An optimal solution for the Uncapacitated Facility Location Problem using the dual formulation

Eva Barendse

360542

Supervisor: R. Spliet

Abstract

In this paper we develop a simple method which provides a solution for the Uncapacitated Facility Location Problem (UFLP). This method is based on two procedures, namely the dual ascent procedure and the dual adjustment procedure. The dual ascent procedure is executed first, if this procedure does not find an optimal solution, then this solution is improved by the dual adjustment procedure. We test this method on three standard problem instances of the UFLP, namely the Bilde-Krarup, the Galvão Raggi and the Gap B instances in order to check if this method works for different instances. We also check what happens to the number of established facilities and the total costs if the opening costs and the transportation costs increase or decrease.

Contents

1. Introduction

This paper describes and analyses a dual-based algorithm for the Uncapacitated Facility Location Problem (UFLP) based on the paper of Erlenkotter (1978). The UFLP is the decision problem where facilities or warehouses should be opened and where customers are assigned to these facilities in such a way that the total costs are minimized. In this case, the total costs consist of the costs for the transportation of each unit from facility location *i* to customer *j* and the fixed costs for opening the facilities that are necessary to satisfy the demand of the customers. There are *m* uncapacitated facilities or warehouses available and *n* customers or demand locations for which the demand should be satisfied by the warehouses.

The UFLP is one of the most studied location problems and its applications are used in a wide variety of settings. Examples of these settings are, according to Lazic, Frey and Aarabi (2010), a distribution system design (Klose and Drexl, 2003), self-configuration in wireless sensor networks (Frank and Romer, 2007), computational biology (Dueck et al., 2008) and computer vision (Li, 2007; Lazic et al., 2009). There are also several different types of the UFLP, depending on, for example, the number of facilities and customers, the objective function and the time horizon which is considered. A lot of extensions are possible for this problem, for instance the p-median location problem, where the number of opened facilities is restricted, and the capacitated facility location problem where the facilities can supply a capacitated amount of units.

A solution to the UFLP can be obtained by solving a mixed integer problem (MIP), from which an integer solution is obtained. Another way to solve the UFLP is to solve the dual problem which gives a dual solution that corresponds with a lower bound for the solution of the UFLP. From this dual solution it is possible to obtain a feasible integer primal solution using complementary slackness relationships for the solutions obtained by the linear programming problem, this feasible integer primal solution corresponds with an upper bound for the solution of the UFLP. In general, dual problems are solved with a so-called simplex method, but because the specific dual problem has a very simple structure and has multiple solutions, a simpler method might be possible.

This simpler method is described in the paper of Erlenkotter (1978) and is based on two components. The dual ascent procedure is the first component and with this procedure, a lower bound on the solution of the UFLP is obtained and from this solution we can obtain an integer primal solution that corresponds with an upper bound on the solution of the UFLP. The second component is the dual adjustment procedure, which is only used if the solution obtained by the dual ascent procedure violate the complementary slackness conditions, hence when this solution is not optimal. With the dual adjustment procedure, the solution obtained by the dual ascent procedure is improved such that we can found a higher lower bound on the solution. When we obtain a primal solution from this procedure, this gives a lower upper bound, hence the gap between the lower bound and the upper bound becomes smaller due to the dual adjustment procedure. If both components of the method do not result in an optimal integer solution for the UFLP, then a branch-and-bound procedure is executed where the solutions obtained by the two components are used as bounds.

In this paper, the simpler method is performed and the results are observed and analysed. The method is tested on three standard instances of the UFLP, namely the Bilde-Krarup, the Galvão Raggi and the Gap B instances. It is checked whether the solutions found by the algorithm are the same as the benchmark solutions given in the standard problem instances.

It is also checked what happens to the number of established facilities and the total costs if the opening costs and transportation costs decrease or increase with a certain percentage. It is interesting to observe what happens to the total costs and the number of established facilities if a company increases its transportation costs or if it becomes more expensive to open a facility. What we expect in this case is that when the opening costs rise, then there should be less facilities established and the total costs will increase as well. And when the transportation costs increase, then more facilities should be established in order to reduce the distance between the facilities and the customers which results in an increase of the total opening costs. When the opening or transportation costs decrease, then we expect to obtain the same pattern but in reversed order.

2. Problem description

As described earlier, the UFLP is the decision problem where facilities should be opened and where customers are assigned to these facilities in such a way that the total costs are minimized. This problem can be described by a mixed integer problem (MIP), which is done in this Section. The *m* possible facilities are described by the set I and the *n* demand locations or customers are described by the set J. We define two variables, namely x_{ij} and y_i. The variable x_{ij} gives the fraction of the demand of location *j* supplied by facility *i* and the variable y_i is equal to one if facility *i* is opened and zero otherwise. The total costs for supplying all demand of location *j* from facility *i* are defined by the parameter c_{ij}, this includes all different costs such as the storage and transportation costs. The parameter f_i represents the fixed costs for opening facility *i*. Finally, the model is formulated in the following way:

$$\min z_p = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i \quad (1)$$
$$\sum_{i \in I} x_{ij} = 1, \quad j \in J \quad (2)$$
$$y_i - x_{ij} \geq 0, \quad i \in I, j \in J \quad (3)$$
$$x_{ij} \geq 0, \quad i \in I, j \in J \quad (4)$$
$$y_i \in \{0,1\}, \quad i \in I \quad (5)$$

The objective function (1) of this problem minimizes the total costs, which are divided into two parts, namely the total costs for supplying the demand of location *j* from facility *i* and the fixed costs for opening facility *i*. The total costs are given by the primal objective value z_p. Constraint (2) ensures that all demand of location *j* is satisfied. It is only possible to supply from facility *i* if that facility is opened, this is ensured by constraint (3). The variable x_{ij} is a positive fraction because there cannot be a negative demand which is given by (4). Finally, because a facility is opened or not, the variable y_i is a binary variable which is indicated by constraint (5).

In order to obtain an integer solution, the linear programming relaxation has to be solved. In this LP relaxation, constraint (5) is replaced by the weaker constraint:

$$y_i \geq 0, \; i \in I \quad (6)$$

Now, the variable y_i can adopt continuous values above zero. With the LP formulation (1)-(4) and (6), it is possible to define the dual problem of the UFLP which is described in the following Section.

3. Methodology

In order to obtain a lower bound and an upper bound on the solution of the UFLP, the dual formulation of the LP relaxation needs to be defined. The same definition for the parameters c_{ij} and f_i as in the MIP is used also for the dual formulation. Furthermore, we define the dual variables v_j and w_{ij}. The dual variable v_j corresponds to constraint (2) and the dual variable w_{ij} corresponds to constraint (3) in the LP formulation. According to Jain and Vazirani (2001), v_j can be interpreted as the total price paid by demand location *j* in order to get connected to some facilities and w_{ij} defines the contribution of demand location *j* for opening facility location *i*. Thus, the difference between v_j and w_{ij} is that v_j represents the total costs for demand location *j* independent of the facilities to which *j* is connected, and w_{ij} represents the price paid by demand location *j* towards opening a specific facility *i*. Suppose that demand location *j* is only connected to facility *i*, then *j* does not contribute in order to open any other facility except *i*, hence $w_{i'j}$ is equal to zero. Thus, the variable w_{ij} refers to the edge *(i,j)* and v_j refers to the demand location *j*. The dual problem is formulated as follows:

$$\max z_D = \sum_{j \in J} v_j \quad (7)$$

$$\sum_{j \in J} w_{ij} \leq f_i, \; i \in I \quad (8)$$

$$v_j - w_{ij} \leq c_{ij}, \; i \in I, j \in J \quad (9)$$

$$w_{ij} \geq 0, \; i \in I, j \in J \quad (10)$$

The objective (7) is to maximize v_j subject to the constraints (8)-(10). The contribution of all demand locations for opening facility location *i* cannot be larger than the fixed costs for opening that facility, this is ensured by (8). The difference between the total price paid by demand location *j* and the contribution of that demand location for opening facility *i* can never be larger than the total costs for supplying all demand of location *j* from facility *i*, so for using edge *(i,j)*, which is ensured by constraint (9). Because the contribution of demand location *j* for opening facility location *i* cannot be negative, w_{ij} is set as a non-negative value by constraint (10). The variable v_j must increase until demand location *j* is connected to an open facility, this can be seen as a shadow price paid by the demand location in order to get connected to an open facility location. Because we want v_j to be as high as possible, the variable w_{ij} must be as low as possible according to constraint (9), hence we can assume:

$$w_{ij} = \max\{0, v_j - c_{ij}\} \quad (11)$$

The proof of (11) is given as follows:

From constraint (9) we know that: $v_j - w_{ij} \leq c_{ij}$, this is equal to the inequality $v_j - c_{ij} \leq w_{ij}$. According to (10) we know that w_{ij} must be positive or zero, hence the inequality $v_j - c_{ij} \leq w_{ij}$ is always satisfied when $v_j - c_{ij}$ is also positive or zero. When the difference between v_j and c_{ij} is negative, then w_{ij} must be equal to zero. From this we obtain that w_{ij} is equal to the maximum of zero and $v_j - c_{ij}$. This equality holds because when w_{ij} is lower than $\max\{0, v_j - c_{ij}\}$, then it could be that, if the maximum of zero and $v_j - c_{ij}$ is equal to zero, w_{ij} takes a negative value which is in contradiction with (10). And when w_{ij} is higher than $\max\{0, v_j - c_{ij}\}$, then there is room to decrease w_{ij} until constraint (9) is not satisfied anymore. In addition, we want that v_j is maximized so according to constraint (9), w_{ij} should be as low as possible. These reasons indicate that the value of w_{ij} must be exactly equal to the maximum of zero and $v_j - c_{ij}$.

When (11) is substituted in (8), we get the following simplified dual problem:

$$\max z_D = \sum_{j \in J} v_j \qquad (12)$$

$$\sum_{j \in J} \max\{0, v_j - c_{ij}\} \leq f_i, \quad i \in I \qquad (13)$$

This simplified dual problem is exactly the same problem as the problem described by the formulation (7)-(10). The objective function (12) remains the same as (7). Then, a substitution of (11) in (8) results in the constraint (13). Constraints (9) and (10) of the original dual problem are also represented by constraint (13) through equality (11). This is validated because when (9) is rewritten into $v_j - c_{ij} \leq w_{ij}$, it is clear that this constraint is always satisfied when w_{ij} is defined as the maximum of zero and $v_j - c_{ij}$, so constraint (9) disappears. Also constraint (10) is always satisfied when w_{ij} is defined as in (11), because the maximum of zero and $v_j - c_{ij}$ is always equal to zero or larger than zero. Z_D is the dual objective value which has to be maximized subject to the constraint (13).

This dual problem can be solved by, besides using a simplex method, a simpler method which uses the dual ascent procedure and the dual adjustment procedure. With the dual ascent procedure, the dual solution gives a lower bound on the solution of the UFLP and the corresponding primal integer solution gives an upper bound on the solution, hence a range is obtained in which the optimal solution must be located. With the dual adjustment procedure, the solutions obtained by the dual ascent procedure are improved such that a higher, hence better, lower bound and a lower, hence better, upper bound on the solution are found. Thus, this procedure aims to close the gap between the primal and dual solutions, this gap is called the duality gap. This dual adjustment procedure is only used if the dual and primal solutions obtained by the dual ascent procedure violate the following complementary slackness conditions:

$$y_i^*[f_i - \sum_{j\in J} \max\{0, v_j^* - c_{ij}\}] = 0 \qquad (14)$$

$$\left[y_i^* - x_{ij}^*\right] \max\left\{0, v_j^* - c_{ij}\right\} = 0 \qquad (15)$$

Where y_i^* and x_{ij}^* are feasible primal solutions and v_j^* is a feasible dual solution. When the solution found by the dual ascent procedure violates the complementary slackness conditions, then the solution is not optimal and we are going to improve the solution using the dual adjustment procedure. However, if the solution found by the dual ascent procedure satisfies the complementary slackness conditions, then this solution is an optimal solution and there is no need to search for another solution. In that case, the dual adjustment procedure is not executed.

3.1 The dual ascent procedure

The dual ascent procedure provides a lower bound and an upper bound on the solution of the UFLP. This procedure starts with any feasible dual solution V, for example when all v_j are set equal to zero, and tries to increase this solution to the next higher value of c_{ij} by cycling through the demand locations j. In this case, c_{ij} are the total costs of supplying all demand of location j from facility i, and v_j is an element of the solution V. At the beginning of the procedure, we reindex c_{ij} in non-decreasing order for each j as c_j^k with k=1,…,m. We reindex c_{ij} because then all costs are ordered and it is possible to move through the sequence of costs when we go from k to $k+1$ for example. As initial feasible dual solution we use $v_j = c_j^1$, hence the lowest value of c_{ij} for each j. We initialize the subset of demand locations J^+, which we are going to use in the dual adjustment procedure. In this set of demand locations, the locations are included for which we want to increase v_j. But in this case, we initialize J^+ with the set of all demand locations J, hence $J^+ = J$.

In the first lines of the procedure, we introduce s_i and $k(j)$. In this algorithm, the variable s_i checks if there is some slack on constraint (13), hence if it is possible to increase v_j. The variable $k(j)$ is used to move to the next higher value of the costs c_j^k such that we can increase v_j with the difference between these costs and v_j. We describe the algorithm for the dual ascent procedure as follows:

1. $v_j = c_j^1, for\ each\ j \in J$

2. $s_i = f_i - \sum_{j \in J} \max\{0, v_j - c_{ij}\} \geq 0, for\ each\ i \in I$

3. $k(j) = \min\{k: v_j \leq c_j^k\}\ if\ v_j = c_j^{k(j)},\ then\ k(j) = k(j) + 1$

4. $\delta = 1$

5. While $\delta = 1$

6. $j = 1$ and $\delta = 0$

7. If $j \in J^+$

8. $\Delta_j = min_{i \in I}\{s_i: v_j - c_{ij} \geq 0\}$

9. If $k(j) \leq m$

10. If $\Delta_j > c_j^{k(j)} - v_j$

11. $\Delta_j = c_j^{k(j)} - v_j$

12. $\delta = 1$

13. $k(j) = k(j) + 1$

14. end

15. For each $i \in I$:

16. If $v_j - c_{ij} \geq 0$

17. $s_i = s_i - \Delta_j$

18. end

19. end

20. $v_j = v_j + \Delta_j$

21. If $j \neq n$

22. $j = j + 1$

23. end

24. end

25. end

26. end

The initialization of v_j, s_i, $k(j)$ and δ occurs in lines 1 to 4. Our objective is to maximize v_j, thus we have to increase v_j until it is no longer possible to increase v_j because then constraint (13) is violated. Variable s_i in line 2 checks if it is possible to increase v_j without violating constraint (13). As long as δ is equal to one, the procedure in lines 6 to 26 is executed.

The algorithm starts with a value of *k,* for each $j \in J$, for which the difference between c_{ij} and v_j is very small because c_j^k is defined as all c_{ij} in non-decreasing order and the value of *k* is chosen as low as possible. This value of *k* is defined as *k(j)* in line 3. This *k(j)* is an indicator for the next higher value of c_j^k for which c_j^k is larger or equal to v_j. During the procedure, the value of *k(j)* is updated by one several times, such that the algorithm jumps to the next higher value in the sequence of all c_j^k. When the value of v_j is in the beginning exactly equal to the value of $c_j^{k(j)}$ then *k(j)* is updated by one such that there is a difference between v_j and $c_j^{k(j)}$. If the last c_j^k in the sequence of costs is considered, then the procedure proceed to the next value of *j*, or terminates if the last *j* is already considered, because there is no higher c_j^k which can be considered. This is stated in line 9.

The algorithm checks for each demand location $j \in J^+$ if it is possible to increase v_j without violating constraint (13). First, the value of Δ_j is determined in line 8, this value is equal to the minimum value of s_i given that v_j is larger or equal to c_{ij}. If Δ_j is larger than the difference between $c_j^{k(j)}$ and v_j, then Δ_j is set equal to $c_j^{k(j)} - v_j$ according to lines 10 and 11. Furthermore, when Δ_j is adopted in this way, δ is set equal to one in line 12 and *k(j)* is updated by one in line 13 such that we are going to consider a higher value of the costs for demand location *j* in the next iteration. The value of δ is set equal to one because there is an increase in v_j possible, hence the algorithm must continue which is ensured by the while statement in line 5.

For each facility location $i \in I$ it is checked in line 16 whether v_j is larger or equal to c_{ij}. If this is the case, then there is room to decrease the value of s_i. When δ is equal to one, it is already discovered that it is possible to increase v_j. But when v_j becomes larger, and when v_j is larger or equal to c_{ij}, then s_i must decrease according to the definition of s_i in line 2. In line 17, s_i is decreased by Δ_j such that s_i remains always a non-negative value, because the value of Δ_j never exceeds the value of s_i which is ensured by the definition of Δ_j. Then after all those moves, v_j is increased by Δ_j in line 20.

This procedure increases the value of v_j if it is possible, but if constraint (13) of the dual problem blocks the increase of v_j, then the maximal level of v_j allowed by the constraint is reached and the procedure terminates. After performing this procedure, a dual solution V^+ is obtained and an associated set of established facility locations I^+ is constructed, where v_j^+ is an element of the solution V^+ and *i* is an element of the set I^+. The way in which we construct the set I^+ and a corresponding primal solution is described in Section 3.2.

If the solution obtained by the dual ascent procedure violates the complementary slackness condition (15), then the solution is not optimal and we try to improve the solution using the dual adjustment procedure. Note that the complementary slackness condition (14) is always satisfied, which is explained in Section 3.2.

3.2 Opening facilities and obtaining a primal solution

After performing the dual ascent procedure, we obtain the dual solution V^+, which is a sequence of values of v_j^+ for each demand location *j*. Furthermore, we obtain for each facility location the slack variable s_i. Now we want to know which facilities should be opened. Before we determine this, we have chosen to set as goal that the complementary slackness conditions (14) and (15) should be satisfied as much as possible. We have chosen this aim because when the complementary slackness conditions are satisfied, then we obtain an optimal solution, hence when they are satisfied as much as possible, we obtain a solution which is as close as possible to the optimal solution.

We have chosen to store the facilities which we might open in the set I^* and the facilities which we open in reality in the set I^+. By this we mean that in the set I^*, the facilities are included for which the slack variable s_i is equal to zero, because then f_i is exactly equal to the value of $\sum_{j \in J} \max\{0, v_j - c_{ij}\}$ for each facility *i*. This indicates that the complementary slackness condition (14) is always satisfied because $f_i - \sum_{j \in J} \max\{0, v_j^+ - c_{ij}\}$ is equal to zero for each $i \in I^*$. This method of constructing the set I^* is just a choice, there can be several other reasons to store a facility in this set, such as store all facilities in I^* for which v_j^+ is exactly equal to c_{ij} for all demand locations *j* such that complementary slackness condition (15) is always satisfied.

The facilities in the set I^* do not all have to be opened, we can make a selection of the facilities that we could open in order to improve the obtained solution. This set of established facilities I^+ is constructed in the following way. First we are going to check for each facility *i* from the set I^* how many demand locations *j* there are for which v_j^+ is strictly larger than c_{ij}. Then we sort the facilities in descending order on the number of demand locations for which $v_j^+ > c_{ij}$ holds. We want to know this because the complementary slackness condition (15) is satisfied when at most one facility has $v_j > c_{ij}$, for some *j*. The reason for this is that the binary variables x_{ij}^+ and y_i^+, which correspond to the used edges and the opened facilities respectively, are both equal to one with certainty for only the lowest value of c_{ij}, hence the difference $y_i^+ - x_{ij}^+$ is equal to zero in this case, which indicates that complementary slackness condition (15) is satisfied.

This indicates that when there are for some *j* two or more facilities from I^* for which v_j^+ is strictly larger than c_{ij}, then we have to delete some facilities from the set I^* in order to form the set I^+. For us it seems the most reasonable choice to delete the facility for which the number of demand locations for which $v_j^+ > c_{ij}$ is the highest, because then for more *j*, the number of facilities for which $v_j^+ > c_{ij}$ decreases. If for some *j* there was just one facility for which $v_j^+ > c_{ij}$, and we delete that facility, then there are for these demand locations *j* zero facilities for which $v_j^+ > c_{ij}$ holds. If this is the case, then complementary slackness condition (15) is still satisfied because $\max\{0, v_j^+ - c_{ij}\}$ is equal to zero for these *j*.

When we have constructed this set I^+, we determine the total costs, hence the opening costs and transportation costs together, when we open all facilities which are included in I^+. If these total costs are lower than the total costs determined when all facilities in I^* are opened, we have obtained a new solution which is better than the solution originally found by the dual ascent procedure. But if these total costs are not lower than the total costs of the original solution, then we are going to adapt the set I^+ in the following way. We take the set I^* and we delete the facility for which the number of demand locations for which $v_j^+ > c_{ij}$ is the second-highest. Then, we are going to check if this results in a reduction of total costs. If this is the case, a new solution is found, but if this is not the case, then the set I^+ is adapted again through deleting the facility for which the number of demand locations for which $v_j^+ > c_{ij}$ is the third-highest, and so on until a better solution is found or all options are considered. If all options of deleting facilities are considered and none of these options result in a better solution, then the set I^+ equals the set I^*. Note that in this procedure a solution is found that is not necessarily the best solution possible. This is because the procedure stops when a better solution is found than the solution with the set I^* such that for as much as possible demand locations *j*, the number of facilities for which $v_j^+ > c_{ij}$ decreases, hence the complementary slackness condition (15) is satisfied as much as possible.

Since we now have a set of opened facilities I^+, we can derive a integer primal solution from this set. The integer primal solution consists of the variables y_i^+ and x_{ij}^+ . The variable y_i^+ is a vector of length *m* containing all facilities. The facilities which belong to the set I^+, so those which are opened according to the rule described before, are indicated with one and the closed facilities are indicated with zero. Furthermore, the variable x_{ij}^+ indicates which demand locations are connected with a certain facility, hence if the edge *(i,j)* is used or not. But we do not yet know which demand locations are connected to facility *i*. To determine this, we have to make a choice or a rule which we are going to apply. For us, because we want to minimize costs, it seems obvious that for each demand location *j*, we search for an established facility *i* from the set I^+ which is the cheapest in supplying the demand of location *j*. In this way, we choose for each demand location the facility with the lowest costs c_{ij}, hence the transportation costs that are as low as possible. Another way to formulate this is that we use the cheapest edge for that demand location. The edges that we want to use are indicated with one, and the edges that we do not want to use are indicated with zero. If we summarize this, we can obtain an integer primal solution from a dual solution in the following way:

$$y_i^+ = \begin{cases} 1, & i \in I^+ \\ 0, & i \notin I^+ \end{cases} \tag{16}$$

$$x_{ij}^+ = \begin{cases} 1, & i = i^+(j), j \in J \\ 0, & otherwise \end{cases} \tag{17}$$

Where $i^+(j) \in I^+$ is a minimum-cost source facility, defined for each demand location *j*. By this, we mean that for each demand location *j*, there is an opened facility *i* for which the costs of satisfying the demand of *j* are the lowest.

Through our definitions of I^+ and y_i^+, the complementary slackness condition (14) is always satisfied as we have seen earlier. Namely, the facilities in set I^+ are facilities for which the difference between f_i and $\sum_{j\in J} \max\{0, v_j - c_{ij}\}$ is exactly equal to zero and for these facilities, the value of y_i^+ is equal to one. Thus, for the established facilities, the complementary slackness condition (14) is always satisfied. For a closed facility, the value of y_i^+ is equal to zero, hence for the closed facilities condition (14) is also satisfied.

Complementary slackness condition (15) is satisfied when at most one facility $i \in I^+$ has $v_j > c_{ij}$ for some *j* as we have seen earlier. This reasoning does not hold with certainty for all c_{ij} except the lowest one, thus we choose the number of demand locations for which $v_j^+ > c_{ij}$ for $i \in I^+$ as low as possible. It is not guaranteed that complementary slackness condition (15) is always satisfied because it is hardly possible to find a feasible solution where there is at most one facility $i \in I^+$ for which $v_j > c_{ij}$ holds for each *j*, but through our definition of I^+ we try to get as close as possible to such a feasible solution.

3.3 The dual adjustment procedure

The dual adjustment procedure starts with a solution V^+ obtained by the dual ascent procedure for which not all complementary slackness conditions are satisfied. We also start with the set I^+ obtained by the procedure described in Section 3.2. From this solution, there are some *j'* selected for which the complementary slackness condition (15) is violated. Only (15) can be violated because condition (14) is always satisfied as we have seen earlier. We decrease $v_{j'}$ which creates slack on the constraints (13) corresponding to the $i \in I^+$ with $v_{j'} > c_{ij'}$. After the decrease of $v_{j'}$, other v_j, that are limited by these constraints (13), are tried to be increased using the dual ascent procedure. If it is possible to increase more than one v_j as $v_{j'}$ is decreased, then the dual objective will be increased by this change. This process is performed for all *j'* and the procedure is repeated as long as the dual objective improves. This method makes use of the dual ascent procedure in case where v_j is attempted to be increased.

We define the following sets which are used in the dual adjustment procedure: $I_j^* = \{i \in I^*: v_j \geq c_{ij}\}$ and $I_j^+ = \{i \in I^+: v_j > c_{ij}\}$ for each demand location $j \in J$. The set I_j^* consists of all facilities in the set I^* for which $v_j \geq c_{ij}$ holds for each $j \in J$. This set is used to create the set J_i^+, which we will discuss later. Then we have defined the set I_j^+, which is in fact a subset of I_j^*. In the set I_j^+, for each demand location $j \in J$, we store the established facilities from the set I^+ for which v_j is strictly larger than c_{ij}. This set is used to determine if the solution is already optimal or if it is possible to improve the current solution, which we will discuss later.

Furthermore, for each $i \in I$ we define the set $J_i^+ = \left\{j: I_j^* = \{i\}\right\}$. In this set, for each facility *i*, the demand locations are given for which the value of I_j^* is equal to facility *i* which satisfies $v_j \geq c_{ij}$. Hence, this set gives for each facility the demand locations that can be served by this facility and for which $v_j \geq c_{ij}$ holds. In fact, according to the definition of I_j^*, we do not consider each facility *i* in this case, but each *i* in the set I^*, hence each facility for which the slack variable is equal to zero. This set J_i^+ is used to check which v_j we can increase.

For each $j \in J$ with $\left|I_j^+\right| > 1$, we define a second-best source $i'(j) \in I^+$ whereby the following property holds with regard to the costs: $c_{i'(j),j} = min_{i \in I^+, i \neq i^+(j)} c_{ij}$. The second-best source $i'(j)$ for a demand location *j* is not the cheapest possible facility to supply the demand of *j* from, namely the minimum-cost source $i^+(j)$, but it is the cheapest one when the minimum-cost source is not taken into account. The second-best source is only defined for a facility *i* with $\left|I_j^+\right| > 1$, because when the number of elements in I_j^+ is smaller or equal to one, then there is simply no second-best source because there is just one facility that can be opened which forms automatically the minimum-cost source. For each $j \in J$ we define $c_j^- = max_{i \in I}\left\{c_{ij}: v_j > c_{ij}\right\}$. The algorithm for the dual adjustment procedure is described as follows:

1. $j = 1$

2. If $\left|I_j^+\right| > 1$

3. If $J_{i^+(j)}^+ \neq J_{i'(j)}^+$

4. For each $i \in I$:

5. If $v_j > c_{ij}$

6. $s_i = s_i + \left(v_j - c_j^-\right)$

7. end

8. end

9. $v_j = c_j^-$

10. $J^+ = J_{i^+(j)}^+ \cup J_{i'(j)}^+$, and execute the dual ascent procedure

11. $J^+ = J^+ \cup j$, and repeat the dual ascent procedure

12. $J^+ = J$, and repeat the dual ascent procedure

13. If the value of v_j has changed in comparison with its original value

14. Return to Step 2

15. else

16. If $j \neq n$

17. $j = j + 1$

18. Return to Step 2

19. end

20. end

21. end

22. end

For each *j* we are going to execute this procedure. We have a violation of the complementary slackness condition (15) if $\left|I_j^+\right|$ is larger than one for a certain $j \in J$. If $\left|I_j^+\right|$ is smaller or equal to one, then for this *j* the solution which corresponds to I^+ is optimal and it is not possible to improve this solution by the dual adjustment procedure. Therefore, we execute this procedure only for all *j* for which $\left|I_j^+\right|$ is larger than one, which is ensured by line 2. These *j* are denoted by *j'*. Line 3 describes that when the demand locations that can be served by the minimum-cost source of demand location *j* are not equal to the demand locations that can be served by the second-best source of demand location *j*, then the procedure in lines 4 to 21 is executed. Hence, first the minimum-cost source and the second-best source of demand location *j* are determined. Then we check which demand locations can be served by this minimum-cost source and this second-best source. If these two sets of demand locations differ from each other, then the demand locations are not served in the cheapest possible way, hence by executing lines 4 to 21, an improvement can be found, otherwise we proceed to the next *j'*.

In lines 4 to 8, we are going to look if s_i can be increased for some *i*. If $v_{j'}$ is larger than $c_{ij'}$, then for that *i*, the complementary slackness condition (15) is violated such that there is slack created on constraint (13) when $v_{j'}$ is decreased. This makes it possible that the dual objective can be increased when two or more other v_j that are limited by these constraints (13) are increased. When $v_{j'}$ is decreased, then it must be possible to increase s_i according to the definition of s_i in line 2 of the dual ascent procedure. After the increasing of s_i in lines 4 to 8, we decrease the $v_{j'}$ for which the complementary slackness condition (15) is violated, to the value of $c_{j'}^-$ in line 9.

After the decrease of $v_{j'}$, we increase other v_j that are limited by constraint (13). This is done in lines 10 to 12 by executing the dual ascent procedure. First, the dual ascent procedure is executed when J^+ is equal to $J^+_{i^+(j)}$ and $J^+_{i'(j)}$ together. This means that only for these *j,* we are going to increase v_j if that is possible. Then we add *j* to J^+ and we repeat the dual ascent procedure in line 11. In order to obtain a valid solution when the procedure is terminated, we repeat the dual ascent procedure when J^+ is set equal to J in line 12. If, after executing the dual ascent procedure three times, the value of v_j is equal to the original value of v_j, then v_j cannot be improved further and *j* is increased by one if possible and we return to Step 2, given in lines 16 to 19. Otherwise, we return to Step 2 without updating *j* which is stated by line 14. If all values of *j* are considered, then the procedure terminates and a solution is found.

The dual ascent procedure and the dual adjustment procedure result in a solution for the UFLP. This solution is always feasible because the constraints are always satisfied, which is ensured during the performance of these methods. The question is, if these methods result in an optimal solution for the UFLP. In order to obtain an optimal solution, all complementary slackness conditions should be satisfied. According to our definition of I^+, complementary slackness condition (14) is always satisfied, as we have described earlier. However, we cannot guarantee that condition (15) is also always satisfied. This is due to the fact that it is hardly possible to find a feasible solution where for each $j \in J$ it holds that at most one facility from the set I^+ has $v_j > c_{ij}$, which is already concluded in Section 3.2, but it is not impossible that condition (15) is always satisfied. From this reasoning we can conclude that the dual ascent procedure and the dual adjustment procedure do not always result in an optimal solution for the UFLP. In order to obtain an optimal solution, a branch-and-bound phase can be used with the lower and upper bound given by the solutions obtained by these procedures.

4. Results

The dual ascent procedure is executed for the three standard problem instances, namely the Bilde-Krarup, the Galvão Raggi and the Gap B instances. The way in which the dual ascent procedure is executed is described in Section 3.1. For each problem instance we check if all complementary slackness conditions are satisfied by the solutions obtained with the dual ascent procedure. If this is not the case, then the dual adjustment procedure described in Section 3.3 is executed, otherwise an optimal solution to the LP relaxation of the UFLP is found. In this Section, the results are analysed for each standard problem instance.

4.1 Bilde-Krarup (B1.1)

In the Bilde-Krarup instance number B1.1 there are fifty facility locations and hundred demand locations. The transportation costs between these locations differ from zero to 999 with a mean of 499 and the costs of opening a facility vary between 1,244 and 9,907 with a mean of 5,817. In the optimal integer solution of this problem, there are five facility locations established with corresponding total costs of 23,468 according to the standard instance. When we execute the dual ascent procedure for the Bilde-Krarup instance and we construct the set I^+ as described in Section 3.2, we obtain total costs of 24,191 corresponding to six established facilities. The results from the dual ascent procedure and the optimal solution given by the benchmark are given in table 4.1.1.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	5	-	-	23,468	-	0%
Dual Ascent Procedure	6	10,388	13,803	24,191	21,494	12.5%

Table 4.1.1: the results of the dual ascent procedure in comparison with the optimal solution of the Bilde-Krarup instance.

We can see from this table that the total opening costs are lower than the total transportation costs, but both of them are about half of the total costs. The transportation costs per unit are relatively low in comparison with the costs of opening a facility, hence less facilities are opened and from each facility, the demand of relatively many demand locations is supplied. Thus, each facility is connected with on average seventeen demand locations. The difference between the integer primal solution and the dual solution is more than twelve percent in this case, which is given by the duality gap in the table. From the results, we can also observe that the difference between the total costs given by the benchmark and the total costs obtained by the dual ascent procedure is relatively small, so we can conclude that we are close to the optimal solution.

After performing the dual ascent procedure, we have checked if all complementary slackness conditions are satisfied. We observed that complementary slackness condition (15) is violated for some *i* and some *j*, which indicates that this is not an optimal solution and the dual adjustment procedure must be executed in order to improve the solution obtained by the dual ascent procedure. The dual adjustment procedure is initialized with the v_j obtained by the dual ascent procedure. After performing the dual adjustment procedure, we observe that there are still six facilities opened, but these facilities are not the same as the ones found by the dual ascent procedure, hence the opening costs are different. At the same time, the total transportation costs decrease because the facilities are probably more evenly distributed, which on average results in a shorter distance between facilities and demand locations. The results of the dual adjustment procedure together with the optimal solution and the results of the dual ascent procedure are given in table 4.1.2.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	5	-	-	23,468	-	0%
Dual Ascent Procedure	6	10,388	13,803	24,191	21,494	12.5%
Dual Adjustment Procedure	6	11,645	12,252	23,897	21,820	9.5%

Table 4.1.2: The results of the dual ascent procedure and the dual adjustment procedure compared with the optimal solution of the Bilde-Krarup instance.

We observe again that the opening costs and the transportation costs are both about half of the total costs. The total costs are decreased after performing the dual adjustment procedure and are very close to the optimal solution according to the benchmark. The value of $\sum v_j$ is increased, which indicates that the dual adjustment procedure gives a higher lower bound on the solution for the UFLP. For some *i* and *j* complementary slackness condition (15) is violated, hence we have not found an optimal solution. We can conclude that the dual adjustment procedure improved the solution for the Bilde-Krarup instance, because the total costs and the duality gap are decreased. We can also conclude that the solution obtained by this procedure is close to the optimal solution given by the benchmark.

4.2 Galvão Raggi (50.1)

The Galvão Raggi instance number 50.1 consists of fifty demand locations and fifty facility locations. The transportation costs for supplying the demand of location *j* from facility *i* vary between zero and 65,534 with a mean of 9,706. The costs of opening a facility differ from a minimum of zero to a maximum of 6,719 and the opening costs have a mean of 2,975.

According to the benchmark of the Galvão Raggi instance, twenty out of fifty facilities have to be opened corresponding to the optimal integer solution. In this case, the total costs are equal to 175,802. When we execute the dual ascent procedure and construct I^+ for this standard instance, we observe that this procedure founds a solution where twenty-two facilities are opened with total costs of 175,802. The results of the dual ascent procedure are given in table 4.2.1, together with the optimal solution of the Galvão Raggi instance.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	20	-	-	175,802	-	0%
Dual Ascent Procedure	22	15,268	160,534	175,802	175,802	0%

Table 4.2.1: the results of the dual ascent procedure in comparison with the optimal solution of the Galvão Raggi instance.

In this case, approximately ninety percent of the total costs consists of the transportation costs and the other ten percent consists of the opening costs. The explanation for this is that the costs for the transportation of one unit from facility *i* to demand location *j* are on average much higher than the costs for opening one facility *i*.

We can confirm this thought when we compare the Galvão Raggi instance with the Bilde-Krarup instance. Both instances consist of fifty facility locations. In the Bilde-Krarup instance, there are only six facilities opened while in the Galvão Raggi instance, there are twenty-two facilities established. This difference is relatively large when we take into account that in the Bilde-Krarup instance, there are twice as many demand locations as in the Galvão Raggi instance. This large difference is due to the fact that in the Bilde-Krarup instance, the opening costs are relatively high in comparison with the transportation costs, and in the Galvão Raggi instance this relationship is reversed.

For the solution obtained by the dual ascent procedure, all complementary slackness conditions are satisfied, hence the optimal solution is found by the dual ascent procedure and we do not execute the dual adjustment procedure. This can also be confirmed by the fact that the total costs found by the dual ascent procedure are exactly the same as the total costs given by the benchmark, hence the duality gap is equal to zero percent and the solution cannot be improved.

4.3 Gap B (1031)

Number 1031 of the Gap B instance consists of hundred possible facility locations and hundred demand locations. The costs of transportation between facilities and demand locations are in most cases equal to 48,029, which is the maximum of the transportation costs, and in some cases the transportation costs are very low, for example one or two. For this reason, the mean of the transportation costs is relatively high, namely 43,226. Furthermore, the costs of opening a facility are for each facility equal to 3,000.

In the optimal integer solution given in the benchmark, there are fourteen facilities established corresponding to the total costs of 42,165. In table 4.3.1 the results are given for the Gap B instance after performing the dual ascent procedure and constructing the set I^+ as described in Section 3.2, the values for the benchmark are also given in this table.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum \mathrm{v_j}$	Duality gap
Optimal solution	14	-	-	42,165	-	0%
Dual Ascent Procedure	22	66,000	113	66,113	24,191	173.3%

Table 4.3.1: the results of the dual ascent procedure in comparison with the optimal solution of the Gap B instance.

Executing the dual ascent procedure results in a solution with total costs of 66,113 corresponding to twenty-two established facilities. For this standard instance of the UFLP we observe that the transportation costs are very low in comparison with the total costs. On the other hand, the contribution of the opening costs to the total costs is very high. The explanation for this is that the opening costs are the same for each facility, hence for each demand location, the facility with the lowest transportation costs is established. For some facilities, the transportation costs to certain demand locations are between zero and five, hence these facilities are established to supply the demand of the demand locations with the lowest transportation costs.

The difference between the total costs corresponding to the optimal solution given in the benchmark and the total costs observed by the dual ascent procedure is relatively large, namely more than twenty thousand. This difference is explained by the fact that in the optimal solution, there are fourteen facilities opened while in the solution obtained by the dual ascent procedure, there are twenty-two facilities established, so eight facilities more. These eight extra facilities provide additional opening costs of twenty-four thousand, which is approximately the difference in total costs between the two cases. Furthermore, we obtain that the duality gap is very large, namely more than hundred seventy percent. This large difference between the primal integer solution and the dual solution also indicates that we are not close to the optimal solution.

We have checked if the complementary slackness conditions are satisfied for the solution obtained by the dual ascent procedure. We observed that some complementary slackness conditions (15) are violated, hence the solution is not optimal. Thus, for the Gap B instance, we are going to perform the dual adjustment procedure described in Section 3.3.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	14	-	-	42,165	-	0%
Dual Ascent Procedure	22	66,000	113	66,113	24,191	173.3%
Dual Adjustment Procedure	24	72,000	101	72,101	24,210	197.8%

Table 4.3.2: The results of the dual ascent procedure and the dual adjustment procedure compared with the benchmark for the Gap B instance.

The results of the dual adjustment procedure for the Gap B instance are given in table 4.3.2, together with the optimal solution and the results of the dual ascent procedure. The value of $\sum v_j$ is increased when the dual adjustment procedure is performed, which is expected because the dual adjustment procedure gives an improvement for the lower bound, hence a higher lower bound, on the solution for the UFLP. Furthermore, the total costs are increased because there are more facilities established, namely twenty-four instead of twenty-two. Because of this increasing number of established facilities, the transportation costs are decreased a bit. The increase of total costs is not expected because when the value of $\sum v_j$ increases, the total costs must decrease in order to obtain a better, hence lower, upper bound.

We observe that the duality gap has increased to almost two hundred percent after performing the dual adjustment procedure, this indicates that we have obtained a solution which is far from the optimal solution. The dual adjustment procedure results in total costs of 72,101 which also indicates that this solution is not close to the optimal solution given by the benchmark and which is not an improvement of the solution found by the dual ascent procedure. From this reasoning we can conclude that the dual ascent procedure and especially the dual adjustment procedure perform not very well for the Gap B instance.

4.3.1 Changes in transportation costs

We want to explain this bad performance of the dual ascent procedure and the dual adjustment procedure for the Gap B instance. It seems as if the algorithm is focused on keeping the total transportation costs as low as possible, but then the number of facilities rises as we can see in table 4.3.2. In order to check the impact of the relatively low transportation costs on the performance of the two procedures, we increase the transportation costs with a large amount, and we execute the dual ascent procedure. The results are shown in table 4.3.1.1, where we assume that the opening costs remain the same in each case.

We have executed the dual ascent procedure when we increase the transportation costs with one thousand, two thousand, three thousand, four thousand and five thousand times the original transportation costs. We have chosen this scale because then, the transportation costs of the cheapest edges, which are generally used and are in most of the cases equal to one, are close to the opening costs of three thousand when we multiply the original costs with three thousand. And because we want to analyse what happens to the solution if the transportation costs are a bit lower and a bit higher than the opening costs, we have also multiplied the transportation costs by one thousand, two thousand, four thousand and five thousand.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	14	-	-	42,165	-	0%
Dual Ascent Procedure	22	66,000	113	66,113	24,191	173.3%
TC*1000	42	126,000	28,000	154,000	116,000	32.8%
TC*2000	41	123,000	44,000	167,000	145,000	15.2%
TC*3000	54	162,000	45,000	207,000	171,000	21.1%
TC*4000	54	162,000	60,000	222,000	186,000	19.4%
TC*5000	54	162,000	75,000	237,000	201,000	18.0%

Table 4.3.1.1: The results of the dual ascent procedure executed for the Gap B instance when we increase the transportation costs (TC) with a large amount.

From table 4.3.1.1 we observe that for all increases of costs the duality gaps are decreased with a large percentage. The solution where the transportation costs are increased with two thousand times the original transportation costs, is the solution which is closest to the optimal solution according to the duality gap, which is around fifteen percent. The difference between this duality gap and the duality gap obtained by the solution with the original costs is very large, which indicates that the dual ascent procedure performs better for values of transportation costs that are closer to the opening costs.

We also execute the dual adjustment procedure for the different values of transportation costs, because for some *i* and *j* complementary slackness condition (15) is violated. The results of this procedure, together with the optimal solution and the results obtained by the procedures with the original costs, are given in table 4.3.1.2.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	14	-	-	42,165	-	0%
Dual Ascent Procedure	22	66,000	113	66,113	24,191	173.3%
Dual Adjustment Procedure	24	72,000	101	72,101	24,210	197.8%
TC*1000	48	144,000	22,000	166,000	116,000	43.1%
TC*2000	43	129,000	40,000	169,000	146,000	15.8%
TC*3000	55	165,000	45,000	210,000	171,000	22.8%
TC*4000	55	165,000	60,000	225,000	186,000	21.0%
TC*5000	55	165,000	75,000	240,000	201,000	19.4%

Table 4.3.1.2: The results of the dual adjustment procedure executed for the Gap B instance when we increase the transportation costs (TC) with a large amount.

From this table we observe that in general, the value of $\sum v_j$ is increased or is remained the same in comparison with the results from table 4.3.1.1, which we expected because this procedure provides an improvement for the lower bound on the solution for the UFLP. We also observe that after performing the dual adjustment procedure, there are more facilities established than after performing the dual ascent procedure, hence the total opening costs are increased. In all cases of different transportation costs, the total costs are increased in comparison with table 4.3.1.1, which is quite strange because when the summation of v_j is increased or is remained the same, then the total costs should decrease or remain the same according to the objective functions (1) and (12) of the primal and dual problems.

In general, the values of the duality gaps in each case are approximately the same as in table 4.3.1.1, except when the transportation costs are multiplied by one thousand times the original costs. In that case, the duality gap is increased from around thirty three to forty three percent, which is a relatively large difference. An explanation for this is that there are forty eight facilities established after performing the dual adjustment procedure instead of forty two after performing the dual ascent procedure. This difference is relatively large, hence the total opening costs are increased with a relatively large amount. The total transportation costs are decreased in this case, but altogether the total costs are increased with twelve thousand while the value of $\sum v_j$ is not changed, this indicates why the duality gap is increased with such a large difference.

We can conclude that the dual ascent procedure performs better for the Gap B instance when the difference between the transportation costs and the opening costs is smaller. But we can also conclude that the dual adjustment procedure performs not very well because the total costs increase and the duality gap increases, hence this procedure does not provide an improvement of the solution found by the dual ascent procedure, which it should actually do. When we compare this with the results in Section 4.3, we obtain the same pattern, namely that the dual adjustment procedure does not provide an improvement on the solution found by the dual ascent procedure. From this, we could conclude that the bad performance of both procedures together is particularly due to the bad performance of the dual adjustment procedure.

Now, we want to know what is the cause of the bad performance of the dual adjustment procedure in the case in which there is a large difference between the opening costs for one facility and the transportation costs for one edge, and when there is a large difference between transportation costs themselves, so in case of the Gap B instance. We have to look into the steps of the procedure for something that is not going well.

From the dual ascent procedure, we obtain values of v_j that are very low, between zero and seven, or very high, namely around three thousand. Because most of the transportation costs are equal to 48,029, there are very few v_j that are strictly larger than c_{ij}. Hence, when we look at the definition of c_j^-, we observe that the values of c_{ij} for which $v_j > c_{ij}$ holds are always between zero and five. This indicates that the value of c_j^- is also always between zero and five. In line 9 of the dual adjustment procedure, v_j is decreased to c_j^-, hence to a value between zero and five. Because of this step in the procedure, it might be that the value of $\sum v_j$ does not increase so much, hence the dual objective value will not improve much in this case such that the dual adjustment procedure does not provide an improvement for the solution found by the dual ascent procedure. For us, through the definition of c_j^-, it seems most likely that the dual adjustment procedure cannot handle large differences between transportation costs themselves very well.

5. Changes in costs

For each standard problem instance, we have checked what happens to the number of established facilities and the total costs when the opening costs per facility and transportation costs per unit increase or decrease. The opening costs and transportation costs are decreased or increased by twenty-five percent of the original costs and then the dual ascent procedure is executed and the set I^+ is determined as described in Section 3.2. We have done this for each type of costs separately where we take into account that the other type of costs remains the same, but also when both opening costs and transportation costs have increased or decreased at the same time. In this Section we analyse the results of these changes in costs.

5.1 Bilde-Krarup (B1.1)

For the Bilde-Krarup instance, we have executed the dual ascent procedure for different values of opening costs and transportation costs. In table 5.1.1 the results are shown, where OC means opening costs and TC means transportation costs.

	Established facilities	Opening costs	Transportation costs	Total costs	Percentage change in total costs	Duality Gap
Original costs	6	10,388	13,803	24,191	0%	12.5%
OC +25%	6	13,499	14,209	27,708	+14.5%	18.5%
OC -25%	7	10,012	11,571	21,583	-10.8%	12.0%
TC +25%	7	13,349	14,464	27,813	+15.0%	13.1%
TC -25%	6	10,799	10,657	21,456	-11.3%	18.8%
OC +25% TC -25%	4	7,774	14,027	21,801	-9.9%	9.3%
OC -25% TC +25%	8	11,888	12,628	24,516	+1.3%	10.7%

Table 5.1.1: results of the dual ascent procedure for the Bilde-Krarup instance when the costs are changed. OC are the opening costs, TC are the transportation costs.

In general, we observe that when the opening costs are decreased, then more facilities are established and the total transportation costs decrease. This is quite obvious because opening a facility becomes cheaper in comparison with the transportation costs, hence it is cheaper to open more facilities in order to reduce the average distance to demand locations, which results in a reduction of total transportation costs. If the opening costs are increased, then it becomes more expensive to open a facility, hence less or the same number of facilities are established.

When the transportation costs are increased, then there are in general more facilities opened, such that the distance between facilities and the demand locations becomes smaller. Because of this reason, the total opening costs increase while the total transportation costs do not differ very much from the original total transportation costs.

From this table, we observe that an increase in the transportation costs of twenty-five percent results in the largest relative increase in total costs, namely an increase of fifteen percent. Furthermore, a decrease of twenty-five percent in the transportation costs results in the largest relative reduction in total costs, namely a reduction of more than eleven percent. Both observations imply that changes in transportation costs have the greatest impact on the total costs for this standard instance. Furthermore, we observe that the duality gap is the smallest when the opening costs are increased and the transportation costs are decreased with twenty-five percent. This indicates that the dual ascent procedure results in a solution which is the closest to the optimal solution when the transportation costs are decreased and the opening costs are increased at the same time with twenty-five percent.

5.2 Galvão Raggi (50.1)

For the Galvão Raggi instance, we have also increased and decreased the opening and transportation costs in order to observe what happens to the number of established facilities and the total costs. The dual ascent procedure is executed for the Galvão Raggi instance for the different costs given in table 5.2.1. The results are also given in this table.

Again, we observe that when the opening costs decrease, then there are more facilities established because it becomes cheaper to open a facility. When the opening costs increase, then there are less or the same number of facilities established. This is because it becomes more expensive to open a facility and it might be more profitable to increase the distance between a facility and a demand location while opening less facilities.

	Established facilities	Opening costs	Transportation costs	Total costs	Percentage change in total costs	Duality Gap
Original costs	22	15,268	160,534	175,802	0%	0%
OC +25%	22	19,085	160,534	179,619	+2.2%	0%
OC -25%	24	16,313	155,234	171,547	-2.4%	0%
TC +25%	23	20,205	195,543	215,748	+22.7%	0%
TC -25%	22	15,268	120,401	135,669	-22.8%	0.04%
OC +25% TC -25%	21	13,838	124,825	138,663	-21.1%	0.3%
OC -25% TC +25%	24	16,313	194,042	210,355	+19.7%	0%

Table 5.2.1: results of the dual ascent procedure for the Galvão Raggi instance when the costs are changed. OC are the opening costs, TC are the transportation costs.

When the transportation costs increase, there are more facilities established. This is due to the fact that it becomes more expensive to transport goods on a longer distance than to open an additional facility, such that the distances becomes smaller. And when the transportation costs decrease, then the opposite happens, namely less or the same number of facilities are opened, because it is more expensive to open an additional facility than to transport goods on a longer distance.

From table 5.2.1 it is clear that an increase in transportation costs results in the largest relative increase in total costs, namely an increase of almost twenty-three percent. The largest relative decrease in total costs is observed when the transportation costs are decreased by twenty-five percent. The decrease in total costs is almost twenty-three percent in this case. From these observations, we can conclude again, that a change in transportation costs has the largest impact on the total costs.

If we compare this with the Bilde-Krarup instance, we observe a somewhat contradictory pattern. In the Bilde-Krarup instance, the transportation costs are on average lower than the opening costs per facility what we have described in Section 4.1. In the Galvão Raggi instance, however, the transportation costs are on average higher than the opening costs for each facility, according to Section 4.2. The paradox is that for both standard problem instances, a change in transportation costs has the greatest impact on the total costs, while for both problem instances the relation between transportation costs and opening costs is different. A possible explanation for this is that in the Galvão Raggi instance, ninety percent of the total costs consists of the transportation costs, hence a change in transportation costs results in a relatively large change in the total costs in this case.

Another possible explanation for this paradox is the difference in number of demand locations for each standard instance. In the Bilde-Krarup instance, there are hundred demand locations, so hundred edges used. In the Galvão Raggi instance, there are fifty demand locations, thus there are fifty edges used in this case.

Thus, when the transportation costs increase with twenty-five percent, then in the Bilde-Krarup instance, this has a large effect on the total transportation costs because there are relatively many edges to which the costs are related, but the opening costs are on average higher than the transportation costs in this case. When the transportation costs increase with the same percentage in the Galvão Raggi instance, this has a large effect on the total transportation costs, because the transportation costs are on average higher than the opening costs and also because the transportation costs contribute for approximately ninety percent to the total costs for this standard problem instance.

In addition, in the Galvão Raggi instance, there are on average two demand locations connected with one facility. Hence, it is hardly possible to decrease the average distance between a facility and a corresponding demand location further, without increasing the opening costs too much. So, when the transportation costs increase, the average distance between facilities and demand locations cannot be reduced so much, hence the total transportation costs increase with almost twenty-two percent, which is relatively much in comparison with the relative increase of total transportation costs of fifteen percent in the Bilde-Krarup instance.

The original solution found by the dual ascent procedure for this instance is already an optimal solution, hence with a duality gap of zero percent. When we change some costs, the dual ascent procedure still finds solutions that are optimal or very close to an optimal solution, which we can conclude from the duality gaps. Thus, also when the costs change, the dual ascent procedure performs well for the Galvão Raggi instance.

5.3 Gap B (1031)

The dual ascent procedure is executed for the Gap B instance for different levels of costs. The results are given in table 5.3.1.

	Established facilities	Opening costs	Transportation costs	Total costs	Percentage change in total costs	Duality Gap
Original costs	22	66,000	113	66,113	0%	173.3%
OC +25%	22	82,500	113	82,613	+25.0%	173.6%
OC -25%	22	49,500	113	49,613	-25.0%	172.8%
TC +25%	22	66,000	141	66,141	+0.04%	172.9%
TC -25%	22	66,000	85	66,085	-0.04%	173.7%
OC +25% TC -25%	22	82,500	85	82,585	+24.9%	174.0%
OC -25% TC +25%	22	49,500	141	49,641	-24.9%	172.2%

Table 5.3.1: results of the dual ascent procedure for the Gap B instance when the costs are changed. OC are the opening costs, TC are the transportation costs.

One thing that is immediately clear when we look at table 5.3.1, is that the number of established facilities is not changed when the opening costs and transportation costs are changed. In each case there are twenty-two facilities established. An explanation for this is that the opening costs are equal for each facility and are on average much higher than the transportation costs for the edges that are used. The costs for using these edges are namely between zero and five, which is relatively low in comparison with 3,000 which are the costs of opening one facility, hence it is not very cheap to open an additional facility also when the transportation costs increase with twenty-five percent.

If the opening costs for each facility are increased with twenty-five percent, then the number of opened facilities does not change, hence the total transportation costs remain the same. The total opening costs increase also with twenty-five percent in this case, thus the total costs increase also with the same percentage which is due to the fact that the total costs consist of almost hundred percent of the total opening costs. When the opening costs are decreased, then the total transportation costs remain the same again, and the total costs decrease also with twenty-five percent.

An increase in transportation costs of twenty-five percent causes an increase in total costs of almost a half percent. The total opening costs remain the same in this case, but the total transportation costs rise with twenty-five percent. Because the total transportation costs are less than one percent of the total costs, the total costs rise with much less than twenty-five percent, namely a half percent. A decrease in transportation costs with twenty-five percent causes a decrease in total costs of almost a half percent, this is due to the same reason as described above.

The combination of an increase in opening costs and a decrease in transportation costs of twenty-five percent causes an increase in total opening costs of twenty-five percent and a decrease in transportation costs which is almost negligible when we compare it with the total costs, hence the total costs increase with almost twenty-five percent. When the opening costs are decreased and the transportation costs are increased with twenty-five percent, the total opening costs decrease also with twenty-five percent and the total transportation costs increase with almost twenty-five percent of the original transportation costs. But because the transportation costs are just a very small percentage of the total costs, the total costs decrease with almost twenty-five percent.

We observed that the opening costs have the greatest impact on the total costs for this problem instance. The explanation for this observation is that the total costs consist for almost hundred percent of the opening costs, hence a change in transportation costs matters relatively little for the change in total costs.

6. Another structure of I^+

We think it is possible to improve the solutions of the Bilde-Krarup and the Gap B instance in Section 4 by constructing the set I^+ in another way. We construct the set I^* in the same way as before, namely we include in I^* all facilities for which the slack variable s_i is equal to zero. From that set, we construct I^+ in the following way. First, we delete the first facility from the set I^*, and the remaining facilities are stored in the set I^+. Then, we determine the total costs when we open the facilities in set I^+ and we store these total costs. After that, we take the original set I^* and we delete the second facility. The remaining facilities are stored in the set I^+. Again, we determine the total costs and we store them. Due to this procedure, it is always the case that in the set I^+, there is one facility less than in the set I^*.

Thus, each time we remove one facility from I^* and we store the remaining facilities in the set I^+. We determine each time the total costs for different sets I^+ and when all facilities from the set I^* are removed once, we choose the set I^+ with the lowest total costs. If none of the sets I^+ results in lower total costs, then set I^+ is equal to the set I^*. The difference with this approach and the approach in Section 3.2 is that in this approach we do not sort the facilities in descending order on the number of demand locations for which $v_j^+ > c_{ij}$ holds, but we remove all facilities once where the sequence of removing does not make sense in this case. The approach in Section 3.2 is focused on satisfying complementary slackness condition (15), while the approach in this Section is focused on selecting the set I^+ for which the total costs are the lowest. We have executed the dual ascent procedure and the dual adjustment procedure for the Bilde-Krarup and the Gap B instance, because for these instances we have not yet found an optimal solution in contrast to the Galvão Raggi instance from which we cannot improve the solution further.

6.1 Bilde-Krarup (B1.1)

In table 6.1.1 the results of the dual ascent procedure and the dual adjustment procedure are given together with the values of the optimal solution. In order to obtain these results, the set I^+ is constructed as described above. From Section 4.1 we observe a solution with total costs of 24,191 and a duality gap of more than twelve percent after performing the dual ascent procedure. When the dual adjustment procedure is executed, we observe a solution with total costs of 23,897 and a duality gap of nine and a half percent.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	5	-	-	23,468	-	0%
Dual Ascent Procedure	6	11,645	12,252	23,897	21,494	11.2%
Dual Adjustment Procedure	6	11,645	12,252	23,897	21,885	9.2%

Table 6.1.1: Results of the dual ascent and the dual adjustment procedure for the Bilde-Krarup instance when I^+ is constructed as described above.

When the set I^+ is constructed in another way, we observe for both procedures the same solutions, but with another dual objective value. In comparison with Section 4.1, the total costs and the duality gap are decreased when the dual ascent procedure is performed, hence this indicates that the solution is improved. But after performing the dual adjustment procedure, the total costs remain the same and the duality gap decreases a bit which indicates that the gap between the dual solution and the primal solution became smaller.

This other way of constructing I^+ results in lower total costs after performing the dual ascent procedure in comparison with the total costs obtained in Section 4.1. For both procedures a solution is obtained which is closer to the optimal solution than the solution found in Section 4.1 according to the values of the duality gap. Thus, we can conclude that this way of constructing I^+ works quite well for the Bilde-Krarup instance.

6.2 Gap B (1031)

In Section 4.3 we saw that the dual ascent procedure and the dual adjustment procedure fail to find a solution for the Gap B instance which is close to the optimal solution. We found for the dual ascent procedure a solution with total costs of 66,113 and a duality gap of more than one hundred seventy percent. After the execution of the dual adjustment procedure, a solution with total costs of 72,101 and a duality gap of almost two hundred percent was found. In table 6.2.1 we can find the results of these two procedures when the set I^+ is constructed in another way, namely the way which we described above.

	Established facilities	Opening costs	Transportation costs	Total costs	$\sum v_j$	Duality gap
Optimal solution	14	-	-	42,165	-	0%
Dual Ascent Procedure	22	66,000	109	66,109	24,191	173.3%
Dual Adjustment Procedure	26	78,000	95	78,095	24,214	222.5%

Table 6.2.1: Results of the dual ascent and the dual adjustment procedure for the Gap B instance when I^+ is constructed as described above.

From this table we observe that the total costs are decreased from 66,113 to 66,109 after performing the dual ascent procedure. But after executing the dual adjustment procedure, the total costs are increased with a large amount, namely from 72,101 to 78,095 and because of this, the duality gap is increased to more than two hundred twenty percent. Notable in this case is that the transportation costs are decreased for both procedures in comparison with Section 4.3. This is again an indication that the dual ascent procedure and the dual adjustment procedure are focused on decreasing the total transportation costs which comes at the expense of decreasing the number of established facilities, hence a reduction of the total opening costs.

We can conclude from the results in table 6.2.1 that the other way of constructing I^+ results in lower transportation costs but it does not result in a solution which is closer to the optimal solution. This is probably caused by the bad performance of both procedures for the Gap B instance in general, as we have analysed earlier.

7. Conclusion

The Uncapacitated Facility Location Problem consists of *m* uncapacitated facility locations and *n* customers or demand locations. The costs of opening a certain facility *i* and the costs of supplying all demand of demand location *j* from facility *i* are known and the total of these costs have to be minimized. In this paper we described that this problem can be formulated as a mixed integer linear programming problem from which we can derive a linear programming relaxation in order to obtain an integer solution. We want to find a lower bound and an upper bound on the solution for the UFLP, hence we have defined a dual formulation of this problem using the LP relaxation. This dual formulation provides a dual solution for the UFLP where the dual objective value corresponds with a lower bound and the corresponding primal objective value corresponds with an upper bound for the solution for the UFLP.

In general, a dual problem can be solved by using a simplex method, but we have performed a simpler method that contains two methods. The first method we used is called the dual ascent procedure. This method is based on increasing the initial feasible solution V to the next higher value of the transportation costs by cycling through all demand locations in the set J until constraint (13) is violated. After this procedure, a lower bound and an upper bound on the solution for the UFLP and the corresponding set of established facilities are obtained. When all complementary slackness conditions are satisfied, then the solution is optimal. If this is not the case, we try to improve the obtained solution by executing the dual adjustment procedure.

The dual adjustment procedure starts with the solution obtained by the dual ascent procedure if the complementary slackness conditions are not satisfied. For some j' for which the complementary slackness conditions are not satisfied, we try to decrease $v_{j'}$ and increase two or more other v_j at the same time such that the dual objective increases. In order to increase v_j, the dual ascent procedure, which is described earlier, is used. This procedure is repeated for all j' for which the complementary slackness conditions are violated, as long as the dual objective continues to increase.

We have performed the dual ascent procedure and the dual adjustment procedure for three standard instances of the UFLP, namely the Bilde-Krarup, the Galvão Raggi and the Gap B instances. After performing the dual ascent procedure for the Bilde-Krarup instance, we observe that we are almost close to the optimal solution given by the benchmark. It appeared that not all complementary slackness conditions are satisfied, hence we executed the dual adjustment procedure. This procedure provides a solution which is very close to the optimal solution, which indicates that this method based on the dual ascent procedure and the dual adjustment procedure works quite well for the Bilde-Krarup instance.

After performing the dual ascent procedure for the Galvão Raggi instance, we observe that all complementary slackness conditions are satisfied, hence an optimal solution is found. The total costs corresponding with our solution are exactly the same as the total costs given by the benchmark, namely 175,802, hence the dual ascent procedure works very well for the Galvão Raggi instance and the dual adjustment procedure do not have to be executed.

For the Gap B instance, the story is a bit different from the other instances. When we perform the dual ascent procedure for the Gap B instance, some complementary slackness conditions are violated and the obtained solution is not very close to the optimal solution given by the benchmark. We found a solution where twenty-two facilities are established with total costs of 66,113 while the optimal solution given by the benchmark is based on fourteen established facilities with total costs of 42,165. Because there are some complementary slackness conditions violated, we have executed the dual adjustment procedure in order to improve the solution. The dual adjustment procedure gives a solution where twenty-four facilities are opened with total costs of 72,101, which is larger than the total costs found by the dual ascent procedure. We can conclude that the dual ascent procedure and especially the dual adjustment procedure do not work very well for the Gap B instance. We have confirmed this thought by increasing the transportation costs with a large amount. After that we obtain that the dual adjustment procedure provides an improvement of the solution obtained by the dual ascent procedure. We think that this bad performance of the dual adjustment procedure is caused by the fact that the value of c_j^- remains always a low value between zero and five.

For all standard instances, we have checked what would happen if the opening costs and the transportation costs are increased or decreased. We have increased or decreased one of the two types of costs with twenty-five percent and we have increased the opening costs with that percentage while decreasing the transportation costs and vice versa. After executing the dual ascent procedure for the different values of costs and for the three instances, we observed that changes in transportation costs have the largest impact on the total costs in the Bilde-Krarup and the Galvão Raggi instance. For the Gap B instance, changes in opening costs have the largest impact for the total costs, namely, when the opening costs increase with twenty-five percent, then the total costs also increase with twenty-five percent. The reason for this is that in the Gap B instance, the total costs consist for almost hundred percent of the opening costs.

We tried another definition of the set I^+ in order to improve the solutions found by the dual ascent procedure and the dual adjustment procedure. This other formulation of I^+ results in an improvement of the solution for the Bilde-Krarup instance. For the Gap B instance it only results in an improvement of the total transportation costs but not in an improvement of the total costs. This is also caused by the bad performance of the both procedures for the Gap B instance. From this we can conclude that another definition of I^+ works very well for the Bilde-Krarup instance but not for the Gap B instance.

From this research we can conclude that the dual ascent procedure and the dual adjustment procedure work quite well for some of the standard instances, namely the Bilde-Krarup and the Galvão Raggi instance. The combination of the dual ascent procedure and the dual adjustment procedure is a fast method to find a feasible solution for these instances. But for the other instance, namely the Gap B instance, we observed that the combination of the dual ascent procedure and the dual adjustment procedure has a bad performance in obtaining an optimal solution.

8. Discussion and further research

This research is done in a relatively short time period, hence there are some topics that we do not have considered yet or not completely. In this Section we mention some things in this research that we could have done differently, and some things for further research.

First, our research is based on the three standard instances of the UFLP. Our choice was to analyse the Bilde-Krarup, the Galvão Raggi and the Gap B instance, but we could choose several other standard instances of this problem in order to obtain other results. Next, another choice we made is the way in which we construct the sets I^* and I^+. In our case, we have defined the set I^* and after that we have removed one facility each time, according to a specific rule, in order to obtain the set I^+. But maybe it was better to remove two or three facilities instead of one in order to obtain a solution with lower total costs. It is just a choice and the construction of I^* and I^+ can be done in several other ways. For further research, it might be interesting to define some other ways in which we construct I^* and I^+ in order to check if it is possible to improve the solutions that we have found so far.

For the Bilde-Krarup and the Gap B instances, we have not found an optimal solution after executing the dual ascent procedure and the dual adjustment procedure. As mentioned earlier, when we do not find an optimal solution using this procedures, we can perform a branch-and-bound procedure with the bounds given by the solutions that we have found. Because of the time limits, we do not have executed a branch-and-bound procedure, but this might be an interesting topic for further research.

We have also looked at what happens to the number of established facilities and the total costs if the transportation costs or the opening costs change with a certain percentage. We have chosen a percentage of twenty-five percent because we thought that this would make a significant difference. It might be interesting for further research to observe what happens with the number of established facilities and the total costs when the costs are changed with other percentages than twenty-five.

References

Dueck, D., Frey, B.J., Jojic, N. and Jojic, V. (2008). Constructing treatment portfolios using affinity propagation. *International conference on research in computational molecular biology.*

Erlenkotter, D. (1978). A dual-based procedure for uncapacitated facility location. *Operations Research*, **Vol. 26**, No. 6, pp. 992-1009.

Frank, C. and Romer, K. (2007). Distributed facility location algorithms for flexible configuration of wireless sensor networks. *Distributed computing in sensor systems.* Springer.

Jain, K. and Vazirani, V.V. (2001). Primal-dual approximation algorithms for metric facility location and k-median problems. *College of computing, Georgia institute of technology*.

Klose, A. and Drexl, A. (2003). Facility location models for distribution system design. *European Journal of Operations Research.*

Lazic, N., Frey, B.J. and Aarabi, P. (2010). Solving the uncapacitated facility location problem using message passing algorithms. *Proceedings of the 13th international conference on artificial intelligence and statistics (AISTATS),* **Vol. 9**, pp. 429-436.

Lazic, N., Givoni, I., Frey, B.J. and Aarabi, P. (2009). Floss: Facility location for subspace segmentation. *International conference on computer vision.*

Li, H. (2007). Two-view motion segmentation from linear programming relaxation. *Computer vision and pattern recognition.*