The role of the manager in the theory of the firm

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Abstract

This study develops a theory of the firm in which the knowledge distribution of the manager plays a key role. First, some important determinants of the optimal knowledge distribution are identified. Both the relative importance of the task and the observability of the task appear to be relevant. Second, building on these insights, the make-or-buy decision is studied. There exists a trade-off between transaction cost when outsourcing a task and informational rent when supervising employees. It is never optimal to outsource a task if there is a less important task not outsourced. When transaction costs go down, more tasks get outsourced.

Keywords: boundaries of the firm, manager, knowledge distribution, principal-agent

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[1]This paper is based on the idea of Dr. J. Delfgaauw. I am grateful to him for sharing his thoughts with me.

1 Introduction

In order to produce one single unit of output a firm typically needs several different inputs. Probably the most well known example is Adam Smith's famous pin factory (1776). The production of a seemingly simple pin already requires the knowledge of a multitude of different tasks. But, if the production of one unit of output already requires a lot of different inputs, which transactions are performed inside a firm and which are negotiated on the market? In other words, what determines the boundaries of the firm?

This question about the make-or-buy decision was introduced by Coase (1937) and became, after several decades, the main question of interest of different studies. In an essay, Gibbons (2005) structured the literature dealing with this question. He distinguishes four different theories in the literature with each their own approach towards this question, namely "rent-seeking", "property-right", "incentive-system" and "adaptation" theory.

Surprisingly, the role of the manager has had little attention in the literature so far. As a central challenge, the need for a role of the manager into the theory of the firm has been explicitly identified in a recent paper by Dessein (2013). This paper has accepted this challenge and provides a limited, but first step towards integration of the manager's role into the theory of the firm.

In this paper a manager is characterized by his knowledge distribution over the different tasks which are needed to produce a unit of output. Key assumptions are that (1) more knowledge about a task makes the manager more able to observe effort of employees performing that task and (2) managers are bounded in their total knowledge. Combining these assumptions, interesting questions arise. What determines the optimal knowledge of a manager about different tasks in the production process of a firm? And what does this knowledge of the manager mean for the boundaries of the firm?

The analysis is split-up in two steps, section 4 will deal with the first question and section 5 with the second one. The key of this analysis of the boundaries of the firm is exactly this bounded knowledge of the manager, which leads to asymmetric information between the manager and the employees. This idea is formalized in a principal-agent model. The use of a principal-agent model seems a logical approach, as the role of the manager in the production process is typically different from that of the employee. Where the employee is needed for production, the manager is not. Mookherjee (2013) lists three important tasks of the manager, namely: (1) information processing necessary for decision making, (2) supervision of employees and suppliers and (3) decision making concerning production, marketing and contracting. This paper will touch upon all these core tasks of the manager.

A manager is responsible for supervising employees performing different tasks. The more the manager knows about a task performed by his employee, the higher the probability he observes whether the employee worked or shirked. If the manager does not always observe the effort of the employee, he needs to pay the employee an informational rent to prevent the employee from shirking. Since the manager is limited in the total knowledge he can acquire, there exists a problem of knowledge allocation over the different tasks. In this principal-agent model, I find that both the relative importance of the task and the observability of the task are important determinants for the optimal knowledge distribution. The more employees are working on the same task, the more knowledge the manager optimally should have about that task. Also, if extra knowledge increases the probability of observing the effort relatively more, the manager optimally should have more knowledge about that task.

Up to this point, the paper assumes that all tasks need to be done by employees supervised by the manager. However, section 5 proceeds by allowing for tasks to be outsourced. Now, the role of the manager is extended to also include the make-or-buy decision. This section builds on the results of section 4. There exists a trade-off between transaction costs associated with outsourcing and the informational rent due to the bounded knowledge of the manager. It is never optimal to outsource a task if there is a less important task not outsourced. When the transaction costs associated with outsourcing are above a threshold, no task will be outsourced. The lower the transaction costs are, the more tasks will get outsourced. It is never optimal to outsource a task if there is a less important task carried out by employees.

The rest of the paper is organized as follows. Section 2 provides the related literature. Section 3 presents the model. Then section 4 uses this model to identify some important determinants of the optimal knowledge distribution of the manager. Building on these insights, section 5 continues by addressing the make-or-buy decision. Section 6 concludes. The proofs are gathered in the appendix.

2 Related literature

This paper is related to, and borrows from, different strands of literature. The model in this paper uses the basic ingredients common to all principal-agent literature, that is, a conflict of interest between a principal and agent together with information asymmetry. The contracting problem in this paper arises because of limited liability of the employees. Widely discussed are incentive schemes which align the self-interest of the agent with the interest of the principal, among which the contracting problem that's arising from limited liability. Prendergast (1999) provides an overview of incentives within firms.

The first part of the paper identifies some determinants of the optimal knowledge distribution of the manager. Related to this article with respect to this topic is the strand of literature that studies the role of task specialization in hierarchical job assignments (e.g., Rosen, 1982; Waldman, 1984; Garicano, 2000). Belonging to this strand of literature, most closely related are the work of Prasad (2009) and Ferreira and Sah (2012), as both these studies acknowledge that abilities have multiple dimensions. Prasad (2009) shows that there exist a trade-off between task complementarities and relative abilities and Ferreira and Sah (2012) show that there exist interaction between the breadth of expertise and the process of communications across different types of expertise.

This paper, however, differs in some important ways from the hierarchical job assignment literature. First, these studies abstract from any agency problems between higher and lower layers in the hierarchy, while this is the driving force of this paper. A second difference is that the focus of this literature is about assigning personnel to a hierarchy layer. The question they study is how to allocate the knowledge resources of their employees to the right layer of the firm, treating the different knowledge distributions of employees exogenous. Therefore, the conclusion is often based on relative abilities to determine the managers of the firm. Also, these studies can not explain the long run and why different kind of managers exist for different firms. In the long run people can get educated for a task, acquire a certain kind of knowledge distribution and even the education system can adapt. Therefore, the personnel pool is subject to changes in the long run. This paper, in contrast, determines an optimal knowledge distribution given the task of the manager. To the best of my knowledge, this paper is first in endogenously examining the optimal knowledge distribution of a manager, instead of assigning a given personnel pool to a task.

In the second part of this paper the boundaries of the firm are the main topic of interest. There exists a large literature focusing on this topic. I will follow Gibbons (2005) in the four categories he distinguishes to briefly discuss the literature. First, there exists a literature that predicts that larger appropriable quasi-rents make integration more likely (Williamson, 1971, 1979, 1985, 1991, 2010; Klein et al., 1978). These appropiable quasi-rents exist because assets cannot be alternatively used without a loss in value, i.e., due to asset specificity. This literature is categorized by Gibbons as the "rent-seeking" theory of the firm. Second, there is the property right theory as defined by Grossman and Hart (1986), Hart and Moore (1990) and Hart (1995).[2] Here, asset ownership determines ex-ante investment incentives. Therefore, this theory predicts that the one who should make an investment in an asset, should own the asset. Third, there exist the agency theory of the firm (Holmstrom and Milgrom, 1991, 1994). Both asset ownership and contracts provide incentives to agents. If effort is not contractible, asset ownership does not only change incentives but also the optimal incentive contract. Therefore, asset ownership and incentive contracts should both be simultaneously optimized. Fourth, there exists an adaptation theory (Hart and Holmstrom, 2010; Baker et al., 2011). These models predict that the one who should be the owner, is the one who's self-interested incentives are best aligned with possible first-best adaptations to uncertainties over time.

Though the topic is the same, the approach taken in this paper is totally different from all these theories. This paper does not address the question of the boundaries of the firm related to asset ownership, a common feature in all the above mentioned theories. Instead it is concerned with the limited supervising capabilities of the manager and it's implications for the boundaries of the firm. Although all these different approaches can be used to define the boundaries of the firm, there is no role for a manager in the theory.

3 The model

Consider a firm where a manager supervises a total of N employees, with $N \geq 1$. These employees are partitioned into L classes, with $L \leq N$. Each class of employees is specialized in performing a task. The number of employees in class $i \in \{1, ..., L\}$ is represented by n_i.

While working on their task, each employee can choose whether or not to exert effort, i.e., work or shirk. Exerting effort is costly for the employee while shirking is not. Hence, the costs of shirking for each employee is equal to 0 and the costs of working for an employee of class i is given by $c_i > 0$.

All employees supervised by the manager together produce one final output. One could think about a consultancy or software company, where employees work together on

[2]For an overview of this literature see Segal and Whinston (2013).

a project. If all classes of employees work, there is a probability the project succeeds. In other words, all employees are required to exert effort in order to have a success. The final output will have a market value Q. When some of the employees shirk, the project will fail.

The task of the manager is to offer a laborcontract to the employees and to monitor them. Managers can differ in their knowledge about the different tasks carried out by the different classes of employees. The knowledge of the manager on the task performed by class i is denoted by $a_i \geq 0$. A key assumption of the model is that humans are bounded in the total knowledge they can acquire. Therefore, the total ability of a manager cannot be bigger than a certain threshold $\bar{a}$, modeled formally as $\sum_{i=1}^{L} a_i = \bar{a}$. A manager is characterized by his knowledge distribution $A = (a_1, ..., a_L)$. If $a_i = \bar{a}$ for any i, then the manager is called a specialist in task i. On the other hand, if $a_i = \frac{\bar{a}}{L}$ for all $i \in \{1, ..., L\}$ then the manager is called a pure generalist. In between these two extremes there exists a continuum of expertise types.

The knowledge of the manager is important since the knowledge distribution of the manager determines the probabilities with which he observes whether the employees exerted effort or not. The probability with which the manager observes whether an employee of class i worked or shirked is given by $P_i(a_i) = \beta_i + \gamma_i a_i$ with $\beta_i > 0$, $\gamma_i > 0$ and $P_i(\bar{a}) = 1$ for all i. Here, β_i represents how easy it is for a manager without any knowledge about task i to observe the effort level of the employee performing that task and γ_i is a measure of how strong the probability changes with a change in knowledge. The fact that$P_i(\bar{a}) = 1$ implies that a manager who is fully specialized in task i will always observe whether the employee performing task i exerted effort or not. Taken together this essentially means that there is an inverse relationship between β_i and γ_i. On the one hand there are tasks which are easy to observe for any level of knowledge and more specific knowledge about the task has little impact on the probability to observe and on the other hand there are tasks for which effort is hard to observe without any knowledge but where more knowledge does improve the capability of the manager to observe effort a lot. In case the manager observes the effort, this information also becomes verifiable.

3.1 Contract and timing

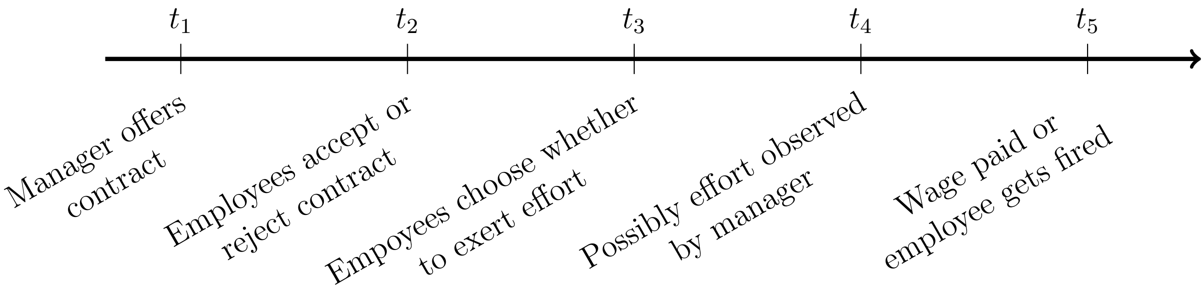
This game considers one period of work, that is, one pay period. This makes the game a one-shot interaction. I will focus on the case where a fixed wage W will be offered to each employee. For employees in the same class, the wage will be the same. However, there can be a difference in wage between employees belonging to different classes. The specified wage will only be received by the employee if he is not caught shirking. Hence, the contract consists of a wage and an effort level.[3]

The game is summarized in figure 1. First, the manager will offer a contract to the employees of the different classes, consisting of a fixed wage and the requirement to exert effort. Then the employees will, after seeing the contract, decide whether to accept or to reject the contract. If the employee participates in the contract the employee will decide

[3]Next to this conditional wage (which is in the literature often referred to as bonus), there are no other streams of payment between the manager and the employee. Since the employee could not be charged upfront, the employee faces limited liability.

whether to work or not. At the end of the work period, the manager might have observed whether the employee exerted effort or not. Lastly the employee will get paid, unless the manager verifiably knows the worker shirked, i.e. the manager observed that the worker did not exert effort.

Figure 1: Timeline



3.2 Utilities

Employees All employees will face a von Neumann Morgernstern utility function. The utility of an employee belonging to class i will be given by:

$$U_i = W_i - c_i \qquad \forall\, i \in \{1, ..., L\}$$

Furthermore, I will assume that the outside option of all the employees is the same and equal to zero.

Manager Employing workers leads to output. The manager cares about output. Assuming that the only cost the manager has to incur to produce output is the compensation of the employees, then the utility of the manager is given by:

$$U_M = Q - \sum_{i=1}^{L} n_i W_i$$

4 The optimal knowledge distribution

In this section I consider a world wherein it is not possible to buy the output produced by a class of employees on the market. In order to produce the final product, all tasks need to be done within the organization by employees supervised by the manager. What knowledge distribution of the manager would be optimal given this situation? In this section I will identify some determinants to answer this question. Both the relative importance of the tasks (modeled as the number of employees in a class) and the observability of the tasks appear to be relevant.

In what follows, I will assume that the interest of the manager is completely aligned with the interest of the owners of the firm. It could for example be that the manager is the owner or that he is the residual claimant. Although the differences between management and the owners of the firm have been discussed in much research in corporate finance, I will thus ignore any incentive problem between owners and managers. Therefore, the manager who is able to attain the highest utility given his knowledge distribution over the different tasks, would be the optimal manager from the firm's perspective.

Employees' behavior First, suppose the employees accept the labor contract offered. In order for an employee to exert effort his incentive compatibility must hold. This means that the expected utility of exerting effort must be higher than of not exerting effort, i.e., $W_i - c_i \geq \big(1 - P_i(a_i)\big)W_i$, implying that: $W_i \geq \frac{c_i}{P_i(a_i)}$. Given the fact that the employee exerts effort if he accepts the contract, the employee must be willing to accept the contract, i.e., the participation constraint must hold. Given the outside option of zero and no other streams of payment, the participation constraint will always be satisfied if the incentive compatibility constraint is satisfied.

Manager's behavior The manager will maximize his expected utility while anticipating the employees' behavior. Clearly, it will always be optimal for the manager to require the employees to exert effort. The manager must offer each class of employees enough wage as to satisfy all the incentive compatibility constraints. Therefore the wage offered by the manager solves:

$$\begin{aligned} \underset{\forall W_i}{\text{maximize}} \quad & E\left(Q - \sum_{i=1}^{L} n_i W_i\right) \\ \text{subject to} \quad & W_i \geq \frac{c_i}{P_i(a_i)} \qquad\qquad \forall\, i \in \{1, ..., L\} \end{aligned}$$

Since wage is negative in the utility function of the manager, it is easy to see that setting all incentive compatibility constraints binding is optimal for the manager. Hence, the wage offered to an employee belonging to class i is given by: $W_i = \frac{c_i}{P_i(a_i)} = \frac{c_i}{\beta_i + \gamma_i a_i}$. Although the wage is paid at the end of the period, the manager can not renege on paying it because the contract offered is legally enforceable.[4]

Hence, if the manager always observes the effort of an employee (that is, if $P_i(a_i) = 1$) then the employee only needs to be compensated for his costs of effort. The wage paid to an employee increases the lower the probability the manager observes the effort of the employee. A lower probability of observing the effort means that the information asymmetry between the manager and the employee is bigger, and that the manager needs to compensate the employee more in order to prevent the employee from shirking. This extra wage the manager pays to the employee is formally known as informational rent. Here we find that for tasks that are easy to observe by it's nature (i.e., tasks with a high β_i) and for tasks the manager knows a lot about (i.e., high a_i), the informational rent is smaller.

[4]There is also a strand of literature that argues that even though the contract is not enforceable, the manager will still not renege due to reputational concerns (see e.g. Baker et al. (1994))

Optimal manager To find the optimal knowledge distribution of the manager, we must take into account the behavior of the manager given his knowledge distribution. Also, we are constrained by the bounded knowledge of the manager. This problem looks like:

$$\begin{aligned}
&\underset{\forall\, a_i}{\text{maximize}} && E\left(Q - \sum_{i=1}^{L} n_i \frac{c_i}{\beta_i + \gamma_i a_i}\right) \\
&\text{subject to} && \sum_{i=1}^{L} a_i = \bar{a}
\end{aligned}$$

As we know from above, an increase in a_i will always lead to an reduction of informational rent the manager pays to an employee of class i. Ceteris paribus, the manager would always prefer to know more about a task. But, given the constraint, an increase in knowledge will result in a less knowledge about all the other classes. Hence, determining the optimal knowledge distribution leads to a tradeoff.

Proposition 1. *If $\beta_i = \beta$ and $\gamma_i = \gamma$ for all $i \in \{1, ..., L\}$ and we are not in a corner solution, then the optimal knowledge distribution of the manager is given by:*

$$a_i = \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}} \bar{a} + \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}} \frac{L\beta}{\gamma} - \frac{\beta}{\gamma}$$

Proof: See appendix.

Intuitively, proposition 1 implies that the optimal manager knows more about tasks which are more important for the business, modeled by relatively large n_i. This can be seen in the first term, which states that a_i gets a share of the total $\bar{a}$ based on it's weight given by $\frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}}$. Interestingly, the dispersion of the knowledge distribution gets bigger for larger $\frac{\beta}{\gamma}$, represented by the second and third term. This means that firms where each of the tasks are relatively easy to observe without knowledge, the knowledge dispersion of the optimal manager is bigger compared to a firm where each of the tasks are relatively hard to observe without knowledge. Thus, it is possible that there exist two firms, where in each case the manager has to observe the same number of classes with the same relative importance, but the optimal knowledge set of the manager differs. The firm where the tasks are harder to observe without any knowledge requires a more generalist manager. Since I assumed that a_i can not be negative, it is possible that the optimal knowledge distribution is a corner solution. In case $\frac{\beta}{\gamma}$ is large and/or the difference in number of employees of different classes is big, the more likely we are in a corner solution. A corner solution implies that it is optimal that the manager doesn't have any knowledge about at least one task, i.e. a less generalist manager. See table 1 for a numerical example.

Proposition 2. *If $n_i c_i = k$ for all $i \in \{1, ..., L\}$ and we are not in a corner solution, then the optimal knowledge distribution of the manager is given by:*

$$a_i = \frac{1}{\sqrt{\gamma_i} \sum_{j=1}^{L} \frac{1}{\sqrt{\gamma_j}}} \bar{a} + \frac{1}{\sqrt{\gamma_i} \sum_{j=1}^{L} \frac{1}{\sqrt{\gamma_j}}} \sum_{j=1}^{L} \frac{\beta_j}{\gamma_j} - \frac{\beta_i}{\gamma_i}$$

Table 1: Numerical example

Suppose $\bar{a} = 1$, $c_i = 1 \,\forall\, i$ *and* $L = 3$, *with* $n_1 = 1$, $n_2 = 4$, $n_3 = 49$			
$P(a_i)$	a_1	a_2	a_3
$\beta = 0$, $\gamma = 1$	1/10	2/10	7/10
$\beta = \frac{1}{4}$, $\gamma = \frac{3}{4}$*	0	1/27	26/27
$\beta = \frac{9}{10}$, $\gamma = \frac{1}{10}$*	0	0	1

Note: * contains a corner solution.

Proof: See appendix.

Consider two different tasks i and k. From proposition 2 it follows that $a_i \geq a_k$ if $\gamma_i > \gamma_k$ (or $\beta_i < \beta_k$). The intuition for this result is as follows: the higher the γ of a task, the more extra knowledge about that task contributes to the probability of observing the effort of an employee working on that task. Therefore, extra knowledge of the manager has more influence on reducing the information asymmetry for that task and, hence, on reducing the informational rent.

We have seen that the manager minimizes the total informational rent he has to pay to his employees. This can be done via two different mechanisms, namely minimizing the rent you have to pay to a lot of employees (large n_i) or gathering knowledge for tasks for which you can reduce the information asymmetry the most (relatively small β_i and large γ_i). From this, we can derived the following simple implications: (i) The manager should know more about tasks that are relatively more important for the firm's business (ii) The manager should know more about tasks that are by it's nature harder to observe, i.e., tasks for which the marginal contribution of extra knowledge on the probability of observing effort is relatively high.

5 The make-or-buy decision: expensive labor

The previous section identified some important determinants of the optimal knowledge distribution of the manager. However, the previous section did not consider the option to buy the result of a task from another firm, i.e., to use suppliers in the production process. Introducing the possibility to outsource a task to the market will make it possible to use the framework to study a whole different question, namely: What are the boundaries of the firm? Although there exists a whole strand of literature which focuses on exactly this question, the approach to the question will be new. As pointed out by Dessein (2013) there is need for a formal theory in which the manager plays a key role in the make-or-buy decision. What follows can be seen as a very limited but first step towards such a theory.

In order to model a world in which tasks can be outsourced, some more assumptions are needed. Not only the costs of producing the task yourself matter, but also at what cost this service is available on the market. I will assume that the suppliers offering their services, will be specialized in that task. For example, one could think about free-lancers offering their services. Moreover, and in contrast to an obligation to use one's best effort as typically is the case for white collar jobs with a fixed wage, the outsourcing contract will be a duty to achieve a given result. Because of this difference between the effort-based laborcontract and the result-based contract with the supplier, the contract with the supplier needs to be more advanced. This will result in a higher transaction cost τ.

Manager's behavior Let's first consider the make-or-buy decision of a manager with a given knowledge distribution. It is easy to see that for a given manager the make-or-buy decision can be considered for each task separately. If the manager decides to buy task i on the market it will cost $n_i(c_i + \tau_i)$. From the previous section we know that if the manager supervises employees to produce task i it will cost $n_i \frac{c_i}{\beta_i + \gamma_i a_i}$. Therefore, the make-or-buy decision simply boils down to the following decision rule:

$$\begin{aligned} \text{Buy task } i \text{ if:} \quad n_i(c_i + \tau_i) &\leq n_i \frac{c_i}{\beta_i + \gamma_i a_i} \\ \Leftrightarrow \qquad \tau_i &\leq \frac{c_i}{\beta_i + \gamma_i a_i} - c_i \end{aligned}$$

That is,

$$\boxed{\text{Buy task } i \text{ if: transaction cost}_i \leq \text{informational rent}_i}$$

Thus, the manager faces a trade-off between costs he has to pay in order to incentivize his employees and costs associated with outsourcing. Since we know that managers with different knowledge distributions pay different amounts of informational rent to their employees, it could well be that one kind of manager would outsource a task while a different kind of manager would not. So, the knowledge distribution of the manager plays an important role in the make-or-buy decision.

Optimal organization In this framework, the organization that is able to produce the output at the lowest total costs is the optimal organization. This firm will be characterized by the knowledge distribution of it's manager and a description which tasks are done by employees supervised by the manager and which are outsourced.

The remainder of this section will focus on the situation for which the different tasks that are needed to produce the final output only differ in their relative importance, i.e., the number of employees working on the task. In order to do so, I assume the same probability function of observing effort for all tasks. More specific, I assume $\beta_i = 0$ and $\gamma_i = 1$ for all i together with $\bar{a} = 1$, that is, $P_i(a_i) = a_i$. Moreover, let the cost of effort be the same for all the different classes and be given by $c_i = 1$ for all i. These assumptions will be convenient as they assure that the optimal knowledge distribution of the manager will never be a 'corner solution'. In this case, and as we have seen in the previous section, the optimal knowledge distribution is given by: $a_i = \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}}$. Furthermore, I'll assume that $\tau_i = \tau$ for all i. This means that the transaction cost of outsourcing is equal for all tasks.

Lemma 1. *Order the tasks such that $n_1 < n_2 < ... < n_L$ and let $k \in \{1, 2, ..., L\}$. If it is optimal to outsource k tasks, then it is optimal to outsource tasks 1 to k.*

Proof: See appendix.

Lemma 1 shows that the manager will never outsource a task if a less important task can be outsourced instead.

Note that this is only the case if all possible combinations of outsourcing and supervising employees are considered. This is not true, however, if the maximum number of tasks that can be outsourced is limited. For example, suppose that the maximum number of tasks a firm can outsource is limited to one. In this case it could well be optimal to outsource a task with a relative large n for low values of τ. To see this, note that the total labor costs can be split up in two parts. On the one hand the wage costs of the employees and on the other hand the costs of the tasks that are outsourced. Clearly, the wage costs are lower if a task with higher n is outsourced and these are independent of τ. However, the costs of the task that is outsourced are higher for a task with higher n and are (linear) increasing in τ. Therefore, only if τ passes a threshold it will be optimal to outsource the task with the least relative importance.

Proposition 3. *Order the tasks such that $n_1 < n_2 < ... < n_L$ and let $k \in \{1, 2, ..., L-1\}$. No task will be outsourced if $\tau > 2\left(\frac{\sum_{i=1}^{L}\sqrt{n_i}}{\sqrt{n_1}} - 1\right)$*
It is optimal to outsource k tasks if $2\left(\frac{\sum_{i=k+1}^{L}\sqrt{n_i}}{\sqrt{n_{k+1}}} - 1\right) < \tau < 2\left(\frac{\sum_{i=k}^{L}\sqrt{n_i}}{\sqrt{n_k}} - 1\right)$
It is never optimal to outsource all L tasks.

Proof: See appendix.

From proposition 3 it follows that if the mark-up on outsourcing is above a certain threshold, no task will be outsourced. This threshold will be higher if L get's bigger and if n_1 get's smaller. It is optimal to not outsource a task if the transaction cost are at least higher than the highest informational rent. As $n_1 < n_2 < ... < n_L$ we know from the previous section that the informational rent paid on task 1 is more than on the other tasks. This informational rent on task 1 is bigger the lower the relative importance of task 1. Therefore, if there are more other tasks or if n_1 becomes smaller, the threshold goes up.

Moreover, proposition 3 shows that if τ is getting smaller it becomes optimal to outsource more tasks. This can be seen by the fact that both $2(\frac{\sum_{i=k}^{L}\sqrt{n_i}}{\sqrt{n_k}} - 1)$ and $2(\frac{\sum_{i=k+1}^{L}\sqrt{n_i}}{\sqrt{n_{k+1}}} - 1)$ are lower for higher k. Intuitively, this means that the cheaper outsourcing is, it is optimal to outsource more tasks.

Combining this with the result form lemma 1 it implies the following process. If transaction costs are close to zero it is optimal to have a manager fully specialize in the most important task (the task with largest n) and all other tasks being outsourced. In this case the manager always observes the effort of the employees working on the same task. Therefore, total wage costs do not include an informational rent but only the (small) transaction cost for the outsourced tasks. When transaction costs are getting higher, on some moment it will become optimal to start paying an informational rent instead of the transaction costs. For the task with the largest number of people working on it, the savings in transaction cost will be highest. Thus, for higher transaction cost the firm will start hiring employees of the task with largest n. As long as transaction cost rise, it is optimal to have more and more tasks performed by employees. This process continues until transaction costs are so high, that it will be optimal not to outsource any task.

6 Conclusion

This study adds a model to the literature of the theory of the firm, in which the knowledge distribution of the manager plays a key role. Both the relative importance of a task and the observability of a task are relevant for the optimal knowledge distribution of the manager. All else equal, the manager should know more about tasks that are more important for the firm's business as this minimizes the informational rent the managers has to pay to a lot of employees. Also, the manager should know more about tasks that are harder to observe, as for these tasks an increase in knowledge has the largest marginal reduction in informational rent. There exists a trade-off between transaction costs paid when outsourcing a task and informational rent when supervising employees. It is never optimal to outsource a task if there is a less important task not outsourced. When transaction costs go down, it is optimal to outsource more tasks.

This analysis provides a limited step towards incorporation of a manager with cognitive limitations into the theory of the firm. There remain several ways to extend this analysis.

First, this paper is limited in that it has no integrated explanation where both costs and benefits from integration come from. More specific, it does not give an explanation for the existence of the transaction costs. Clearly, this theory can make statements like, it is never optimal to outsource a task if there is a less important task not outsourced. And for these statements, it is not necessary to know the source of the transaction costs. But for the theory to be a more formal theory, both costs and benefits should be explained.

Second, as this model is a first attempt towards a theory of the firm in which the manager plays a key role, the model presented is reduced to a basic model. In the analysis of the make-or-buy decision all tasks are assumed to be the same except for the number of people needed to fulfill a task. However, other differences between tasks can also influence the knowledge distribution of the manager. This paper, for example, shows that the observability of the task influences the knowledge distribution. Hence, other differences between the tasks would also influence the make-or-buy decision. Future work can extend the model by allowing for other differences between the tasks.

Third, this paper provides a theoretical model for a theory of the firm, but totally ignores empirical testing. Of course, however, theory and empirical testing should go hand in hand. Therefore, next to expanding the theoretical work, the propositions from this theoretical model should be empirically tested in new work.

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Appendix

Proposition 1. *If $\beta_i = \beta$ and $\gamma_i = \gamma$ for all $i \in \{1, ..., L\}$ and we are not in a corner solution, then the optimal knowledge distribution of the manager is given by:*

$$a_i = \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}} \bar{a} + \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}} \frac{L\beta}{\gamma} - \frac{\beta}{\gamma}$$

Proof. Suppose $\beta_i = \beta$ and $\gamma_i = \gamma$ for all $i \in \{1, ..., L\}$. Given this situation, the optimal knowledge distribution is defined by:

$$\begin{aligned} \underset{\forall\, a_i}{\text{maximize}} \quad & E\Bigg(Q - \sum_{i=1}^{L} n_i \frac{c_i}{\beta + \gamma a_i}\Bigg) \\ \text{subject to} \quad & \sum_{i=1}^{L} a_i = \bar{a} \end{aligned}$$

This maximization could be rewritten to:

$$\underset{\forall\, a_i}{\text{minimize}} \quad \sum_{i=1}^{L} \frac{n_i c_i}{\beta + \gamma a_i}$$

$$\text{subject to} \quad \sum_{i=1}^{L} a_i = \bar{a}$$

The Lagrangian for this problem is given by:

$$\Lambda = \sum_{i=1}^{L} \frac{n_i c_i}{\beta + \gamma a_i} + \lambda\left(\sum_{i=1}^{L} a_i - \bar{a}\right)$$

Leading to the following first order conditions:

$$\frac{\partial \Lambda}{\partial a_i} = -\frac{\gamma n_i c_i}{(\beta + \gamma a_i)^2} + \lambda = 0 \qquad \forall\, i \in \{1, ..., L\} \tag{1}$$

$$\frac{\partial \Lambda}{\partial \lambda} = \sum_{i=1}^{L} a_i - \bar{a} = 0 \tag{2}$$

Solving (1) for a_i gives:

$$a_i = \frac{1}{\gamma}\left(\sqrt{\frac{\gamma n_i c_i}{\lambda}} - \beta\right) = \frac{1}{\gamma}\left(\frac{1}{\sqrt{\lambda}}\sqrt{\gamma n_i c_i} - \beta\right) \tag{3}$$

Substituting (3) in (2) gives:

$$\sum_{i=1}^{L} \frac{1}{\gamma}\left(\sqrt{\frac{\gamma n_i c_i}{\lambda}} - \beta\right) - \bar{a} = 0$$

Which we can rewrite to:

$$\frac{1}{\sqrt{\lambda}} = \frac{\gamma \bar{a} + L\beta}{\sum_{i=1}^{L} \sqrt{\gamma n_i c_i}} \tag{4}$$

Substitution of the result from (4) in (3) gives:

$$\begin{aligned} a_i &= \frac{1}{\gamma}\left(\frac{\gamma \bar{a} + L\beta}{\sum_{j=1}^{L} \sqrt{\gamma n_j c_j}} \sqrt{\gamma n_i c_i} - \beta\right) \\ &= \frac{1}{\gamma} \frac{\gamma \bar{a} \sqrt{\gamma n_i c_i}}{\sum_{j=1}^{L} \sqrt{\gamma n_j c_j}} + \frac{1}{\gamma} \frac{L\beta \sqrt{\gamma n_i c_i}}{\sum_{j=1}^{L} \sqrt{\gamma n_j c_j}} - \frac{1}{\gamma}\beta \\ &= \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}} \bar{a} + \frac{\sqrt{n_i c_i}}{\sum_{j=1}^{L} \sqrt{n_j c_j}} \frac{L\beta}{\gamma} - \frac{\beta}{\gamma} \end{aligned}$$

□

Proposition 2. *If $n_i c_i = k$ for all $i \in \{1, ..., L\}$ and we are not in a corner solution, then the optimal knowledge distribution of the manager is given by:*

$$a_i = \frac{1}{\sqrt{\gamma_i}\sum_{j=1}^{L}\frac{1}{\sqrt{\gamma_j}}}\bar{a} + \frac{1}{\sqrt{\gamma_i}\sum_{j=1}^{L}\frac{1}{\sqrt{\gamma_j}}}\sum_{j=1}^{L}\frac{\beta_j}{\gamma_j} - \frac{\beta_i}{\gamma_i}$$

Proof. Suppose $n_i c_i = k$ for all $i \in \{1, ..., L\}$. Given this situation, the optimal knowledge distribution is defined by:

$$\begin{aligned} \underset{\forall\, a_i}{\text{maximize}} \quad & E\Bigg(Q - \sum_{i=1}^{L}\frac{k}{\beta_i + \gamma_i a_i}\Bigg) \\ \text{subject to} \quad & \sum_{i=1}^{L} a_i = \bar{a} \end{aligned}$$

This maximization could be rewritten to:

$$\begin{aligned} \underset{\forall\, a_i}{\text{minimize}} \quad & \sum_{i=1}^{L}\frac{k}{\beta_i + \gamma_i a_i} \\ \text{subject to} \quad & \sum_{i=1}^{L} a_i = \bar{a} \end{aligned}$$

The Lagrangian for this problem is given by:

$$\Lambda = \sum_{i=1}^{L}\frac{k}{\beta_i + \gamma_i a_i} + \lambda\Bigg(\sum_{i=1}^{L} a_i - \bar{a}\Bigg)$$

Leading to the following first order conditions:

$$\frac{\partial \Lambda}{\partial a_i} = -\frac{\gamma_i k}{(\beta_i + \gamma_i a_i)^2} + \lambda = 0 \qquad \forall\, i \in \{1, ..., L\} \tag{5}$$

$$\frac{\partial \Lambda}{\partial \lambda} = \sum_{i=1}^{L} a_i - \bar{a} = 0 \tag{6}$$

Solving (5) for a_i gives:

$$a_i = \frac{1}{\gamma_i}\Bigg(\sqrt{\frac{\gamma_i k}{\lambda}} - \beta_i\Bigg) = \frac{1}{\gamma_i}\Bigg(\frac{1}{\sqrt{\lambda}}\sqrt{\gamma_i k} - \beta_i\Bigg) \tag{7}$$

Substituting (3) in (2) gives:

$$\sum_{i=1}^{L}\frac{1}{\gamma_i}\Bigg(\sqrt{\frac{\gamma_i k}{\lambda}} - \beta_i\Bigg) - \bar{a} = 0$$

Which we can rewrite to:

$$\frac{1}{\sqrt{\lambda}} = \frac{\bar{a} + \sum_{i=1}^{L} \frac{\beta_i}{\gamma_i}}{\sum_{i=1}^{L} \frac{1}{\gamma_i}\sqrt{\gamma_i k}} \tag{8}$$

Substitution of the result from (8) in (7) gives:

$$\begin{aligned} a_i &= \frac{1}{\gamma_i}\left(\frac{\bar{a} + \sum_{j=1}^{L} \frac{\beta_j}{\gamma_j}}{\sum_{j=1}^{L} \frac{1}{\gamma_j}\sqrt{\gamma_j k}}\sqrt{\gamma_i k} - \beta_i\right) \\ &= \frac{1}{\gamma_i}\frac{\bar{a}\sqrt{\gamma_i k}}{\sum_{j=1}^{L} \frac{1}{\gamma_j}\sqrt{\gamma_j k}} + \frac{1}{\gamma_i}\frac{\sum_{j=1}^{L} \frac{\beta_j}{\gamma_j}\sqrt{\gamma_j k}}{\sum_{j=1}^{L} \frac{1}{\gamma_j}\sqrt{\gamma_j k}} - \frac{1}{\gamma_i}\beta_i \\ &= \frac{1}{\sqrt{\gamma_i}\sum_{j=1}^{L} \frac{1}{\sqrt{\gamma_j}}}\bar{a} + \frac{1}{\sqrt{\gamma_i}\sum_{j=1}^{L} \frac{1}{\sqrt{\gamma_j}}}\sum_{j=1}^{L} \frac{\beta_j}{\gamma_j} - \frac{\beta_i}{\gamma_i} \end{aligned}$$

□

Lemma 1. *Order the tasks such that $n_1 < n_2 < ... < n_L$ and let $k \in \{1, 2, ..., L\}$. If it is optimal to outsource k tasks, then it is optimal to outsource tasks 1 to k.*

Proof. Suppose $P_i(a_i) = a_i$, $c_i = 1$ and $\tau_i = \tau$ for all i and order the tasks such that $n_1 < n_2 < ... < n_L$. An organizational structure is optimal if the total wage cost are lower than for all other possible structures. The total minimum costs for the organization that only outsources task 1 are given by:

$$n_1(1+\tau) + \sum_{i=2}^{L} n_i W_i = n_1(1+\tau) + \sum_{i=2}^{L} n_i \frac{\sum_{j=2}^{L}\sqrt{n_j}}{\sqrt{n_i}} = n_1(1+\tau) + \sum_{i=2}^{L}\sqrt{n_i}\sum_{j=2}^{L}\sqrt{n_j}$$

The total minimum costs for the organization that only outsources task $m > 1$ are given by:

$$n_m(1+\tau) + \sum_{i=1, i\neq m}^{L} n_i W_i = n_m(1+\tau) + \sum_{i=1, i\neq m}^{L}\sqrt{n_i}\sum_{j=1, j\neq m}^{L}\sqrt{n_j}$$

The organization that outsources task m has lower total costs than the organization that outsources task 1 if:

$$n_m(1+\tau) + \sum_{i=1, i\neq m}^{L}\sqrt{n_i}\sum_{j=1, j\neq m}^{L}\sqrt{n_j} < n_1(1+\tau) + \sum_{i=2}^{L}\sqrt{n_i}\sum_{j=2}^{L}\sqrt{n_j}$$

$$n_m(1+\tau) + 2\sqrt{n_1}\sum_{i=1, i\neq m}^{L}\sqrt{n_i} - n_1 < n_1(1+\tau) + 2\sqrt{n_m}\sum_{i=2}^{L}\sqrt{n_i} - n_m$$

$$(n_m - n_1)\tau < 2\left(\sqrt{n_m}\sum_{i=2}^{L}\sqrt{n_i} - \sqrt{n_1}\sum_{i=1, i\neq m}^{L}\sqrt{n_i} + (n_1 - n_m)\right)$$

$$\tau < \frac{2\left(\sqrt{n_m}\sum_{i=2}^{L}\sqrt{n_i} - \sqrt{n_1}\sum_{i=1, i\neq m}^{L}\sqrt{n_i} + (n_1 - n_m)\right)}{(n_m - n_1)}$$

The total minimum costs for the organization that outsources both task 1 and task m are given by:

$$(n_1 + n_m)(1+\tau) + \sum_{i=2, i\neq m}^{L} n_i W_i = (n_1 + n_m)(1+\tau) + \sum_{i=2, i\neq m}^{L}\sqrt{n_i}\sum_{j=2, j\neq m}^{L}\sqrt{n_j}$$

The organization that only outsources task m has lower total costs than the organization that outsources both task 1 and m if:

$$n_m(1+\tau) + \sum_{i=1, i\neq m}^{L}\sqrt{n_i}\sum_{j=1, j\neq m}^{L}\sqrt{n_j} < (n_1 + n_m)(1+\tau) + \sum_{i=2, i\neq m}^{L}\sqrt{n_i}\sum_{j=2, j\neq m}^{L}\sqrt{n_j}$$

$$2\sqrt{n_1}\sum_{i=1, i\neq m}^{L}\sqrt{n_i} - n_1 < n_1(1+\tau)$$

$$\frac{2\left(\sqrt{n_1}\sum_{i=1, i\neq m}^{L}\sqrt{n_i} - n_1\right)}{n_1} < \tau$$

$$\tau > \frac{2\sum_{i=2, i\neq m}^{L}\sqrt{n_i}}{\sqrt{n_1}}$$

The organization that only outsources task m can only be an optimal organization if there exist a τ for which this organization has both lower costs than the organization that outsources task 1 and the organization that outsources both task 1 and m. This τ exists if:

$$\frac{2\left(\sqrt{n_m}\sum_{i=2}^{L}\sqrt{n_i} - \sqrt{n_1}\sum_{i=1, i\neq m}^{L}\sqrt{n_i} + (n_1 - n_m)\right)}{(n_m - n_1)} > \frac{2\sum_{i=2, i\neq m}^{L}\sqrt{n_i}}{\sqrt{n_1}}$$

$$\frac{2\left(\sqrt{n_m}\sum_{i=2, i\neq m}^{L}\sqrt{n_i} - \sqrt{n_1}\sum_{i=2, i\neq m}^{L}\sqrt{n_i}\right)}{(n_m - n_1)} > \frac{2\sum_{i=2, i\neq m}^{L}\sqrt{n_i}}{\sqrt{n_1}}$$

$$\frac{2(\sqrt{n_m} - \sqrt{n_1})\sum_{i=2, i\neq m}^{L}\sqrt{n_i}}{(n_m - n_1)} > \frac{2\sum_{i=2, i\neq m}^{L}\sqrt{n_i}}{\sqrt{n_1}}$$

$$\frac{(\sqrt{n_m}-\sqrt{n_1})}{(n_m-n_1)} > \frac{1}{\sqrt{n_1}}$$

$$\sqrt{n_1}(\sqrt{n_m}-\sqrt{n_1}) > (n_m-n_1)$$

$$\sqrt{n_1}\sqrt{n_m} > n_m$$

$$n_1 > n_m$$

Since we started with $n_1 < n_m$, this condition can never be satisfied. Thus, there does not exist a τ for which an organization that only outsources task $m > 1$ can be optimal.

Now consider all the $\binom{L}{k}$ possible firms that outsource k tasks, with $1 \leq k < L$. Since we assumed that tasks are identical except for n_i, we can well think about all the k task that are outsourced as one big task. Formally, if k tasks are outsourced, let $m_{kj} = \sum_{i \in \{outsourced\, tasks\}} n_i$ with $j \in \{1, 2, ..., \binom{L}{k}\}$. For each k, order m_{kj} such that $m_{k1} < m_{k2} < ... < m_{k\binom{L}{k}}$. Since $n_1 < n_2 < ... < n_L$, it follows that $m_{k1} = \sum_{i=1}^{k} n_i$. This situation reduces to the same situation as the situation above. Following the same logic, comparing m_{kj} with both m_1 and $m_{(k+1)1}$ for $j \neq 1$, m_{kj} can never be optimal.

Lastly, consider $k = L$. For this situation it is trivial that if L tasks are outsourced this need to be tasks 1 to L. Hence, it can never be optimal if k tasks are outsourced that this are not tasks $\{n_1, n_2, ..., n_k\}$.

□

Proposition 3. *Order the tasks such that* $n_1 < n_2 < ... < n_L$ *and let* $k \in \{1, 2, ..., L-1\}$*.*
No task will be outsourced if $\tau > 2\left(\frac{\sum_{i=1}^{L}\sqrt{n_i}}{\sqrt{n_1}} - 1\right)$
It is optimal to outsource k tasks if $2\left(\frac{\sum_{i=k+1}^{L}\sqrt{n_i}}{\sqrt{n_{k+1}}} - 1\right) < \tau < 2\left(\frac{\sum_{i=k}^{L}\sqrt{n_i}}{\sqrt{n_k}} - 1\right)$
It is never optimal to outsource all L tasks.

Proof. Order the tasks such that $n_1 < n_2 < ... < n_L$ and let $k \in \{1, 2, ..., L\}$. From lemma 1 we know that if it is optimal to outsource k tasks, then it is optimal to outsource tasks 1 to k. Therefore, the wage costs if it is optimal to outsource k tasks are given by

$$\sum_{i=1}^{k} n_i(1+\tau) + \sum_{i=k+1}^{L} n_i W_i = \sum_{i=1}^{k} n_i(1+\tau) + \sum_{i=k+1}^{L} n_i \frac{\sum_{j=1}\sqrt{n_j}}{\sqrt{n_i}}$$
$$= \sum_{i=1}^{k} n_i(1+\tau) + \sum_{i=k+1}^{L}\sqrt{n_i}\sum_{j=k+1}^{L}\sqrt{n_j}$$

.

It is optimal to outsource no task if the following conditions hold:

$$\sum_{i=1}^{L}\sqrt{n_i}\sum_{j=1}^{L}\sqrt{n_j} < \sum_{i=1}^{k} n_i(1+\tau) + \sum_{i=k+1}^{L}\sqrt{n_i}\sum_{j=k+1}^{L}\sqrt{n_j} \quad \forall\, k \in \{1,2,...,L\}$$

This is equivalent to:

$$2\sum_{i=1}^{k}\sqrt{n_i}\sum_{j=1}^{L}\sqrt{n_j} - \sum_{i=1}^{k} n_i < \sum_{i=1}^{k} n_i(1+\tau) \quad \forall\, k \in \{1,2,...,L\}$$

$$2\left(\sum_{i=1}^{k}\sqrt{n_i}\sum_{j=1}^{L}\sqrt{n_j} - \sum_{i=1}^{k} n_i\right) < \sum_{i=1}^{k} n_i\tau \quad \forall\, k \in \{1,2,...,L\}$$

$$\tau > 2\left(\frac{\sum_{i=1}^{k}\sqrt{n_i}\sum_{j=1}^{L}\sqrt{n_j}}{\sum_{i=1}^{k} n_i} - 1\right) \quad \forall\, k \in \{1,2,...,L\}$$

As $\frac{\sqrt{n_1}}{n_1} = \frac{\sqrt{n_1}\sum_{i=1}^{k}\frac{n_i}{n_1}}{\sum_{i=1}^{k} n_i} = \frac{\frac{1}{\sqrt{n_1}}\sum_{i=1}^{k} n_i}{\sum_{i=1}^{k} n_i}$ and since $n_1 < n_2 < ... < n_L$ gives $\frac{1}{\sqrt{n_1}}\sum_{i=1}^{k} n_i > \sum_{i=1}^{k}\sqrt{n_i}$ for $k \in \{2,3,...,L\}$ it follows that $\frac{\sqrt{n_1}}{n_1} > \frac{\sum_{i=1}^{k}\sqrt{n_i}}{\sum_{i=1}^{k} n_i}$ for $k \in \{2,3,...,L\}$.

Therefore all conditions

$$\tau > 2\left(\frac{\sum_{i=1}^{k}\sqrt{n_i}\sum_{j=1}^{L}\sqrt{n_j}}{\sum_{i=1}^{k} n_i} - 1\right) \quad \forall\, k \in \{1,2,...,L\}$$

are satisfied if

$$\tau > 2\left(\frac{\sqrt{n_1}\sum_{j=1}^{L}\sqrt{n_j}}{n_1} - 1\right)$$

It is optimal to outsource k tasks if the following conditions hold:

$$\sum_{i=1}^{k} n_i(1+\tau) + \sum_{i=k+1}^{L}\sqrt{n_i}\sum_{j=k+1}^{L}\sqrt{n_j} <$$

$$\sum_{i=1}^{m} n_i(1+\tau) + \sum_{i=m+1}^{L}\sqrt{n_i}\sum_{j=m+1}^{L}\sqrt{n_j} \quad \forall\, m \in \{1,2,...,L\} \text{ and } m \neq k$$

That is:

$$\sum_{i=m+1}^{k} n_i(1+\tau) + \sum_{i=k+1}^{L}\sqrt{n_i}\sum_{j=k+1}^{L}\sqrt{n_j} < \sum_{i=m+1}^{L}\sqrt{n_i}\sum_{j=m+1}^{L}\sqrt{n_j} \quad \forall m \in \{1,2,...,L\} \text{ and } m < k$$

$$\sum_{i=k+1}^{L}\sqrt{n_i}\sum_{j=k+1}^{L}\sqrt{n_j} < \sum_{i=k+1}^{m} n_i(1+\tau) + \sum_{i=m+1}^{L}\sqrt{n_i}\sum_{j=m+1}^{L}\sqrt{n_j} \quad \forall m \in \{1,2,...,L\} \text{ and } m > k$$

$$\Leftrightarrow$$

$$\tau < 2\left(\frac{\sum_{i=m+1}^{k}\sqrt{n_i}\sum_{j=m+1}^{L}\sqrt{n_j}}{\sum_{i=m+1}^{k}n_i} - 1\right) \quad \forall\, m \in \{1,2,...,L\} \text{ and } m < k$$

$$\tau > 2\left(\frac{\sum_{i=k+1}^{m}\sqrt{n_i}\sum_{j=k+1}^{L}\sqrt{n_j}}{\sum_{i=k+1}^{m}n_i} - 1\right) \quad \forall\, m \in \{1,2,...,L\} \text{ and } m > k$$

$$\Leftrightarrow$$

$$\tau < 2\left(\frac{\sqrt{n_k}\sum_{j=k}^{L}\sqrt{n_j}}{n_k} - 1\right)$$

$$\tau > 2\left(\frac{\sqrt{n_{k+1}}\sum_{j=k+1}^{L}\sqrt{n_j}}{n_{k+1}} - 1\right)$$

$\square$