Real Options and Ambiguity Aversion: a Theoretical Approach

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Master Thesis

Author: Johan Lourens
Studentnumber: 377281
Supervisor: Aurelien Baillon
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Abstract

How Ambiguity affects Real Options is a recent research topic. Earlier research incorporated the maxmin model to estimate the effects, but the maxmin model groups together the perceived ambiguity level and the decision makers' ambiguity preference. Here I create a Real Option model using a naïve α-Maxmin model, effectively separating the preference and the ambiguity level. A decision maker expresses his or her preference by creating a weighted average of an extremely ambiguity averse and extremely ambiguity loving preference. I show the merit of the α-Maxmin model by comparing it to the Maxmin model and the ambiguity neutral model normally used. From an academic perspective the results are informative and interesting for future development. But in the field of economic implementation of the Maxmin and α-Maxmin model work needs to be done, if real life implementation is what we seek.

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1 Introduction

Real options (RO) are a valuable tool in dynamic decision-making processes. Real option theory is an application of option pricing theory, as economists noticed the similarities between the uncertainty faced for a financial assets and company investments in its various forms. These models were based upon the notion that we know the outcome probabilities for every future period, e.g. from the start we know how the future probabilities for every event looks like. Thus, using a Wiener process or lattice to simulate the future possible states, we can predict the expected optimal point in time to make investments in a project or terminate a project and we can take future decisions into account when valuing a project (see for example Brennan and Schwartz, 1985; Quigg, 1993). The option value and the value of waiting then follow from the simulation. The method focuses solely on a risky environment, although we do not know with certainty the future outcomes, the probability measures describe it quite precisely.

Often decision makers do not know the exact probabilities for certain events. These unknown, hard to predict events do have a real impact on future results. Black and Scholes (1973) and Cox et al. (1979) type RO pricing do not take into account that invest/terminate decisions are sometimes made without precise knowledge on a probability measure. For example, an oil company can assess how likely it is that demand increases, but these estimates are surrounded with uncertainty regarding this estimate. Furthermore, not everyone has the same taste preference for this ambiguity. In the literature maxmin utility is utilized to incorporate ambiguity, as developed by Gilboa and Schmeidler (1989). In this thesis I develop six models: (1) a binomial RO model without ambiguity, (2) a naïve maxmin binomial RO model, (3) a naïve α-maxmin binomial RO model, (4) a quadrinomial RO model without ambiguity, (5) a naïve maxmin quadrinomial RO model and (6) a naïve α-maxmin quadrinomial RO model. Within the real option literature the maxmin expected utility model is used to model ambiguity attitude and when extended to a dynamic model also called the multiple priors model (Epstein, 1999). Instead of using one probability measure for all periods, a set of probability measures (priors) is taken. This makes sure that the probability measure can vary in every period. The most pessimistic measure is selected from this set to calculate the option value for a company with ambiguity aversion. The model with one known probability measure is ambiguity neutral - or more precisely, the conjunction of ambiguity loving and ambiguity averse. The priors set size describe the level of ambiguity aversion present.

In the decision theory literature alternatives have been proposed to measure ambiguity aversion, which can be incorporated in the RO theory. The theoretical addition to the RO literature will be the development of an α-maxmin model (Ghirardato and Marinacci, 2002). The added value of an α-maxmin model for ROs is the direct measure of ambiguity aversion, distinct from the ambiguity level. I keep the set size of the priors equal for all companies, while distinguishing between the levels of ambiguity aversion between companies. This increases the flexibility when modeling RO. An applied framework of the α-maxmin model is not yet developed for ROs. Here three different ambiguity models are developed, using discrete time and a finite horizon approach. The differences in approach are compared using the ambiguity neutral model, the maxmin model and the α-maxmin model.

In section 2 the existing literature on the topic will be discussed, section 3 introduces the baseline model and the extensions into the other two models, section 4 compares the resulting effects of the three models and section 5 concludes and discusses some possible extensions.

2 Review of literature

2.1 Theoretical approaches to ambiguity and ambiguity aversion

Decision-making processes and ambiguity is a topic of interest with a long history within economics. Knight (1921) was one of the first to make a distinction between measurable uncertainty and immeasurable uncertainty with respect to uncertain future events. In more recent literature, the former is risk and the latter is ambiguity; I will adapt these terms. Often many neglect the differences between risk, ambiguity and uncertainty. Risk describes uncertainty over future outcomes with known probabilities; ambiguity describes uncertainty about both probabilities and outcomes; and both ambiguity and risk are a type of uncertainty. Venezia (1983) was one of the earlier authors trying to describe a Real Option model with unknown future growth. Ambiguity aversion is not explicitly mentioned and the prior beliefs were still in the realm of the rational approach. Note that the strongest result is for the Bayesian approach, while the non-Bayesian approach is mostly described in terms relative to the Bayesian.

Two inputs that impact ambiguity aversions that are often used, deal with the application of Real Options and the applied utility function. When trying to analyse a dynamic decision making process, there are several sources of ambiguity. If we consider an investment problem: waiting to invest in a project involves ambiguity/risk about payoffs, but the same applies to the project profits itself (McDonald and Siegel, 1986).

Risk aversion is a well-studied and well-defined phenomenon. A similar approach to ambiguity aversion is still being debated. Gilbao and Schmeidler (1989) developed the maxmin model. It describes extreme ambiguity aversion for a given set of priors, as it considers ambiguity as a set of priors/probability measures on future outcomes and takes the probability measure that describes the worst-case scenario. The relative ambiguity aversion is then covered by the size of the set and the convexity of the sets.

Convexity is not a necessary requirement to have an aversion towards ambiguity. An exertion to define ambiguity aversion in a more rigorous fashion was made by Epstein (1999). Ambiguity aversion is then defined alongside risk aversion on a utility function. Risk aversion is defined relative to some risk neutral case, to measure the level of risk aversion and to have a baseline. Likewise, ambiguity aversion needs some baseline case, but this obfuscated through the effects of risk aversion. To keep ambiguity aversion and risk aversion apart, we need an ambiguity neutral baseline measurement incorporating risk aversion. Epstein proposes a probabilistic sophisticated decision preference as ambiguity neutral. Ghirardato and Marinacci (2002) use expected utility as the ambiguity neutral baseline.

For this thesis a more appealing definition is based on a more intuitive notion of ambiguity. Ellsberg (1961) did an experiment using two urns, showing the impact of ambiguity on decision-making. There are 100 black and red marbles in the urns. The distribution of black and red marbles is known for urn 1, 50 black marbles and 50 red marbles. For urn 2 the distribution of black and red marbles is unknown. Two groups of experiment participants are asked to choose whether they want to select a marble from urn 1 or urn 2. The first group gets paid €10 for a black marble and otherwise nothing. The second group gets paid €10 for a red marble and otherwise nothing. In the experiment both groups prefer to select a marble from the unambiguous urn 1, in other words they perceive urn 1 as the better bet. As the possible payoffs are the same, this implies that the participants perceive the subjective probability for black and red marbles for the ambiguous urn to be smaller than for the unambiguous urn.

$$P^{\mathrm{amb}}(\mathrm{black}) < 0.5 \tag{1}$$

$$P^{\mathrm{amb}}(\mathrm{red}) < 0.5 \tag{2}$$

$$P^{\mathrm{amb}}(\mathrm{black}) + P^{\mathrm{amb}}(\mathrm{red}) < 1 \tag{3}$$

Where $P^{\mathrm{amb}}(\cdot)$ is the probability measure for the ambiguous urn. Expected utility dictates that probabilities should add up to unity, which is not the case here. Therefore, an improved model is necessary to describe behaviour.

Two models that describe ambiguity aversion shown in Ellsberg's experiment are Choquet expected utility and multiple-priors expected utility, which are both generalisations of expected utility. Here the focus will be on the latter model, which has an intuitive explanation for the ambiguity aversion effect. For the ambiguous urn numerous distributions of black and red marbles are possible, e.g. 20 black and 80 red, 40 black and 60 red, etc. The maxmin model selects the worst expected case of all possible distributions that are deemed possible by the decision maker - called the set of priors from here on. This allows the probabilities for black marbles and red marbles not add up to one, because the probabilities for either the black or red marble are considered separate from each other and the probabilities could be taken from a different distribution for both colours.

The model developed by Epstein and others have as a disadvantage that ambiguity aversion and the ambiguity level is modelled as one element. It would enhance insights when we separate them, as one is a preference and the other is a perceived level of ambiguity. Developments come from behavioural and model robustness concerns, but establishes similar results. Cagetti et al. (2002) are not directly concerned about ambiguity aversion, but show that if a hidden Markov jump model is introduced as the true process then traditional smooth - continuous - models do not perform optimally. Ambiguity about this unknown jump process is modelled through relative entropy. Some boundaries are introduced to create a reasonable deviation from the estimated probability. Nearness is used to ensure a reasonable deviation, which is defined as having the same null events as the estimated probability. The null events are those events that are not described by the set of possible outcomes and therefore do not exist for this particular description of the world. When the deviation from the estimated probability measure is larger, the ambiguity about the correct model is larger. To handle ambiguity aversion, a penalty function is introduced on this deviating measure. Robustness is then further developed for continuous time and - using recursive utility - an intertemporal model is produced (Cagetti et al., 2001; Skiadas, 2003)

But criticism is also cast on the robustness approach using the nearness idea, because this technical requirement can be quite restrictive. Some models may incorporate some events that are hard to predict, for example extreme weather. This is not necessary included in all outcome spaces. In risk management, the coherent and convex measures of risk have a representation very similar to the maxmin model. Exploiting these risk measures leads to robust models without the necessity for nearness to some estimated probability measure (Cont, 2003).

Ju and Miao (2011) make use of an initial model by Klibanoff et al. (2005, 2006) for smooth ambiguity measurement of asset returns. It does not show the kink in the indifference curve that the maxmin model shows and the model makes a distinction between ambiguity and ambiguity aversion, e.g. between beliefs and taste. They are able to explain the first moment of the equity premium puzzle, making a distinction between ambiguity and ambiguity aversion. As a disadvantage the method is hard to apply empirically and calibration remains tricky. Learning and intertemporal substitution over time can also be folded into the analysis (Hayashi and Miao, 2011). It seems that a more intuitive and easily implemented method could be applied to estimate ambiguity and ambiguity aversion.

A final approach was developed using Epstein's work, where the authors rely on identical utility functions when comparing relative ambiguity aversion for decision makers (Ghirardato and Marinacci, 2002). The results are less strong/defined than the work of Epstein, but easier to implement as the baseline set has to be less rich. As an alternative maxmin utility function they introduce the α-maxmin utility function. The attractive trait of the model is its simplicity relative to the smooth ambiguity model or even choquet expected utility. Furthermore, it does measure ambiguity aversion and ambiguity separately. It uses a weighted average of the maxmin and the maxmax model, where the weight factor α is the aversion to ambiguity. Maxmax is the best possible outcome given the set of priors. Thus, it measures a weighted average of the worst-case scenario and the best-case scenario. Its mathematical tractability allows easy implementation into the real option theory.

2.2 Ambiguity aversion in empirical applications

Chen and Epstein (2002) use the recursive multiple prior utility model and the earlier developed definitions for ambiguity aversion by Ghirardato and Marinacci (2002) to develop a continuous time model for asset pricing where risk aversion and ambiguity aversion are separate input factors. The authors do not make a difference between ambiguity aversion and ambiguity.

A portfolio selection experiment, by Ahn, Choi, Gale and Kariv (2007), tests rival measures of utility. Here the α-maxmin model performs well, but does not fit every subjects preferences. Therefore the evidence shows a mixed image.

Rigotti, Ryan and Vaithianathan (2008) use the Hurwicz criterion to measure levels of ambiguity aversion in an economy with ambiguity and risks. The economy is made up of old industry and new industry, where the former there are known risks and the latter shows ambiguity. They model how agents with heterogeneous ambiguity aversion choose between worker/entrepreneur and old/new industries. The model is compared to empirical findings and shows reasonably comparable results.

2.3 Real options

There are many similarities between financial options and real options; it both tries to model future opportunities with a right but not the obligation to some action. A financial option gives the right to buy or sell a stock or some other financial asset, meanwhile a real option gives the right to invest or disinvest in a project. The latter could be any decision where timing and uncertainty matters: when to go from renting to buying a house, the optimal time to marry and when to buy a new car. The costs involved analysing these decisions in such a fashion might be too high for such decision, but the possibility exists. Certain decisions with a structure that reflects more exotic options can easily be replicated using option theory. Nonetheless, real options are not perfect replications of financial options at all times.

Financial option pricing is based upon arbitrage arguments therefore a risk neutral pricing method can be used. The arbitrage argument states that there is a portfolio replicating the payoffs of the option. When the option price deviates from this price in a complete market - the assets necessary to create the replicating portfolio are available - a risk free profit could theoretically be made. This approach does not take into account any ambiguities and risks that occur in practice, subsection 2.4 deals with the former and a limited description of the latter will be treated in this subsection.

Mining operations, R&D, land development and supply chains are project types that are often taken as example for the use if real options. It could be the option to invest, disinvest, grow and shrink operations. Further, it could be an option on an option, an option with more than one source of uncertainty and with boundaries on payoffs. This list is not exhaustive, but gives a clear picture how real options help us analyse decisions on dynamic subjects.

Some real options have a closed form solution often these are in a sense the 'simpler' models. Whenever it gets more complicated some type of simulation is necessary. Cox (1979) introduced the relation between the Black-Scholes model and the binomial tree model. The latter model made it possible to price American options, which are quite interesting from a real option point of view. A European option can only be exercised/used at the end of its maturity T, when the maturity of an option is 10 years a project can only be initiated in 10 years. An American option can be exercised/used whenever the holder wants to exercise, the option with maturity of 10 years can also be exercised in 5 years, 3 years or tomorrow.

For stocks that do not pay dividends it is never interesting to terminate an option before maturity, but with stocks that do pay dividends early exercise can be a good opportunity. Real options often have costs imbedded in the underlying project that can be described as a form of dividends. Individual projects have a limited time where (excess) profits can be made, while a stable financial asset keeps its value and potential for capital gains for a much longer time. Permits to mine or drill often have a limited lifetime before the permit has to be renewed. Otherwise, companies face competitors that invest in the same R&D, which makes the maximum delay to invest in R&D limited. Therefore, exercise before maturity is often a good opportunity. Another approach when pricing real options could be Monte Carlo simulations. This makes direct use of the Wiener process, to model many possible paths. The Monte Carlo simulation has the advantage that many more decision moments and degrees of up and down are taken into account, but the binomial tree is easier to work with to model complicated processes. Both binomial trees and Monte Carlo simulations are computer intensive - a good reason to try and find closed form solutions when possible - but more flexible than closed form solutions.

Finally, risk neutral valuation is not always the best route to take. Brennan shows that if the project to be valuated has resources that are publicly traded in the form of futures, risk neutral valuation is possible using a replicating portfolio. Without doubt this is will regularly not be the case, think of R&D for products that do not exist yet and processed products that are partially correlated, not correlated or correlated with many resource futures. Whenever this happens and assuming a risk averse investor, one has to calculate the option value using a utility functions that reflect the investor's risk averseness.

2.4 Real options and ambiguity aversion

Within the real option literature ambiguity aversion is just emerging as a promising modelling tool. For financial options Cont (2003) is an early developer of ambiguity in the real option literature. More recent examples are Trojanowska and Kort (2010) and Miao and Wang (2011).

The former use a method likewise to Cagetti et al. (2002) and Skiadas (2003) for ambiguity aversion, e.g. using probability measures related to a reference probability measure P. Waiting becomes less valuable when ambiguity is introduced and when ambiguity goes over a certain level a company might even not invest with a positive probability. This non-investment is related to time, as time goes towards infinity the probability to not invest is always larger than zero. Thus, the longer a project runs, the higher the probability that a company invests. An alternative model is developed using an endogenous risk factor, described by a Brownian motion.

Miao and Wang (2011) use the multiple-prior model to see how uncertainty affects Real Option exercise, set in an infinite time horizon, discrete time setting. If we use a certain payment at exercise, therefore all uncertainty is resolved at that point; ambiguity pulls the exercise/stopping point forward. E.g. exercise is sooner and the overall uncertainty is resolved sooner in a continuation RO. If there are two prior sets $\mathcal{P}_1$ and $\mathcal{P}_2$ where $\mathcal{P}_2 \geq \mathcal{P}_1$, then as ambiguity is larger for $\mathcal{P}_2$ the point of stopping is reached earlier. The worst-case scenario is worse for $\mathcal{P}_2$ then $\mathcal{P}_1$, therefore potential future outcomes are not as profitable in a maxmin type model. In the more realistic case where ambiguity is also present for the payoff, the results of ambiguity are less obvious. The ambiguity of payoffs counteracts the ambiguity of waiting, but if ambiguity is very large a myopic Net Present Value (NPV) rule should be used. Myopic NPV is calculated using the standard NPV model, but including a negative bias on future values thereby making future potential profits smaller then the expected value.

Following Miao and Wang, I propose two models using the α-maxmin model to conveniently separate ambiguity and ambiguity aversion and show how it performs relative to the ambiguity neutral model and the maxmin model.

3 Model setup

ROs show us the value of future opportunities and the decisions that are related to these opportunities. If decision makers contemplate whether to wait or to act immediately, ROs help them quantitatively analyse the future uncertainty and its effects on decision making. All models are set up in discrete time and with a finite time frame. Modern companies all face projects that might run for a long time or a short time, say respectively an energy plant and a mobile phone. The lack of an infinite time horizon unites these projects and therefore it seems reasonable to develop a model that has a finite time horizon.

I develop three models in a logical progression; first the ambiguity neutral set up, second the maxmin set up and third the α-maxmin set up.

3.1 Baseline set up

3.1.1 Probability space

The subject of interest is what value future decisions have, whether to suspend, start, grow or cut a project. Due to future uncertainty - here indicating a general term for future outcomes that we do not know whatever the source - it can only be described in terms of possible outcomes. Define an outcome space $(\Omega, \mathcal{F})$ with Ω the set of possible states of the world ω and $\mathcal{F}$ the set of all possible outcomes. In the baseline set up, there is one probability measure P for the outcomes $\mathcal{F}$ as there is only uncertainty about outcomes and not about the probabilities. An intertemporal process requires a filtration $\mathcal{F}_t$, that leads to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $\mathcal{F}_T = \mathcal{F}$ and $0 \leq t \leq T$. An example will be given for all described technical requirements.

Take an urn with blue, red and green marbles. Ω contains the set of possible individual states when taking out a marble, therefore Ω: $\{\mathrm{b}, \mathrm{r}, \mathrm{g}\}$. $\mathcal{F}$ reflects all possible outcomes, note that the outcomes could be combinations of individual states. If we are interested in taking one marble from the urn $\mathcal{F}$, coincides with Ω as the outcome set consists of only individual states, therefore $\mathcal{F}$: $\{\mathrm{b}, \mathrm{r}, \mathrm{g}\}$. Matters change whenever we take two marbles from the urn; we create a set containing combinations of states, therefore $\mathcal{F}\colon\{\text{bb, br, bg, rr, rg, rb, gg, gb, gr}\}$. The more marbles we take from the urn, $\mathcal{F}$ changes in a similar fashion. Conveniently, the above description leads to the filtration $\mathcal{F}_t$: assume there are two time periods where $\mathcal{F}_2 = \mathcal{F}$. In each period there are a limited amount of possible added states as described by Ω. In period one a marble is selected from the urn and $\mathcal{F}_1\colon\{\text{b, r, g}\}$; in period two - the final period - a marble is again selected from the urn and $\mathcal{F} = \mathcal{F}_2\colon\{\text{bb, br, bg, rr, rg, rb, gg, gb, gr}\}$. Summarising, the filtration describes the development of the possible outcomes over time, until the final period is reached.

Extending from the outcome space, when we select marbles from the urn there is uncertainty about the exact outcome. Before a marble is selected from the urn the outcome can only be described in terms of likeliness. To describe this we need a probability measure. For the baseline model there are 100 marbles in the urn with 60 green marbles, 30 red marbles and 10 blue marbles. This yields the following probabilities for green, red and blue marbles:

$$P(g) = \frac{60}{100} = 0.6 \tag{4}$$

$$P(r) = \frac{30}{100} = 0.3 \tag{5}$$

$$P(b) = \frac{10}{100} = 0.1 \tag{6}$$

The function P measures how likely some state ω is or more succinct $P\colon \Omega \to \mathbb{R}$.

3.1.2 From continuous Wiener process to discrete lattice

Wiener processes are often used to model movements of random processes. It assumes independence between periods, such that past results do not influence the current result. A Wiener process is not a perfect representation of reality and often denounced when tested, but the process can be expanded to be more realistic (Lüders and Schröder, 2004). Here a simple Wiener process will suffice to show the effects of ambiguity compared to the ambiguity neutral model, ceteris paribus.

If the process of interest is the optimal point of investment in a project, we are interested whether it pays off to wait or act now. The simplest case is when we have just one variable process and the rest is fixed, e.g. uncertainty is resolved at the moment of exercise and the investment has a fixed price. In that case the process describes how a profit varies over time and when exercised it materializes immediately. If profit is denoted by π, the Wiener process is described by

$$\frac{d\pi}{\pi} = r_\pi dt + \sigma_\pi dB_t \tag{7}$$

With a return r, an infinitsimal change in time dt, the return standard deviation σ and the Brownian motion B_t, which is a random variable with a standard normal distribution $\mathcal{N}(0,1)$. If the option to invest is exercised, the turnover at that point will be π_t at the costs of investment I.

To simplify matters I create a discrete variant of the Wiener process, using a binomial tree. The question is how I transform the Wiener process in a binomial discrete process with an upstate U and a downstate D. The discrete and continuous models are required to have matching expectations and variance. Furthermore, in the binomial model upstate U is reached with an objective probability q. Figure 1 shows the graphic representation of the binomial model.

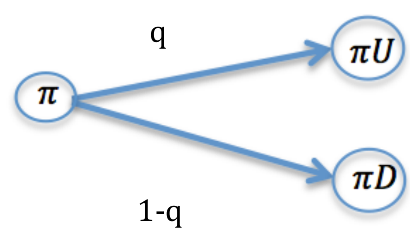

Figure 1: Binomial tree

If the continuous process has an expected return r, the expected return for the discrete process should equal that return:

$$1 + r = qU + (1 - q)D \tag{8}$$

and similar the variance σ^2 for the continuous process should equal the variance of the discrete process:

$$\begin{aligned} \sigma^2 &= q\big(U - (qU + (1-q)D)\big)^2 + (1-q)\big(D - (qU + (1-q)D)\big)^2 \\ &= q\big((1-q)U - (1-q)D\big)^2 + (1-q)(qD - qU)^2 \\ &= q(1-q)^2(U-D)^2 + (1-q)q^2(U-D)^2 \\ &= q(1-q)(U-D)^2 \end{aligned} \tag{9}$$

In the case we know the true objective probability q was, there were two unknown variables - U and D - and two functions with the unknown variables in it. This gives us the opportunity to calculate the required values for U and D. When we maximise the uncertainty, q is 0.5. Throughout the thesis we will assume that q = 0.5. Note that this objective imposed probability is not the probability used when calculating the option value. Using equations 8 and 9, enter 0.5 for q and rewrite we obtain:

$$2(1+r) = U + D \tag{10}$$

$$2\sigma = U - D \tag{11}$$

Rewriting functions 10 and 11 in terms of U and D gives

$$U = 1 + r + \sigma \tag{12}$$

$$D = 1 + r - \sigma \tag{13}$$

Because variance and returns are not always scaled at the correct step size - e.g. lattice steps are monthly and the continuous parameters r and σ are yearly - we scale them using the correct time step size. Return is scaled with respect to time with equation 14

$$(1+r)^t \tag{14}$$

As the returns at each point are assumed to be independent of each other, we can simply use a linear scaling for variance with respect to time using equation 15

$$t\sigma^2 \tag{15}$$

Plugging in equation 14 and 15 for the return and variance into equation 12 and 13 we obtain equation 16 and 17. This in effect makes U and D dependent on step size.

$$U(t) = (1 + r)^t + \sigma\sqrt{t} \tag{16}$$

$$D(t) = (1 + r)^t - \sigma\sqrt{t} \tag{17}$$

Whenever the time interval nears zero, $t \to 0$, $(1 + r)^t$ approximates $1 + rt$. This leads to

$$U(t) \cong 1 + rt + \sigma\sqrt{t} \tag{18}$$

$$D(t) \cong 1 + rt - \sigma\sqrt{t} \tag{19}$$

The first term of a Taylor expansion for an exponential function is equal to our outcome, therefore equations 18 and 19 can be approximated by

$$U(t) \cong e^{rh+\sigma\sqrt{h}} \tag{20}$$

$$D(t) \cong e^{rh-\sigma\sqrt{h}} \tag{21}$$

The binomial tree model, introduced by Cox et al. (1979), did not use the drift term, but the model can be expanded to create a certain drift up or down[1]. As noted earlier, the probability q will not be used, but a risk neutral probability. Following Cox et al. this probability has the form

[1] Derivation adapted from reader created by João Amaro de Matos, Professor at Nova University Lisbon.

$$p = \frac{e^{rh} - D}{U - D} \tag{22}$$

The risk neutral probability is the probability that makes the expected return equal to the risk free return. Nonetheless, it will be necessary that $D < e^{rh} < U$, otherwise $p \notin [0,1]$.

3.1.3 Option value

Time t might not be the optimal point of entry; a company should neither be too late nor too early to invest in a project. A company has a limited horizon when it can invest; in such a case an American option is the obvious choice to value a project and I assume that less units θ can be sold as time progresses. The latter gives the opportunity to model a certain necessity to invest within a reasonable time frame, as standard American calls will never exercise before maturity. Thus, $\theta_0 > \theta_t > \theta_T$ with $t < T$ and $\theta_t = \theta_{t-1} - \delta$ where δ is a predetermined and fixed number. Here I assume constant investment costs. At time t one wants to maximize payoffs

$$F_t = \max\left(\pi_t \theta_t - I, \beta \mathbb{E}_t^p (F_{t+1})\right) \tag{23}$$

And at time T the maximizing function is

$$F_T = \max(\pi_T \theta_T - I, 0) \tag{24}$$

Where $\beta = [0,1]$ is the discount rate, π_t is the turnover at time t and $\mathbb{E}_t^p$ is the expectation operator at time t under the risk neutral probability measure described earlier. We are already taking into account what happens when a project becomes unprofitable, therefore the option could be seen as a risk-decreasing tool. Nonetheless risk neutral valuation will in some cases not be the right method. In such a case the expectation operator becomes more complicated. As the process for calculating options is recursive[2] the value function for the real option at the initial decision time 0 is

[2] Example: at any node in the binomial tree at time $T - 1$ the recursive maximising function has the form: $F_{T-1} = \max\left(\pi_{T-1}\theta - I, pF_{T,u} + (1-p)F_{T,d}\right)$. Where the second term is the risk neutral expected value.

$$\mathrm{F}_0 = \max\left(\pi_0\theta_0 - I, \beta\mathbb{E}_0^p(F_1)\right) \qquad \text{with some predetermined } \pi_0 \tag{29}$$

Note that the above model is an American option, but it could easily be transformed into a European option or something in between when decision moments are not as plentiful.

Figure 2 gives a graphical description of the option pricing process and how it is reversed from the earlier price process.

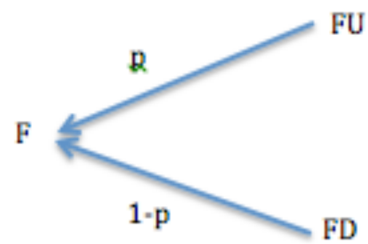

Figure 2: the recursive process for the option pricing using the risk neutral probabilities

3.1.4 Uncertain investments

Naturally, this is not a very realistic case. Not only future turnovers are affected by changes over time, costs and project outcomes can be affected by uncertainty about the future. Investment costs I and project profits are not necessarily certain at time of investment, prices for raw resources might fluctuate and customers might not like the product the company sells. But project turnovers for a started project and a potential future turnover stream are determined from the same source. If we assume that an investment is only made at the start of a project, investments are only variable over time for the determination of the potential future costs. In that case turnover is determined by[3]

$$\frac{d\pi}{\pi} = r_\pi dt + \sigma_\pi dB_{\pi,t} \tag{25}$$

[3] Subscript added for clarity.

and future investment costs

$$\frac{dI}{I} = r_I dt + \sigma_I dB_{I,t} \tag{26}$$

Using the same method described above, $\frac{dI}{I}$can be made discrete

$$U_I(t) \cong e^{r_I + \sigma_I \sqrt{t}} \tag{27}$$

$$D_I(t) \cong e^{r_I - \sigma_I \sqrt{t}} \tag{28}$$

Calculating the risk neutral probability s is the same for the investment, but can differ due to differences in drift and standard deviation. Moreover, there are two processes now enlarging the space of possible outcomes at any time, e.g. investment costs can be in up state and profits can be in down state. We expand the binomial model to a quadrinomial model where the processes are assumed to be independent, which gives us the risk neutral probabilities for every combination of states shown in table 1

		Investment costs	
		Up	Down
Profits	Up	ps	$p(1-s)$
	Down	$(1-p)s$	$(1-p)(1-s)$

Table 1: Risk neutral probabilities for states with two processes

This gives me the opportunity to create a quadrinomial model; the ordering for the tree is clear for the top and bottom, lower investment costs and higher profits dominate over all other results and higher investment costs and lower profits are dominated by all others. The ordering in the middle depends on the parameters. The quadrinomial model is shown in figure 2 with objective probabilities q and k for the up and down states.

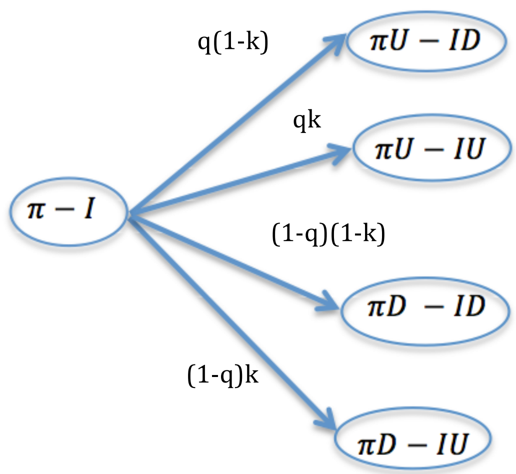

Figure 3: Quadrinomial model

The cut-off point is important; the turnover process does not affect the option on the future the same as the project in implemented state. This is due to having a choice to invest or not for the former case; e.g. when the economic climate turns out negative for the project no real losses will be made and the project will not be implemented. In that case, the option value at time t is determined by

$$F_t = \max\left(\pi_t\theta_t - I_t, \beta\mathbb{E}_t^p(F_{t+1})\right) \tag{29}$$

And at time T the maximizing function is

$$F_T = \max(\pi_T\theta_T - I_T, 0) \tag{30}$$

And at the decision moment F_0 the option value is determined by

$$F_0 = \max\left(\pi_0\theta_0 - I_0, \beta\mathbb{E}_0^p(F_1)\right) \text{ with some predetermined } \pi_0 \text{ and } I_0 \tag{31}$$

3.2 Models with ambiguity

3.2.1 Maxmin model

Most decisions inherently contain more uncertainty than merely future outcomes and therefore ambiguity about the probability measure will be introduced in this subsection. Intuitively it seems straightforward to think of this ambiguity in a sense of having a vague idea about the exact parameters needed to put in the real option model. But the mathematics requires more than a vague idea to set up the real option model. How could one represent ambiguity? Any probability measure could be a valid predictor, but often there are some guidelines about the behaviour of the uncertain subject, some historical numbers that could help.

Say there are reasonable guesses based upon historical data or internal research on the subject matter, but not boiled down to one probability measure. These different measures could be because different time frames are researched, large deviations in the mean, noise or reasonable fitting for several models. These measures can be bundled in a set - within the maxmin model also called a set of priors - and say that these are the measures that can be used when valuating a real option or find the optimal timing. The maxmin assumes that a decision maker wants to be at the safe side when making a decision. This leads them to take the worst-case scenario when estimating the results.

The model has a lot of similarities with the baseline model, the outcome space and the filtration remains the same. The probability measure becomes a prior. In essence the probability space changes from $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, where $\mathcal{P}$ is the set of priors. Note that it generalises the baseline model, because when the prior contains one element the model collapses into the baseline model: $\mathcal{P} = \{P\}$. When the set $\mathcal{P}$ is larger there is more ambiguity surrounding the decision: if decision 1 contains more ambiguity than decision 2, than $\mathcal{P}^2 \subset \mathcal{P}^1$.

The maxmin model selects the least favourable case - calculate all results and pick the least favourable result. The set of priors is reduced with every step as more information becomes available. Variance in the drift estimate is assumed to drive the ambiguity - I could also vary other variables like the variance or change the complete underlying distribution, but it is not necessary to show the results - and the variance in drift affect the risk neutral probabilities for the binomial or quadrinomial tree.

Before briefly describing the models used, I like to discuss the updating rule. In reality new information becomes available over time and the information can be incorporated if the investment is not yet done. Before hand we do not know this, thus one by one drift estimations are taken from the set, until there is only one probability measure left at the end. At the end it will be clear what the trend has been over time. This is obviously easy for the binomial model, but the quadrinomial model is more involved. Here the both processes have a set of priors evolving independently.

Model descriptions can be found below, starting with a price at some time t, at time T and at time 0. Calculations coincide with each other, except that there is a selection procedure for the minimum function. The functions are given for the more general quadrinomial case, but return to the Binomial when I is constant.

$$F_t^{maxmin} = \max\left(\pi_t\theta_t - I_t\theta_t, \beta \min_{p\in\mathcal{P}_{\pi,I}} \mathbb{E}_t^p(F_{t+1})\right) \tag{32}$$

$$F_T^{maxmin} = \max(\pi_T\theta_T - I_T\theta_T, 0) \tag{33}$$

$$F_0^{maxmin} = \max\left(\pi_0\theta_0 - I_0\theta_0, \beta \min_{p\in\mathcal{P}_{\pi,I}} \mathbb{E}_0^p(F_1)\right) \tag{34}$$

with some predetermined π_0 and I_0

For modelling purposes I create a naïve version of the maxmin model where no learning effects are taken into account. In that case I can use several probability measure to calculate the option prices and take the worst case at time 0. I feel this assumption is a reasonable one, as a good updating rule for the priors is quite complex to create and implement for a decision maker. Over time new information will determine how the priors will develop, but estimating this development at time 0 will be near impossible for most decision makers.

3.2.2 α-Maxmin model

Section 2 already mentioned that the maxmin model does not take into account the issue that ambiguity and ambiguity aversion are not separated and that it assumes that ambiguity is very much disliked by decision makers. The α-maxmin gives us the ability to separate these and at the same it is an easy application as the data is already available through the maxmin model. The model descriptions for the α-maxmin are

$$F_t^{\alpha maxmin} = \max\left(\pi_t\theta_t - I_t\theta_t, \beta\left[\alpha\left(\min_{p\in\mathcal{P}_{\pi,I}} \mathbb{E}_t^p(F_{t+1})\right) + (1-\alpha)\left(\max_{p\in\mathcal{P}_{\pi,I}} \mathbb{E}_t^p(F_{t+1})\right)\right]\right)$$

$$F_T^{maxmin} = \max(\pi_T\theta_T - I_T\theta_T, 0)$$

$$F_0^{\alpha maxmin} = \max\left(\pi_0\theta_0 - I_0\theta_0, \beta\left[\alpha\left(\min_{p\in\mathcal{P}_{\pi,I}} \mathbb{E}_0^p(F_1)\right) + (1-\alpha)\left(\max_{p\in\mathcal{P}_{\pi,I}} \mathbb{E}_0^p(F_1)\right)\right]\right)$$

The α-maxmin model developed here is naïve in the sense that there are no learning effects over time and it is the weighted average over the maxmin and maxmax model at (and only at) time 0.

4 Model analysis

The models have some specific effects; I will start with the basic models with respect to prices. Some limitations for the maxmin model and the α-maxmin will be discussed. General price and optimal timing will be discussed and the differences between the standard model and the other two models. Finally, a numerical example will be given to see how the model could be applied. Before presenting the analysis some notation. T is time, N are time steps in the tree, I are investments, r is the risk free rate, π are profits, θ_t are expected units sold, δ is the decrease in expected units sold, μ are drifts, σ is volatility, $\mathcal{P}$ is the set of priors.

4.1 Analysis for the binomial model

4.1.1 Value effect

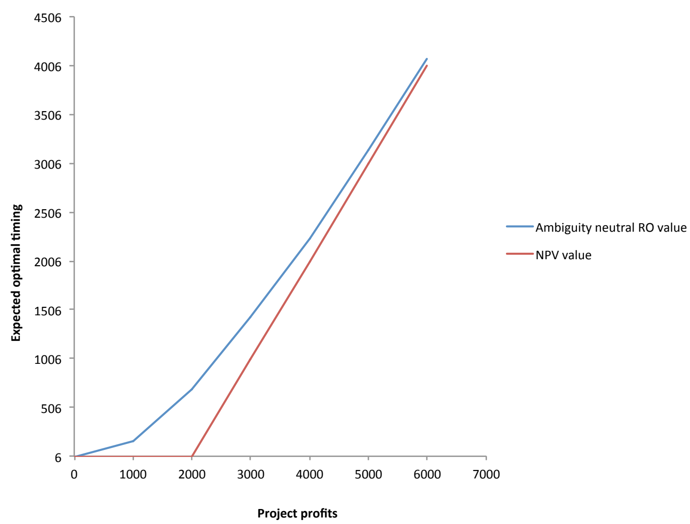

Figure 4: Value comparison between baseline investment option and immediate investment. T=5, N=10, r=0.04, σ=0.3, μ=0 I=2000, θ_0=100, δ=1, π_0 is varied between 0 and 60.

Figure 4 shows how a standard type model operates. It yields what I would expect: waiting to invest is a profitable strategy at many levels compared to immediate investing. The lower line is the expected NPV without the option to wait, the upper line is the option value combined with immediate investment. The added value for waiting is the difference between the upper and lower lines. But when varying the different parameters a richer image is created. Varying the drift has relatively little impact on the price of the option as can be seen in figure 5, where I took a drift rate of 0, 0,3 and -0,3. The option has the highest value with a 0 drift rate, following intuition a negative drift performs worst in value terms and a positive drift is in between the zero and negative drift. The latter seems surprising, but an extremely positive outlook and decreasing units

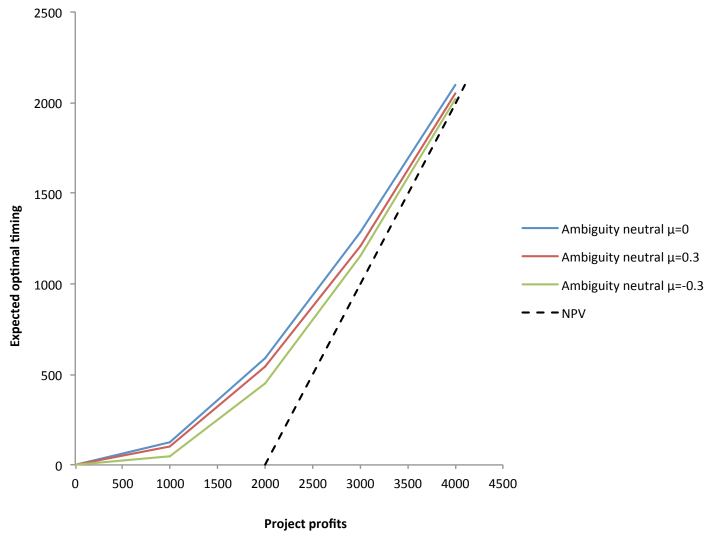

Figure 5: Varying drifts and the resulting values. T=5, N=10, r=0.04, σ=0.3, I=2000, θ_0=100, δ=1, π_0 is varied between 0 and 40, μ is varied between -0.3 and 0.3.

sold over time will favour a rapid deployment reaping the immediate benefits and the volatility has less of a downward grip on future profits.

When varying volatility the effects are larger and more impressive, but the results obviously follow classical theory: increased volatility makes waiting more valuable. As can be seen in figure 6 the speed of convergence towards the boundary line is faster for the low variance option than the high variance option, where the dotted line is the boundary line for immediate investment.

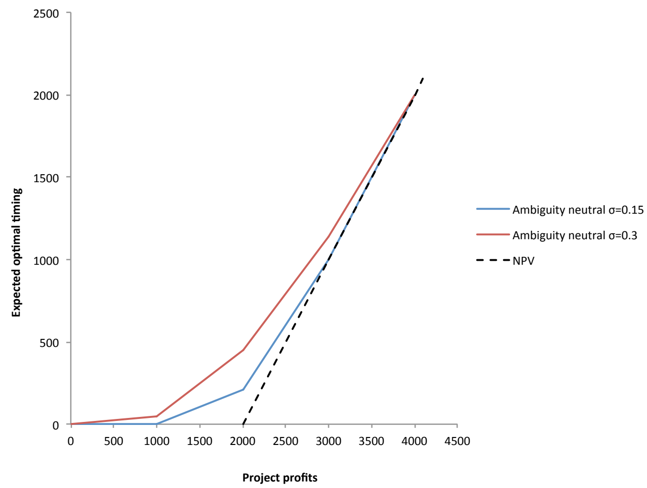

Figure 6: T=5, N=10, r=0.04, μ=0, I=2000, θ_0=100, δ=1, π_0 is varied between 0 and 40, σ is varied between 0.15 and 0.3.

As expected, unit sales deterioration has a large impact on the option value to wait, later exercise is less attractive due to this decrease. It is similar to a dividend payment and though it is of little interest how the decreasing attractiveness is modelled, I find it the most realistic. Figure 7 is illustrative for the effects of decreasing units sales over time, when there is no such effect the convergence towards the boundary line is slow indicating that it is always profitable to wait. When there is just a small effect present, 1 unit sale less every period, convergence happens much faster. Whenever I increase this to 3 unit sales less every period it becomes even more pronounced. If it goes above 4 unit sales decrease, the option is only more attractive when profits in the present are negative or zero.

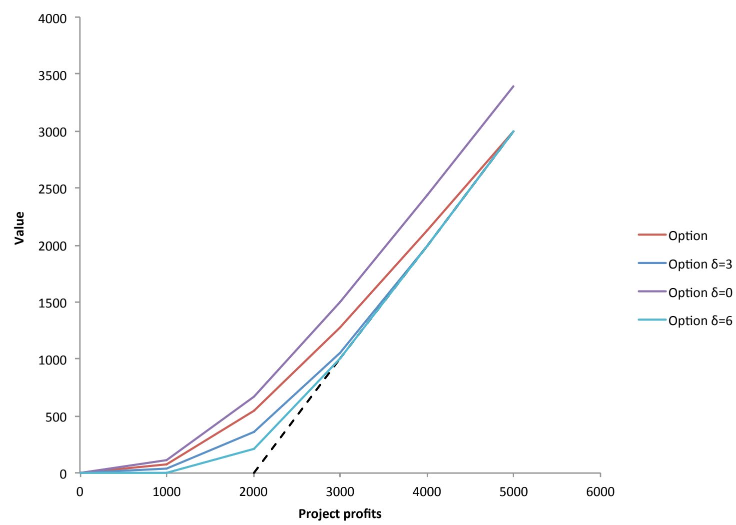

Figure 7: Value difference when depreciation is varied. T=5, N=10, r=0.04, μ=0, σ=0.3, I=2000, θ_0=100, π_0 is varied between 0 and 40, δ is varied between 0 and 0,6.

As I use a naïve versions of the maxmin and α-maxmin model, such that the probability measures do not change over time and there is no interaction between the best case of the worst case, the general findings as described above will be similar. Nonetheless, ambiguity aversion does have an impact. Note that to measure ambiguity only the drift varies and what we saw earlier is that the value function does not purely shift parallel to each other. Instead the curvature can change. Depending on input parameters it is not obvious that the highest or lowest option value is produced by respectively the highest or lowest drift. This could give us some interesting insights.

The difference between outcomes for the two lines are indeed not constant. At the bottom and top of the graphs the differences are smaller, while in the middle the effect is larger. If volatility would also be varied the effects would be larger than what we see here. When implementing the maxmin model, it seems that it matters what causes the ambiguity. Ambiguity about decreasing sales and volatility in profits have a larger impact than ambuigity about the drift direction.

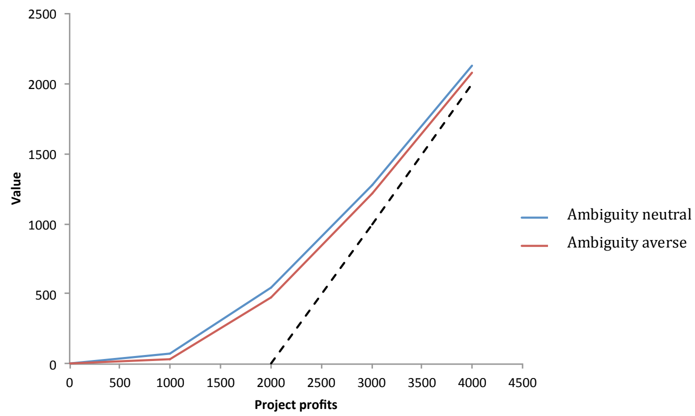

Figure 8: differences between maxmin and standard option. T=5, N=10, r=0.04, σ=0.3, μ=0 I=2000, θ_0=100, δ=1, π_0 is varied between 0 and 60. Ambiguity range is 0.2 with steps of 0.05.

Figure 8 shows the effects of a ambiguity neutral RO price and a maxmin ambiguity RO. The added value of the RO - recall it as the difference between the NPV and the RO price - becomes smaller when ambuigity is introduced. Ambuigity here is the uncertainty about the estimated drift. Note that the effects are quite small for the ambiguity model, although if multiplied by €100,000 the effects would be more pronounced and influence decision making drastic.

Ambiguity aversion affects the value of waiting, the boundary profit level at which the decision maker immediately invests is lower compared to the standard model. Due to the changes in curvature of the RO value function simply taking the lowest and/or highest level of some range is even for the naïve version not a wise decision. Of course, at any starting profit π_0 a different probability measure might be the worst or best measure, while it is static for the standard model adding a certain flexibility. Figure 9 is a comparison for the standard option, the maxmax ($\alpha = 1$) and the maxmin ($\alpha = 0$) model. Figure 10 shows the effects for the maxmin, maxmax, lowest drift and highest drift and how they are related. The lowest drift exactly matches the the maxmin model, but the highest drift does not match the maxmax. The maxmin and lowest drift are therefore combined to one line in figure 10.

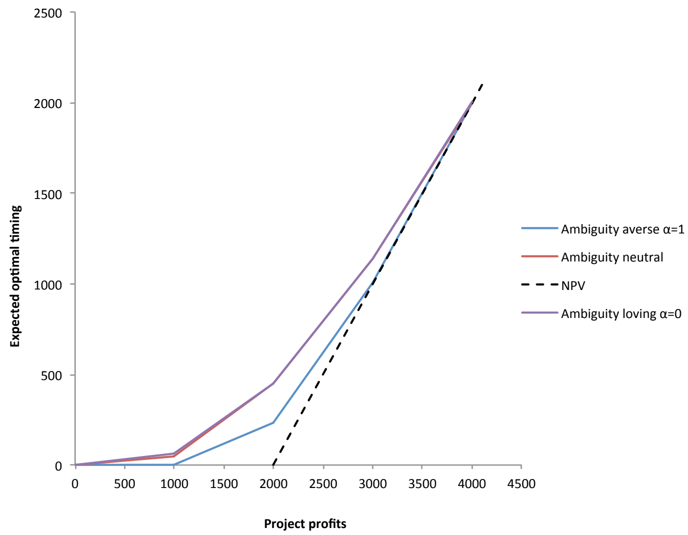

Figure 9: Comparison maxmax, maxmin and ambiguity neutral RO. Same input as figure 8.

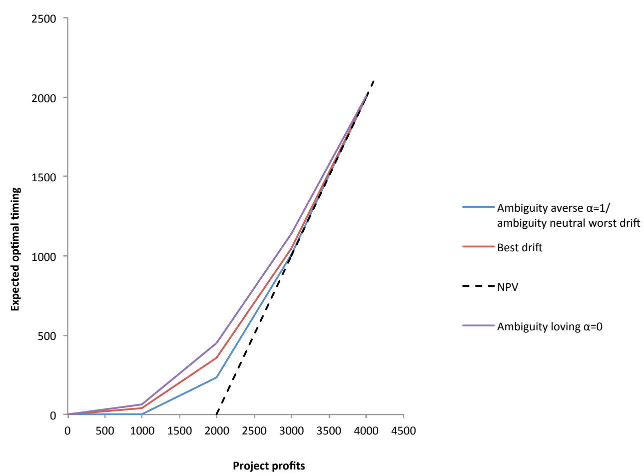

Figure 10: Comparison for maxmax, maxmin, worst drift and best drift models. Not obvious that the highest drift commands the best option value.

A static α-maxmin model with $\alpha = 0.5$ matches an ambiguity neutral model, as shown by Ghirardato and Marinacci (2002). The intertemporal naïve α-maxmin RO model presented here does not match the ambiguity neutral RO model and there is no reason to assume that it has to due to the simplifications assumed and interactions in the dynamic model. The results reflect this: the ambiguity neutral RO model is in this case almost the most optimistic case[4], therefore the RO model with $\alpha = 0.5$ cannot be equal to the ambiguity neutral model with regards to the models presented in this thesis. This is in line when varying the drift rate for the standard model.

Finally there is the case of ambiguity levels, which the α-maxmin model helps to resolve in terms of the difference between the level of ambiguity and taste in ambiguity. Figure 11 gives an example how ambiguity levels affects the option value for an ambiguity averse decision maker. There is no difference for the ambiguity loving decision makers when ambiguity is increased or decreased. The most optimistic case lies near a drift of 0, therefore enlarging the ambiguity's range does not affect the valuation.

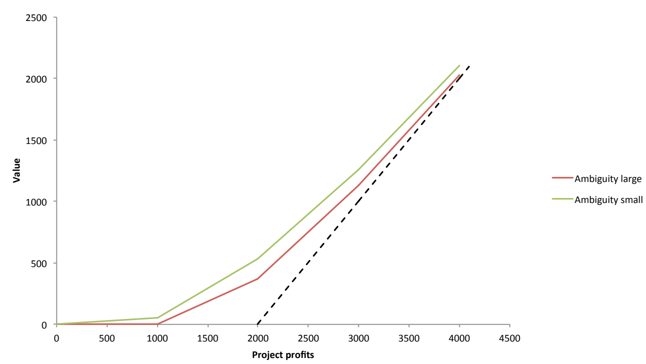

Figure 11: Difference between high ambiguity and low ambiguity for an ambiguity averse decision maker. Range ambiguity large 0.6, range ambiguity small 0.2.

4.1.2 Optimal timing

When considering optimal expected timing effects between the models I will focus only on the differences between the standard model, maxmin and α-maxmin model. As the valuations indicate, the expected optimal timing of the option decreases when an agent is

[4] Almost indiscernible in the graph, but there is a difference.

ambiguity averse. Expected timing shows us that agents who are more ambiguity averse will wait a shorter period compared to agents who do not take into account ambiguity/are ambiguity neutral. While ambiguity loving agents clearly prefer to wait for a longer period before they will invest. When the initial project profits are large enough the larger certainty about a positive future outcome let the outcomes almost converge to one point. The results seem more potent for timing an investment than for valuing an investment. The optimal timing functions for the ambiguity preferences have different sensitivities changes in the initial present value (PV) profits. An ambiguity averse decision maker changes their optimal timing with a smaller amount when the PV increases compared to an ambiguity loving decision maker. This makes sense, because ambiguity about profitability is higher when the PV is low. The ambiguity averse will exercise the option earlier and the ambiguity loving will exercise later. When the PV is high a project will remain profitable most of the time even if circumstances worsen. This reduces the ambiguity about potential losses and the preferences of the ambiguity averse and ambiguity loving will converge.

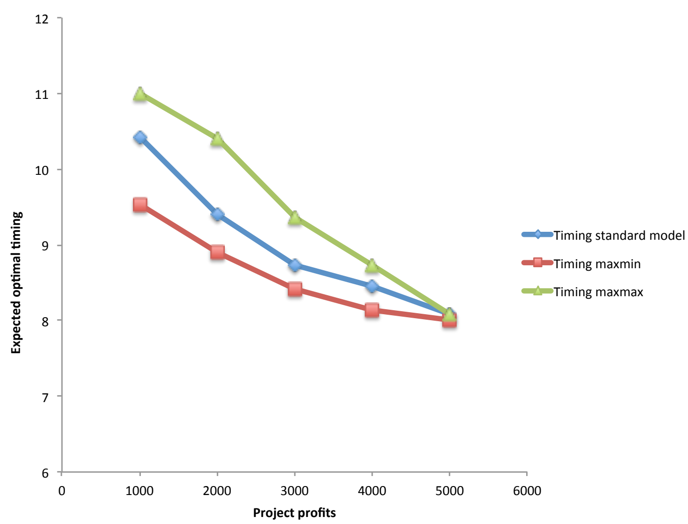

Figure 12: Optimal expected timing for the standard model, maxmin and the maxmax model. Timing is per semester.

4.2 Analysis for the quadrinomial model

4.2.1 Value effect

Varying investment costs independently from profits introduces more risk and ambiguity for all models involved. It would also be more realistic: a static exercise price is a fact for a financial option due to the contracts involved, but not for an investment. These costs could be hedged, but hedges are often imperfect. This does introduce a challenge with respect to the cost development. In section 4.1 I assumed throughout the section fixed cost, I again assume fixed costs instead of variable costs. The difference is that the fixed costs vary over time.

The positive influence from volatility is retained when introducing varying costs over time. What is interesting though is that with low cost variance the quadrinomial model returns lower values compared to the binomial model that is otherwise the same, but when this volatility increases the option becomes more valuable relative to the binomial model. The rest of the ideas that hold for the binomial model, also hold for the quadrinomial model (figure 13 and 14).

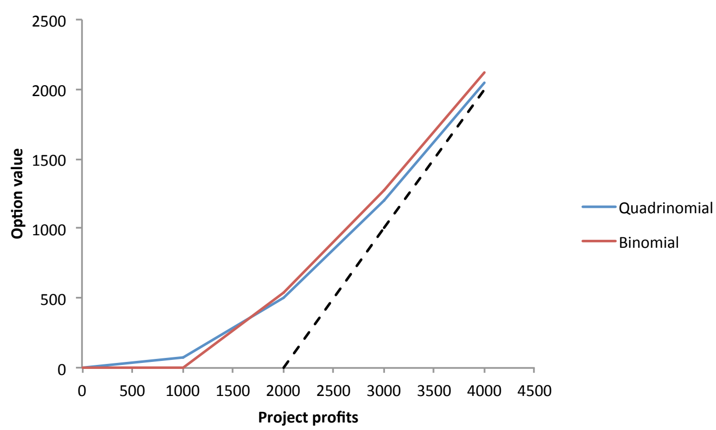

Figure 13: Binomial model vs quadrinomial model, variance 0.2 for costs.

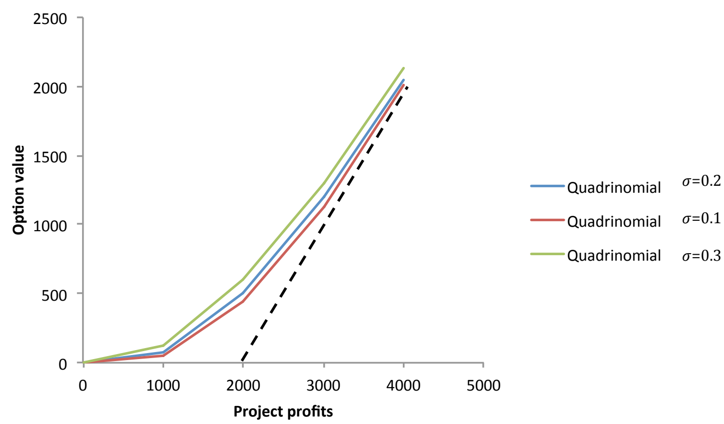

Figure 14: Different volatility levels for the quadrinomial model

The maxmin model and α-maxmin model performs the same as in the case of the binomial model. Maxmin has a more pronounced effect for the quadrinomial model than for the binomial model. Obviously, many more combinations of worst-case scenarios can be created compared for the quadrinomial model relative to the binomial model. If there are four different measures in the set of priors for profits and costs, the binomial model will have four combinations to compare, while the quadrinomial has 16 combinations to compare. This leads to more pronounced effects as costs could have a greater negative weight (see figure 15).

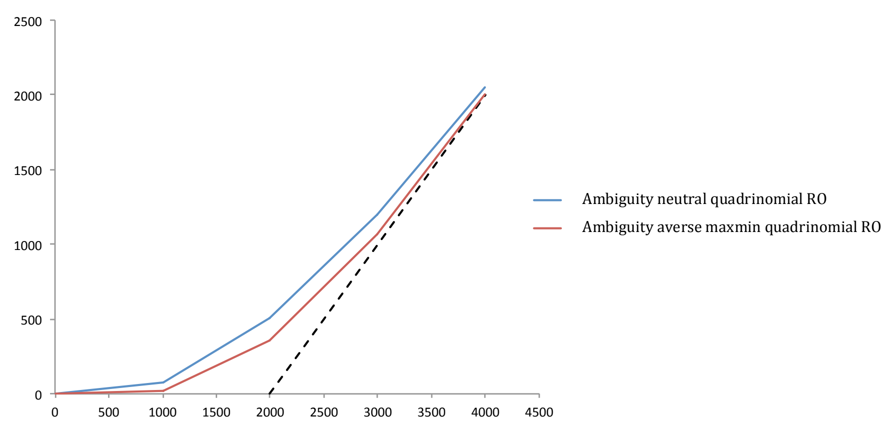

Figure 15: ambiguity neutral quadrinomial and maxmin.

The α-maxmin model than gives us the opportunity to see how the other extreme looks like. As is shown in figure 16.

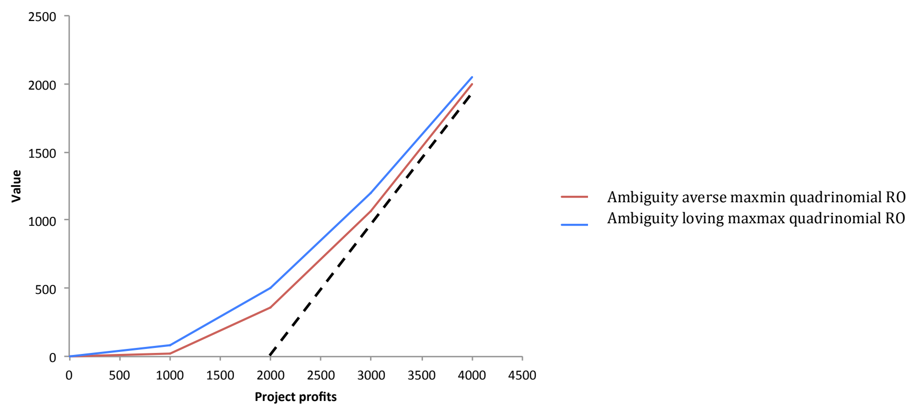

Figure 16: Maxmax and maxmin

4.2.2 Timing effects

Timing effects for the quadrinomial model are across the board larger than for the binomial model. What we see is that an expected optimal timing is quite a few periods ahead, around four years there is the tendency to implement the strategy, at that moment enough information has reached the decision maker. Again the standard model is in the middle in terms of waiting, ambiguity-averse agents will prefer earlier exercise and ambiguity-loving agents will prefer later exercise. It is a decreasing function with respect to initial profits. Convergence over time is similar to the binomial model, but runs a little smoother when considering project profits.

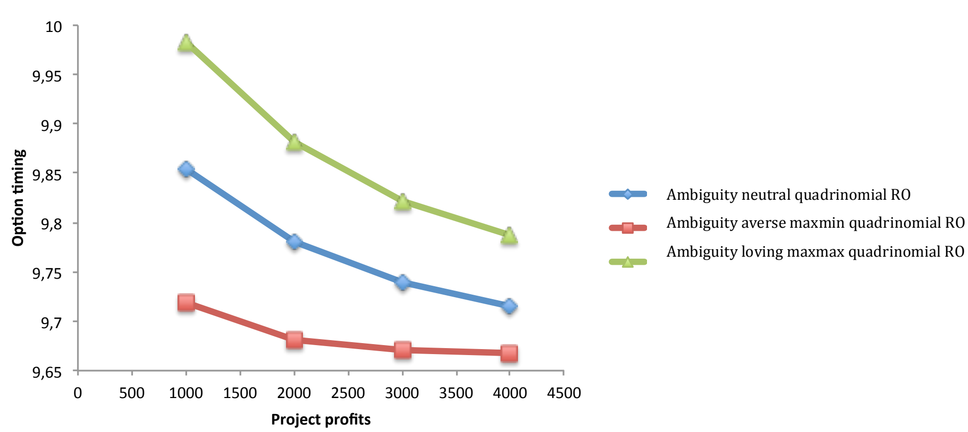

Figure 17: Optimal expected timing for quadrinomial model, timing is per semester

4.3 Economic relevance

The models presented here are technical in nature and economical relevance/practical application is a serious issue to deal with. Implementing a more complicated model is only interesting when the model performance is significantly better, reflects the preferences of the agents and has an intuitive feel for the final decision maker. Agents could be satisfied with a limited quantitative estimate - not representing their full preferences - due to the costs involved in finding how ambiguity averse they are or have difficulties with understanding what information the model does and does not convey.

I will give a numerical example, giving results for all models in terms of waiting time and valuation. Thereby illuminating how a practical application looks like and reaching out a hand from academics to practice.

Mini case: Mining venture

A company has bought the rights to exploit a goldmine. The mining permit has a limited lifetime; during the next 10 years the company is allowed to mine. The mine contains (effectively) 550 tons of gold; the yearly excavation is 50 tons of gold. Every year they do not invest they lose a capacity of 50 tons of excavation. The volatility of the profits is 40% and the company expects no drift. They now estimate an average profit of $20 million per tonne, which is what they expect to earn over time - $11 billion if they invest immediately. They have estimated investment costs of $4 billion, these costs can be assumed constant or vary over time. The costs have a volatility of 20%. The company is not completely sure; it is hard to estimate how prices will develop. Costs are more stable over time, thus costs show a symmetric range of 10% drift differential from the expectations. Profits have been eradicate lately and there could be 20% drift differential. Steps of 5% are taken as different levels of probability measures. Commodities require large investments and are general older/conservative companies. After some assessment the company estimated their ambiguity aversion parameter α at 0.3. Finally, the risk free rate is 4%.

The valuation of the project yields the same results using any model. The cause can be found in the fact that from a valuation point of view, the waiting value does not weigh against immediate investment profits. My assessment is that depreciation of tonnes sold dominates all other effects. From this point of view the standard model performs quite well and the agent could save the time of developing more complicated models. On the other hand, if we take a look at the option's expected optimal timing it becomes more interesting. Postponing investments yield seriously different results and it seems worthwhile to look into possible preferences and cost structures creating drastically different results for the company. Timing an investment is important for optimal company performance and these models could support decision makers when facing complicated projects with long-term effects.

	Value	Timing
Binomial ambiguity neutral	7000	3 years 6 months
Binomial maxmin	7000	3 years
Binomial α-maxmin	7000	3 years 2 months
Quadrinomial ambiguity neutral	7000	4 years 9 months
Quadrinomial maxmin	7000	4 years 6 months
Quadrinomial α-maxmin	7000	4 years 7 months

Table 2: Values project and timing project

Ambiguity aversion as a differentiated measure has a real effect when observing the difference for the maxmin and α-maxmin model. The mining corporation has the same ambiguity, but in the α-maxmin model they can also express their preferences about the ambiguity. They are expected to wait longer to invest, giving us a hint that the ambiguity level is not the same as the ambiguity preference. The differences are minor though, compared to the ambiguity neutral model. I do not expect the ambiguity effects on drifts to be economic relevant when calculating the expectations for the project. With different input and more uncertainty the models might gain more traction. Within this framework the standard model would be precise enough and easier to understand for the mining company executives. Furthermore, a standard NPV model can also be applied for calculating the mine value, but the added value for the RO model lies in the field project timing.

5 Conclusion and discussion

Concluding I can say that the real option models analysed in this thesis have very distinct uses. Especially when the uncertainty is great and the project complex, there is a payoff to do an analysis combining varying costs and ambiguity levels. From the point of view for optimal timing the results were most useful and salient. For optimal timing we see obvious differences between the ambiguity preferences. The extreme ambiguity averse maxmin preference decision maker pull forward the decision to invest relative to the ambiguity neutral. The extreme ambiguity lovers maxmax preference decision makers push backward the decision to invest relative to the ambiguity neutral. The quadrinomial model further pushes back the decision to invest when compared to the more standard binomial model. This is not strange, due to the extra information gained by waiting using the quadrinomial model. For certain parameters the valuations still matter though and even if the results are less striking, information can be gathered from it. It might be more interesting if there is an investment delay before profits can be made. The pricing of the ROs decreased with the ambiguity level for the ambiguity averse, the ambiguity loving decision makers show no effect with respect to increases in ambiguity. There can be several sources of ambiguity; here I investigated the influence of drift ambiguity. Nonetheless, research can be expanded when delving into several forms of ambiguity changing the shape of the probability function and even parameters not directly linked to the probability measure. An example for the former could be the standard deviation and an example for the latter could be the expected sales reduction over time. Drift ambiguity shows us that the maxmin model and the ambiguity neutral model with the worst drift coincide, but the maxmax model and the ambiguity neutral model with the best drift do not coincide. Suggesting it is too easy to just take what seems intuitively the worst case or the best case. Thereby, showing the added economic value of the model for ambiguous projects. The models give a nice distinction between ambiguity and ambiguity preference. The α-maxmin model shows that even for a naïve version it has an added value, due to the range of possible outcomes while maintaining one level of ambiguity. This preference can drastically influence project values and timing for decision makers, significantly impacting business decisions when implemented. An interesting research topic would be how α could be estimated. Here I assumed some level, but I cannot be sure how real life preferences are related to the α. The interpretation of α and its relation to the ambiguity neutral case is another interesting topic. The ambiguity neutral RO is often at the top end of the ambiguity preference spectrum, thereby skewing the measure α downward. E.g. measuring mostly aversion attitudes and little loving attitudes. Often decision makers are averse, but the extreme skewedness in the measure is a striking phenomenon.

The model developed in this thesis has its limitations. The model I used for the maxmin is naïve, where the agent does not update upon receiving new information. This assumption works, as we do not know in the present what information we receive in the future. Often we only know what kind of profits are reasonable when the project is implemented, which makes it safer to keep the ambiguity constant over time. Further, I only try to understand what happens with the valuations and optimal timing in the present when I change the parameters. If one would compare a more dynamic approach, then some kind of information reflection could be incorporated. One does not invest at time 0 and waits to invest. At time 1 one could use an updating rule based upon best/worst performance and eliminate one of elements in the prior. This updating rule could be based upon several distance rules and estimation procedures. On the other hand, introducing learning within a RO model is a practice of making many assumptions about how the profit or cost processes develop and then imply how the decision maker would react on that. Learning effects are more interesting if we incorporate multi period RO calculations and at every period new, objective market information becomes available to incorporate in the model. Note this does not mean a sophisticated model, but better naïve models. Without doubt this new information does not necessarily decrease ambiguity and the model may survive without adaptations.

The α-maxmin model as implemented in this thesis has an inherent challenge, due to the dynamic inconsistency of the model (Schröder, 2011). Dynamic consistency indicates that a dynamic model can be solved in a recursive fashion. As implemented in this thesis, it is a weighted average of the maxmax and maxmin model at time 0. This does not take into account the effects of weighted averages at times $t > 0$. Effectively, I average a model for the extreme ambiguity loving and the extreme ambiguity averse. But a decision maker who is somewhere between those extremes should average the extremes at every point in time, taking into account interactions between extreme ambiguity aversion and extreme ambiguity love.

A further limitation in my model is that I limit it to a discretisation of the lognormal model. It does not take into account skewness or fat tails, when considering worst cases. This also relates to limiting the ambiguity to the drift. Nonetheless, the possibility probably exists to adapt the discretisation to the third and fourth moments.

Ambiguity aversion has a reasonable influence - albeit mostly an influence for the academic world - on a real option and the differentiation between ambiguity aversion and the ambiguity level has a real life impact. The α-maxmin model should be more extensively developed. While I only made a weighted average at time 0, an improvement would be to find the worst/best option at every node and create a weighted average. Nonetheless, it is a promising field to do further research in and helps us shape the ideas about how true human preferences produce the best possible decisions. Finally, both the maxmin and the α-maxmin model should be further developed if we strive for real life implementation, as the results for these model do not deviate a lot from the ambiguity neutral model.

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Appendix - Algorithm for models.

Binomial model

```
Function Binomial_Standard(P, K, T, r, mu1, v1, N, Init, St)

S0 = P
I0 = K
Tijd = 0
Num = 0


dt = T / N
U1 = Exp(mu1 * dt + v1 * dt ^ 0.5)  'size of up jump for profit
D1 = Exp(mu1 * dt - v1 * dt ^ 0.5) 'size of down jump for profit
P1 = (Exp(r * dt) - D1) / (U1 - D1) 'risk free probability of up jump for profit
P2 = 1 - P1  'risk free probability of down jump for profit


ReDim Smat(1 To N + 1, 1 To N + 1) 'holds profits

Smat(1, 1) = S0

For i = 1 To UBound(Smat, 2) - 1 ‘fills the profit tree
   Smat(1, i + 1) = (Smat(1, i) * U1)
   For j = 2 To i + 1
      Smat(j, i + 1) = Smat(j - 1, i) * D1
   Next j
Next i

For q = 1 To UBound(Smat, 2) ‘differentiates for every period how much is sold
   For x = 1 To q
      Smat(x, q) = Smat(x, q) * Init
   Next x
   Init = Init - St
Next q

ReDim Cmat(1 To N + 1, 1 To N + 1)

For h = N + 1 To 1 Step -1 ‘recursive calculation of the RO
   For g = 1 To h
```

```
    If h = N + 1 Then
      Cmat(g, h) = WorksheetFunction.Max(Smat(g, h) - K, 0)
      If Smat(g, h) - K > 0 Then
        Tijd = Tijd + h
        Num = Num + 1
      End If
    Else
      op = Exp(-r * dt) * (P1 * Cmat(g, h + 1) + P2 * Cmat(g + 1, h + 1))
      Cmat(g, h) = WorksheetFunction.Max(Smat(g, h) - K, op)
      If op < Smat(g, h) - K Then
        Tijd = Tijd + h
        Num = Num + 1
      End If
    End If
  Next g
Next h

Binomial_Standard = Tijd / Num ‘expected timing
OR ALTERNATIVELY
Binomial_Standard=Cmat(1,1) ‘expected value

End Function
```

α-maxmin and maxmin binomial model

```
Public Sub alpha_Minmax_Binomial_model()

Range("B14") = ""

P = Range("B1")
K = Range("B2")
T = Range("B3")
r = Range("B4")
mu1 = Range("B5")
v1 = Range("B6")
N = Range("B7")
RAp = Range("B8")
SAp = Range("B9")
alpha = Range("B10")
Init = Range("B11")
St = Range("B12")


minmu = mu1 - RAp  'creates range
maxmu = mu1 + RAp

steps = (maxmu - minmu) / SAp + 1
nmbr = 1

ReDim ambiguity(1 To steps + 1)

For i = minmu To maxmu Step SAp  'creates all possibilities
   option_price = Binomial_Standard(P, K, T, r, i, v1, N, Init, St) 'creates value
   ambiguity(nmbr) = option_price
   nmbr = nmbr + 1
   Init = Range("B11")
Next i

optlo = WorksheetFunction.Min(ambiguity)
opthi = WorksheetFunction.Max(ambiguity)

Range("B14") = alpha * optlo + (1 - alpha) * opthi 'weighted average
End Sub
```

Standard Binomial model

```
Public Sub Binomial_model()

Range("B12") = ""

P = Range("B1")
K = Range("B2")
T = Range("B3")
r = Range("B4")
mu1 = Range("B5")
Init = Range("B6")
v1 = Range("B7")
St = Range("B8")
N = Range("B9")


option_price = Binomial_Standard(P, K, T, r, mu1, v1, N, Init, St) ‘creates the necessary price
or timing
Range("B12") = option_price

End Sub
```

Quadrinomial model

```
Public Function Quadrinomial(P, K, T, r, mu1, mu2, v1, v2, N, Init, St)

 S0 = P
  I0 = K

   dt = T / N
  U1 = Exp(mu1 * dt + v1 * dt ^ 0.5)  'size of up jump for profit
  D1 = Exp(mu1 * dt - v1 * dt ^ 0.5) 'size of down jump for profit
  U2 = Exp(mu2 * dt - v2 * dt ^ 0.5) 'size of up jump for costs
  D2 = Exp(mu2 * dt + v2 * dt ^ 0.5) 'size of down jump for costs
  P1 = (Exp(r * dt) - D1) / (U1 - D1) 'risk free probability of up jump for profit
  P2 = 1 - P1  'risk free probability of down jump for profit
  S1 = (Exp(r * dt) - D2) / (U2 - D2) 'risk free probability of up jump for costs
  S2 = 1 - S1 'risk free probability of down jump for costs
  Nodes = (4 ^ N) ‘total nodes


  ReDim Smat(1 To Nodes + 1, 1 To N + 1) 'holds profits
  ReDim Imat(1 To Nodes + 1, 1 To N + 1) 'holds costs

  Smat(1, 1) = S0
  Imat(1, 1) = I0
  subnodes = ""

  For i = 1 To UBound(Smat, 2) - 1 ‘loops creates all necessary filling of tree for profits and
costs
    ex = 1
    ec = 1
    subnodes = 4 ^ (i - 1)
    For j = 1 To subnodes
        Smat(ex, i + 1) = Smat(j, i) * U1
        Smat(ex + 1, i + 1) = Smat(j, i) * D1
        Smat(ex + 2, i + 1) = Smat(j, i) * U1
        Smat(ex + 3, i + 1) = Smat(j, i) * D1

        ex = ex + 4

        Imat(ec, i + 1) = Imat(j, i) * U2
        Imat(ec + 1, i + 1) = Imat(j, i) * U2
        Imat(ec + 2, i + 1) = Imat(j, i) * D2
        Imat(ec + 3, i + 1) = Imat(j, i) * D2

        ec = ec + 4
    Next j
```

```
    subnodes = ""
  Next i

For x = 1 To UBound(Smat, 2) - 1 'for every period expected sales is plugged in
    subnodes = 4 ^ (x - 1)
    For y = 1 To subnodes
        Smat(y, x) = Smat(y, x) * Init
    Next y
    Init = Init - St
    subnodes = ""
Next x


ReDim Cmat(1 To Nodes + 1, 1 To N + 1)
subnodes = ""
tijd = 0

For h = N To 1 Step -1 'creates option value using cost and profit tree in recursive fashion
    ev = 1
    subnodes = 4 ^ (h - 1)
    For g = 1 To subnodes
        If h = N Then
            Cmat(g, h) = WorksheetFunction.Max(Smat(g, h) - Imat(g, h), 0)
            If Smat(g, h) - Imat(g, h) > 0 Then
                tijd = tijd + h
                nmbr = nmbr + 1
            End If
        Else
            op = Exp(-r * dt) * (P1 * S1 * Cmat(ev, h + 1) + P2 * S1 * Cmat(ev + 1, h + 1) + P1 * S2 *
Cmat(ev + 2, h + 1) + P2 * S2 * Cmat(ev + 3, h + 1))
            ev = ev + 4
            Cmat(g, h) = WorksheetFunction.Max(Smat(g, h) - Imat(g, h), op)
            If Smat(g, h) - Imat(g, h) > op Then
                tijd = tijd + h
                nmbr = nmbr + 1
            End If
        End If
    Next g
Next h

Quadrinomial = Cmat(1, 1)

End Function
```

Standard quadrinomial model

```
Sub Simple_Quadrinomial()

Range("B14") = ""

P = Range("B1")
K = Range("B2")
T = Range("B3")
r = Range("B4")
mu1 = Range("B5")
mu2 = Range("B6")
v1 = Range("B7")
v2 = Range("B8")
N = Range("B9")
Init = Range("B10")
St = Range("B11")

option_price = Quadrinomial(P, K, T, r, mu1, mu2, v1, v2, N, Init, St) ‘option price is created
Range("B14") = option_price

End Sub
```

α-maxmin and maxmin model

```
Sub alpha_Minmax_Quadrinomial()

Range("B18") = ""

P = Range("B1")
K = Range("B2")
T = Range("B3")
r = Range("B4")
mup = Range("B5")
muc = Range("B6")
vp = Range("B7")
vc = Range("B8")
N = Range("B9")
RAp = Range("b10")
SAp = Range("b11")
RAc = Range("b12")
SAc = Range("b13")
alpha = Range("B14")
Init = Range("B15")
St = Range("B16")

minmup = mup - RAp ‘range is established
maxmup = mup + RAp

minmuc = muc - RAc
maxmuc = muc + RAc

StepsP = (maxmup - minmup) / SAp + 1
StepsC = (maxmuc - minmuc) / SAc + 1
Total_Possibles = StepsP * StepsC
nmbr = 1

ReDim Ambiguity(1 To Total_Possibles) ‘possibilites are put into array and then the best or worst is found for value or timing.

For i = minmup To maxmup Step SAp
  For j = minmuc To maxmuc Step SAc
    option_price = Quadrinomial(P, K, T, r, i, j, vp, vc, N, Init, St)
    Ambiguity(nmbr) = option_price
    nmbr = nmbr + 1
    Init = Range("B15")
  Next j
Next i
```

```
alphaoption = WorksheetFunction.Min(Ambiguity) * (1 - alpha) +
WorksheetFunction.Max(Ambiguity) * alpha
Range("B18") = alphaoption

End Sub
```