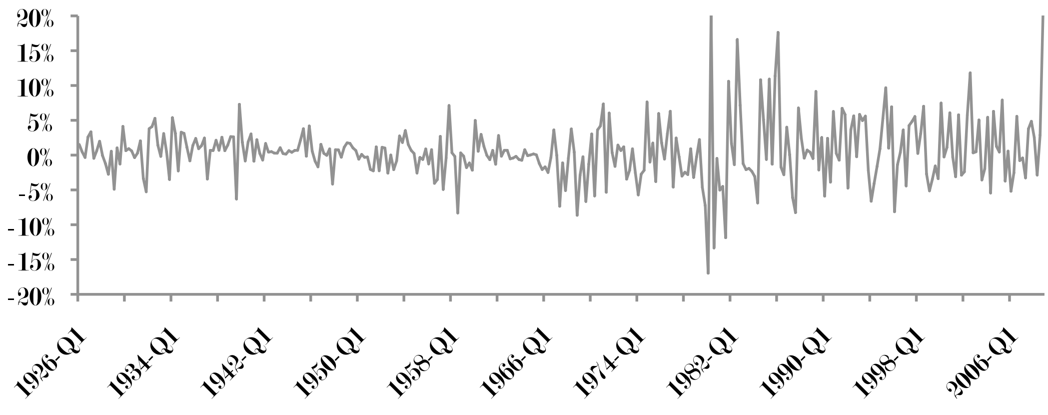

Figure B.4: The log of the dividend-price ratio for 1926Q1-2008Q4

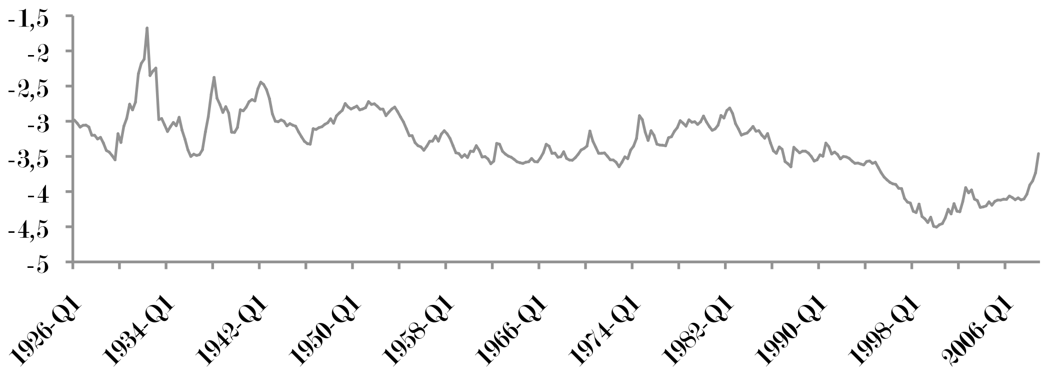

Figure B.5: The log of the Treasury Bill rate for 1926Q1-2008Q4

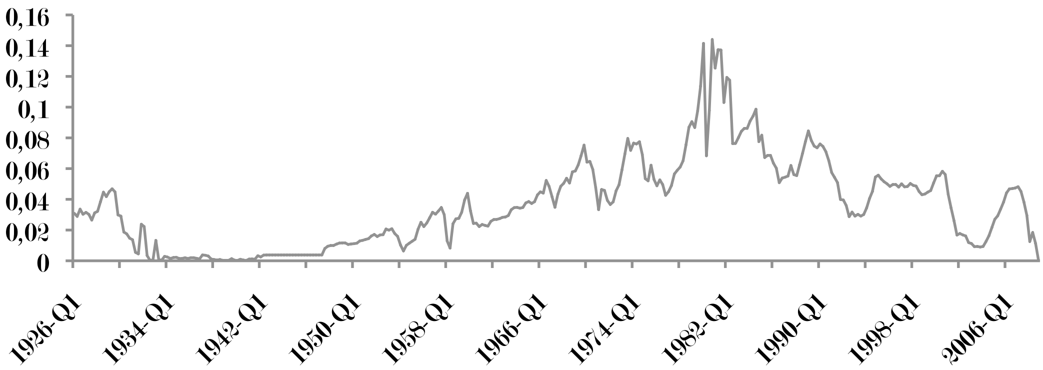

Figure B.6: The log of the Term Spread for 1926Q1-2008Q4

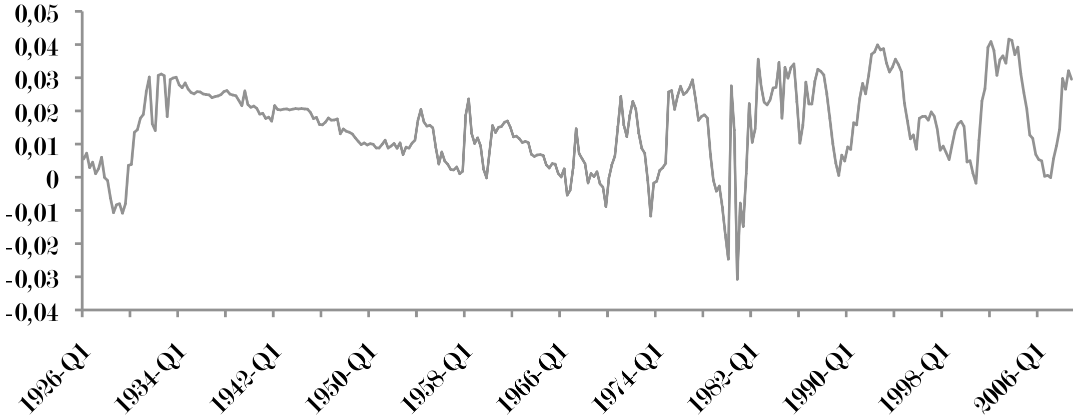

Figure B.7: Mean weights as a percentage of wealth for the bond index when $\gamma = 10$, investment horizon of 4 periods and *DP, TBL & TMS* included, using 1926Q1-2008Q4

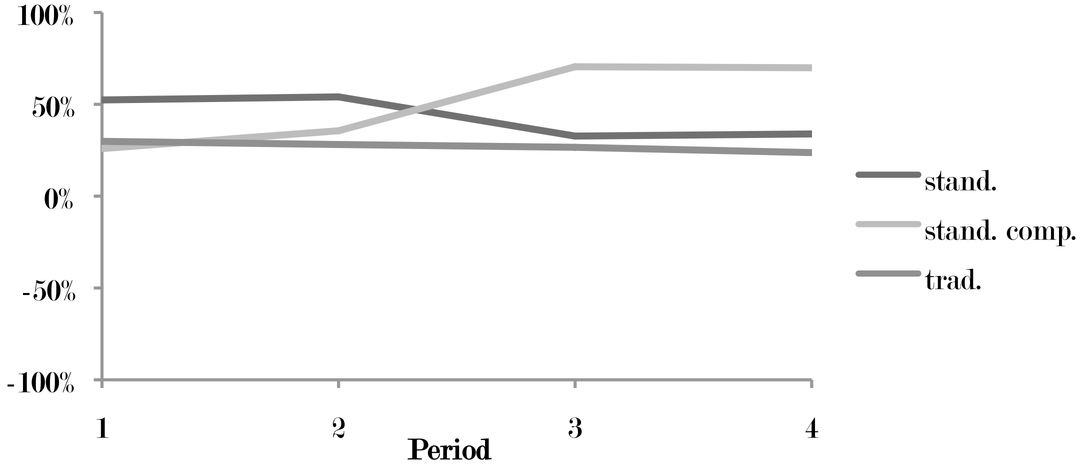

Figure B.8: Mean weights as a percentage of wealth for the risk-free rate when $\gamma = 10$, investment horizon of 4 periods and *DP, TBL & TMS* included, using 1926Q1-2008Q4

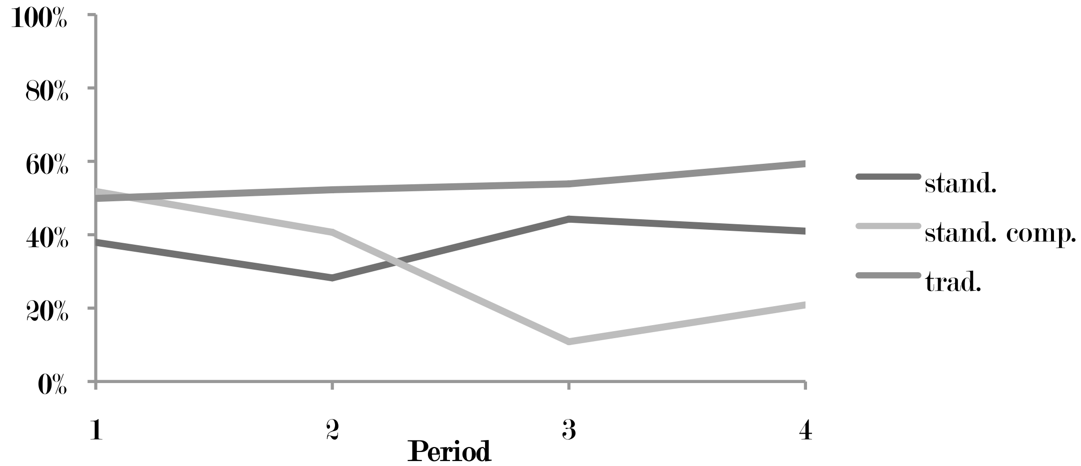

Figure B.9: Mean weights as a percentage of wealth for the bond index when $\gamma = 5$, investment horizon of 20 periods and *DP, TBL & TMS* included, using 1926Q1-2008Q4

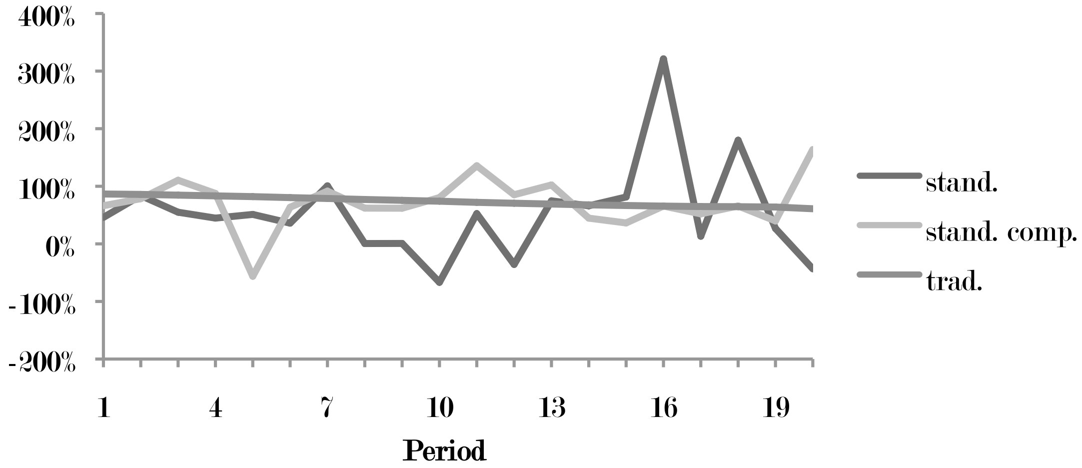

Figure B.10: Mean weights as a percentage of wealth for the risk-free rate when $\gamma = 5$, investment horizon of 20 periods and *DP, TBL & TMS* included, using 1926Q1-2008Q4

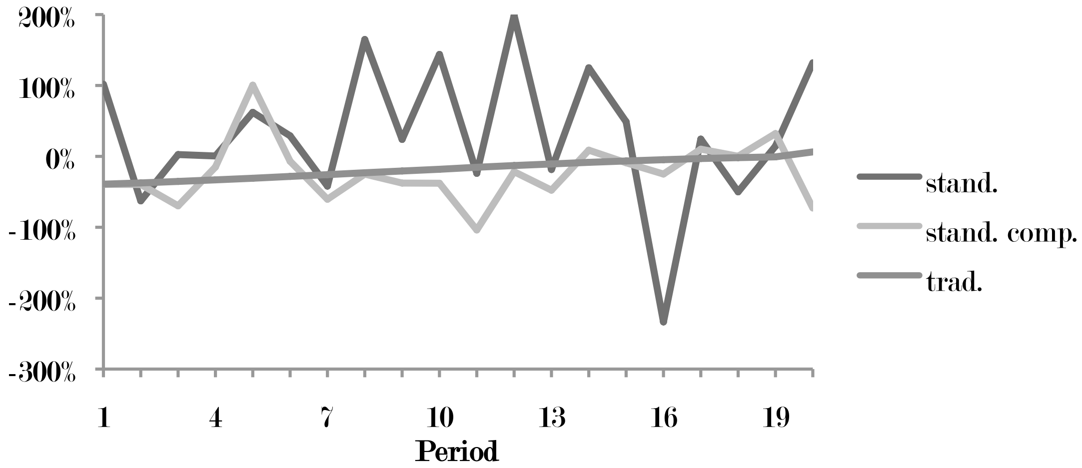

Figure B.11: Mean weights as a percentage of wealth for the bond index when $\gamma = 5$, investment horizon of 20 periods and DP,TBL & TMS included, using 1926Q1-2008Q4

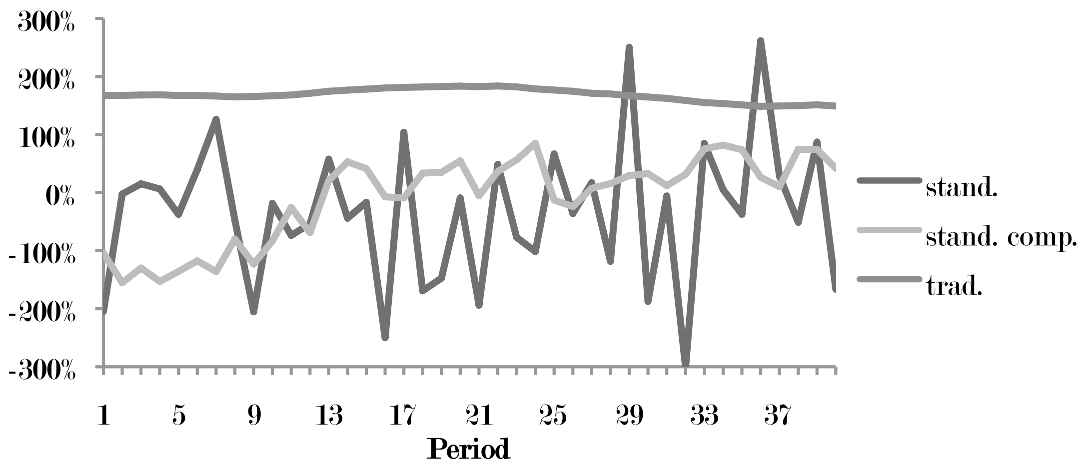

Figure B.12: Mean weights as a percentage of wealth for the risk-free rate when $\gamma = 5$, investment horizon of 40 periods and *DP,TBL & TMS* included, using 1926Q1-2008Q4