Measuring A Priori Voting Power

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1. Introduction

"Legislators represent people, not trees or acres. Legislators are elected by voters, not farms or cities or economic interests."

So said Chief Justice Earl Warren in his majority decision in the US Supreme Court case of Reynolds v. Sims (1964). In what can be seen as the first time the "one person, one vote" principle was formally recognised in the United States, the Supreme Court decided that Congressional districts within a state should have approximately equal populations. The idea of a voter in one district having substantially greater influence than one in another district was deemed unconstitutional and undemocratic. This conclusion brought the issue of fairness to the forefront of public debate, but not so much as to trigger significant adjustments to the electoral system. In the decades that followed, the issue would come to the fore every now and then, but never in a way that produced meaningful change. A particularly poignant example is the 2000 presidential election, in which the democratic candidate received roughly half a million votes more than his republican opponent, but still lost the election to him. This too caused a stir, but as of today "one man, one vote" is still an unrealised ideal.

On the other side of the Atlantic, reforms of the political system have been rather more common – at the supranational level, at least. The European Economic Community had just six members in 1958, but periodic expansion has resulted in what is now a European Union with 28 member states. With every new accession, voting weights and quota's used in the primary decision-making bodies were reconsidered and adjusted based on existing rules and negotiations between members. As politicians and policy makers looked for appropriate ways to organise Europe's institutions, academics from various countries started to study relevant issues of voting power, leading to what in the last few decades has become a boom in literature on the topic.

So what is voting power exactly? Despite all the research, there is no definite consensus on this question. Consequently, neither is there an established method of measuring it. In their seminal work 'The measurement of voting power', Felsenthal and Machover (1998a) created some order from the pre-existing chaos by systematically describing the evolution of voting power theories, the most prominent ones that are still used, as well as some applications. It has come to be seen as the most comprehensive review of voting power theory to date, and will be used as a guiding framework for this thesis.

However, the overview presented by Felsenthal and Machover is not complete, and some of the classifications they make, as well as some of the arguments they present, have been criticised by other authors. It will be the purpose of this thesis to provide a more up-to-date overview of a priori voting power theory that includes both a historical overview, as well as criticisms and other developments published in the past 15 years.

2. Defining Voting Power

The literature on voting power is far from a coherent whole. Sometimes authors have used different terms for the same concepts (not so problematic); other times they have used the same terms for different concepts (much more so). In a number of cases, authors have apparently ignored or even misinterpreted previous research. All this confusion makes it difficult to provide a definition of ‘voting power’ itself that can be used when discussing different kinds of theories. Since that means it has to be sufficiently broad, the following definition will be used in this thesis:

> *“Voting power is the degree to which an actor in a group can influence the outcome of a decision made by that group”*

Although it might seem rather vague, the rest of this chapter will show that such a general definition is indeed necessary to accommodate the many types of voting power theorists write about. To make sense of all these different types, we will now present some classifications of voting power used in the literature.

2.1. A priori versus a posteriori

The most basic distinction to make is between a priori and a posteriori theories. A priori theories try to predict voting power in a particular voting situation, without taking into account any information other than the ‘rules of the game’ and the number of actors involved. If other factors are needed in the analysis, assumptions are made to account for them. For instance, when considering decision-making in the EU’s Council of Ministers, an a priori analysis will most likely limit itself to the number of ministers, the number of inhabitants of the countries they represent, the weights assigned to their votes according, and the quota needed for proposals to pass. Generally, the voting behaviour of (other) actors is then assumed to be completely independent and random. Such a limited view is of course quite far removed from the day-to-day practice in the Council of Ministers. *Actual* voting power, then, depends on far more factors than those just mentioned. Preferences of other actors, negotiating skills, institutional arrangements beyond the voting procedure, and relations between countries aside from the matter at hand are just a few examples of what in reality are hundreds, if not thousands of relevant factors. This makes clear an important point about a priori theories: they do not purport to say anything about voting in practice. Rather, they aim to judge systems on their own merits, regardless of the people that use them, or the complex environment in which they operate in reality.

A posteriori theories, on the other hand, often try to incorporate as much relevant empirical data as possible. Such theories aim to capture exactly that part of reality that allows them to say something about the amount of voting power actors have in practice. This in some ways makes them more relevant to everyday decision-making, but there are severe downsides as well. The variables involved are legion, and many of them are difficult – if not impossible – to measure. For instance, how would one rate the negotiating skills of ministers, or the degree to which they respect one another? Moreover, conclusions based on a posteriori theories are far more restricted with respect to time than are those based on a priori theories. The latter are in a way 'timeless': if a voting system has certain merits on its own, that will still be true 100 years later. But if an analysis uses, for instance, demographic data, its conclusions may become outdated in just a few years.

As has already become clear in the introduction, this thesis will focus on theories of a priori voting power. There is a critical point to be made about the usefulness of such theories and the analyses produced with them; this will be touched upon in chapter 5.

2.2. I-Power versus P-Power

The distinction between I-Power and P-power is one that was made by Felsenthal and Machover specifically to make clear that there is a fundamental difference between the most prominent theories of voting power. This differences hinges on what is meant by the 'outcome' of a collective decision. Voting power theories such as those by Penrose and Banzhaf (par. 3.x) take the outcome to be the passing of a bill (or failure thereof), or the election of a person to a certain office. Voters are assumed to be policy-seeking, and interested merely in the outcome of the procedure as described above. Their payoffs (e.g., a subsidy created by an accepted bill) can differ substantially from voter to voter, but this is *irrelevant* to the decision rule and the measure of voting power in question. Voting power here is measured not just a priori, but also without taking into account the possible consequences of the decision that is made. It is this type of voting power that Felsenthal and Machover have termed 'I-Power'. In their own words: "a member's voting power is the degree to which that member's vote is able to *influence* the outcome of a division: whether the bill in question will pass or fail" (p.36).

P-Power on the other hand, has much more to do with the actual division that is created. Here voters are assumed to be office-seeking, and the outcome as described before (failure or passage of a bill) is no longer their ultimate goal. Rather, it is a step towards the actual result: a share in the 'prize' that is at stake. If we take again the example of a subsidy created by a bill, the prize is the monetary amount voters (potentially) gain by getting it passed. Or alternatively, if we look at the election of a person to a certain office, the prize is the collection of advantages which that person will create for the people who voted for him. Voting power as P-Power can therefore best be described the extent to which a voter has control over the *ultimate* outcome of a decision, or in other words: the share of the prize which that person can be expected to get as a result of the decision.

2.3. Absolute versus relative voting power

Throughout the literature, varying opinions can be found on whether voting power measures should be absolute or relative. Both have their pros and cons, so there is no definitive 'right' way. Interestingly, the distinction between I-Power and P-Power as described in the previous section plays an important role here. I-Power is primarily an absolute measure: it essentially assigns to each voter a probability of that voter being critical in realising a certain outcome. This is definitely informative, but makes it difficult to compare voting power between different voting situations. If the total amount of voting power (the added probabilities for all voters) differs significantly between two cases, it is basically impossible to compare voting power for individual voters between those cases. To make this possible, absolute voting power indices are often normalised to create relative indices. P-Power indices, on the other hand, are inherently relative measures. As described previously, the basis for P-Power is the share of the 'prize' voters get as a result of the created division. Voting power in this case, therefore, is not an absolute number, but a ratio. The ratios for all voters combined of course add up to one, creating a measure that is (at least numerically) similar to relative I-Power measures, making cross-case comparisons possible.

3. Historical overview

As mentioned in the introduction, the history of voting power theories is far from a linear timeline. For much of the past 60 years, confusion has reigned; ignorance and misinterpretations were common. Nevertheless, following the lead of Felsenthal and Machover (1998a), this chapter attempts to give an overview of the field from the very earliest thoughts on a priori voting power to the most recent work.

3.1. Pre-20th century

Although the first beginnings of voting power theory are rather murky and relatively unknown, the first mathematical work on voting procedures as such – work that touches the boundaries of voting power theory – is quite established. The accomplishments of Borda and Condorcet in the 1780s are significant to this day, and cannot be ignored when studying voting power.

The first mention in the literature of something akin to voting power is not by an academic, but by a politician. In 1787, Luther Martin represented Maryland at the Constitutional Convention in Philadelphia. He protested against ratification of the US constitution, arguing that the proposed method of weighting votes in proportion to population size was unfair to smaller states. In a pamphlet he published later, he not only showed that smaller states would be at a disadvantage if states always voted as a bloc, but actually made a crude attempt to measure voting power (Riker, 1986). Felsenthal and Machover consider his approach an example of I-Power, similar to Holler’s index (Felsenthal & Machover, 2005). Although one can explain away his concerns (Martin was from a small state himself), Riker explains their relevance: although his ideas were ultimately put aside in the political arena, he deserves recognition for being the first on record to fight the incorrect assumption that weighting in proportion to size is fair, and for providing a mathematical basis to support his position.

3.2. L.S. Penrose

The first (properly) scientific attempt at developing a measure of voting power came much later, at the hand of L.S. Penrose. In a revolutionary paper, he laid out a statistical model to determine the influence of individuals on the outcomes of collective decisions (Penrose, 1946). Geared toward solving issues related to vote weighting at the United Nations, which had been established just the year before, his approach focuses on simple majority voting in an assembly. Since then, however, it has been shown to be much more generally applicable. His approach is an a priori one, and would later be categorised as an I-Power measure by Felsenthal and Machover. Penrose used a Bernoullian model to determine the likelihood of voting outcomes, albeit not explicitly. In his model, voters only have two choices: either they vote 'yes', or they vote 'no'. Abstention is therefore not an available option. In order to predict outcomes, each voter is assumed to have a ½ chance of voting 'yes', and an equal chance of voting 'no'. His main conclusion comes down to this:

> *"In general, the power of the individual vote can be measured by the amount by which his chance of being on the winning side exceeds one half. [..] It follows that the power of the individual vote is inversely proportional to the square root of the number of people in the committee" (Penrose, 1946)*

Unfortunately, since his paper did not receive much attention (and was in fact soon forgotten), his important conclusions failed to have an impact on the weighted voting system used by the United Nations and on the European institutions that were founded in the following decades.

3.3. <u>Shapley & Shubik</u>

In 1952, Lloyd Shapley published a paper in which he describes a *value* for n-person cooperative games (Shapley, 1952). This value (later called the Shapley value) expresses the share of a fixed prize which a player is expected to receive on playing the game in question. Two years after his paper, Shapley joined forces with Martin Shubik to develop the Shapley value into an index of a priori voting power, which came to be known as the Shapley-Shubik index (Shapley & Shubik, 1954). Contrary to Penrose's a priori measure (of which Shapley and Shubik were apparently unaware), the S-S index was presented as a branch of cooperative game theory (CGT), and for most of the 20^{th} century was seen as the primary way of looking at voting power. Rather than just being interested in the passage or failure of a bill, voters were thought to maximize their share in some prize, or payoff. In the words of Shapley (1952): "[..] the acquisition of power is the payoff". This makes the S-S index a prime example of a P-Power theory. This in itself would not have been so problematic, were it not for the fact that for nearly half a century, the S-S index was seen by many as the *only* way of measuring voting power. Felsenthal and Machover (2005) list five difficulties caused by this, one of which was a reason for Coleman to come up with an alternative (see par. 3.5).

3.4. Banzhaf

Despite the work of Penrose being largely forgotten and the S-S index gaining prominence the way it did, there was in fact a measure of I-Power developed that gained some traction. John F. Banzhaf was a lawyer who contributed greatly to the thinking on the US electoral system. In his first paper on the topic of voting power, he tries to steer clear of any probabilistic assumptions: “No assumptions are made as to the relative likelihood of any combination” (Banzhaf, 1964). Here, he merely presents a *score* (later dubbed the Banzhaf *count*), in order to compare voting power between individual voters. This score is the number of combinations in which a voter is critical (i.e. when his vote can change the outcome of the election). Banzhaf does imply here that all possible voting combinations are equally likely, a statement he makes explicit in his next paper on the topic (Banzhaf, 1966). In it, he also explains how his measure of voting power is specifically a relative one. But it wouldn’t be until his third and most famous paper that he combined these aspects into his definitive measure of a priori voting power: “Since, a priori, all voting combinations are equally likely and therefore equally significant, the number of combinations in which each voter can change the outcome by changing his vote serves as the measure of his voting power” (Banzhaf, 1968). Interestingly, the relative Banzhaf (Bz) *index* (which was to become more common than the absolute Banzhaf count) was developed later by his followers, rather than by Banzhaf himself.

3.5. Coleman

One of the most important objections to P-Power measures such as the S-S index is that it is not always realistic to view the payoff or prize as a private good, to be divided among the winners. Often, it makes more sense to see it as a public good that benefits mostly people not involved in the decision-making process (consider decisions made by the UN Security Council, for instance). Coleman latched on to precisely this criticism in 1971, when he published his own alternative measure of voting power. His approach centres around three concepts: the power of a collectivity to act, the power of a member to prevent action, and the power of a member to initiate action. The first can be seen as the probability of a bill being passed, the second as the probability of a voter being critical. Coleman himself does not use these terms, but does acknowledge he implicitly uses a probabilistic model. His work, then, can be considered an I-Power measure, similar but not identical to that of Banzhaf (of whose work he seems to have been unaware). It is probably due to this last fact, in combination with the popularity of the S-S index, that Coleman's work on voting power was mostly ignored.

3.6. Deegan-Packel, Johnston and Holler

Although the voting power measures developed by Penrose, Banzhaf and Shapley & Shubik are by far the most used, over the years some other indices have been proposed that should be mentioned. The Deegan-Packel (D-P) index is explicitly of the P-Power variety (Deegan Jr & Packel, 1978, 1983). What is special about this index is firstly that instead of all winning coalitions, only *minimally* winning coalitions are counted. The reason given for this by Deegan and Packel is that any non-critical member of a winning coalition would be rejected by the others and so excluded from sharing in the payoff. This, in addition to two unexplained assumptions (considering all minimal winning coalitions equally likely and sharing the payoff equally between members of the winning coalition), unfortunately creates significant problems for this index (Felsenthal & Machover, 1998a, pp. 211-214).

The Johnston index, also created in 1978, was basically a result of the confusion between I-Power and P-Power indices that existed at the time. Johnston at one point used the I-Power Bz index (Johnston, 1977), but was severely (and mistakenly) criticised on this by Laver (1978). Laver wrongly interpreted the Bernoullian model used by Johnston as some kind of bargaining model, and compared it unfavourably to the S-S index. But instead of dealing with the objection head-on, Johnston decided to modify the Bz index to accommodate the criticism (Johnston, 1978). The result – the Johnston index – is an I-Power index with a modification based on a P-Power perspective, which makes it rather incoherent.

The last index to mention here is the one created by Holler, which he named the 'public good index' (Holler, 1982). It is based on the D-P index, but with one modification. Whereas Deegan and Packel divided the payoff (a single unit) equally between the members of a (minimally) winning coalition, Holler now awards each member the whole payoff. His reason for doing this is that the payoff is actually a public good (and therefore non-rivalrous). This seems sensible, but does carry the consequence that the reason Deegan and Packel used to justify their focus on *minimally* winning coalitions (the exclusion of non-critical members by the others, so they don't have to share the payoff with them) no longer holds. Holler does not give his own reason for limiting himself to minimally winning coalitions, nor for assigning them equal probability – which hurts the credibility of his index.

3.7. Turnovec

So far in this chapter, the I-Power/P-Power distinction made by Felsenthal and Machover (Felsenthal & Machover, 1998a; Felsenthal, Machover, & Zwicker, 1998) has been followed. It is useful in classifying both older and newer voting power indices, while making clear different indices do not necessarily measure the same thing. But the fundamental difference between I-Power and P-Power has since been disputed by Turnovec, in a number of publications (Turnovec, 2007; Turnovec, Mercik, & Mazurkiewicz, 2004, 2008).

As explained before, the difference between I-Power and P-Power is supposed to lie in what is considered to be the outcome of the voting procedure. In the first case, the outcome is simply the passing or failing of a bill. In the latter, it is the division of a 'prize', or payoff. What Turnovec does is provide his own 'generalised' power index, and subsequently show that the prime examples of I-Power and P-Power indices – the Penrose-Banzhaf and Shapley-Shubik index respectively – can be seen as applications of this generalised index under particular circumstances. While a full analysis of his approach would take too much space, a short summary can perhaps shed some light on it.

The way Turnovec tries to fuse I-Power and P-Power together is by treating the first as being about 'swings' and the second as being about 'pivots'. Both these terms describe a situation in which a voter can change the outcome, but they use a slightly different approach. For a pivot, one looks at a given permutation of voting preferences, and first orders the voters according to the intensity of their preference. Then, if the group of voters with a stronger preference than the voter in question does not have enough votes to pass the proposal, and at the same time the group of voters with a weaker preference does not have enough votes to block it, the voter is said to be pivotal. Assuming all preference orderings equally likely, the a priori voting power (i.e. the probability of a voter being pivotal) is measured by the S-S index. For a swing, on the other hand, one simply looks at every (unordered) permutation of voters. If the proponents of the bill do not have enough votes to pass it, but would have enough if the voter in question joined them, that voter is said to have a swing. The probability of this occurring comes is measured by the Penrose-Banzhaf index.

Given this interpretation of the P-B and S-S indices, the differences indeed do not seem to be as significant as before. However, it is unclear if the way Turnovec treats these indices is entirely correct. As it is, his work does not seem to have been picked up by other authors in the field, so a proper review of his methods is still lacking.

3.8. A small excursion

All theories discussed until now have been explicitly a priori, looking at the voting process before it has taken place and without using any real-world information. The work done in this field has sometimes proven useful beyond its boundaries, however. As an interesting example, this section describes some studies that have aimed to assign responsibility to voters. Especially in committee settings, with different weights for different voters, it can be informative to determine how much responsibility for a certain outcome each voter carries. But how can this be measured?

A framework to do this was developed a few years ago by Beisbart and Bovens (2009). They take as their starting point I-Power measures, as described by Felsenthal & Machover. Specifically, they focus on the Banzhaf index and the Bernoulli probability model underlying it. Next, they look at the most important factor in their model: counterfactuals. Counterfactuals describe alternate versions of events (alternate worlds, if you will), and so are vital in trying to accommodate various what-if scenarios. Ultimately, they present a general model that uses a conceptualisation of counterfactuals that is based on causal connections – not directly using the Banzhaf index anymore, but still owing much to it.

A different approach was taken by Braham and van Hees (2009). Starting out with the same goal – measuring the degree of responsibility of individual voters – they try to incorporate causality in a different way. They use what is called a NESS-test (necessary element of a sufficient set) to determine causal contributions, by applying it to a game theoretical framework. It quickly becomes clear that here too there is quite a bit of overlap with the existing I-Power index formulated by Banzhaf. And although their model is different from the one built by Beisbart and Bovens, Braham and Van Hees also present their conclusion as “a generalised version of the normalized Penrose-Banzhaf index of voting power” (Braham & van Hees, 2009). As an interesting side note, Braham and Van Hees actually received some criticism on an earlier version of their paper from Felsenthal and Machover (2009), leading them to make some changes to the way they dealt with the NESS-test.

All of this makes clear that the indices described are powerful, and carry influence beyond just those wishing to measure a priori voting power. Since all the discussion so far has necessarily been rather superficial, the next chapter will explain the workings the two most prominent indices – the Penrose-Banzhaf index and the Shapley-Shubik index – in greater detail.

4. Voting power measures in detail

As has been stated earlier in this thesis, there are two fundamentally different ways to look at voting power – as I-Power, and as P-Power. This chapter explains for each the most prominent index: the Penrose-Banzhaf index and the Shapley-Shubik index, respectively. Most of what is presented in this chapter is based on Felsenthal and Machover (1998a). Before starting describing the indices, however, it is important to explain some basics.

The voting power theories discussed here make use of (and are best explained by using) a simple mathematical structure called a *simple voting game* (SVG). An SVG is a collection *W* of subsets of a finite set *N*. It is defined by the following criteria (Felsenthal & Machover, 1998a, p. 11):

1. $N \in W$
2. $\emptyset \notin W$
3. *Monotonicity*: whenever $X \subseteq Y \subseteq N$ and $X \subseteq W$ then also $Y \in W$.

W is considered to be a proper SVG if it also satisfies a fourth criterion (and *improper* if not):

4. Whenever $X \in W$ and $Y \in W$ then $X \cap Y \neq \emptyset$.

N is also called *W*'s *assembly* or *grand coalition*, and is in fact the set of all voters. Any subset of *N* is called a *coalition* of *W*. Whether a coalition *S* is *winning* or *losing* depends on whether it is a member of *W*: if $S \in W$, it is a winning coalition; if $S \notin W$, it is a losing coalition.

Finally, the *characteristic function* (CF) of *W* is the map *w* from the set of all coalitions of *W* (the power set of *N*) to {0,1} such that, for any coalition *X*:

$wX = 1$ if $X \in W$

$wX = 0$ if $X \notin W$

wX is called the *worth* of *X*.

Now that the SVG structure and terminology is clear, we can turn to the indices.

4.1. I-Power: Penrose & Banzhaf

Since he was the first to mathematically formalise a notion of a priori voting power, it makes sense to start this detailed overview with L.S. Penrose's 1946 paper. In this, he assumes other voters to be 'indifferent' (i.e., independent, not forming blocs) and to vote 'randomly'. As stated before, Penrose implicitly uses a Bernoullian probability model, in which voters vote either 'yes' or 'no' with probability $\frac{1}{2}$. This all implies that each of 2^n possible divisions of a set of *n* voters is equally likely. So what is the amount of voting power of a given voter? Penrose defines this as "the amount by which his chance of being on the winning side exceeds one half" (Penrose, 1946). 'Being on the winning side' should here be interpreted as the collectivity voting the same way the voter in question votes. Penrose goes on to say that this measure of voting power is exactly equal to half the probability of the voter in question being critical. Using the notation of Felsenthal & Machover (2005), we can now write the voting power of a voter a (which they call 'Penrose's identity') as follows:

$$r_a - \frac{1}{2} = \frac{\psi_a}{2}$$

A few years after his 1946 paper, however, Penrose apparently changed his mind somewhat, and decided to multiply his measure by two, so that we get:

$$2r_a - 1 = \psi_a$$

This means Penrose's measure is now exactly equal to the probability of a voter being critical. To see how this relates to that other prominent measure of a priori voting power, we turn to Banzhaf's work. As mentioned in the previous chapter, his voting power measures centre around what is called the Banzhaf *score* or *count*: the number of coalitions in which a voter is critical. This *Bz score* therefore can be defined as a function η, which assigns to any voter a in an SVG W a value $\eta a[W]$ (called the Bz score of a in *W*). $\eta a[W]$ represents the number of coalitions of W in which voter a is critical. Using the Bz score, we can define the (relative) *Bz index of voting power*:

$$\beta_a[W] = \frac{\eta_a[W]}{\sum_{x \in N} \eta_x[W]}$$

In addition, we can use $\eta a[W]$ to build the (absolute) *Bz measure of voting power*:

$$\beta'_a[W] = \frac{\eta_a[W]}{2^{n-1}}$$

What may not be obvious at first sight is that the latter measure β'_a is equal to the probability to voter a being critical, and therefore also to Penrose's measure ψ. For a proof of this equivalence, see Felsenthal & Machover (1998a, pp. 45-46).

Some of the confusion that existed in the literature (mostly before the 1990s) had to do with the question which of these measures was the primary one. Authors sometimes mentioned only one, and (apparently) chose to ignore the other. In recent years, most authors seem to agree that it is the second measure, β', which is primary. The reason for this is that β can simply be obtained by *normalising* the values of β' for all voters, so that they add up to 1. β' then can be seen as the primary, absolute I-Power measure of a priori voting power, while the relative β is merely a derivative: it measures not the amount of power a voter has, but his share of total power.

The importance of β' can be further illustrated by two intuitive explanations of what it represents. Firstly, it describes the a priori probability that, if *a*'s vote were to be switched once the votes have been cast, the result would switch along with it. A similar explanation is that β' describes the conditional probability, given that we know how *a* will vote, that the result would be switched if *a* were to change his mind. So if we were to put it crudely, β' measures the amount of power a voter has, whereas β only measures the voter's share of total power.

Example

Suppose *W* is a Weighted Voting Game (WVG) with assembly *N* = {a,b,c,d}, weights of (respectively) 3,2,1,1, and a quota of 5. To calculate the Bz measure (which as mentioned is equal to Penrose's measure) for each voter, we first have to list all possible coalitions which have at least one critical member. The following five coalitions are the only ones (out of a total of 16) that meet this criterion:

{a,b}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}

The critical members have been underlined for clarity. To calculate β' for each voter, we simply count the number of coalitions in which that voter is a critical member, and divide that by 2^{n-1}. This gives us:

$$\beta'_a = \frac{5}{8}\ ,\ \ \beta'_b = \frac{3}{8}\ ,\ \ \beta'_c = \frac{1}{8}\ ,\ \ \beta'_d = \frac{1}{8}$$

From this we can calculate the Bz index value for each voter, by normalising the β' values so they all add up to 1. Since the total amount of power is $\frac{10}{8}$, we simply divide each value by that number, so we get:

$$\beta_a = \frac{1}{2}\ ,\ \ \beta_b = \frac{3}{10}\ ,\ \ \beta_c = \frac{1}{10}\ ,\ \ \beta_d = \frac{1}{10}$$

4.2. P-Power: Shapley & Shubik

Having seen how Penrose's and Banzhaf's measures of work, we can now turn to the other approach to measuring voting power: the P-Power index of Shapley & Shubik. The vital difference here, as stated before, is that voters are no longer assumed to be policy-seeking (as in I-Power), but office-seeking. If the outcome of the voting procedure is positive, the voters belonging to the winning coalition (the ones who voted 'yes') divide among themselves a fixed amount of transferable utility: the *prize*. These voters all receive a non-negative payoff, whereas the rest of the voters receive 0. If the outcome was negative to begin with, none of the voters receives any payoff. P-Power can now be described as a voter's expected payoff (a priori). Note that, as stated by Coleman (1971), this is scenario is clearly not a very realistic one. However, that is not needed for the theory to be effective. One can even assume the bill to be decided upon is proposed by a certain coalition *S*, and states explicitly which voter in coalition *S* gets what (i.e., which part of the total prize) if it is accepted. Since it only benefits members of coalition *S*, no other voter would support the bill (on assumption of self-interested behaviour). And because the bill only benefits *S*'s members if it is passed, it would only be proposed if *S* is a winning coalition. This again is not a realistic description of actual voting processes, but it does offer an intuitive way to understand the workings of a P-Power theory.

As explained in the previous chapter, the theory developed by Shapley & Shubik centres first of all around the so-called Shapley value (ϕ). This is a general concept, which can be applied to many types of games – not just voting games. For present purposes, relevant definitions are used as follows Felsenthal and Machover (1998a, p. 177):

> *By a* game *on a nonempty finite set* N *we shall mean a real-valued function* w *whose domain is the power set (that is, the set of all subsets) of* N *such that* $w\emptyset = 0$. *We refer to any member of* N *as a* coalition *[of* w*].* N *itself is called the* grand coalition *[of* w*]. If* S *is a* coalition*, the real number w*S *is called the* worth of S *[in* w*].*

Using this as a framework, the Shapley value for a player *a* in a game w can be calculated via the following formula:

$$\phi_a(w) \ = \frac{1}{n!}\sum_{X\subseteq N}(|X|-1)!\,(n-|X|)!\,(wX - w(X-\{a\}))$$

Although based on the Shapley value, the S-S index is not a general measure, but one specifically aimed at measuring voting power. This is because it is, in fact, an *application* of the Shapley value to the case of voting games. This becomes clear immediately when looking at the formulas used to calculate the S-S index. First, there is the *S-S score*. This is a function κ which assigns to any voter *a* in an SVG W a value $\kappa_a[W]$. This is called the *S-S score of a* [in *W*]:

$$\kappa_a[W] \ = \sum_{X\subseteq N}(|X|-1)!\,(n-|X|)!\,(wX - w(X-\{a\}))$$

Here, *w* if the CF of *W*.

Using the S-S score, the S-S index is now simply calculated as follows:

$$\phi_a[W] = \frac{\kappa_a[W]}{n!}$$

The S-S index is, then, a sensible way to calculate the voting power of individual voters – as long as that is considered to be the expected share in a payoff they will receive. The index is a straightforward application of the Shapley value calculation; the only added requirement is that *w* is the CF of an SVG *W*.

Example

To calculate the S-S index of the voting game from the previous section's example, we have to consider *w*. This can only take two values: 1 for a winning coalition, 0 for a losing one. Therefore, a voter *x* can only contribute to the worth of coalition *X* if *x* is critical in that coalition. The consequence of this is that the only relevant coalitions are the one with at least one critical voter:

{a,b}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}

As the contribution of any critical member to a coalition here is always 1, multiplying those contributions by $(|X| - 1)!\,(4 - |X|)!$ (*n* = 4 in this case) gives us this sum for voter *a*:

$$\kappa_a[W] = 2 + 2 + 2 + 2 + 6 = 14$$

So we can now calculate the S-S index for voter *a*, and, using the same method, for the other voters as well:

$$\phi_a[W] = \frac{14}{24} = \frac{7}{12}\ ,\ \ \phi_b[W] = \frac{1}{4}\ ,\ \ \phi_c[W] = \frac{1}{12}\ ,\ \ \phi_d[W] = \frac{1}{12}$$

As should be, the S-S index values add up to 1, which corresponds to the fact that a voter's voting power is here his expected share of the prize.

5. Criticisms

Although the theories discussed in chapter 3, and the ones explained in chapter 4 in particular, have been applied to real-world scenarios on many occasions, they are not beyond criticism. Various arguments have been made in the literature against voting power indices; some aimed at a particular index, but most aimed at the general basis for such indices. This chapter will discuss some of the criticisms that have been levelled at voting power theories, along with some general suggestions on how these should be dealt with. The first few sections relate to the distinctions made in chapter 2.

5.1. Relevance of a priori theories

The first criticism is also the most fundamental one. As explained in chapter 2, there is a vital difference between *a priori* and *a posteriori* voting power. A priori voting power theories take into account only the (formal) system in which the voting takes places; they ignore many factors that influence voting in real-world situations. One way to view this issue is as Albert (2003) did: as a fundamental scientific shortcoming. Albert rejects a priori voting power theories on the basis that they do not meet the criteria for (positive) scientific theories. According to him, the central flaw here is that voting power theories do not have any factual content, and are therefore unsuited for purposes of prediction or explanation – which he considers the goal of any scientific theory. He finds terminology such as ‘measuring voting power’ misleading, again because there is supposedly nothing in the world that is measured in the strict, empirical sense of the word. In addition, he objects to the *Principle of Insufficient Reason* (PIR), which lies at the heart of theories such as Penrose’s. PIR is the basis for the assumption that voters vote randomly, and is one that has been shown not to be a good approximation of real-world voting (Gelman, Katz, & Bafumi, 2002). However, the conclusion that either of these arguments mean that a priori voting power measures cannot be used in real-world situations has been disputed (Felsenthal, Leech, List, & Machover, 2003).

Another way to look at this is to focus on the specific factors that a priori measures fail to incorporate. An important example of this are voters’ preferences, which are in fact ignored – a point strongly made by some authors (Garrett & Tsebelis, 1999a, 1999b). These authors also point to other aspects that are not accounted for, such as connected coalitions, and agenda-setting. However, as with the previous arguments, these have more or less been rebutted (Holler & Widgrén, 1999). It is hard, therefore, to properly weigh the arguments made with respect to the (perceived) shortcomings of a priori theories. Perhaps the clearest argument is a general one: the relevance of a priori voting power depends on the context in which it is measured. It is most relevant in situations where the institutions are yet to be designed (e.g. when a new supra- or international organisation is founded). It is then that there really is no previous data on voter behaviour. However, even then one might want to consider aspects that may very well occur in the long term, such as relatively stable coalitions. When looking at real-world voting situations that do have a history (say, U.S. presidential elections), it would be too simplistic to *only* look at an a priori measure to see if the system is fair. Once certain divisions are known, and they turn out to be (in many places) quite stable, it makes sense to consider this when analysing voting power. Perhaps the system was much fairer on the drawing board than in reality. In all, one should be aware that a priori theories have certain shortcomings, but should not discard them entirely for that reason.

5.2. I-Power or P-Power

In previous chapters, much has been said about the distinction between I-Power and P-Power, made first by Felsenthal and Machover (1998a). Leaving the criticism on the correctness of the distinction itself by Turnovec et al. (2004, 2008) aside, it is interesting to consider the implications of such a fundamental division of the theoretic body on voting power. Since I-Power and P-Power approaches purport to measure two different things, it seems sensible to suppose there is only one 'proper' type to apply to any given voting situation. Either voters care about a bill being passed, or they care about the share in some payoff they will receive after voting. In the first case one would use an I-Power index, in the second a P-Power index. However, reality is far less clear-cut. Perhaps the only clear case is if the voting is actually *about* some definite payoff to be divided among the winners. In every other case, voters are likely to experience a little bit of both sentiments: they want the bill passed, but they also care about what that means for them. This ambiguity is exacerbated by phrasing which suggests the payoff in P-Power indices (specifically the S-S index) can also be abstract 'power'. This implies that even in cases where motivations for voting seem to be restricted to one's views on a specific issue, the consequences in terms of power are relevant as well. Members of parliament might vote for a bill on abortion, say, not just because they agree or disagree with the bill itself. They might at the same time be considering what it means for them and their party in terms of power to win or lose the vote in question. This makes it quite clear that in hardly any situation does the distinction between I-Power and P-Power, which seems so sensible when looking at theories in isolation, dictate a choice for a specific type of theory to use. This calls into question the relevance of any application of voting power theory: if both types of theory describe some aspect of the voting, how can conclusions be drawn at all? Such a problem seems to call for a new theory that somehow combines the two types, or at least analyses that use both and subsequently try to weigh the importance of each. Or perhaps it will turn out Turnovec is right in saying there really *is* no distinction between I-Power and P-Power, and voting power theorists will converge on a single view. Either way, the distinction as it exists is problematic when trying to judge the outcomes of voting power research.

5.3. Critical or minimal coalitions

Another distinction that deserves some extra attention is that between *critical* and *minimal* coalitions. Most theories discussed consider critical coalitions to be the only ones that matter: as long as some division of voters contains at least one voter who could change (or could have changed) the outcome by switching his vote, that division is relevant to measuring the amount of voting power he has. However, depending on the context, there is something to be said for focussing on minimal coalitions instead of critical ones. In minimal coalitions, *every* member is critical: were any member to change his vote, the outcome would be changed. It was first shown by Riker (1962) that politicians might rationally choose to form minimal winning coalitions instead of trying to gain as many votes as possible. The voting power index invented by (Holler, 1982, 1998), which was discussed in section 3.6, actually uses minimal coalitions to measure voting power. His approach has been criticised by Machover (2000), but those criticisms have themselves been criticised as well (Bertini, Gambarelli, & Stach, 2008; Holler, 2001).

What seems clear is that there are some contexts in which it does *not* make much sense to prefer minimal over critical coalitions. In situations where voters act independently and there is no form of bargaining whatsoever, critical coalitions are what matter. However, in any context where bargaining does take place (say, the Council of Ministers of the EU), one could very well argue that minimal coalitions are so much more likely to occur than coalitions with ‘surplus’ votes, that those should be the starting point of any voting power measurement. Interestingly, the literature seems divided on the question of whether a index such as Holler’s should be categorised as I-Power or P-Power. This gives extra force to the argument made in section 5.1 that that distinction is somewhat questionable. Now we find that the common choice for critical coalitions is not as unproblematic as previously thought, either.

5.4. Voting paradoxes

The last section of this chapter has a topic seemingly less close connected to voting power theories: voting paradoxes. Although written about by many authors, they are usually limited to counterintuitive or illogical outcomes of certain voting procedures. As such, voting paradoxes as they are normally conceptualised are not entirely the same as paradoxes in voting power measures. Nonetheless, they have to be considered when judging the merits of such measures. Some authors have studied the intrinsic problems that voting power indices can have in this respect (Felsenthal & Machover, 1998a, 1998b; Laruelle & Valenciano, 2005; van Deemen & Rusinowska, 2003), and although it is not possible to dissect such studies here, we can say that the results have been mixed. Some indices suffer more than others, but no index is wholly free from (apparent) paradoxes.

This adds to the points made in the previous sections of this chapter: no index is perfect. When using any index, its shortcomings should be taken into careful consideration. Does it not leave out too many factors as an a priori theory? Does it neglect part of the voters' behaviour by being squarely in either the I-Power or the P-Power camp? Does it merely consider critical coalitions, when it should also (or only) count minimal ones? Does it potentially produce unacceptable paradoxes? Researchers should consider answering such questions answered when performing a study of voting power using any particular index. But perhaps some of these criticisms can be resolved by a new voting power index, yet to be invented. We can assume that no theory would ever be completely free of criticisms, but it might be possible to improve existing measures in a way that would greatly enhance their credibility.

6. Conclusion

The field of a priori voting power theories is not an uncomplicated one. Partly due to past confusion among authors, and partly due to rivalling theories addressing only part of the issue, there has never been a single leading approach to measuring voting power. This thesis has shown that various indices use different ways to try and accomplish the same thing. Felsenthal and Machover made a noteworthy attempt to establish a rough categorisation of theories using their concepts of I-Power and P-Power. Following this, one can definitely see the fundamental differences that seem to exist between some indices. And so some order is created in the long historical progression from author to author as presented in chapter 3. Laying the most prominent examples of I-Power and P-Power (the Penrose-Banzhaf index and Shapley-Shubik index respectively) side by side in chapter 4, it became clear that both approaches are plausible and mathematically coherent. This does not mean there are no shortcomings to consider, however. Chapter 5 latched on to this and showed that no aspect of voting power theory is entirely unproblematic. The indices can be powerful, and applicable to scenarios beyond what one would consider their home turf. But it in hardly any situation is it entirely clear which index should be used. The I-Power/P-Power distinction is useful in categorising theories, but questions can be raised on whether it does not also imply any single voting power index leaves something out. Moreover, without exception each index carries with it certain shortcomings, which cannot be avoided. If an index uses critical coalitions, it is possible that using minimal coalitions would have been better. Every measure is vulnerable to at least some voting paradoxes, but those are usually not given attention when actually applying it. Finally, there is the fundamental criticism to a priori theories: they do not take into account any real-world information beyond the bare minimum. Unless one is designing a new voting procedure from scratch, this is a significant objection to consider.

It is not all doom and gloom for a priori voting power theories, however. Over the past 60 years or so, the field has managed to grow from non-existent to (sometimes) quite influential in matters of voting. Theories and indices have been invented that provide a valuable insight into the positions individual voters actually have in voting situations. And although their prescriptions are not considered law yet, it seems unlikely any new institution would ‘accidentally’ set up its voting procedures in such a way that a member has absolutely no influence on the outcome (as happened to Luxemburg in the EEC in 1958). One can only hope that these theories will continue to develop, so that they can address some of their shortcomings and gain an even stronger and more valuable place in decisions on voting systems.

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