Mitigating end-effects in Production Scheduling

Bachelor Thesis
Econometrie en Operationele Research

Ivan Olthuis
359299

Supervisor: Dr. Wilco van den Heuvel

June 30, 2014

Abstract

In this report, a solution for the common problem of end-effects in production scheduling is investigated. Fisher et. al (2001) developed a method based on ending inventory valuation. This algorithm will be compared with the classical Wagner-Whitin algorithm and the Silver-Meal heuristic, where a rolling horizon setting is used. Furthermore, this algorithm is compared with an algorithm, as described by Stadtler (2000), that is specifically used for mitigating end-effects.

(End-effects; Rolling horizon; Ending Inventory Valuation algorithm)

Contents

1 Introduction and Problem Description

It is important for companies to know how they need to plan their production. When a lot-sizing model is used, an appropriate model horizon T should be chosen in order to plan this production. In this report, a dynamic lot-sizing model is used, where also the demand is time varying. The chosen horizon should be as large as the expected time that the company will be in business, but often the model horizon is smaller than this expected time. This means that production decisions are made for a time period that contains only several months or is based on a yearly time horizon. The main reason why these short horizons are used for the production planning is that there is great uncertainty about the future demand, as it is difficult to obtain accurate forecasts of demands in the future.

However, as one can imagine this approach has also some disadvantages. This planning is based only on the short run and that does not mean that this planning is also optimal in the long run. As a consequence, the use of short model horizons leads to end-effects, as described by Grinold (1983). In the short run it is optimal to end with zero inventory, but in the long run this can be inefficient as there can be high setup costs in the future. Wagner and Whitin (1958) and Federgruen and Tzur (1994) obtain a solution that is optimal for an infinite horizon, by formulating sufficient conditions on the values of demand for a particular horizon T. The goal of this report is to describe another approach in order to deal with these end-effects in an efficient manner and to compare this method to other methods that can be used in order to mitigate these end-effects. In this case the ideas as presented by Fisher et al. (2001) are used.

The main idea of their approach is to include a valuation term for end-of-horizon inventory in the objective function of the short horizon model. When such a valuation term is used, it is not necessarily optimal to end with zero inventory at the end of the horizon, because this valuation term takes this level of ending inventory into account. This valuation function is defined as follows:

$$V(I_T) = K - \frac{h}{2D}(x^* - I_T)^2 \tag{1}$$

This term is based on the classical EOQ framework. Here, D is the constant demand rate, K is the cost for placing an order, h the holding cost, T the model horizon, and x^* follows from the EOQ formula as $x^* = \sqrt{(2KD/h)}$. This valuation term captures the future setup costs that are avoided by having this ending inventory. The inventory cost for depleting the I_T units is equal to $hI_T^2/2D$. Subtracting this cost from the optimal cost under the optimal EOQ policy, gives a reduction of $(I_T/D)C^* - hI_T^2/2D$. Rewriting this term results in the valuation function as given in formula (1).

The goal of this report is to implement the valuation term in order to avoid the end-effects when solving the dynamic lot-sizing problem (DLSP). The dynamic lot-sizing problem has as purpose to find a production schedule where the total costs of storing the products and producing these products is minimized. In this case, there is only one type of product, with a variable demand in each period. In order to solve the DLSP, Wagner and Whitin (1958) developed an algorithm based on dynamic programming. However, in this algorithm it is optimal to end with zero inventory at the end of the model horizon, as time beyond this model horizon T is not taken into account. This algorithm is modified such that it contains this valuation function.

The solution obtained by this Ending-Inventory Valuation (or EIV) algorithm will be compared with two other well-known procedures: The Wagner-Whitin algorithm (1958) and the Silver Meal heuristic, as enhanced by Blackburn and Millen (1980). Also, this EIV algorithm will be compared with an algorithm as enhanced by Stadtler (2000), which is specifically used to mitigate end effects. When solving, a rolling horizon scheme is implemented. This means that the DLSP is solved for the model horizon T, but only the first production period is taken into account. After that, the schedule start is rolled foreward to the first period of which the demand is not covered yet, and again the DLSP is solved for the model horizon. This procedure is repeated until all demand is covered.

The rest of this report is divided as follows: In Section 2 the relevant literature is investigated. In Section 3, the four different algorithms are described. In Section 4, the performance of these algorithms is analysed, where four different demand patterns are used. Also, in this section a sensitivity analysis on the EIV algorithm is performed. Finally, in Section 5 the findings of this report will be concluded and ideas for further research will be proposed.

2 Literature

The idea behind the use of the valuation function is to mitigate the end-effects occurring when a short horizon is chosen. However, some other methods to deal with this end effects are proposed in the literature. Stadtler (2000) makes use of a modification of the shortest route representation of the Single-level Lot-sizing problem (SLLSP) as described by Eppen and Martin (1987). Stadtler modifies the model by looking beyond the planning horizon, in order to mitigate the end effects.

Van den Heuvel and Wagelmans (2005) show that the superior performance of the EIV algorithm is due to the fact that this algorithm uses more information than the other algorithms, as it makes use of accurate future demand estimates. Van den Heuvel and Wagelmans show that a simple modification to the Wagner-Whitin algorithm will gain similar results. The idea is that they extend the T-period horizon of the Wagner-Whitin algorithm in order to avoid these end-effects.

3 Methodology

In this section, four different procedures to solve the DLSP are discussed. The optimal solution for the 300 periods is computed, where different demand patterns are used. However, it is important to test if the EIV algorithm performs better than the other algorithms when a shorter model horizon is chosen, where the horizon T ranges from 2 to 20 periods. We also make use of a rolling horizon setting, which means that the DLSP is solved for the whole period until horizon T, but only the first production lot size is implemented. After that, the start of the next period is rolled forward, until the first period of which the demand is not covered yet. Then, the problem is solved for the model horizon and again only the first production decision is implemented. This process is repeated until the demand of all 300 periods is covered. The found cost when making use of the shorter horizon and the rolling horizon scheme for each of the different horizon lengths between 2 and 20 and for each of the different algorithms is compared to the optimal 300 periods solution. From this, the percentage increase in cost from the optimal solution is calculated.

3.1 Silver Meal heuristic

The general idea of this heuristic is that there occurs a production in a certain period, covering the demand for a number of following periods, if the average costs of this production is lowest. For this heuristic we define $A[t, s]$, which is the average cost for periods t until s, when production took place in period t. We can calculate this average cost as:

$$A[t,s] = \frac{1}{s+1-t}\left(K + \sum_{i=t+1}^{s-1}(i-t)hd_i\right) \tag{2}$$

Here, K is the fixed setup cost for placing an order, h is the holding cost and d_i is the demand in period i.

Because we assume that the starting inventory is zero, the setup of this heuristic occurs in period 1. Next, let t_1 be the first period for which the average cost $A[1, t_1 + 1]$ is larger than $A[1, t_1]$. This means that we produce in period 1 an amount of $\sum_{i=1}^{t_1} d_i$, which exactly suffices the amount of products required for the periods 1 through t_1. Let t_2 be the first period after t_1 for which the demand in a period differs from zero. Then, the next setup occurs in period t_2 and the previous steps are repeated until the model horizon is reached. However, as Blackburn and Millen (1980) described it makes more sense to define $A[t, s]$ as the average cost for the period t until period $s+h$, where h is the number of consecutive periods with zero demand. This adaption only influences the found solution when there are periods with zero demand, and in that case often an improved solution is found.

3.2 Wagner-Whitin algorithm

The Wagner-Whitin (in the following denoted as WW) algorithm is based on dynamic programming. The idea behind dynamic programing is to solve the whole problem by solving subproblems of this large problem separately, where the solution of a certain subproblem is needed to solve another of these sub problems.

The idea of the WW algorithm is as follows. Let $C_{t,s}$, for $1 \leq t \leq s$, denote the minimum cost of producing in period t the demand for periods t through $s-1$. This means that it is optimal to produce exactly the demand of the periods t through $s-1$, and only produce when the inventory at the beginning of period t equals zero. $C_{t,s}$ can be calculated as follows:

$$C_{t,s} = K + \sum_{i=t+1}^{s-1} (i-t)hd_i \tag{3}$$

Next, let f_t be the optimal total cost for periods t until T. f_t can be obtained by the following recursion:

$$f_t = \min_{t<s\leq T+1} (C_{t,s} + f_s) \tag{4}$$

This recursion starts with initializing $f_{T+1} = 0$. Then, for f_T the optimal cost is determined, and this is recursively repeated until f_1 is reached, which is the minimum cost for a schedule for the entire horizon T. The use of this recursive formula decreases the number of possible combinations that needs to be checked quite drastically. For example, suppose we have a model horizon of $T = 4$. Then, if f_3 and f_4 are known, only three possible combinations need to be checked to determine f_2 . These three combinations are either producing all demand for periods 2 through 4, or producing in period 2 for period 2 only and then use f_3, or produce for periods 2 and 3, and use f_4. Especially when the length of the horizon grows, the decrease in number of possible combinations will be particularly large, which is the main advantage of the use of dynamic programming. When instead of using dynamic programming, we would just compute all possible combinations, there would be an exponentially number of combinations that needs to be checked: $n = 2^{t-1}$. When the problem instances are getting larger, this would result in an enormous amount of combinations. However, when dynamic programming is used, only a quadratic number of combinations needs to be checked, namely: $n = \sum_{i=1}^{n} i$. This can be written as $n = \frac{1}{2}t(t+1)$.

3.3 Ending-Inventory Valuation algorithm

The Ending-Inventory Valuation (or EIV) algorithm is a modification of the WW algorithm as described before. The idea is exactly the same; although the valuation function given by formula (1) is implemented here. Because in this case the ending inventory can differ from zero, the cost function and optimal ordering quantity differ from the WW algorithm. We can calculate $C_{t,s}$ as given in formula (3), except when $s = T + 1$. Then, the valuation function is taken into account and that leads to the following formulation:

$$C_{t,T+1} = \sum_{j=t}^{T} h(x_t - \sum_{i=t}^{j} d_i) + \frac{h}{2D}(x^* - x_t + \sum_{i=t}^{j} d_i)^2 \tag{5}$$

Here x_t is the quantity produced in period t. This cost can be seen as the cost incurred in periods $t, ..., T$, given that an amount of x_t has been produced, less the value of the ending inventory I_T. Minimizing this function results in the following expression for x_t, which is the optimal production quantity:

$$x_t = \begin{cases} \sum_{i=t}^{j} d_i + x^* - (T+1-t)D & ,\text{if } \ x^* > (T+1-t)D \\ \sum_{i=t}^{j} d_i & ,\text{if } \ x^* \le (T+1-t)D \end{cases} \tag{6}$$

This means that with this EIV algorithm the produced quantity in a period t can be either zero or $\sum_{i=t}^{j} d_i$ just as with the WW algorithm, but in case the end of the horizon is reached, the produced quantity will depend on the value of x^*. The production if $x^* > (T + 1 - t)D$ is such that there is enough produced in order to fulfill the demand of the same number of periods as the periodic order quantity (POQ). The recursion as described by formula (4) remains the same for this EIV algorithm.

It is to be expected that this algorithm will outperform the WW algorithm and the SM heuristic when a rolling horizon is used. However, this is because this algorithm makes use of extra information, because future demand is also known and used in an effective manner in order to obtain improved results. For this reason, it is quite unfair to compare these algorithms, as they do not use the same amount of information. However, Stadtler (2000) also describes an algorithm that can be used to mitigate end-effects, and a fair comparison between these algorithms can be made as both algorithms require the same knowledge about future demand. This algorithm will be described in the next section.

3.4 Stadtler's algorithm

Another algorithm that should mitigate the end-effects is the algorithm as proposed by Stadtler (2000). The general idea behind this algorithm is to determine the number of periods for which the demand should be produced, and then only the costs for the periods that are within the model horizon are taken into account. For this algorithm, first the time between order (TBO) τ_t is determined for the periods $t = 1, ..., T$. The time between an order is the number of periods of which the demand is fulfilled when producing in period t. Stadtler uses Groff's stopping criterion in order to determine these TBOs in linear time.

Algorithm 1 Calculating TBOs with Groff's stopping criterion

1: begin
2: $\tau_0 = 1$
3: **for** $t = 1, ..., T$ **do**
4:     $\tau = \max\{1, \tau_{t-1} - 1\}$
5:     **while** $t + \tau \leq T^{max}$ *and* $\frac{K}{\tau(\tau+1)} > \frac{h}{2} D_{t+\tau}$ **do**
6:         $\tau = \tau + 1$
7:         $\tau_t = \tau$
8:     **end while**
9: **end for**

Groff's stopping criterion is line (5) as described in the algorithm above. The idea is that the TBO is increased, unless $\frac{K}{\tau(\tau+1)} > \frac{h}{2} D_{t+\tau}$. This can be seen that the TBO is increased until the marginal setup costs are larger than the marginal holding costs. Of course, in that case it is beneficial to start a new production period.

When using Groff's stopping criterion an important factor is T^{max}. For periods beyond the horizon T, forecasted demand is used, where T^{max} gives an upper bound for the number of periods for which the demand should be forecasted. The choice of T^{max} is the same as suggested by Van den Heuvel and Wagelmans (2005), and equals $T^{max} = T + \sqrt{(2K/Dh)} - 1$. The idea behind this choice of T^{max} is that it only contains one cycle of forecasted demand in this case. This is because $\sqrt{(2K/Dh)}$ is the periodic order quantity.

For this algorithm, also $D_{t+\tau}$ needs to be known. However, it is possible that $t + \tau$ exceeds the value of the model horizon T. For this reason, we need to simulate the demand for periods beyond the model horizon T. How this demand is simulated depends on the type of demand pattern and will be described later in Section 4 for each of the demand patterns separately.

When these TBOs are calculated, the costs are calculated as follows:

$$C_{t,T+1} = \begin{cases} \frac{T-t+1}{\tau_t}(K + \sum_{j=t}^{t+\tau_t-1} h(j-t)d_j) & \text{, if } t+\tau_t-1 > T \\ K + \sum_{j=t}^{t+\tau_t-1} h(j-t)d_j & \text{, if } t+\tau_t-1 \leq T \end{cases} \tag{7}$$

The idea behind these costs is that we only take the costs into account that fall within the horizon T. So, for example if production occurred in period $T-1$, and the demand produced covers the following 3 periods, then only the costs for periods $T-1$ and T are taken into account, as period $T+1$ exceeds the planning horizon.

The costs $C_{t,s}$, where $t < s \leq T$ are the same as given in formula (3). After the costs are calculated, the recursion from the WW algorithm can be used.

4 Results

In order to test the performance of the four algorithms, demand is generated for 300 periods. For the generation of this demand, several demand patterns are used, which are the same demand patterns as described by Fisher et al. (2001):

- Stationary demand
- Linearly increasing or decreasing demand
- Seasonal demand
- Correlated demand

It is interesting to see whether the chosen demand pattern influences the found solution by each of the algorithms. It could be that for some of these patterns the algorithms perform worse. Also, it is interesting to see how the algorithms perform compared to each other. Especially, how these algorithms perform for different lengths of the model horizon T. This analysis is performed in Section 4.1 through Section 4.4.

Further, in Section 4.5 sensitivity analysis will be performed in order to determine how the performance of the EIV algorithm is when the forecasted demand pattern is more inaccurate. Also, the performance of all algorithms will be compared when the setup costs are either increased or decreased.

4.1 Stationary demand

In this case, two different distributions are distinguished: the normal and uniform distribution. First we will consider the normal distribution. The mean μ is set equal to 100, and the standard deviation $\sigma = 0, 10, 22, 43$. Furthermore, $h = 1$ and $K = 800$, and the long run average demand $D = \mu$. This long run average demand is required in order to make forecasts for the EIV algorithm and Stadtler's algorithm. Also, the value of K is such that the EOQ equals 400 in this case, which are four periods of demand. For the uniform distribution, the mean μ is also 100, and the range is $R = 0, 35, 75, 150$. The range of the uniform distribution is chosen such that the standard error of the normal distribution and the uniform distribution are comparable. For example, if $R = 75$, the standard error of the uniform distribution can be calculated as:

$$\sigma(U) = \sqrt{\frac{1}{12}R^2} \tag{8}$$

This means that the standard error equals $\sqrt{468.75}$, which equals the standard error of the normal distribution of 22.

As said before, 300 periods of demand are generated, and for each different choice of input parameters, eight different problem instances containing 300 periods of demand are generated. We need to mention that the demand is rounded, such that integer values of demand are obtained. Further, in case of the normal distribution, it is possible that a negative demand is generated in some period. In that case, the demand in this period is set equal to 0.

For each problem instance, the optimal 300 periods solution is computed. This means that we solve the problem instance by making use of the WW algorithm, where the rolling horizon setting is not implemented. After the optimal solution is determined for each of the problem instances, the percentage above these optimal costs is calculated for each of the algorithms in a rolling horizon setting, for the short model horizon varying from $T = 2, ..., 20$. Finally, the results represent the average deviation from the optimal solution for the eight problem instances. In Table 1, the results are represented for the normal distribution. The results as obtained for the uniform distribution are comparable with these of the normal distribution and can be found in the Appendix.

From this table it is clear that the EIV algorithm and Stadtler's algorithm outperform both the SM and WW algorithm for almost all values of T. Except when $T = 3$, and sometimes when $T = 4$ these algorithms perform worse. Of course, the superior results are caused by the fact that more information is used of future demand, which especially for $T = 2$ results in a large improvement. It is noticeable that SM and WW perform the same for $T = 2$, which is logical, as both algorithms will produce the demand for these 2 periods, ending with zero inventory.

When $T = 3$, the performance of the EIV and Stadtler algorithm is weak. For $T = 3$,

Table 1: Percentage deviation from optimality for normally distributed demand

	$\sigma = 0$				$\sigma = 10$				$\sigma = 22$				$\sigma = 43$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	28.57	28.57	0.00	0.00	29.23	29.23	15.51	15.51	31.64	31.64	18.41	18.41	37.33	37.33	22.01	22.01
3	4.76	4.76	0.00	0.00	5.40	5.40	15.13	15.13	7.38	7.38	16.95	16.95	11.75	11.58	21.71	21.75
4	0.00	0.00	0.00	0.00	0.79	0.57	0.70	0.70	1.94	2.39	2.03	2.03	4.55	7.25	4.27	4.26
5	0.00	2.86	0.00	0.00	0.59	3.51	0.56	0.53	1.07	5.45	0.92	0.97	2.33	7.42	2.01	2.07
6	0.00	4.76	0.00	0.00	0.59	5.33	0.48	0.46	0.98	5.19	0.58	0.64	1.86	5.47	1.35	1.22
7	0.00	4.76	0.00	0.00	0.59	1.58	0.38	0.38	0.98	1.94	0.53	0.63	1.73	2.53	0.59	0.55
8	0.00	0.00	0.00	0.00	0.59	0.52	0.34	0.34	0.98	1.12	0.41	0.47	1.73	1.05	0.37	0.33
9	0.00	0.00	0.00	0.00	0.59	1.24	0.27	0.27	0.98	1.14	0.30	0.29	1.73	1.24	0.26	0.27
10	0.00	4.67	0.00	0.00	0.59	1.64	0.31	0.29	0.98	1.29	0.24	0.24	1.73	0.81	0.27	0.30
11	0.00	4.67	0.00	0.00	0.59	0.85	0.23	0.20	0.98	0.85	0.21	0.22	1.73	0.67	0.17	0.17
12	0.00	0.00	0.00	0.00	0.59	0.45	0.19	0.16	0.98	0.60	0.16	0.13	1.73	0.31	0.11	0.12
13	0.00	0.00	0.00	0.00	0.59	0.54	0.16	0.16	0.98	0.34	0.12	0.12	1.73	0.32	0.08	0.06
14	0.00	4.57	0.00	0.00	0.59	0.76	0.13	0.13	0.98	0.44	0.10	0.09	1.73	0.25	0.05	0.05
15	0.00	4.57	0.00	0.00	0.59	0.42	0.13	0.13	0.98	0.28	0.07	0.08	1.73	0.10	0.03	0.02
16	0.00	0.00	0.00	0.00	0.59	0.28	0.14	0.14	0.98	0.16	0.08	0.08	1.73	0.12	0.07	0.06
17	0.00	0.00	0.00	0.00	0.59	0.30	0.11	0.12	0.98	0.18	0.06	0.04	1.73	0.12	0.04	0.05
18	0.00	4.57	0.00	0.00	0.59	0.43	0.08	0.08	0.98	0.13	0.03	0.03	1.73	0.05	0.02	0.02
19	0.00	4.57	0.00	0.00	0.59	0.28	0.08	0.08	0.98	0.11	0.02	0.03	1.73	0.03	0.01	0.01
20	0.00	0.00	0.00	0.00	0.59	0.21	0.04	0.04	0.98	0.10	0.03	0.05	1.73	0.04	0.00	0.00

Note: $\mu = 100$, $K = 800$, $h = 1$, T = model horizon, *SM* = Silver-Meal, *WW* = Wagner-Whitin, *EIV* = Ending Inventory Valuation, *ST* = Stadtler

$x^* = 400$ and $D = 100$, and usually there will be a production which is around 400 units. This results in an expected ending inventory of $x^* - 3D = 100$. This is often not sufficient to cover the demand in period $T + 1$, such that there are high costs because there are extra inventory costs, which are not compensated by a reduction in setup costs. For $T = 4$ the performance of the algorithms is roughly the same, probably due to the fact that this is the value of the POQ.

When we compare the EIV algorithm and Stadtler's algorithm, it can be seen that there are barely any differences between these algorithms, both showing the same pattern. Comparing WW and SM, WW outperforms SM in case the model horizon T is extended. However, the performance of the SM heuristic deteriorates for larger values of the standard error σ, while, surprisingly, the performance of the WW algorithm improves. This phenomenon was already observed by Federgruen and Tzur (1994), and they explain this improved performance by the fact that it becomes more easy to identify the optimal first production period after period 1 when the standard error increases.

Finally, a strange pattern occurs for the WW algorithm when $\sigma = 0$. Normally, as the value of T increases, the percentage deviation form optimality decreases. However, the pattern for this case is somewhat strange. For example, for $T = 4$, the optimal solution is found, while for $T = 5$, the found solution exceeds the cost of the optimal solution with almost 3 percent. This seems to be strange, but can be explained. The cost per period for $T = 4$ will be $1400/4 = 350$, as it is optimal to fulfill demand of all periods by producing in the first period. For $T = 5$ it is optimal to produce for all 5 periods, which results in a cost per period of $1800/5 = 360$, an increase of 2.86 percent. For other values of T, the same reasoning can be used to explain the found solutions. Note that the results differ, depending on how the WW algorithm is implemented. Here, the first production period is chosen to be as small as possible. For example, for $T = 7$ it is optimal to produce for 3 and for 4 periods. As the first production period is smallest, this results in production for 3 periods. Note that this results in a worse solution in this case, because producing for 4 periods would be optimal in this case.

4.2 Linearly increasing or decreasing demand

In their report, Fisher et al.(2001) also include the case of a linearly increasing or decreasing demand. However, this type of demand pattern seems rather superfluous. The idea is that the demand for the linearly increasing case is generated as:

$$d_t = \mu + \sigma z_t + c(t-1) \tag{9}$$

Here, Fisher et al. suggest to take $c = 1, 10, 20, 40$. However, as one can imagine if $c = 40$ the demand will explode in the 300 periods. For example, the expected demand is around 12,060 units for the 300th period. In that case, it is optimal to produce in each period. This production for one period is also which is actually observed for each of the algorithms, and for this reason the performance of the algorithms will be very comparable, as the production occurs only for one period. For this reason, this demand pattern is not further analysed in this report, as it will not provide any useful information.

4.3 Seasonal demand

In this case, the demand is generated with a seasonal pattern. The choice of a seasonal demand pattern is quite natural, because seasonality occurs a lot with actual demand patterns. The following formula is used to obtain a seasonal pattern:

$$d_t = \mu + \sigma z_t + a \sin\left(\frac{2\pi}{b}(t + b/4)\right) \tag{10}$$

Here, z_t is a standard normal variable, a denotes the amplitude and b the length of the cycle. The mean μ is again set equal to 100, $a = 20, 40, 60, 80$, $b = 4, 12, 52$, which will create either a quarterly, monthly or weakly pattern, and $\sigma = 10$. As before, $h = 1$ and $K = 800$.

However, in contrary to the stationary demand pattern, setting the long run average demand D_t equal to μ will not be sufficient in this case, as the seasonality is not taken into account in this way. For this reason, D_t needs to be adjusted in order to capture the seasonality, and hence D_t is as follows:

$$D_t = \mu + a \sin\left(\frac{2\pi}{b}(t + b/4)\right) \tag{11}$$

From these demand forecasts, we can determine which demand estimate needs to be used when calculating the end-of-horizon inventory, which will be denoted by $D_{t,T}$:

$$D_{t,T} = \frac{1}{t^*} \sum_{j=t+T}^{j=t+T+t^*-1} D_j \tag{12}$$

where $t^* = \sqrt{(2K/Dh)}$, the optimal number of periods between two orders. The idea is that in formula (6) D is replaced by $D_{t,T}$ and x^* is replaced by $x^*_{t,T}$. This $x^*_{t,T}$ is calculated with the use of the adjusted value $D_{t,T}$. The idea behind the calculation of $D_{t,T}$ in this way is that during peaks in the seasonal cycle, the production will be higher than the average production when $D = 100$ is used, while the opposite will be true for the troughs in the seasonal cycle. This is accomplished because the future demand after period $t+T$ is used in order to determine $D_{t,T}$. By using t^*, the value of $D_{t,T}$ will be the average of the demands from period $t+T$ through $t+T+t^*-1$.

Also for this demand pattern, for each combination of input parameters eight different problem instances containing 300 periods of demand are generated. The results for $b = 12$ and all values of a are reported in Table 2. In the Appendix, the results for other values of b are reported.

Table 2: Percentage deviation from optimality for seasonal demand, with $b = 12$

	$a = 20$				$a = 40$				$a = 60$				$a = 80$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	29.84	29.84	16.96	15.00	31.11	31.11	16.50	16.51	35.21	35.21	15.65	17.16	40.73	40.73	13.52	21.29
3	5.77	5.77	16.93	14.35	6.88	6.88	16.80	11.77	10.00	10.00	13.75	11.17	14.66	14.66	13.60	8.34
4	1.09	0.97	4.10	1.25	2.03	2.06	6.07	4.75	4.40	5.01	4.50	4.34	8.23	9.43	6.19	5.17
5	0.77	3.84	0.65	0.67	1.50	4.25	0.70	0.92	3.55	6.61	3.97	4.10	7.83	9.76	4.53	4.93
6	0.80	4.42	0.52	0.54	1.35	2.36	0.56	0.81	2.71	1.88	1.03	1.40	2.61	2.55	2.00	5.20
7	0.80	1.91	0.45	0.46	1.50	2.44	0.50	0.62	3.29	2.47	0.86	1.62	4.48	2.22	1.12	2.15
8	0.80	0.61	0.35	0.37	1.51	1.17	0.46	0.69	3.76	2.23	0.32	0.90	3.71	2.91	0.38	2.52
9	0.80	1.24	0.28	0.26	1.51	1.26	0.32	0.35	4.24	1.68	0.18	0.26	4.63	1.90	0.27	0.67
10	0.80	1.64	0.18	0.24	1.51	1.64	0.22	0.31	4.24	1.08	0.15	0.24	4.68	0.92	0.10	0.17
11	0.80	0.74	0.25	0.27	1.51	1.02	0.14	0.19	4.24	0.76	0.18	0.22	4.68	0.49	0.15	0.21
12	0.80	0.39	0.23	0.22	1.51	0.30	0.14	0.22	4.24	0.24	0.20	0.27	4.68	0.29	0.36	0.42
13	0.80	0.62	0.14	0.15	1.51	0.60	0.15	0.18	4.24	0.40	0.15	0.22	4.68	0.38	0.28	0.52
14	0.80	0.74	0.15	0.17	1.51	0.56	0.10	0.20	4.24	0.34	0.09	0.28	4.68	0.47	0.11	0.63
15	0.80	0.38	0.09	0.11	1.51	0.35	0.08	0.14	4.24	0.30	0.07	0.21	4.68	0.56	0.08	0.53
16	0.80	0.25	0.12	0.10	1.51	0.24	0.07	0.10	4.24	0.29	0.06	0.11	4.68	0.42	0.05	0.33
17	0.80	0.21	0.07	0.10	1.51	0.20	0.09	0.08	4.24	0.18	0.04	0.08	4.68	0.21	0.14	0.13
18	0.80	0.35	0.07	0.10	1.51	0.19	0.04	0.10	4.24	0.12	0.03	0.07	4.68	0.06	0.04	0.06
19	0.80	0.28	0.05	0.06	1.51	0.23	0.05	0.06	4.24	0.12	0.04	0.08	4.68	0.11	0.04	0.07
20	0.80	0.17	0.06	0.05	1.51	0.13	0.04	0.04	4.24	0.13	0.02	0.05	4.68	0.10	0.02	0.02

Note: $\mu = 100$, $K = 800$, $h = 1$, $\sigma = 10$

The percentage deviation from optimality in case of a seasonal demand pattern is particularly large compared to the stationary demand. Especially the SM heuristic performs poorly compared to that case. Of course, this was to be expected as the SM heuristic does not take this seasonal pattern into account and hence does not take into account that it would be better to produce for more periods when the demand is lowest. As the SM heuristic does not take any demand beyond period t into account, it makes sense that the performance is worse. The performance of the EIV and Stadtler is roughly the same, due to the adjusted long-run demand pattern. Stadtler and EIV outperform WW and SM just as with the stationary demand pattern. Also, WW outperforms SM, especially for larger values of T and for larger amplitudes - which means for higher values of a.

However, in this case there is a difference in performance between Stadtler's algorithm and the EIV algorithm. For small values of a, Stadtler's algorithm provides better results compared to the EIV algorithm, while for larger values of a, the EIV algorithm will give better results. A possible explanation for the observation that Stadtler outperforms EIV is due to the fact that Stadtler is able to describe the seasonal trend more efficient, which leads to improved results, especially for smaller values of T. However, when the amplitude increases, the use of Groff's heuristic results in worse forecasts of the TBOs.

Again, EIV particulary performs poorly for $T = 3$, also in comparison to Stadtler's algorithm. Despite the fact that the EIV algorithm uses quite accurate demand forecasts, the performance remains bad. Again, this is mainly due to the fact that it occurs quite often that the required production for period $T + 1$ cannot be satisfied with this production, such that extra inventory costs are incurred, while no reduction in setup costs compensates these extra costs.

4.4 Correlated demand

The last demand pattern used is a correlated demand pattern. This pattern is generated by the use of a Markov process. In this case, we assume that there are three different states, where in each state the demand in that specific state differs from the other states. The mean of each state equals respectively $\mu_L = 60$, $\mu_M = 100$ and $\mu_H = 140$. The transition probabilities of this Markov process, where $P_{i,j}$ is the probability that state j will be the next state given that the current state is i, are given in Table 3.

For example, if the current state is L ,with probability 0.70 state L is also the next state, with probability 0.25 the next state is state M, and with probability 0.05 the next state will be state H. In this case, $K = 800$, $h = 1$ and $\sigma = 0, 5, 10, 15$.

Now, there are two possible methods that can be used to calculate the long-run average

Table 3: Markov process for correlated demand

		State in period $t+1$		
		L	M	H
	L	0.70	0.25	0.05
State in period t	M	0.15	0.70	0.15
	H	0.05	0.25	0.70

demand. First, the long-run average demand D is the demand in the steady state. In order to be able to determine this demand in the steady state, it is required to determine the steady state probabilities. The steady state probabilities can be seen as the proportion of the total time in which the process is in a certain state i. The steady state probabilities can be calculated by solving the following set of equations:

$$\pi_L = 0.70\pi_L + 0.15\pi_M + 0.05\pi_H \tag{13}$$

$$\pi_M = 0.25\pi_L + 0.70\pi_M + 0.25\pi_H \tag{14}$$

$$\pi_H = 0.05\pi_L + 0.15\pi_M + 0.70\pi_H \tag{15}$$

$$\sum_{i=L,M,H} \pi_i = 1 \tag{16}$$

Solving these set of equations results in $\pi_L = 3/11$, $\pi_M = 5/11$ and $\pi_H = 3/11$. With these limiting probabilities, the long-run average demand D can be calculated:

$$D = \pi_L\mu_L + \pi_M\mu_M + \pi_H\mu_H \tag{17}$$

This method can work quite effectively when the ending inventory covers multiple periods. However, if the ending inventory covers only one period, this forecast of D might be inappropriate.

The second method uses a more myopic demand estimation. For a model horizon of length T that starts in period t, a forecast of the demand in period $t+T$ can be obtained.

$$D_{t+T} = P_{i,L}\mu_L + P_{i,M}\mu_M + P_{i,H}\mu_H \tag{18}$$

This formula requires knowledge about the state in period $t+T-1$, which is state i in this formula. Also for this demand pattern, for each combination of input parameters eight different problem instances containing 300 periods of demand are generated. However, in order to be able to compare both average long-run demand methods, the same problem instances are used for those methods. The results for all values of σ will be reported below, as well as the two different methods to obtain the average long-run demand D. In Table 4 the long-run average demand method is used, while in Table 5 the myopic average long-run demand method is used.

Table 4: Percentage deviation from optimality for correlated demand, making use of long-run average demand

	$\sigma = 0$				$\sigma = 5$				$\sigma = 10$				$\sigma = 15$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	31.48	31.48	11.56	11.56	31.52	31.52	16.86	16.86	33.26	33.26	16.60	16.60	34.13	34.13	18.33	18.33
3	7.27	7.27	11.57	11.57	7.29	7.29	19.31	19.31	8.48	8.48	18.93	18.93	8.87	8.87	20.71	20.71
4	1.89	2.67	1.89	1.89	2.05	2.72	2.05	2.05	2.71	3.33	2.75	2.75	3.25	3.97	3.20	3.20
5	0.83	5.83	0.83	0.98	1.08	4.73	1.10	1.00	1.50	4.60	1.45	1.47	1.72	5.49	1.64	1.47
6	0.82	3.65	0.77	0.71	0.99	3.83	0.95	1.03	1.27	4.27	0.91	0.78	1.46	4.66	1.06	1.16
7	0.82	2.83	0.50	0.88	0.99	2.26	0.61	0.84	1.20	2.72	0.71	0.96	1.46	2.74	0.61	0.77
8	0.82	1.40	0.46	0.52	0.99	1.45	0.52	0.66	1.20	1.52	0.59	0.66	1.41	1.79	0.49	0.49
9	0.82	1.55	0.51	0.54	0.99	1.50	0.35	0.48	1.20	1.19	0.44	0.48	1.41	1.36	0.39	0.42
10	0.82	1.39	0.26	0.30	0.99	1.18	0.36	0.41	1.20	1.21	0.30	0.33	1.41	1.17	0.31	0.31
11	0.82	0.94	0.19	0.26	0.99	0.77	0.26	0.33	1.20	0.89	0.25	0.34	1.41	0.94	0.21	0.27
12	0.82	0.73	0.20	0.24	0.99	0.65	0.28	0.27	1.20	0.67	0.19	0.22	1.41	0.69	0.22	0.22
13	0.82	0.50	0.20	0.20	0.99	0.47	0.20	0.19	1.20	0.51	0.16	0.15	1.41	0.48	0.15	0.18
14	0.82	0.45	0.15	0.18	0.99	0.38	0.14	0.13	1.20	0.33	0.11	0.12	1.41	0.37	0.10	0.14
15	0.82	0.30	0.14	0.16	0.99	0.29	0.09	0.09	1.20	0.33	0.06	0.06	1.41	0.36	0.04	0.08
16	0.82	0.26	0.09	0.09	0.99	0.22	0.07	0.07	1.20	0.18	0.11	0.10	1.41	0.24	0.08	0.07
17	0.82	0.18	0.06	0.06	0.99	0.19	0.04	0.05	1.20	0.14	0.06	0.07	1.41	0.13	0.06	0.09
18	0.82	0.16	0.04	0.03	0.99	0.12	0.05	0.03	1.20	0.17	0.05	0.05	1.41	0.14	0.03	0.04
19	0.82	0.17	0.06	0.07	0.99	0.15	0.04	0.04	1.20	0.12	0.04	0.04	1.41	0.08	0.02	0.02
20	0.82	0.12	0.06	0.05	0.99	0.09	0.03	0.04	1.20	0.07	0.03	0.03	1.41	0.08	0.03	0.03

Note: $\mu_L = 60$, $\mu_M = 100$, $\mu_H = 140$, $K = 800$, $h = 1$

Table 5: Percentage deviation from optimality for correlated demand, making use of a myopic average long-run demand

	$\sigma = 0$				$\sigma = 5$				$\sigma = 10$				$\sigma = 15$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	31.48	31.48	15.19	14.91	31.52	31.52	17.53	19.18	33.26	33.26	16.87	18.09	34.13	34.13	18.26	19.61
3	7.27	7.27	12.84	14.94	7.29	7.29	17.74	19.47	8.48	8.48	18.15	19.00	8.87	8.87	18.78	19.08
4	1.89	2.67	5.44	3.47	2.05	2.72	5.48	3.59	2.71	3.33	5.72	4.79	3.25	3.97	6.17	5.28
5	0.83	5.83	0.98	0.83	1.08	4.73	1.17	0.93	1.50	4.60	1.55	1.16	1.72	5.49	1.55	1.47
6	0.82	3.65	0.67	0.64	0.99	3.83	0.86	0.69	1.27	4.27	0.70	0.72	1.46	4.66	0.86	0.76
7	0.82	2.83	0.54	0.47	0.99	2.26	0.61	0.46	1.20	2.72	0.57	0.47	1.46	2.74	0.50	0.49
8	0.82	1.40	0.39	0.26	0.99	1.45	0.48	0.40	1.20	1.52	0.54	0.39	1.41	1.79	0.40	0.34
9	0.82	1.55	0.36	0.40	0.99	1.50	0.41	0.34	1.20	1.19	0.41	0.31	1.41	1.36	0.31	0.41
10	0.82	1.39	0.26	0.21	0.99	1.18	0.31	0.28	1.20	1.21	0.21	0.24	1.41	1.17	0.28	0.26
11	0.82	0.94	0.18	0.18	0.99	0.77	0.26	0.26	1.20	0.89	0.25	0.18	1.41	0.94	0.17	0.20
12	0.82	0.73	0.20	0.19	0.99	0.65	0.25	0.24	1.20	0.67	0.21	0.15	1.41	0.69	0.23	0.16
13	0.82	0.50	0.15	0.14	0.99	0.47	0.16	0.14	1.20	0.51	0.16	0.13	1.41	0.48	0.12	0.15
14	0.82	0.45	0.13	0.13	0.99	0.38	0.12	0.07	1.20	0.33	0.10	0.09	1.41	0.37	0.11	0.08
15	0.82	0.30	0.13	0.11	0.99	0.29	0.09	0.07	1.20	0.33	0.05	0.06	1.41	0.36	0.04	0.04
16	0.82	0.26	0.09	0.07	0.99	0.22	0.05	0.03	1.20	0.18	0.09	0.08	1.41	0.24	0.06	0.05
17	0.82	0.18	0.06	0.06	0.99	0.19	0.03	0.04	1.20	0.14	0.05	0.05	1.41	0.13	0.06	0.05
18	0.82	0.16	0.04	0.03	0.99	0.12	0.04	0.05	1.20	0.17	0.03	0.03	1.41	0.14	0.04	0.03
19	0.82	0.17	0.05	0.04	0.99	0.15	0.04	0.04	1.20	0.12	0.03	0.04	1.41	0.08	0.03	0.02
20	0.82	0.12	0.06	0.06	0.99	0.09	0.03	0.04	1.20	0.07	0.02	0.03	1.41	0.08	0.02	0.02

Note: $\mu_L = 60$, $\mu_M = 100$, $\mu_H = 140$, $K = 800$, $h = 1$

From both tables it is clear that the EIV algorithm and Stadtler's algorithm outperform WW and SM. First, when the long-run average demand is used, the performance of Stadtler and EIV is almost the same for each value of T. Of course, because both algorithms use $D = 100$ for all values of t, there are barely differences (the same can be concluded when stationary demand is used). However, when a myopic average long-run demand is used, there are differences between Stadtler and EIV, especially for small values of the model horizon T ($T \leq 4$). It seems that Stadtler performs worse for $T = 2$ and $T = 3$, while EIV performs worse for $T = 4$. Again, for these short horizons the amount produced is important, as sometimes the demand for period $T + 1$ cannot be satisfied while extra inventory costs are incurred.

It is interesting to compare both tables with each other, in order to find out whether there are major differences in performance when another type of average long-run demand D is used. Of course, SM and WW have exactly the same values, because the average long-run demand D does not influence the found solution of these algorithms. For the EIV algorithm, the differences are small between both methods. For $T = 3$, the performance of the myopic average long-run demand method is better compared to the other method. This difference is probably due to the fact that there is usually 1 period of inventory left for $T = 3$, and this ending-inventory for this period is more precisely determined when the myopic demand pattern is used. However, for $T = 4$ the opposite is true. The performance of the long-run average demand method is a lot better compared to the myopic method. This observation is quite unexpected, as the performance should have been comparable. The difference is due to higher inventory costs in case of the myopic average long-run demand method. Only when $\sigma = 0$ these patterns do not hold, and for this case the long-run average demand method outperforms the myopic demand method. Because in this case, the demand in a period always equals either 60, 100 or 140, it is quite effective to assume that $D = 100$. This results in less extra inventory and for this reason a solution with lower costs is found.

For Stadtler's algorithm the differences are smaller, although it seems to be that the long-run average demand method gives better results. This can be the result of how the TBOs are determined. Especially for $T = 2$ there are some differences.

4.5 Sensitivity analysis

In this section, sensitivity analysis of the algorithms is performed. First, the robustness of the EIV algorithm is investigated when the long-run demand forecasts are less accurate compared to the previous sections. Then, the performance of each of the algorithms is investigated when the setup costs are varied.

4.5.1 Effect of inaccurate demand forecasts

In the results as obtained in the previous sections, the assumption was made that the demand can be forecasted accurately. However, in practice it is highly unlikely that such accurate demand forecasts can be obtained. Either because the patterns in real demand will not be as clear as assumed here, or because it often occurs that the pattern as observed in the past will change in the future. It is interesting to see whether the superior results in comparison to the SM heuristic and the WW algorithm still exist when these demand patterns are less accurate.

To test the robustness of this algorithm, the problem instances as described in Section 4.1 will be executed again, although D is replaced by D^*, where $D^* = \alpha D$. Here, α is a coefficient, such that the demand is either underestimated or overestimated. For this, $\alpha = 0.8, 0.9, 1.1, 1.2$. Table 6 reports the results of these cases, for all different values of σ, $\sigma = 0, 10, 22, 43$. This table also contains columns with $\alpha = 1$, this is just the same case as described in Section 4.1.

Table 6: Performance of the EIV algorithm when the demand is inaccurate

T			$\sigma = 0$					$\sigma = 10$					$\sigma = 22$					$\sigma = 43$		
α	0.8	0.9	1.0	1.1	1.2	0.8	0.9	1.0	1.1	1.2	0.8	0.9	1.0	1.1	1.2	0.8	0.9	1.0	1.1	1.2
2	32.48	32.76	0.00	0.00	32.48	16.61	15.96	15.51	15.51	16.61	18.82	18.50	18.41	18.41	18.82	21.76	21.70	22.01	22.01	21.76
3	5.07	2.54	0.00	30.22	26.82	7.10	8.07	15.13	24.22	26.95	13.78	15.54	16.95	20.70	23.65	19.98	20.16	21.71	23.20	22.73
4	10.71	5.36	0.00	0.00	0.00	9.99	5.64	0.70	0.76	0.63	9.58	6.01	2.03	1.86	2.00	10.89	7.87	4.27	4.31	4.90
5	0.00	0.00	0.00	0.00	0.00	0.67	0.55	0.56	0.72	0.95	1.19	1.01	0.92	0.92	0.98	2.05	1.81	2.01	2.16	2.14
6	0.00	0.00	0.00	0.00	0.00	0.46	0.49	0.48	0.51	0.53	0.61	0.57	0.58	0.60	0.68	1.25	1.30	1.35	1.24	1.33
7	0.00	0.00	0.00	0.00	0.00	0.41	0.42	0.38	0.46	0.61	0.54	0.51	0.53	0.54	0.64	0.84	0.66	0.59	0.54	0.64
8	0.00	0.00	0.00	0.00	0.00	0.46	0.44	0.34	0.35	0.37	0.51	0.42	0.41	0.50	0.65	0.43	0.47	0.37	0.38	0.35
9	0.00	0.00	0.00	0.00	0.00	0.35	0.30	0.27	0.27	0.37	0.39	0.40	0.30	0.32	0.38	0.28	0.29	0.26	0.29	0.29
10	0.00	0.00	0.00	0.00	0.00	0.29	0.28	0.31	0.28	0.27	0.28	0.25	0.24	0.25	0.29	0.29	0.29	0.27	0.26	0.27
11	0.00	0.00	0.00	0.00	0.00	0.23	0.20	0.23	0.23	0.27	0.25	0.24	0.21	0.24	0.21	0.19	0.20	0.17	0.16	0.19
12	0.00	0.00	0.00	0.00	0.00	0.25	0.20	0.19	0.16	0.17	0.18	0.16	0.16	0.16	0.16	0.17	0.14	0.11	0.11	0.11
13	0.00	0.00	0.00	0.00	0.00	0.22	0.22	0.16	0.16	0.20	0.14	0.13	0.12	0.13	0.12	0.08	0.10	0.08	0.07	0.06
14	0.00	0.00	0.00	0.00	0.00	0.12	0.12	0.13	0.14	0.20	0.10	0.10	0.10	0.07	0.09	0.06	0.07	0.05	0.05	0.07
15	0.00	0.00	0.00	0.00	0.00	0.12	0.11	0.13	0.15	0.22	0.09	0.07	0.07	0.09	0.10	0.03	0.05	0.03	0.04	0.04
16	0.00	0.00	0.00	0.00	0.00	0.17	0.12	0.14	0.14	0.14	0.09	0.09	0.08	0.08	0.06	0.05	0.05	0.07	0.07	0.07
17	0.00	0.00	0.00	0.00	0.00	0.15	0.12	0.11	0.10	0.14	0.05	0.05	0.06	0.04	0.03	0.03	0.04	0.04	0.03	0.03
18	0.00	0.00	0.00	0.00	0.00	0.09	0.07	0.08	0.10	0.12	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02
19	0.00	0.00	0.00	0.00	0.00	0.09	0.07	0.08	0.10	0.12	0.03	0.02	0.02	0.04	0.05	0.01	0.01	0.01	0.01	0.01
20	0.00	0.00	0.00	0.00	0.00	0.10	0.07	0.04	0.06	0.08	0.04	0.04	0.03	0.04	0.04	0.01	0.00	0.00	0.00	0.01

Note: $\mu = 100$, $K = 800$, $h = 1$

From this table there are some noticeable changes from the case where $\alpha = 1.0$. First, for $T = 2$ it seems that for each value of α that the performance becomes worse. However, for $\alpha = 1.1$ this is not the case, as the results obtained for this value are exactly the same compared to $\alpha = 1.0$. The explanation for this strange observation follows from formula (6). When $T = 2$, the amount produced in the first period will be equal to $\sum_{i=T-1}^{T} d_i + x^* - 2D$. This is for $\alpha = 1.1$ equal to $\sum_{i=T-1}^{T} d_i + 419.5 - 420$, which is the same as when $\alpha = 1.0$, because only integer values can be produced. However, for other values of α the expected amount of production equals either 398 or 399 units. Of course, for $\sigma = 0$ this results in a huge change in results, as the fourth demand period (so period $T + 2$) cannot be sufficed from the produced demand.

For $T = 3$ we observe a remarkable pattern: In case the demand is underestimated, the obtained solution outperforms the case where the demand is accurately forecasted. Although this result seems to be counterintuitive, an explanation follows from formula (6). For example, for $\alpha = 0.8$ it is optimal to produce in the first period for all periods until the model horizon. This means that the production in the first period is $\sum_{i=T-1}^{T} d_i + 357.8 - 240$, which exceeds the production for the case where $\alpha = 1.0$ with 18 units. This increase in production is due to the fact that the optimal periodic order quantity (POQ) is larger compared to the basic case where POQ $= 4$, as can be deducted from the formula of the POQ: $\sqrt{(2K/Dh)}$. Because, D is 80 instead of 100, the value of the POQ increases. This leads to an increase in inventory costs, but this increased production will often result in a decrease in setup costs. This is an improvement to the normal case, because in that case it occurred quite often that this production did not satisfy the demand in period $T + 1$.

An overestimation of demand forecasts gives opposite results. A decrease in the POQ leads to a decrease in production, such that less periods of demand can be covered with this production. For $\alpha = 1.2$ the production is $\sum_{i=T-1}^{T} d_i + 438.2 - 360$, a decrease of 22 units compared to the standard case. Of course, for the cases where the demand in period $T + 1$ could not have been satisfied in the normal case a decrease in inventory costs is realized. However, in that case it is also quite often possible to satisfy that demand, resulting in a decrease in setup costs. However, for $\alpha = 0.8$ is this case not very likely, and hence a reduction in setup costs almost never occurs.

Finally, there are some differences for $T = 4$. In case the demand is underestimated, the obtained solution is poor compared to the normal case. This is because the production is higher compared to this case. Where in case for $T = 3$ this increase in production results in better results, in this case it gives poor results. This is due to an increase in inventory costs, as the production only should satisfy these 4 periods of demand. However, in case $\alpha = 0.8$ the expected production is around 438 units. These 38 extra units are not sufficient to cover the demand in period $T + 1 = 5$, and hence only an increase in inventory costs is incurred. In case the demand is overestimated, the results are almost equal to the normal case. This is due to the fact that the production equals $\sum_{i=T-1}^{T} d_i$, because $4D > x^*$.

For all other values of T, the differences are fairly small. Hence, it seems that the EIV algorithm produces robust solutions, except for small ($T \leq 4$) values of T.

4.5.2 Effect of changing values of setup costs

In the previous sections, the setup costs were chosen such that the POQ equals 4. However, it is interesting to see how the algorithms perform when the setup costs are varied. Here, it is especially interesting to see whether some observed patterns remain visible and if the values of the percentage deviation from the optimal solution deviate from the ones as described in Table 1. The POQ is defined as $t^* = \sqrt{(2K/Dh)}$. Now, we choose K such that this POQ equals 3 periods or 5 periods, implying that $K = 450$ and $K = 1250$. In Table 7 the results are shown for the case of normally distributed demand where $K = 450$, in Table 8 the case of normally distributed demand where $K = 1250$.

Table 7: Percentage deviation from optimality for setup costs $K = 450$

	$\sigma = 0$				$\sigma = 10$				$\sigma = 22$				$\sigma = 43$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	10.00	10.00	0.00	0.00	10.45	10.45	21.69	21.69	12.59	12.59	23.18	23.18	17.77	17.77	29.29	29.29
3	0.00	0.00	0.00	0.00	0.56	0.56	0.69	0.69	1.90	2.51	1.91	1.91	5.04	6.55	4.64	4.64
4	0.00	5.00	0.00	0.00	0.49	5.34	0.58	0.58	1.23	5.62	0.92	1.01	2.28	6.40	1.74	1.76
5	0.00	9.93	0.00	0.00	0.49	2.92	0.52	0.55	1.15	2.70	0.54	0.62	1.64	3.28	0.87	0.89
6	0.00	0.00	0.00	0.00	0.49	0.57	0.42	0.41	1.15	1.39	0.41	0.41	1.60	1.63	0.45	0.50
7	0.00	0.00	0.00	0.00	0.49	1.95	0.36	0.36	1.15	1.37	0.32	0.33	1.60	1.01	0.26	0.26
8	0.00	9.80	0.00	0.00	0.49	1.17	0.33	0.32	1.15	0.86	0.21	0.22	1.60	0.45	0.18	0.19
9	0.00	0.00	0.00	0.00	0.49	0.49	0.26	0.26	1.15	0.53	0.18	0.16	1.60	0.29	0.06	0.08
10	0.00	0.00	0.00	0.00	0.49	0.93	0.20	0.20	1.15	0.38	0.10	0.10	1.60	0.19	0.05	0.04
11	0.00	9.73	0.00	0.00	0.49	0.71	0.20	0.18	1.15	0.35	0.09	0.09	1.60	0.17	0.04	0.04
12	0.00	0.00	0.00	0.00	0.49	0.38	0.14	0.17	1.15	0.16	0.05	0.06	1.60	0.08	0.02	0.02
13	0.00	0.00	0.00	0.00	0.49	0.53	0.12	0.13	1.15	0.16	0.06	0.05	1.60	0.07	0.02	0.03
14	0.00	9.60	0.00	0.00	0.49	0.39	0.11	0.10	1.15	0.11	0.05	0.04	1.60	0.03	0.01	0.00
15	0.00	0.00	0.00	0.00	0.49	0.26	0.11	0.11	1.15	0.08	0.03	0.03	1.60	0.02	0.00	0.00
16	0.00	0.00	0.00	0.00	0.49	0.38	0.10	0.11	1.15	0.08	0.03	0.02	1.60	0.02	0.01	0.01
17	0.00	9.53	0.00	0.00	0.49	0.27	0.10	0.10	1.15	0.06	0.01	0.00	1.60	0.01	0.01	0.01
18	0.00	0.00	0.00	0.00	0.49	0.15	0.07	0.06	1.15	0.03	0.02	0.02	1.60	0.00	0.01	0.01
19	0.00	0.00	0.00	0.00	0.49	0.20	0.08	0.08	1.15	0.04	0.03	0.03	1.60	0.00	0.01	0.01
20	0.00	9.40	0.00	0.00	0.49	0.20	0.07	0.07	1.15	0.03	0.00	0.00	1.60	0.01	0.00	0.00

Note: $\mu = 100$, $K = 450$, $h = 1$

From Table 7, there are some noticeable changes compared to Table 1. First, for $T = 2$ WW and SM outperform EIV and Stadtler, while this was not the case previously. This observation requires the same kind of explanation as provided before where it concerned the poor performance of EIV for $T = 3$. Because the POQ is 3 instead of 4, the same reasoning for $T = 2$ should be given here, as for $T = 3$ before. If $T = 2$, production only occurs in the first period, and the production is usually around 300 units. However, quite often this is not sufficient to satisfy the demand in period $T + 1$, such that extra inventory costs are incurred and no reduction in setup costs compensates these extra inventory costs. Because the POQ is reduced from 4 to 3, the observed patterns in Table 1 are moved in this case, such that specific observations for $T = x$ in Table 1 are now observed for $T = x - 1$.

Also, the absolute differences from optimality are interesting. Here, it seems that SM and WW perform especially better for $T = 2$ and $T = 3$ than before. This improved performance is due to the decrease of the POQ. Previously, it was optimal to produce for four periods of demand, while in this case it is optimal to produce only for three periods of demand. As SM and WW always end with zero inventory, the percentage deviation from the optimal solution decreases for $T = 2$ in this case, as producing for two periods is closer to the optimal of producing for three periods than it was before compared to the four periods production.

Table 8: Percentage deviation from optimality for setup costs $K = 1250$

	$\sigma = 0$				$\sigma = 10$				$\sigma = 22$				$\sigma = 43$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	50.00	50.00	0.00	0.00	50.77	50.77	11.92	11.92	53.38	53.38	14.13	14.13	59.48	59.48	17.72	17.72
3	14.81	14.81	0.00	0.00	15.48	15.48	11.98	11.98	17.50	17.50	14.30	14.30	21.94	21.94	17.29	17.29
4	2.78	2.78	0.00	0.00	3.35	3.35	12.17	12.17	5.10	5.10	14.66	14.70	9.36	9.75	17.41	17.65
5	0.00	0.00	0.00	0.00	0.62	0.61	0.66	0.66	1.79	2.52	1.68	1.69	4.36	6.17	3.77	3.70
6	0.00	1.85	0.00	0.00	0.48	2.51	0.44	0.51	1.07	4.46	0.90	0.95	2.33	6.99	1.99	2.05
7	0.00	6.26	0.00	0.00	0.48	6.04	0.44	0.44	0.98	6.07	0.86	0.86	2.16	6.61	1.52	1.32
8	0.00	2.78	0.00	0.00	0.48	3.24	0.39	0.41	0.98	3.75	0.65	0.59	2.10	3.65	0.99	0.99
9	0.00	2.78	0.00	0.00	0.48	1.04	0.41	0.41	0.98	1.51	0.59	0.56	1.95	1.98	0.52	0.53
10	0.00	0.00	0.00	0.00	0.48	0.43	0.34	0.33	0.98	0.93	0.53	0.48	1.95	1.03	0.48	0.48
11	0.00	0.00	0.00	0.00	0.48	0.98	0.24	0.23	0.98	1.16	0.34	0.33	1.95	1.35	0.35	0.32
12	0.00	1.85	0.00	0.00	0.48	1.80	0.22	0.23	0.98	1.42	0.29	0.26	1.95	1.02	0.24	0.31
13	0.00	2.74	0.00	0.00	0.48	1.36	0.20	0.23	0.98	1.19	0.22	0.20	1.95	0.72	0.24	0.20
14	0.00	2.74	0.00	0.00	0.48	0.55	0.18	0.14	0.98	0.75	0.19	0.15	1.95	0.57	0.17	0.14
15	0.00	0.00	0.00	0.00	0.48	0.37	0.12	0.12	0.98	0.41	0.11	0.10	1.95	0.42	0.14	0.11
16	0.00	0.00	0.00	0.00	0.48	0.45	0.14	0.15	0.98	0.47	0.14	0.13	1.95	0.25	0.06	0.06
17	0.00	0.00	0.00	0.00	0.48	0.66	0.12	0.12	0.98	0.46	0.11	0.14	1.95	0.28	0.05	0.08
18	0.00	2.67	0.00	0.00	0.48	0.62	0.13	0.14	0.98	0.39	0.09	0.10	1.95	0.22	0.04	0.06
19	0.00	2.67	0.00	0.00	0.48	0.36	0.11	0.10	0.98	0.26	0.07	0.08	1.95	0.13	0.05	0.05
20	0.00	0.00	0.00	0.00	0.48	0.15	0.10	0.11	0.98	0.23	0.06	0.06	1.95	0.13	0.06	0.06

Note: $\mu = 100$, $K = 1250$, $h = 1$

When the setup costs are increased, such that the POQ is increased from 4 to 5, the opposite from the case where the setup costs were decreased is true. Instead of an improved percentage deviation for the SM and WW algorithm for the small values of T, the percentage deviation is quite dramatic for $T = 2$, up to 60 percent. Of course, producing only two periods of demand, while it is optimal to produce for around five periods of demand should lead to these huge differences.

The EIV algorithm and Stadtler's algorithm outperform WW and SM in all cases, except for $T = 4$. Of course, this poor performance for $T = 4$ has the same reasoning as in the case where $K = 800$ for $T = 3$.

The effect of changing the setup costs can be quite dramatic, especially when it is optimal to produce for a large amount of periods and a short model horizon is used. In that case, the obtained solutions with either the SM heuristic or WW algorithm are very poor, and can be improved by using Stadtler or the EIV algorithm. Furthermore, it is noticeable that the patterns as observed in case $K = 800$ remain visible, even for different values of K. The only difference is that these patterns move, where the direction of this movement is dependent on the increase or decrease of the setup costs K.

5 Conclusion

It is important that an appropriate model horizon T is chosen when planning production in case a lot-sizing model is used. This model horizon T should equal the expected lifetime of a company, but often much smaller values of T are used in order to make actual production plannings. However, using these smaller values for T leads to so called end-effects. This means that the solution found in the short run does not need to be optimal in the long run. In order to mitigate these end-effects, Fisher et al. (2001) use a valuation term as given in formula (1). The idea is that this term takes ending inventory into account, such that it is not necessarily optimal to end with zero inventory. This Ending Inventory Valuation (EIV) algorithm is compared with three other algorithms: The Silver Meal heuristic, the Wagner-Whitin algorithm and Stadtler's algorithm. The idea behind Stadtler's algorithm is to take only costs into account for periods that are within the model horizon.

In order to test the performance of this EIV algorithm, eight problem instances containing 300 periods of demand are generated. There are four different demand patterns used, the same as described by Fisher et al. (2001). In general, the EIV algorithm outperforms the WW algorithm and the SM heuristic. However, making a comparison between these algorithms is quite unfair. The EIV uses extra information compared to SM and WW, because forecasted demand is used. Also, SM and WW will always end with zero inventory, while EIV is able to have ending inventory due to this valuation term. The performance of the EIV algorithm can be compared with Stadtler's algorithm, and the performance is in most cases comparable.

However, accurate demand forecasts were used in order to obtain the results for the EIV algorithm. In reality, it seems unlikely that accurate demand forecasts can be obtained. For this reason, a sensitivity analysis is performed in order to investigate the performance of the EIV algorithm when demand forecasts are less accurate. Even if the demand is underestimated with twenty percent, EIV still outperforms SM and WW, except for really small model horizons ($T \leq 4$). However, in practice it seems unlikely that these very short model horizons are used. Hence, it seems that the performance of the EIV algorithm remains superior, despite less accurate demand forecasts are used. It can be concluded that the EIV algorithm gives quite robust results, and if a reasonable demand forecast can be made, is preferred over SM and WW.

6 Further research and Improvements

The biggest improvement that should be made is to generate a greater amount of different demand patterns compared to the eight problem instances for each different demand pattern that has been used. The absolute values of percentage deviation from optimality are too volatile when only eight problem instances are used. For example, in case of the normal distribution, the optimal 300 period solution ranges from 95224 euros to 99725 euros, a difference of almost five percent. In order to obtain more robust absolute differences from optimality, a larger amount of problem instances is required. Fisher et al. (2001) probably used a small number of problem instances due to the rapidly increasing running times. However, nowadays this should not be a problem, mainly because the running times are decreased in general due to the improved processors of personal computers.

Further research on this topic should be performed, in order to find an effective algorithm to mitigate end-effects. As said before, the performance of the EIV algorithm and Stadtler's algorithm is particulary poor for the model horizon $T = POQ - 1$. In this case, WW and SM outperform this algorithm.

References

Blackburn, J.D., R.A. Millen. 1980. *Heuristic lot-sizing performance in a rolling-schedule environment.* Decision Science 11, 691-701.

Federgruen, A., M.Tzur. 1994. *Minimal forecast horizons and a new planning procedure for the general dynamic lot sizing model: nervousness revisited.* Operations Research 42, 456-468.

Fisher, M., K.Ramdas, Y.Zheng. 2001. *Ending Inventory Valuation in Multiperiod Production Scheduling.* Management Science 45, 679-692.

Grinold, R.C. 1983. *Model building techniques for the correction of end effects in multistage convex programs.* Operations Research 31, 407-431.

Groff, G.K. 1979. *A lot sizing rule for time-phased component demand.* Production and Inventory Management 20, 47-53.

van den Heuvel, W., A.P.M. Wagelmans. 2005. *A comparison of methods for lot-sizing in a rolling horizon environment.* Operations Research Letters 33, 486-496.

Silver, E.A., H.C. Meal. 1973. *A heuristic selecting lot size requirements for the case of a deterministic time-varying demand rate and discrete oppurtunities for replenishment.* Production and Inventory Management 14, 64-77.

Stadtler, H. 2000. *Improved rolling schedules for the dynamic single-level lot-sizing problem.* Management Science 46(2), 318-326.

Wagner, H.M., T.M. Whitin. 1958. *Dynamic version of the economic lot size problem.* Management Science 5, 89-96.

Appendices

Table 9: Percentage deviation from optimality for the uniform distribution

	$R = 0$				$R = 35$				$R = 75$				$R = 150$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	28.57	28.57	0.00	0.00	29.41	29.41	15.36	15.36	31.62	31.62	17.73	17.73	38.09	38.09	21.62	21.62
3	4.76	4.76	0.00	0.00	5.41	5.41	14.86	14.86	7.15	7.15	17.85	17.85	12.36	12.36	21.90	21.90
4	0.00	0.00	0.00	0.00	0.57	0.57	0.90	0.90	1.75	2.20	1.70	1.70	4.66	7.29	4.74	4.74
5	0.00	2.86	0.00	0.00	0.75	3.37	0.57	0.69	1.07	5.00	0.90	1.05	2.26	7.63	2.00	2.07
6	0.00	4.76	0.00	0.00	0.75	5.35	0.53	0.57	1.09	5.18	0.80	0.77	1.76	4.84	0.86	0.87
7	0.00	4.76	0.00	0.00	0.75	1.56	0.45	0.48	1.09	2.18	0.66	0.70	1.70	2.12	0.68	0.65
8	0.00	0.00	0.00	0.00	0.75	0.58	0.35	0.34	1.09	1.20	0.41	0.44	1.68	1.22	0.44	0.44
9	0.00	0.00	0.00	0.00	0.75	1.10	0.24	0.25	1.09	1.25	0.27	0.29	1.68	1.33	0.33	0.37
10	0.00	4.67	0.00	0.00	0.75	1.70	0.24	0.21	1.09	1.50	0.24	0.23	1.68	0.72	0.36	0.39
11	0.00	4.67	0.00	0.00	0.75	0.85	0.17	0.19	1.09	0.92	0.19	0.21	1.68	0.48	0.22	0.24
12	0.00	0.00	0.00	0.00	0.75	0.43	0.14	0.13	1.09	0.54	0.20	0.20	1.68	0.38	0.16	0.14
13	0.00	0.00	0.00	0.00	0.75	0.54	0.12	0.12	1.09	0.50	0.11	0.13	1.68	0.40	0.06	0.07
14	0.00	4.57	0.00	0.00	0.75	0.80	0.11	0.12	1.09	0.41	0.12	0.13	1.68	0.23	0.05	0.04
15	0.00	4.57	0.00	0.00	0.75	0.48	0.13	0.14	1.09	0.31	0.10	0.10	1.68	0.14	0.04	0.06
16	0.00	0.00	0.00	0.00	0.75	0.24	0.11	0.11	1.09	0.24	0.07	0.08	1.68	0.06	0.04	0.03
17	0.00	0.00	0.00	0.00	0.75	0.36	0.07	0.08	1.09	0.22	0.06	0.05	1.68	0.12	0.02	0.03
18	0.00	4.57	0.00	0.00	0.75	0.38	0.07	0.07	1.09	0.17	0.03	0.03	1.68	0.09	0.02	0.03
19	0.00	4.57	0.00	0.00	0.75	0.29	0.08	0.08	1.09	0.13	0.03	0.03	1.68	0.05	0.01	0.01
20	0.00	0.00	0.00	0.00	0.75	0.17	0.06	0.06	1.09	0.13	0.04	0.04	1.68	0.02	0.01	0.01

Note: $\mu = 100$, $K = 800$, $h = 1$

Table 10: Percentage deviation from optimality for seasonal demand, where $b = 4$

	$a = 20$				$a = 40$				$a = 60$				$a = 80$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	32.44	32.44	10.29	17.66	36.29	36.29	12.72	15.71	40.75	40.75	19.68	15.14	44.97	44.97	26.16	17.42
3	7.94	7.94	7.66	18.49	11.14	11.14	12.57	15.80	14.53	14.53	18.85	15.75	18.02	18.02	22.61	15.82
4	0.48	5.98	0.35	0.32	0.33	12.05	0.35	0.35	0.36	18.69	0.30	0.34	0.33	25.68	0.31	0.33
5	0.55	5.88	0.65	0.39	0.32	9.00	0.55	0.29	0.33	11.20	0.49	0.30	0.29	6.41	0.59	0.30
6	0.55	6.56	0.31	0.25	0.32	0.95	0.28	0.33	0.33	0.39	0.31	0.32	0.29	0.33	0.46	0.28
7	0.55	0.41	0.25	0.22	0.32	0.39	0.28	0.35	0.33	0.38	0.30	0.30	0.29	0.31	0.50	0.30
8	0.55	1.11	0.16	0.17	0.32	0.33	0.16	0.21	0.33	0.34	0.17	0.24	0.29	0.33	0.20	0.25
9	0.55	0.61	0.17	0.10	0.32	0.36	0.19	0.18	0.33	0.76	0.26	0.20	0.29	1.18	0.36	0.18
10	0.55	1.31	0.15	0.10	0.32	0.82	0.25	0.18	0.33	0.38	0.26	0.18	0.29	0.28	0.38	0.20
11	0.55	0.25	0.12	0.09	0.32	0.32	0.19	0.15	0.33	0.29	0.22	0.15	0.29	0.24	0.31	0.13
12	0.55	0.26	0.10	0.11	0.32	0.23	0.08	0.08	0.33	0.23	0.07	0.13	0.29	0.20	0.05	0.10
13	0.55	0.24	0.10	0.09	0.32	0.21	0.12	0.08	0.33	0.32	0.16	0.11	0.29	0.48	0.23	0.08
14	0.55	0.47	0.09	0.08	0.32	0.29	0.14	0.11	0.33	0.20	0.17	0.10	0.29	0.18	0.31	0.08
15	0.55	0.14	0.12	0.07	0.32	0.21	0.11	0.08	0.33	0.17	0.13	0.09	0.29	0.11	0.19	0.07
16	0.55	0.14	0.06	0.05	0.32	0.14	0.05	0.06	0.33	0.15	0.03	0.06	0.29	0.13	0.05	0.06
17	0.55	0.16	0.06	0.04	0.32	0.13	0.08	0.05	0.33	0.17	0.07	0.05	0.29	0.24	0.09	0.05
18	0.55	0.24	0.07	0.04	0.32	0.17	0.10	0.03	0.33	0.11	0.08	0.04	0.29	0.06	0.19	0.06
19	0.55	0.10	0.06	0.05	0.32	0.10	0.06	0.02	0.33	0.11	0.06	0.05	0.29	0.07	0.11	0.05
20	0.55	0.09	0.05	0.04	0.32	0.09	0.03	0.02	0.33	0.08	0.02	0.04	0.29	0.08	0.02	0.05

Note: $\mu = 100$, $K = 800$, $h = 1$, $\sigma = 10$

Table 11: **Percentage deviation from optimality for seasonal demand, where $b = 12$**

	$a = 20$				$a = 40$				$a = 60$				$a = 80$			
T	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST	SM	WW	EIV	ST
2	29.83	29.83	17.24	16.02	31.63	31.63	15.66	16.40	34.24	34.24	13.07	16.09	38.95	38.95	14.42	17.03
3	5.69	5.69	16.98	15.96	7.02	7.02	15.85	11.44	9.07	9.07	13.43	8.74	12.79	12.79	10.50	7.43
4	1.07	0.82	4.72	1.12	1.89	1.91	5.68	4.07	3.17	3.78	5.26	4.45	5.71	7.30	4.33	3.88
5	0.61	3.63	0.50	0.47	0.73	3.98	0.73	0.67	1.43	4.42	2.88	2.53	3.17	5.83	2.96	2.61
6	0.61	4.57	0.44	0.46	0.70	3.34	0.45	0.42	0.91	2.56	0.69	0.67	1.98	2.97	1.80	2.17
7	0.61	1.87	0.34	0.35	0.70	2.40	0.40	0.43	0.73	2.62	0.46	0.41	1.44	2.71	1.31	1.49
8	0.61	0.75	0.32	0.34	0.70	1.67	0.37	0.38	0.75	2.45	0.36	0.38	1.26	2.57	0.87	1.09
9	0.61	1.19	0.27	0.32	0.70	1.05	0.25	0.27	0.74	1.70	0.25	0.27	1.08	2.17	0.47	0.69
10	0.61	1.51	0.26	0.24	0.70	0.94	0.26	0.22	0.74	1.10	0.24	0.32	1.08	1.37	0.40	0.71
11	0.61	0.90	0.23	0.19	0.70	0.91	0.19	0.18	0.74	1.07	0.24	0.28	1.08	1.46	0.31	0.45
12	0.61	0.54	0.19	0.19	0.70	0.71	0.12	0.11	0.74	0.86	0.21	0.19	1.08	1.28	0.27	0.40
13	0.61	0.72	0.13	0.14	0.70	0.57	0.08	0.10	0.74	0.63	0.19	0.16	1.08	1.08	0.23	0.31
14	0.61	0.59	0.13	0.12	0.70	0.41	0.09	0.10	0.74	0.75	0.12	0.12	1.08	0.72	0.18	0.25
15	0.61	0.52	0.11	0.13	0.70	0.41	0.08	0.08	0.74	0.58	0.09	0.12	1.08	0.61	0.15	0.20
16	0.61	0.34	0.10	0.10	0.70	0.34	0.07	0.08	0.74	0.40	0.11	0.16	1.08	0.55	0.12	0.17
17	0.61	0.37	0.09	0.09	0.70	0.30	0.07	0.07	0.74	0.40	0.10	0.11	1.08	0.46	0.09	0.15
18	0.61	0.30	0.09	0.08	0.70	0.22	0.07	0.07	0.74	0.41	0.09	0.13	1.08	0.45	0.07	0.11
19	0.61	0.21	0.09	0.08	0.70	0.16	0.06	0.06	0.74	0.29	0.10	0.10	1.08	0.42	0.05	0.05
20	0.61	0.19	0.09	0.09	0.70	0.21	0.05	0.04	0.74	0.27	0.09	0.09	1.08	0.25	0.06	0.05

Note: $\mu = 100$, $K = 800$, $h = 1$, $\sigma = 10$