Time-Varying Volatility in the Dynamic Nelson-Siegel Model

Bram Lips (306176)
Erasmus University Rotterdam
MSc Econometrics & Management Science
Quantitative Finance
June 21, 2012

Abstract

This thesis looks into various extensions of the Dynamic Nelson-Siegel (DNS) model that allow for time-varying volatility. A common shock component with time-varying variance in the measurement equation of the state space framework greatly improves model fit. The inclusion of a second common component gives insight in how general interest rate and stock market volatility are both priced in the yield curve. The total volatility in the two component model is regressed on the VIX and a measure of general interest rate market volatility to show this for different maturities. Furthermore, various GARCH-type processes are considered to account for the time-varying volatility in the yields and in the factors of the DNS model. I study asymmetric volatility extensions and allow for influences of exogenous macroeconomic and financial factors in the GARCH equation. The alternative specifications to capture the dynamics of the common volatility turn out to improve model fit in volatile periods, but do not outperform the standard GARCH in stable times. Allowing for time-varying volatility results in predictions that significantly outperform the naive random walk forecast for short maturity yields at medium and long horizons. Parsimoniousness is key when forecasting is concerned and a model with the common shock component in the state equation produces the most accurate predictions.

Keywords: Nelson-Siegel, Time-varying volatility, GARCH, Kalman Filter, State-space model

Supervisor:	Dr. Michel van der Wel
Co-reader:	Dr. Kees Bouwman

Contents

1 Introduction **1**

2 Models and Methodology **5**
2.1 The Dynamic Nelson-Siegel Model 5
2.1.1 The DNS in State Space Framework 6
2.1.2 State Space Estimation of DNS 7
2.2 Dynamic Nelson-Siegel Model with Time-Varying Volatility 8
2.2.1 Time-Varying Volatility 9
2.2.2 The DNS-TVV in State Space Framework 12
2.2.3 State Space Estimation of DNS-TVV 14

3 Data **16**
3.1 Interest Rate Data 16
3.2 Macroeconomic and Financial Data 18

4 In-Sample Results **20**
4.1 Dynamic Nelson-Siegel Model (DNS) 21
4.2 Time-Varying Volatility (DNS-GARCH) 22
4.3 Two Common Volatility Components (DNS-2GARCH) 28
4.4 Alternative Volatility Dynamics (DNS-TGARCH and DNS-EGARCH) 33
4.5 Time-Varying Volatility in the Factors (DNS-FactorGARCH) 38
4.6 Volatility and Macroeconomic and Financial Variables (DNS-GARCHX-DRA and DNS-GARCHX-VIX) 39

5 Out-of-Sample Forecasting **45**

6 Robustness of the Results **51**
6.1 Results Based on a Subsample 51
6.2 Parameter Robustness to Initial Values 52

7 Conclusion **53**

8 Suggestions for Further Research **54**

References **56**

A Finding the Unconditional Covariance Matrix of the State Vector **59**

B Coefficients in the General State Space Form **59**

C Additional Tables and Figures **61**

1 Introduction

Modelling and forecasting the term structure of interest rates is of great importance in many areas of finance such as derivatives pricing, asset allocation and debt restructuring. Not surprisingly, a vast amount of literature is devoted to research in this part of academia in order to find optimal methods and models to fit past and predict future interest rates. In this thesis I present several extensions of the dynamic Nelson-Siegel model while allowing for time-varying volatility in interest rates. I use a state space framework and Kalman filter estimation for all models.

The literature on term structure models is generally divided in two different classes, namely between the theoretically based models and models of a statistical nature. The first class was introduced with the work of Vasicek (1977) and consists of models derived from economic theory, usually under the assumption of absence of arbitrage. Other influential contributions in this class are Cox, Ingersoll, and Ross (1985), Hull and White (1990) and Duffie and Kan (1996). However, the appealing characteristics of no-arbitrage and a sound economic foundation often come at the cost of poor fit and this class of models is therefore empirically found to be unable to beat a naive random walk forecast on interest rates, see Duffee (2002). Furthermore, estimation of these models is repeatedly found to be challenging, requiring additional restrictions that are often not well motivated statistically or theoretically, see for example Duffee (2011).

The second class of models, on the other hand, is based merely on statistical grounds and is known for its relatively good empirical fit. One of the most popular subclasses of models within the statistical class is based on the Nelson and Siegel (1987) model. The Nelson-Siegel model thanks its popularity for a large part to its relative simplicity, ease of estimation and to the fact that there is some underlying economic interpretation in the three factors it is based on, which represent level, slope and curvature of the yield curve, see De Pooter (2007). However, the downside of the statistical class of models is that they often lack theoretical support and do not assume absence of arbitrage. Several papers try to overcome this disadvantage and close the gap at least partially by investigating the interaction of the models with the macroeconomy and imposing no-arbitrage restrictions, see for example Ang and Piazzesi (2003) and Rudebusch and Wu (2008). However, Coroneo, Nyholm, and Vidova-Koleva (2011) find despite of the fact that the Nelson-Siegel yield curve model does not ensure absence of arbitrage theoretically, it is compatible with no-arbitrage constraints in the US interest rate market.

As an evolution of the Nelson-Siegel branch of term structure models, Diebold and Li (2006) introduce the Dynamic Nelson-Siegel (DNS) model by estimating the classical one with time-varying factors and model them using (V)AR specifications. Moreover, Diebold, Rudebusch, and Aruoba (2006) put the DNS model in a state space format and include macroeconomic indicators to fit and forecast the yield curve. Both papers show their forecasts outperform standard time series models and have therefore brought the focus of academics back to the Nelson-Siegel class. In order to overcome the disadvantage of no absence of arbitrage, Christensen, Diebold, and Rudebusch (2011) derive the Nelson-Siegel model under absence of the riskless arbitrage assumption and introduce the Arbitrage Free Nelson-Siegel (AFNS) model, thereby partly bridging the divide between the theoretical and statistical classes of term structure models. De Pooter (2007) discusses various other extensions to the Nelson-Siegel model as well as two different estimation approaches. He finds that a model with an extra slope factor added to the standard Nelson-Siegel model outperforms forecasts of competitor models across horizons and maturities, especially when estimated via a one-step state space approach.

Koopman, Mallee, and van der Wel (2010) point out that volatility in interest rates is assumed constant over time in most empirical papers on term structure modelling. There are only a few exceptions in the literature where models with time-varying volatility are considered, see for example Engle, Ng, and Rothschild (1990) and Bianchi, Mumtaz, and Surico (2009). Both discuss a term structure model that allows for heteroskedasticity in yields. Koopman, Mallee, and van der Wel (2010) introduce the concept of time-varying volatility to the DNS model. They use a standard GARCH specification to describe the volatility process of a common shock in the yields or the latent factors of the DNS model while adopting a state space approach. Adding a common component allows the model to capture latent exogenous shocks that affect the entire yield curve and are not captured by the three factor structure of the level, slope and curvature factors. This expansion increases the flexibility of the term structure model and enables it to better fit more complex shapes of the yield curve, as Koopman, Mallee, and van der Wel (2010) show by plotting some fitted curves. They find that allowing for time-varying volatility significantly increases the likelihood value relative to the traditional DNS model. Therefore their extended specification is a valuable addition to the literature on term structure models.

Prior research on interest rate volatility is supportive of the comments by Koopman, Mallee, and van der Wel (2010) on constant variances. Brenner, Harjes, and Kroner (1996) and Koedijk, Nissen, Schotman, and Wolff (1997) both study models for short term interest rate volatility, thereby implicitly arguing that it is not constant over time. Furthermore, Litterman, Scheinkman, and Weiss (1991) and Christiansen and Lund (2005) discuss the effect of interest rate volatility on the shape of the yield curve. The first paper examines theoretical models that highlight the link between the two. It shows how these models can be used to rationalise the shape of a zero-coupon yield curve, estimated from coupon bearing US Treasury bonds. The second paper uses a VAR model for the level, slope and curvature factors that describe the yield curve, combined with a GARCH-in-mean for the error term. Inclusion of the short rate volatility in the mean specification enables analysis of the effect of interest rate volatility on the factors shaping the term structure. Both studies stress the importance of the role of time-varying volatility in interest rates and therefore encourage further research to continue on the path where Koopman, Mallee, and van der Wel (2010) took the first steps.

This thesis expands the work done by Koopman, Mallee, and van der Wel (2010) by looking at more elaborate specifications to account for time-varying volatility in the DNS model. I do this in two main directions. First of all, I investigate whether there are different dynamics that influence volatility in the short and long ends of the yield curve. Therefore, I consider a dataset that includes maturities up to 30 years. Different factors influence rates at the short and long ends of the yield curve and this can also be the case for volatilities in both parts. I introduce a second common volatility component to account for possible diverse dynamics in short and long end volatility. Koopman, Mallee, and van der Wel (2010) use a dataset that only considers maturities up to 10 years, thereby leaving out the true long end of the yield curve. Based on this dataset they conclude that volatility patterns seem to differ in magnitude, but show similar dynamics for varying maturities. Hence they introduce a single common volatility component for all maturities to which different yields have varying sensitivities. Furthermore, Koopman, Mallee, and van der Wel (2010) look at two methods to implement the time-varying volatility in the DNS model, namely by modelling the disturbance term of the yields or the noise term in the factors via a GARCH specification. They find the first to improve the fit of the model much more than the latter. However, this finding might be different when a longer set of maturities is considered due to a more pronounced variation in importance of the three different factors in the DNS model for varying yields.

Secondly, Koopman, Mallee, and van der Wel (2010) implement their idea using a standard GARCH specification (see Bollerslev (1986)). Yet, in the literature on volatility in financial markets a wider variety of GARCH specifications is often considered to better model empirics. Therefore I look into the performance of asymmetric volatility models by replacing the standard GARCH process by GJR-GARCH (see Glosten, Jagannathan, and Runkle (1993)) and Exponential GARCH (see Nelson (1991)), following the approach by Koopman, Mallee, and van der Wel (2010). Furthermore I discuss the inclusion of exogenous macroeconomic and financial variables in the volatility equation using a GARCH-X model similar to the one presented by Brenner, Harjes, and Kroner (1996). These more sophisticated volatility processes to extend the standard DNS model can give more insight in the volatility dynamics of the common component that Koopman, Mallee, and van der Wel (2010) find to strongly improve model fit. For example, I discuss the possibly differing effects of positive and negative shocks to the yield curve, releases of new information on macroeconomic indicators and the influence of tensions in the stock market on common volatility in the term structure of interest rates.

The importance of further research on extensions to the standard Nelson-Siegel model lies in the popularity of its class of term structure models amongst practitioners at central banks and in financial markets, see for example Svensson (1995) and De Pooter (2007). Time-varying volatility provides insight in confidence intervals surrounding estimates and increased flexibility of volatility modelling can improve forecasting performance. Moreover, the existence of a common volatility component can be of great importance to for example interest rate option traders who manage risk in an entire book of interest rate volatility positions. Knowledge of a common component that determines volatilities in different parts of the yield curve allows traders to mitigate overall risk in the trading book by taking offsetting positions in different yields along the curve. Managing common interest rate volatility risk for the entire term structure can therefore be done more effectively compared to the case where individual rates or parts of the yield curve are looked at separately.

The empirical results in this thesis show that volatility in the short and long end of the yield curve is not governed by completely different dynamics. Yet, adding a second common component to the time-varying DNS model of Koopman, Mallee, and van der Wel (2010) leads to an important finding. Besides being affected by shocks to the entire term structure, yields also seem to share a common sensitivity to shocks in the stock market. In the model with two separate common shock components, different volatility dynamics are explored of which one appears to roughly represent tensions in interest rates in general and the other in the stock market. This finding implies that stock market volatility is, at least partially, priced in the term structure of interest rates.

Allowing for asymmetric response of the variance of the common component to shocks turns out to increase in-sample fit of the time-varying volatility DNS model. Volatility reacts more heavily to negative than to positive shocks in the GJR-GARCH and E-GARCH specifications. Also the extension of the standard GARCH process with macroeconomic and financial variables is useful. The exogenous factors improve the fit of the models and show that common volatility in the yield curve can be better explained using links to the macroeconomy and stock market volatility.

The encouraging findings for the alternative volatility specifications, however, do not seem to be robust to changing the sample to a more calm period with gradually declining interest rates. During times as such, the extensions of the standard GARCH process do not seem to cause notable improvements compared to the model of Koopman, Mallee, and van der Wel (2010). Yet, the addition of a second common shock component is still of large value.

Random walk forecasts turn out to be difficult to beat in the short term, as also noted by Duffee (2002). For the medium and long term the DNS models with time-varying volatility components seem to be able to significantly outperform the naive forecasting method at the short end of the yield curve. However, in the long end of the curve the random walk forecasts are relatively accurate and stay very hard to beat. Parsimoniousness turns out to be important in making predictions on future interest rates. The DNS model with a common shock component in the factors, which has the smallest number of parameters among the time-varying volatility models, performs best when forecasting is concerned.

The remainder of this thesis is structured as follows. First, chapter 2 discusses the models and methodology used in the research. Subsequently, chapter 3 describes the data used in the empirical study. Third, chapter 4 presents the in-sample results. The forecasting performance of the different models presented in this thesis is assessed in chapter 5 and chapter 6 examines the robustness of the results. Chapter 7 summarises and concludes the study. Last, chapter 8 finalises the thesis by giving some suggestions for further research.

2 Models and Methodology

In this chapter of the thesis I first discuss the Dynamic Nelson-Siegel (DNS) model as in Diebold and Li (2006), its representation in state space form and the estimation procedure in section 2.1. Thereafter, in section 2.2, I explain how time-varying volatility is incorporated using one or two common components and I describe the different GARCH specifications to capture volatility dynamics. Furthermore in the same section I discusses the class of DNS models with time-varying volatility (DNS-TVV) in state space form and the estimation method that uses the Kalman filter and maximum likelihood.

2.1 The Dynamic Nelson-Siegel Model

Following on Nelson and Siegel (1987), Diebold and Li (2006) introduce the DNS model to fit the term structure of interest rates. A set of N yields $y_t(\tau_i)$ for $i = 1, ..., N$ at time $t = 1, ..., T$, where τ_i is the time to maturity, is fitted in the DNS model given by

$$y_t(\tau_i) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda\tau_i}}{\lambda\tau_i}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda\tau_i}}{\lambda\tau_i} - e^{-\lambda\tau_i}\right) + \varepsilon_{i,t}$$

$$\varepsilon_{i,t} \sim \mathrm{N}\left(0, \sigma^2 \mathbf{I}_N\right), \tag{1}$$

where the coefficients $\beta_{j,t}$ for $j = 1, 2, 3$ are representing the factors level, slope and curvature, respectively. The constant parameter λ is the decay parameter of the factor loading of the slope of the yield curve and determines the optimum of the curvature factor loading.

Examination of the limits of the DNS model shows where the interpretations of the factors come from. When time to maturity goes to infinity, we find the infinitely long end of the curve which is given by

$$\lim_{\tau\to\infty} y_t(\tau_i) = \beta_{1,t}. \tag{2}$$

Given the fact that the first factor loading is equal to 1, the first factor gets the interpretation of the level factor. Letting time to maturity go towards zero, the infinitely short end of the curve is obtained as

$$\lim_{\tau\downarrow 0} y_t(\tau_i) = \beta_{1,t} + \beta_{2,t}, \tag{3}$$

meaning that the short rate is influenced by the first and second factor. Defining the slope of the yield curve as the long end minus the short end, it can be seen from the equations above that it is given by $-\beta_{2,t}$. The third factor loading in (1) approaches zero in both cases, when time to maturity goes to zero or infinity and is positive for intermediate values of τ. Therefore, $\beta_{3,t}$ affects the middle part of the yield curve and hence is interpreted as the curvature factor in the DNS model.

In this thesis I use the DNS model as the standard and regard it as a benchmark against which the performance of the time-varying volatility models introduced here, is measured. The theoretical foundation of the DNS serves as the basis for the extended models and the empirical results for the benchmark model are presented for comparative purposes.

2.1.1 The DNS in State Space Framework

Diebold and Li (2006) find that the time series of estimated coefficients in the DNS model show high autocorrelation, which implies they can easily be modelled and forecast using a simple framework. This finding allows the DNS model itself to be used to predict future interest rates in a two-step procedure. In this approach Diebold and Li (2006) first estimate the time series of $\beta_{1,t}, \beta_{2,t}$ and $\beta_{3,t}$ in the cross-section of yields using least squares. Subsequently, in the second step they adopt a (V)AR(1) specification to model the persistence in the time series of these three coefficients, given by

$$\boldsymbol{\beta}_{t+1} = (\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} + \boldsymbol{\Phi}\boldsymbol{\beta}_t + \boldsymbol{\nu}_t \qquad \boldsymbol{\nu}_t \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}\right), \tag{4}$$

where $\boldsymbol{\beta}_t = (\beta_{1,t}, \beta_{2,t}, \beta_{3,t})'$, $\boldsymbol{\Phi}$ is a 3×3 matrix of coefficients, $\boldsymbol{\mu}$ a 3×1 vector of constants, $\boldsymbol{\nu}_t$ is a 3×1 vector of random disturbances with constant covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\nu}}$. For the AR(1) specification $\boldsymbol{\Phi}$ is a diagonal matrix whereas it is a full matrix in case of a VAR(1). After the second step the yields can be forecast by plugging in the predictions from (4) into (1).

As Diebold and Li (2006) note, the most important stylized facts of the yield curve can be captured in the two-step framework. For example, the short end of the curve is more volatile than the long end as it depends on two factors instead of one, as can be seen from (2) and (3)[1].

Diebold, Rudebusch, and Aruoba (2006) take a different approach and put the DNS model into a state space form, treating the factors as latent variables. They use the Kalman filter to obtain estimates of the factors. In the state space approach of Diebold, Rudebusch, and Aruoba (2006) the measurement equation is given by

$$\boldsymbol{y}_t = \boldsymbol{\Lambda}(\lambda)\boldsymbol{\beta}_t + \boldsymbol{\varepsilon}_t \qquad \boldsymbol{\varepsilon}_t \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right), \tag{5}$$

where the i-th element of $\boldsymbol{y}_t$ contains the yield $y_t(\tau_i)$ for $i = 1, ..., N$ and the i-th row of $\boldsymbol{\Lambda}(\lambda)$ is given by $\boldsymbol{\Lambda}(\lambda)_i = \left[1, \left(\frac{1-e^{-\lambda\tau_i}}{\lambda\tau_i}\right), \left(\frac{1-e^{-\lambda\tau_i}}{\lambda\tau_i} - e^{-\lambda\tau_i}\right)\right]$, where τ_i is the time to maturity of yield i. The $N \times 1$ vector $\boldsymbol{\varepsilon}_t$ contains random disturbances with a constant diagonal covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}$. The state equation in Diebold, Rudebusch, and Aruoba (2006) is equal to (4). In this thesis I use a state space approach and Kalman filter estimation for all models.

[1] See the paper by Diebold and Li (2006) for a more elaborate discussion on how other stylized facts are captured in the DNS model with a (V)AR(1) specification for the coefficients.

2.1.2 State Space Estimation of DNS

Estimation via the Kalman filter is done in a single step that incorporates all uncertainty in the entire framework, coming from estimating the measurement and the state equation. In contrast, in the two-step approach of Diebold and Li (2006), the estimation uncertainty from the (V)AR(1) model is not taken into account when estimating the measurement equation. The book by Kim and Nelson (1999), which I closely follow, explains state space model estimation.

The Kalman filter consists of two steps to find a minimum mean squared error estimate of the latent factors $\boldsymbol{\beta}_t$, namely the prediction and the update step. At a given time t I form an optimal prediction of $\boldsymbol{y}_t$ based on all information available up to time $t-1$, denoted by $\boldsymbol{y}_{t|t-1}$. This prediction can be made using (5) and $\boldsymbol{\beta}_{t|t-1}$, which can be calculated using (4) and $\boldsymbol{y}_{t-1|t-1}$. After obtaining the prediction on $\boldsymbol{y}_t$, the prediction error ($\boldsymbol{\eta}_{t|t-1}$) and its variance ($\boldsymbol{F}_{t|t-1}$) can be calculated to obtain information on $\boldsymbol{\beta}_t$ that is not yet contained in $\boldsymbol{y}_{t|t-1}$. In the update step the estimate of $\boldsymbol{\beta}_t$ at time t using information up to time $t-1$ ($\boldsymbol{\beta}_{t|t-1}$) is updated by incorporating the new information from the prediction error to obtain $\boldsymbol{\beta}_{t|t}$. The estimate $\boldsymbol{\beta}_{t|t}$ contains information up to time t. The prediction step is summarised by the following four equations,

$$\boldsymbol{\beta}_{t|t-1} = (\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} + \boldsymbol{\Phi}\boldsymbol{\beta}_{t-1|t-1}, \tag{6}$$

$$\boldsymbol{P}_{t|t-1} = \boldsymbol{\Phi}\boldsymbol{P}_{t-1|t-1}\boldsymbol{\Phi}' + \boldsymbol{\Sigma}_{\boldsymbol{\nu}}, \tag{7}$$

$$\boldsymbol{\eta}_{t|t-1} = \boldsymbol{y}_t - \boldsymbol{y}_{t|t-1} = \boldsymbol{y}_t - \boldsymbol{\Lambda}(\lambda)\boldsymbol{\beta}_{t|t-1}, \tag{8}$$

$$\boldsymbol{F}_{t|t-1} = \boldsymbol{\Lambda}(\lambda)\boldsymbol{P}_{t|t-1}\boldsymbol{\Lambda}(\lambda)' + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} \tag{9}$$

and the update step is described by the two equations given as follows,

$$\boldsymbol{\beta}_{t|t} = \boldsymbol{\beta}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{\Lambda}(\lambda)'\boldsymbol{F}_{t|t-1}^{-1}\boldsymbol{\eta}_{t|t-1}, \tag{10}$$

$$\boldsymbol{P}_{t|t} = \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{\Lambda}(\lambda)'\boldsymbol{F}_{t|t-1}^{-1}\boldsymbol{\Lambda}(\lambda)\boldsymbol{P}_{t|t-1}. \tag{11}$$

Here $\boldsymbol{P}_t$ is the variance of $\boldsymbol{\beta}_t$ in the prediction and update step. The equations enable the Kalman filter to recursively estimate all latent variables for $t = 1, ..., T$[2].

In order to start the recursion, the initial value for $\boldsymbol{\beta}_t$ is set equal to the unconditional mean, $\boldsymbol{\beta}_{1|0} = \mathbb{E}[\boldsymbol{\beta}_t] = \boldsymbol{\mu}$, and the initial covariance matrix of the state vector, $\boldsymbol{P}_{1|0}$, is set equal to $\boldsymbol{\Sigma}_{\boldsymbol{\beta}}$, which is chosen such that $\boldsymbol{\Sigma}_{\boldsymbol{\beta}} - \boldsymbol{\Phi}\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\boldsymbol{\Phi}' = \boldsymbol{\Sigma}_{\boldsymbol{\nu}}$[3]. This initiation

[2] Derivation of the equations (6)-(9) is straightforward, see the book by Kim and Nelson (1999) for the derivations of (10) and (11).

[3] See Appendix A for an explanation on how to solve for $\boldsymbol{\Sigma}_{\boldsymbol{\beta}}$.

enables the Kalman filter to provide a minimum mean squared error estimate of $\boldsymbol{\beta}_t$ at every time $t = 1, ..., T$ given information up to time $t - 1$.

I obtain the latent variables using the Kalman filter, conditional on the hyperparameters of the state space framework. However, some of these hyperparameters are unknown and have to be estimated using maximum likelihood. Let all unknown parameters of the measurement and state equation now be put into $\theta = (\boldsymbol{\mu}, \boldsymbol{\Phi}, \lambda, \boldsymbol{\Sigma_{\varepsilon}}, \boldsymbol{\Sigma_{\nu}})$. Given that $\{\boldsymbol{\varepsilon}_t, \boldsymbol{\nu}_t\}_{t=1}^{T}$ are assumed to be Gaussian distributed, the distribution of $\boldsymbol{y}_t$ conditional on the information up to time $t-1$ (denoted by Ψ_{t-1}) is also Gaussian. It is then found that

$$\boldsymbol{y}_t | \Psi_{t-1} \sim \mathrm{N}\left(\boldsymbol{y}_{t|t-1}, \boldsymbol{F}_{t|t-1}\right), \tag{12}$$

and hence the log likelihood is given by

$$\ell(\theta) = -\frac{NT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^{T} \ln |\boldsymbol{F}_{t|t-1}| - \frac{1}{2} \sum_{t=1}^{T} \boldsymbol{\eta}'_{t|t-1} \boldsymbol{F}_{t|t-1}^{-1} \boldsymbol{\eta}_{t|t-1}. \tag{13}$$

Numerically optimizing the log likelihood function, $\ell(\theta)$, yields maximum likelihood estimates of the hyperparameters. The process to find the latent factors and consistent estimates of the hyperparameters is a recursive one. The procedure is started by initiating the recursion using certain starting values for the hyperparameters ($\theta^{(0)}$) that enable the Kalman filter to obtain estimates of the latent factors, conditional on the initial choice for the parameters ($\boldsymbol{\beta}_t^{(0)}$). Subsequently, given $\boldsymbol{\beta}_t^{(0)}$, the likelihood function (13) is maximised in the optimisation step to obtain new estimates of the hyperparameters, $\theta^{(1)}$, that yield a higher likelihood. These estimates are used in the Kalman filter again to obtain newly estimated latent factors, $\boldsymbol{\beta}_t^{(1)}$ and the corresponding likelihood value. The likelihood value is then maximised again by choosing θ optimally. These recursive steps in the algorithm continue until the estimates of the hyperparameters converge and I find the optimum of the likelihood function.

2.2 Dynamic Nelson-Siegel Model with Time-Varying Volatility

In this section I discuss the concept of time-varying volatility in the DNS model as introduced by Koopman, Mallee, and van der Wel (2010), who follow the common GARCH specification of Harvey, Ruiz, and Sentana (1992). First, in subsection 2.2.1, I describe the basic model in the DNS-TVV class relying on the methodology described by Koopman, Mallee, and van der Wel (2010), the inclusion of a second common shock component, alternative specifications of the GARCH process and the possibility to include the common component in the state equation. Second, I discuss the state space representation of the DNS-TVV class in subsection 2.2.2. Last, subsection 2.2.3 discusses the estimation procedure of the DNS-TVV models in a state space framework using the Kalman filter.

2.2.1 Time-Varying Volatility

To allow for time-varying volatility in the DNS model, Koopman, Mallee, and van der Wel (2010) follow Harvey, Ruiz, and Sentana (1992). They consider a state space framework like the one in subsection 2.1.2, but set $\boldsymbol{\varepsilon}_t$ as follows

$$\boldsymbol{\varepsilon}_t = \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}} \varepsilon_t^* + \boldsymbol{\varepsilon}_t^+ \qquad \boldsymbol{\varepsilon}_t^+ \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^+\right), \tag{14}$$

where $\boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\varepsilon}_t^+$ are $N \times 1$ vectors of loadings and noise components, respectively, and ε_t^* is a scalar representing the common disturbance term. In this model $\boldsymbol{\varepsilon}_t^+$ and ε_t^* are independent. The loading factor, $\boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}$, determines how sensitive the different yields are to the common shock. Koopman, Mallee, and van der Wel (2010) argue that magnitudes of volatility differ across yields and find shorter maturity yields in general to be more heavily loaded on the common shock component than longer maturity yields. The distribution of the common volatility component, ε_t^*, given the information up to time $t-1$ is

$$\varepsilon_t^* | \Psi_{t-1} \sim \mathrm{N}\left(0, h_t\right), \tag{15}$$

where h_t follows a GARCH specification as introduced by Bollerslev (1986), which is given by

$$h_t = \gamma_0 + \gamma_1 \varepsilon_{t-1}^{*2} + \gamma_2 h_{t-1} \qquad t = 2, ...T. \tag{16}$$

The GARCH specification is subject to restrictions $\gamma_0, \gamma_1, \gamma_2 > 0$ and $\gamma_1 + \gamma_2 < 1$ on its parameters in order to guarantee that h_t is positive. The variance of the common component at $t = 1$ is set equal to $h_1 = \frac{\gamma_0}{1-\gamma_1-\gamma_2}$ which is the unconditional variance. It can now easily be seen that the covariance matrix of $\boldsymbol{\varepsilon}_t$ is time-varying through h_t and given by

$$\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}(h_t) = h_t \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}} \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}' + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^+, \tag{17}$$

where $t = 1, ..., T$.

In the setting discussed above, a restriction is required to overcome identification issues and there are several possibilities. Koopman, Mallee, and van der Wel (2010) note that a normalization $\boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}} \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}' = 1$ is an option, but choose to fix γ_0 at a very small value close to zero. I choose to fix the first element of $\boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}$ at 1 as this also prevents problems and in addition to that it provides an intuitive method to distinguish two common components in case the specification in (14) is extended with a second common disturbance term (discussed later in this subsection). The actual choice for the restriction to prevent identification problems is irrelevant to the results of the analysis as the outcomes of all methods are equal up to a scaling factor.

Two Common Volatility Components

As explained in section 2.1, the long end of the yield curve in the DNS model is governed only by the level factor, whereas the short end is also affected by the slope factor. The curvature factor, in turn, mainly influences the middle part of the yield curve. Hence different factors influence interest rates in different parts of the curve. This can also be the case for volatilities in both ends of the curve as variances of the different factors may vary over time. To possibly account for this multiplicity of influences, I introduce a second common time-varying volatility component into the measurement equation (5) by adding an extra term to (14). In this setting, $\boldsymbol{\varepsilon}_t$ is decomposed as

$$\begin{gathered}\boldsymbol{\varepsilon}_t = \boldsymbol{\Gamma}_{1,\varepsilon}\varepsilon^*_{1,t} + \boldsymbol{\Gamma}_{2,\varepsilon}\varepsilon^*_{2,t} + \boldsymbol{\varepsilon}^+_t, \\ \boldsymbol{\varepsilon}^+_t \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}^+_{\boldsymbol{\varepsilon}}\right), \qquad \varepsilon^*_{i,t} \sim \mathrm{N}\left(0, h_{i,t}\right) \text{ with } i = 1, 2,\end{gathered} \tag{18}$$

where $\boldsymbol{\Gamma}_{i,\varepsilon}$ for $i = 1, 2$ and $\boldsymbol{\varepsilon}^+_t$ are $N \times 1$ loading and noise component vectors as before and $\varepsilon^*_{i,t}$ for $i = 1, 2$ are independent common disturbance scalars. All disturbance terms, $\varepsilon^*_{1,t}, \varepsilon^*_{2,t}$ and $\boldsymbol{\varepsilon}^+_t$ are mutually independent. The variances of the common components are both modelled by a separate GARCH process as in (16), subject to the same parameter restrictions as mentioned. The variance matrix of $\boldsymbol{\varepsilon}_t$ is given by

$$\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}(h_{1,t}, h_{2,t}) = h_{1,t}\boldsymbol{\Gamma}_{1,\varepsilon}\boldsymbol{\Gamma}'_{1,\varepsilon} + h_{2,t}\boldsymbol{\Gamma}_{2,\varepsilon}\boldsymbol{\Gamma}'_{2,\varepsilon} + \boldsymbol{\Sigma}^+_{\boldsymbol{\varepsilon}}, \tag{19}$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}(h_{1,t}, h_{2,t})$ varies over time, depending on the variances from the two GARCH specifications, $h_{1,t}$ and $h_{2,t}$.

As in the case of a single common volatility component, restrictions on some parameters are needed to prevent identification issues. I put the same restrictions on $\boldsymbol{\Gamma}_{1,\varepsilon}$ as in the specification in which there is only one common component, namely I set the first element of the $N \times 1$ vector equal to 1. For $\boldsymbol{\Gamma}_{2,\varepsilon}$, on the other hand, I fix the last element at 1. This way an interpretation of the two time-varying volatility components is provided intuitively as $\boldsymbol{\Gamma}_{1,\varepsilon}$ is forced to at least apply to common shocks in the shortest maturity yield and $\boldsymbol{\Gamma}_{2,\varepsilon}$ to common shocks in the long end of the curve (the 30Y yield). Similar to the case in which we have only a single common shock component, the outcomes of the model are insensitive to the restriction used to overcome identification issues. Other possibilities can be implemented, but will lead to identical results, up to a scaling factor.

Alternative Volatility Dynamics

Financial markets respond in different ways to positive and negative shocks and it is common knowledge that volatility tends to increase quickly when negative news reaches traders and investors whereas positive news usually has a much less pronounced effect. Along that line of reasoning some asymmetric volatility models were introduced in prior studies of which two important and well known examples are the threshold model GJR-GARCH (further referred to as T-GARCH) from Glosten, Jagannathan, and Runkle (1993) and Exponential GARCH (E-GARCH) introduced by Nelson (1991). These are two extensions of the popular GARCH model, as given in (16). As an alternative to the standard GARCH specification to account for time-varying volatility in the DNS model, I introduce the T-GARCH and E-GARCH specifications to model the common volatility dynamics. The T-GARCH specification to model h_t in (15) is given by

$$h_t = \gamma_0 + \gamma_1\varepsilon^{*2}_{t-1} + \psi\mathbb{I}[\varepsilon^*_t < 0]\varepsilon^{*2}_{t-1} + \gamma_2 h_{t-1}, \tag{20}$$

where $\mathbb{I}[a]$ takes the value 1 if a occurs and 0 otherwise. The parameters are restricted in a similar manner as for the standard GARCH process, meaning that $\gamma_0, \gamma_1, \gamma_2, \psi > 0$ and $\gamma_1 + \gamma_2 + \frac{1}{2}\psi < 1$. The volatility at $t = 1$ is set equal to the unconditional variance which is $\frac{\gamma_0}{1-\gamma_1-\gamma_2-\frac{1}{2}\psi}$. The alternative E-GARCH specification is

$$\ln(h_t) = \gamma_0 + \gamma_1 \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} + \psi \left(\left| \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} \right| - \mathbb{E}\left[\left| \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} \right| \right] \right) + \gamma_2 \ln(h_{t-1}), \tag{21}$$

where $\mathbb{E}\left[\left|\frac{\varepsilon^*_t}{\sqrt{h_t}}\right|\right]$ is the expectation of the absolute value of a standard normally distributed random variable, which is equal to $\sqrt{\frac{2}{\pi}}$. No restrictions on the parameters are required in the E-GARCH specification and the unconditional expectation of the log variance, $\mathbb{E}[\ln(h_t)]$, is found to be equal to $\frac{\gamma_0}{1-\gamma_2}$. The alternative specifications for variance dynamics enable the common volatility component in the DNS model to account for asymmetric response to positive and negative shocks.

Time-Varying Volatility in the Factors

As an alternative to incorporating a common volatility component in the measurement equation, Koopman, Mallee, and van der Wel (2010) and Harvey, Ruiz, and Sentana (1992) propose a method to include it in the state equation. In that way, the common volatility component does not directly influence the yields, but applies to the estimated latent factors of the DNS model and indirectly affects the estimated yields. The term $\boldsymbol{\nu}_t$ in (4) is then decomposed as

$$\boldsymbol{\nu}_t = \boldsymbol{\Gamma}_\nu \nu^*_t + \boldsymbol{\nu}^+_t \qquad \boldsymbol{\nu}^+_t \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}^+_\nu\right), \tag{22}$$

where $\boldsymbol{\Gamma}_\nu$ and $\boldsymbol{\nu}^+_t$ are 3×1 vectors of loadings and noise terms, respectively, and ν^*_t is a scalar representing the common disturbance component where $\nu^*_t|\Psi_{t-1} \sim \mathrm{N}(0, g_t)$ and g_t is given as h_t in (16). The common disturbance and the noise term vector elements are all mutually independent. In order to prevent identification issues for this DNS-TVV model specification I set γ_0 equal to 0.0001 and estimate the elements of $\boldsymbol{\Gamma}_\nu$ freely. Koopman, Mallee, and van der Wel (2010) regard the specification of a common volatility component in the state equation as a restriction compared to including it in the measurement equation.

Volatility and Macroeconomic and Financial Factors

In prior research GARCH models have been extended to also include other sources of information, coming from exogenous explanatory factors. An example is given by Brenner, Harjes, and Kroner (1996), who describe a GARCH process for short-term interest rate volatility that also depends on the level of this short rate, they name the result GARCH-X. Combining the idea of time-varying volatility in the DNS model with the result of Diebold, Rudebusch, and Aruoba (2006) who find support for extending the DNS by several exogenous variables, I introduce a GARCH-X model to describe the variance dynamics of the common volatility component in the DNS-TVV model. This extension of the GARCH model in (16) is given by

$$h_t = \gamma_0 + \gamma_1 \varepsilon^{*2}_{t-1} + \gamma_2 h_{t-1} + \boldsymbol{\phi}' \boldsymbol{z}_{t-1} \qquad t = 2, ...T, \tag{23}$$

where γ_0, γ_1 and γ_2 are the standard GARCH parameters as before, $\boldsymbol{\phi}$ is a $K \times 1$ vector of parameters and $\boldsymbol{z}_t$ a $K \times 1$ vector of K exogenous variables. The exogenous variables are included in the model with a lag of one period such that the resulting model is conditional and better applicable to forecast. The unconditional expectation of the variance in (23) is equal to $\frac{\gamma_0 + \boldsymbol{\phi}\mathbb{E}[\boldsymbol{z}_t]}{1-\gamma_1-\gamma_2}$, where $\mathbb{E}[\boldsymbol{z}_t]$ is the $K \times 1$ vector with unconditional expectations of the exogenous variables.

In this study I regard two different GARCH-X models to describe the volatility process of the common component, namely one that includes the macroeconomic variables used to extend the DNS in Diebold, Rudebusch, and Aruoba (2006) and one in which the Chicago Board Options Exchange Market Volatility Index (VIX), a measure of the S&P 500 option implied volatility, is used. The first extension enables to link tension and volatility in interest rate markets to the economy using the macroeconomic variables capacity utilization (CU), the federal funds rate (FFR) and annual price inflation (INFL). Diebold, Rudebusch, and Aruoba (2006) say these variables are often regarded as the minimum set of fundamental variables to describe the basic macroeconomic condition. Therefore they are a good start to introduce the GARCH-X concept to the DNS-TVV class of models. In order to prevent negative estimates for the volatility, the macro variables are included in the GARCH-X process as the squared values of their first differences, I call this first specification the GARCHX-DRA. This way the finding of Lee (2002), that changes in the federal funds rate target affect volatility in interest rates, can be investigated in the DNS framework as well. The second GARCH-X model, including the VIX, allows for linking stock market volatility to interest rate volatility. The VIX is often referred to as Wall Street's fear gauge and could therefore have a significant impact on the variance in the common shock component in the DNS-TVV model. From here on this second exogenous variable GARCH specification is called GARCHX-VIX.

2.2.2 The DNS-TVV in State Space Framework

Introducing time-varying volatility, as discussed in section 2.2.1, to the DNS model discussed in section 2.1 requires some adjustments to the structure of the model. The time-varying variance, h_t, in (16), (20), (21) and (23) depends on past values of the unobserved common disturbance term ε^*_t which therefore has to be treated as a latent variable. Hence ε^*_t should be included in the state vector, together with the DNS factors. In contrast to Koopman, Mallee, and van der Wel (2010), who also allow for time-varying volatility through a GARCH specification and use a non-linear state space representation of the DNS model, I take the loading parameter (λ) to be constant over time, leading to a linear state-space model. In this model the measurement equation is given by

$$\boldsymbol{y}_t = \mathbf{Z}_t(\boldsymbol{\alpha}_t) + \boldsymbol{\varepsilon}_t^+, \qquad \boldsymbol{\varepsilon}_t^+ \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}_\varepsilon^+\right), \qquad t = 1, ..., T, \tag{24}$$

where $\boldsymbol{Z}_t(\boldsymbol{\alpha}_t)$ is a $N \times 1$ vector function defined as

$$\boldsymbol{Z}_t(\boldsymbol{\alpha}_t) = \begin{bmatrix}\boldsymbol{\Lambda}(\lambda) & \boldsymbol{\Gamma}_\varepsilon\end{bmatrix} \boldsymbol{\alpha}_t = \boldsymbol{\Lambda}(\lambda)\boldsymbol{\beta}_t + \boldsymbol{\Gamma}_\varepsilon \varepsilon_t^*, \qquad \varepsilon_t^* \sim \mathrm{N}\left(0, h_t\right), \tag{25}$$

and $\boldsymbol{\alpha}_t = (\boldsymbol{\beta}_t', \varepsilon_t^*)' = (\beta_{1,t}, \beta_{2,t}, \beta_{3,t}, \varepsilon_t^*)'$ is the state vector. In this state vector, coefficients $\beta_{i,t}$ for $(i = 1, 2, 3)$ are the factors of the DNS model. Furthermore, $\boldsymbol{\Lambda}(\lambda)$ is an $N \times 3$ matrix containing the three constant factor loadings in the rows, as before. The factors of the DNS model are again modelled as a VAR(1) process, hence the state equation is given by

$$\begin{gathered}\boldsymbol{\alpha}_{t+1} = \begin{bmatrix}(\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} \\ 0\end{bmatrix} + \begin{bmatrix}\boldsymbol{\Phi} & \mathbf{0}_3 \\ \mathbf{0}_3' & 0\end{bmatrix} \boldsymbol{\alpha}_t + \begin{bmatrix}\boldsymbol{\nu}_{t+1} \\ \varepsilon_{t+1}^*\end{bmatrix}, \\ \begin{bmatrix}\boldsymbol{\nu}_{t+1} \\ \varepsilon_{t+1}^*\end{bmatrix} \sim \mathrm{N}\left(0, \begin{bmatrix}\boldsymbol{\Sigma}_\nu & 0 \\ 0 & h_{t+1}\end{bmatrix}\right), \\ t = 1, ..., T,\end{gathered} \tag{26}$$

where $\boldsymbol{\Phi}$ is a 3×3 coefficient matrix, $\boldsymbol{\mu}$ and $\mathbf{0}_3$ are 3×1 vectors of coefficients and zeros, respectively, and h_{t+1} is modelled as in (16), (20), (21) or (23). I refer to the models using the state equation in (26) as DNS-GARCH, DNS-TGARCH, DNS-EGARCH, DNS-GARCHX-DRA and DNS-GARCHX-VIX.

When two common volatility components are included, the vector of latent variables is augmented with another extra element compared to (24), so the measurement equation becomes

$$\boldsymbol{y}_t = \begin{bmatrix}\boldsymbol{\Lambda}(\lambda) & \boldsymbol{\Gamma}_{1,\varepsilon} & \boldsymbol{\Gamma}_{2,\varepsilon}\end{bmatrix} \boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_t^+, \qquad \boldsymbol{\varepsilon}_t^+ \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}_\varepsilon^+\right), \tag{27}$$

where $\boldsymbol{\alpha}_t = (\boldsymbol{\beta}_t', \varepsilon_{1,t}^*, \varepsilon_{2,t}^*)'$ and the other parameters are as explained before. The state equation in this case is given by

$$\begin{gathered}\boldsymbol{\alpha}_{t+1} = \begin{bmatrix}(\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} \\ 0 \\ 0\end{bmatrix} + \begin{bmatrix}\boldsymbol{\Phi} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3' & 0 & 0 \\ \mathbf{0}_3' & 0 & 0\end{bmatrix} \boldsymbol{\alpha}_t + \begin{bmatrix}\boldsymbol{\nu}_{t+1} \\ \varepsilon_{1,t+1}^* \\ \varepsilon_{2,t+1}^*\end{bmatrix}, \\ \begin{bmatrix}\boldsymbol{\nu}_{t+1} \\ \varepsilon_{1,t+1}^* \\ \varepsilon_{2,t+1}^*\end{bmatrix} \sim \mathrm{N}\left(0, \begin{bmatrix}\boldsymbol{\Sigma}_\nu & 0 & 0 \\ 0 & h_{1,t+1} & 0 \\ 0 & 0 & h_{2,t+1}\end{bmatrix}\right), \\ t = 1, ..., T,\end{gathered} \tag{28}$$

where $h_{1,t+1}$ and $h_{2,t+1}$ are modelled by separate GARCH processes, as in (16). I name this model DNS-2GARCH.

In case the time-varying volatility component is incorporated in the state equation, as in (22), the state space model structure is again slightly different from the two above. The vector of latent variables is now augmented by the common disturbance term for the factors of the standard DNS model. The measurement equation does not include a common disturbance term and is given by

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{\Lambda}(\lambda) & \boldsymbol{0}_N \end{bmatrix} \boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_t \qquad \boldsymbol{\varepsilon}_t \sim \mathrm{N}\left(0, \boldsymbol{\Sigma}_{\varepsilon}\right), \tag{29}$$

where $\boldsymbol{\alpha}_t = (\boldsymbol{\beta}_t', \nu_t^*)'$ and $\boldsymbol{0}_N$ is a $N \times 1$ vector of zeros. An extra coefficient matrix in the state equation is the main difference for this model compared to the previously presented two. The dynamics of the latent variables are modelled as

$$\begin{aligned} \boldsymbol{\alpha}_{t+1} = & \begin{bmatrix} (\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0}_3 \\ \boldsymbol{0}_3' & 0 \end{bmatrix} \boldsymbol{\alpha}_t + \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{\Gamma}_\nu \\ \boldsymbol{0}_3' & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_{t+1}^+ \\ \nu_{t+1}^* \end{bmatrix}, \\ & \begin{bmatrix} \boldsymbol{\nu}_{t+1}^+ \\ \nu_{t+1}^* \end{bmatrix} \sim \mathrm{N}\left(0, \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\nu}}^+ & 0 \\ 0 & h_{t+1} \end{bmatrix}\right), \\ & t = 1, ..., T, \end{aligned} \tag{30}$$

where all parameters are as defined before and h_{t+1} is again modelled by a GARCH process as in (16). This model is further referred to as DNS-FactorGARCH.

2.2.3 State Space Estimation of DNS-TVV

In this subsection the estimation procedure, based on the Kalman filter, for the DNS-TVV class of models (see section 2.2.2) is explained. In subsection 2.1.2, the steps in the Kalman filter are discussed for estimating the standard DNS model in state space form. The basics in the approach to estimate the DNS-TVV model are similar, but some adjustments need to be made. First of all, in the standard DNS model, the state vector containing the latent variables is equal to $\boldsymbol{\beta}_t$ whereas this vector is augmented in the DNS-TVV model and denoted by $\boldsymbol{\alpha}_t$. For convenience I rewrite the measurement equations (24), (27) and (29) and the state equations (26), (28) and (30) and introduce some new notation to obtain the general DNS-TVV state space form

$$\begin{aligned} \boldsymbol{y}_t &= \boldsymbol{H}\boldsymbol{\alpha}_t + \boldsymbol{\omega}_t, \\ \boldsymbol{\alpha}_{t+1} &= \boldsymbol{C} + \boldsymbol{K}\boldsymbol{\alpha}_t + \boldsymbol{G}\boldsymbol{v}_{t+1}, \\ \boldsymbol{\omega}_t \sim \mathrm{N}\left(0, \boldsymbol{R}\right), &\qquad \boldsymbol{v}_{t+1}|\Psi_t \sim \mathrm{N}\left(0, \boldsymbol{Q}_{t+1}\right), \end{aligned} \tag{31}$$

where the expressions of $\boldsymbol{\alpha}_t, \boldsymbol{H}, \boldsymbol{K}, \boldsymbol{C}, \boldsymbol{G}, \boldsymbol{v}_{t+1},$ and $\boldsymbol{Q}_{t+1}$ are given in appendix B in the case of one or two common volatility components in the yields and for inclusion of it in the state equation. The equations of the prediction step in the Kalman filter are then given by the following

$$\boldsymbol{\alpha}_{t|t-1} = \boldsymbol{C} + \boldsymbol{K}\boldsymbol{\alpha}_{t-1|t-1}, \tag{32}$$

$$\boldsymbol{P}_{t|t-1} = \boldsymbol{K}\boldsymbol{P}_{t-1|t-1}\boldsymbol{K}' + \boldsymbol{G}\boldsymbol{Q}_t\boldsymbol{G}', \tag{33}$$

$$\boldsymbol{\eta}_{t|t-1} = \boldsymbol{y}_t - \boldsymbol{H}\boldsymbol{\alpha}_{t|t-1}, \tag{34}$$

$$\boldsymbol{F}_{t|t-1} = \boldsymbol{H}\boldsymbol{P}_{t|t-1}\boldsymbol{H}' + \boldsymbol{R} \tag{35}$$

and the update step is summarised by

$$\boldsymbol{\alpha}_{t|t} = \boldsymbol{\alpha}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{H}'\boldsymbol{F}_{t|t-1}^{-1}\boldsymbol{\eta}_{t|t-1}, \tag{36}$$

$$\boldsymbol{P}_{t|t} = \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{H}'\boldsymbol{F}_{t|t-1}^{-1}\boldsymbol{H}\boldsymbol{P}_{t|t-1}. \tag{37}$$

Matrix $\boldsymbol{Q}$ contains h_{t+1} or $h_{1,t+1}$ and $h_{2,t+1}$ which are modelled by GARCH processes and rely on latent shocks at time t which are unobservable. The book by Kim and Nelson (1999) suggests taking expectations of the latent variables in (16) which gives

$$h_t = \gamma_0 + \gamma_1 \mathbb{E}[\varepsilon_{t-1}^{*2}|\Psi_{t-1}] + \gamma_2 h_{t-1} \qquad t = 2, ...T. \tag{38}$$

where $\mathbb{E}[\varepsilon_{t-1}^{*2}|\Psi_{t-1}]$ can straightforwardly be calculated as it is trivial that

$$\varepsilon_{t-1}^{*} = \mathbb{E}[\varepsilon_{t-1}^{*}|\Psi_{t-1}] + (\varepsilon_{t-1}^{*} - \mathbb{E}[\varepsilon_{t-1}^{*}|\Psi_{t-1}]) \tag{39}$$

and it can therefore easily be shown that

$$\mathbb{E}[\varepsilon_{t-1}^{*2}|\Psi_{t-1}] = \mathbb{E}[\varepsilon_{t-1}^{*}|\Psi_{t-1}]^2 + \mathbb{E}[(\varepsilon_{t-1}^{*} - \mathbb{E}[\varepsilon_{t-1}^{*}|\Psi_{t-1}])^2] \tag{40}$$

where $\mathbb{E}[\varepsilon_{t-1}^{*}|\Psi_{t-1}]$ is the last element of $\boldsymbol{\alpha}_{t-1|t-1}$ and $\mathbb{E}[(\varepsilon_{t-1}^{*} - \mathbb{E}[\varepsilon_{t-1}^{*}|\Psi_{t-1}])^2]$ is the last diagonal element of $\boldsymbol{P}_{t-1|t-1}$[4].

Starting values for $\boldsymbol{\alpha}_t$ and $\boldsymbol{P}_t$ in the Kalman filter recursion are taken to be the unconditional mean and covariance matrix, as before. In the time-varying volatility case this initiation means that $\boldsymbol{\alpha}_{0|1} = \mathbb{E}[\boldsymbol{\alpha}_t] = \boldsymbol{C}$ and

$$\boldsymbol{P}_{0|1} = \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\beta}} & \boldsymbol{0}_3 \\ \boldsymbol{0}_3' & h_1 \end{bmatrix}.$$

Now the Kalman filter is able to provide a minimum mean squared error estimate of $\boldsymbol{\alpha}_t$ for $t = 1, ..., T$ given information up to time $t-1$ and given the hyperparameters.

As discussed in subsection 2.1.2, the Kalman filter provides estimates for the latent variables and the unknown hyperparameters have to be estimated using maximum

[4]When two common components are considered, we need the last two elements of $\boldsymbol{\alpha}_{t-1|t-1}$ and of the diagonal of $\boldsymbol{P}_{t-1|t-1}$. In case the common volatility component is included in the state equation, the last elements of $\boldsymbol{\alpha}_{t-1|t-1}$ and of the diagonal of $\boldsymbol{P}_{t-1|t-1}$ contain $\mathbb{E}[\nu_{t-1}^{*}|\Psi_{t-1}]$ and $\mathbb{E}[(\nu_{t-1}^{*} - \mathbb{E}[\nu_{t}^{*}|\Psi_{t-1}])^2]$, respectively.

likelihood. Compared to the DNS model, the DNS-TVV model has some additional unknown hyperparameters and therefore we have $\theta = (\boldsymbol{\mu}, \boldsymbol{\Phi}, \lambda, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^{+}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}, \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}, \gamma_0, \gamma_1, \gamma_2)$[5]. Because $\{\boldsymbol{\varepsilon}_t^{+}, \boldsymbol{\nu}_t\}_{t=1}^{T}$ follow a Gaussian distribution, the distribution of $\boldsymbol{y}_t$ conditional on information up to time $t-1$ is again Gaussian in the DNS-TVV class of models and hence (12) again holds, hence the likelihood is also given by (13). However, $\boldsymbol{F}_{t|t-1}$ and $\boldsymbol{\eta}_{t|t-1}$ are now given by equations (34) and (35), respectively, and they depend on (32), (33), (36) and (37) which are clearly different from the prediction and update equations in subsection 2.1.2.

3 Data

This chapter describes the data used in the empirical study. Section 3.1 first discusses the interest rate data and gives summary statistics for the different yields in the sample. Secondly, section 3.2 describes the macroeconomic and financial dataset that is used for the GARCH-X models.

3.1 Interest Rate Data

For the empirical analysis in this thesis I use monthly data consisting of constant maturity yields of US government zero-coupon bonds obtained from the United States Department of Treasury, similar to Bekker and Bouwman (2009), who use daily data[6]. The dataset in this thesis consists of end-of-month yields for the period from October 1993 until December 2011 and includes maturities of 3 and 6 months and 1, 2, 3, 5, 7, 10, 20 and 30 years. In contrast to Bekker and Bouwman I leave out the 1 month maturity due to its high sensitivity to the federal funds rate and its poor availability (it is only available from July 2001 onwards).

Except for the 30 year yield, all series are available over the entire sample period. The 30 year yield series starts years before the beginning of the sample, but auctioning of this Treasury bond was ceased in February 2002 and reintroduced in February 2006. During this period of discontinuity of the series, the US Treasury Department published extrapolation factors to derive 30 year yield estimates, which I use to find a substitute for the non-existing data. The extrapolation factors are calculated by determining the slope of the long end of the yield curve and extrapolating it to the 30Y maturity[7]. For the period after the reintroduction of the 30 year bonds I again use the regular series of constant maturity yields.

Figure I presents a plot of the cross section of yields over the sample period. The

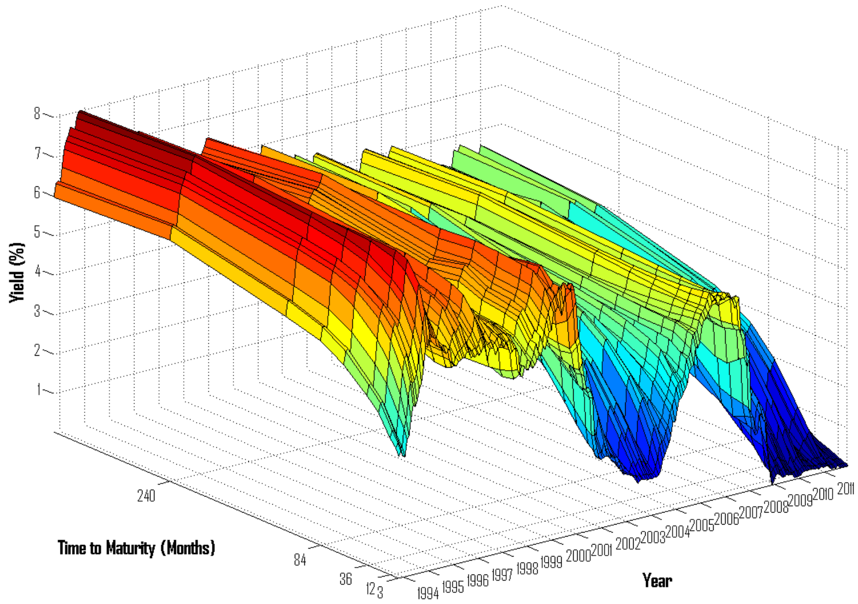
[5] In case two common volatility components are included we find $\theta = (\boldsymbol{\mu}, \boldsymbol{\Phi}, \lambda, \boldsymbol{\Sigma}_{\varepsilon}^{+}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}, \boldsymbol{\Gamma}_{1,\varepsilon}, \boldsymbol{\Gamma}_{2,\varepsilon}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ where $\boldsymbol{\gamma}_i = (\gamma_{i,0}; \gamma_{i,1}; \gamma_{i,2})$ for $i = 1, 2$ and when the common volatility component is included in the state equation we find $\theta = (\boldsymbol{\mu}, \boldsymbol{\Phi}, \lambda, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}^{+}, \boldsymbol{\Gamma}_{\boldsymbol{\nu}}, \gamma_0, \gamma_1, \gamma_2)$. If the GARCH specification in (16) is replaced by a T-GARCH (20) or E-GARCH (21) an additional unknown parameter is added and we have $\theta = (\boldsymbol{\mu}, \boldsymbol{\Phi}, \lambda, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^{+}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}, \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}, \gamma_0, \gamma_1, \gamma_2, \psi)$ and for the DNS-GARCHX (23) model the parameter vector is given by $\theta = (\boldsymbol{\mu}, \boldsymbol{\Phi}, \lambda, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^{+}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}}, \boldsymbol{\Gamma}_{\boldsymbol{\varepsilon}}, \gamma_0, \gamma_1, \gamma_2, \boldsymbol{\phi})$.

[6] See http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/default.aspx.

[7] See http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=longtermrate for more information on the extrapolation method.

Figure I: **Cross Section of Yields**

The figure shows the cross section of the 3M, 6M, 1Y, 2Y, 3Y, 5Y, 7Y, 10Y, 20Y and 30Y yields from October 1993 until December 2011. During the period between February 2002 and February 2006, the 30Y yield is obtained using an extrapolation factor provided by the United States Department of Treasury.



long term trend is downwards, with short term interest rates currently near zero. However, interest rates have varied significantly over time. The yield curve is concave and upward sloping most of the time, but that it can also take a downward sloping or humped shape, as the figure clearly shows.

Table I presents the summary statistics for the ten yields in the dataset as well as for a slope and curvature proxy. Here the slope of the yield curve over time is defined as $y_t(360) - y_t(3)$, where $y_t(\tau)$ is the yield for maturity τ measured in months, and the curvature of the yield curve is given by $[y_t(60) - y_t(3)] - [y_t(360) - y_t(60)]$. The longest maturity on the curve, the 30-year yield, is assumed to proxy for the level of the yield curve (see section 2.1). Some of the stylized facts of the yield curve become clearly present from the table. The average yield curve, represented by the means of the different yields, is concave and upward sloping. The lower average yield in the 30-year maturity compared to the 20-year is explained by Litterman, Scheinkman, and Weiss (1991) who argue that volatility has a larger impact on the long end of the curve. They reason that due to the convexity of the function relating interest rates to discount factors, the longer yields are tilted downwards. The stylized fact of shorter maturity yields being more volatile than those in the long end of the curve is also present in table I, as shown by the decreasing standard deviations for longer maturities. An exception is the 3-month yield which has lower volatility than the 6-month yield, a remark also made by Koopman, Mallee, and van der Wel (2010) for their dataset. Furthermore, the high autocorrelations for all maturities at different horizons illustrate the persistence of the

Table I: **Yield Summary Statistics**
Summary statistics for end-of-month constant maturity yield data for US government bonds from October 1993 until December 2011. $\rho(\tau)$ represents the τ month autocorrelation.

Maturity	Mean	St. Dev.	Min.	Max.	$\rho(1)$	$\rho(3)$	$\rho(12)$	$\rho(24)$
3M	3.181	2.100	0.010	6.380	0.994	0.972	0.753	0.372
6M	3.330	2.131	0.050	6.510	0.994	0.971	0.756	0.384
1Y	3.470	2.118	0.100	7.200	0.993	0.968	0.776	0.446
2Y	3.754	2.066	0.200	7.690	0.991	0.961	0.795	0.537
3Y	3.957	1.953	0.300	7.800	0.989	0.955	0.800	0.592
5Y	4.334	1.700	0.710	7.830	0.985	0.945	0.798	0.665
7Y	4.631	1.524	1.240	7.840	0.982	0.939	0.793	0.708
10Y	4.849	1.336	1.830	7.910	0.978	0.930	0.778	0.736
20Y	5.398	1.178	2.570	8.100	0.977	0.933	0.802	0.817
30Y (Level)	5.344	1.091	2.690	7.990	0.975	0.924	0.777	0.803
Slope	2.161	1.521	-0.640	4.570	0.979	0.933	0.554	0.012
Curvature	-0.612	1.077	-3.120	2.030	0.938	0.822	0.486	0.391

yield dynamics. It is strongest in the long end of the curve, as can be seen from the autocorrelations still being high even after two years. The slope and curvature proxies also show high persistence. Curvature still has an autocorrelation similar to that of the shorter maturity yields after a two year period. The autocorrelation of the slope proxy, however, goes quickly towards zero for longer horizons.

3.2 Macroeconomic and Financial Data

The data for the macroeconomic and financial explanatory variables in the GARCH-X DNS-TVV models in this study is obtained from Datastream. The variables that are used are the capacity utilisation (CU), the federal funds rate (FFR), annual price inflation (INFL) and the Chicago Board Options Exchange Market Volatility Index (VIX)[8]. All four variables concern the US economy or US financial markets as Treasury yields are studied here. I choose to use the macroeconomic factors CU, FFR and INFL following Diebold, Rudebusch, and Aruoba (2006). They argue that these three make up the minimum set of variables to describe the basic macroeconomy. Table II shows the summary statistics of the exogenous factors and the correlation matrix of the variables as they are used in the GARCH-X models and figure II presents plots of the series over time. Capacity utilisation and annual inflation rate follow roughly similar patterns. They both fall sharply during crisis periods, for example after the dot-com bubble burst in 2000 or following the credit crunch in 2008, and return to more stable levels in times of economic expansion. This behaviour leads to a positive correlation, as can be seen from table II(b), but it is only small in absolute terms. Hence both factors certainly also seem to signal different macroeconomic developments. As we see from table II(a) and figure II(c), the sample period also includes a period of deflation with a low of -3.5%.

[8]The FFR is taken as its monthly average and the annual price inflation is calculated as the 12-month change in the price deflator for personal consumption.

Table II: **Summary Statistics Macroeconomic and Financial Factors**
The table shows the summary statistics of the macroeconomic and financial factors, capacity utilisation (CU), the federal funds rate (FFR), annual price inflation (INFL) and the volatility index (VIX) in panel (a). Panel (b) presents the correlation matrix for the squared first differences of the macroeconomic variables (CU, FFR and INFL), the squared monthly VIX, the squared differences in the 3M, 2Y and 30Y yields and those of the empirical slope (30Y-3M) and curvature proxies ([5Y-3M]-[30Y-5Y]). The exogenous variables and changes in yields and curve characteristics are all squared to ensure they only take positive values and represent variances (in case of the VIX).

(a) Summary Statistics

	CU	FFR	INFL	VIX
Mean	77.20	3.38	0.050	21.21
St. Dev.	4.88	2.16	0.022	8.24
Minimum	63.58	0.07	-0.035	10.42
Maximum	84.74	6.54	0.091	59.89

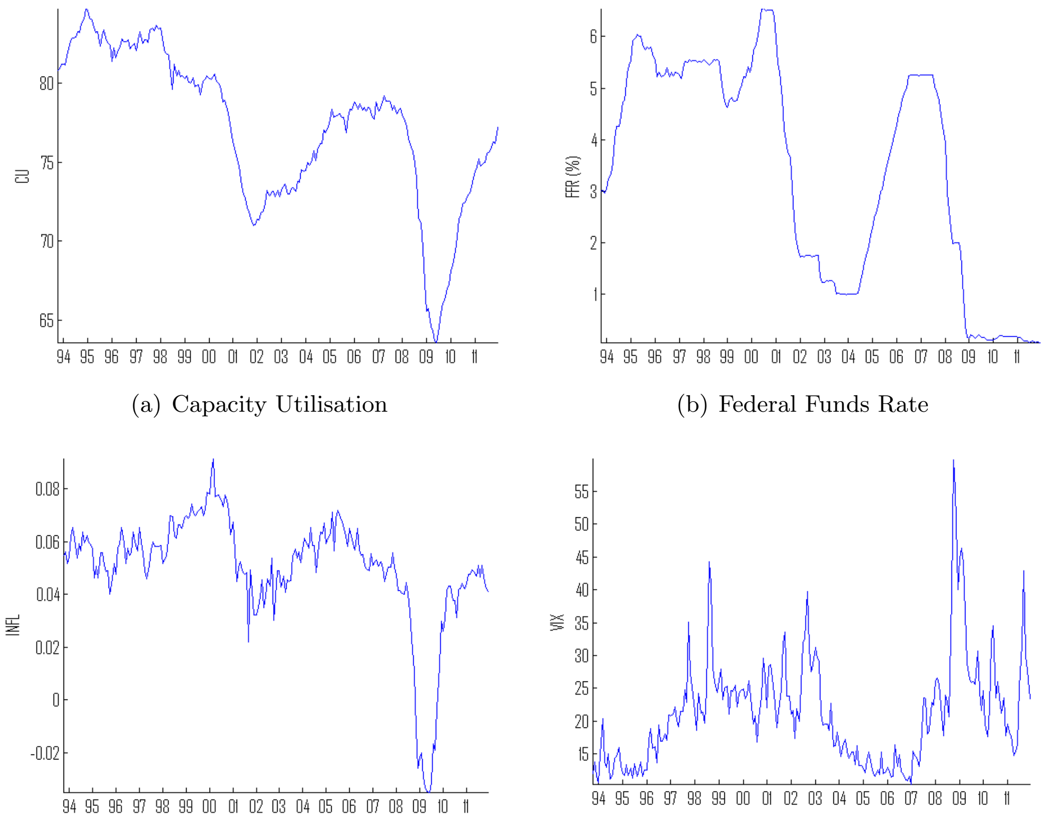
(b) Correlation Matrix

	CU	FFR	INFL	VIX	3M	2Y	30Y	Slope	Curvature
CU	1	0.026	0.137	0.388	0.124	0.058	0.358	0.231	-0.024
FFR		1	0.323	0.330	0.237	0.291	0.072	0.212	0.140
INFL			1	0.319	0.171	0.211	0.260	0.223	0.013
VIX				1	0.137	0.164	0.417	0.249	0.080
3M					1	0.562	-0.038	0.793	-0.042
2Y						1	0.265	0.477	0.569
30Y							1	0.367	0.188
Slope								1	0.095

This period followed the latest financial crisis. In figure II(b) we see the federal funds rate decreasing sharply during economic downturns. Rate cuts usually coincide with increasing stock market volatility, as can be seen from the VIX in figure II(d). The VIX reached an all time high level in October 2008 as a consequence of the panic in stock markets following the credit crunch and the default of Lehman Brothers. The FFR is at record low levels at the end of the sample, close to zero percent. Hence the sample period used in this thesis includes stable, but also turbulent and extreme periods in the economy and financial markets. Table II(b) shows the correlations between the four exogenous variables. They are all positive and relatively low, indicating they can all signal different information in the volatility processes of the DNS-GARCHX models.

Panel (b) in table II also shows correlations for the macroeconomic variables and the VIX with three different yields (3M, 2Y and 30Y) and the empirical slope and curvature proxies. As we are interested in the relation between macroeconomic and financial variables and the volatility in yields, the squared first differences are taken for the three maturities, the slope and the curvature before calculating the correlations. Remember this transformation is also made for the macroeconomic factors. The squared changes in capacity utilisation relate mostly to the long end of the curve whereas for the federal funds rate this is more the case with the short end. Correlation between variation in annual inflation and in yields increases with maturity. The variance in different parts of the curve therefore seem to be affected by different macroeconomic

Figure II: **Macroeconomic and Financial Factors**
The figure presents time series plots for the macroeconomic and ainancial factors used as exogenous variables in the GARCH-X volatility specifications in the DNS-GARCHX-DRA and DNS-GARCHX-VIX models. The US capacity utilisation (CU), the federal funds rate (FFR), the US annual price inflation (INFL) and the Chicago Board Options Exchange Market Volatility Index (VIX) are plotted in panels(a), (b), (c) and (d), respectively.



(a) Capacity Utilisation

(b) Federal Funds Rate

(c) Annual Price Inflation

(d) VIX

variables. The 30Y yield and the squared monthly VIX show the highest correlation (0.417) between the interest rates and exogenous factors, indicating that surges in stock market volatility tend to coincide with increases in volatility in the 30Y maturity. For the slope proxy I find all correlations corresponding to the macro variables and the VIX of similar magnitude, slightly above 20% and they are mostly near zero for the curvature of the yield curve.

4 In-Sample Results

This chapter describes the empirical in-sample results on the various Nelson-Siegel models. First, section 4.1 discusses the standard DNS model. Section 4.2 then gives the results for the DNS-GARCH model. Subsequently, the addition of a second common component is examined in section 4.3. Section 4.4 describes the findings on the alternative volatility specifications, T-GARCH and E-GARCH, for the common component in the yields and 4.5 reviews the results on the DNS-FactorGARCH model. Lastly, in 4.6

Table III: **VAR Estimates for the DNS Model**

In the table the parameter estimates of the VAR model for the latent factors are shown. The results correspond to the standard DNS model estimated in state space form. Panel (a) reports the estimated coefficients in $\boldsymbol{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\boldsymbol{\Sigma_\nu}$ is given. Asymptotic standard errors are obtained using the information matrix.

(a) Constant and Coefficients - DNS Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.936***	0.002	0.015	4.621***
	0.02822	0.01396	0.01332	1.10319
$\beta_{2,t}$ (Slope)	-0.034	0.922***	0.072***	-3.908
	0.03754	0.01859	0.01766	2.69615
$\beta_{3,t}$ (Curvature)	0.123	0.058	0.894***	-3.899
	0.08275	0.04087	0.03857	3.40847

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.069***	-0.071***	0.023
	0.00795	0.00918	0.01649
$\beta_{2,t}$ (Slope)		0.124***	-0.050**
		0.01310	0.02065
$\beta_{3,t}$ (Curvature)			0.595***
			0.06556

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

I consider the empirical findings for the addition of exogenous variables to the GARCH equation.

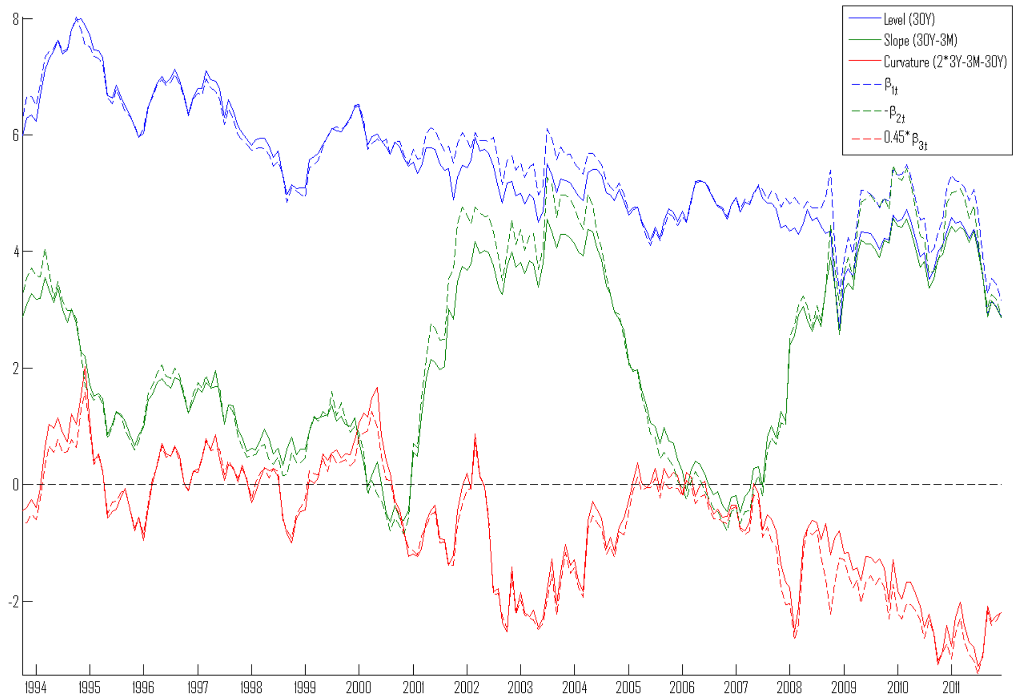
4.1 Dynamic Nelson-Siegel Model (DNS)

The estimation results for the vector autoregression (VAR) of the latent factors in the standard DNS model from section 2.1, are presented in table III. The high persistence in the yields and empirical factor proxies, shown in table I, is also present in the DNS factors in panel III(a) as can be seen from the diagonal elements of the coefficient matrix all being close to one. Moreover, the lagged value of the third factor, which proxies for the curvature, has a significant influence on the slope factor. This significant relation encourages the use of a VAR model to describe the dynamics of the latent factors in the DNS instead of the more parsimonious AR(1) model. The parameter λ in the standard DNS model is estimated at 0.0495 with a standard error of 0.00059, indicating that the estimate is highly significant.

In figure III the filtered latent factors of the standard DNS model, obtained from the Kalman filter, are plotted together with their empirical proxies for level, slope and curvature of the yield curve. The time series of $\beta_{2,t}$ is multiplied by -1 to obtain a proxy for slope. This multiplication is done because, as I explained in section 2.1, the slope in the DNS model is given by $-\beta_{2,t}$. The series of $\beta_{3,t}$ is multiplied by a factor 0.45 in order to better match the scale of the empirical curvature factor. You can see why

Figure III: **Empirical and DNS Factors**

The figure shows the empirical level, slope and curvature factors together with the filtered latent factors from the DNS model. The empirical level is proxied by the 30Y yield, slope is defined as 30Y-3M yield and curvature as 2*3Y-3M-30Y. The negative of the second factor is taken and the third latent factor is scaled by 0.45.



the factors in the DNS are said to represent the three main characteristics of the term structure of interest rates.

The log likelihood, AIC and BIC for the standard DNS model are given in table IV and table VI shows the averages and standard deviations of the filtered errors for all maturities. The filtered errors are defined as the difference between the observed yield and the filtered estimate obtained from the Kalman filter. Both ends of the curve seem hardest to fit for the DNS model as becomes clear from the large mean filtered errors and standard deviations in the 3M, 20Y and 30Y maturities. In contrast, the 3Y yield is fit with high accuracy and the 6M interest rate is even fit perfectly. Overall the model seems to provide best filtered estimates for the intermediate maturities and tends to have problems with the very short and long end of the curve.

4.2 Time-Varying Volatility (DNS-GARCH)

In this thesis I use the DNS-GARCH model introduced by Koopman, Mallee, and van der Wel (2010) as the baseline model of the DNS-TVV class and regard it as a benchmark to compare the relative performance of the more extensive models. The inclusion of a common component in the DNS-GARCH model provides extra flexibility compared to the standard DNS. In this standard model the yield curve dynamics are thought to be captured by a level, slope and curvature factor and the residual is assumed to be white noise. The DNS-GARCH adds a common shock component to the measurement equa-

Table IV: **Log Likelihood, AIC and BIC for the Models**
The Log Likelihood, Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are reported in the table together with the number of parameters ($\#\theta$) of the different models. The test statistics of Likelihood Ratio (LR) tests to compare the different model extensions to the standard DNS and the DNS-GARCH are reported in the last two columns, respectively.

	$\ell(\theta)$	$\#\theta$	AIC	BIC	LR-Statistics	
DNS	1477.75	29	-2897.50	-2732.44		
DNS-GARCH	1741.26	41	-3400.52	-3167.16	527.0	
DNS-2GARCH	1938.74	53	-3771.49	-3469.83	922.0	395.0
DNS-TGARCH	1759.36	42	-3434.72	-3195.67	563.2	36.2
DNS-EGARCH	1758.75	42	-3433.50	-3194.45	562.0	35.0
DNS-FactorGARCH	1506.62	34	-2945.24	-2751.72	57.7	
DNS-GARCHX-DRA	1752.79	44	-3417.58	-3167.15	550.1	23.1
DNS-GARCHX-VIX	1766.54	42	-3449.09	-3210.04	577.6	50.6

tion of the state space framework to account for latent shocks that are not captured by the three factor structure and could for example be caused by exogenous factors. Yet, the shocks do not necessarily affect all yields with similar magnitudes as the loadings $\mathbf{\Gamma}_{\boldsymbol{\varepsilon}}$ in (14) determine the sensitivities of different maturities to them. Furthermore, the shocks are assumed to arrive in clusters and their variance exhibits GARCH-type characteristics. The measurement error therefore is not assumed to be a pure white noise process, but can consist of a common shock component *and* a white noise term, see (14). In times when the GARCH volatility of the common shock component is low and relatively constant, the measurement error is close to a white noise process. However, in periods when volatility surges, latent shocks affect the yield curve and the measurement error cannot be characterised as such. This specification could indicate that the three factor structure does not suffice in fitting the shape of the yield curve during these periods in time.

The VAR of the three latent factors in the DNS-GARCH model show similar dynamics and persistence as in the standard DNS, as can be seen from panel (a) in table XIV in the appendix. Panel (b) in the same table shows the corresponding covariance matrix. A remarkable difference to the DNS model is that the lagged value of the level factor has a significant positive effect on the curvature of the yield curve in the model presented here. The parameter λ in the DNS-GARCH model is estimated at 0.0546 (standard error of 0.00074), which is about 10% larger than in the standard DNS model, hence the introduction of a common component changes the optimal factor loadings via λ. Table V shows the estimated coefficients of the GARCH specification in (16) for the volatility of the common shock component in the yields. The high and significant estimate of the γ_1 parameter indicates that much weight is put on recent shocks. The lag coefficient (γ_2) in the GARCH equation is low and not significantly different from zero. Therefore the volatility of the common component is highly sensitive to the latest innovations, it increases quickly with large shocks and reverts back soon thereafter.

The time series plots of the filtered latent factors in the DNS-GARCH model again closely resemble their empirical counterparts that proxy for level, slope and curvature,

Table V: **GARCH Parameters of the DNS-GARCH Model**
The table presents the estimates of γ_0, γ_1 and γ_2 in the volatility process (16) of the common shock component in the DNS-GARCH model. The parameters are estimated using maximum likelihood and asymptotic standard errors, obtained using the information matrix, are reported.

γ_0	γ_1	γ_2
0.003*	0.835***	0.110
0.00169	0.17189	0.11602

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

as can be seen from figure XV in the appendix. In panel (a) of figure IV h_t is plotted over time. Some historical events are clearly illustrated in the graph. Firstly, the spike in December 1994 coincides exactly with the Mexican Peso crisis. The Russian default and the problems of Long-Term Capital Management (LTCM) as a consequence of that caused tensions in worldwide financial markets starting in August 1998 which is also visible in the plot. During the global financial crisis of 2008 the time series shows an increase in volatility too. Yet, the two largest and latest spikes, corresponding to the start of the problems concerning the Eurozone in early 2010 and the US downgrade from its AAA credit status in August 2011 are most pronounced. Apparently the panic following these last two events caused extreme surges in volatility of the common component in the DNS-GARCH model. Yet, this should not be surprising as these events actually concerned sovereign debt and therefore directly translate into Treasury yields.

Table VI: **Filtered Errors**

The table shows the mean filtered errors and their standard deviations (measured in basis points) for the standard and extended DNS models. The filtered errors are the difference between the observed yield and the filtered estimate obtained from the Kalman filter at every point in time. Panel (a) presents the results for the standard DNS, the DNS-GARCH and DNS-2GARCH models, panel (b) gives them for the DNS-TGARCH, DNS-EGARCH and DNS-FactorGARCH and panel (c) shows the errors for the DNS-GARCHX-DRA and DNS-GARCHX-VIX models. The median and averages of the means and standard deviations as well as the number of maturities for which the mean or standard deviation of the filtered errors is lower than for the DNS model (# Lower) are reported.

(a) Filtered errors DNS, DNS-GARCH and DNS-2GARCH

	DNS		DNS-GARCH		DNS-2GARCH	
Maturity	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
3M	-3.219	8.781	-3.156	5.868	-1.158	1.325
6M	0.000	0.000	0.000	0.000	0.410	2.014
1Y	0.218	5.270	-0.453	4.224	-0.286	1.546
2Y	0.262	2.699	0.000	0.000	-0.193	1.177
3Y	-0.235	0.372	-1.047	1.722	-0.306	0.539
5Y	-0.492	4.678	0.000	0.000	-0.224	2.170
7Y	1.548	3.506	0.992	5.396	-0.100	2.923
10Y	-0.863	4.292	-1.026	5.166	-0.001	0.005
20Y	20.292	13.388	8.629	10.693	0.001	0.003
30Y	2.886	12.588	-0.935	4.332	-1.731	10.635
Median	0.109	4.485	-0.227	4.278	-0.208	1.435
Mean	2.040	5.557	0.300	3.740	-0.359	2.234
# Lower			6	6	7	8

(b) Filtered errors DNS-TGARCH, DNS-EGARCH and DNS-FactorGARCH

	DNS-TGARCH		DNS-EGARCH		DNS-FactorGARCH	
Maturity	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
3M	-3.153	6.074	-2.715	6.490	-4.275	7.999
6M	0.000	0.001	0.053	0.130	0.000	0.000
1Y	-0.391	4.263	-0.475	4.157	0.258	5.008
2Y	0.000	0.000	-0.001	0.017	0.263	2.667
3Y	-1.023	1.742	-0.948	1.645	0.000	0.001
5Y	0.000	0.000	0.004	0.030	-0.273	4.792
7Y	0.977	5.301	0.948	5.300	1.541	3.220
10Y	-1.008	5.235	-1.074	5.206	-1.004	4.974
20Y	8.684	10.322	8.823	10.375	19.973	13.496
30Y	-1.026	4.521	-1.067	4.489	2.573	12.763
Median	-0.195	4.392	-0.238	4.323	0.129	4.883
Mean	0.306	3.746	0.355	3.784	1.906	5.492
# Lower	6	6	6	6	5	5

(c) Filtered errors DNS-GARCHX-DRA and DNS-GARCHX-VIX

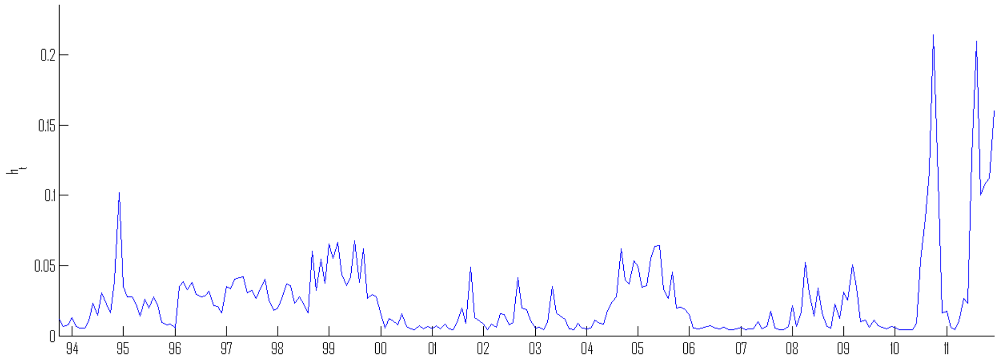
	DNS-GARCHX-DRA		DNS-GARCHX-VIX	
Maturity	Mean	St. Dev.	Mean	St. Dev.
3M	-3.137	5.700	-3.182	5.784
6M	0.000	0.000	0.000	0.000
1Y	-0.474	4.199	-0.508	4.231
2Y	0.000	0.000	0.000	0.000
3Y	-1.022	1.681	-1.039	1.705
5Y	0.000	0.000	0.000	0.000
7Y	0.935	5.381	0.968	5.317
10Y	-1.064	5.119	-1.005	5.130
20Y	8.486	10.654	8.616	10.563
30Y	-0.957	4.543	-0.905	4.456
Median	-0.237	4.371	-0.254	4.344
Mean	0.277	3.728	0.294	3.719
# Lower	7	7	6	6

Panel (b) of figure IV plots the loadings in the vector $\mathbf{\Gamma}_\varepsilon$ against maturity. For the 3M maturity the loading is fixed at the value of 1 in order to overcome identification issues (see section 2.2.1). The overall pattern of loadings against maturity is roughly similar to that of Koopman, Mallee, and van der Wel (2010) who find a remarkably lower sensitivity of the 1Y and 9Y maturities. The 9Y is the before last maturity in their sample and I therefore compare it to the 20Y yield, for which the loading is estimated at nearly zero here. Hence it seems that regardless of the choice of maturities included in the sample, the DNS-GARCH fits the loadings to the common component in such a manner that the shortest, medium-term and longest maturities are most sensitive to common shocks and the maturities in between much less.

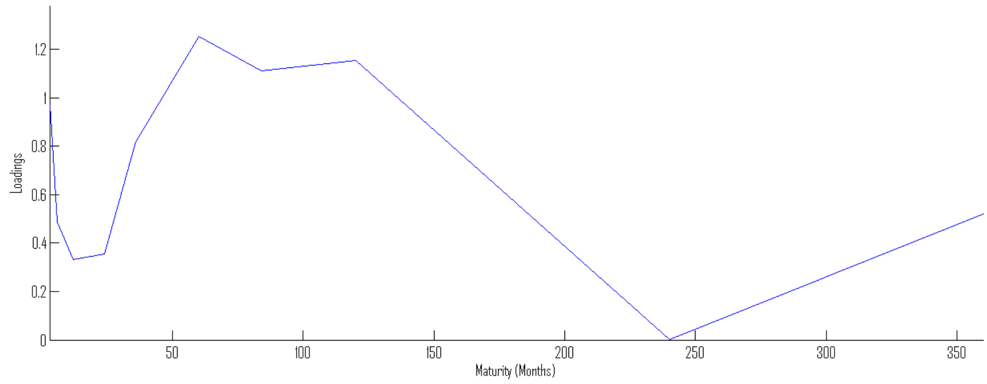
Table IV confirms the finding of Koopman, Mallee, and van der Wel (2010). Allowing for time-varying volatility leads to a large increase in the log likelihood value. The likelihood ratio (LR) statistic to compare the DNS-GARCH model to the standard DNS is equal to 527.0, indicating a large improvement in fit. After correction for the addition of extra parameters the AIC and BIC also conclude that allowing for time-varying volatility is a useful extension to the standard DNS model. Judgement on the significance of the LR-statistics should be conducted with caution. They do not follow the regular Chi-squared distribution with degrees of freedom equal to the difference in number of parameters between the two models that are compared. Due to the presence of nuisance parameters the actual distribution of the statistics is unknown, see Hansen (1996) for a further discussion on this problem.

The filtered errors of the DNS-GARCH model are presented in panel (a) of table VI. The average of the mean filtered error and the standard deviation is much lower than in the standard DNS model, which is mostly due to the better fit of the shortest and longest maturities. Especially the fit of the 20Y yield improves when the DNS model is extended with a common volatility component governed by a GARCH specification.

Figure IV: **DNS-GARCH Common Volatility and Loadings**
The figure shows plots of the volatility (h_t) of the common shock component (ε_t^*) over time for the DNS-GARCH model in panel (a). Panel (b) plots the loadings for the different yields against maturity (in months). The loadings are the elements of the vector $\mathbf{\Gamma}_\varepsilon$ and are defined as the sensitivities of the various yields to the common shock.



(a) Common Volatility (h_t)

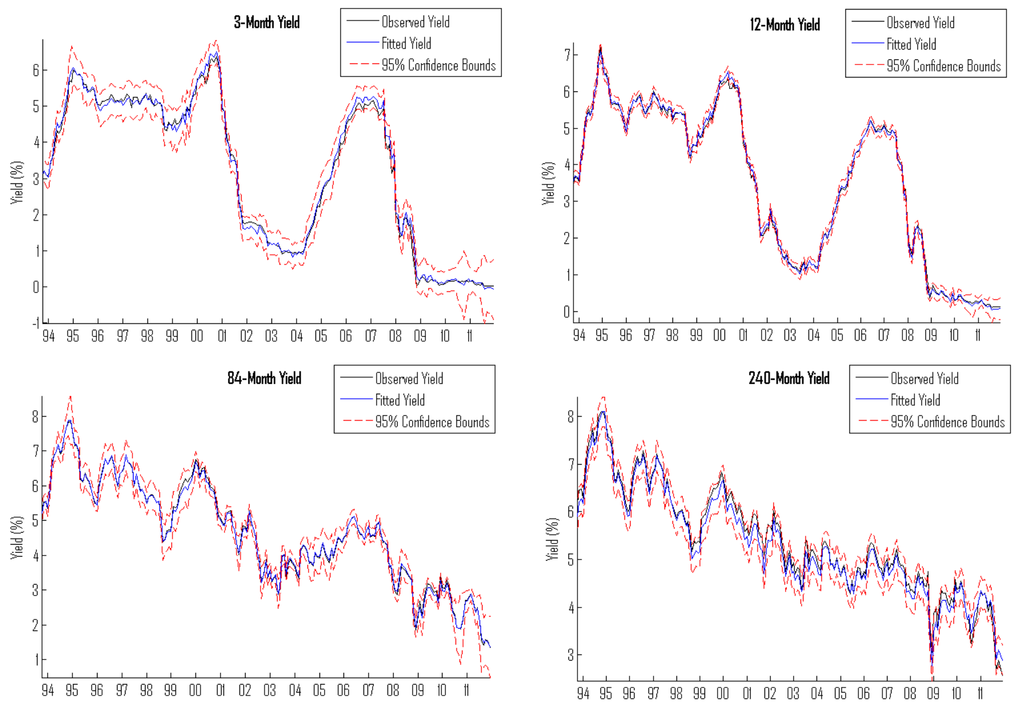


(b) Common Component Loadings ($\mathbf{\Gamma}_\varepsilon$)

Furthermore, not only the 6M yield, but also the 5Y yield is fit perfectly (up to 3 decimals) by the filtered yield curves from this model. In total the filtered estimates are better using the DNS-GARCH than with the standard DNS model for 6 out of 10 maturities and the standard deviations of the filtered errors are lower in an equal number of cases.

In figure V the filtered 3M, 1Y, 7Y and 20Y yields are plotted over time together with the 95% confidence bounds of the estimates and the observed series. The confidence intervals give much additional insight in the accuracy of estimates and can therefore be of great importance in practice. Increased uncertainty in financial markets leads to less accurate estimations of the interest rates which should therefore be interpreted and used more prudently. The width of the confidence intervals depends on the variance of and sensitivity to the common component (ε_t^*) and on the individual variance of every yield (diagonal elements of $\mathbf{\Sigma}_{\boldsymbol{\varepsilon}}^+$ in (14)). Due to the low sensitivity to the common component, for which the variance is time-varying, the confidence ranges for the 1Y and 20Y

Figure V: **Observed and Estimated Yields**
The figure shows observed and estimated yields from the DNS-GARCH model for the 3M, 1Y, 7Y and 20Y maturities. The estimated yields are obtained using the filtered latent factors from the state space model. Time-varying confidence bounds depend on the sensitivity of the yield to the common volatility (h_t) and the variance of the residual in the measurement equation (diagonal elements of $\mathbf{\Sigma}_{\varepsilon}^{+}$).



maturities in the figure are nearly constant. On the other hand, for the 3M and 7Y rates the width of the confidence intervals changes severely over time. For example at the end of the sample period the two sequential surges in volatility as a result of Eurozone and US credit rating problems lead to wider confidence bounds on the filtered estimates.

4.3 Two Common Volatility Components (DNS-2GARCH)

In order to obtain insight in the possible existence of additional common shocks in the yield curve, the DNS-2GARCH model contains a second common component in the measurement equation, see (18). Table VII presents the results from the two separate GARCH processes for the volatilities of the common components in this model. The dynamics of the second GARCH equation are similar to that of the DNS-GARCH model as the process is also highly reactive to new shocks due to the estimate of γ_1 being close to 1. Only little weight is put on lagged levels of volatility, as the small, but significant estimate of γ_2 shows. Therefore, the effect of shocks in the second common component die out quickly. The GARCH process of the first common component shows different dynamics. Much more weight is put on the lagged level of volatility and the sensitivity to the most recent shock is lower. These differing dynamics are nicely illustrated in panel (a) of figure VI, where the volatilities of both common components of the model are plotted over time. Due to the larger weight on lagged levels, the volatility stays high for a more prolonged period after a spike in the first component compared to the second.

Table VII: **GARCH Parameters for the DNS-2GARCH Model**
The figure shows the estimates for $\gamma_{i,0}, \gamma_{i,1}$ and $\gamma_{i,2}$ (for $i = 1, 2$) of the volatility processes in the DNS-2GARCH model. The DNS model is extended with two common shock components for which the variance is governed by separate standard GARCH processes as in (16). Robust standard errors are obtained using the sandwich estimator.

	$\gamma_{i,0}$	$\gamma_{i,1}$	$\gamma_{i,2}$
$i = 1$	0.0003***	0.6483***	0.3365***
	0.00004	0.01813	0.00901
$i = 2$	0.0002***	0.9195***	0.0648***
	0.000003	0.01334	0.00508

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

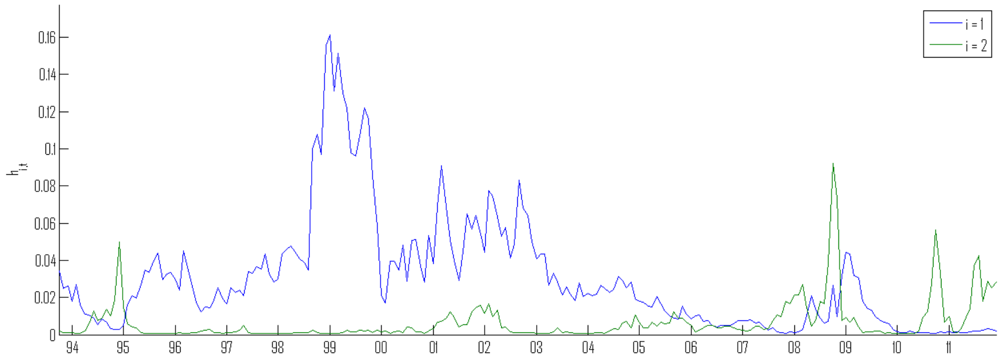
Furthermore, both time series seem to indicate different events in financial history with differing intensities. For example the first component captures the increased level of volatility that was present in the markets during the end of the nineties and at the start of the new century whereas the second component really captures the increased uncertainty as a result of the global financial crisis, the Eurozone problems and the US downgrade.

In panel (b) of figure VI the loadings of the ten yields are plotted against maturity. Remember that for the first component the loading of the 3M yield was fixed at 1 and for the second component this was done for the loading of the 30Y yield. It does not seem to be the case that one component captures mostly the common volatility in the short end of the curve and the other in the long end. In contrast, the patterns of the loadings plots show rough similarities. The most noteworthy difference is that the sensitivity to the second component for the 6M yield is near zero, but around 1 for the first. Apparently the 6M yield is mostly sensitive to the more persistent common volatility component. The loading of the 20Y yield is shown to be lower than the 10Y and 30Y for both common components and therefore stands out in the overall pattern of sensitivities. This remarkable observation was also noted in the DNS-GARCH model earlier. It could possibly be caused by the fact that no 20Y maturity bonds are issued by the US Treasury Department. Hence the yield is estimated from prices at which securities are trading in the secondary market which is a less transparent over-the-counter (OTC) market. Therefore it is subject to a much larger error than the yields for which the US Treasury auctions bonds and bills.

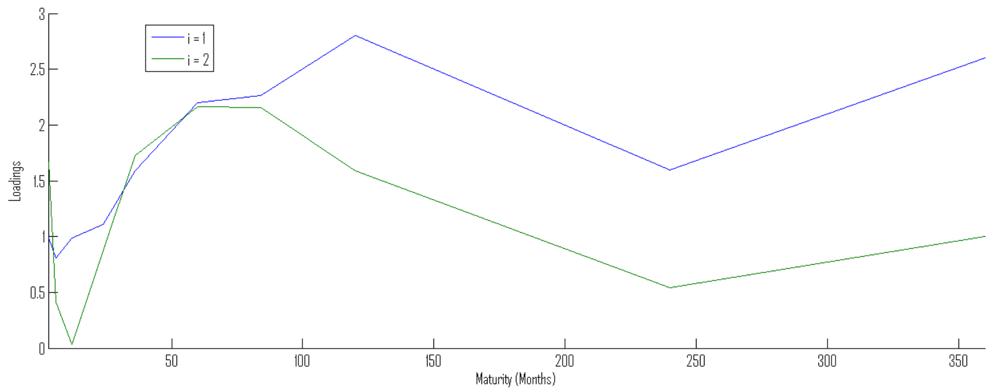
Similar to the DNS and DNS-GARCH model the factors characterising the yield curve show high levels of persistence in the VAR framework of the state space model in which the standard DNS is extended with two common components in the measurement equation. Table XV in appendix C shows the results. The lagged level factor has a significant negative effect on the slope whereas the lagged curvature has a significant positive effect on this factor. The parameter λ in the DNS-2GARCH model is estimated at 0.0563 with a standard error of 0.00002. Overall there are no large differences in the VAR model for the latent factors between the DNS-GARCH and DNS-2GARCH model. Figure XV in the appendix plots the time series of the latent factors. Compared to the DNS and DNS-GARCH models the first factor, representing the level of the yield

Figure VI: **DNS-2GARCH Factors and Volatility**

The figure shows plots of the volatilities ($h_{i,t}$ for $i = 1, 2$) of the common shock components ($\varepsilon^*_{i,t}$) over time for the DNS-2GARCH model in panel (a). Panel (b) plots the loadings, corresponding to the two common components, for the different yields against maturity (in months). The loadings are the elements of the vectors $\mathbf{\Gamma}_{1,\varepsilon}$ and $\mathbf{\Gamma}_{2,\varepsilon}$ and are defined as the sensitivities of the various yields to the common shock.



(a) Common Volatilities ($h_{1,t}$ and $h_{2,t}$)



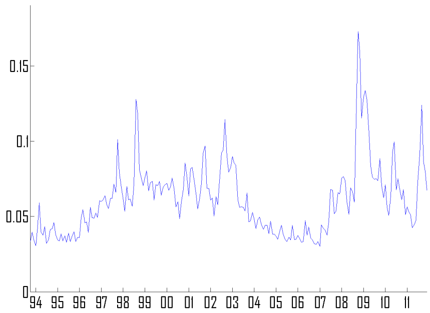
(b) Common Component Loadings ($\mathbf{\Gamma}_{1,\varepsilon}$ and $\mathbf{\Gamma}_{2,\varepsilon}$)

curve, deviates more from its empirical proxy. This deviation is especially large in the middle part of the sample where we also see increased levels of steepness in the curve, as indicated by the higher $\beta_{2,t}$ estimate.

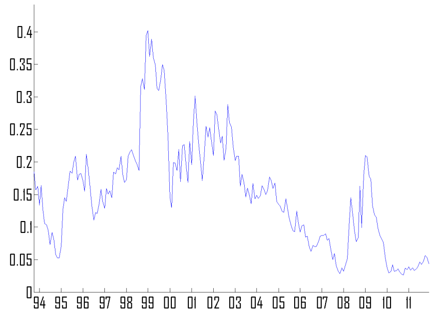
Figure VII shows a comparison of the volatilities of the two common components in the DNS-2GARCH model with the monthly VIX and the common component volatility from the DNS-GARCH model. Surprisingly the volatility of the first common component in the DNS-2GARCH model, for which the element corresponding to the 3M yield was fixed at 1, shows roughly a similar pattern as the VIX. The variance of the second component, for which the 30Y loading was fixed, resembles globally the dynamics of the volatility of the common component in the DNS-GARCH model. There is not a one-to-one match between the series as can for example be seen by the peak in the VIX during the credit crunch in 2008. During that period $h_{1,t}$ shows increased volatility, but the real spike is present in the second component. Similar to that, the period of increased volatility between 1996 and 2000 in the DNS-GARCH model, visible in the series h_t,

Figure VII: **Comparison of Volatilities**

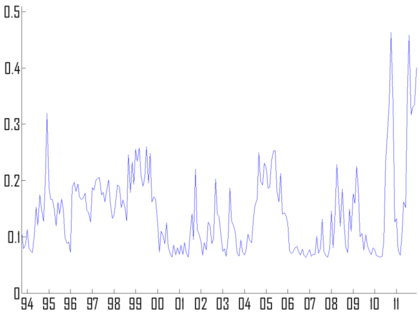
The figure shows plots of the transformation of VIX to monthly volatility in decimal notation ($\frac{VIX}{100\sqrt{12}}$) in panel (a), the standard deviation of the common component in the DNS-GARCH model in (c) and the standard deviations of the two common components in the DNS-2GARCH model, (b) and (d). The VIX shows dynamics which are roughly similar to the variance of the first component in DNS-2GARCH and the time series of the standard deviation of the second common component in DNS-2GARCH shows correspondence to the standard deviation of the common component in the DNS-GARCH model.



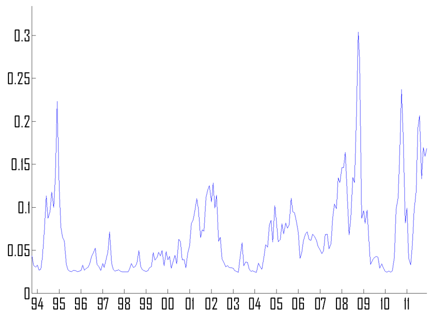
(a) VIX (Monthly)



(b) $\sqrt{h_{1,t}}$



(c) $\sqrt{h_t}$



(d) $\sqrt{h_{2,t}}$

is not present in $h_{2,t}$. However, the first component in the DNS-2GARCH model picks up this increase, especially during the last two years of the previous century. Yet, the overall fluctuations of $h_{1,t}$ and $h_{2,t}$ seem to coincide with those of the VIX and h_t. Therefore it is interesting to see whether the standard deviation of $\boldsymbol{\varepsilon}_t$ in (18), expressed as $\sqrt{\mathbf{\Gamma}^2_{1,\varepsilon}(i)h_{1,t} + \mathbf{\Gamma}^2_{2,\varepsilon}(i)h_{2,t} + \mathbf{\Sigma}^+_{\boldsymbol{\varepsilon}}(i,i)}$, where $i = 1, ..., 10$ indicates the element of the vector or matrix, can be explained by the VIX and the square root of h_t from the DNS-GARCH model. Table VIII shows the results of regressions to analyse this relation for all maturities. Almost all coefficient estimates are significantly different from zero at the 99% level. Figure VIII shows plots of the estimated coefficients β_{VIX} and scaled loadings in vector $\mathbf{\Gamma}_{1,\varepsilon}$ against maturity in the left graph and of β_{GARCH} and scaled $\mathbf{\Gamma}_{2,\varepsilon}$ in the right. As the first element of vector $\mathbf{\Gamma}_{1,\varepsilon}$ is fixed at 1 in the maximum likelihood estimation to overcome identification issues, the loadings are now scaled by the β_{VIX} estimate of the 3M yield, 1.350, to allow for comparison. For the same reason $\mathbf{\Gamma}_{2,\varepsilon}$ is scaled by the 30Y β_{GARCH} estimate, 0.221, as the loading for the longest maturity is fixed at 1 for the second common component. As the graphs show, the coefficients and loadings are remarkably close, up to the scaling factor. The similarities indicate that $h_{1,t}$ accounts in a way for stock market volatility and $h_{2,t}$ for volatility in interest

Table VIII: **Volatility Component Regressions**
The table shows regression results for the standard deviation of $\boldsymbol{\varepsilon}_t$ in the DNS-2GARCH model, defined as the square root of (19), for every maturity. All regressions include a constant and have the square root of the variance of the common component in the DNS-GARCH model (h_t), given in (16), and the VIX (transformed to monthly volatility and scaled to decimal notation) as explanatory variables. Hence the regression model is $\sqrt{\boldsymbol{\Gamma}^2_{\mathbf{1},\boldsymbol{\varepsilon}}(i)h_{1,t} + \boldsymbol{\Gamma}^2_{\mathbf{2},\boldsymbol{\varepsilon}}(i)h_{2,t} + \boldsymbol{\Sigma}^+_{\boldsymbol{\varepsilon}}(i,i)} = c + \beta_{GARCH}\sqrt{h_t} + \beta_{VIX}\frac{VIX_t}{100\sqrt{12}}$ where $i = 1, ..., 10$ indicates the elements of the vectors and matrices corresponding to the 10 different maturities, c is a constant and β_{GARCH} and β_{VIX} are the regression coefficients. The R^2 values for the 10 regression models are shown in the last column and standard errors are presented below the coefficient estimates.

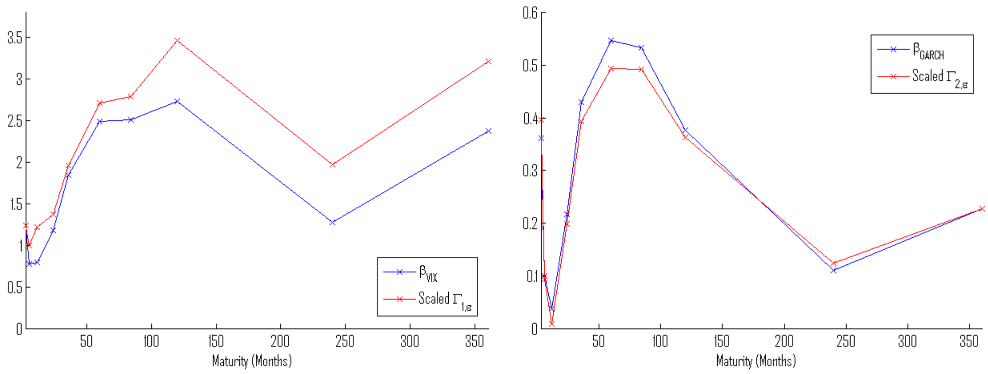
	Constant	β_{GARCH}	β_{VIX}	R^2
3M	0.081***	0.413***	1.350***	0.395
	0.0039	0.0245	0.0601	
6M	0.082***	0.088***	0.740***	0.114
	0.0037	0.0229	0.0562	
1Y	0.103***	0.030	0.826***	0.066
	0.0051	0.0319	0.0783	
2Y	0.090***	0.210***	1.163***	0.175
	0.0049	0.0306	0.0749	
3Y	0.112***	0.435***	1.853***	0.258
	0.0067	0.0413	0.1013	
5Y	0.161***	0.540***	2.477***	0.230
	0.0094	0.0581	0.1424	
7Y	0.168***	0.536***	2.521***	0.221
	0.0097	0.0601	0.1473	
10Y	0.230***	0.386***	2.754***	0.123
	0.0135	0.0841	0.2061	
20Y	0.142***	0.126**	1.455***	0.084
	0.0083	0.0514	0.1261	
30Y	0.251***	0.221***	2.340***	0.093
	0.0127	0.0791	0.1939	

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

rate markets in general, assuming that this is represented by the common volatility component in the DNS-GARCH model. The loadings of the first component and the estimated coefficients β_{VIX} are generally lower for shorter maturities than for the rest. Hence the volatility of the short end of the curve is less sensitive to tensions in the stock market than the long end. On the other hand, from the loadings of the second component and the estimated coefficients of β_{GARCH} we see that especially the medium-term yields are sensitive to general interest rate volatility. Furthermore, the R^2 statistics show that especially for the shortest maturity (3M) and for the medium-term maturities (3Y, 5Y and 7Y) a relatively large part of the variance of ε_t can be explained by the monthly VIX and the volatility of the common component in the DNS-GARCH model. These observations imply that the DNS-2GARCH model not only captures information from shocks to interest rate markets, but also from shocks to stock markets and hence this information is priced in the term structure[9].

[9]I also looked at one-on-one regressions in which $h_{1,t}$ was regressed on a constant and VIX and $h_{2,t}$ on a constant and h_t, but the results were less satisfactory, in terms of explanatory power, than what

Figure VIII: **Volatility Component Loadings and Regression Coefficients**
The figure shows the estimated coefficients β_{GARCH} from table VIII and scaled loading parameters $\mathbf{\Gamma}_{1,\varepsilon}$ plotted against maturity in the left graph and the coefficients β_{VIX} and loadings $\mathbf{\Gamma}_{2\varepsilon}$ in the right. The loadings in vector $\mathbf{\Gamma}_{1,\varepsilon}$ are scaled by 1.350, which is the β_{VIX} estimate for the 3M maturity and $\mathbf{\Gamma}_{2,\varepsilon}$ is scaled by the estimate of β_{GARCH} for the 30Y yield, 0.221.



The log likelihood increases largely when I add a second common component to the DNS-GARCH model, as shown in table IV. An additional 12 parameters are estimated to achieve an increase of 197.48 likelihood points over the benchmark in the DNS-TVV class of models, leading to a LR-statistic of 395.0. This increase indicates that an additional common shock component is a valuable extension of the DNS-GARCH model. Similar conclusions can be drawn from the AIC and BIC criterions which are both substantially lower for the DNS-2GARCH than for the DNS-GARCH.

The DNS-2GARCH model also performs better than the DNS-GARCH model in terms of fitting the yields, as is shown in table VI. For 7 out of 10 maturities the average filtered error is lower than for the standard DNS model and 8 out of 10 times the standard deviation of the filtered errors is lower. Remarkable is the very good fit of the 20Y yield using the DNS-2GARCH model which totally contrasts the results for the two models discussed earlier. On average the filtered error is only 0.001 basis points with a standard deviation of 0.003 for this maturity. The overall standard deviation of the filtered errors is also substantially lower compared to the DNS and the DNS-GARCH models.

4.4 Alternative Volatility Dynamics (DNS-TGARCH and DNS-EGARCH)

Financial market volatility in many prior studies is characterised by asymmetric volatility rather than symmetric. For example stock market volatility tends to surge when indices are falling and revert back to normal levels only gradually when prices increase. This phenomenon is also present in interest rate markets, as studied by Dungey, McKenzie, and Tambakis (2009), who find US Treasuries to increase (and hence yields fall) in volatile times. Yet, the standard GARCH model is not able to allow for different re-

is presented here. A possible explanation for this finding is that there is no one-to-one match between the series, but only the patterns of both $h_{1,t}$ and $h_{2,t}$ are roughly coinciding with those of the VIX and h_t, as discussed.

Table IX: **GARCH Parameters for the DNS-TGARCH and DNS-EGARCH Models**
The table presents the estimates of $\gamma_0, \gamma_1, \gamma_2$ and ψ for the asymmetric volatility processes (see (20) and (21)) of the common components in the DNS-TGARCH (panel (a)) and DNS-EGARCH (panel (b)) models. Robust standard errors are obtained using the sandwich estimator.

(a) T-GARCH Parameters

γ_0	γ_1	γ_2	ψ
0.003***	0.702***	0.122***	0.136***
0.00008	0.05039	0.00218	0.02446

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) E-GARCH Parameters

γ_0	γ_1	γ_2	ψ
-1.463***	-0.140***	0.735***	1.041***
0.22743	0.00692	0.04300	0.04639

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

sponses of volatility to negative and positive shocks and therefore implies a perfectly symmetric structure on the volatility process.

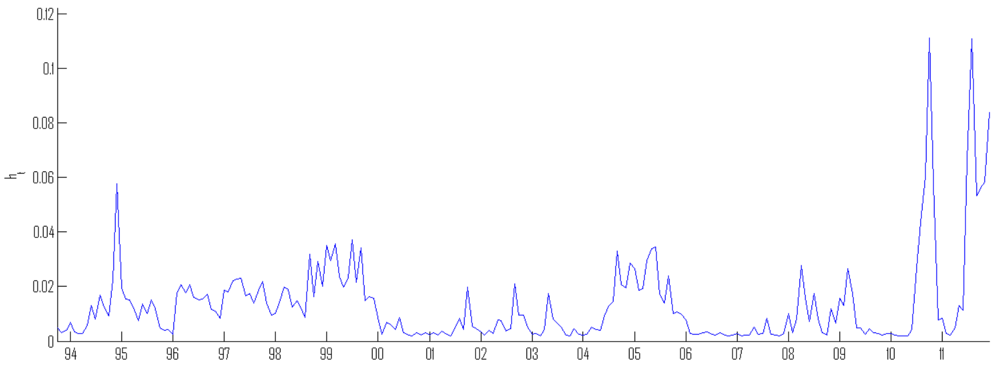
In order to allow for asymmetric dynamics I estimate two alternative specifications of the volatility process for the common component in the DNS-TVV model. The results of the VAR of the DNS-TGARCH and DNS-EGARCH are shown in tables XVI and XVII in the appendix, respectively. Both are very similar to each other in terms of magnitude and significance of coefficients, but there are some differences compared to the DNS, DNS-GARCH and DNS-2GARCH models. Using robust standard errors, all coefficients in the E-GARCH VAR and almost all in the T-GARCH VAR are found to be significantly different from zero. This finding implies a much richer structure among the three latent factors characterising the yield curve than in the case of the three models discussed earlier. The parameter λ in the DNS-TGARCH model is estimated at 0.0544 (standard error of 0.00003) and 0.0541 (standard error of 0.000003) in the DNS-EGARCH model. The difference between the two estimates is not large, but given the small standard errors it is significant. The same holds when compared to the estimates of λ in the DNS, DNS-GARCH and DNS-2GARCH models which were 0.0495 (0.00059), 0.0546 (0.00074) and 0.0563 (0.00002), respectively[10]. The time series of filtered factors of the DNS-TGARCH and DNS-EGARCH models are very close to the ones from the DNS-GARCH model (see figure XV in the appendix) and are therefore not presented here.

Table IX(a) presents the estimates of the parameters for the T-GARCH specification given by equation (20). The results support the hypothesis of asymmetric volatility dynamics in the common shock component as all parameters, including ϕ are significant. The weight on the lagged variance (γ_2) in the T-GARCH model is similar to that in the DNS-GARCH. Also in this specification recent shocks play the most important role, only now the sign of a shock has an effect on the magnitude of the change in volatility.

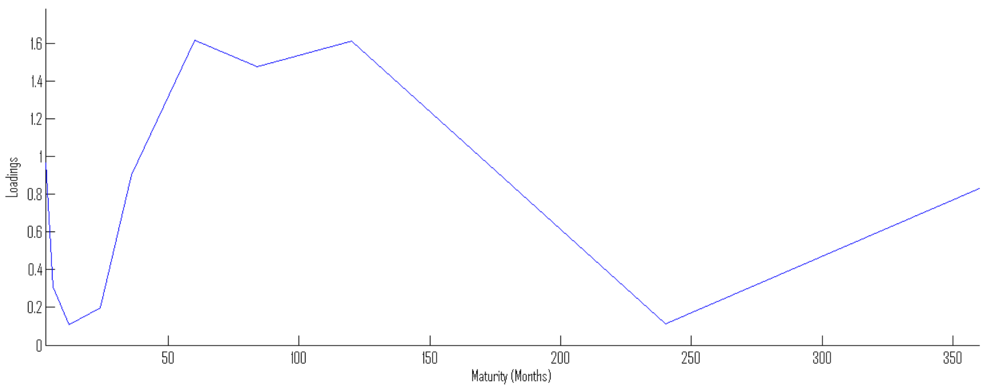
[10]Standard errors of the λ estimates in parentheses.

Figure IX: **DNS-TGARCH Common Volatility and Loadings**

The figure shows a plot of the volatility (h_t) of the common shock component (ε_t^*) over time for the DNS-TGARCH model in panel (a). Panel (b) plots the loadings for the different yields against maturity (in months). The loadings are the elements of the vector $\mathbf{\Gamma}_\varepsilon$ and are defined as the sensitivities of the various yields to the common shock.



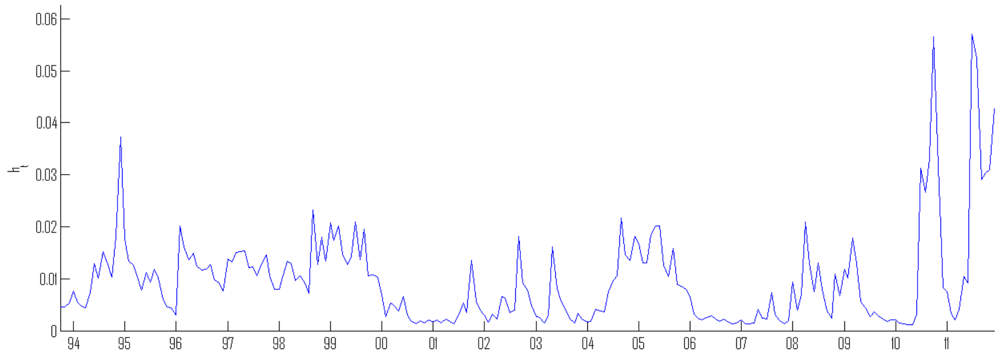
(a) Common Volatility (h_t)



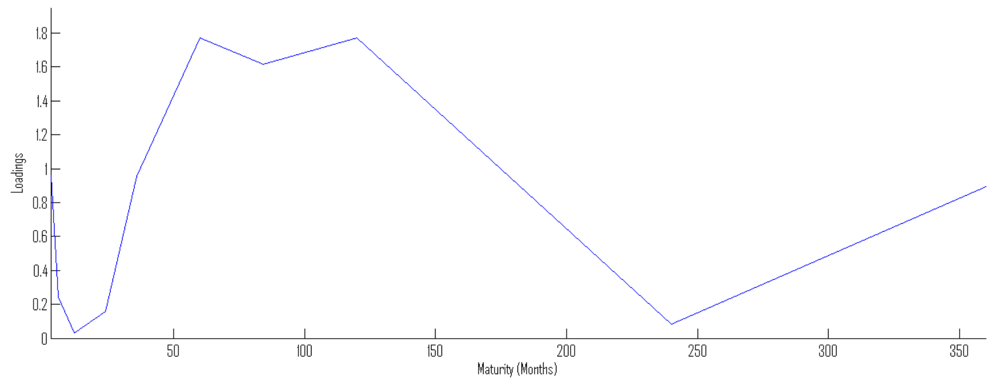
(b) Common Component Loadings ($\mathbf{\Gamma}_\varepsilon$)

All parameters for the E-GARCH specification, given by (21), in the corresponding DNS-TVV model turn out to be significant, as shown in panel (b) of table IX. Hence also the DNS-EGARCH model allows for asymmetry in the volatility process of the common component, again supporting the finding of Dungey, McKenzie, and Tambakis (2009). As the standard GARCH model is not nested in the E-GARCH specification, the results from the table cannot directly be compared as in the case of T-GARCH. Yet, figures IX(a) and X(a) show the time series of h_t in the DNS-TGARCH and DNS-EGARCH models, respectively. Not surprisingly the overall pattern is not very different from that of the common volatility in the DNS-GARCH model, but there are some deviations between the three. For example the spike in 2001 in the asymmetric models is not as large as in the symmetric model and in the beginning of 1996 the sudden jump in volatility is more pronounced in the DNS-EGARCH than in the other two. Figures IX(b) and X(b) show the common factor loadings for both asymmetric volatility models. Also these are similar to the loadings of the DNS-GARCH model, see IV(b), except for the fact

Figure X: **DNS-EGARCH Common Volatility and Loadings**
The figure shows a plot of the volatility (h_t) of the common shock component (ε_t^*) over time for the DNS-EGARCH model in panel (a). Panel (b) plots the loadings for the different yields against maturity (in months). The loadings are the elements of the vector $\mathbf{\Gamma}_\varepsilon$ and are defined as the sensitivities of the various yields to the common shock.



(a) Common Volatility (h_t)

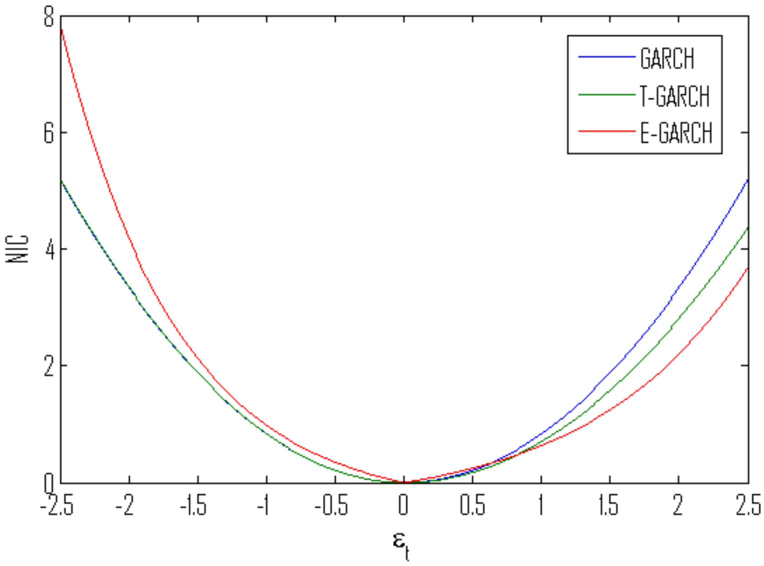


(b) Common Component Loadings ($\mathbf{\Gamma}_\varepsilon$)

that the 20Y maturity now shows a small sensitivity whereas it is nearly zero in the benchmark DNS-TVV model. Despite the loading for this yield still being low, it turns out to be exposed to the common shock component as well when I allow for asymmetric volatility, as opposed to symmetric.

In order to obtain a better insight in the effect of to new shocks in the two asymmetric volatility models compared to the symmetric GARCH, figure XI visualises the response by showing the news impact curves as described in Engle and Ng (1993) for the estimated volatility processes in the three models. The standard GARCH specification is symmetric and only allows positive and negative shocks to have an equal impact on the level of volatility. In contrast, in the T-GARCH and E-GARCH specifications it is also possible for the volatility to respond different to positive and negative shocks, leading to asymmetric news impact curves. The responses to negative shocks in the DNS-GARCH and DNS-TGARCH are nearly equal, because γ_1 in the DNS-GARCH model is about equal to $\gamma_1 + \psi$ in DNS-TGARCH. Hence γ_1 in the DNS-TGARCH is

Figure XI: **News Impact Curves**
The figure shows the news impact curves of the volatility processes in the DNS-GARCH, DNS-TGARCH and DNS-EGARCH models. Here ε_t denotes a random normal shock of a certain magnitude. The curves show how changes in volatility in the different models are related to shocks. In the GARCH model, the shape of the curve depends only on γ_1, leading to a symmetric curve, and for T-GARCH and E-GARCH it depends on γ_1 and ψ, allowing for the asymmetry.



lower than in the DNS-GARCH and the model responds less heavily to positive shocks, as can be seen from the figure. The same holds for the DNS-EGARCH model, which also reacts stronger to negative than to positive shocks and in fact the asymmetry is even more pronounced than in the DNS-TGARCH model. Note that the GARCH process is nested in the T-GARCH, but not in the E-GARCH specification. It is therefore not possible to obtain exactly equal news impact curves for the GARCH and E-GARCH models as would be possible for GARCH and T-GARCH.

Table IV shows a higher log likelihood compared to the DNS-GARCH model when I allow for asymmetry in the volatility model of the common component. The difference in likelihood is 18.10 points for DNS-TGARCH and 17.49 for DNS-EGARCH with just one additional parameter and hence the AIC and BIC also indicate preference for the extended models, as can be seen from the lower values. The average filtered errors in table VI show that the fit of the DNS-TGARCH and DNS-EGARCH models is not much different from the fit of the DNS-GARCH. Based on the mean of the average filtered error and of the standard errors among all maturities, the symmetric model even shows a better performance. Further it is noteworthy that in contrast to the DNS-GARCH and DNS-TGARCH, the DNS-EGARCH does not fit any of the yields perfectly.

In general it can be concluded that allowing for asymmetry in the volatility specification of the common component improves the DNS-TVV model in terms of fit. Yet, when the likelihood values and fit for both asymmetric models are concerned, there is no clear preference for one or the other as they are very close. However, due to the simpler structure and because of the fact that it nests the standard DNS, the T-GARCH specification seems to be the more appealing extension.

Table X: **GARCH Parameters for the DNS-FactorGARCH Model**

The table presents the estimates of γ_0, γ_1 and γ_2 in the volatility process of the common shock component in the DNS-FactorGARCH model. In this model the common component is included in the state equation of the state space framework. The parameters are estimated using maximum likelihood and robust standard errors are obtained using the sandwich estimator.

γ_0	γ_1	γ_2
0.0001	0.337	0.655***
Fixed	0.24443	0.20519

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

4.5 Time-Varying Volatility in the Factors (DNS-FactorGARCH)

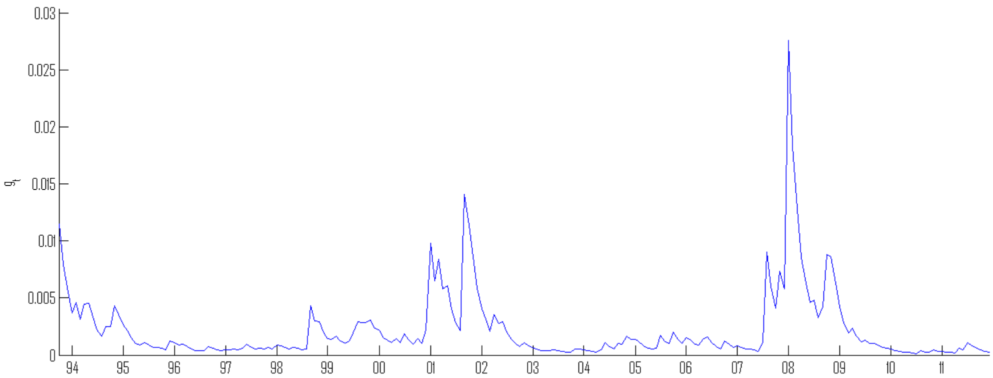
In addition to the DNS-GARCH model, Koopman, Mallee, and van der Wel (2010) present an alternative model in which the common shock component is included in the state equation, see (30). In this framework the common component, for which the variance is governed by a GARCH process, does not affect the yields directly, but only indirectly via the latent factors. Koopman, Mallee, and van der Wel (2010) find the model to have a higher log likelihood than the standard DNS model, but the increase is only small compared to the case in which the common term is included in the measurement equation of the state space model.

Table IV presents similar results as in Koopman, Mallee, and van der Wel (2010) for the DNS-FactorGARCH model in this empirical study. I find an increase in the likelihood value compared to the standard DNS, but it is about 10 times smaller than the increase obtained when the common component is included in the measurement equation. The use of a dataset that includes the true long end of the yield curve therefore does not change the relative performance of this model in the DNS-TVV class as was earlier held possible. The estimates of the VAR are shown in table XVIII and the latent factors are plotted in figure XV in the appendix. Again the persistence in the factors is very high and as in the DNS-2GARCH model, the only significant off-diagonal elements are those for the lagged level and curvature on the slope. The slope of the yield curve is negatively affected by the level and positively by the curvature in the previous period. The parameter λ in the DNS-FactorGARCH model is estimated at 0.0499 (standard error of 0.00017) which is close to the estimate of the standard DNS model and slightly lower than that of the other time-varying volatility models. Hence the factor loadings in this model are closer to that of the standard DNS than to the other extended models.

Table X gives the parameter estimates of the GARCH process for the volatility of the common component in the factors. In contrast to the other DNS-TVV models, the coefficient of the lagged volatility (γ_2) is relatively high compared to the weight that is put on new shocks (γ_1). Therefore the GARCH specification in the DNS-FactorGARCH model is much less responsive to new shocks and past jumps in volatility die out more gradually. To prevent identification issues γ_0 is fixed at 0.0001, hence the three loadings in $\boldsymbol{\Gamma_\nu}$ are estimated freely. Figure XII plots the volatility of the common component in the DNS-FactorGARCH model over time. Surprisingly, the overall pattern is completely different from what is found for the other DNS-TVV models. High levels are

Figure XII: **DNS-FactorGARCH Factors and Volatility**

The figure shows the volatility (h_t) of the common shock component (ε_t^*) over time for the DNS-FactorGARCH model. In this model the common shock component is included in the state equation.



observed after the burst of the internet bubble in 2001 and during the credit crunch in 2008. Furthermore, the peaks in volatility coinciding with the US downgrade and the emerging problems in the Eurozone that are present in all other DNS-TVV models, are not visible in the volatility plot presented here. Hence the volatility dynamics of the common shock component in the factors are completely different from that in the yields, as the process is much more persistent and clearly affected by different shocks.

The vector of estimated loadings for the common volatility component of the three factors is equal to $\mathbf{\Gamma}_{\boldsymbol{\nu}} = (-0.296, 5.926, 0.324)'$. A common shock therefore affects the latent factors, characterising the yield curve, in different directions. For example a positive shock leads to a lower yield in the long end of the curve (level decreases), an even stronger decrease in the short interest rate (slope increases) and a relative increase in the medium term yields (increase in curvature). The result is in line with the findings of Christiansen and Lund (2005) who find slope and curvature of the yield curve to be positively dependent on the short rate volatility in their combined model with a VAR for the factors and a GARCH-in-mean for the error term.

Table VI shows the average filtered errors and their standard deviations for the DNS-FactorGARCH model. These are close to the errors in the standard DNS model. The DNS-FactorGARCH model also has difficulties fitting the yields in both ends of the curve, as indicated by the large standard deviations. The mean average filtered error and the mean standard deviation across the ten maturities are slightly lower than in the standard DNS model, but overall the filtered estimates do not seem to improve much from including a common volatility component in the factors.

4.6 Volatility and Macroeconomic and Financial Variables (DNS-GARCHX-DRA and DNS-GARCHX-VIX)

Section 3.2 discusses the correlation between the squared changes in capacity utilisation, the federal funds rate, annual inflation, the VIX and yields for some maturities. Different maturities relate to different macroeconomic or financial variables with varying magnitudes. Despite none of the correlations being really high, exogenous factors can possibly partly capture additional dynamics in common volatility in the yield curve. Therefore a GARCH-X specification, as in (23), is introduced to present another new branch of models in the DNS-TVV class which I named DNS-GARCHX. As shown by table IV, the inclusion of macroeconomic and financial factors using the explained methodology, improves model fit. For both the DNS-GARCHX-DRA and the DNS-GARCHX-VIX model, the log likelihood is reasonably higher compared to that of the DNS-GARCH model. The second GARCH-X specification, including the VIX, even has the highest likelihood of all models that include only one common component. Adding only the VIX as an explanatory complement to the GARCH model increases the log likelihood by 25.28. The three macroeconomic variables also turn out to be a valuable addition. However, they perform less well according to the statistical information criteria, especially because this GARCHX-DRA extension uses three extra parameters compared to DNS-GARCH whereas this is only one in the DNS-GARCHX-VIX. However, in terms of filtered errors the model including macroeconomic factors does best of all single common component DNS-TVV models, as presented in table VI. It outperforms the standard DNS in terms of average filtered error and standard deviation for 7 out of 10 maturities. This result contrasts that of the DNS-GARCHX-VIX model, which outperforms in 6 out of 10 for both, similar to the other DNS-TVV models with a single common component. However, the result should be interpreted with caution. The average filtered error for the 6M maturity is very close, but not equal to zero (rounded to 0.000 in the table) for all models, except for the DNS-2GARCH and DNS-EGARCH. Yet, the DNS-GARCHX-DRA is the only model for which it is actually smaller (in absolute terms) than the average filtered error in the standard DNS model, but the difference is very minor[11]. Hence even though there is in fact a statistical outperformance by the DNS-GARCHX-DRA model concerning the filtered errors for the 6M yields, it is not economically relevant.

Now I first turn further attention to the DNS-GARCHX-DRA model in specific. Diebold, Rudebusch, and Aruoba (2006) find real economic activity relative to potential, annual inflation and the monetary policy instrument to have an influence on future movements in the yield curve. They add the three variables as additional factors to the Nelson-Siegel model in order to link the term structure to the macroeconomy. The three exogenous factors are proxied by capacity utilisation (CU), annual price inflation (INFL) and the monthly average federal funds rate (FFR). They are often seen as the minimum set of variables to explain the basic macroeconomic condition. I use the squared changes of the variables proposed by Diebold, Rudebusch, and Aruoba (2006) to gain additional explanatory power on the volatility dynamics of the common shock component in the measurement equation of the state space framework. The estimation results of the latent factor VAR model are presented in table XIX in the appendix. No remarkable differences with the other models are present in the VAR, except for the magnitude of

[11]The same holds for the standard deviation of the filtered error.

Table XI: **GARCH Parameters for the DNS-GARCHX-DRA and DNS-GARCHX-VIX Models**

The table presents the parameter estimates of the volatility processes of the common components in the DNS-GARCHX-DRA (panel (a)) and DNS-GARCHX-VIX (panel (b)) models. In the DNS-GARCHX-DRA model, the common volatility component is modelled via a GARCH-X (23) in which the squared changes of the macroeconomic variables used in Diebold, Rudebusch, and Aruoba (2006) are added to the volatility process as exogenous variables. These variables are capacity utilisation (CU), the federal funds rate (FFR) and annual price inflation (INFL). Panel (a) shows the estimates for $\gamma_0, \gamma_1, \gamma_2$, ϕ_{CU}, ϕ_{FFR} and ϕ_{INFL}. In the DNS-GARCHX-VIX model, the squared monthly VIX $\left(\frac{VIX}{100\sqrt{12}}\right)^2$ is used as exogenous variable to extend the volatility specification. Panel (b) shows the estimates for $\gamma_0, \gamma_1, \gamma_2$ and ϕ_{VIX}. Robust standard errors are obtained using the sandwich estimator.

(a) GARCHX-DRA Parameters

γ_0	γ_1	γ_2	ϕ_{CU}	ϕ_{FFR}	ϕ_{INFL}
0.002***	0.877***	0.054***	0.002***	0.017***	0.0001
0.000004	0.00211	0.00099	0.0000001	0.00001	0.00084

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) GARCHX-VIX Parameters

γ_0	γ_1	γ_2	ϕ_{VIX}
0.001***	0.814***	0.085***	0.560***
0.00006	0.01024	0.00490	0.01208

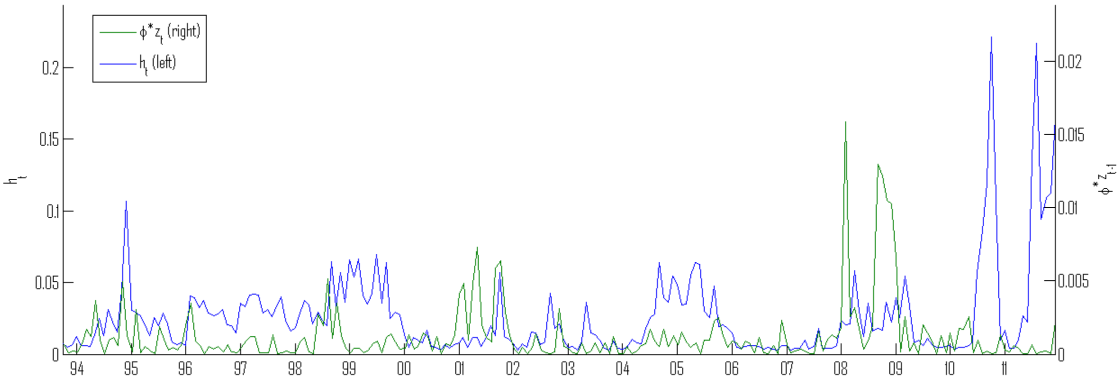
Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

the constants. They are slightly higher, in absolute terms, than the constants in the other DNS models and in contrast to those, they are all significantly different from zero here because of the relatively small size of the standard errors. Plotting the time series of the latent factors along with those of the DNS-GARCH (see figure XV in the appendix) model showed that they are very close. Therefore time series of the factors of the DNS-GARCHX-DRA model are not presented. The estimate of λ is 0.0547 (standard error of 0.0000001), close to the estimate in the DNS-GARCH model. Panel (a) in table XI presents the estimated coefficients of the GARCH-X process. All standard GARCH parameters turn out to be significant in the GARCH-X-DRA model as well as the coefficients of CU and FFR. The parameters γ_0, γ_1 and γ_2 take similar values as in the DNS-GARCH model and the complementing coefficients for the exogenous variables are relatively small. Therefore the plot of the volatility time series (h_t) in panel (a) of figure XIII has only minor differences compared to that of the DNS-GARCH model in figure IV(a). Figure XIII(a) also shows the influence of the exogenous factors ($\boldsymbol{\phi}'\boldsymbol{z}_{t-1}$) in the GARCH process given by (23). These mostly cause the differences in common volatility between the benchmark DNS-TVV model and the model discussed here. The influence of the macroeconomic variables is most pronounced at the height of the global financial crisis in 2008 where we find two large spikes. Hence the increase in volatility in the yield curve at that time can for a significant part be accredited to changes in the macroeconomy. The loadings of the common component are nearly identical in the DNS-GARCH and DNS-GARCHX-DRA model and hence highest for the medium term maturities, see figure XIII(b).

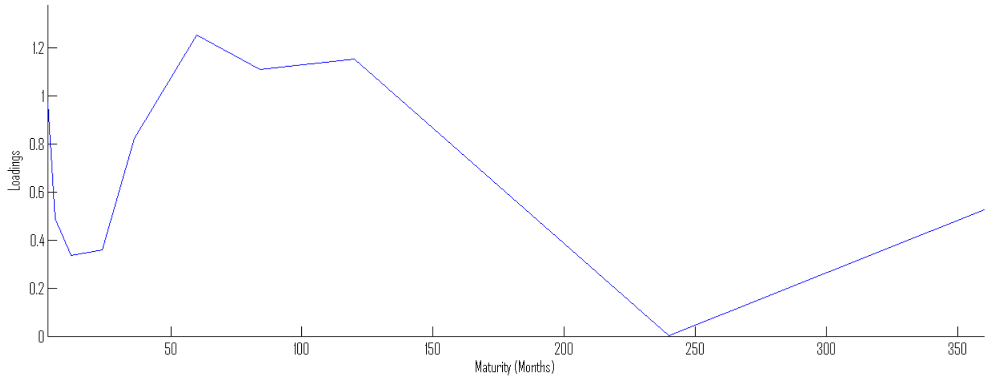
The finding of significant influence of two out of three variables that are regarded as the minimal set of fundamental variables to describe the macroeconomy, does not automatically imply that there is a direct link between yield curve volatility and the macroeconomic condition. The exogenous variables are included in the GARCH process as squared first differences and therefore do indicate that changes in observed economic indicators affect volatility of interest rates. However, it is the shock to financial markets in general as a consequence of new information flows that causes a change in volatility rather than a direct effect of a change in the macro variables. Changes in the FFR and CU can come as a surprise to the market, reveal new information and possibly change views on the economic future to which investors respond, leading to increased volatility in the term structure. A positive change in CU can indicate improving prospects on real activity, boosting stocks. Investors then possibly shift funds from bonds to stocks, affecting interest rate volatility. In that way the change in CU indirectly affects interest rate volatility rather than it being the actual and direct cause of an increase in common volatility in yields. The result that FFR significantly affects h_t supports the findings of Lee (2002) who finds that interest rate volatility is sensitive to changes in the FFR. Although in this thesis no distinction is made between unanticipated and anticipated changes and rate hikes and decreases, which turns out to be important in Lee (2002), there is evidence that the Federal Reserve policy actions do affect common volatility in the term structure. The estimated coefficient for annual price inflation (ϕ_{INFL}) is not significant. Monthly changes in annual inflation do not significantly affect the monthly common volatility in the yield curve. This result is related to that of Mishkin (1992) who finds no presence of the Fisher effect in the short run. He argues that during certain periods, inflation and interest rates trend together in the long run and Fisher's theory strongly holds, but his analysis does not support presence of a relation between the two in the short run. Moreover, the periods of a breakdown in the trending behaviour of inflation and interest rates and the consequent weak support of the Fisher effect are those where there is increased volatility. In other words, even the long run Fisher effect only finds strong evidence in times of low volatility. Given that the relevance of the common shock component is largest in periods of increased volatility, I relate the insignificant effect of inflation and common volatility to this result.

The second DNS-GARCHX model incorporates the VIX as exogenous variable in the GARCH equation for the common volatility. We have seen from the DNS-2GARCH model that stock market volatility is also priced in the yield curve and is partially captured through a second common volatility component in the DNS model. However, from the noteworthy increase in likelihood for the DNS-GARCHX-VIX model, we see that additional explanatory power can also be obtained by incorporating a measure of stock market volatility in the single common component model via a GARCH-X specification. Yet, the difference is that in the latter case the common component still resembles general interest rate volatility, as in the DNS-GARCH model, and the VIX only slightly affects the dynamics of it. In contrast, in the DNS-2GARCH model, the stock market volatility is governed by a completely separate and own GARCH process,

Figure XIII: **DNS-GARCHX-DRA Common Volatility and Loadings**
The figure shows plots of the volatility (h_t) of the common shock component (ε_t^*) and the influence of exogenous factors ($\boldsymbol{\phi}'\boldsymbol{z}_{t-1}$) on the GARCH process over time for the DNS-GARCHX-DRA model in panel (a). Panel (b) plots the loadings for the different yields against maturity (in months). The loadings are the elements of the vector $\boldsymbol{\Gamma}_\varepsilon$ and are defined as the sensitivities of the various yields to the common shock.



(a) Common Volatility (h_t) and Exogenous Influence ($\boldsymbol{\phi}'\boldsymbol{z}_{t-1}$)



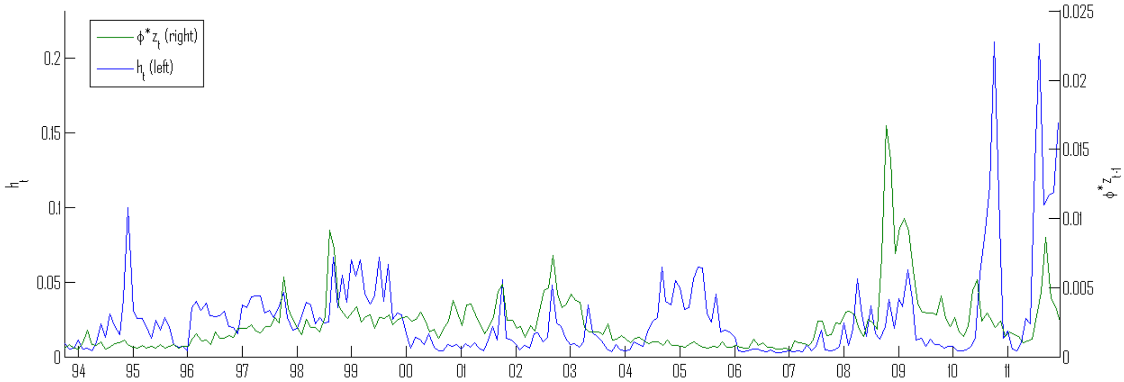
(b) Common Component Loadings ($\boldsymbol{\Gamma}_\varepsilon$)

leaving much more freedom and resulting in much more improvement in model fit.

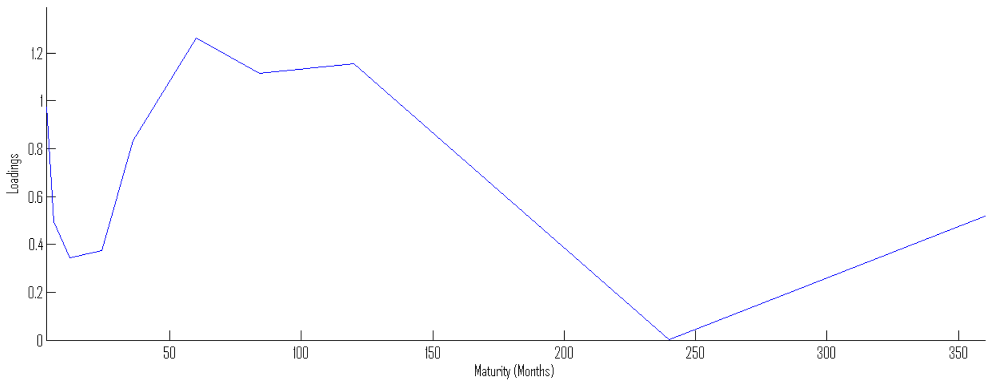
Table XX in the appendix gives the results of the VAR for the DNS-GARCHX-VIX model. The time series of the latent factors turn out to be very close to those from the DNS-GARCH model (see figure XV in the appendix) and are therefore not separately shown here. The parameter λ is estimated at 0.0547 (standard error of 0.00053), equal to the estimate in the DNS-GARCHX-DRA model. Panel (b) in table XI presents the estimated GARCH-X coefficients which are all significant. As in the other models the responsiveness of h_t to new shocks is high and the weight on the lagged level of volatility low. The standard GARCH specification is complemented by the squared monthly VIX (in decimal terms) for which the coefficient ϕ_{VIX} takes the value 0.560[12]. In panel (a)

[12]The squared monthly VIX in decimal terms is given by $\left(\frac{VIX}{100\sqrt{12}}\right)^2$. The VIX roughly translates to the annualized expected 30-day movement in the S&P 500 in percentage terms, hence it concerns a standard deviation. As h_t represents the monthly variance of the common shock component in the yields in decimal terms, a transformation is required.

Figure XIV: **DNS-GARCHX-VIX Common Volatility and Loadings**
The figure shows plots of the volatility (h_t) of the common shock component (ε_t^*) and the influence of exogenous factors ($\boldsymbol{\phi}'\boldsymbol{z}_{t-1}$) on the GARCH process over time for the DNS-GARCHX-VIX model in panel (a). Panel (b) plots the loadings for the different yields against maturity (in months). The loadings are the elements of the vector $\boldsymbol{\Gamma}_\varepsilon$ and are defined as the sensitivities of the various yields to the common shock.



(a) Common Volatility (h_t) and Exogenous Influence ($\boldsymbol{\phi}'\boldsymbol{z}_{t-1}$)



(b) Common Component Loadings ($\boldsymbol{\Gamma}_\varepsilon$)

of figure XIV the volatility dynamics of the common component (h_t) again resemble the general interest rate volatility from the common component in the DNS-GARCH model with only minor deviations. The influence of the VIX on the GARCH process ($\boldsymbol{\phi}'\boldsymbol{z}_{t-1}$) is plotted in the same graph and, as with the macroeconomic variables, is largest at the height of the financial crisis in 2008[13]. Furthermore, increased tensions in the stock market following the US downgrade in the summer of 2011 also seem to significantly affect the common yield curve volatility. However, altogether the inclusion of a stock market volatility index in the GARCH equation does not dramatically change the volatility of the common shock component in the yields and make it show similar dynamics as $h_{1,t}$ in panel (a) in figure VI. That is because the shock is filtered out of the data using the Kalman filter. The volatility dynamics of this shock do not completely

[13]The plot of the influence of the VIX on the GARCH process here is exactly equal to the plot of the VIX itself in figure 3(d) up to a scaling factor as it is the only exogenous variable included in the volatility process of the DNS-GARCHX-VIX model.

change the filtered estimates. Hence including the VIX in the GARCH-X model does not force a stock market volatility component, as in the DNS-2GARCH, to be filtered out first. The filter extracts the largest common factor from the data as the common shock component, for which the variance turns out to mostly capture general interest rate market volatility. The loadings in panel (b) of figure XIV, are also nearly equal to those in the benchmark DNS-TVV model, which is not surprising given that the volatility of the common shock component in the DNS-GARCHX-VIX model is very close to that of the DNS-GARCH model.

5 Out-of-Sample Forecasting

This chapter assesses the forecasting ability of all models presented in this thesis for short (1-month), medium (6-months) and long (12-months) horizons. Koopman, Mallee, and van der Wel (2010) only report in-sample fit statistics for their DNS-GARCH model and no out-of-sample forecasting results. However, besides fitting current and describing past yield curve dynamics, term structure models are also used for what Diebold and Li (2006) call a key practical problem, namely to predict future interest rates. Hence it is useful to also evaluate the forecasting performance of the various models presented here.

In the forecasting procedure I first estimate the parameters of the different models over a subsample period for the state space framework as in (31). From the Kalman filter I obtain the filtered latent factors for the full sample period and subsequently predict the h-month ahead forecast at every point in the out-of-sample period by iterating forward the state equation h periods. Therefore the h-month ahead forecast of the state vector is given by

$$\hat{\boldsymbol{\alpha}}_{t+h|t} = \left(\boldsymbol{I}_d + \sum_{i=1}^{h-1} \boldsymbol{K}^i\right) \boldsymbol{C} + \boldsymbol{K}^h \boldsymbol{\alpha}_{t|t}, \tag{41}$$

with h being the forecast horizon, d is equal to the dimension of $\boldsymbol{K}$ and the other parameters are as defined before. For the 1-month ahead forecast (41) is equal to the first equation in the prediction step in the Kalman filter (see (32)) and therefore automatically follows from the recursions, but for the 6- and 12-month the iterations are needed to obtain the estimates. After the state vector is forecast, the h-month ahead yield forecasts follow from

$$\hat{\boldsymbol{y}}_{t+h|t} = \boldsymbol{H} \hat{\boldsymbol{\alpha}}_{t+h|t}, \tag{42}$$

where $\hat{\boldsymbol{y}}_{t+h|t}$ is the $N \times 1$ vector of h-month ahead yield forecasts for all N maturities, made at time t, and the other parameters are as defined before. As the common shock component is zero in expectation, it does not play a direct role in the forward iterations to forecast the yield curve and hence the predictions only depend on the three Nelson-Siegel factors representing level, slope and curvature. These roles are ensured by the fact that only the first three elements of $\boldsymbol{C}$ and the upper 3×3 block of $\boldsymbol{K}$ contain non-zero values. However, the time-varying volatility is accounted for in the filtering steps in the Kalman filter and therefore affects the estimates of the factors. Hence, the common shock does have an indirect influence on the predictions through $\boldsymbol{\alpha}_{t|t}$.

I use a simple random walk (RW) forecast as the benchmark for comparison of the predictive accuracy of the models in this evaluation. Duffee (2002) states that most term structure models already encounter difficulties beating this naive prediction method which makes it a useful and simple benchmark model. The RW model is given by

$$y_t(\tau_i) = y_{t-1}(\tau_i) + \varepsilon_{i,t}, \quad \varepsilon_{i,t} \sim \mathrm{N}\left(0, \sigma_i^2\right), \tag{43}$$

where $y_t(\tau_i)$ is the yield at time t for maturity τ_i, with $i = 1, ..., N$ being the different maturities in the sample. The RW h-month ahead forecast for the yield is equal to the last observation and hence given by

$$\hat{y}_{t+h}(\tau_i) = y_t(\tau_i), \tag{44}$$

which simply implies that interest rates do not change over the forecast horizon.

For the forecast evaluation, the estimation period runs from October 1993 until November 2003, leaving an out-of-sample evaluation period from December 2003 until December 2011. The parameters of the various models are estimated over the subsample period once and held constant for the entire out-of-sample evaluation period. The reason for not choosing a dynamic forecasting method with a moving or expanding window is because the parameter estimation is demanding and especially for the more advanced models it can be hard to find the optimum of the likelihood function. When using a dynamic forecasting method, the possibility of ending up in a local maximum if a recursive procedure is followed probably leads to less optimal forecasts compared to using a static forecasting method as I do in this study.

Table XII shows the root mean squared errors (RMSE) of the RW model, and the ratios of those of the standard DNS and the DNS-TVV models relative to the RW for out-of-sample forecasts at 1-,6- and 12-month horizons. Outperformance of the RW model is indicated by the shaded cells. In order to assess the significance of the relative forecast accuracy of the DNS models compared to the benchmark, table XXI in appendix C gives the Diebold and Mariano (1995) (DM) statistics of the predictions of all models for every maturity at the three different horizons. The loss differentials are calculated assuming that the squared forecast error is the relevant loss function. The test statistics are asymptotically standard normally distributed[14]. In the table a positive DM-statistic indicates an outperformance of the RW by a particular DNS model. The loss function of the DM-statistics is chosen such that it allows them to indicate significance of out- or underperformance of a certain model based on the RMSE. Due to the fact that the root of the mean is a monotonically increasing function, the indicators in tables XII and XXI point in the same direction and hence the tables are complementary.

For the 1-month ahead forecasts the RMSEs of the RW model are of similar mag-

[14]The h-step ahead forecasts are assumed to be $h-1$ dependent when calculating the DM-statistics. Diebold and Mariano (1995) and Giacomini and White (2006) note that this assumption may not hold in reality, but argue it works well in practical applications and therefore assume it a reasonable benchmark.

nitude for all maturities, indicating an equal forecasting performance across the yield curve. The standard DNS model produces predictions that are close to the benchmark in terms of accuracy and outperforms, although not significantly, in 4 out of 10 maturities. Considering RMSEs, the only model that is able to do better is the DNS-FactorGARCH model, which beats the benchmark for 5 maturities. All DNS-TVV models, except for the DNS-2GARCH, outperform the RW forecast for the 6M and 1Y maturities. The in-sample fit of these models was also good for these points on the yield curve, as was indicated in table VI. Remarkable is, however, that none of the models is able to significantly outperform the benchmark at even a single maturity.

Predictions at the medium horizon show a better performance than for the short 1-month horizon. All DNS-TVV models outperform the benchmark at the three shortest maturities and the outperformances for the 3M are significant at the 5% or 10% level. Also for the 1Y yield the relative forecast accuracy is significantly better for all DNS-TVV models, except for DNS-TGARCH. The best performing models in terms of RMSE are the standard DNS, DNS-EGARCH and DNS-FactorGARCH as they beat the benchmark for 4 maturities. Yet, if we also consider the significance of the relative accuracy, the DNS-2GARCH and DNS-FactorGARCH show the most promising results. The first shows significantly positive DM-statistics in table XXI for the 3M, 6M and 1Y maturities at 95% and 99% levels. However, the performance of the longer maturity predictions is poor, with highly significant underperformance relative to the RW model for the six longest maturities. The DNS-FactorGARCH does not achieve as good results in the short end of the curve, although it significantly beats the benchmark, but does only significantly underperform in the long end for the 7Y, 10Y, 20Y and 30Y yields. Moreover, this underperformance is not as large as for the DNS-2GARCH and the other DNS-TVV models. Remarkable is that despite it does not significantly beat the forecast accuracy of the benchmark at any maturity, the standard DNS model only significantly underperforms for the 30Y yield. Overall the DNS-TVV models seem to do best at the short end of the curve, which is where the RW model shows the largest RMSEs, and worst at the long end, where the benchmark shows a better performance.

The most promising results come from the long horizon forecasts. All models, except for the DNS-GARCHX-DRA, outperform the benchmark for the 3M, 6M, 1Y and 2Y maturities, based on RMSE. Furthermore, the relative accuracy of all DNS-TVV models for the shortest three maturities is significantly higher than that of the RW forecast at the 99% level. The outperformances based on RMSE are roughly in the range of 20-25%. The RMSEs of the RW model predictions are again highest in the most volatile yields which are those at the short end of the curve. At a long horizon a naive forecasting method does not turn out to produce good results for short maturities. The standard DNS model does somewhat better and shows significantly higher accuracy for the most volatile yields at the 12-month horizon. The results in the tables show that incorporating a second common volatility component, allowing for asymmetric volatility or adding exogenous variables to the GARCH equation is not notably changing the forecasting performance of the DNS-GARCH model. The highest forecast accuracy is again achieved by the most parsimonious DNS-TVV model, namely the DNS-FactorGARCH. Its forecasts outperform the RW in terms of RMSE for the five shortest maturities. According to the DM-statistics the accuracy is significantly higher for the 3M, 6M, 1Y and 2Y yields and significantly lower for the longest four maturities, thereby balancing the out- and underperformances.

Similar to Moench (2008), who uses a term structure model with a broad macroeconomic information set, I find that it is very difficult to beat the RW forecasts at the 1-month horizon. The RMSEs for all maturities are relatively small, indicating that the naive prediction method provides reasonable forecasts. At the medium and long horizon the performance of the DNS-TVV models is better for the shorter maturities. These are the ones for which the accuracy of the RW model predictions is low, hence leaving more room for improvement. The reason for the low accuracy of the naive benchmark for these yields is probably the higher volatility, as is shown in table I. Allowing for time-varying volatility in the DNS model turns out to improve the forecast accuracy in this part of the curve as the models apparently are better able to capture purely the dynamics of yields in the short end. The common volatility component therefore seems to be a valuable addition to the DNS model when short maturity yield forecasting is concerned at longer horizons. However, care should be taken for the longer maturities as the relative accuracy compared to the RW model at the long end of the curve is poor. This poor performance is probably caused by the fact that the longer maturity yields stayed fairly stable over the entire out-of-sample period, as can be seen from figure I, which makes it very hard to beat RW predictions.

Overall it can be concluded that, as is often the case with forecasting models, parsimoniousness is best. The standard DNS model does not suffer from large underperformances in parts of the yield curve at the three horizons and it outperforms the benchmark for some short maturities. Yet, the DNS-FactorGARCH wins the horse race, with especially good results at the long horizon. Hence the most parsimonious DNS-TVV model turns out to be the best forecasting model.

Table XII: **Root Mean Squared Error of Forecasts**

The table shows the (relative) root mean squared error (RMSE) of the 1-month, 6-months and 12-months ahead forecasts of a random walk forecast (RW), of the standard DNS model and those of the DNS-TVV models. The RMSE for the RW forecasts are reported (in basis points) at every forecast horizon and a ratio of the relative RMSE is presented for the different DNS models. The shaded cells highlight outperformance of the RW forecast, meaning a relative RMSE below 1.

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	20Y	30Y
1 Month Ahead										
Random Walk	25.62	23.76	23.13	25.41	27.19	27.91	28.04	27.47	26.49	25.73
DNS	1.071	0.998	1.060	1.063	0.998	0.990	1.005	0.979	1.131	1.137
DNS-GARCH	1.150	0.922	0.914	1.354	1.813	2.143	2.127	2.020	1.474	1.817
DNS-2GARCH	1.316	1.036	1.178	1.835	2.363	2.653	2.542	2.328	1.672	1.855
DNS-TGARCH	1.146	0.926	0.916	1.359	1.823	2.160	2.147	2.044	1.498	1.849
DNS-EGARCH	1.150	0.916	0.898	1.333	1.784	2.112	2.100	1.996	1.449	1.806
DNS-FactorGARCH	1.115	0.930	0.966	1.028	1.004	0.993	0.999	0.978	1.258	1.176
DNS-GARCHX-DRA	1.169	0.933	0.940	1.416	1.888	2.223	2.205	2.098	1.529	1.896
DNS-GARCHX-VIX	1.164	0.929	0.929	1.388	1.853	2.186	2.170	2.064	1.507	1.862
6 Months Ahead										
Random Walk	86.88	86.88	83.17	78.79	77.32	73.39	69.98	64.97	58.52	60.04
DNS	0.968	1.007	0.995	0.992	0.984	1.007	1.029	1.055	1.035	1.112
DNS-GARCH	0.884	0.895	0.901	1.048	1.213	1.373	1.406	1.429	1.308	1.377
DNS-2GARCH	0.873	0.861	0.871	1.037	1.216	1.371	1.392	1.384	1.243	1.282
DNS-TGARCH	0.893	0.907	0.909	1.054	1.223	1.389	1.426	1.453	1.336	1.406
DNS-EGARCH	0.895	0.905	0.887	0.980	1.110	1.249	1.283	1.302	1.190	1.264
DNS-FactorGARCH	0.883	0.891	0.910	0.976	1.013	1.046	1.056	1.071	1.110	1.091
DNS-GARCHX-DRA	0.886	0.897	0.904	1.064	1.239	1.409	1.446	1.472	1.349	1.422
DNS-GARCHX-VIX	0.884	0.895	0.904	1.059	1.230	1.395	1.431	1.456	1.335	1.407

Table XII: *(continued)*

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	20Y	30Y
12 Months Ahead										
Random Walk	153.88	150.99	139.76	121.09	108.04	86.47	74.76	64.11	54.11	57.98
DNS	0.881	0.903	0.897	0.912	0.937	1.033	1.118	1.236	1.274	1.373
DNS-GARCH	0.757	0.774	0.814	0.983	1.1711	1.452	1.575	1.702	1.625	1.626
DNS-2GARCH	0.765	0.771	0.792	0.924	1.081	1.318	1.414	1.498	1.401	1.398
DNS-TGARCH	0.765	0.783	0.821	0.989	1.181	1.473	1.605	1.742	1.677	1.677
DNS-EGARCH	0.785	0.798	0.804	0.897	1.021	1.227	1.319	1.411	1.326	1.360
DNS-FactorGARCH	0.782	0.793	0.819	0.901	0.968	1.069	1.118	1.189	1.228	1.210
DNS-GARCHX-DRA	0.760	0.778	0.822	1.001	1.199	1.495	1.628	1.765	1.694	1.695
DNS-GARCHX-VIX	0.760	0.778	0.820	0.994	1.187	1.477	1.607	1.742	1.672	1.674

6 Robustness of the Results

In this chapter I discuss the robustness of the results presented in this thesis in twofold. First, by considering the model fit on a subsample in section 6.1. Second, in section 6.2, by commenting on the sensitivity of the model outcomes to choices of the initial parameter values.

6.1 Results Based on a Subsample

In order to assess the robustness of the results of the different DNS-TVV models in terms of model fit, I consider a smaller subsample of the full sample to evaluate the different common volatility extensions. This subsample runs from October 1993 until November 2003 and hence is equal to the estimation window used for forecasting, see chapter 5. Table XIII presents the log likelihood, AIC, BIC and LR-statistics of the DNS-TVV models compared to the standard DNS and where possible also relative to the DNS-GARCH. All DNS-TVV models again have much higher likelihood values, and lower AIC and BIC, than the standard DNS model and hence the earlier conclusion of time-varying volatility being a valuable extension, is robust to changing the sample. This observation is in line with Koopman, Mallee, and van der Wel (2010) who also estimate their DNS-GARCH model on a smaller subsample and find it to have significantly better fit than the standard DNS model. However, comparison of the asymmetric volatility and GARCH-X models to the DNS-GARCH model is discouraging. The likelihood values are only minimally higher and the increases do not outweigh the punishment from parameter addition in the AIC and BIC statistics[15]. A reason for this disappointing result could be the chosen time frame. During almost this entire period the volatility in the term structure was relatively low, the yield curve was concave and upward sloping and the general level of interest rates was gradually declining, as can be seen from figure I. In such a period, the addition of a common component still is a valuable extension of the standard DNS. However, the exact choice for the specification of the volatility process to govern the variance of the component is not resulting in striking differences.

In contrast, the DNS-2GARCH model does also have a significantly higher likelihood value than the DNS-GARCH for the subsample, indicated by the large LR-statistic. According to the AIC and BIC the addition of the twelve extra parameters to model the volatility dynamics of and sensitivity to the second common component is more than compensated for by the increase in model fit. So the finding that the second component has additional explanatory power as a complement to the first common component turns out to be robust to changing the evaluation window. As the second common component is found to roughly represent shocks to the stock market, it is not surprising that it is a valuable addition to the DNS-GARCH model also in the smaller time frame since panels (a) and (b) in figure VII show that stock market volatility has seen increased levels for a large part of the subsample. Furthermore, $h_{1,t}$ even has its peak during the subperiod

[15] The DNS-EGARCH model even has a lower log likelihood value than the DNS-GARCH over the subsample which is possible because the GARCH model is not nested in the E-GARCH specification, as is explained in section 4.4.

Table XIII: **Log Likelihood, AIC and BIC for the Models**
The Log Likelihood, Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are reported in the table together with the number of parameters ($\#\theta$) of the different models estimated over the sample from October 1993 until November 2003. The test statistics of Likelihood Ratio (LR) tests to compare the different model extensions to the standard DNS and the DNS-GARCH are reported in the last two columns, respectively.

	$\ell(\theta)$	$\#\theta$	AIC	BIC	LR-Statistics	
DNS	874.27	29	-1690.53	-1542.44		
DNS-GARCH	1057.83	41	-2033.67	-1824.29	367.1	
DNS-2GARCH	1206.31	53	-2306.61	-2035.96	664.1	296.95
DNS-TGARCH	1057.88	42	-2031.75	-1817.27	367.2	0.08
DNS-EGARCH	1054.37	42	-2024.73	-1810.25	360.2	
DNS-FactorGARCH	883.08	34	-1698.16	-1524.54	17.6	
DNS-GARCHX-DRA	1058.20	44	-2028.41	-1803.72	367.9	0.74
DNS-GARCHX-VIX	1058.50	42	-2033.00	-1818.53	368.5	1.34

and we can therefore not speak of a relatively calm period in the stock market from October 1993 until November 2003.

6.2 Parameter Robustness to Initial Values

Estimating the parameters of the various DNS models in the state space framework is done by finding parameter values that optimise the likelihood function. Due to the large number of parameters in the models presented in this thesis, the optimisation problems are highly dimensional and the likelihood functions can have multiple local maxima. In order to start the optimisation procedure, I choose certain initial values for the model parameters that I expect are most likely to lead to the global optimum. However, the sensitivity of the optimisation outcome to the initial values can be large if certain dynamics are not strongly present in the data and hence the algorithm can encounter difficulties finding the global maximum of the likelihood function. This problem is also recognised by Gauthier and Simonato (2012). They try to solve the issue by linearising Nelson-Siegel and Svensson models and including prior information on the parameter values. Their procedure decreases the dimension of the optimisation, produces more stable estimates and converges quicker than standard approaches. Using US coupon bond prices from 1987 to 1996 they find that their methodology improves the probability of finding the optimal parameter estimates.

In the estimation procedure I did not experience large difficulties in finding the optimum for the standard DNS, the DNS-GARCH, the DNS-2GARCH and the DNS-FactorGARCH model. The dynamics of the latent factors and the common volatility components seem to be strongly present in the data and the dynamics of the standard GARCH process are not hard to explore by the algorithm. Repeating the optimisation procedure with different initial values should lead to the same optimum every time in order to be sure that the global maximum of the likelihood function is found. Experimenting with different starting values for the parameters and using estimates of local optima as new initial parameter choices leads to quick convergence to the global optimum for the four models.

For the asymmetric volatility and GARCH-X models the optima are much harder to find. The problem with these is that the alternative volatility dynamics of the common component are not as strongly present in the data and because of the high dimensionality of the problem, the algorithm easily ends up in local maxima. The outcome of the optimisation procedure therefore is very sensitive to the initial parameter values chosen for the DNS-TGARCH, DNS-EGARCH, DNS-GARCHX-DRA and DNS-GARCHX-VIX. A lot of repetition of the optimisation procedure at different starting values is required to ensure finding the global optimum. This issue is concerning for practitioners. Despite the fact that the models in this study are used to fit and forecast only monthly yield curves, it is undesirable if calibration requires much effort. I use MATLAB to estimate all models, but other software packages might be more efficient in finding the optima of the high dimensional optimisation problems of these four DNS-TVV models. The linearisation methodology of Gauthier and Simonato (2012) for Nelson-Siegel and Svensson models can also provide a (partial) solution to the issue. They only introduce the concept for the basic models, but it seems to be a good starting point to apply the techniques on more advanced models as the DNS of Diebold and Li (2006), in a state space framework and on the models presented in this thesis. This is a topic of further research which enables to make the models workable in practice.

7 Conclusion

In this thesis I extend the work of Koopman, Mallee, and van der Wel (2010) who introduce the concept of time-varying volatility in the Dynamic Nelson-Siegel model using a common volatility component, modelled via a standard GARCH process. I use a new dataset of US Treasury yields, including maturities up to 30 years, in order to consider possible different dynamics in common volatility in the short and long end of the yield curve. This approach leads to my first contribution. By introducing a second common volatility component to the measurement equation of the state space framework I find that shocks to interest rate and stock markets are both priced in the term structure. Long maturities tend to be more sensitive to the common component that seems to resemble volatility in stocks than short maturities and common shocks to interest rate markets in general most strongly affect medium term yields. The extension greatly improves model fit and the result is robust to different sample choice.

My second contribution focuses on alternative specifications for the volatility process of the common shock component. I consider asymmetric models and possible influence of macroeconomic and financial factors. Volatility in financial markets is often found to be asymmetrically affected by positive and negative shocks. Using Threshold-GARCH and Exponential-GARCH models I find that this asymmetry is also present in the common shock component of the DNS model with time-varying volatility (DNS-TVV). Following Diebold, Rudebusch, and Aruoba (2006) who extend the standard DNS model by including macroeconomic variables, I use the same exogenous factors in a GARCH-X specification in the DNS-TVV class of models. Changes in real economic activity and in the federal funds rate turn out to significantly affect the volatility of the common component. Furthermore, in a different model, I find that a measure of stock market volatility also has a significant impact on the common volatility process when modelled via GARCH-X. All proposed extensions of the standard GARCH improve the fit of the baseline DNS-TVV model. However, when a smaller and less volatile subperiod of the dataset is analysed, the results for alternative specifications of the GARCH equation turn out not to be robust. In periods of calmness and absence of large movements in the yield curve, the standard GARCH process in the DNS-TVV class is not outperformed by the alternative volatility models in terms of fit.

The DNS-TVV models are able to significantly outperform a naive random walk forecast for short maturities at medium and long term horizons. Allowing for time-varying volatility in the DNS enables the model to better capture dynamics in the most volatile yields and produce relatively accurate 6- and 12-month ahead forecasts. However, the naive model forecasts again turn out to be very difficult to beat at the short horizon, also see Duffee (2002). For longer maturity yields the predictions of the DNS-TVV models are poor compared to those of the random walk. Parsimoniousness turns out to be key, the time-varying volatility model with the smallest number of parameters produces the best results. Incorporating the common volatility component in the state equation of the state space framework seems to be the most interesting alternative when forecasting is concerned.

8 Suggestions for Further Research

The extensions on the work of Koopman, Mallee, and van der Wel (2010) presented in this thesis offer several directions for further research. I discuss three suggestions I find particularly interesting. First, the DNS-2GARCH model is introduced here using standard GARCH processes for the two volatility components. However, as the dynamics of both turn out to be quite different and given the interpretations of the volatilities to represent stock market and general interest rate market volatility, it can be worthwhile to specify the GARCH models differently. For example, the variance of the component relating to stock market volatility is possibly better described by the GARCHX-VIX process. On the other hand, the interest rate market volatility dynamics might be captured more efficiently by an asymmetric model or a GARCH-X including macroeconomic factors. Second, in previous literature yield volatility often depends on the level of interest rates, as in the famous model by Cox, Ingersoll, and Ross (1985). Brenner, Harjes, and Kroner (1996) discuss a framework that allows interest rate volatility to depend on yield levels and on information shocks. As the level factor affects all maturities equally in the DNS model, it could also affect volatility in different yields similarly. Therefore the general level of interest rates can possibly yield additional explanatory power on the common volatility component. The DNS-GARCHX model introduced in this thesis easily allows to implement this idea in the DNS-TVV class of models. Third, the concept of time-varying volatility in the DNS class can be introduced to some popular extensions. De Pooter (2007) notes that a lot of practitioners use the Svensson (1995) model to construct zero-coupon yield curves. It is interesting to see whether allowing for time-varying volatility also improves fit for this more flexible extension of the standard DNS model as it might by itself be better able to accommodate to exogenous shocks. More recently, Christensen, Diebold, and Rudebusch (2011) proposed the arbitrage-free Nelson-Siegel (AFNS) model. The absence of arbitrage is key in for example using term structure models to price derivatives which makes the AFNS a very valuable contribution to practice. Time-varying volatility in the AFNS possibly improves the accuracy of this interesting model.

References

Ang, A. and M. Piazzesi (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary Economics 50*, 745–787.

Bekker, P. and K. Bouwman (2009). Arbitrage smoothing in fitting a sequence of yield curves. *International Journal of Theoretical and Applied Finance 12*, 577–588.

Bianchi, F., H. Mumtaz, and P. Surico (2009). The great moderation of the term structure of UK interest rates. *Journal of Monetary Economics 56*, 856–871.

Bollerslev, T. (1986). Generalised autoregressive conditional heteroskedasticity. *Journal of Econometrics 51*, 307–327.

Brenner, R., R. Harjes, and K. Kroner (1996). Another look at models of the short-term interest rate. *Journal of Financial and Quantitative Analysis 31*, 85–107.

Christensen, B. and M. van der Wel (2010). An asset pricing approach to testing general term structure models including Heath-Jarrow-Morton specifications and affine subclasses. *Working Paper*.

Christensen, J., F. Diebold, and G. Rudebusch (2011). The affine arbitrage-free class of Nelson-Siegel term structure models. *Journal of Econometrics 164*, 4–20.

Christiansen, C. and J. Lund (2005). Revisiting the shape of the yield curve: The effect of interest rate volatility. *Working Paper*.

Coroneo, L., K. Nyholm, and R. Vidova-Koleva (2011). How arbitrage-free is the Nelson-Siegel model? *Journal of Empirical Finance 18*, 393–407.

Cox, J., J. Ingersoll, and S. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*, 385–407.

De Pooter, M. (2007). Examining the Nelson-Siegel class of term structure models. *Working Paper*.

Diebold, F. and C. Li (2006). Forecasting the term structure of governmentbond yields. *Journal of Econometrics 130*, 337–364.

Diebold, F. and R. Mariano (1995). Comparing predictive accuracy. *Journal of Business & Economic Statistics 13*, 134–144.

Diebold, F., G. Rudebusch, and S. Aruoba (2006). The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of Econometrics 131*, 309–338.

Duffee, G. (2002). Term premia and interest rate forecasts in affine models. *Journal of Finance 57*, 405–443.

Duffee, G. (2011). Information in (and not in) the term structure. *Review of Financial Studies 24*, 2895–2934.

Duffie, D. and R. Kan (1996). A yield-factor model of interest rates. *Mathematical Finance 6*, 379–406.

Dungey, M., M. McKenzie, and D. Tambakis (2009). Flight-to-quality and asymmetric volatility responses in US Treasuries. *Global Finance Journal 19*, 252–267.

Engle, R. and V. Ng (1993). Measuring and testing the impact of news on volatility. *Journal of Finance 48*, 1749–1778.

Engle, R., V. Ng, and M. Rothschild (1990). Asset pricing with a factor-ARCH covariance structure: Empirical estimates for treasury bills. *Journal of Econometrics 45*, 213–237.

Gauthier, G. and J. Simonato (2012). Linearized Nelson-Siegel and Svensson models for the estimation of spot interest rates. *European Journal of Operational Research 219*, 442–451.

Giacomini, R. and H. White (2006). Tests of conditional predictive ability. *Econometrica 74*, 1545–1578.

Glosten, L., R. Jagannathan, and D. Runkle (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance 48*, 1779–1801.

Hansen, B. (1996). Inference when a nuisance parameter is not identified under the null hypothesis. *Econometrica 64*, 413–430.

Harvey, A., E. Ruiz, and E. Sentana (1992). Unobserved component time series models with ARCH disturbances. *Journal of Econometrics 52*, 129–157.

Hull, J. and A. White (1990). Pricing interest-rate-derivative secturities. *The review of financial studies 3*, 573–592.

Kim, C. and C. Nelson (1999). *State-space models with regime swithcing.* Cambridge, Massachusetts: The MIT Press.

Koedijk, K., F. Nissen, P. Schotman, and C. Wolff (1997). The dynamics of short-term interest rate volatility reconsidered. *European Finance Review 1*, 105–130.

Koopman, S., M. Mallee, and M. van der Wel (2010). Analyzing the term structure of interest rates using Dynamic Nelson-Siegel model with time-varying parameters. *Journal of Business and Economic Statistics 28*, 329–343.

Lee, J. (2002). Federal funds rate target changes and interest rate volatility. *Journal of Economics and Business 54*, 159–191.

Litterman, R., J. Scheinkman, and L. Weiss (1991). Volatility and the yield curve. *Journal of Fixed Income 1*, 49–53.

Mishkin, F. (1992). Is the Fisher effect for real? *Journal of Monetary Economics 30*, 195–215.

Moench, E. (2008). Forecasting the yield curve in a data-rich environment: A no-arbitrage factor-augmented VAR approach. *Journal of Econometrics 146*, 26–43.

Nelson, C. and A. Siegel (1987). Parsimonious modelling of yield curves. *Journal of Business 60*, 473–489.

Nelson, D. (1991). Heteroskedasticity in asset returns: A new approach. *Econometrica 59*, 347–370.

Rudebusch, G. and T. Wu (2008). A macro-finance model of the term structure, monetary policy and the economy. *The Economic Journal 118*, 906–926.

Svensson, L. (1995). Estimating forward interest rates with the extended Nelson-Siegel method. *Sveriges Riksbank, Quarterly Review 3*, 13–26.

Vasicek, O. (1977). An equilibrium characterizatoin of the term structure. *Journal of Financial Economics 5*, 177–188.

Appendix

A Finding the Unconditional Covariance Matrix of the State Vector

In order to solve $\boldsymbol{\Sigma_\beta} - \boldsymbol{\Phi}\boldsymbol{\Sigma_\beta}\boldsymbol{\Phi'} = \boldsymbol{\Sigma_\nu}$ for $\boldsymbol{\Sigma_\beta}$ I make use of the properties of the vectorization operator, following Christensen and van der Wel (2010). First the vectorization operator is applied to both sides, giving

$$vec(\boldsymbol{\Sigma_\beta}) - vec(\boldsymbol{\Phi}\boldsymbol{\Sigma_\beta}\boldsymbol{\Phi'}) = vec(\boldsymbol{\Sigma_\nu}),$$

this can be rewritten as

$$\boldsymbol{I}_{d^2} vec(\boldsymbol{\Sigma_\beta}) - (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi}) vec(\boldsymbol{\Sigma_\beta}) = \left[\boldsymbol{I}_{d^2} - (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi})\right] vec(\boldsymbol{\Sigma_\beta}) = vec(\boldsymbol{\Sigma_\nu}),$$

where d is the dimension of $\boldsymbol{\beta}$, which is equal to 3 in the case of the Nelson-Siegel model. The above can now easily be solved to arrive at the solution

$$vec(\boldsymbol{\Sigma_\beta}) = \left[\boldsymbol{I}_{d^2} - (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi})\right]^{-1} vec(\boldsymbol{\Sigma_\nu}).$$

B Coefficients in the General State Space Form

One common volatility component in the yields

$$\begin{aligned}
\boldsymbol{H} &= \begin{bmatrix} \boldsymbol{\Lambda}(\lambda) & \boldsymbol{\Gamma}_\varepsilon \end{bmatrix} & \boldsymbol{\alpha}_t &= \begin{bmatrix} \boldsymbol{\beta}_t \\ \varepsilon_t^* \end{bmatrix} \\
\boldsymbol{C} &= \begin{bmatrix} (\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} \\ 0 \end{bmatrix} & \boldsymbol{K} &= \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0}_3 \\ \boldsymbol{0}_3' & 0 \end{bmatrix} \\
\boldsymbol{G} &= \boldsymbol{I}_4 & \boldsymbol{\upsilon}_{t+1} &= \begin{bmatrix} \boldsymbol{\nu}_{t+1} \\ \varepsilon_{t+1}^* \end{bmatrix} \\
\boldsymbol{R} &= \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^+ & \boldsymbol{Q}_{t+1} &= \begin{bmatrix} \boldsymbol{\Sigma}_\nu & 0 \\ 0 & h_{t+1} \end{bmatrix}
\end{aligned}$$

Two common volatility components in the yields

$$\begin{aligned}
\boldsymbol{H} &= \begin{bmatrix} \boldsymbol{\Lambda}(\lambda) & \boldsymbol{\Gamma}_{1,\varepsilon} & \boldsymbol{\Gamma}_{1,\varepsilon} \end{bmatrix} & \boldsymbol{\alpha}_t &= \begin{bmatrix} \boldsymbol{\beta}_t \\ \varepsilon^*_{1,t} \\ \varepsilon^*_{2,t} \end{bmatrix} \\
\boldsymbol{C} &= \begin{bmatrix} (\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} \\ 0 \\ 0 \end{bmatrix} & \boldsymbol{K} &= \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3' & 0 & 0 \\ \mathbf{0}_3' & 0 & 0 \end{bmatrix} \\
\boldsymbol{G} &= \boldsymbol{I}_5 & \boldsymbol{\upsilon}_{t+1} &= \begin{bmatrix} \boldsymbol{\nu}_{t+1} \\ \varepsilon^*_{1,t+1} \\ \varepsilon^*_{2,t+1} \end{bmatrix} \\
\boldsymbol{R} &= \boldsymbol{\Sigma}^+_{\boldsymbol{\varepsilon}} & \boldsymbol{Q}_{t+1} &= \begin{bmatrix} \boldsymbol{\Sigma}_\nu & 0 & 0 \\ 0 & h_{1,t+1} & 0 \\ 0 & 0 & h_{2,t+1} \end{bmatrix}
\end{aligned}$$

One common volatility component in the factors

$$\begin{aligned}
\boldsymbol{H} &= \begin{bmatrix} \boldsymbol{\Lambda}(\lambda) & \mathbf{0}_N \end{bmatrix} & \boldsymbol{\alpha}_t &= \begin{bmatrix} \boldsymbol{\beta}_t \\ \nu^*_t \end{bmatrix} \\
\boldsymbol{C} &= \begin{bmatrix} (\boldsymbol{I}_3 - \boldsymbol{\Phi})\boldsymbol{\mu} \\ 0 \end{bmatrix} & \boldsymbol{K} &= \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{0}_3 \\ \mathbf{0}_3' & 0 \end{bmatrix} \\
\boldsymbol{G} &= \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{\Gamma}_\nu \\ \mathbf{0}_3' & 1 \end{bmatrix} & \boldsymbol{\upsilon}_{t+1} &= \begin{bmatrix} \boldsymbol{\nu}^+_{t+1} \\ \nu^*_{t+1} \end{bmatrix} \\
\boldsymbol{R} &= \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} & \boldsymbol{Q}_{t+1} &= \begin{bmatrix} \boldsymbol{\Sigma}^+_\nu & 0 \\ 0 & h_{t+1} \end{bmatrix}
\end{aligned}$$

C Additional Tables and Figures

Table XIV: **VAR Estimates for the DNS-GARCH Model**
In the table the parameter estimates of the VAR model and the covariance matrix of the latent factors for the DNS-GARCH model are shown. The results correspond to the Vector Autoregression of the DNS factors in the model with a common volatility component modelled via GARCH. Panel (a) reports the estimated coefficients in $\boldsymbol{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\boldsymbol{\Sigma_\nu}$ is given. Asymptotic standard errors are obtained using the information matrix.

(a) Constant and Coefficients - DNS-GARCH Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.972***	0.001	0.004	4.692***
	0.02470	0.01238	0.01200	1.68538
$\beta_{2,t}$ (Slope)	-0.050	0.918***	0.080***	-4.132
	0.03542	0.01797	0.01733	3.25970
$\beta_{3,t}$ (Curvature)	0.178**	0.066	0.877***	-4.245
	0.08088	0.04076	0.03917	4.57929

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-GARCH Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.055***	-0.059***	0.051***
	0.00612	0.00751	0.01426
$\beta_{2,t}$ (Slope)		0.116***	-0.088***
		0.0117	0.01967
$\beta_{3,t}$ (Curvature)			0.611***
			0.06284

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

Table XV: **VAR Estimates for the DNS-2GARCH Model**

In the table the parameter estimates of the VAR model for the latent factors and the corresponding covariance matrix are reported for the the DNS-2GARCH model. The results correspond to the Vector Autoregression of the DNS factors in the model with two common volatility components modelled via separate GARCH processes. Panel (a) reports the estimated coefficients in $\mathbf{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\nu}}$ is given. Robust standard errors are obtained using the sandwich estimator.

(a) Constant and Coefficients - DNS-2GARCH Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.995***	0.003	-0.002	1.911
	0.00344	0.01310	0.02347	10.2761
$\beta_{2,t}$ (Slope)	-0.078***	0.918***	0.087***	1.591
	0.00945	0.01480	0.02795	3.49429
$\beta_{3,t}$ (Curvature)	0.047	0.052	0.914***	-1.915
	0.05033	.06170	.01197	13.6418

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-2GARCH Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.060***	-0.066***	0.024***
	0.00097	0.00124	0.00119
$\beta_{2,t}$ (Slope)		0.116***	-0.035***
		.00156	0.00189
$\beta_{3,t}$ (Curvature)			0.533***
			0.01012

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

Table XVI: **VAR Estimates for the DNS-TGARCH Model**

In the table the parameter estimates of the VAR model for the latent factors and the corresponding covariance matrix of the T-GARCH model are shown. The results correspond to the Vector Autoregression of the DNS factors in the model with a common volatility component modelled via T-GARCH. Panel (a) reports the estimated coefficients in $\mathbf{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\nu}}$ is given. Robust standard errors are obtained using the sandwich estimator.

(a) Constant and Coefficients - DNS-TGARCH Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.996***	0.003	-0.005***	2.158
	0.00021	0.00289	0.00132	1.53767
$\beta_{2,t}$ (Slope)	-0.082***	0.919***	0.089***	2.128
	0.00099	0.00362	0.00194	1.80510
$\beta_{3,t}$ (Curvature)	0.078***	0.083***	0.892***	-1.065***
	0.00115	0.00369	0.00129	0.11451

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-TGARCH Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.054***	-0.059***	0.046***
	0.00010	0.00010	0.00010
$\beta_{2,t}$ (Slope)		0.117***	-0.083***
		0.00010	0.00008
$\beta_{3,t}$ (Curvature)			0.618***
			0.00038

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

Table XVII: **VAR Estimates for the DNS-EGARCH Model**

In the table the parameter estimates of the VAR model for the latent factors and the corresponding covariance matrix the E-GARCH model are given. The results correspond to the Vector Autoregression of the DNS factors in the model with a common volatility component modelled via E-GARCH. Panel (a) reports the estimated coefficients in $\boldsymbol{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\boldsymbol{\Sigma_\nu}$ is given. Robust standard errors are obtained using the sandwich estimator.

(a) Constant and Coefficients - DNS-EGARCH Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.992***	-0.001***	-0.002**	1.700
	0.00399	0.00000	0.00081	2.14623
$\beta_{2,t}$ (Slope)	-0.069***	0.926***	0.083***	2.004
	0.00323	0.00168	0.00141	2.14512
$\beta_{3,t}$ (Curvature)	0.061***	0.072***	0.902***	-1.286***
	0.00000	0.00130	0.00130	0.07398

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-EGARCH Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.055***	-0.061***	0.047***
	0.00124	0.00177	0.00018
$\beta_{2,t}$ (Slope)		0.119***	-0.084***
		0.00188	0.00048
$\beta_{3,t}$ (Curvature)			0.607***
			0.00190

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

Table XVIII: **VAR Estimates for the DNS-FactorGARCH Model**

In the table the parameter estimates of the VAR model for the latent factors, the covariance matrix and the parameters of the FactorGARCH model for time-varying volatility are shown. The results correspond to the Vector Autoregression of the DNS factors in the model with a common volatility component in the state equation of the state space framework, modelled via a GARCH process. Panel (a) reports the estimated coefficients in $\mathbf{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\nu}}$ is given. Robust standard errors are obtained using the sandwich estimator.

(a) Constant and Coefficients - DNS-FactorGARCH Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.994***	0.001	0.006	1.247
	0.03185	0.01976	0.03042	7.37507
$\beta_{2,t}$ (Slope)	-0.052**	0.941***	0.045***	2.403
	0.02176	0.02290	0.00894	9.41943
$\beta_{3,t}$ (Curvature)	0.040	0.041	0.945***	-1.460
	0.19052	0.15524	0.19459	3.33920

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-FactorGARCH Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.080***	-0.086***	0.024***
	0.00156	0.00720	0.00449
$\beta_{2,t}$ (Slope)		0.094***	-0.078***
		0.01749	0.00189
$\beta_{3,t}$ (Curvature)			0.589***
			0.04918

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

Table XIX: **VAR Estimates for the DNS-GARCHX-DRA Model**

In the table the parameter estimates of the VAR model for the latent factors and the corresponding covariance matrix of the GARCHX-DRA are presented. The results correspond to the Vector Autoregression of the DNS factors in the model with a common volatility component modelled via a GARCH-X in which the squared first differences of the macroeconomic variables used in Diebold, Rudebusch, and Aruoba (2006) are used as exogenous variables. These variables are capacity utilisation (CU), the federal funds rate (FFR) and annual price inflation (INFL). Panel (a) reports the estimated coefficients in $\mathbf{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\nu}}$ is given. Robust standard errors are obtained using the sandwich estimator.

(a) Constant and Coefficients - DNS-GARCHX-DRA Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.999***	0.006***	-0.006***	3.176***
	0.00033	0.00062	0.00078	0.32862
$\beta_{2,t}$ (Slope)	-0.080***	0.912***	0.091***	-6.037***
	0.00118	0.00101	0.00128	0.47762
$\beta_{3,t}$ (Curvature)	0.190***	0.069***	0.872***	-7.537***
	.00050	0.00066	0.00100	0.76322

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-GARCHX-DRA Model

	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.056***	-0.060***	0.052***
	0.00001	0.00002	.00004
$\beta_{2,t}$ (Slope)		0.117***	-0.088***
		0.00002	0.00004
$\beta_{3,t}$ (Curvature)			0.607***
			0.00005

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

Table XX: **VAR Estimates for the DNS-GARCHX-VIX Model**

In the table the parameter estimates of the VAR model for the latent factors and the corresponding covariance matrix of the DNS-GARCHX-VIX model are shown. The results correspond to the Vector Autoregression of the DNS factors in the model with a common volatility component modelled via a GARCH-X in which the squared monthly VIX $\left(\frac{VIX}{100\sqrt{12}}\right)^2$ is used as exogenous variable. Panel (a) reports the estimated coefficients in $\mathbf{\Phi}$ and the constants in $\boldsymbol{\mu}$. In panel (b) the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\nu}}$ is given. Robust standard errors are obtained using the sandwich estimator.

(a) Constant and Coefficients - DNS-GARCHX-VIX Model

	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	μ (Constant)
$\beta_{1,t}$ (Level)	0.997***	0.0002	-0.001	1.424
	0.00524	0.00318	0.00215	8.18681
$\beta_{2,t}$ (Slope)	-0.081***	0.919***	0.087***	1.612
	0.00617	0.00535	0.00661	3.07192
$\beta_{3,t}$ (Curvature)	0.070***	0.068**	0.903***	-2.053
	0.01009	0.02654	0.00392	4.67776

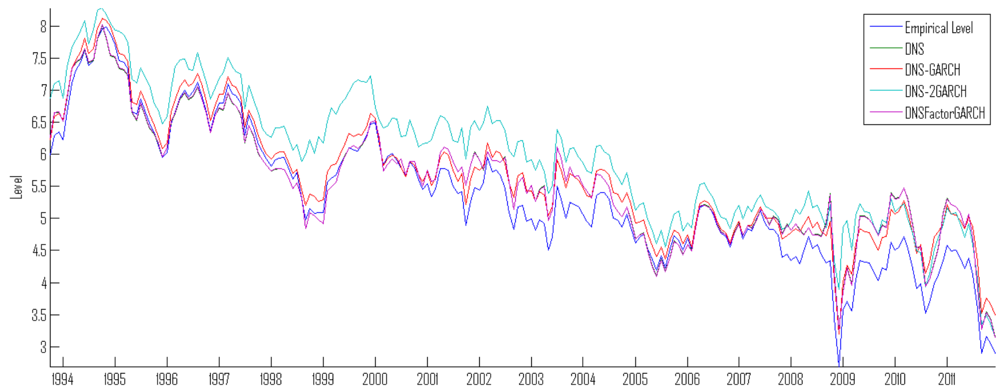
Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

(b) Covariance Matrix of VAR - DNS-GARCHX-VIX Model

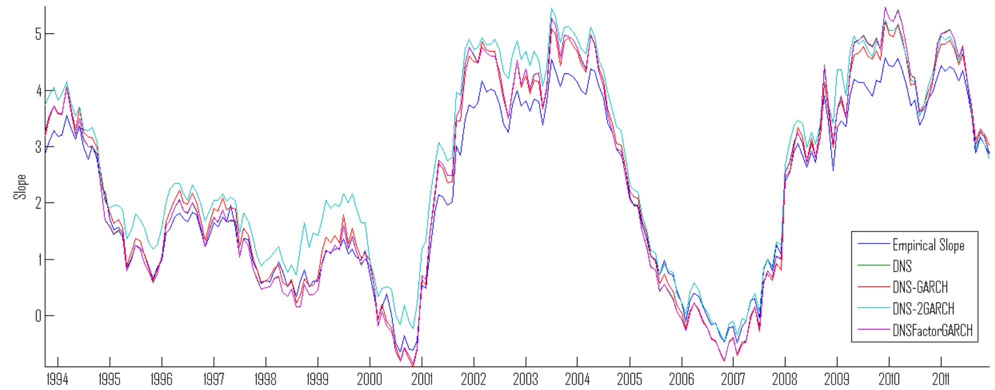
	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$
$\beta_{1,t}$ (Level)	0.056***	-0.060***	0.050***
	0.00010	0.00005	0.00023
$\beta_{2,t}$ (Slope)		0.117***	-0.086***
		0.00004	.00034
$\beta_{3,t}$ (Curvature)			0.615***
			0.00302

Standard errors are shown below the estimates and significance at 90%/95%/99% is indicated by */**/***.

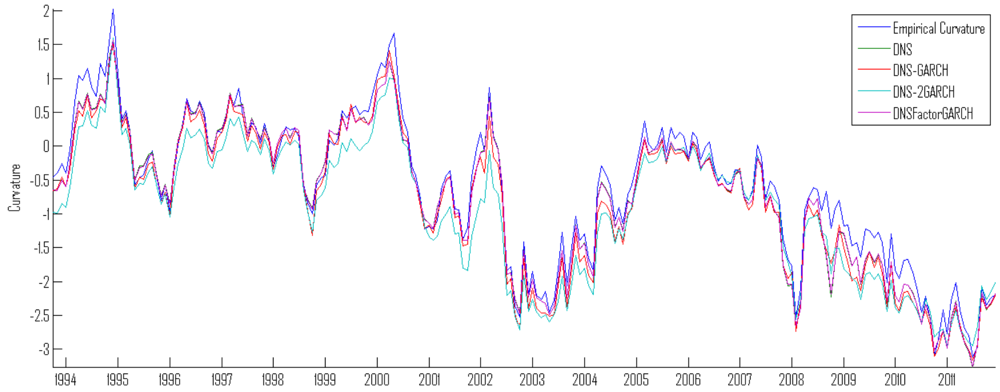
Figure XV: **Empirical and Filtered Factors**
The figure shows the empirical factors level (a), slope (b) and curvature (c) together with the filtered latent factors of the DNS, DNS-GARCH, DNS-2GARCH and DNS-FactorGARCH models. The filtered estimates of the DNS-TGARCH, DNS-EGARCH, DNS-GARCHX-DRA and DNS-GARCHX-VIX models are very close to those of the DNS-GARCH and are therefore omitted in all three plots. In section 2.1 it is shown that the slope of the yield curve is given by $-\beta_{2,t}$ in the DNS model. Therefore the $\beta_{2,t}$ estimates are plotted as their mirror images. The $\beta_{3,t}$ series are scaled by 0.45 in order to better match the empirical factors and make comparison easier.



(a) Level



(b) Slope



(c) Curvature

Table XXI: **Diebold-Mariano Statistics of Forecasts**

The table shows the Diebold-Mariano (DM) statistics of the 1-month, 6-months and 12-months ahead forecasts of the standard DNS model and the DNS-TVV models compared to a random walk forecast (RW). The relevant loss function is the squared prediction error and the DM-statistics are assumed to be standard normally distributed. Positive DM statistics indicate an outperformance of the model prediction in comparison to the RW forecast. On the contrary, a negative DM statistic indicates a better performance of the RW model.

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	20Y	30Y
1 Month Ahead										
DNS	-0.936	0.025	-0.620	-1.245	0.172	0.335	-0.175	0.535	-2.245**	-1.371
DNS-GARCH	-1.682*	0.973	1.067	-4.452***	-6.272***	-7.062***	-6.854***	-6.661***	-3.856***	-4.375***
DNS-2GARCH	-3.237***	-0.556	-2.178**	-5.997***	-6.925***	-7.059***	-6.842***	-6.396***	-3.789***	-5.414***
DNS-TGARCH	-1.643	0.929	1.050	-4.461***	-6.286***	-7.106***	-6.919***	-6.760***	-3.990***	-4.492***
DNS-EGARCH	-1.697*	1.094	1.346	-4.303***	-6.194***	-7.057***	-6.882***	-6.698***	-3.820***	-4.330***
DNS-FactorGARCH	-1.287	0.933	0.367	-0.648	-0.291	0.244	0.059	0.615	-3.278***	-2.026**
DNS-GARCHX-DRA	-1.845*	0.868	0.781	-4.829***	-6.600***	-7.409***	-7.228***	-7.081***	-4.260***	-4.655***
DNS-GARCHX-VIX	-1.808*	0.906	0.898	-4.662***	-6.431***	-7.227***	-7.034***	-6.872***	-4.073***	-4.536***
6 Months Ahead										
DNS	0.595	-0.107	0.090	0.212	0.679	-0.319	-0.878	-1.425	-0.859	-1.951*
DNS-GARCH	1.896*	1.591	1.715*	-0.967	-3.551***	-5.007***	-5.109***	-5.378***	-4.169***	-4.781***
DNS-2GARCH	2.409**	2.509**	2.586***	-0.756	-3.480***	-4.872***	-4.868***	-4.833***	-3.364***	-4.024
DNS-TGARCH	1.708*	1.381	1.536	-1.088	-3.661***	-5.111***	-5.242***	-5.530***	-4.366***	-4.997***
DNS-EGARCH	1.824*	1.525	2.127**	0.526	-2.386**	-4.093***	-4.247***	-4.566***	-3.294***	-3.974***
DNS-FactorGARCH	2.407**	2.080**	1.938*	0.754	-0.475	-1.484	-1.745*	-2.062**	-2.615***	-2.116**
DNS-GARCHX-DRA	1.857*	1.574	1.657*	-1.267	-3.868***	-5.327***	-5.450***	-5.747***	-4.547***	-5.172***
DNS-GARCHX-VIX	1.907*	1.604	1.674*	-1.182	-3.760***	-5.196***	-5.313***	-5.600***	-4.409***	-5.034

Table XXI: *(continued)*

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	20Y	30Y
12 Months Ahead										
DNS	2.037**	1.590	1.828*	2.059**	2.043**	-1.063	-2.511**	-3.443***	-3.413***	-4.041***
DNS-GARCH	3.952***	3.646***	3.181***	0.282	-2.443**	-5.082***	-5.919***	-6.464***	-5.876***	-5.974***
DNS-2GARCH	4.584***	4.432***	4.322***	1.595	-1.352	-3.897***	-4.531***	-4.946***	-4.318***	-4.542***
DNS-TGARCH	3.703***	3.392***	2.987***	0.187	-2.539**	-5.219***	-6.093***	-6.675***	-6.141***	-6.257***
DNS-EGARCH	3.698***	3.475***	3.796***	2.395**	-0.439	-3.478***	-4.287***	-4.831***	-4.149***	-4.174***
DNS-FactorGARCH	4.637***	4.295***	4.046***	2.632***	0.810	-1.533	-2.450**	-3.385***	-4.192***	-3.524***
DNS-GARCHX-DRA	3.892***	3.580***	3.052***	-0.017	-2.791***	-5.458***	-6.331***	-6.884***	-6.288***	-6.402***
DNS-GARCHX-VIX	3.921***	3.599***	3.085***	0.099	-2.648***	-5.307***	-6.173***	-6.735***	-6.171***	-6.280***

Significance at 90%/95%/99% level is indicated by */**/***.