Discrimination on the Work Floor, When Being the Best is no Guarantee for Promotion

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Abstract

This article develops a model of discrimination in organizations where a manager makes a decision to promote one of two employees. This model involves informational asymmetry, with the manager observing the employees abilities, but the employees themselves not. If employees expect discrimination to occur, the manager who anticipates on the effect of his promotion decision on the employees self esteem, rationally follows a strategy that involves discrimination as expected from him. When abilities are independently uniformly distributed on the unit interval, all stable equilibria involve discrimination. One of the employees is only promoted if he is twice as talented as the other. The remainder of the article analyses information systems where employees obtain private signals, employees who care about fairness of decisions and different shapes of the distributions. Discriminative equilibria are shown to exist under a large variety of settings.

Keywords: *Discrimination, Promotion, Intrinsic Motivation, Confidence Management*

1 Introduction

Major victories has been booked in fighting discrimination in organizations. Importantly, the extent of racial and gender disparity has been considerably reduced. Still, the presence of discrimination in organizations remains a major issue of concern. A world where managers treat all their employees systematically in an equal way remains far away. A main concern is that discrimination is highly persistence. Once a group becomes victim, inequalities tend to persist for long time and are difficult to reverse as history shows with many examples. This article provides a new explanation for the persistence of discrimination, in a context of a manger having to decide which employee to promote. Specifically, if discrimination against a group prevails for some time in society, employees judgements about their chance to receive the promotion, is influenced by the knowledge of the group to which they belong. If the manager promotes an employee belonging to a group which is known to be unfavoured, self esteem and motivation of employees belonging to the group, known to be favoured is considerably hurt. Although they are members of the favoured group, they still did not get the promotion. Apparently their ability and prospects are really poor. The manager anticipates and will have a tendency to promote an employee of the favoured group, even if other employees are slightly better. Therefore he is discriminating, even if he has no intrinsic aversion against the unfavoured group, but only aims in maximizing profits. In this way the continuation of discrimination results from self fulfilling expectations. There are several requirements for this mechanism to work. First, it is required that the manager has knowledge about employees abilities, otherwise his promotion decision contains no information. Second employees should have an imperfect self knowledge, otherwise there is no room for signalling. Lastly, a lower perception of ability should demotivate and as a result lead to a reduction of exerted effort. Intuitively, many real world environments fit these requirements. Literature to motivate this view will be discussed in the next section.

This article studies the described mechanism and its interaction with other factors that are present in real world environments. In section 3 the mechanism is analysed in isolation. We develop a model, where a manger supervises two employees and decides to promote one of them. Employees' abilities are not known to themselves, but are observed by the manager. Initially abilities are drawn independently from each other, both from a uniform distribution on the unit interval. It is shown that besides the fair equilibrium, two discriminating Bayesian Nash Equilibria are present, where one of the employees is only promoted if his ability is at least two times as large as the others. Moreover, only the discriminating equilibria are stable.

Hereafter, we study changes in the informational set-up in section 4. The aim is to embed the mechanism in environments which incorporate more realistic features. First, real world employees have, after years of live experience, at least some educated view about their abilities. We make use of private *personal signals* to model this issue and find that discrimination increases against the employee who receives more precise information about his ability. Intuitively, he is already more certain about his prospects and consequently information in the promotion decisions has less additional meaning to him. Therefore he can be exploited without severe consequences. Second, people have a tendency to occupy themselves with what others do. We study environments where employees have some knowledge about how their abilities compare relative to others. This is modelled by means of private *relative signals*, which tell an employee how his ability compares to the other, that is whether it is higher, lower, or equal to. It turns out that discrimination against an employee increases if he is better informed about how his prospects compare. Being better informed makes an employees' self perception less sensitive to the promotion decision.

An important factor that could counteract the severity of discrimination is the extent to which employees care about the fairness of decisions. This issue is analysed in section 5. Specifically, we assume that employees have an aversion against unfairness that is to their disadvantage. Employees observe relative signals, which are modelled as in section 4.2. If an employee learns that his ability is highest, while the other employee is promoted, he loses motivation due to being treated in an unfair way and consequently exerts less effort. The magnitude of this effect depends on his care about fairness, which enters his utility function in form of an exogenous parameter. It is shown that concerns about fairness counteract the effect of relative signals. An increase in the precision of the unfavoured employees' relative signal increases the extent of discrimination. However at the same time he is more likely to learn that he was treated in an unfair way. The manager anticipates and tends to discriminate less.

Section 6 analyses different shapes of the distribution of abilities and consists of two parts. First, distributions are kept equal between employees, and are analysed in a general way. Abilities are assumed to be continuous random variables. We consider distributions which have the unit interval as support. The main result is that discrimination is likely to be less of concern if most employees have moderate abilities and high abilities are less frequent although not negligible at the same time. The second part investigates differences in the distributions. Specifically we study the case where one of the groups has typically more favourable prospects and the case where in one of the groups abilities are more concentrated (less volatile) compared to the other group. It is found that discrimination increases against the employee with better ex ante prospects, and increases against the employee with lower ex ante uncertainty. Section 7 concludes.

2 Literature

For long time, typical economic models of incentives in organizations assumed perfect knowledge of employees about their types, be it the cost to achieve some desired end, or their ability to perform a task. The supervisor on contrary lacked knowledge about attributes. The issue of discrimination was also analysed in these type of frameworks and elicited some interesting insights (Arrow 1973, Coate and Loury, 1993). Arrow (1973) explains the persistence of discrimination as follows. First, the manager offers different wages if he expects that a different fractions of workers in the favoured and unfavoured group invest in skills. Second, the wage differences indeed induce different incentives to invest in skills for the two otherwise equal groups.

Self knowledge of employees and of individuals in general is questionable in many situations. For some attributes, like the colour of one's eye, self knowledge is almost evident, for others, like abilities and prospects that are involved in a complex job setting, it is less clear. Moreover, a supervisor, who has seen a lot of different employees in his career is probably in a better position to make assessments about employees abilities (Ishida, 2006). The possibility of imperfect self knowledge has been utilized increasingly by economists and has lead to a new variety of models and insights (Prendergast 1992, Ishida, 2006, Ishida, 2012, Bénabou and Tirole, 2003). Also this article belongs to this type of research, since it is the manager who knows abilities and it are the employees who have to infer them from his actions. The main idea is as follows. We are aware that others treat us according to the perceptions that they hold about us.

In this way, their actions partly reveal their intentions and views. As a result we learn about the self by witnessing the unfolding in our daily life. This tendency to increase our understanding of the self by inferring what others know about us from their actions, is what Cooley (1902) called the "looking glass self".

Individuals often enjoy performing a certain task, regardless of (monetary) compensation. The resulting motivation is called intrinsic and has been analysed extensively both by psychologists and economists (Ryan and Deci, 2000, Bénabou and Tirole, 2003). We make use of intrinsic motivation to account for complementarity of ability and effort. If ability and monetary pay-offs are positively related, complementarity seems naturally adequate. However, in many organizations, differences in abilities are, for employees performing similar tasks and with the same seniority, unlikely to be a strong predictor of wages. It is at this point where intrinsic motivation comes in as an alternative. The intensity of intrinsic motivation increases with perceived ability from an intuitive point of view. Indeed, if we belief that it would take an infinite time before we master a certain skill to a level that is barely satisfactory, would we spend less of our time and energy on its development? Ryan and Deci (2000) argue that a basic psychological need for competence can evoke intrinsic motivation in activities that reward with a feeling of competence. This feeling can results from anticipation on future achievements if one beliefs to be talented. The reason is that people often derive feelings from states of affairs that lie in the future by means of anticipation (Loewenstein, 1987). For example, when we first play on a new music instrument, the reward of being able to play and enjoy the music after days of learning, can be felt already in the present, although with less intensity.

The framework used in this article is related to Crutzen, Swank and Visser (Crutzen, Swank and Visser, 2007). The authors examine the role of comparative talk in motivating employees. Like in our model, there are two employees (juniors), who's abilities are independently drawn from the same distribution and only observed by the manager. In this setting, the manager can not tell credibly to an employee what his ability is, since he has an incentive to exaggerate (assuming that ability and effort are complementary). However, with two or more employees he can make comparative statements instead, which do not face this credibility problem (see also Prendergast and Topel 1993). Importantly, the manager can announce which employee's ability is higher, but he can also choose to refrain from any comparative massage. This leads to equilibria without discrimination in the sense that the manager reveals for a pair of abilities that the first employee is better if and only if he reveals that the second employee is better if abilities are interchanged. A different way to signal information credibly is by means of promoting one of the employees before the production period, which we analyse in this article. In our model, the manager has to promote exactly one of the employees, therefore he can not choose to refrain from any action. It is for this reason that discriminative equilibria arise, while they are not seen in Crutzen, Swank and Visser.

Lastly, the article also relates to the fairness literature. First, we assume that employees only care about unfairness if it is to their disadvantage. There are several ways to motivate this assumption. Intuitively, if unfairness is disadvantageous, powerful negatively loaded emotions like envy are more likely to be invoked. Loewenstein, Thompson and Bazerman (1989) estimate social utility functions in an experimental set-up. Two disputants order outcomes in various hypothetical situations. The authors find a strong an persistent aversion of disputants to disadvantageous inequity and a less clear, even ambiguous attitude to advantageous inequality. Although the role of unfairness was not analysed explicitly, the results suggest that unfairness leading to advantageous inequity, is likely to hurt one less than unfairness leading to disadvantageous inequity, since individuals are quite insensitive to the former outcome. Second, we assume that employees have less will to work if they feel treated in an unfair way. The crowd out of intrinsic motivation if decision processes are experienced as unfair is documented by experimental research (Zapata-Phelan, Colquitt, Brent, Livingston, 2009). The authors let participants solve word anagrams and the fairness of the grading was manipulated. Participants in the unfair condition devoted significantly less of their voluntary (ungraded) time to the puzzle.

Third, we argue that the setting of ultimatum games is surprisingly adequate to represent the environment of our model. An ultimatum game consists of two players, a proposer and a receiver. The proposer is given the task to divide a certain amount of money between him and the receiver. After he has done so, the receiver can either reject the offer, in which case both parties receive nothing, or accept it. The game has been repeated many times in laboratory settings (see Fehr, Smith, 2009 for an overview and discussion of the empiric results). Importantly, receivers reject offers that are unfair, and as a result proposers tend to divide the sum of money in a reasonably fair way. Both in our model as in the ultimatum there is a reason for the authority to be careful with unfair decisions. In the ultimatum game, there is a risk that the receiver rejects the offer. In our model, the disadvantaged employee loses motivation at the cost of profits. However, the authority has in both settings a strong incentive to behave in an unfair way if agents do not care about fairness. These similarities, together with the empiric findings on the ultimatum game, suggest the importance of taking into account fairness concerns in our model.

3 The Model

There are two employees and their abilities $\alpha_i, i = 1, 2$ are drawn independently from the uniform distribution on the unit interval. At the start of the game abilities are drawn by nature, after which they are observed by the manager. The employees do not observe their type, but are aware of the manager's informational superiority. The manager makes, after observing the abilities a promotion decision $m \in \{1, 2\}$. For example, if $m = 1$ the first employee is promoted. Let e_i be the effort level exerted by employee i. The production function is given by:

$$\pi = \alpha_1 e_1 + \alpha_2 e_2$$

Employees are risk neutral and choose effort to maximize expected utility. Effort and ability are assumed to be complementary. The utility function is, for $\mathscr{Y}$ being the set of information available to employee i specified as:

$$U(e_i) \quad = \quad \mathbb{E}[\alpha_i \mid \mathscr{Y}]e_i - \frac{1}{2}e_i^2$$

It follows that the optimal level of effort increases with believed ability and is given by:

$$e^* \quad = \quad \mathbb{E}[\alpha_i \mid \mathscr{Y}]$$

The strategy of the manager assigns a probability p to each pair of abilities, which is the probability that the second employee is promoted. The game is analysed by computing Perfect Bayesian Nash equilibria, where the manager has no incentives to change his strategy given the beliefs of the employees, which are true given the managers strategy, and are updated according to Bayes' rule. Babbling equilibria, where the strategy of the manager does not contain information about abilities are ignored. Moreover, te following definitions are used to analyse the stability of equilibria:

Definition 1 *Let x be a fixed point of the transformation $f : D \mapsto \mathbb{R}$ where $D \subseteq \mathbb{R}$, that is $f(x) = x$. Then:*

- *x is called globally stable if for every sequence $\{x_i\}_{i=1}^{\infty}$ in D, with $x_{i+1} = f(x_i)$ for $i > 1$ it holds that $\lim_{i\to\infty} x_i = x$.*
- *x is called stable or locally stable if there exists an open interval $O \subseteq D$, with $x \in O$, such that for every sequence $\{x_i\}_{i=1}^{\infty}$ with $x_1 \in O$ and $x_{i+1} = f(x_i)$ for $i > 1$ it holds that $\lim_{i\to\infty} x_i = x$.*
- *x is called unstable if it is not locally stable.*
- *The basin of attraction is defined as the set S such that the sequence $\{x_i\}_{i=1}^{\infty}$ with $x_{i+1} = f(x_i)$, $i > 1$ converges to x if and only if $x_1 \in S$.*

Intuitively, stability gives an indication, whether the system restores to an equilibrium as result of a small perturbation. An equilibrium where discrimination occurs is more of concern if it is stable. It suggests discrimination which is persistent. Besides establishing stability we will also investigate its meaning. Specifically, we will analyse the basin of attraction, which comprises all values y_1 around a point of equilibrium x such that the sequence that result by iteratively applying the best response operator on y_1 converges to x. A large basin of attraction means that the equilibrium is immune for even larger distortions. A small basin of attraction on the contrary suggests that although the equilibrium restores if perturbations all small, it likely collapses when the push is harder[1].

[1]Of course 'large' and 'small' have to be interpreted in an economic context, not just by looking purely at the numerical boundaries of the basin. For example a basin of attraction of $(1, 2)$ may be more meaningful than one of $(0, 4)$ depending on the interpretation of the unit of measurement in both cases.

We now compute the equilibria of the game. Suppose employees expect the manager to follow strategy f. Therefore, given a promotion decision $m = j$, their expected ability is given by $\mathbb{E}[\alpha_i \mid m = j, f]$. To simplify notation, we will implicitly assume the strategy f and suppress it in the notation. Therefore, expected abilities given promotion decisions will be written as $\mathbb{E}[\alpha_i \mid m = j]$. After he has observed abilities, the manager knows that profits given the promotion m are equal to:

$$\pi(m) \quad = \quad \alpha_1 \mathbb{E}[\alpha_1 \mid m] + \alpha_2 \mathbb{E}[\alpha_2 \mid m]$$

The second employee is promoted if $\pi(m = 2) > \pi(m = 1)$, which is the case if:

$$\alpha_2 \quad > \quad \frac{\mathbb{E}[\alpha_1 \mid m = 1] - \mathbb{E}[\alpha_1 \mid m = 2]}{\mathbb{E}[\alpha_2 \mid m = 2] - \mathbb{E}[\alpha_2 \mid m = 1]}\alpha_1$$

This condition leads to an important insight about the type of strategies that can occur in equilibrium. It follows that, for any combination of values that the beliefs can take, the best what the manager can do is to follow a promotion rule which is described by a linear function $\alpha_2 = c\alpha_1$, where the constant c depends on the beliefs of the employees. The second employee is promoted for pairs of abilities above the graph of this linear function, and the first employee for pairs of abilities below the graph. To find the equilibria, it suffices to search within these type of strategies, which we will denote by the slope c, as other strategies are never best response. Suppose that the employees expect the manager to play the strategy c. The best response $q(c)$ of the manager is given by:

$$q(c) \quad = \quad \frac{\mathbb{E}[\alpha_1 \mid m = 1] - \mathbb{E}[\alpha_1 \mid m = 2]}{\mathbb{E}[\alpha_2 \mid m = 2] - \mathbb{E}[\alpha_2 \mid m = 1]}$$

With the expression for $q(c)$ known for $0 < c \leq 1$, the best response function can be automatically extended to $c > 1$ by using a symmetry argument that we will apply in this article repeatedly. Note that the choice on which axis to place each employee was arbitrary. Suppose that we are interested in the best response of the manager to $c > 1$. We could than revise our decision of the coordinate system and place the second employee on the horizontal axis. Now, the second employee becomes the horizontal employee and the strategy c becomes $\tilde{c} = \frac{1}{c} < 1$ in the new coordinate system (the α_2, α_1 plane). The expression of $q(c)$ that is valid for $c < 1$ can be readily evaluated to determine the best response. After this, the resulting best response strategy is interpreted in the original coordinate system (the α_1, α_2 plane) to obtain the best response for c. Formally, the best response function satisfies the functional relation $q(c) = q(c^{-1})^{-1}$.

Equilibria correspond to fixed points of the transformation given by q, that is c is an equilibrium if $q(c) = c$. The immediate consequence of symmetry is that if a strategy c is an equilibrium, the strategy $\frac{1}{c}$ is also an equilibrium. A second consequence is that it suffices to search the interval $0 < c \leq 1$ for equilibria, as the other equilibria are easily constructed from these. Expected abilities of the employees given their belief about c and the promotion decision, are given by:

$$\mathbb{E}[\alpha_1 \mid m = 1] = \frac{2}{3} \qquad \mathbb{E}[\alpha_1 \mid m = 2] = \frac{1 - \frac{2}{3}c}{2 - c}$$

$$\mathbb{E}[\alpha_2 \mid m = 1] = \frac{1}{3}c \qquad \mathbb{E}[\alpha_2 \mid m = 2] = \frac{1 - \frac{1}{3}c^2}{2 - c}$$

For example, the first statement is derived by conditioning that α_1 is contained in the region bounded by the lines $\alpha_2 = 0$, $\alpha_1 = 1$ and $\alpha_2 = c\alpha_1$. Indeed, this gives:

$$\mathbb{E}[\alpha_1 \mid m = 1] \quad = \quad \frac{\int_0^1 \int_0^{ct} tdsdt}{\int_0^1 \int_0^{ct} dsdt} \quad = \quad \frac{2}{3}$$

Substituting these expressions in the best response function gives:

$$q(c) \quad = \quad \frac{1}{3 - 2c}$$

The equilibria on the interval $0 < c \leq 1$ are $c = 0.5$ and $c = 1$. By symmetry $c = 2$ is also an equilibrium. Therefore we find that there exists a fair equilibrium $c = 1$ where the best employee is promoted, which is an intuitive result. What is more surprising, is that besides this equilibrium, there are two other ones which involve discrimination in the sense that not always the best employee is promoted. For each of these two other equilibria one of the employees is only promoted if he is at least twice as able as the other, while the other is in many cases promoted even when he in fact has a lower ability. Moreover, only the discriminative equilibria are stable. This result is proved in Appendix B. Figure 1 shows the graph of the best response function $q(c)$, together with the 45 degrees line on the interval $(0, 2.5]$. The points of intersection correspond to the equilibria $c = 0.5$ and $c = 1$ and $c = 2$. Theorem 1 summarizes the findings of this section.

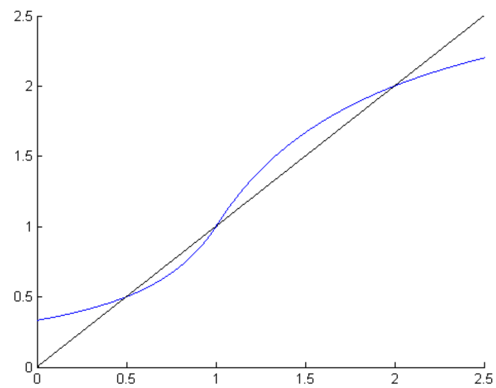


Figure 1: $q(c)$ and 45 degrees line on interval $0 < c \leq 2.5$

Theorem 1 *Suppose employees abilities are uniformly distributed on the unit interval and are independent. The set of equilibria c is given by $\{\frac{1}{2}, 1, 2\}$. Only $c = 1$ and $c = 2$ are stable. The equilibrium $c = 1$ has basin of attraction $(0, 1)$, and $c = 2$ has basin of attraction $(1, \infty)$.*

4 Information Systems

In this section we extend the previous analysis to different informational settings. First we allow employees to have a certain amount of self knowledge. Second, we study the case where employees have knowledge about how their abilities compare to others. Lastly, we study environments where managers have imperfect assessments of employees abilities.

4.1 personal signals

We assume that employees receive a fully accurate private signal with probability ρ_i about their ability. The manager can not observe or infer whether an employee has received the signal. However the probabilities ρ_i are common knowledge. Expected profit as a function of the promotion decision is, after the manager observes abilities, given by:

$$\pi(m) = \sum_{i=1}^{2} \rho_i \alpha_i^2 + \sum_{i=1}^{2} (1 - \rho_i) \mathbb{E}[\alpha_i \mid m] \alpha_i$$

The second employee is promoted if $\pi(m = 2) > \pi(m = 1)$. This is the case when:

$$\alpha_2 \geq \frac{1 - \rho_1}{1 - \rho_2} \frac{\mathbb{E}[\alpha_1 \mid m = 1] - \mathbb{E}[\alpha_1 \mid m = 2]}{\mathbb{E}[\alpha_2 \mid m = 2] - \mathbb{E}[\alpha_2 \mid m = 1]} \alpha_1$$

The ratio of $1-\rho_1$ to $1-\rho_2$ is new compared to the expression that was found when personal signals were absent. The difference results from the fact that the manager now faces uncertainly about the effort of employees and needs to condition on whether they obtained a personal signal. To find the equilibria it again suffices to analyse strategies of the form $\alpha_2 = c\alpha_1$. Moreover we have derived the expressions for $\mathbb{E}[\alpha_i \mid m = j]$ earlier. Given that employees expect the manager to play a strategy $0 < c \leq 1$ the best response is:

$$q(c;\rho_1,\rho_2) \quad = \quad \frac{1-\rho_1}{1-\rho_2}\frac{1}{3-2c}$$

The candidate equilibria are:

$$\begin{aligned} c(\rho_1,\rho_2)_1 &= \frac{3}{4}+\frac{1}{2}\sqrt{2}\sqrt{\frac{9}{8}-\frac{1-\rho_1}{1-\rho_2}} \\ c(\rho_1,\rho_2)_2 &= \frac{3}{4}-\frac{1}{2}\sqrt{2}\sqrt{\frac{9}{8}-\frac{1-\rho_1}{1-\rho_2}} \end{aligned}$$

From symmetry it follows that $c(\rho_1,\rho_2)_{i+2} = c(\rho_2,\rho_1)_i^{-1}$ are also candidate equilibria. To be an equilibrium the first two candidates should satisfy the restriction $0 < c \leq 1$ and the latter two the restriction $c \geq 1$. Theorem 2, which is proved in Appendix C presents existence results.

Theorem 2 *Suppose that employee i receives a personal signal with probability ρ_i. The following equilibrium results can be distinguished:*

- $c(\rho_1,\rho_2)_1$ *is an equilibrium if* $1 \leq \frac{1-\rho_1}{1-\rho_2} \leq \frac{9}{8}$
- $c(\rho_1,\rho_2)_2$ *is an equilibrium if* $0 < \frac{1-\rho_1}{1-\rho_2} \leq \frac{9}{8}$
- $c(\rho_1,\rho_2)_3$ *is an equilibrium if* $\frac{8}{9} \leq \frac{1-\rho_1}{1-\rho_2} \leq 1$
- $c(\rho_1,\rho_2)_4$ *is an equilibrium if* $\frac{8}{9} \leq \frac{1-\rho_1}{1-\rho_2}$

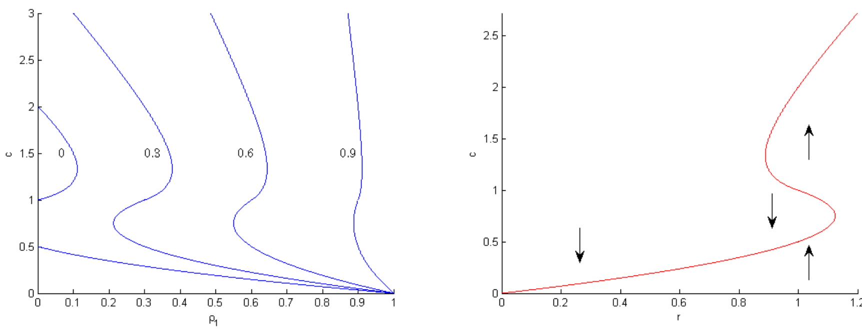
It follows that the number of distinct equilibria is at most three. Theorem 3 presents the stability properties, and is proved in Appendix D. Here m_i is defined as the closest equilibrium smaller than $c(\rho_1,\rho_2)_i$ if this exists, and $m_i = 0$ else, and p_i is defined as the closest equilibrium larger than $c(\rho_1,\rho_2)_i$ if this exists, and $p_i = \infty$ else.

Theorem 3 *Suppose employee i receives a personal signal with probability ρ_i. The stability properties of the equilibria are as follows:*

- *If* $c(\rho_1,\rho_2)_1$ *is an equilibrium, then it is unstable*
- *If* $c(\rho_1,\rho_2)_2$ *is and equilibrium the it is stable if:* $\frac{1-\rho_1}{1-\rho_2} < \frac{9}{8}$
- *If* $c(\rho_1,\rho_2)_3$ *is an equilibrium, then it is unstable.*

- *If $c(\rho_1,\rho_2)_4$ is an equilibrium, then it is stable if:* $\frac{8}{9} < \frac{1-\rho_1}{1-\rho_2}$
- *$c(\rho_1,\rho_2)_2$ has basin of attraction $(0, d_2)$ and $c(\rho_1,\rho_2)_4$ has basin attraction (m_4, ∞).*

Panel A in Figure 2 displays the equilibria as a correspondence (set values function) of ρ_1 for selected values of ρ_2, specifically $\rho_2 \in \{0, 0.3, 0.6, 0.9\}$. Graphs lying more to the right correspond to higher values of ρ_2.



Panel A Panel B

Figure 2: Equilibria personal signals

A perhaps even clearer graphical summary of the equilibrium results utilizes the fact that the equilibria only depend on the ratio:

$$r(\rho_1,\rho_2) \quad = \quad \frac{1-\rho_1}{1-\rho_2}$$

Panel B in Figure 2 shows the equilibria as a correspondence of $r = r(\rho_1,\rho_2)$. Note that r increases if either ρ_1 decreases or ρ_2 increases. The sign of $q(c) - c$ is the same for all points that can be connected without crossing or touching the graph of the correspondence. There are two different regions of such points. In the figure each of these regions is accompanied with arrows, pointing downward if $q(c) < c$ and upwards if $q(c) > c$. An equilibrium c is stable if at both sides the arrows point towards c and unstable else.

Several observations can be made. If we keep ρ_2 constant and increase ρ_1, which means that we increase the relative strength of the first employees' signal, $c(\rho_1,\rho_2)_2$ decreases. This result can be seen as surprising on the first sight as it suggests that, by means of the mechanism that is discussed in this article, discrimination *against* employees who are more aware of their ability is stronger. The intuition is that if the first employee already has a strong signal, any information contained in the promotion decision is of minor value to him. In the extreme case that the signal is perfect, the promotion decision can not change the belief of this employee, independent of its content. Moreover we see that the extent of discrimination against the first employee decreases if the other employee receives relatively stronger signals. The mechanism is the mirror image of the one discussed.

4.2 relative signals

In competitive environments, where employees care relatively more about how they compare to others, the affect of social comparisons on self perception will likely interact with instruments like discrimination which convey information to employees about their ability trough the "looking glass self". In which way could these interactions occur? Take as an example the scenario, where a majority of students does not pass a course. Many of us are familiar with the tendency of students to justify their results, even when insufficient, as being good, because they compare in a positive way to others results. They tend to ignore the informational value contained in their grades which is, when the course is set up in a way that is though out well, likely to be negative. We consider an extension of the original model where employees receive relative private signals. Specifically, each employee receives a signal with some probability ϕ_i which tells with full precision whether his ability is higher, lower or equal to the ability of the other employee. The parameter ϕ_i can both be interpreted as reflecting the characteristics of the environment that catalyse comparative behaviour between employees, or as reflecting characteristics of the individual, that is how sensitive he is to information that tells how his talents compare relatively to others.

First we analyse the conditions under which the fair equilibrium $c = 1$ exists. Suppose employees expect the manager to follow $c = 1$. Then $c = 1$ is an equilibrium if the best response of the manager is to promote the employee with the highest ability. If $\alpha_1 > \alpha_2$, then irrespective of relative signals, promoting the first employee results in perceives abilities equal to $\frac{2}{3}$ and $\frac{1}{3}$. Promoting the second employee makes employees think that $\alpha_2 > \alpha_1$ if they do not observe a relative signal while a relative signal confirms to an employee that the promotion decision is inconsistent with his belief, since he learns that not the best employee is promoted. In this latter case the relative signal is the only source of information to condition on. Therefore promoting the best employee is optimal if:

$$\begin{aligned} \frac{2}{3} + \frac{1}{3} &\geq \phi_1 \frac{2}{3} + (1 - \phi_1)\frac{1}{3} + \phi_2 \frac{1}{3} + (1 - \phi_2)\frac{2}{3} \\ \phi_1 - \phi_2 &\leq 0 \end{aligned}$$

Similarly, if $\alpha_2 > \alpha_1$, promoting the best employee is optimal if $\phi_1 - \phi_2 \geq 0$. Therefore a fair equilibrium only exists is $\phi_1 = \phi_2$. However, this equilibrium is unstable (see Appendix F). We now analyse the other equilibria. Since the set where $\alpha_1 = \alpha_2$ is of zero probability, we consider the promotion decision for $\alpha_1 \neq \alpha_2$. Let I_1 be the indicator function on the set $\{\alpha_1 > \alpha_2\}$, and I_2 the indicator function on the set $\{\alpha_2 > \alpha_1\}$. Expected profit as function of the promotion decision is, after observing abilities, given by:

$$\pi(m) = \sum_{i=1}^{2} \phi_i \sum_{l=1}^{2} \sum_{k \neq l} I_l \mathbb{E}[\alpha_i \mid m, \alpha_l > \alpha_k]\alpha_i + \sum_{i=1}^{2} (1-\phi_i)\mathbb{E}[\alpha_i \mid m]\alpha_i$$

For notational purposes, define the operator Δ_{ij}, $i, j \in \{1, 2\}$ that acts on a function $h(m)$ by:

$$\Delta_{ij} h(m) = h(m = i) - h(m = j)$$

The second employee is promoted if $\pi(m = 2) \geq \pi(m = 1)$. This occurs if:

$$\alpha_2 \geq \frac{\Delta_{12}\phi_1 \mathbb{E}[\alpha_1 \mid m, \alpha_1 > \alpha_2] + \Delta_{12}(1-\phi_1)\mathbb{E}[\alpha_1 \mid m]}{\Delta_{21}\phi_2 \mathbb{E}[\alpha_2 \mid m, \alpha_1 > \alpha_2] + \Delta_{21}(1-\phi_2)\mathbb{E}[\alpha_2 \mid m]} I_1 \alpha_1 + \frac{\Delta_{12}\phi_1 \mathbb{E}[\alpha_1 \mid m, \alpha_2 > \alpha_1] + \Delta_{12}(1-\phi_1)\mathbb{E}[\alpha_1 \mid m]}{\Delta_{21}\phi_2 \mathbb{E}[\alpha_2 \mid m, \alpha_2 > \alpha_1] + \Delta_{21}(1-\phi_2)\mathbb{E}[\alpha_2 \mid m]} I_2 \alpha_1$$

For each of the regions $A_1 = \{\alpha_1 > \alpha_2\}$ and $A_2 = \{\alpha_1 < \alpha_2\}$ a linear function $\alpha_2 = c_i \alpha_1$ is specified. This strategy has the following interpretation. First, the manager evaluates to which region the pair (α_1, α_2) belongs, then promotes the second employee if his ability exceeds the required threshold corresponding to the region. This opens the possibility that the second employee is promoted based on A_1, but would not be promoted if his ability were somewhat higher, based on the regime applied in A_2. At least from an intuitive point of view, these type of situations seem unreasonable, therefore we impose some restrictions. Specifically, we assume that employees never expect these type of strategies to occur, and that the manager indeed never plays them[2]. This means that we restrict us to strategies were $c_1 > 1$ if and only if $c_2 > 1$. This implies that the regime in precisely one of the sets A_1 and A_2 is trivial. For example, if sometimes the second employee is promoted in A_1 then he is always promoted in A_2. As a result we can characterize relevant strategies with a single value c as before, with the interpretation that the second employee is promoted if $\alpha_2 > c\alpha_1$. Suppose that a discriminating equilibrium exists where $0 < c < 1$. Then, the second employee is promoted if:

$$\alpha_2 \geq \frac{\Delta_{12}\phi_1 \mathbb{E}[\alpha_1 \mid m, \alpha_1 > \alpha_2] + \Delta_{12}(1-\phi_1)\mathbb{E}[\alpha_1 \mid m]}{\Delta_{21}\phi_2 \mathbb{E}[\alpha_2 \mid m, \alpha_1 > \alpha_2] + \Delta_{21}(1-\phi_2)\mathbb{E}[\alpha_2 \mid m]} \alpha_1$$

The expectations $\mathbb{E}[\alpha_i \mid m]$ has been computed in the previous section. The expectations conditional on the relative events are new. These are equal to:

$$\mathbb{E}[\alpha_1 \mid m = 1, \alpha_1 > \alpha_2] = \frac{2}{3} \qquad \mathbb{E}[\alpha_1 \mid m = 2, \alpha_1 > \alpha_2] = \frac{2}{3}$$

$$\mathbb{E}[\alpha_2 \mid m = 1, \alpha_1 > \alpha_2] = \frac{1}{3}c \qquad \mathbb{E}[\alpha_2 \mid m = 2, \alpha_1 > \alpha_2] = \frac{1}{3}(1+c)$$

[2] A more precise analysis is likely to yield the same results in the sense that in equilibrium, in one of the two subsets a single employee is always promoted.

The first employee apparently sees through "the veil" of promotion decisions, and is not affected by them if he observes the relative signal. It does not matter which promotion decision is taken, his conditional beliefs are as if he only conditioned on the information that his ability is higher. This does not hold for the second employee. If he would only condition on his ability being lower, his estimate would be $\frac{1}{3}$. However, if he is promoted, his expected ability becomes $\frac{1}{3}(1+c)$, which means that his self perception is to some extent restored. The best response function for $c < 1$ after some algebraic manipulations becomes:

$$q(c;\phi_1,\phi_2) = \frac{1-\phi_1}{(3-\phi_2)-(2-\phi_2)c}$$

Solving $q(c) = c$ by using the expression for $q(c)$ that is valid for $c < 1$ gives the following candidate equilibria:

$$c(\phi_1,\phi_2)_1 = \frac{1}{2}\left(\frac{3-\phi_2}{2-\phi_2}\right) + \sqrt{\frac{1}{4}\left(\frac{3-\phi_2}{2-\phi_2}\right)^2 - \frac{1-\phi_1}{2-\phi_2}}$$

$$c(\phi_1,\phi_2)_2 = \frac{1}{2}\left(\frac{3-\phi_2}{2-\phi_2}\right) - \sqrt{\frac{1}{4}\left(\frac{3-\phi_2}{2-\phi_2}\right)^2 - \frac{1-\phi_1}{2-\phi_2}}$$

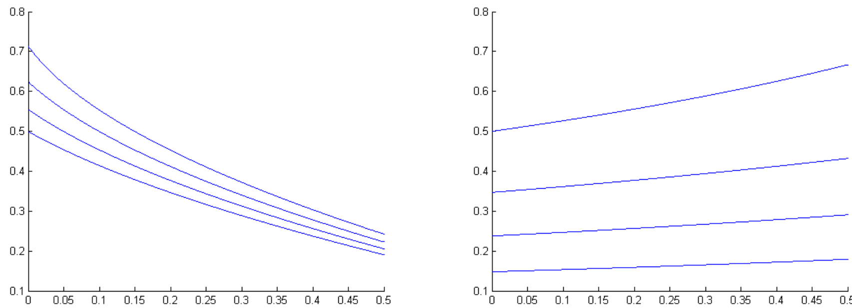
By symmetry it follows that $c(\phi_1,\phi_2)_{i+2} = c(\phi_2,\phi_1)_i^{-1}$ are also candidate equilibria. To be an equilibrium the first two should satisfy the domain restriction $0 < c < 1$ and the latter two the restriction $c > 1$. Existence results are derived in Appendix E. It turns out that the domain restrictions are not satisfied by $c(\phi_1,\phi_2)_1$ and $c(\phi_1,\phi_2)_3$. The remaining candidates are equilibria if[3] if $0 < \phi_i < 1$. Moreover, the discriminating equilibria are stable (Appendix F) Figure 3 displays how the discriminating equilibrium against the first employee varies with the information parameters ϕ_1, ϕ_2. More specifically the graphs of the ϕ_i sections[4] are displayed for selected values of the other parameter $\phi_j \in \{0, 0.2, 0.4, 0.6\}$. To which value of ϕ_2 the different graphs belong, can be seen by using (Appendix G):

$$\frac{\partial c(\phi_1\phi_2)_2}{\partial \phi_1} < 0 \quad \frac{\partial c(\phi_1\phi_2)_2}{\partial \phi_2} > 0 \quad \frac{\partial c(\phi_1\phi_2)_4}{\partial \phi_1} < 0 \quad \frac{\partial c(\phi_1\phi_2)_4}{\partial \phi_2} > 0$$

Discrimination against the unfavoured employee increases with the precision of his relative signal and decreases with the precision of the favoured employee's signal in each of the discriminating equilibria.

[3] $0 < \phi_i < 1$ is sufficient for existence. It holds that $c(\phi_1,\phi_2)_2$ exists if and only if $\phi_1 < 1$ and in case $\phi_1 = 0$, $\phi_2 < 1$. Existence for $c(\phi_1,\phi_2)_4$ follows from symmetry.

[4] The y-section $f_y(x)$ of a function $f(x,y)$, is defined as $f_y(x) = f(x,y)$. The x-section is defined analogously.



Equilibrium $c(\phi_1, \phi_2)_2$,
ϕ_1 sections for selected values of ϕ_2

Equilibrium $c(\phi_1, \phi_2)_2$,
ϕ_2 sections for selected values of ϕ_1

Figure 3: Equilibria relative signals

The results have the following interpretation. First, if the unfavoured employee is promoted, his perceived ability is $\frac{2}{3}$, whether he observed a relative signal or not. However, if he is not promoted his self esteem is hurt less if he does observe a relative signal. Therefore if he is more likely to observe a relative signal, discrimination is indeed expected to be larger. A similar story holds for the favoured employee. If he is not promoted his belief is $\frac{1}{3}$, whether he receives a relative signal or not. If he is promoted his self esteem is lower if he observes the relative signal and as a result, the benefits of discrimination are less. Therefore discrimination is expected to be lower if the favoured employee receives more precise signals.

Theorem 4 *Suppose employee i receives a relative signal with probability ϕ_i which tell with full precision whether $\alpha_1 > \alpha_2$, $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$. With probability $1 - \phi_i$ the signal is not received. Then the following results hold:*

- *The fair equilibrium $c = 1$ exists if $\phi_1 = \phi_2$, but is unstable.*
- *Two stable discriminating equilibria exists if $0 < \phi_i < 1$, one in favour of each employee. The extent of discrimination against an employee increases with the precision of his signal and decreases with the precision of the other employees signal.*

5 Concerns about Fairness

In this section we analyse fairness concerns of employees. Specifically we assume for simplicity that employees only care about unfairness which is to their disadvantage. There are two natural ways to extend the model. One way to extend the model is by allowing the employee who is not promoted to consider the probability that his ability was best, and the expected ability difference given this event. A different approach is to work with the framework for relative signals. If the best employee is not promoted he learns that he was treated in an unfair way if he observes a relative signal. Both approaches yield qualitatively similar results, namely that discrimination does not disappear completely.

We now describe the model. First, expected utility of employee $i = 1, 2$ is, for I_i as defined before, $I(A)$ being the indicator function on a set A, $\mathscr{Y}$ the set of information available to employee i and ξ_i being a fairness parameter, equal to:

$$U_i = \mathbb{E}[\alpha_i \mid \mathscr{Y}]e_i - \xi_i I_i \sum_{j \neq i} I(\{m = j\})\mathbb{E}[\alpha_i - \alpha_j \mid m = j, \alpha_i > \alpha_j]e_i - \frac{1}{2}e_i^2$$

The utility function implies that employees do not like violations from fairness if it is against them, and that their dislike increases if the expectation of the difference between their and the other employees ability is higher. The optimal choice of effort level is:

$$e_i = \mathbb{E}[\alpha_i \mid \mathscr{Y}] - \xi_i I_i \sum_{j \neq i} I(\{m = j\})\mathbb{E}[\alpha_i - \alpha_j \mid m = j, \alpha_i > \alpha_j]$$

We assume that $0 \leq \xi_i \leq 1$. The lower bound implies that disadvantageous unfairness never makes an employee better of. The upper bound ensures that the selected effort choice is non-negative. However if $\xi_i > 1$, effort choice of employee i can still be positive. It should be kept in mind that the upper bound is somewhat arbitrary. A value of ξ_i larger than unity would imply that individual i is hurt to such an extent by an unfair decision, that he would be happier if his ability were lower. His averseness to unfairness is stronger than his desire to be talented. Intuitively such an averseness is quite extreme and $\xi_i \leq 1$ seems from this viewpoint adequate.

The analysis of the managers best response strategies is similar to section 4.2, with the slight difference that in a discriminating equilibrium, not promoting the best employee reduces his effort and consequently profit if he observes a relative signal. As before, the managers strategies that are best response to some beliefs are characterized by a single number c, such that the second employee is promoted if $\alpha_2 > c\alpha_1$. The best response function, given that employees expect the manager to follow the strategy $c < 1$, is given by:

$$q(c; \phi_1, \phi_2, \xi_1, \xi_2) = \frac{\phi_1\xi_1(1-c)(2-c) + (1-\phi_1)}{(3-\phi_2) - (2-\phi_2)c}$$

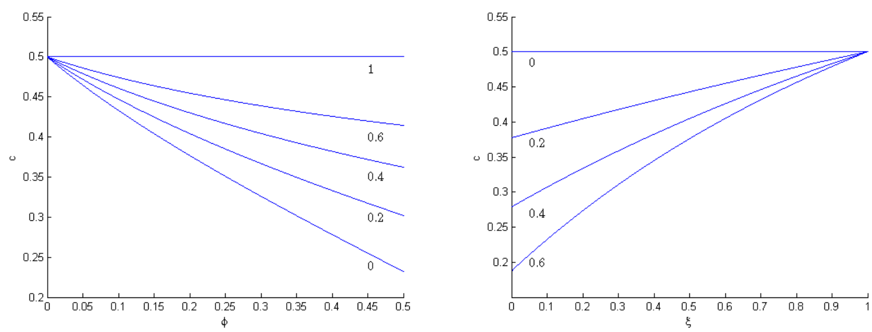
We now analyse the discriminative equilibria. The candidates are given by:

$$c(\phi_1\phi_2, \xi_1, \xi_2)_1 = \frac{1}{2}\frac{(3-\phi_2) + 3\phi_1\xi_1}{(2-\phi_2) + \phi_1\xi_1} + \sqrt{\frac{1}{4}\left(\frac{(3-\phi_2) + 3\phi_1\xi_1}{(2-\phi_2) + \phi_1\xi_1}\right)^2 - \frac{(1-\phi_1) + 2\phi_1\xi_1}{(2-\phi_2) + \phi_1\xi_1}}$$

$$c(\phi_1\phi_2, \xi_1, \xi_2)_2 = \frac{1}{2}\frac{(3-\phi_2) + 3\phi_1\xi_1}{(2-\phi_2) + \phi_1\xi_1} - \sqrt{\frac{1}{4}\left(\frac{(3-\phi_2) + 3\phi_1\xi_1}{(2-\phi_2) + \phi_1\xi_1}\right)^2 - \frac{(1-\phi_1) + 2\phi_1\xi_1}{(2-\phi_2) + \phi_1\xi_1}}$$

By symmetry, $c(\phi_1, \phi_2, \xi_1, \xi_2)_{2+i} = c(\phi_2, \phi_1, \xi_2, \xi_1)_i^{-1}$ are also candidate equilibria. We assume for simplicity that $\phi_i > 0$, $i = 1, 2$. In Appendix H it is shown that the domain restrictions are only satisfied by the candidates $c(\phi_1\phi_2, \xi_1, \xi_2)_2$ and $c(\phi_1\phi_2, \xi_1, \xi_2)_4$. The discriminating equilibria do not depend on the parameter ξ_i of the favoured employee. This is no surprise since we have only modelled disadvantageous unfairness.

Next we analyse the case where the abilities of the employees to observe relative signals are equal, that is $\phi_1 = \phi_2$. We define ϕ as the common value of ϕ_1 and ϕ_2 and $c(\phi, \xi_i)_i$ as the discriminating equilibrium against employee i. By symmetry it suffices to analyse $c(\phi, \xi_1)_1$. In Appendix J it is shown that this equilibrium is stable with basin of attraction $(0, 1)$ if $\phi < 1$. Figure 4 displays the ϕ sections of $c(\phi, \xi_1)_1$ for $\xi \in \{0, 0.2, 0.4, 0.6, 1\}$ and ξ sections for $\phi \in \{0, 0.2, 0.4, 0.6\}$.



ϕ sections for selected values of ξ $\qquad$ ξ sections for selected values of ϕ

Figure 4: Equilibria with concerns about fairness

The following conclusions can be drawn. In section 4.2 it was shown that discrimination against the unfavoured employee increases if his relative signal is more precise. If he cares about fairness, this effect is counteracted. The strength of this force in the opposite direction also depends positively on the precision his signal. The intuition is that the signals are the only source of information that tell whether an unfair decision has been made against them. Stronger relative signals result in a tendency towards less discrimination due to fairness concerns, but also in a tendency towards more discrimination due to signal precision. As a result, discrimination does not disappear altogether. As ξ_1 tend to one, the unfavoured employee attaches more and more importance to fairness and the discriminating equilibrium converges point-wise to $\frac{1}{2}$.

6 Distributions

In the preceding analysis, abilities were assumed to be distributed uniformly on the unit interval. It is natural to ask to which extent discriminative outcomes would arise when the distributions take various shapes. Our aim is to draw qualitative insights in the equilibrium results that arise with different features of the underlying distributions. The analysis is carried out in two steps. First equal distributions of a general type are studied. It is shown that the fair equilibrium always exits. A criterion is derived for the existence of at least one other discriminating equilibrium. Moreover, if this criterion applies, the non-discriminating equilibrium is not stable. The second part of the analysis considers the case where the distributions are different. Insights are derived from studying the family of uniform distributions with arbitrary rectangular supports.

6.1 General Distributions

Abilities now are drawn independently from each other from a common general distribution with the unit interval as support, with cumulative density function G and probability density function g. Besides this change, the set-up of the original model is taken: the manager observes abilities perfectly, and the promotion decision is the only source of information to the employees. The manager promotes the second employee if:

$$\alpha_2 \quad > \quad \frac{\mathbb{E}[\alpha_1 \mid m=1] - \mathbb{E}[\alpha_1 \mid m=2]}{\mathbb{E}[\alpha_2 \mid m=2] - \mathbb{E}[\alpha_2 \mid m=1]}\alpha_1$$

The beliefs are computed as before. For example $\mathbb{E}[\alpha_1 \mid m=1]$ is calculated by conditioning on the fact that α_1 is contained in the region bounded by the lines $\alpha_2 = 0$, $\alpha_1 = 1$ and $\alpha_2 = c\alpha_1$, which gives:

$$\mathbb{E}[\alpha_1 \mid m=1] \quad = \quad \frac{\int_0^1 \int_0^{c\alpha_1} \alpha_1 g(\alpha_1) d\alpha_2 d\alpha_1}{\int_0^1 \int_0^{c\alpha_1} g(\alpha_1) d\alpha_2 d\alpha_1} \quad = \quad \frac{\int_0^1 \alpha_1 G(c\alpha_1) g(\alpha_1) d\alpha_1}{\int_0^1 G(c\alpha_1) g(\alpha_1) d\alpha_1}$$

After some algebraic manipulations we obtain the following expression for the best response function:

$$q(c) \quad = \quad \frac{\int_0^1 \alpha_1 G(c\alpha_1) g(\alpha_1) d\alpha_1 - \mathbb{E}[\alpha_1] \int_0^1 G(c\alpha_1) g(\alpha_1) d\alpha_1}{\mathbb{E}[\alpha_2] \int_0^1 G(c\alpha_1) g(\alpha_1) d\alpha_1 - \int_0^c \alpha_2 (1 - G(c^{-1}\alpha_2)) g(\alpha_2) d\alpha_2}$$

We now show the existence of the equilibrium $c = 1$. Let $\alpha_{1:2}$ be the minimum and $\alpha_{2:2}$ the maximum of the abilities. The density function of the minimum is given by $g_{1:2}(y) = 2g(y)(1 - G(y))$ and the density of the maximum by $g_{2:2}(y) = 2G(y)g(y)$. It follows that $c = 1$ is an equilibrium if and only if:

$$1 \quad = \quad -\frac{\mathbb{E}[\alpha_1] - \mathbb{E}[\alpha_{2:2}]}{\mathbb{E}[\alpha_2] - \mathbb{E}[\alpha_{1:2}]}$$

In fact this condition always holds. To see note that $\alpha_1 + \alpha_2 = \alpha_{1:2} + \alpha_{2:2}$. Taking expectations and rewriting afterwards gives the equilibrium condition.

Therefore we conclude that $c = 1$ is an equilibrium. We now make use of this result to derive a condition for the existence of discriminating equilibria. In Appendix K we establish the following property, assuming that $g(0) < \infty$:

$$\lim_{c \downarrow 0} q(c) \quad = \quad \frac{\text{Var}[\alpha_1]}{\mathbb{E}[\alpha_1]}$$

This limit is strictly larger than zero. We have just established the property $q(1) = 1$. If the derivative of q, evaluated at $c = 1$ is more than unity, the graph of $q(c)$ lies below the 45 degrees line at the left of the point of intersection $c = 1$, for all c in some non-empty interval. Together with the fact that initially $q(c) > c$ and the continuity of q, this implies the existence of another point of intersection for a certain $c < 1$, which is a discriminating equilibrium.

There are several points of attention, which are illustrated by the hypothetical example in Figure 5. First, the criterion of the derivative evaluated at $c = 1$ larger than unity, is sufficient but need not to be necessary. Second, nothing is said about uniqueness, only existence. The Figure, which is not based on any actual distribution analysed, shows a case where the derivative at $c = 1$ is less than unity. Nonetheless two discriminative equilibria exist. The 'derivative test' becomes necessary and uniqueness is ensured in case the best response function is convex. Theorem 5 summarizes the discussion and shows the criterion that results from the derivative at $c = 1$ being strictly larger than unity (see Appendix L for a formal proof of the last two statements).

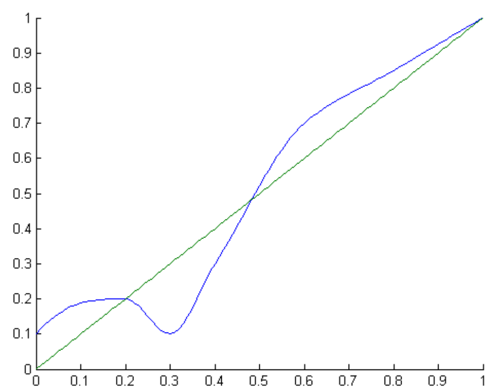


Figure 5: Example

Theorem 5 *Suppose that abilities of the employees are distributed identically, and are drawn from a distribution on the unit interval with probability density function g, such that $g(0) < \infty$. Then:*

- *A sufficient condition for a discriminating equilibrium to exists is:*

$$\mathbb{E}[\alpha_{1:2}] + 4\int_0^1 \alpha_1^2 g(\alpha_1)^2 d\alpha_1 \quad > \quad \left(1 + 4\int_0^1 \alpha_1 g(\alpha_1)^2 d\alpha_1\right)\mathbb{E}[\alpha_1]$$

- *If the best response function of the manager is strictly convex on (0,1], the criterion is also necessary and the equilibrium with $c < 1$ is unique.*

- *If the best response function is strictly convex and increasing on (0,1], the unique equilibrium with $c < 1$ is stable and the fair equilibrium is unstable.*

There are some points of attention. First, a quick investigation by the authors did not find a counterexample of a distribution where the best response function is not convex. Clarifying the conditions for convexity is an interesting point for further research. Second, best response functions need not to be always increasing. However, the equilibrium could be still (and in fact is likely to be) stable, even if the best response function is not increasing. The requirement for local stability is that at the point of equilibrium the tangent to the best response function is between -1 and 1.

In finding distributions without discriminating equilibria, a reasonable starting point is to consider distributions which do not satisfy the property stated in Theorem 5. First note that $\mathbb{E}[\alpha_{1:2}]$ is always smaller than $\mathbb{E}[\alpha_2]$. Therefore, the criterion will not be satisfied if the integral on the left hand side is sufficiently low. Given the convexity of the quadratic function $y = \alpha_1^2$, this is accomplished by concentrating the probability mass of the distribution more towards the left end of the unit interval. However care should be given in this process, and taking a distribution that is almost degenerate will not work, because the difference between $\mathbb{E}[\alpha_{1:2}]$ and $\mathbb{E}[\alpha_2]$ collapses to zero, with the result that the condition remains satisfied. A more clever approach is to concentrate as much of the distribution towards the left as possible, while still holding a small but meaningful part on the right end, so that the difference between the expectations stays sufficiently large. The resulting distributions will have their peaks somewhere at the left of the unit interval and will decay gradually when getting closer to the right end. These type of distributions have the following interpretation. The majority of employees has low or moderate ability. While being a talent is an exception rather than a rule, talents are still in enough number, to make an encounter with them a realistic possibility. This observation corresponds to the fact that the distribution is not allowed to decay to quickly to keep the difference in expectations large enough.

To explore the issue further, we construct a continuum of density functions by gradually transforming the uniform distribution on the unit interval into the uniform distribution on the half interval $(0, \frac{1}{2})$ and follow the position of the equilibria in the meantime. First, starting with the $U(0,1)$ distribution, the amount of density on the right is gradually placed towards the left. The distributions become skewed and at a certain point the discriminative equilibrium is, for the reasons discussed, expected to disappear. However if the concentration towards the left half of the interval is continued, the tail and skewness become less prominent. The difference between $\mathbb{E}[\alpha_{1:2}]$ and $\mathbb{E}[\alpha_1]$ decreases and discriminative equilibria are likely to be observed again as a result. The distributions g_η are, for η being a parameter describing the transitions and $I[0 \leq \alpha_i \leq 1]$ the indicator function on the unit interval, given by:

$$g_\eta(\alpha_i) \quad = \quad I[0 \leq \alpha_i \leq 1] \frac{1}{1 + \exp \eta(\alpha_i - 0.5)}$$

As η tends to zero, g converges to the uniform distribution with support $[0, 1]$, and with η increasing to infinity g converges to the uniform distribution with support $[0, \frac{1}{2}]$. Moreover, the distributions flow smoothly into each other as η is varied.

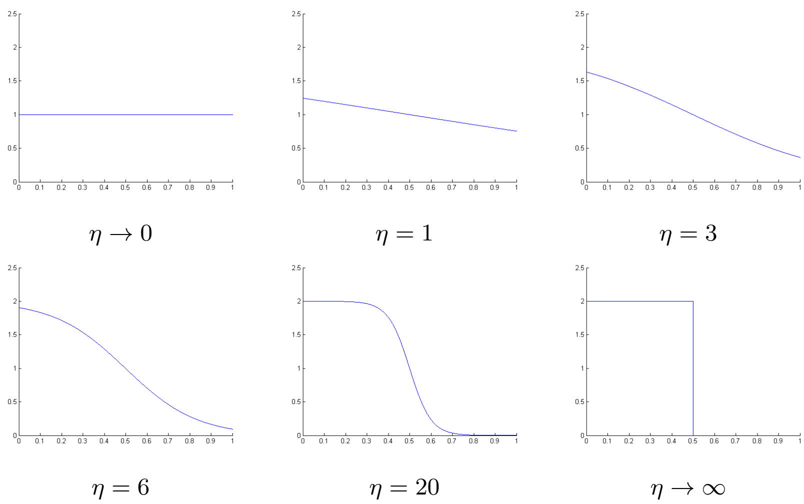


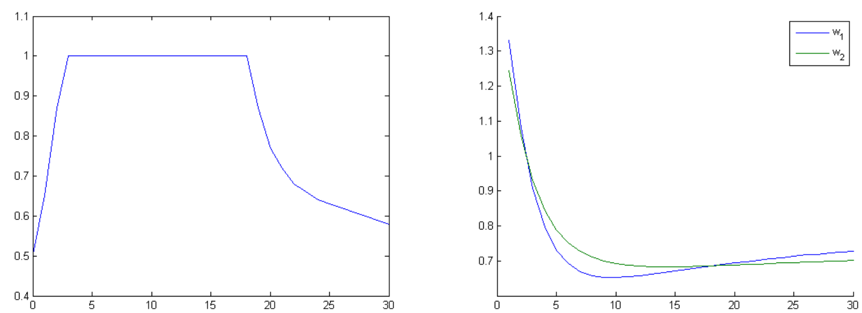
Figure 6: Shape of g_η for selected values of η

Define $c(\eta)$ as the discriminative equilibrium against the first employee, if this equilibrium exists and as the fair equilibrium if it does not exist. Panel A in figure 7 displays $c(\eta)$ against the first employee as a function of η. The discriminating equilibrium against the second employee is equal to $1/c(\eta)$, which follows from the fact that the distributions are the same. Panel B shows $c(\eta)$ as a function of η. The right figure shows the graphs of w_1 and w_2 which are defines as follows:

$$w_1 \quad = \quad \mathbb{E}[\alpha_{1:2}] + 4 \int_0^1 \alpha_1^2 g(\alpha_1)^2 d\alpha_1$$

$$w_2 \quad = \quad \left(1 + 4 \int_0^1 \alpha_1 g(\alpha_1)^2 d\alpha_1\right) \mathbb{E}[\alpha_1]$$

According to Theorem 5 a sufficient condition for the existence of discriminative equilibria is the property $w_1 > w_2$. This is in line with the results as shown in the Figure. When the graph of w_1 lies above w_2, it holds that $c(\eta) < 1$, which means that a discriminative equilibrium exists. Moreover there are no discriminative equilibria when $w_1 \leq w_2$.



Panel A Panel B

Figure 7: Existence of discriminative equilibrium

6.2 Rectangular Uniform Distributions

Suppose now, that abilities of the two employees are drawn independently from different distribution. In practice this is the case if the employees belong to two different groups, where group entity is observed easily and contains valuable information about production. As the class of general pairs of distributions is too large to lend itself for analysis without the arise of mathematical complexities, we consider the class of uniform distributions $U(l_i, u_i)$, $i = 1, 2$, where $0 \leq l_i < u_i$. Two issues that arise when the distributions are unequal are the following. First, the employees can face different ex ante expectations about their ability and second, they can face different amounts of ex-ante uncertainty about their abilities. The first is conveniently analysed by holding the length of the support $u_i - l_i$ equal, but varying the difference between l_1 and l_2. We find, which comes to some surprise, that discrimination against an employee increases as his ex ante expectation increases relatively with respect to the other employee. As his ex ante expectation increases with respect to the other employee, he becomes less sensitive to discrimination. The second issue is analysed by holding ex ante expectations the same and varying the difference between the supports $u_1 - l_1$ and $u_2 - l_2$. We find that discrimination against the employee with the smallest support increases. In fact there is a similarity with personal signals. As with a stronger personal signal less variance in the ex ante distribution allows an employee to be better informed about the position of his ability than the other employee. The joint distributions correspond to uniform distributions with rectangular support:

$$\{\alpha_1, \alpha_2 \mid l_i \leq \alpha_i \leq u_i, \quad 0 \leq l_i < u_i\}$$

A rectangle of this form will be denoted as $R = (l_1, u_1, l_2, u_2)$. Again only informative equilibria are considered. Therefore, the line $\alpha_2 = c\alpha_1$ passes through the interior of the rectangle. The vertexes of a rectangle are labelled as follows, $A = (l_1, l_2)$, $B = (u_1, l_2)$, $C = (u_1, u_2)$ and $D = (l_1, u_2)$. The sides of the rectangle are given by AB, BC, CD, AD. Strategies c can be characterized according to which sides the line $\alpha_2 = c\alpha_1$ intersects. The line has two points of intersection with the rectangle. The first point can be on AB or AD and the second on BC or CD. Therefore, four different cases can be distinguished, although not all cases necessarily appear within the *same* rectangle. Figure 8 shows the labelling of the cases by means of examples.

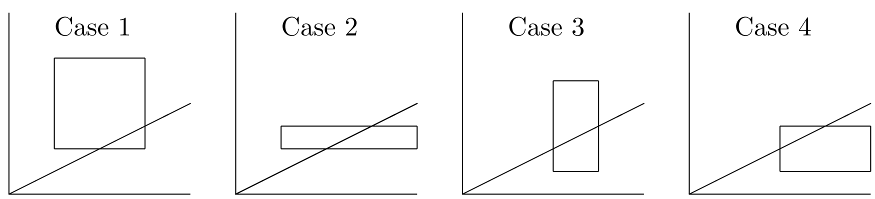


Figure 8: Characterization of strategies c according to points of intersection

If suffices to compute the conditional beliefs $\mathbb{E}[\alpha_i \mid m = j]$ and the resulting best response function for the first and second cases explicitly. This follows by applying the same symmetry argument as before. Formally, let $q(c, l_1, u_1, l_2, u_2)$ be the best response function of the manager, given that employees beliefs are such that they expect him to follow the strategy c. Then it satisfies the functional relation:

$$q(c; l_1, u_1, l_2, u_2) \quad = \quad q(c^{-1}; l_2, u_2, l_1, u_1)^{-1}$$

Given a strategy c, the conditional beliefs $\mathbb{E}[\alpha_i \mid m = j]$ are given by:

- *Case 1*

$$\mathbb{E}[\alpha_1 \mid m = 1] = \frac{\frac{1}{3}c(u_1^3 - \frac{l_2^3}{c^3}) - \frac{1}{2}l_2(u_1^2 - \frac{l_2^2}{c^2})}{\frac{1}{2}(u_1 - \frac{l_2}{c})(cu_1 - l_2)}$$

$$\mathbb{E}[\alpha_1 \mid m = 2] = \frac{\frac{1}{2}(u_2 - l_2)(\frac{l_2^2}{c^2} - l_1^2) + \frac{1}{2}u_2(u_1^2 - \frac{l_2^2}{c^2}) - \frac{1}{3}c(u_1^3 - \frac{l_2^3}{c^3})}{(u_1 - l_1)(u_2 - l_2) - \frac{1}{2}(u_1 - \frac{l_2}{c})(cu_1 - l_2)}$$

$$\mathbb{E}[\alpha_2 \mid m = 1] = \frac{\frac{1}{2}u_1(c^2u_1^2 - l_2^2) - \frac{1}{3}\frac{1}{c}(c^3u_1^3 - l_2^3)}{\frac{1}{2}(u_1 - \frac{l_2}{c})(cu_1 - l_2)}$$

$$\mathbb{E}[\alpha_2 \mid m = 2] = \frac{\frac{1}{3}\frac{1}{c}(c^3u_1^3 - l_2^3) - \frac{1}{2}l_1(c^2u_1^2 - l_2^2) + \frac{1}{2}(u_1 - l_1)(u_2^2 - c^2u_1^2)}{(u_1 - l_1)(u_2 - l_2) - \frac{1}{2}(u_1 - \frac{l_2}{c})(cu_1 - l_2)}$$

- *Case 2*

$$\mathbb{E}[\alpha_1 \mid m = 1] = \frac{\frac{1}{3}(\frac{u_2^3}{c^2} - \frac{l_2^3}{c^2}) - \frac{1}{2}l_2(\frac{u_2^2}{c^2} - \frac{l_2^2}{c^2}) + \frac{1}{2}(u_2 - u_1)(u_1^2 - \frac{u_2^2}{c^2})}{\frac{1}{2}\frac{1}{c}(u_2 - l_2)^2 + (u_1 - \frac{u_2}{c})(u_2 - l_2)}$$

$$\mathbb{E}[\alpha_1 \mid m = 2] = \frac{\frac{1}{2}(u_2 - l_2)(\frac{l_2^2}{c^2} - l_1^2) + \frac{1}{2}u_2(u_2^2 - l_2^2)\frac{1}{c^2} - \frac{1}{3}(u_2^3 - l_2^3)\frac{1}{c^2}}{(u_1 - l_1)(u_2 - l_2) - \frac{1}{2}\frac{1}{c}(u_2 - l_2)^2 - (u_1 - \frac{u_2}{c})(u_2 - l_2)}$$

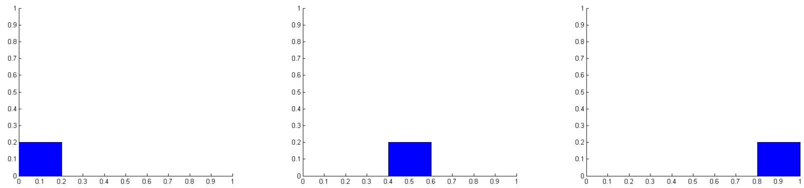
$$\mathbb{E}[\alpha_2 \mid m = 1] = \frac{\frac{1}{2}u_1(u_2^2 - l_2^2) - \frac{1}{3}\frac{1}{c}(u_2^3 - l_2^3)}{\frac{1}{2}\frac{1}{c}(u_2 - l_2)^2 + (u_1 - \frac{u_2}{c})(u_2 - l_2)}$$

$$\mathbb{E}[\alpha_2 \mid m = 2] = \frac{\frac{1}{3}\frac{1}{c}(u_2^3 - l_2^3) - \frac{1}{2}l_1(u_2^2 - l_2^2)}{(u_1 - l_1)(u_2 - l_2) - \frac{1}{2}\frac{1}{c}(u_2 - l_2)^2 - (u_1 - \frac{u_2}{c})(u_2 - l_2)}$$

Equilibria are given by the solutions of the equation $c = q(c, l_1, u_1, l_2, u_2)$, which are computed numerically. This is done by partitioning a rectangle into subsets, so that rays $\alpha_2 = c\alpha_1$ passing through a subset correspond to strategies c of the same case, and rays passing to different subsets correspond to strategies of different cases. Each of the subsets is analysed separately.

6.3 Ex ante expectations

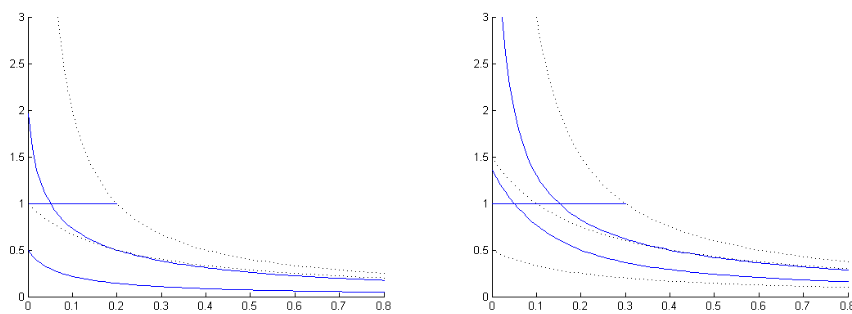
The question that is investigated now, is how discrimination is affected if employees have different ex ante expectations about their abilities. We consider uniform distributions where the length of the support is equal for the two employees. This is done to control for the effect of different amounts of uncertainty. More specifically, we take the length of the support equal to $u_i - l_i = 0.2$, and fix l_2. As a result we get a continuum of rectangles $R = (l_1, l_1 + 0.2, l_2, l_2 + 0.2)$. which are described by the parameter l_1. If l_1 increases, the difference between the ex ante abilities of the first and second employee gets larger. Figure 9 illustrates the rectangles $R = (l_1, l_1 + 0.2, 0, 0.2)$ for selected l_1.



$l_1 = 0$ $l_1 = 0.4$ $l_1 = 0.8$

Figure 9: $R = (l_1, l_1 + 0.2, 0, 0.2)$ for selected values of l_1

We define θ_B, θ_C and θ_D as the slope of the rays which connect the origin to the vertexes B, C and D. Figure 10 shows how the equilibria vary with the parameter l_1 given $l_2 = 0$ and $l_2 = 0.1$. We will discuss the case where $l_2 = 0$. As suggested by the figure, other values of l_2, like $l_2 = 0.1$ give more or less results that are qualitatively similar. Moreover θ_C and θ_D are also displayed as dotted lines. Note that $\theta_C < \theta_D$ and that any non babbling equilibrium c should satisfy $c < \theta_D$. For interpretation it is helpful to use the fact that $\alpha_2 = c\alpha_1$ has its point of intersection along the size BC if $c \leq \theta_C$ and along CD if $c \geq \theta_C$. By comparing c to θ_B, θ_C and θ_D one gets an indication of the location of the point of intersection. For example if c is relatively close to θ_B than to θ_C, this indicates that $\alpha_2 = c\alpha_1$ intersects BC at a point closer to B than to C.



$R = (l_1, l_1 + 0.2, 0, 0.2)$, $0 \leq l_1 \leq 0.8$ $R = (l_1, l_1 + 0.2, 0.1, 0.3)$, $0 \leq l_1 \leq 0.8$

Figure 10: Dependence equilibria on position of rectangle

We define the equilibria c_1, c_2 and c_3 as follows. If there are three equilibria, c_1 is the least favourable to the first employee, c_2 the least favourable to the second employee and c_3 is the third equilibrium. If there are only two equilibria, c_2 is left undefined, c_1 is defined as before and c_3 is defined as the least favourable to the second employee. Note that c_2 is discriminating against the second employee for small l_1, but discrimination decreases quickly if l_1 increases. For larger l_1, c_2 becomes a fair equilibrium. The equilibrium c_1 is discriminating against the first employee on the entire range studied and the extent of discrimination increases with l_1. Lastly c_3 is initially a fair equilibrium, but discriminates against the first employee for larger l_1 and the extent of discrimination increases with l_1. There are several observations to make. First, there exists a fair equilibrium, provided that the 45 degrees line $\alpha_2 = \alpha_1$ passes through the interior of the rectangle[5]. For large l_1, which means that the first employees ex ante belief is higher, all equilibria (c_1 and c_3) involve discrimination against him. The equilibrium c_3 converges to θ_C, which means that the point of intersection between $\alpha_2 = c_3\alpha_1$ and BC lies close to C. Therefore if l_1 is large, the first employee is promoted most of the time. In the equilibrium c_1, the point of intersection converges to the right lower corner B, which implies that the first employee is almost newer promoted.

Visually, an equilibrium is stable if the best response function intersects the 45 degrees from above, and does not decline too quickly (the slope is larger than -1 at the point of intersection). The graph of the best response function together with the line $\alpha_2 = \alpha_1$ is shown in Figure 11, for selected values of l_1. It turns out that c_1 and c_2 are stable but c_3 not. This implies that the fair equilibrium is unstable for l_1 small but stable if l_1 is larger than some threshold l_1^*, which lies somewhere around 0.05.

The result that discrimination against the first employee strengthens when his ex ante belief increases is somewhat surprising on first sight, especially the fact that c_1 is the only stable equilibrium if l_1 is sufficiently large. To explain this result, note that for any "non-degenerate" strategy, the coefficient c is necessarily small for large l_1, since $\alpha_2 = c\alpha_1$ passes through the interior of the rectangle. As a result the graph of this line is nearly flat. This implies that the first employee's belief is largely insensitive to the promotion decision. The impact on the beliefs of the second employee is considerable. Not being promoted is hard for him to swallow. He learns even with certainty that his ability lies below a certain threshold (cu_1). The reason why discrimination arises in the first place is to avoid the situation that the employee who is expected to be favoured is not promoted. The fact that it does not really matter for the first employees beliefs how the promotion decision turns out, makes it possible to increase the extent of discrimination which accounts for the results found.

[5] This leads to the conjecture that the fair equilibrium exists in general for uniform distributions where the supports have the same length.

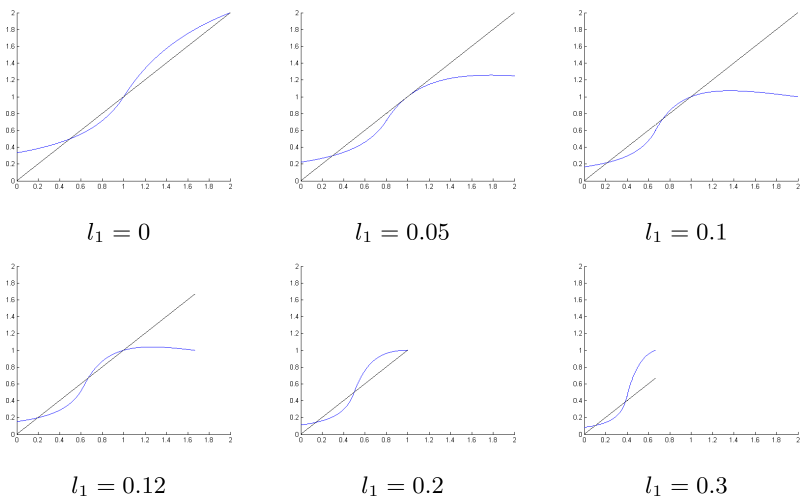


Figure 11: Best response function for selected values of l_1

6.4 Ex ante uncertainty

In this section we analyse differences in ex ante uncertainty. As a real world example, consider the group of young employees who just have entered the labour market, and experienced professionals, who have had the opportunity to learn about themselves for years. Individuals belonging to the former group intuitively face uncertainty to a larger extent. Differences in ex ante uncertainty are modelled in the following way. First, ex ante expectations of the two employees are kept equal, to control for their influence studied in the previous section. This is achieved by studying a continuum of rectangles $R = (0.4, 0.6, 0.4-\kappa, 0.6+\kappa)$, parametrized by κ which varies over the interval $-0.1 < \kappa \leq 0.4$. As κ increases the first employee faces increasingly less uncertainty compared to the second employee. Figure 12 shows the rectangles for selected values of κ.

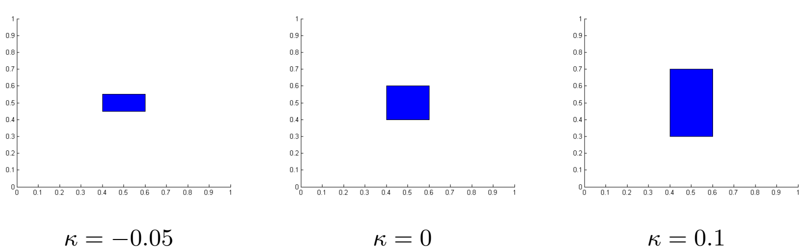
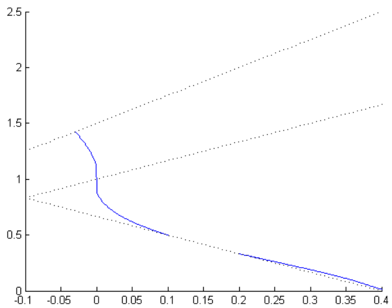
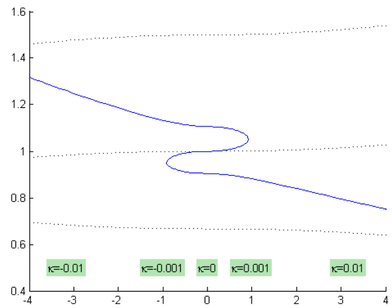


Figure 12: $R = (0.4, 0.6, 0.4 - \kappa, 0.6 + \kappa)$ for selected values of κ

Figure 13 shows the dependence of the equilibria on κ. Moreover θ_B, θ_C and θ_D as defined in the preceding section are also shown, which give an indication about the point of intersection of $\alpha_2 = c\alpha_1$ with the rectangle's sides.



Panel A



Panel B

Figure 13: Dependence equilibria on κ

In Panel A, κ varies from -0.1 to -0.4. However, an interesting property can not be seen in Panel A. For values of κ close to zero, three equilibria exists, where two of them collapse, if $|\kappa|$ increases. This is shown in Panel B. To emphasize both the behaviour of the equilibria for $|\kappa|$ small and the way how they translate when $|\kappa|$ increases, a different scale of measurement has been used along the horizontal axis, given by the following transformation:

$$y = 10\sqrt{10}\sqrt{\kappa}I[\kappa \geq 0] - 10\sqrt{10}\sqrt{-\kappa}I[\kappa \leq 0]$$

This means that the equilibria are displayed as (set valued) function of y instead of κ directly, but the corresponding values of κ are also shown. The transformation is concave on both half intervals and pushes values of κ close to zero more to the outside, while contracting values of κ further away. Perhaps the most prominent feature that follows from the graphs, is the fact that discrimination against the first employee is stronger if his uncertainty about his ability is less compared to the uncertainty surrounding the second employee. What is also remarkable is the sensitivity of the fair equilibrium. This equilibrium exists when the distributions are equal but even the smallest perturbation in uncertainties of the distributions causes the equilibrium to cease. Specifically, it suffices to make the support of one distribution longer by a few percent compared to the support of the other distribution. Lastly non babbling equilibria in pure strategies do not always exist on the entire range, in particular they do not exist for values of κ below (approximately) -0.03 and between 0.2 and 0.3.

The main finding of this section is the fact that discrimination against an employee increases if he is ex ante more certain about his ability. The results closely resemble the analysis of perfect signals, where was found that discrimination against an employee strengthens, if his signals are more precise. Both cases share in common that employees perceptions about their ability can not be changed in a meaningful way, if they already possess valuable information about the position of their abilities. With personal signals, the informational superiority resulted from having a stronger signal. Here it results from the fact that an employee that originates from a distribution where the length of the support is small does not even require personal signals since he knows his ability already with quite some accuracy.

7 Concluding Remarks

This article studies discrimination in an environment where a manager has private knowledge about the abilities of two employees, and promotes one of them. The manager takes into account that the promotion decision influences employees' self assessments and as a result profit. Importantly, the favoured employee is hurt considerably if he is not promoted, thinking that his ability must be especially low. Consequently, the manager who aims at maximizing profits will be especially careful with him and even promote him in fact the unfavoured employee is somewhat better. These type of discriminating equilibria were found in various settings. In practice, a promotion system that involves discrimination is not only morally undesirable, but also costly. Often, the new position of an employee who is promoted is more ability intensive. Promoting the wrong employee can have a considerable adverse impact on profits. Although ending up with a fair equilibrium is also a possibility if employees have no reason to expect discrimination to occur, it is not likely to be persistent. In reality, it is almost always not hard to find some reasons to expect that the parties are not treated in a completely equal way, be it a difference in attributes, stereotypes or the result of a social setting. The forces discussed in this article seem to be persistent and meaningful, partly because they seem to operate in the world around us at least intuitively and partly following from the fact that modelling them jointly with factors that could be interfering, reinforces most of the time the initial conclusion that they are prevalent.

References

[1] Arrow, K., J., 1973. The Theory of Discrimination. *In Discrimination in Labor Markets, edited by Orley Ashenfelter, Princeton, Princeton University Press*, 3-33.

[2] Bènabou, R., Tirole, J., 2003. Intrinsic and Extrinsic Motivation. *Review of Economic Studies*, **70**, 489-520.

[3] Coate, S., Loury, G., 1993a Will Affirmative Action Policies Eliminate Negative Stereotypes? *American Economic Review*, **83**, 1220-42.

[4] Coate, S., Loury, G., 1993b. Antidiscrimination Enforcement and the Problem of Patronization. *American Economic Review Papers and Proceedings*, **83 (2)**, 92-98.

[5] Cooley, C., 1902. Human Nature and the Social Order *New York: Scribner's.*

[6] Crutzen, B., Swank, O., Visser, B., 2007. Confidence management: on interpersonal comparisons in teams. *Tinbergen Discussion Paper*, **TI 2007-040/1**, 26.

[7] Ryan, R., Deci, E., 2000, Intrinsic and Extrinsic Motivations: Classic Definitions and New Directions. *Contemporary Educational Psychology*, **25**, 54-67.

[8] Fehr, E., Schmidt, K., 1999. A Theory of Fairness, Competition and Cooperation. *Quarterly Journal of Economics* , **114**, 817-868.

[9] Ishida, J., 2006. Optimal promotion policies with the looking-glass effect. *Journal of Labor Economics*, **24**, 857-877.

[10] Ishida, J., 2012. Contracting with Self Esteem Concerns. *Journal of Economic Behavior and Organization* , **81(2)** , 329-340.

[11] Loewenstein, G., 1987. Anticipation and the Valuation of Delayed Consumption. *The Economic Journal* , **97(387)**, 666-684.

[12] Loewenstein, G., Thompson, L., Bazerman, M., 1989. Social Utility and Decision Making in Interpersonal Contexts, *Journal of Personality and Social Psychology* **57(3)**, 426-441.

[13] Prendergast, C. 1992. Career Development and Specific Human Capital Collection. *Journal of the Japanese and International Economies*, **6**, 207-227.

[14] Prendergast, C., Topel, R., 1993. Discretion and bias in performance evaluation. *European Economic Review*, **37**, 355-365.

[15] Zapata-Phelan, C., Colquitt, J., Brent, S., Livingston, B., 2009. procedural justice and task performance, the mediating role of intrinsic motivation, *Organizational Behavior and Decision Processes*, **108**, 93-105.

Appendix A

In this Appendix, we state and proof a Lemma that will be applied repeatedly in subsequent stability proofs. The statement is as follows:

Lemma 1 *Let $f : (a, b) \mapsto \mathbb{R}$ be increasing, with $a < b$ and possibly $b = \infty$ and let $x \in (a, b)$ be a unique fixed point of f on its domain, that is $f(x) = x$. Suppose that f satisfies the property:*

- *$f(y) > y$ if $y < x$ and $f(y) < y$ is $y > x$*

Then x is stable, and every sequence $\{x_i\}_{i=1}^{\infty}$ with $x_1 \in (a, b)$ and $x_{i+1} = f(x_i)$, $i > 1$ converges to x.

proof: Take an arbitrary y such that $x < y < b$, and express y as $x + d$. It holds that

$$x < f(x + d) < x + d$$

The first inequality follows since f is strictly increasing, and the second from the fact that $f(y) < y$ if $0.5 < y < 1$. Subtracting x on both sides gives:

$$0 < f(x+d) - x < d$$

This means that for $h < b$ there exists some constant $\rho_h < b$ such that for *all* choices of y in $x < y \leq h$ it holds that[6]

$$0 < f(x+d) - x < \rho_h d$$

Now, define a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_1 = y$ and $x_{i+1} = f(c_i)$, $i > 1$. It can be shown easily with full induction that:

$$0 < x_i - x < \rho_h^{i-1} d$$

Since $\rho_h < 1$, this implies that the sequence converges to x. Since h is arbitrary, it follows that the sequence converges for every $x < y < b$. Analogously, the result can be proven for the case $y < 0.5$. In particular these results imply that x is stable. Therefore the desired results are established. ■

Appendix B

We prove the stability results shown in Theorem 1. We will make use of Lemma 1 (Appendix A). If suffices to establish that every sequence $\{c_i\}_{i=1}^{\infty}$ with $c_{i+1} = q(c_i)$ converges to $c = 0.5$ if $0 < c_1 < 1$. To see, note that this implies that $c = 0.5$ is stable and that $c = 1$ can not be stable. By using symmetry, it also holds that every sequence $\{c_i\}_{i=1}^{\infty}$ with $c_{i+1} = q(c_i)$ converges to $c = 2$ if $c_1 > 1$, which means that $c = 2$ is stable. The best response function on the interval $0 < c < 1$ is given by:

$$q(c) = \frac{1}{3-2c}$$

It holds that $q(c) > c$ if $0 < c < 0.5$ and $q(c) < c$ if $0.5 < c < 1$. Moreover $q(c)$ is increasing on the interval $0 < c < 1$ since:

$$\frac{\partial q}{\partial c} = \frac{2}{(3-2c)^2} > 0$$

Therefore, we can apply Lemma 1 with $f = q$, $a = 0$ and $b = 1$. Immediately, it follows that $c = 0.5$ is stable and that every sequence $\{c_i\}_{i=1}^{\infty}$ with $c_{i+1} = q(c_i)$, $i > 1$ converges to $c = 0.5$ if $0 < c_1 < 1$. This completes the proof. ■

Appendix C

(Proof Theorem 2) The candidate equilibria on the interval $0 < c \leq 1$ are given by:

$$c(\rho_1, \rho_2)_1 = \frac{3}{4} + \frac{1}{2}\sqrt{2}\sqrt{\frac{9}{8} - \frac{1-\rho_1}{1-\rho_2}}$$

$$c(\rho_1, \rho_2)_2 = \frac{3}{4} - \frac{1}{2}\sqrt{2}\sqrt{\frac{9}{8} - \frac{1-\rho_1}{1-\rho_2}}$$

To be an equilibrium the values should satisfy the domain restriction. In particular this implies that the candidate equilibria are real valued, that is, the imaginary part should equal zero. We now show the result for the candidate equilibrium $c(\rho_1, \rho_2)_1$. Clearly, the imaginary part is zero if:

$$\frac{1-\rho_1}{1-\rho_2} \leq \frac{9}{8}$$

[6] h needs to be chosen, because it is possible that a constant ρ which applies for all choices of y in $x < y < b$ does not exist *uniformly*, this occurs for example if f is continuous and satisfies $f(y) \to b$ if $y \to b$, which is exactly the case that we frequently encounter in this article.

Note, that $c(\rho_1, \rho_2)_1 \geq \frac{3}{4}$. Therefore, the domain restriction is satisfied if:

$$\begin{aligned} \frac{1}{4} &\geq \frac{1}{2}\sqrt{2}\sqrt{\frac{9}{8} - \frac{1-\rho_1}{1-\rho_2}} \\ 1 &\leq \frac{1-\rho_1}{1-\rho_2} \end{aligned}$$

This establishes the stated result for $c(\rho_1, \rho_2)_1$. The result for $c(\rho_1, \rho_2)_2$ is derived in a similar way. From $c(\rho_1, \rho_2)_{2+i} = c(\rho_2, \rho_1)_i^{-1}$, it follows that $c(\rho_1, \rho_2)_{2+i}$ exists if and only if $c(\rho_2, \rho_1)_i$ exists, since all equilibria are strictly positive. For example, $c(\rho_1, \rho_3)_3$ exists if:

$$\begin{aligned} 1 &\leq \tfrac{1-\rho_2}{1-\rho_1} \leq \tfrac{9}{8} \\ \tfrac{8}{9} &\leq \tfrac{1-\rho_2}{1-\rho_1} \leq 1 \end{aligned}$$

Therefore, the desired results are established and the proof is complete. ■

Appendix D

(Proof Theorem 3) We will make use of Lemma 1 (Appendix A). For simplicity of notation we will denote the equilibria $c(\rho_1, \rho_2)_i$ by c_i. Moreover we define for two equilibria c_i, c_j such that $c_i < c_j$ the interval $I_{i,j}$ as $c_i < c < c_j$. The best response function will be written as $q(c)$ most of the time, but when its dependence on the parameters is needed for the argument, the notation $q(c; \rho_1, \rho_2)$ will be used. Let r be the ratio of $1 - \rho_1$ and $1 - \rho_2$. First we show that the best response function is increasing. If $0 < c \leq 1$ it holds that:

$$q'(c) = \tfrac{1-\rho_1}{1-\rho_2}\tfrac{2}{(3-2c)^2} > 0$$

For $c > 1$ the best response function is obtained by using $q(c; \rho_1, \rho_2) = q(c^{-1}; \rho_2, \rho_1)^{-1}$. This gives:

$$q(c) = 3\tfrac{1-\rho_1}{1-\rho_2} - \tfrac{1-\rho_1}{1-\rho_2}\tfrac{2}{c}$$

Therefore it holds that:

$$q'(c) = \tfrac{1-\rho_1}{1-\rho_2}\tfrac{2}{c^2} > 0$$

We conclude that q is increasing. Moreover q is differentiable on $(0, \infty)$, in particular at $c = 1$. Next, we derive some auxiliary results. First, note that $q(0) > 0$. Second, it holds that $q(c; \rho_1, \rho_2) < c$ if and only if $q(c^{-1}; \rho_2, \rho_1) > c^{-1}$. This follows directly from the identity for q. Since $q(0; \rho_2, \rho_1) > 0$, there exist $\epsilon > 0$ such that $q(c; \rho_2, \rho_1) > c$ for all $c < \epsilon$, which implies $q(c; \rho_1, \rho_2) < c$ for all $c > \frac{1}{\epsilon} = M$. Ultimately $q(c)$ lies below the 45 degrees line for all parameters. To summarize, it holds that:

1. $q(0) > 0$
2. There exists $M > 1$ such that $q(c) < c$ for all $c > M$

Suppose $r < \frac{8}{9}$. From Theorem 2 it follows that c_2 is the only equilibrium. Property 1 implies that $q(c) > c$ if $c < c_2$. Property 2 implies that $q(c) < c$ if $c > c_2$, since c_2 is the only equilibrium. We can now apply Lemma 1 with $f = q$, $a = 0$ and $b = \infty$. It follows that c_2 is globally stable.

Next we examine the case $\frac{8}{9} < r < 1$. The equilibria are c_2, c_3 and c_4. Property 1 implies that $q(c) > c$ if $c < c_2$. We show by contradiction that $q(c) < c$ if $c \in I_{23}$. Suppose that $q(c) > c$ if $c \in I_{23}$. From property 2 it follows $q'(c_4) \leq 1$. We distinguish two cases. First, assume that the derivative is strictly less than 1. It follows that q crosses the 45 degrees line from above at this point. Therefore it must hold that $q(c) > c$ if $c \in I_{34}$. However since $q(c) = c$ at both endpoints of I_{34}, it follows from the mean value theorem that there exists some $\xi \in I_{34}$ such that $q'(\xi) = 1$. Since $q(c) > c$ if $c \in I_{23}$, and $q(c) > c$ if $c \in I_{34}$, it follows that the derivative of q at c_3 is also equal to unity. This leads to a contradiction since $q'' < 0$, which means that only one point c can exist such that $q'(c) = 1$. In the other case it holds that $q'(c_4) = 1$. Since $q'' < 0$ this implies that $q(c) < c$ if $c \in I_{34}$. The contradiction is similar as in the fist case. Since $q(c) = c$ at both endpoints there is at least one $\xi \in I_{34}$ such that $q'(\xi) = 1$. This is in contradiction with $q'' < 0$. Therefore, we conclude that $q(c) < c$ if $c \in I_{23}$. We can now apply Lemma 1 with $f = q$, $a = 0$, $b = c_3$. It follows that c_2 is stable. Moreover every sequence $\{x_i\}_{i=1}^{\infty}$ with $x_i = q(x_{i-1})$, $i > 1$ converges to c_2 if $0 < x_1 < c_3$. Since $c < q(c)$ if $c < c_3$, it must hold that $c > q(c)$ if $c \in I_{34}$. Otherwise $q'(c_3) = q'(c_4) = 1$, which would lead to a contradiction. Therefore we can apply Lemma 1 with $f = q$, $a = c_3$, $b = \infty$, and it follows that c_4 is stable and that every sequence $\{x_i\}_{i=1}^{\infty}$ with $x_i = q(x_{i-1})$, $i > 1$ converges to c_4 if $x_1 > c_3$. From the converge results for c_2 and c_4 it follows that c_3 is unstable. Lastly the stability results for $r > 1$ follow by symmetry (switching the axes and using $\frac{1}{r} < 1$). ■

Appendix E

In this Appendix we proof the existence results of the equilibria stated in section 4.2 (relative signals). Specifically, we analyse the candidate equilibria $c(\phi_1, \phi_2)_i$, $i = 1, 2$, since the results for $c(\phi_1, \phi_2)_i$, $i = 3, 4$ follow from symmetry. First, we show that $c(\phi_1, \phi_2)_1$ is newer an equilibrium. By using the expression, we get:

$$
\begin{aligned}
\frac{1}{2}\left(\frac{3-\phi_2}{2-\phi_2}\right) + \sqrt{\frac{1}{4}\left(\frac{3-\phi_2}{2-\phi_2}\right)^2 - \frac{1-\phi_1}{2-\phi_2}} &< 1 \\
(3-\phi_2)^2 - 4(1-\phi_1)(2-\phi_2) &< (1-\phi_2)^2 \\
4(2-\phi_2)\phi_1 &< 0
\end{aligned}
$$

Since $0 \leq \phi_i \leq 1$, $i = 1, 2$, this implies the desired result. We now show that $c(\phi_1, \phi_2)_2$ is an equilibrium if and only if (i) $\phi_1 < 1$ and (ii) in case $\phi_1 = 0$, $\phi_2 < 1$. The best response function is given by:

$$
q(c; \phi_1, \phi_2) = \frac{1-\phi_1}{(3-\phi_2) - (2-\phi_2)c}
$$

Therefore $q(0, \phi_1, \phi_2) > 0$ unless $\phi_1 = 1$. If $\phi_1 = 1$, it holds that $q = 0$, which means that $c(\phi_1, \phi_2)_2$ is a babbling equilibrium. Now suppose that $0 < \phi_1 < 1$. Note that $q(c) \to 1 - \phi_1$, if $c \to 1$. Since on the interval $0 < c < 1$ the graph of q lies for small c above the 45 degrees line, but for large c below the 45 degrees line, there must be a point of intersection somewhere in the interval as q is clearly continuous on $0 < c < 1$. The only candidate on the interval is $c(\phi_1, \phi_2)_2$ and therefore is necessarily an equilibrium. Lastly suppose that $\phi_1 = 0$. In this case, we obtain:

$$
c(\phi_1, \phi_2)_2 = \frac{1}{2-\phi_2}
$$

Therefore, $c(\phi_1, \phi_2)_2$ satisfies the domain restriction if $\phi_2 < 1$. This establishes the desired result and the proof is complete. ■

Appendix F

In this Appendix we prove stability results for the equilibria in section 4.2 (relative signals). We will make use of Lemma 1 (Appendix A). The best response function for $0 < c < 1$ is given by:

$$
q(c; \phi_1, \phi_2) = \frac{1-\phi_1}{(3-\phi_2) - (2-\phi_2)c}
$$

It follows that q is increasing, since $q' > 0$. Moreover, $q(0) > 0$ if $\phi_1 < 1$ (If $\phi_1 = 1$, $q = 0$ and c_2 would be a babbling equilibrium which we rule out). Note that $q(c; \phi_1, \phi_2) \to 1 - \phi_1$

as $c \to 1$ from below. Since c_2 is the only equilibrium on the interval $0 < c < 1$ it follows that $q(c;\phi_1,\phi_2) > c$ if $c < c_2$ and $q(c;\phi_1,\phi_2) < c$ if $c > c_2$. Therefore we can apply Lemma 1 with $f = q$, $a = 0$, $b = 1$ which implies that c_2 is stable and that every sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_{i+1} = q(x_i;\phi_1,\phi_2)$, $i > 1$ converges to c_2 if $0 < x_1 < 1$. From this convergence result it follows that the fair equilibrium is unstable if it exists. The corresponding results for c_3 and c_4 follow by symmetry. ■

Appendix G

We prove the following statements presented in section 4.2 (relative signals):

$$\frac{\partial c(\phi_1\phi_2)_2}{\partial\phi_1} < 0 \qquad \frac{\partial c(\phi_1\phi_2)_2}{\partial\phi_2} > 0 \qquad \frac{\partial c(\phi_1\phi_2)_4}{\partial\phi_1} < 0 \qquad \frac{\partial c(\phi_1\phi_2)_4}{\partial\phi_2} > 0$$

The first result is trivial, as can be seen from the expression of $c(\phi_1,\phi_2)_1$. The last two statements follows from the first two by using symmetry. Therefore it suffices to establish the second result. To increase the simplicity of notation we will write c_2 instead of $c(\phi_1,\phi_2)_2$. Let q_{ϕ_2} be the partial derivative of q with respect to ϕ_2. It holds that:

$$q_{\phi_2} \quad = \quad \frac{(1-c)(1-\phi_1)}{[(3-\phi_2)-(2-\phi_2)c]^2}$$

Therefore $q_{\phi_2} > 0$ on the interval $0 < c < 1$ if $\phi_1 \neq 0$. Moreover it has been shown in Appendix F that $q(c;\phi_1,\phi_2) > c$ if $c < c_2$ and $q(c;\phi_1,\phi_2) > c$ if $c > c_2$. Since $q_{\phi_2} > 0$, any increase $\epsilon > 0$ of ϕ_2 results in a upward shift of the entire best response curve, which implies that $q(c;\phi_1,\phi_2) > c$ for $c \leq c_2$. The new equilibrium $\tilde{c}_2$ can not be on the interval $(0, c_2]$. This implies $\tilde{c}_2 > c_2$. This establishes the second statement and the proof is complete. ■

Figure 14 illustrates graphically the argument that the new equilibrium $\tilde{c}_2$ must be on the right of the initial equilibrium c_2 if ϕ_1 increases. An increase in ϕ_2 leads to an upward shift of q to $\tilde{q}$ and the equilibrium c_2 necessarily moves to the right.

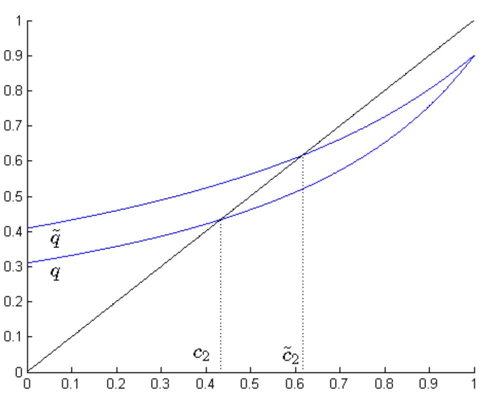


Figure 14: Proof of second statement

Appendix H

In this Appendix we show existence results stated in section 5. First we show that $c(\phi_1,\phi_2,\xi_1,\xi_2)_1$ and $c(\phi_1,\phi_2,\xi_1,\xi_2)_3$ are no equilibria, since they do not satisfy the domain restrictions. By symmetry, it suffices to derive this result for the first candidate equilibrium. This gives:

$$\frac{1}{2}\frac{(3-\phi_2)+3\phi_1\xi_1}{(2-\phi_2)+\phi_1\xi_1} + \sqrt{\frac{1}{4}\left(\frac{(3-\phi_2)+3\phi_1\xi_1}{(2-\phi_2)+\phi_1\xi_1}\right)^2 - \frac{(1-\phi_1)+2\phi_1\xi_1}{(2-\phi_2)+\phi_1\xi_1}} \quad < \quad 1$$

$$\sqrt{((3-\phi_2)+3\phi_1\xi_1)^2 - 4\left((1-\phi_1)+2\phi_1\xi_1\right)\left((2-\phi_2)+\phi_1\xi_1\right)} \quad < \quad 1-\phi_2-\phi_1\xi_1$$

Squaring both sides and rearranging gives:

$$4\phi_1(2-\phi_2+\phi_1\xi_1) \quad < \quad 0$$

Since $0 < \phi_i \leq 1$ this gives the desired result. Next, note that $q(c;\phi_1,\phi_2,\xi_1,\xi_2) \to 1-\phi_1$ as $c \to 1$ from below, and $q(0;\phi_1,\phi_2,\xi_1,\xi_2) > 0$. This implies that there exist at least one equilibrium on the interval $0 < c < 1$ if $\phi_1 > 0$. However, the only candidate equilibrium on this interval is $c(\phi_1,\phi_2,\xi_1,\xi_2)_2$, which implies that it exist. Existence of $c(\phi_1,\phi_2,\xi_1,\xi_2)_4$ follows by symmetry. ■

Appendix I

Lemma 1 (Appendix A) assumes that the function of question is increasing. However, the best response functions in section 5 about fairness considerations are not always increasing on the interval $0 < c < 1$. To derive stability results for this section, we propose a slightly different Lemma.

Lemma 2 *Let $f : (a,b) \mapsto \mathbb{R}$ be differentiable, with $a < b$ and possibly $b = \infty$ and let $x \in (a,b)$ be a unique fixed point of f on its domain, that is $f(x) = x$. Suppose that f satisfies the property:*

- *$f(y) > y$ if $y < x$ and $f(y) < y$ is $y > x$*
- *$f'(y) > -1$*

Then x is stable, and every sequence $\{x_i\}_{i=1}^{\infty}$ with $x_1 \in (a,b)$ and $x_{i+1} = f(x_i)$, $i > 1$ converges to x.

Proof: Take $x < y < b$ arbitrary. It holds that:

$$f(y) \quad = \quad x + \int_x^y f'(t)dt > x + \int_x^y (-1)dt = 2x - y$$

On the other hand, $q(y) < y$ by assumption. This gives:

$$2x - y < f(y) < y$$

Subtracting x on both sides gives:

$$x - y < f(y) - x < y - x$$

This is equivalent with:

$$|f(y) - x| \quad < \quad |y - x|$$

Similarly, it can be proved that this result also holds for $a < y < x$. The remaining part of the proof goes along the similar way as the proof of Lemma 1, by writing $y = x + d$, but working with absolute differences in showing that a sequence converges. ■

Appendix J

This Appendix provides a proof for the stability results stated in section 5. The idea of the proof is as follows. First we show that q is convex on the interval $0 \leq c \leq 1$, hence q' takes its minimum value at $c = 0$. We show that the value of this minimum is larger than -1 if $\phi < 1$ and apply Lemma 2 (Appendix I). For convenience of notation, we will repress the dependence of q on the parameters in the notation. The best response function is given by:

$$q(c) = \frac{\phi\xi(1-c)(2-c) + (1-\phi)}{(3-\phi) - (2-\phi)c}$$

We show that $q(c)$ is convex by computing the second derivative. The first derivative is:

$$q'(c) = \frac{-(2-\phi)\phi\xi c^2 + 2(3-\phi)\phi\xi c - (5-\phi)\phi\xi + (2-\phi)(1-\phi)}{((3-\phi)-(2-\phi)c)^2}$$

The second derivative becomes, after some algebraic manipulations:

$$q''(c) = \frac{(1-\phi)\left((2-\phi)^2 - \phi\xi\right)}{((3-\phi)-(2-\phi)c)^3}$$

Clearly, $q''(c)$ is non-negative on the interval $0 \leq c \leq 1$. Therefore, the smallest value of q' occurs at $c = 0$. This is equal to:

$$q'(0) = \frac{-(5-\phi)\phi\xi + (2-\phi)(1-\phi)}{(3-\phi)^2}$$

Next we minimize $q'(0)$ over ϕ and ξ. First note that the partial derivative of $q'(0)$ with respect to ξ is strictly less than zero if $\phi > 0$ and zero if $\phi = 1$. In both cases $\xi = 1$ minimizes $q'(0)$ given ϕ. Therefore we obtain a reduced form equation by substitution of $\xi = 1$. Note that $f_1(x) = (2-x)(1-x)$ is a parabola that opens upward with zeros $x = 1$ and $x = 2$. Therefore $\phi = 1$ minimizes $f_1(\phi)$. Moreover $f_2(x) = -(5-x)x$ is also a parabola that opens upward, and achieves its maximum $x = 2\frac{1}{2}$. Therefore $\phi = 1$ minimizes $f_2(\phi)$. Lastly the denominator $f_3 = (3-\phi_2)^2$ is also minimized at $\phi = 1$. Since the minimum of the numerator is less than zero (the best response function is for some parameters decreasing), and since f_1, f_2 and f_3 all take their minimum at $\phi = 1$, it follows that $q'(0)$ is minimized if $\phi = 1$ and $\xi = 1$. If $\phi_i < 1$, q is strictly convex. Substituting $\xi = 1$ and $\phi = 1$ in $q'(0)$ learns that the value of the minimum is equal to -1. From the preceding discussion we conclude that $q'(c) > -1$ on the interval $0 < c < 1$ if $\phi < 1$. The only equilibrium on this interval is $c(\phi, \xi)_1$. Moreover it holds that:

$$q(0) = \frac{2\phi\xi + (1-\phi)}{(3-\phi)}$$

$$\lim_{c \uparrow 1} q(c) = 1 - \phi$$

Therefore, $q(c) > c$ is $c < c(\phi, \xi)_1$ and $q(c) < c$ is $c > c(\phi, \xi)_1$. We can now apply Lemma 2 with $f = q$, $a = 0$ and $b = 1$ and it follows that $c(\phi, \xi)_1$ is stable with basin of attraction $(0, 1)$. The result for the basin of attraction implies that a fair equilibrium is unstable if it exists. By symmetry $c(\phi, \xi)_2$ is stable with basin of attraction $(1, \infty)$. ■

Appendix K

We proof the following result, that was stated in the main text:

$$\lim_{c \downarrow 0} q(c) = \frac{\mathrm{Var}[\alpha_1]}{\mathbb{E}[\alpha_1]^2}$$

The best response function is given by:

$$q(c) = \frac{\int_0^1 \alpha_1 G(c\alpha_1)g(\alpha_1)d\alpha_1 - \mathbb{E}[\alpha_1]\int_0^1 G(c\alpha_1)g(\alpha_1)d\alpha_1}{\mathbb{E}[\alpha_1]\int_0^1 G(c\alpha_1)g(\alpha_1)d\alpha_1 - \int_0^c \alpha_1(1-G(c^{-1}\alpha_1))g(\alpha_1)d\alpha_1}$$

Next we make use of the assumption $g(0) < \infty$. The Tailor expansion of $G(c\alpha_1)$ around $c = 0$, gives, by using $G(0) = 0$:

$$G(c\alpha_1) = cg(0)\alpha_1 + r(c)$$

Where $\frac{r(c)}{c} \to 0$ if $c \to 0$. Substituting into $q(c)$ gives:

$$q(c) = \frac{cg(0)\int_0^1 \alpha_1^2 g(\alpha_1)d\alpha_1 - cg(0)\mathbb{E}[\alpha_1]\int_0^1 \alpha_1 g(\alpha_1)d\alpha_1}{cg(0)\mathbb{E}[\alpha_1]\int_0^1 \alpha_1 g(\alpha_1)d\alpha_1 - \int_0^c \alpha_2(1-G(c^{-1}\alpha_2))g(\alpha_2)d\alpha_2} \cdots$$

$$\cdots \frac{+r(c)\int_0^1 \alpha_1 g(\alpha_1)d\alpha_1 - r(c)\mathbb{E}[\alpha_1]\int_0^1 g(\alpha_1)d\alpha_1}{+r(c)\mathbb{E}[\alpha_1]\int_0^1 g(\alpha_1)d\alpha_1}$$

This is equal to (using that g is a density function):

$$q(c) = \frac{g(0)c\text{Var}[\alpha_1] + r(c)\mathbb{E}[\alpha_1] - r(c)\mathbb{E}[\alpha_1]}{cg(0)\mathbb{E}[\alpha_1]^2 - \int_0^c \alpha_2(1-G(c^{-1}\alpha_2))g(\alpha_2)d\alpha_2 + r(c)\mathbb{E}[\alpha_1]}$$

Note that two terms in the numerator vanish. Now define:

$$u(c) = \int_0^c \alpha_2(1-G(c^{-1}\alpha_2))g(\alpha_2)d\alpha_2$$

Taking the right limit of c to zero, and using the calculus for limits of ratios and sums together with the facts that $r(c) \to 0$ as $c \to 0$ and $u(c) \to 0$ if $c \to 0$ gives:

$$\lim_{c\downarrow 0} q(c) = \frac{\lim_{c\downarrow 0} cg(0)\text{Var}[\alpha_1]}{\lim_{c\downarrow 0} cg(0)\mathbb{E}[\alpha_1]^2 + \lim_{c\downarrow 0} r(c)\mathbb{E}[\alpha_1] - \lim_{c\downarrow 0} u(c)}$$

$$= \frac{\lim_{c\downarrow 0} cg(0)\text{Var}[\alpha_1]}{\lim_{c\downarrow 0} cg(0)\mathbb{E}[\alpha_1]^2}$$

$$\frac{\text{Var}[\alpha_1]}{\mathbb{E}[\alpha_1]^2}$$

This establishes the desired result. ■

Appendix L

(Proof Theorem 5) The criterion stated in the Theorem corresponds to the derivative evaluated at $c = 1$ being larger than unity. Suppose that the best response function is strictly convex. First, we show that the criterion $q'(1) > 1$ is also necessary in this case for the existence of a discriminating equilibrium $c < 1$. Suppose that $q'(1) \leq 1$. Strict convexity implies that q' is monotone increasing. Therefore $q'(c) < q'(1)$ if $c < 1$. This gives for $c < 1$, using that $q(1) = 1$:

$$q(c) = 1 - \int_x^1 q'(x)dx > 1 - \int_x^1 dx = c$$

This gives the desired result. Next, we show by contradiction that a unique discriminating equilibrium with $c < 1$ exists if the best response function is strictly convex. Suppose that the discriminating equilibrium is not unique. Then, we can choose discriminating equilibria $x_1 < x_2 < 1$. From the mean value theorem it follows that there exists a point ξ_1 in the interval (x_1, x_2) such that $q'(\xi_1) = 1$. Similarly it follows that there exists ξ_2 in $(x_2, 1)$ such that $q'(\xi_2) = 1$. This is in contradiction with the strict convexity of q.

Lastly, suppose that the best response function is strictly convex and increasing. We have already established that there exists a unique discrimination equilibrium $c < 1$. Since $q'(1) > 1$, it follows that $q(y) < y$ if $c < y < 1$. Moreover, $q(y) > y$ if $y < c$, which follows from the property (Appendix **??**):

$$\lim_{c \downarrow 0} q(c) \quad = \quad \frac{\mathrm{Var}[\alpha_1]}{\mathbb{E}[\alpha_1]^2}$$

We can now apply Lemma 1 with $f = q$, $a = 0$, $b = 1$. It follows that c is stable and that every sequence $\{y_i\}_{i=1}^{\infty}$ with $y_{i+1} = q(y_i)$, $i > 1$ converges to c if $0 < y_1 < 1$. From this, it follows that $c = 1$ is unstable, and the proof is complete. ■