An improved proof of the truth telling equilibrium of the alternative ascending bid auction

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Section 1

Introduction

Auctions have received significant interest in the past few years. An example are the recent radio frequency auctions in the Netherlands, through which the Dutch government allocated 4G frequencies to bidders in the telecom market. These auctions, in which multiple objects are auctioned simultaneously, can be difficult to model, especially when bidders have dependent values for the objects.

The broader area of *mechanism design*, the field in game theory where we think about the design of games (mechanisms) that allocate goods in a proper (economically efficient) way, has been a topic receiving significant academic interest in the past few years.

The central topic in this thesis is the (alternative) ascending bid auction introduced in [1]. It is an auction for multiple identical goods, in which participants submit a bid (a quantity) in each round. The price for the goods is then increased in each round. What makes the alternative ascending bid auction special is the *payment rule*. It is structured such that the price to be payed by each player only depends on the bids of all the other players (and not on his own). [1] shows that in this setting, the set of strategies where each player bids *truthfully* forms an ex post perfect equilibrium. In words this means, that when all players $j \neq i$ bid truthfully, then player i does not have an incentive to deviate from his truthful strategy.

This thesis is a critique on the actual proof that is given in [1]. We will argue that the proof given is incomplete and a more thorough mathematical analysis is necessary. We do not contest the result itself.

The rest of the thesis is organised as follows. In section 2, we will first discuss the basic auction framework itself, with its precise mathematical definitions. In section 3 we will introduce the uniform-price ascending bid auction, in which a uniform price is payed by each participant (a simpler version of the alternative introduced in [1]). We will show the concept of demand reduction (an example of a technique through which players can gain advantage by 'not telling the truth') in an auction. In section 4, we will turn to Ausubel's alternative ascending bid auction, discuss the precise proof as given in [1] and state our alternative.

Section 2

Theory of multiple object auctions

In the design of auctions it is all about distributing objects to participants (i.e. bidders) of the auction in a 'proper' (efficient) way. A particular property of a combinatorial auction is the fact that multiple objects (as opposed to a single object) must be simultaneously distributed. Examples of goods that are distributed with a combinatorial auction are radio frequency rights, airport slots or network bandwidth.

A simple example, as introduced in [2] is the auction of four chairs and a table. It is possible to auction the five items separately, or the four chairs together and the table separately. In fact, there are many possibilities to group the items.

Players can have many preferences for the items. For instance, one bidder might only be interested in the complete set, while another bidder might only be interested in the table. This poses problems in the allocation mechanism.

2.1 Basics

For the basics of auction theory (and in particular the theory of multiple object auctions) we refer to [3] as the main source. We will repeat some of the examples and theorems from this work.

Auctions can have a single stage where participants submit information, or there can be an open (or *dynamic*) format where there is an interaction of information through multiple rounds. Participants can have private values for objects, usually modeled by some probability distribution, or they can be public, or something in between. In the case of a multiple object auction, it is important to consider the possibility that the objects to be auctioned can be identical, or different. In this thesis we will focus on auctions with private values and identical objects.

First of all, there are two key ingredients for an auction. It needs an *allocation rule*, and a *pricing rule*. An auction usually requires the participants to submit a piece of information (a bid) to the leader of the auction (the auctioneer). The auctioneer then allocates the goods according to the predefined allocation rule, based on the information transmitted by the participants. The auctioneer then determines the price that is to be payed by the participants, according to the pricing rule. We assumed here that there is only one instance where the participants interact with the auctioneer. Let us discuss a few examples of those type of auctions, that are called *sealed bid auctions*.

2.1.1 Sealed bid auctions: Pay-your-bid, Uniform-Price

Suppose there are K identical items to be allocated to N players. Player $i \in \{1, \dots N\}$ submits a bid vector $\mathbf{b} = (b_1, \dots, b_K)$ such that $b_1 \geq \dots \geq b_K$ that indicates how many he is willing to pay for each additional item (the marginal bid). In each of the following three auctions, the allocation rule is as follows. The auctioneer determines which K of the NK bids are the highest. He then allocates the objects to the players that submitted those K highest bids. There is of course a case where the auctioneer cannot decide if bids are equal. This could be solved by a lottery, but we will assume throughout this thesis that ties do not exist.

We have yet to determine what price the participants should pay, based on their bids. Let us show three examples.

Pay-your-bid

In the pay-your-bid auction (in [3] called discriminatory auction), if participant i receives k objects, the price payed by this participant is equal to the sum of the highest k bids he submitted. In words, we could describe the auction as “each player pays what he bids”.

Uniform-price auction

In the uniform-price auction the auctioneer sets one ‘market-clearing’ price p at which the total demand is equal to the supply of items. This is a price between the highest losing bid and the lowest winning bid. We can choose the price p equal to the highest losing bid. Each player then pays k times price p, if he receives k items.

Bidding behaviour depends on the auction rules

It is important to realise that the auction formats shown above have different pricing rules, but the behaviour of the participants will be different in each case. Therefore it is not yet determined which is the formats is to be preferred. We will get back to that issue later.

2.1.2 Open auctions: Dutch and English auctions

The previous auction formats all consisted of one round in which participants transmitted information to the auctioneer, and the rest was settled. In the next three examples, the auction consists of multiple rounds, in which the participants can transmit information (and also receive information).

Dutch auction: open descending price

In the Dutch auction, the auctioneer announces a price that is large enough so that the total demand for the items is zero. In each round, the auctioneer decreases the price, until a participant is willing to buy a unit at that price. The participant is then awarded one item and pays the price that was announced. The auction then continues with the remaining items.

The Dutch auction is similar to the discriminatory auction in the following sense. If the participants behave according to the bid vectors described in the discriminatory auction, the allocation of items will be the same in the Dutch auction. We could say that the two auction formats are *outcome equivalent*. However, in the case of the Dutch auction, there is some information revealed to the participants in each round. Each round tells the participants whether or not an item has been sold against a certain price. This information can be useful to the players when the values of the participants for the items are dependent. Therefore the behaviour of the participants can be different in the Dutch auction versus the discriminatory auction.

English auction: open ascending price

In the English auction, the auctioneer chooses a price low enough for the bidders to have more demand than the supply of items. He raises the price each round, until the total demand for the items drops to K items. The price payed by each participant is the received items k^i times the price when the demand dropped from $K+1$ to K.

This auction format is outcome equivalent to the uniform price auction (in the sense that was explained before).

2.1.3 Values and efficiency

In the previous sections we have not spoken about the reason for the preferences of participants to the auction. We define a *marginal value vector* $\mathbf{x^i} = (x_1, \ldots, x_K)$ for each participant i, with each component drawn from some distribution function F.

A desirable property for an auction is that it is *efficient*. That means, it distributes the items (regardless of any monetary transaction that happens in the meantime) to the participants with the highest value for those items. In other words, it *maximizes* the sum of values associated with the objects distributed to players. When an auction is *standard* in the sense that it distributes items to those who bid highest, the efficiency condition can be mathematically stated as

$$x_k^i > x_l^j \quad \text{if and only if} \quad \beta_k^i(\mathbf{x^i}) > \beta_l^j(\mathbf{x^j})$$

for every i, j, k, l, where β_k^i means the bid of player i for object k. The meaning of this, is that the ranking of values should be the same as the ranking of the bids of each player. There are two important implications of this definition.

One implication is that $\beta_k^i(\mathbf{x^i})$ can only depend on x_k^i, not on the other values of the participant. Suppose that that would be the case instead. Then there would exist a vector δ such that $\delta_i = 0$ if $i = k$, but $\delta_i \neq 0$ for at least one $i \neq k$ and either $\beta_k^i(\mathbf{x^i}) > \beta_k^i(\mathbf{x^i} + \delta)$ or $\beta_k^i(\mathbf{x^i}) < \beta_k^i(\mathbf{x^i} + \delta)$. But if the auction is efficient, then either $x_k^i > x_k^i$ or $x_k^i < x_k^i$, which are both impossible.

Another implication of this is symmetry between the bidding functions of participants. Suppose that $\beta_k^i(\mathbf{x}) > \beta_k^j(\mathbf{x})$ for some value vector $\mathbf{x}$, participants i, j and item k. Then for efficiency to hold, we should have $x_k > x_k$ which is clearly not possible. Therefore we must have

$$\beta_k^i(\mathbf{x}) = \beta_k^j(\mathbf{x}).$$

We conclude that a standard auction (where the allocation is made to the highest bids) is efficient if and only if there exists a single increasing function β such that

$$\beta_k^i(\mathbf{x^i}) = \beta(x_k^i).$$

The fact that we only need to find one function β can greatly simplify any search of an efficient equilibrium.

2.1.4 Demand reduction

A reason why a some auctions are inefficient, is *demand reduction*. This is a technique, where bidders lower their demand near the end of the auction, to keep prices low. An example of demand reduction in a uniform-price auction (a closed bid auction) is given in [3], section 13.4.1. We will repeat that argument (in a lengthier manner, so that all the steps involved are clear). We wish to clarify this phenomenon in a slightly easier way before we move in to the more general setting of the ascending bid auction. Suppose there are two identical units ($K = 2$) and two players. Assume equilibrium strategies $\beta = (\beta_1, \beta_2)$ exist. Player 2 is assumed to play according to this equilibrium. Furthermore assume player 1 to have values $\mathbf{x} = (x_1, x_2)$ for the two objects (such that $x_1 >= x_2$). Call $\mathbf{b} = (b_1, b_2)$ the bid of player 1, and $\mathbf{c} = (c_1, c_2)$ to be the competing bid. Call H_1, H_2 the distribution functions of the random variables c_1, c_2 respectively, and h_1, h_2 are their density functions (we assume they exist).

In section A.1 of the Appendix, we have analysed the optimal bidding strategy of the first player, given that the second player bids truthfully. The value of b_2 that maximizes the expected value for the first player is equal to

$$b_2 = x_2 - \frac{H_2(b_1) - H_1(b_2)}{h_1(b_2)}.$$

We have only looked at the second bid of the first player, b_2, but we can already see that it is optimal to bid less than the true value x_2. Concluding, the (closed-bid) uniform price auction does not have an equilibrium where the players bid their true values. This generally leads to an inefficient outcome where the items are not fullly distributed to the players with the most value for it.

Section 3

The uniform-price ascending bid auction

The uniform-price ascending bid auction is a simplification of the later introduced alternative ascending bid auction. We investigate properties of the uniform-price ascending bid auction by conditioning on the different rounds (which does not require a constraint on the bidding strategies). This approach will prove to be effective in solving equilibrium problems in both the uniform-price ascending bid auction, as will it in the alternative ascending bid auction.

3.1 Two players: Definition

Our game is an auction set up to distribute $K \in \mathbb{N}$ identical objects. Two players, denoted by the index $i \in \{1, 2\}$ have private values for the K objects. These values are drawn from a probability distribution as follows.

First, each player i draws a random variable $\mathbf{x_i} \in \mathbb{R}^K$, the *marginal values vector*, where each component $x_{i,j}$ is independently and identically distributed with distribution function F. The vectors $\mathbf{x_i}$ are also mutually independent. The players are aware of these facts.

Second, for each player we define a vector $\mathbf{v_i}$ as the sorted version of $\mathbf{x_i}$, with the first component being $v_{i,1} = \max(\mathbf{x_i})$, the second component the second highest value of $\mathbf{x_i}$, and so on to $v_{i,K} = \min(\mathbf{x_i})$. The vector $\mathbf{v_i}$ represents the *sorted marginal values* for player i.

The auction consists of $N \in \mathbb{N}$ rounds, defined as followed. Before round r commences, the auctioneer publicly announces a price $p(r)$. The players then privately submit a 'bid' β_i saying how many objects they would buy for the announced price (their demand). Once every player has made the bid, the auctioneer privately calculates the aggregate demand $D_r = \sum_i \beta_i(p(r))$ (the sum of the demands) and compares this to the supply of objects K. The players do not know the aggregate demand.

If the demand is greater than the supply, the auction remains unsettled and we continue to a new round. The auctioneer then sets a higher price ($p(r)$ is increasing in r) and the process is repeated.

If the demand is smaller than or equal to the supply, the auction is settled. Players receive the exact amount of objects they had bid for, and pay the price announced before that round.

Finally, if the auction remains unsettled after the Nth round, that is if the demand exceeds supply, neither player will receive any objects, nor will they pay anything.

Players are assumed to be risk neutral. Assume the price function to be $p(r) = r\delta$ with $\delta > 0$.

3.2 Two players: Demand reduction

A bid function β_i for player i is defined as a mapping from the real numbers to a number of items

$$\beta_i(p) : \mathbb{R} \to \{0, \ldots, S\}.$$

The *truthful bidding function* d_i for player i (we use Krishna's [3] notation) is a bidding function defined as

$$d_i(p) = \max\{k : v_{i,k} > p\}.$$

This is precisely the amount of objects for which the player has a value that is above the current price.

The profit π_i of player i is defined as the random variable that represents the difference between the values for the obtained objects and the price payed for the objects,

$$\pi_i = \sum_{i=1}^{\beta_i(p(R))} v_{ij} - p(R)\beta_i(p(R))$$

where the random variable R is defined as the round in which the game ends. In the definition of the game we have defined a maximum number of N rounds in which bids are submitted. The expectation of the profit π_i of the player can be written as a sum of conditional expectations

$$\mathbb{E}[\pi_i] = \sum_{r=1}^{N} \mathbb{E}[\pi_i | R = r] \Pr[R = r].$$

With a slight abuse of notation, from now on we call π_i the players 'strategy' as well as the players profit under that strategy. For convenience, we use the following notation:

$$\begin{aligned} f(r) &= \Pr[R = r] \\ g(r) &= \mathbb{E}[\pi_i | R = r]. \end{aligned}$$

So that the profit expectation becomes

$$\mathbb{E}[\pi_i] = \sum_{r=1}^{N} f(r)g(r).$$

Let us investigate $f(r)$ further. We note the discreteness of the variable R and split $f(r)$ into two components:

$$f(r) = \Pr[R \le r] - \Pr[R \le r - 1] = h(r) - h(r - 1)$$

where we have defined $h(r) = \Pr[R \le r]$. In terms of the bid functions, $h(r)$ can be written in terms of the bidding functions as

$$h(r) = \Pr[\beta_1(p(k)) + \beta_2(p(k)) \le K \text{ for at least one } k = 1, \ldots, r].$$

Let us assume that the price increases linear with a stepsize $\delta > 0$ in each round

$$p(r) = \delta r.$$

So that $h(r)$ becomes

$$h(r) = \Pr[\beta_1(\delta k) + \beta_2(\delta k) \le K \text{ for at least one } k = 1, \ldots, r].$$

It is important to realize that this probability can be complicated to calculate, especially when the bidding function β is complicated. However, in the (natural) case of a decreasing bidding function, it simplifies:

$$\begin{aligned} h(r) &= \Pr[\beta_1(\delta k) + \beta_2(\delta k) \le K \text{ for at least one } k = 1, \ldots, r] \\ &= \Pr[\beta_1(\delta r) + \beta_2(\delta r) \le K]. \end{aligned}$$

From this point onwards, we will be interested in the properties of the truthful bidding strategy (which is a decreasing function in terms of the price), so we can assume the above simplification.

We will look for a truthful Nash equilibrium, so let us assume that the other player submits his true value $\beta_2(\delta r) = d_2(\delta r)$:

$$\begin{aligned} h(r) &= \Pr[\beta_1(\delta r) + d_2(\delta r) \leq K] \\ &= \Pr[\beta_1(\delta r) + \max\{k : v_{2,k} > \delta r\} \leq K] \\ &= \Pr[\max\{k : v_{2,k} > \delta r\} \leq K - \beta_1(\delta r)]. \end{aligned}$$

We must now see that the event on the left hand side is equivalent to another event. Please mind the step: it is trivial but requires some care.

$$\{\max\{k : v_{i,k} > \delta r\} \leq x\} \Leftrightarrow \{v_{i,x+1} \leq \delta r\} \qquad \text{where } x \text{ is an integer}$$

Now our function $h(r)$ becomes easy:

$$h(r) = \Pr(v_{2,K-\beta_1(\delta r)+1} \leq \delta r).$$

We will now introduce some notation for order statistics.

Definition 3.2.1. *Let $V = (v_1, \ldots, v_K)$ with v_i i.i.d. from the cumulative distribution function $F(x)$. Then the sorted vector $W = (\max(V), \ldots, \min(V))$ has elements w_i with c.d.f. $F_i^{(K)}(x)$ called the i-th order statistic.*

The explicit calculation of these can be difficult for $k > 1$. In our analysis, we will not need to calculate these explicitly, so we will keep this notation.

Applying this definition to our function $h(r)$ gives the simplification

$$h(r) = F^{(K)}_{K-\beta_1(\delta r)+1}(\delta r).$$

Putting this into $f(r)$ gives:

$$f(r) = F^{(K)}_{K-\beta_1(\delta r)+1}(\delta r) - F^{(K)}_{K-\beta_1(\delta(r-1))+1}(\delta(r-1)).$$

Now we turn to analyzing $g(r)$. The profit is defined as the value of the goods obtained minus the total price paid for them - of course from a perspective of the first player.

$$\begin{aligned} g(r) &= \mathbb{E}[\pi_i | R = r] \\ &= \mathbb{E}[v_1 + \ldots + v_{\beta_1(\delta r)} - p(r)\beta_1(\delta r) | \beta_1(\delta r) + \beta_2(\delta r) \leq K] \\ &= \mathbb{E}[v_1 + \ldots + v_{\beta_1(\delta r)} - \delta r \beta_1(\delta r) | \beta_1(\delta r) + \beta_2(\delta r) \leq K] \end{aligned}$$

Since we know that the game ends in round r, the rules of the game fix the outcome such that it is certain:

$$g(r) = v_1 + \ldots + v_{\beta_1(\delta r)} - \delta r \beta_1(\delta r).$$

Concluding, given the fact that the other player bids truthfully, our expected profit given a bid function β_1 is:

$$\mathbb{E}[\pi_i] \;=\; \sum_{r=1}^{N} \left(F^{(K)}_{K-\beta_1(\delta r)+1}(\delta r) - F^{(K)}_{K-\beta_1(\delta(r-1))+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{\beta_1(\delta r)} - \delta r \beta_1(\delta r) \right].$$

Now define $\mathbb{E}[\hat{\pi}_i]$ as the profit expectation where the bid function β_1 is replaced by the truthful bidding function d_1. We can then write the profit equation as

$$\mathbb{E}[\hat{\pi}_i] \;=\; \sum_{r=1}^{N} \left(F^{(K)}_{K-\beta_1(\delta r)+1}(\delta r) - F^{(K)}_{K-\beta_1(\delta(r-1))+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{d_1(\delta r)} - \delta r d_1(\delta r) \right],$$

and formulate the following theorem.

Theorem 3.2.2. *A demand reduction strategy can be optimal in the two player ascending bid auction.*

The proof is found in section A.2 of the Appendix. One might think that our reasoning was not complete as the chosen ϵ_r does not always produce an admissible strategy for β_1 (since it reduces demand for only one round $\hat{r}$). However, if we only consider the value outcomes where the true demand d_1 is reduced in round $\hat{r}+1$ (this is possible), then the above analysis holds. It is enough that the analysis holds for this set of outcomes (as we were only looking for one example where demand reduction is possible).

3.3 Multi-player extension

As we shall see, the model is structured in a way such that it extends very easily to n players. From the viewpoint of player 1, we still have the expected profit equation

$$\mathbb{E}[\pi_i] \;=\; \sum_{r=1}^{N} f(r) g(r),$$

where $f(r)$ and $g(r)$ are defined as in the 2-player case:

$$\begin{aligned} f(r) &= \Pr(R = r) = h(r) - h(r-1) \\ g(r) &= \mathbb{E}[\pi_i | R = r] = v_1 + \ldots + v_{\beta_1(\delta r)} - \delta r \beta_1(\delta r). \end{aligned}$$

The expected profit, given the event that the game ends in round r, $g(r)$, remains the same in the n player case as it was in the two player case. It remains to determine $f(r)$ in terms of the value distribution F (assuming symmetry: that is, the value distribution F is assumed equal for each player).

The function $h(r)$ is adjusted for the n-player case (assuming increasing bids):

$$h(r) = \Pr(\beta_1(p(r)) + \ldots + \beta_n(p(r)) \leq K).$$

Now define $z(p(r)) = \beta_2(p(r)) + \ldots + \beta_n(p(r))$. This is the (random) aggregate demand of the other players. This gives the expression for $h(r)$ in terms of $z(p(r))$

$$\begin{aligned} h(r) &= \Pr(\beta_1(p(r)) + z(p(r)) \leq K) \\ &= \Pr(z(p(r)) \leq K - \beta_1(p(r))) \\ &= Z_r(K - \beta_1(p(r))), \end{aligned}$$

if we call Z_r the distribution function of $z(p(r))$. Now, can we determine this distribution explicitly? We repeat the definition of Z_r:

$$Z_r(x) = \Pr(\beta_2(p(r)) + \ldots + \beta_n(p(r)) \leq x)$$

Now for our analysis we will assume the other players to submit their true demand $\beta_i(p(r)) = d_i(p(r)) = \max\{k : v_{i,k} > p\}$ as defined earlier.

$$Z_r(x) = \Pr(\max\{k : v_{2,k} > p\} + \ldots + \max\{k : v_{n,k} > p\} \leq x)$$

For the unsorted value vectors of the players $\mathbf{x_2}, \ldots, \mathbf{x_n}$, define a concatenated vector $\mathbf{x_{-1}}$ as the unsorted value vector of all the players except the first:

$$\mathbf{x_{-1}} = (\mathbf{x_2} \cdots \mathbf{x_n})$$

Now define the sorted value vector v_{-1} for all players except the first:

$$\mathbf{v_{-1}} = (\max(\mathbf{x_{-1}}), \ldots, \min(\mathbf{x_{-1}}))$$

Now it is easy to see that this is just an ordered set of i.i.d. variables $x_{-1,k} \sim F$. Furthermore

$$\max\{k : v_{-1,k} > p\} = \max\{k : v_{2,k} > p\} + \ldots + \max\{k : v_{n,k} > p\},$$

since both sides represent the number of values greater than p. Returning to equation (3.1), we can simplify it to

$$\begin{aligned} Z_r(x) &= \Pr(\max\{k : v_{-1,k} > p(r)\} \leq x) \\ &= \Pr(v_{-1,x+1} \leq p(r)) \end{aligned}$$

where the last equation is according to an earlier obtained result in equation (3.1). We now use Definition 3.2.1 to obtain a result in terms of the order statistics function.

$$Z_r(x) \quad = \quad F_{x+1}^{((n-1)K)}(p(r))$$

where we recall the definition of $F_k^{(K)}(x)$ as the k-th order statistics function of a vector of length K. We substitute this in equation (3.1) to obtain the final result for $h(r)$:

$$h(r) \quad = \quad F_{K-\beta_1(p(r))+1}^{((n-1)K)}(p(r))$$

We can check the case $n = 2$ and confirm that it is equal to equation (3.1).

Now we have obtained explicit functions for $f(r)$ and $g(r)$ in terms of the values, that are very similar to the case $n = 2$. In the proof of Theorem 3.2.2 we can just exchange these functions and obtain the same results. We shall not repeat the proof here.

Theorem 3.3.1. *A demand reduction strategy can be optimal in the n-player ascending bid auction.*

Proof. Equivalent to the proof of Theorem 3.2.2 with $h(r)$ substituted from equation (3.1). □

Section 4

Ausubel's alternative ascending bid auction

The inefficiency of the ascending bid auction in the above format lies in the fact that participants can reduce their demand so that the price for the items that are still to be won, are reduced. [1] gives a solution for this, called a 'clinching rule', where objects are allocated to participants during the auction (against different prices).

Let there be K identical items to be auctioned to n players in a maximum number of rounds N. The auction has an increasing price $p(r) = \delta r$. Players have independent values $v_{i,k} \sim F$ as in the ascending bid auction introduced earlier.

We introduce the *cumulative clinches* C_i^r up to round r as

$$C_i^r = \max \left\{ 0, \sum_{j \neq i} \beta_j(p(r)) \right\} \tag{4.1}$$

and the *clinch function* c_i^r as

$$c_i^r = C_i^r - C_i^{r-1}$$

which denotes the number of items player i obtains (clinches) in round r. If all K items are clinched, the auction ends. The allocation rule is defined as

$$k_i \quad = \quad C_i^R, \tag{4.2}$$

which means that k_i, the number of units awarded to player i, is equal to the cumulative clinches up to and including time R of the last round. The payoff π_i of player i can be written as

$$\pi_i \quad = \quad \sum_{j=1}^{C_i^R} v_{i,j} - \sum_{r=1}^{N} c_i^r p(r)$$

where R is the random round at which the auction ends. At last, we require the bidding function to adhere to $\beta_i(p(r)) \geq C_i^r$.

4.1 Truth telling forms an ex post perfect equilibrium: proof by Ausubel

In [1], Ausubel proves the alternative ascending bid auction to have a 'truth telling equilibrium': a set of strategies such that one player, given that all the other players bid according to their true demand strategy, does not have an incentive to deviate from the truth telling strategy. The terminology in [1] is slightly different from ours; therefore we shall introduce the concept of an *ex post perfect equilibrium*, as explained in [1].

Definition 4.1.1 (Ex post perfect equilibrium)**.** *The strategy n-tuple $\{\beta_i\}_{i=1}^n$ is said to be an ex post perfect equilibrium if for every time t, following any history h^t, and for every realization $\{U_i\}_{i=1}^n$ of private information, the n-tuple of continuation strategies $\{\sigma_i(\cdot,\cdot|t,h_i^t,U_i)\}_{i=1}^n$ constitutes a Nash equilibrium of the game in which the realization of $\{U_i\}_{i=1}^n$ is common knowledge.*

It is not necessary to discuss this definition in detail. For the exact meaning of a *strategy* and a *history* we refer to [1]. We should see this definition as the generalization of the Nash equilibrium (which is relevant for games where there is no time component), that takes into account a flow of information through time.

We repeat the proof of Ausubel (to be found in Appendix 1 of [1]), where we replace any symbols with the corresponding symbols in our analysis. In between the lines we will discuss it. The first part reads:

> "At every point in the alternative ascending-bid auction up until its end, all of the payoff-relevant events in the auction occur through clinching. The cumulative quantity of clinched units for bidder i at time (and price) t is given by equations (4.1) and (4.2). Observe that the right side of equation (4.1) is independent of bidder i's actions; hence, changing one's own bid strategy can have no effect on payoff, except to the extent that: (i) it leads rival bidders to respond; or (ii) it determines ones own final quantity x_i^*. "

What is very essential to understand here, is that the strategy of any player always determines ones own final quantity. We could certainly view the clinch process C_i^r as 'exogeneous' to player i as player i does not influence its values at each time r. However, player i does in fact influence the (random) time R at which the game ends, and as such the random value C_i^R. The text seems to imply that case (ii) is of minor importance, but because of the stated reason, it is not. The author continues:

"Since marginal utilities were assumed (weakly) diminishing, the sincere bidding strategy given by equation (8) always yields monotonically nonincreasing quantities over time. Moreover, sincere bidding by bidder i always yields a final price p^* and final quantity x_i^* satisfying $x_i^* \in \arg\max_{x_i \in X_i}\{U_i(x_i) - p^* x_i\}$."

The maximization seems to convince us that the strategy is somehow optimal, but we have to be careful. In fact, the maximization shows that for the truthful strategy a player bids for an object if his value for it is above the current price p^*. This implies a maximization of profit for that player in the case that the player obtains all the goods at that price p^*. However, the allocation mechanism is different in this game - it allocates objects as they are clinched, and the prices payed are different for each object! The author does not treat this phenomenon. The last relevant part reads:

If all rival bidders $j \neq i$ bid sincerely, then rivals never respond to bidder i's strategy, except through price. Hence, sincere bidding is a mutual best response for every bidder - for every realization of utilities and after every history - and hence it is an ex post perfect equilibrium. (...)"

The author does not state what is meant exactly with a 'responding' strategy. We could think of this of course as one player having a strategy that is dependent on the strategy of another player. The author ends here by stating that case (i) of the first part never happens, and seems to imply that case (ii) of the first part also never happens, and therefore there is no incentive to deviate from the truthful bidding strategy.

The point here, is that the author has not fully convinced the reader that point (ii) does not occur. The proof lacks mathematical rigor of treating every case systematically. The point is not to state that the proof is incorrect - it is not. It might be that the author considers the matter trivial and assumes the reader to fill in the gaps. This thesis tries to fill that gap by bringing a systematic analysis of this problem.

4.2 Our own proof

As in the ascending bid auction we can write the expectation of the payoff as a sum of conditional expectations:

$$\mathbb{E}[\pi_i] \;=\; \sum_{r=1}^{N} \mathbb{E}[\pi_i | R = r] \Pr[R = r] = \sum_{r=1}^{N} f(r) g(r).$$

and we define (as in the previous case)

$$\begin{aligned} f(r) &= \Pr[R = r] = \Pr[R <= r] - \Pr[R \leq r - 1] = h(r) - h(r-1) \\ g(r) &= \mathbb{E}[\pi_i | R = r]. \end{aligned}$$

with $h(r) = \Pr[R \leq= r]$.

An interesting property of this setup is that the payoff of player i, given that the game ends in round R, is independent of the strategy of player i. This follows from the fact that the clinch function only depends on the other players. We will now investigate whether truth telling forms an equilibrium strategy in this setup.

4.2.1 Truth telling forms a Nash equilibrium

In this analysis we assume ties (players with equal values) not to exist. This prevents some peculiarities in the clinching rule. We take the viewpoint of the first player, while assuming the other players to play according to a truth telling strategy.

The first thing to be determined, is what determines R, the final round of the game. It is the time at which all the objects are clinched:

$$R = \min \left\{ r : \sum_{i=1}^{n} C_i^r \geq K \right\}.$$

In the case of no ties, this is the unique R such that

$$\sum_{i=1}^{n} C_i^R = K.$$

As stated before, the conditional expectation $g(r) = \mathbb{E}[\pi_i | R = r]$ does not depend on player i's strategy. Let us therefore focus on determining $h(r) = \Pr[R \leq r]$. This probability is related to the clinching function by

$$h(r) = \Pr\left[\sum_{i=1}^{n} C_i^r \geq K \right].$$

In the two player case, this simplifies to

$$\begin{aligned} h(r) &= \Pr\left[\max(0, K - \beta_1(p(r))) + \max(0, K - \beta_2(p(r))) \geq K\right] \\ &= \Pr\left[K - \beta_1(p(r)) + K - \beta_2(p(r)) \geq K\right] \\ &= \Pr\left[\beta_2(p(r)) \leq K - \beta_1(p(r))\right]. \end{aligned}$$

Suppose now that the second player has a truth telling strategy $\beta_2(p(r)) = d_2(p(r)) =$

$\max\{k : v_{2,k} > p(r)\}$. Then $h(r)$ simplifies to

$$\begin{aligned} h(r) &= \Pr\left[d_2(p(r)) \le K - \beta_1(p(r))\right] \\ &= \Pr\left[\max\{k : v_{2,k} > p(r)\} \le K - \beta_1(p(r))\right] \\ &= \Pr\left[v_{2,K-\beta_1(p(r))+1} \le p(r)\right] \\ &= F^{(K)}_{K-\beta_1(p(r))+1}(p(r)). \end{aligned}$$

where $F_k^K(x)$ is the kth order statistic function of a vector of length K. The steps taken in the last equations are the same as in the derivation of the ascending bid auction probabilities. Now define a deviating (demand reducing) strategy π_1 by $\beta_1(p(r)) = d_1(p(r)) - \epsilon_r$ and

$$\epsilon_r = \begin{cases} 0, & \text{if } r \neq \hat{r} \\ 1, & \text{if } r = \hat{r} \end{cases}$$

for some round $\hat{r}$. Define $\hat{\pi_1}$ as the truth telling strategy and calculate the expected payoff difference of the two strategies:

$$\begin{aligned} \mathbb{E}[\pi_1] - \mathbb{E}[\hat{\pi_1}] &= \sum_{i=1}^{N} f(r)\left(F^{(K)}_{K-d_1(p(r))+\epsilon_r+1}(p(r)) - F^{(K)}_{K-d_1(p(r-1))+\epsilon_{r-1}+1}(p(r-1))\right) \\ &\quad -\sum_{i=1}^{N} f(r)\left(F^{(K)}_{K-d_1(p(r))+1}(p(r)) - F^{(K)}_{K-d_1(p(r-1))+1}(p(r-1))\right) \\ &= (f(\hat{r}) - f(\hat{r}+1)(F^{(K)}_{K-d_1(p(\hat{r}))+2}(p(\hat{r})) - F^{(K)}_{K-d_1(p(\hat{r}))+1}(p(\hat{r}))). \end{aligned}$$

Since $F^{(K)}_{k+1} \ge F^{(K)}_{k}$ for all k, we have $F^{(K)}_{K-d_1(p(\hat{r}))+2}(p(\hat{r})) - F^{(K)}_{K-d_1(p(\hat{r}))+1}(p(\hat{r})) \ge 0$. It remains to show that $f(\hat{r}) \le f(\hat{r}+1)$ for the truth telling strategy to be optimal. This is shown in section A.3 of the Appendix. Concluding, we have found that the deviating strategy cannot improve the truth telling strategy.

We have shown that a deviating strategy with a demand decrease at one point in time is not optimal. To show that any deviation strategy $\beta_1(p(r)) = d_1(p(r)) + \epsilon_r$ is suboptimal, we can show that for each round r, either

$$f(r) - f(r+1) \le 0 \text{ and } F^{(K)}_{K-d_1(p(r))+\epsilon_r+1}(p(r)) - F^{(K)}_{K-d_1(p(r))+1}(p(r)) \ge 0$$

or

$$f(r) - f(r+1) \ge 0 \text{ and } F^{(K)}_{K-d_1(p(r))+\epsilon_r+1}(p(r)) - F^{(K)}_{K-d_1(p(r))+1}(p(r)) \le 0.$$

It is a similar exercise to what we have just shown in the simple demand reduction case.

4.2.2 Auction efficiency

We need to take a look at the efficiency of the ascending bid auction with a clinch rule. The previous section has shown that truth telling forms a Nash equilibrium.

We define every value of a player to be either winning or losing. The precise definition of winning: if a player i gets assigned M objects in the auction, the largest M values of player i are deemed winning and the other values are losing.

The (unique) efficient allocation (assuming no value ties!) is the allocation where the largest K values of all the players are winning.

There are precisely K clinches $\{c_i^r : c_i^r > 0, r \leq R\}$ (equal to the amount of items to be auctioned) during the game. Suppose a 'wrong clinch' occurs, i.e. $c_i^r > 0$ causing a player i to receive more than he would in the unique efficient allocation. Then there are two values, v_i of player i and v_j of player j, such that $v_i < v_j$ but the value v_i is winning (while it is losing in the efficient allocation) and the value v_j loses (while it is winning in the efficient allocation).

Value v_i has some clinch $c_j^{\hat{r}}$ associated with it, while value v_j has clinch c_i^r associated with it. Mind that a clinch can occur after the game ends, and then it does not have any effect (because all the items are already auctioned). Obviously clinch c_i^r has not occurred during the game.

The demand reduction associated with clinch $c_j^{\hat{r}}$ should have happened before the demand reduction associated with c_i^r assuming players bid truthfully. Hence $\hat{r} < r$. But then clinch $c_j^{\hat{r}}$ has happened before clinch c_i^r, which is a contradiction with our assumption. We conclude that 'wrong clinches' cannot exist when all players bid truthfully, and the auction is then efficient.

4.3 Afterword on assumptions

We have obtained a few results about the ascending clock auction. The most important observation is that it is optimal for players to tell the truth (their values for the objects) in the auction with a clinching rule - except possibly for some trivial cases where the round step size δ is large. We have used some important assumptions to simplify our investigation. Let us contemplate on them and try to imagine if our results would still hold if we would relax any of these.

1. Values are independent and identically distributed.

It is very well possible that in the real world values for objects are dependent. This way, players in the game can use the information given by their own values to obtain information about values of the other players. Imagine for instance, a situation where players know values are 'close together'. With 'close' it is meant that before any values are known, values are equally likely to lie somewhere in an interval, but once a value is known, it is more likely for another value to lie close to the first value than far away from it. Now imagine we are in some round of the auction, and the other player drops in demand. Now the probabilities of the game ending in the round after that change, since we expect the other values to be close to that value! So we see something happens that has not happened in our previous investigation: events that occur in one round have influence on the following rounds. This must be important.

2. Players know the number of items, K.

This is a bit hypothetical, but suppose players do not know in advantage how many items are sold. Then they do not know what the probabilities are of rounds ending, and therefore cannot determine their expected profits. Perhaps they can form a belief about the number of items, for instance some probability distribution. Then an analysis is possible.

3. Players do not know the aggregate demand after each round.

This assumption perhaps has the most elegant implication. Suppose there are many items for sale in the auction, and players know after each round how large the total demand was. Now suppose that after a certain round a player knows there is only one unit demand too much. Then this player can strategically lower his demand to prevent paying a higher price for all the other units he would like to have. This probably leads to inefficiencies in the auction. In the next section we explore this possibility.

4. The items are identical.

In this paper we assume the items to be identical, i.e. they are interchangable as the players are concerned. In practice, it might be that several objects that are similar or not similar at all might be auctioned together. The example of the table and four chairs makes it obvious that this introduces a problem: a player attaches a value to each possible *package* instead of each object separately.

Section 5

Conclusion

We have first discussed some existing auction mechanisms. We have discussed the concept of efficiency, and shown that some auction formats do not lead to desirable results.

We have discussed the uniform-price ascending bid auction, a generalized version of the English auction with multiple identical units. We have shown the concept of demand reduction, that can be applicable to the ascending bid auction. With this technique, bidders may be able to deviate from their truthful bidding strategy by letting the auction end sooner than usual, paying a lower price, but possibly losing some value by not obtaining objects they would have won otherwise.

The key contribution in this thesis is the analysis of Ausubel's alternative ascending bid auction, where we analyzed Ausubel's proof of the existence of a truthful bidding equilibrium, showing the weaknesses in this proof, and then showing a direct proof. In this direct proof, dependencies on certain assumptions (such as increasing bids, private and independent values) are made transparant.

The approach taken in the proof might be useful to other more advanced auctions as well. The main lesson learnt is that conditioning on the rounds in a dynamic auction can be a powerful and a precise tool. A drawback in this approach is the mathematical difficulty that a conditional expectation introduces (such as the 'conditional bid' in the main proof).

Appendix A

A.1 Demand reduction example

The expected profit for the first player can be written as a sum:

$$\begin{aligned}
\mathbb{E}[\pi_1] &= \mathbb{E}[\pi_1(1_{\{\text{win two items}\}} + 1_{\{\text{win one item}\}} + 1_{\{\text{win no items}\}})] \\
&= \mathbb{E}[\pi_1 1_{\{\text{win two items}\}}] + \mathbb{E}[\pi_1 1_{\{\text{win one item}\}}] + \mathbb{E}[\pi_1 1_{\{\text{win no items}\}}] \\
&= \mathbb{E}[(x_1 + x_2 - 2c_1)1_{\{\text{win two items}\}}] + \mathbb{E}[(x_1 - \max(b_2, c_2))1_{\{\text{win one item}\}}].
\end{aligned}$$

where 1_X is the random variable that is one if the event X occurs, and zero otherwise. In the case where player 1 wins two items, it is clear that $c_1 < b_2$. Hence

$$\begin{aligned}
\mathbb{E}[(x_1 + x_2 - 2c_1)1_{\{\text{win two items}\}}] &= \mathbb{E}[(x_1 + x_2 - 2c_1)1_{\{c_1 < b_2\}}] \\
&= \int_0^\infty (x_1 + x_2 - 2c_1)1_{\{c_1 < b_2\}} h_1(c_1)dc_1 \\
&= \int_0^{b_2} (x_1 + x_2 - 2c_1)h_1(c_1)dc_1 \\
&= (x_1 + x_2)\int_0^{b_2} h_1(c_1)dc_1 - 2\int_0^{b_2} c_1 h_1(c_1)dc_1 \\
&= (x_1 + x_2)H_1(b_2) - 2\int_0^{b_2} c_1 h_1(c_1)dc_1.
\end{aligned}$$

The second expectation is more tricky. We first split it into two parts:

$$\mathbb{E}[(x_1 - \max(b_2, c_2))1_{\{\text{win one item}\}}] = \mathbb{E}[x_1 1_{\{\text{win one item}\}}] + \mathbb{E}[\max(b_2, c_2)1_{\{\text{win one item}\}}].$$

For the first part, we realise that

$$\{\text{win one item}\} = \{\text{win at least one item}\} \cap \{\text{win two items}\}^c = \{b_1 > c_2\} \cap \{b_2 > c_1\}^c.$$

Hence,

$$\Pr(\text{win one item}) = \Pr(b_1 > c_2) - \Pr(b_2 > c_1) = H_2(b_1) - H_1(b_2).$$

The first part of the expectation simplifies as

$$\begin{aligned}\mathbb{E}[x_1 1_{\{\text{win one item}\}}] &= x_1\mathbb{E}[1_{\{\text{win one item}\}}] = x_1\Pr(\text{win one item})\\ &= x_1(H_2(b_1) - H_1(b_2)).\end{aligned}$$

For the second part of the expectation, we need to realise that the event {win one item} is equal to a disjoint union of events concerning c_1 and c_2:

$$\{\text{win one item}\} = \{b_2 < c_2 < b_1\} \cup \{c_2 < b_2, c_1 > b_1\} \cup \{c_2 < b_2, b_2 < c_1 < b_1\}.$$

This implies that the random variable $1_{\{\text{win one item}\}}$ can be written as a sum:

$$1_{\text{win one item}} = 1_{\{b_2<c_2<b_1\}} + 1_{\{c_2<b_2,c_1>b_1\}} + 1_{\{c_2<b_2,b_2<c_1<b_1\}}.$$

Now using the linearity of expectation, we obtain

$$\begin{aligned}\mathbb{E}[\max(b_2,c_2)1_{\{\text{win one item}\}}] &= \mathbb{E}[\max(b_2,c_2)1_{\{b_2<c_2<b_1\}}] + \mathbb{E}[\max(b_2,c_2)1_{\{c_2<b_2,c_1>b_1\}}]\\ &\quad +\mathbb{E}[\max(b_2,c_2)1_{\{c_2<b_2,b_2<c_1<b_1\}}\\ &= \mathbb{E}[c_2 1_{\{b_2<c_2<b_1\}}] + \mathbb{E}[b_2 1_{\{c_2<b_2,c_1>b_1\}}] + \mathbb{E}[b_2 1_{\{c_2<b_2,b_2<c_1<b_1\}}]\\ &= \int_{b_2}^{b_1} c_2 h_2(c_2)dc_2 + b_2\mathbb{E}[1_{\{c_2<b_2,c_1>b_1\}} + 1_{\{c_2<b_2,b_2<c_1<b_1\}}]\\ &= \int_{b_2}^{b_1} c_2 h_2(c_2)dc_2 + b_2\mathbb{E}[1_{\{c_2<b_2\}} - 1_{\{b_2>c_1\}}]\\ &= \int_{b_2}^{b_1} c_2 h_2(c_2)dc_2 + b_2(H_2(b_1) - H_1(b_2)).\end{aligned}$$

Now the total expectation becomes

$$\begin{aligned}\mathbb{E}[\pi_1] &= (x_1+x_2)H_1(b_2) - 2\int_0^{b_2} c_1 h_1(c_1)dc_1 + x_1(H_2(b_1) - H_1(b_2))\\ &\quad -\int_{b_2}^{b_1} c_2 h_2(c_2)dc_2 - b_2(H_2(b_1) - H_1(b_2)).\end{aligned}$$

Differentiate this with respect to b_2:

$$\begin{aligned}\frac{\partial \mathbb{E}[\pi_1]}{\partial b_2} &= (x_1+x_2)h_1(b_2) - 2b_2h_1(b_2) \\ &\quad -x_1h_1(b_2) + b_2h_2(b_2) - (H_2(b_2) - H_1(b_2)) - b_2(h_2(b_2) - h_1(b_2)) \\ &= (x_2 - b_2)h_1(b_2) - (H_2(b_2) - H_1(b_2)).\end{aligned}$$

Now suppose that $b_2 = x_2$. Then the derivative becomes

$$\frac{\partial \mathbb{E}[\pi_1]}{\partial b_2} = -(H_2(x_2) - H_1(x_2)) < 0$$

by construction of the distributions H_1 and H_2. The fact that this is negative implies that it is optimal for the first player to deviate from the equilibrium strategy, which implies that the truth telling equilibrium does not exist after all. In fact we can calculate the optimal strategy for the first player (assuming that the other player plays by the above strategy). We set $\frac{\partial \mathbb{E}[\pi_1]}{\partial b_2} = 0$ to obtain the value of b_2 that maximizes the expected value for the first player:

$$b_2 = x_2 - \frac{H_2(b_1) - H_1(b_2)}{h_1(b_2)}.$$

A.2 Proof 1: demand reduction is possible in the ascending bid auction

Proof. Define a bidding strategy $\beta_1(\delta r) = d_1(\delta r) - \epsilon_r$ that deviates from the truthful bidding strategy with ϵ_r in each round r. Now we determine the difference between the expected profits of this strategy and the truthful strategy.

$$\begin{aligned}
\mathbb{E}[\pi_i] - \mathbb{E}[\hat{\pi}_i] &= \sum_{r=1}^{N} \left(F^{(K)}_{K-\beta_1(\delta r)+1}(\delta r) - F^{(K)}_{K-\beta_1(\delta(r-1))+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{\beta_1(\delta r)} - \delta r \beta_1(\delta r) \right] \\
&\quad - \sum_{r=1}^{N} \left(F^{(K)}_{K-d_1(\delta r)+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{d_1(\delta r)} - \delta r d_1(\delta r) \right] \\
&= \sum_{r=1}^{N} \left(F^{(K)}_{K-d_1(\delta r)+\epsilon_r+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+\epsilon_{r-1}+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{d_1(\delta r)-\epsilon_r} - \delta r d_1(\delta r) + \delta r \epsilon_r \right] \\
&\quad - \sum_{r=1}^{N} \left(F^{(K)}_{K-d_1(\delta r)+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{d_1(\delta r)} - \delta r d_1(\delta r) \right] \\
&= \sum_{r=1}^{N} \left(F^{(K)}_{K-d_1(\delta r)+\epsilon_r+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+\epsilon_{r-1}+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{d_1(\delta r)} - \delta r d_1(\delta r) \right] \\
&\quad - \sum_{r=1}^{N} \left(F^{(K)}_{K-d_1(\delta r)+\epsilon_r+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+\epsilon_{r-1}+1}(\delta(r-1)) \right) \left[v_{d_1(\delta r)-\epsilon_r+1} + \ldots + v_{d_1(\delta r)} - \delta r \epsilon_r \right] \\
&\quad - \sum_{r=1}^{N} \left(F^{(K)}_{K-d_1(\delta r)+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+1}(\delta(r-1)) \right) \left[v_1 + \ldots + v_{d_1(\delta r)} - \delta r d_1(\delta r) \right] \\
&= \sum_{r=1}^{N} A_r B_r - \sum_{r=1}^{N} C_r D_r .
\end{aligned}$$

Where we have defined

$$\begin{aligned}
A_r &= \left(F^{(K)}_{K-d_1(\delta r)+\epsilon_r+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+\epsilon_{r-1}+1}(\delta(r-1)) \right) - \\
&\quad \left(F^{(K)}_{K-d_1(\delta r)+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+1}(\delta(r-1)) \right) \\
B_r &= v_1 + \ldots + v_{d_1(\delta r)} - \delta r d_1(\delta r) \\
C_r &= F^{(K)}_{K-d_1(\delta r)+\epsilon_r+1}(\delta r) - F^{(K)}_{K-d_1(\delta(r-1))+\epsilon_{r-1}+1}(\delta(r-1)) \\
D_r &= v_{d_1(\delta r)-\epsilon_r+1} + \ldots + v_{d_1(\delta r)} - \delta r \epsilon_r .
\end{aligned}$$

We need to show that if the opposite player bids truthfully, the player has an incentive to deviate from the truthful strategy. This corresponds to $\mathbb{E}[\pi_i] - \mathbb{E}[\hat{\pi}_i] = \sum_{r=1}^{N} A_r B_r + \sum_{r=1}^{N} C_r D_r > 0$.

First, we look at B_r. This is equal to the sum of the values of the objects included in the truthful bid $d_1(\delta r)$ minus the costs associated with this package. Since the truthful bid always bids such that the values of the bid are larger than the total price, we must have $B_r \geq 0$.

The term D_r is the sum of the values of the objects removed by the demand reduction ϵ_r minus the price for these objects. From the definition of the truthful bid, we also conclude that $D_r \geq 0$.

We conclude, that the investigation of the possibility that $\sum_{r=1}^{N} A_r B_r > \sum_{r=1}^{N} C_r D_r$, fully depends on the probabilities defined in the order statistics function F. The structure of A_r and C_r can be complicated, since a strategy is a sequence ϵ_k defined for all k. Therefore, let us investigate a simple strategy. Define the deviation strategy ϵ_r by

$$\epsilon_r = \begin{cases} 0, & \text{if } r \neq \hat{r} \\ 1, & \text{if } r = \hat{r} \end{cases}$$

for some round $\hat{r}$. This means, that the bid remains the truthful bid for rounds other than $\hat{r}$, but the bid is reduced by one at round $\hat{r}$.

For rounds $r \neq \hat{r}$, we see that $A_r = 0$ and $D_r = 0$, as $\epsilon_r = 0$. Hence, for the deviating strategy to be better than the truth telling strategy, we only need to show that $A_{\hat{r}} B_{\hat{r}} > C_{\hat{r}} D_{\hat{r}}$ or:

$$\begin{aligned} &\left(\left(F^{(K)}_{K-d_1(\delta\hat{r})+\epsilon_{\hat{r}}+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+\epsilon_{\hat{r}-1}+1}(\delta(\hat{r}-1))\right) - \left(F^{(K)}_{K-d_1(\delta\hat{r})+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+1}(\delta(\hat{r}-1))\right)\right)\left(v_1 + \ldots + v_{d_1(\delta\hat{r})} - \delta\hat{r}d_1(\delta\hat{r})\right) \\ &> \left(F^{(K)}_{K-d_1(\delta\hat{r})+\epsilon_{\hat{r}}+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+\epsilon_{\hat{r}-1}+1}(\delta(\hat{r}-1))\right)\left(v_{d_1(\delta\hat{r})} - \delta\hat{r}\right). \end{aligned}$$

Mind that $\epsilon_{\hat{r}-1} = 0$ for this particular strategy. We can rewrite this as

$$\begin{aligned} &\frac{\left(F^{(K)}_{K-d_1(\delta\hat{r})+\epsilon_{\hat{r}}+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+\epsilon_{\hat{r}-1}+1}(\delta(\hat{r}-1))\right) - \left(F^{(K)}_{K-d_1(\delta\hat{r})+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+1}(\delta(\hat{r}-1))\right)}{F^{(K)}_{K-d_1(\delta\hat{r})+\epsilon_{\hat{r}}+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+\epsilon_{\hat{r}-1}+1}(\delta(\hat{r}-1))} \\ &> \frac{v_{d_1(\delta\hat{r})} - \delta\hat{r}}{v_1 + \ldots + v_{d_1(\delta\hat{r})} - \delta\hat{r}d_1(\delta\hat{r})}. \end{aligned}$$

We have assumed here that the denominators are unequal to zero. This is not problematic as we are only restricting F to a certain set of functions. We only need to find one example where demand reduction is possible (we are not trying to show that demand reduction is always possible, just that it can be possible in certain cases). Simplify the above to

$$1 - \frac{F^{(K)}_{K-d_1(\delta\hat{r})+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+1}(\delta(\hat{r}-1))}{F^{(K)}_{K-d_1(\delta\hat{r})+\epsilon_{\hat{r}}+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+1}(\delta(\hat{r}-1))} > \frac{v_{d_1(\delta\hat{r})} - \delta\hat{r}}{v_1 + \ldots + v_{d_1(\delta\hat{r})} - \delta\hat{r}d_1(\delta\hat{r})},$$

we notice that this is satisfied if the probability increase that the game will end in the next round $F^{(K)}_{K-d_1(\delta\hat{r})+\epsilon_{\hat{r}}+1}(\delta\hat{r}) - F^{(K)}_{K-d_1(\delta(\hat{r}-1))+1}(\delta(\hat{r}-1))$ is large enough, compared to the loss caused by missing one unit $v_{d_1(\delta\hat{r})} - \delta\hat{r}$.

The right hand side will be between zero and one: if the value of the 'last object' is small (relative to the higher valued objects), then it is close to zero, and the demand reduction strategy will be optimal at some point. This example of a demand reduction strategy shows that the player possibly has an incentive to deviate from the truth telling strategy, and thus truth telling is not an equilibrium strategy. □

A.3 Proof 2: Truth telling is optimal in the alternative ascending bid auction

Assume two players. The profit of the first player can then be written as

$$\begin{aligned}\pi_1 &= \sum_{j=1}^{C_i^R} v_{1,j} - \sum_{r=1}^{R} c_i^r p(r) \\ &= \sum_{j=1}^{K-\beta_2(p(R))} v_{1,j} - \sum_{r=1}^{R} \left(\beta_2(p(r-1)) - \beta_2(p(r))\right) p(r) \end{aligned} \tag{A.1}$$

since $C_i^r = \max\left\{0, K - \sum_{j\neq i} \beta_j(p(r))\right\} = K - \beta_2(p(r))$ in the two player case. The conditional expecation $f(r)$ can be written as

$$\begin{aligned} f(r) &= \mathbb{E}[f(r)|R=r] \\ &= \mathbb{E}\left[\sum_{j=1}^{K-\beta_{2|R=r}(p(r))} v_{1,j} - \sum_{k=1}^{r} \left(\beta_{2|R=r}(p(k-1)) - \beta_{2|R=r}(p(k))\right) p(k)\right] \end{aligned}$$

where $\beta_{2|R=r}$ is the bid function of player 2 conditional on the event $\{R=r\}$. This can be complicated to determine exactly, but we will not need its exact form. Assuming player 2 bids truthfully, we investigate the difference $f(\hat{r}) - f(\hat{r}+1)$,

$$\begin{aligned} f(\hat{r}) - f(\hat{r}+1) &= \mathbb{E}\left[\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}+1))} v_{1,j}\right] \\ &\quad -\mathbb{E}\left[\sum_{k=1}^{\hat{r}+1} \left(\beta_{2|R=\hat{r}+1}(p(k-1)) - \beta_{2|R=\hat{r}+1}(p(k))\right) p(k)\right. \\ &\quad \left. - \sum_{k=1}^{\hat{r}} \left(\beta_{2|R=\hat{r}}(p(k-1)) - \beta_{2|R=\hat{r}}(p(k))\right) p(k)\right]. \end{aligned}$$

Now introduce the events A_0 and A_1

$$\begin{aligned} A_0 &= \{\beta_{2|R=\hat{r}+1}(p(\hat{r})) - \beta_{2|R=\hat{r}+1}(p(\hat{r}+1)) = 0\} \\ A_1 &= \{\beta_{2|R=\hat{r}+1}(p(\hat{r})) - \beta_{2|R=\hat{r}+1}(p(\hat{r}+1)) = 1\} \end{aligned}$$

and the random indicator variables $1 = 1_{A_0} + 1_{A_1}$. Mind that this equality holds with probability one as we assumed ties not to exist and as such the bid difference between the rounds (if the round step size δ is small enough, which we assume) can be at most one.

For the first sum of (A.1), we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}+1))} v_{1,j}\right] &= \mathbb{E}\left[\left(\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}+1))} v_{1,j}\right) 1_{A_0}\right] \\
&\quad + \mathbb{E}\left[\left(\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}+1))} v_{1,j}\right) 1_{A_1}\right] \\
&= \mathbb{E}\left[\left(\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}))} v_{1,j}\right) 1_{A_0}\right] \\
&\quad + \mathbb{E}\left[\left(\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}))+1} v_{1,j}\right) 1_{A_1}\right]
\end{aligned}$$

We can assume the price step size δ to be small enough, so that $\beta_{2|R=\hat{r}+1} = \beta_{2|R=\hat{r}+1}$. The above then simplifies to

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}+1}(p(\hat{r}+1))} v_{1,j}\right] &= \mathbb{E}\left[\left(\sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))} v_{1,j} - \sum_{j=1}^{K-\beta_{2|R=\hat{r}}(p(\hat{r}))+1} v_{1,j}\right) 1_{A_1}\right] \\
&= -v_{1,K-\beta_{2|R=\hat{r}}(p(\hat{r}))}\mathbb{E}[1_{A_1}] = -\Pr(A_1) v_{1,K-\beta_{2|R=\hat{r}}(p(\hat{r}))}
\end{aligned} \tag{A.2}$$

We turn to the second sum of (A.1) and separate it

$$\begin{aligned}
& \mathbb{E}\left[\sum_{k=1}^{\hat{r}+1}\left(\beta_{2|R=\hat{r}+1}(p(k-1)) - \beta_{2|R=\hat{r}+1}(p(k))\right)p(k) - \sum_{k=1}^{\hat{r}}\left(\beta_{2|R=\hat{r}}(p(k-1)) - \beta_{2|R=\hat{r}}(p(k))\right)p(k)\right] \\
= \; & \mathbb{E}\left[\sum_{k=1}^{\hat{r}}\left(\beta_{2|R=\hat{r}+1}(p(k-1)) - \beta_{2|R=\hat{r}+1}(p(k))\right)p(k) - \sum_{k=1}^{\hat{r}}\left(\beta_{2|R=\hat{r}}(p(k-1)) - \beta_{2|R=\hat{r}}(p(k))\right)p(k)\right] \\
& + \mathbb{E}\left[\left(\beta_{2|R=\hat{r}+1}(p(\hat{r})) - \beta_{2|R=\hat{r}+1}(p(\hat{r}+1))\right)p(\hat{r}+1)\right] \\
= \; & \mathbb{E}\left[\sum_{k=1}^{\hat{r}}\left(\beta_{2|R=\hat{r}+1}(p(k-1)) - \beta_{2|R=\hat{r}+1}(p(k))\right)p(k) - \sum_{k=1}^{\hat{r}}\left(\beta_{2|R=\hat{r}}(p(k-1)) - \beta_{2|R=\hat{r}}(p(k))\right)p(k)\right] \\
+ \; & p(\hat{r}+1)\Pr(A)
\end{aligned}$$

Now if we let the round step size δ become very small, the expectation of the sum will go to zero as $\beta_{2|R=\hat{r}+1} = \beta_{2|R=\hat{r}}$ as $\delta \to 0$. Furthermore we will have $p(\hat{r}+1) = p(\hat{r})$ as $\delta \to 0$, and the above is equal to

$$\begin{aligned}
& \mathbb{E}\left[\sum_{k=1}^{\hat{r}+1}\left(\beta_{2|R=\hat{r}+1}(p(k-1)) - \beta_{2|R=\hat{r}+1}(p(k))\right)p(k) - \sum_{k=1}^{\hat{r}}\left(\beta_{2|R=\hat{r}}(p(k-1)) - \beta_{2|R=\hat{r}}(p(k))\right)p(k)\right] \\
= \; & p(\hat{r})\Pr(A)
\end{aligned} \tag{A.3}$$

Combining (A.2) and (A.3) gives

$$f(\hat{r}) - f(\hat{r}+1) \quad = \quad \Pr(A_1)(p(\hat{r}) - v_{1,K-\beta_{2|R=\hat{r}}(p(\hat{r}))}).$$

Of course we know that in the 2-player case

$$K - \beta_{2|R=\hat{r}}(p(\hat{r})) = \beta_1(p(\hat{r}))$$

and we conclude that

$$f(\hat{r}) - f(\hat{r}+1) \quad = \quad \Pr(A_1)(p(\hat{r}) - v_{1,\beta_1(p(\hat{r}))}) = \Pr(A_1)(p(\hat{r}) - v_{1,d_1(p(\hat{r})-1)}) \leq 0.$$

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