Duverger's (f)law: Counterproof to the Osborne Conjecture

Jan-Pieter de Vries (331895)

January 20, 2015

Abstract

Duverger's law implies that elections under plurality rule (winner takes all), tend towards two party systems. This paper investigates this claim for a theoretical, linear model with sincere voting and sequential entry. It finds that Osborne's conjecture regarding an equilibrium of the first and the last pawn at the median voter does not hold if there are at least twelve potential entrants. It remains to be proven however what the new equilibrium might be.

1 Introduction

Duverger once wrote that single ballot majority electorates (plurality rule) tend to lead towards two party systems. Ever since, economists have tried to test this using theoretical and empirical research. One of them, Osborne[1], made a conjecture in the mid 1980's about the equilibrium of a certain model. He argued that under the assumptions of the model (see section 3), only the first and last party would position at the median voter, whereas the rest would refrain from entering the election. This paper shows that this conjecture does not hold once there are at least twelve parties in total to consider. In particular, it shows that the second party may enter to initiate a setup that allows six parties to share victory. This setup is optimal for all participants, and prevents further entry. Furthermore, after the setup is complete, the share of votes that remains for the first party is inferior to that of the six from the setup.

The paper is structured as follows. Section 2 starts by analysing the literature regarding Duverger's law, as well as similar models. This analysis is performed using both theoretical as well as empirical papers. Furthermore, the assumptions of this paper will be discussed, and compared to other models. Section 3 introduces the model, along with some definitions, and the first lemmas. After that, Section 4 disproves any two party equilibrium that includes the first party. It starts with the most ssimple proofs, and ends with disproving Osborne's conjecture. Section 5 concludes the findings, and discusses the results.

[1] http://www.economics.utoronto.ca/osborne/research/CONJECT.HTM (9-jan-2015)

2 Literature Review

This section describes the findings of the literature regarding Duverger's law, as well as the implications of the assumptions regarding the model. It begins by analyzing the literature regarding the law itself, both theoretically, as well as empirically. After that, the assumptions are discussed.

2.1 Duverger's law

The origin of Duverger's law stem from his observations, where Duverger noticed that single ballot majority systems (plurality rule/winner takes all), have a tendency to lead toward two party systems. This claim has been investigated ever since, and some researchers even tried to improve it, based on their their findings. This paper does not intent to judge the correctness of Duverger's law, nor the alternate versions. What will be done, is discuss some of the literature, in order to gain insights into the intuitions from other researchers. These insights will then be used to analyze the robustness and the implications of the assumptions. In order to do so, the next subsection discusses the empirical findings, followed by the theoretical analysis.

2.1.1 Empirical data

The countries that appear to have been analyzed most extensively, are Canada, India, and the USA. From these three countries however, only the USA seems to properly abide to Duverger's law, in the sense that only two parties actively participate in the elections. This is for instance reported by Grofman et al (2009) and Scarrow (1986). They do note however, that the Democrats and the Republicans are not just two parties, but appear to be two large alliances, consisting of many small subgroups. Thus, although the elections are run between only two parties, the political preferences of people appear to diverge more than just one of two options. Furthermore, Grofman et al (2009) argue that the lack of further competition also stems from the high entry barriers, as well as the strategic positioning of the two dominant parties. They thus argue that this is what maintains the two party competition. Gaines (1997), disagrees with these conclusions, as he points out to multiple instances where third parties received well over ten percent of the votes. He then continues to use these percentages gained by third parties to calculate the effective number of parties that may be considered. Although that finding is not relevant for this paper, the notion that third parties do arise, despite all the forces working against them, is worth keeping in mind.

Another country where a third party continues to claim a substantial share of the votes is Canada. All papers analyzed regarding Canada agreed that there is a consistent third party there, with a considerable amount of votes. The explanations for this vary. Dobell (1986) for instance argues that that it is the lack of national focus that allows a third, more regional, party to score impressively. He analyzes both Canada, as well as India, to show that 'local heroes' can be successful in national elections. The persistence of these local heroes, are confirmed by Diwakar (2007), whith his statistical analysis regarding both Canada and India. Gaines (1999) however, partly disagrees in the case of Canada, by explaining the importance of the regard for Canadian institutions, and their effect on the number of parties. Riker (1976), is more in favor of the local hero argument, but also notes that third parties do tend to perish in the long run. Despite the fact that some third parties continuously reach ten percent of the votes or more, as long as victories remain absent, the motivation for those third parties to continue running seems to diminish. The question then thus becomes how serious those third parties can be taken. If they do appear to perish in the long run, then perhaps the equilibrium is a two party system after all.

Several papers regarding countries other than Canada, India or the USA support the idea that third parties tend to perish. In Brazil for instance, the elections for mayors were changed towards a plurality system, after which Fujiwara (2011) reported that this lead towards a substantial decrease in the number of candidates. He further stated that in almost all the cities investigated, the system seemed to create a two party competition. Another analysis, regarding Italy, was performed by Reed (2001). He found that in the district elections, the change of the system towards a plurality rule bi-polarized 80 percent of all districts. This thus suggests that the implementation of a different electorate system changes both the number of candidates, as well as the policies that are being presented, and thereby the policies being implemented. Colomor (2005) disagrees with this reasoning, and argues that it is in fact the other way around. After analyzing 200 elections in over 80 countries, he concluded that it was the mindset of the parties that changed the system, rather than it being the other way around. He found that if the parties wanted a two party system, that they would then implement a plurality rule. If they preferred the presence of multiple parties, then proportional representation seemed to be favored.

As a final remarks regarding the empirics, it might be worth noting that the increasing amount of available data allows for more and more in depth analysis of Duverger's law. This point was also emphasized by Benoit (2006), who argued that, based on the majority of the data he analyzed, Duverger's law is almost as good as an actual law, comparable to laws in the field of physics.

2.1.2 Theoretical models

The theoretical papers, and their models, appear to argue mostly in favor of Duverger's law, as they tend to find two party equilibria under plurality rule. This is thus in contrast with the empirical data, which appears to contain a multitude of papers that describe exceptions to the law. The deviation of the theoretical models might be a results of the assumptions that they make. Therefore, in an attempt to analyze this, the following subsections will investigate the assumptions that are made in the literature, by comparing them to the assumptions that are made in this paper.

2.2 Uniform voter distribution over a linear interval

This model assumes that voters are uniformly distributed over a linear interval. The benefits of these assumptions are that the model remains relatively simple to solve and understand. The question is however, whether or not these assumptions oversimplify reality. According to Blais and Carty (1991), that would be the case. They investigated the psychology behind voter behavior, and discovered that the implementation of the plurality rule tends to polarize peoples political preferences. This suggests that the preferences are not uniformly distributed, nor normally distributed for that matter. A double peaked distribution would be more in order according to their research. Furthermore, their research suggests that the distribution is not exogenous, which may further complicate any realistic modeling.

The linear interval assumption is also not free from critics. Taagepera and Grofman (1985) argue that there is a strong correlation between the number of dimension and the number of parties that survive in equilibrium. They find that the number of parties tends to be one higher than the number of issues (or dimensions). This thus suggests that the linear interval assumption is critical for determining whether or not there is a two party equilibrium. Despite that, most (political) models continue to use the linear interval, even if only for the sake of simplicity or the sake of argument. One ought to keep in mind however, that the (explanatory) power of the model does diminish in the presence of questionable simplifications.

2.3 Sincere voting (no strategic voting)

Another assumption regarding the voters is that they vote sincere. This assumption is harmless if Duverger's law holds, but might be rather critical in cases with multi-party equilibria. The reason why it is harmless under Duverger's law is as follows. If there are only two parties, there is nothing to gain from voting for ones second favorite party. Even if your favorite party loses for sure, a change of vote does not change the outcome. Once there are more than two parties however, as change of vote from you favorite (but losing) party towards your second choice might alter the outcome in favor of your second choice. For this to work however, one additional condition must be true. People must have an impression of what the outcome will be without strategic voting. Usually this is accomplished with polls. The effect of polls has been measured by Endersby and Shaw (2009), who conducted and artificial experiment. They appointed people a political preference, along with the mission to elect a party with a platform as close to that preference as possible. Before the actual election, they provided two rounds of polls, where people were asked to state their choice before the actual election. What they noticed was that these polls quickly reduced the number of viable candidates to two, and that once there were only two realistic options left, that (among them) voting was completely sincere (for the reason mentioned at the beginning of this paragraph). This finding is also supported by Fedderson (1992), who models strategic (but costly) voting. Fedderson argues that it is the strategic voting that reduced the number of parties to at most two, after which people vote sincere. Fey (1997) confirms this finding in and even more in depth analysis, by modeling the polls along with the election outcomes under strategic voting. His model then suggests that the polls enforce the strategic voting, and that, similar to Endersby and Shaw, and Fedderson, the strategic voting leads to a Duvergian outcome.

In the absence of polls, when strategic voting becomes rather difficult, it seems to be more sustainable for smaller parties to survive. Clough (2007) modeled the percentage of knowledge people have regarding the election outcomes, and their effect on the number of parties. He finds that without any information, almost all people vote sincere. This then declines as the fraction of available information increases, until everyone votes strategically when all information is available. The question remains however, to what degree people vote strategically in actual elections.

2.4 Party preferences and positioning

Parties are the players in the model of this paper. They make decisions and try top optimize their outcome, whereas voters only passively respond. It is for instance assumed that parties want to maximize their chance of winning, regardless of the platform required to do so. Furthermore, if politicians do not have a positive chance of winning, they are presumed not to enter the election in the first place. Although this last assumption seems rather harmless, as losing politicians tend to perish, the indifference regarding policies does not. For instance, Lehoucq (1995) investigated Costa Rica, and found that there were politicians willing to take a stand on a losing position. He then argued that it was their intrinsic belief in that policy that made them try despite being in an unfavorable position. This thus also argues against the assumption that politicians only enter with a positive chance of winning (in the short run). The argument regarding platform indifference may however still be defendable, as one might argue that it is the popularity of a certain platform that motivates politicians to run for office. Thus, if a certain platform has a high chance of winning, a politician who believes in that platform might arise.

Osborne's conjecture states that only two parties will enter, and that they will do so at the median voter. This suggests that both parties prefer the same policy, and that votes are given out of sheer indifference, leaving victory up to chance. This result is similar to Fedderson (1990), who uses a model with strategic voting under plurality rule. The final victor in his model is also determined by some randomness in the voting behavior of the indifferent voters. Congleton and Steunenberg (1998) on the other hand, model myopic entry, and find that more than two parties enter at entirely separate locations. An even more extreme case is that of Grosser and Palfrey (2014), who find that only the most extreme parties position themselves, at the two ends of the spectrum. These alternate findings raise the question to what degree the assumptions lead to a realistic outcome, since it is impossible that these opposing points of view are all correct. For this paper, the setup that disproves Osborne's conjecture is a multi-party setup with unique and diverse locations. This paper thus allows for diversity, but does not find extreme outcomes (similar to Congleton and Steunenberg (1998)).

2.5 Sequential decision making regarding party positioning

When it comes to modeling elections, there appears to be a binary choice. One can either assume simultaneous or sequential entry. The argument in favor of simultaneous positioning is generally that parties do not wait for others to present their platform, but that it al happens at a (sufficiently) similar time, and that the presentation of a platform of one party does not imply a radical change of the presentation of the other. Therefore, the decision regarding the platform that they present is sometimes presumed to be made independent, while only the expectations of the positions of other are considered. The logic behind sequential positioning is that incumbents tend to rerun with their old platform, (thereby already being positioned), while others enter after that. Furthermore, since there is some time between the presentation of the platforms, it may also be interpreted as if they are a reaction to one another.

Both approaches are of course different from one another, and thus yield different results when it comes to Duverger's law. Humes (1990) for instance, analyzed both sequential and simultaneous models, and found that there were no multiparty equilibria under sequential entry, whereas they do exist under simultaneous positioning. The finding regarding the sequential entry may however depend on the assumption that no two parties may share the same platform in his model. Callander and Wilson (2007) find that with simultaneous entry, only two parties enter at separate positions, and that they thereby deter further entry. This would thus support Duverger's law. For the case of sequential entry, it may not only be important whether or not parties may share a position (they can copy each other), but it may also be important if it is known up front how many parties may enter the election. If the number is fixed, then backward induction may be applied to reason if and where parties locate. If that number is uncertain or infinite, it may prove difficult to find any equilibrium at all. The next subsection thus discusses this assumption.

2.6 Number of parties

The aspect that distinguishes this paper from (almost) all others, is that it actively considers twelve parties in a sequential setting. Most other theoretical papers are limited to three parties. For instance, Forsythe et al (1993), uses three parties to show that pre-election polls change electorate outcomes, and lead to a two party system. This results from strategic voting for the two parties with the highest chance of winning. Rietz (2003) supports that finding, and argues that with three (initial) parties and strategic voting, only two effective parties remain. Another paper by Osborne (2000), considers three parties with sequential (and costly) entry, where the first two parties position in such a way that the third party is deterred from entering. These three papers thus all start with three parties, and show a process through which that number is (further) reduced to two. The setup from this paper, which has six tied winners, shows that there is potential for multi-party equilibria if the initial number of parties is high enough. This paper thereby disproves Osborne's conjecture of an equilibrium where the first and last party at the median voter. In all fairness however, it must be said that Osborne himself does not exclude a multi-party equilibrium from existing. In Osborne (1995), he argues that adding realistic features to a simple spatial model might allow for new equilibria that deviate from the current findings.

With respect to this paper, the assumption of twelve parties (or potential entrants) is necessary for the setup to be stable, although it may easily be extended to more than twelve parties. This is due to the fact that the initiator of the setup that disproves the conjecture, P_2, may easily be replaced with P_{n-10}, P_3 by P_{n-9} and so on until $P_{12} = P_n$. This goes w.l.o.g., as the conjecture presumed that no parties between P_1 and P_{12} were meant to enter. Therefore, if there are for instance thirteen rather than twelve parties, one may presume P_2 to refrain from entry, while $P3$ then takes it place in the setup as P_{n-10}. Therefore, any number of parties that is at least twelve, disproves the Osborne's conjecture, and might allow for a multi-party equilibrium. The generality of the results may however be lost, if the number of parties is unknown, or infinite. In both those cases, it will no longer be possible to properly apply backward induction, which in turn may undermine some reasoning's of the proof. For this paper therefore, the assumption of a finite and known number is essential, whereas the initial number of twelve may be extended to any fixed number larger than twelve.

3 Model

This section will first describe the assumptions, setup and definitions of the model, followed by its direct implications. These direct implications consist of lemmas, which may be proven without explicit examples, and form the backbone of the proof of this paper.

3.1 Assumptions

3.1.1 The parties

There are twelve parties, who make their decisions sequentially. They are named $P_1, P_2, ..., P_{12}$, and they make their decisions in the order of their number (P_1 is first, P_{12} is last). Parties are fully informed at all times[2]. The parties choose to position themselves on the linear political spectrum, or to refrain from entry; $\in$ [0,1] $\cup$ {OUT}. The pay-offs of the game are then as follows. After all parties have made their decision, the positioned parties receive a share of votes from a continuum of voters. Voter preferences are uniformly distributed over

[2]Party P_k makes it decision, knowing what parties P_1 to P_{k-1} have decided.

the interval [0,1], and votes are sincere[3]. All (k) parties with the largest share receive a pay-off $\frac{1}{k}$. Parties that do not enter receive 0, and parties that entered without chance of winning receive -1. Each party thus wants to maximize its chance of winning, but if winning is not possible, it will not enter. Parties cannot commit to cooperation, as they will always maximize their own chance of winning.

3.2 Definitions

The following definitions will be used throughout the remainder of the paper.

- Master plan: A setup of parties, intended to let all the participating parties receive the exact same share of votes, which is larger than the largest share of the non participating parties.

- Pawn: A party that is part of a master plan.

- Deviator(/Deviating): A pawn that locates on a different position than one of the positions of the master plan.

- Playing nice: A pawn that follows the master plan and locates at one of the intended positions.

- New master plan. A deviator that attempts to construct a new master plan. This new master plan consists of at least two pawns.

- Counter plan. A new master plan, that is initiated after the completion of a previous master plan, or after one party deviated on its own.

- victors' share: the share of votes that a pawn receives by complying to the master plan. This will be denoted by v, v' or v'' throughout the paper.

- Distortion(ary): If the deviation of a pawn implies an altered victors' share, then the deviation is distortionary. If a pawn can deviate whilst keeping the victors' share in tact, then the deviation is non-distortionary.

- Included party: A party that positioned before the (official) beginning of a master plan, that is included in the master plan. If the master plan is successful, this party will end with a victors' share. Additionally, this party will be considered a pawn.

- Excluded party: A party that positioned before the (official) beginning of a master plan, that is not included in the master plan.

- Area: A section or part of the voters that is located between two excluded parties.

- Edge: A section or part of the voters that is located between an excluded party and the edge of the model (either 0 or 1).

- Excess share(s): The votes in an area (or edge) that are not allocated to pawns receiving the victors' share, and that are therefore to be divided among the excluded parties.

- Interfering: -With respect to a party- positioning either on top of, or adjacent to that party. -With respect to an area or edge- positioning either within the area, or on top of an adjacent (excluded) party.

[3]Voters always vote for the party that is closest to their political preference. If two parties are equally close, then an equal chance determines which party gets the vote.

- Potential entrant: A party that comes after the completion of the (initial) master plan, which may then decide whether or not entry is profitable.

3.3 Preliminary Lemma's

The direct implications are the lemmas that follow from the assumptions of the model.

Lemma 1. If two or more pawns with unique locations are to position in an area or edge, then the last pawn may not position adjacent to an excluded party.

Proof: Suppose the last pawn to enter in an area is closest to an excluded party. This party will then deviate marginally away from the excluded party. The reason for this is that it does not provide additional votes for this deviating party, which means that it does not distort any (potential) remainder of the master plan. Moreover, it does take a sufficiently small amount of votes away from the pawn that was supposed to be in the master plan, and gives it to the excluded party. As this only concerns a sufficiently small amount of votes, the excluded party remains excluded (and is still inferior in terms of votes), and the formerly included pawn is now excluded as well. As a result, victory is shared with fewer parties compared to the initial plan, without creating any distortion. The lemma follows.

Corollary 1. It is impossible to have a master plan where an area is covered by exactly two pawns.

Proof: Regardless of which pawn locates first, the second will always be in violation of Lemma 1. The lemma follows.

Lemma 2. An area with an even number of pawns (k), with unique locations, must create two equal excess shares, and is strictly between k and $k+2$ victors' shares large.

Proof: Suppose the pawn closest to the excluded party is $2xv$ (with x being a fraction of the victors' share) away from the excluded party. This implies that it will receive xv votes on this side. It then needs to get $(1-x)v$ votes on the other side, to end up with one victors' share in total. This is the case when the next pawn is at a distance of $2(1-x)v$. That party, in turn, requires xv votes on its other side, implying that the next pawn or excluded party is a distance of $2xv$ away. Consequently, for any even number of pawns, the excluded parties both receive xv votes each. On top of that x must be smaller than one, as otherwise both excluded parties would receive more than v votes, which would make them superior to the master plan. Consequently, within the area, all pawns must receive v, and the two excluded parties receive xv (each), implying that the size of the area must strictly be between k and $k+2$ victors' shares large. The lemma follows.

Note that the excess share that any of the excluded parties from such an area receives is thus always: $\frac{area.size-kv}{2}$.

Lemma 3. An area with an odd number of pawns (k), with unique locations, must be $k+1$ victors' shares large.

Proof: Suppose the pawn closest to the excluded party is $2xv$ (with x being a fraction of the victors' share) away from the excluded party. It then requires $(1-x)v$ votes on the other side, to end up with one victors' share in total. This is the case when the next pawn (or excluded party) is at a distance of $2(1-x)v$. For any odd number of pawns, the two excluded parties of an area combined receive v[4] votes. As all the pawns also claim one victors' share each, the total size of the area must be $k+1$ victors' shares large. The lemma follows.

Note that the excess shares may be freely divided among the excluded parties (in contrast to a situation with an even number of pawns in an area).

Lemma 4. Interfering with an area using three or more pawns, automatically requires interference with areas that are half its size or larger.

Proof: By having three or more pawns in an area, the victors' share is at most one fourth of that area. The excluded parties of the smaller area would get half this area in votes on their respective sides (should there be no entry here), as well as a positive amount on their other side. Combined, they receive more than one fourth of the initial area. This is thus strictly larger than the victors' share. The lemma follows.

Lemma 5. If a certain master plan covers an area or edge with a certain amount of pawns and a certain victors' share, then any reduction in the amount of pawns in that area must be accompanied by an increase in the victors' share.

Proof: Define the size of the area as y.

Suppose the area was intended to be covered by an odd number of pawns (k). This implies that (by Lemma 3):

(1) $y = (k+1) * v$, or

(2) $k+1 = \frac{y}{v}$

Where v represents a victors' share.

Suppose now that one is trying to cover the area with one pawn less. This implies that (by Lemma 2):

(3) $(k-1) * v' < y < (k+1) * v'$, or

(4) $k+1 > \frac{y}{v'}$. Furthermore,

(5) $k-1 < \frac{y}{v'}$.

Where v' represents the new victors' share.

Combining equations (2) and (4) gives:

[4] Both excluded parties receive a share of votes equal to half the distance between them and the nearest pawn. As the distances are 2xv and 2(1-x)v respectively, the excluded parties receive $0.5(2x+2(1-x))v = v$ combined.

(6) $(k+1) = \frac{y}{v} > \frac{y}{v'}$, which gives

(7) $v' > v$

This means that, for the case above, using one less pawn demands a strictly larger new victors' share.

Now suppose that an area was supposed to be covered by an even number of pawns $(k-1)$, and that one is trying to cover that area with one pawn less $(k-2)$. This results in the following equation:

(8) $y = (k-1) * v''$[5], or

(9) $k - 1 = \frac{y}{v''}$

Where v'' represents the newest victors' share.

Combining equations (5) and (9) gives:

(10) $(k-1) = \frac{y}{v''} < \frac{y}{v'}$, or:

(11) $v'' > v'$

This implies that for the case above, using one less pawn demands a strictly larger new victors' share. Furthermore, combining (7) and (11) results in

(12) $v'' > v' > v$

Consequently, one can conclude that any reduction in the number of pawns results in a strict increase in the required victors' share per pawn. The lemma follows.

Lemma 6a. Any master plan with a copied position (two pawns at one location), where both pawns receive an unequal share of votes on both sides, is unstable if there is at least one potential entrant left after the completion of the master plan.

Proof: The two pawns must receive a victors' share each, for $2v$ combined. As both sides are unequal, there must be one side where both pawns receive more than v combined. The potential entrant may then position itself on this side, at a sufficiently small distance away from the two pawns. The potential entrant then captures more than v votes, making it superior. If one were to assume (w.l.o.g.) that the potential entrant is the last party (P_{12}), then there is no threat of further entry, and the lemma follows.

Lemma 6b. Any master plan with a copied position that is not closest to the edge, with unequal shares on both sides, must have at least one other copied position.

Proof: The shares that the two parties at the copied position receive on their largest side is combined more than one victors' share, as both receive more than half of a victors' share there. This then implies that the adjacent position receives more than one victors' share, unless it is also copied. The lemma follows.

Lemma 6c. Any master plan that copies a position between 0 and 1, with equal shares of votes on either side, when thus far no other

[5] $y = (k - 2 + 1) * v''$

position had been copied, requires that all positions between 0 and 1 must be copied.

Proof: For any position that is copied, with equal shares on both sides, both parties are to receive half a victors' share on either side. Combined, this implies that any direct neighbor must be at a distance of $2v$, as it is otherwise not possible to receive the correct amount of votes on either side. This neighbor thus receives v votes on this side. Combined with any positive share it may receive on its other side, this would be superior. This may only be overcome with a copied position for the neighbor(s) as well, or by having a pawn that is located at exactly 0 or 1 (as there are then no votes on the other side to be considered). In the case of a neighbor that requires copying as well, the same reasoning applies as to the initial two pawns. If the neighbors are also to receive a victors' share each, then they must receive another half a victors' share each on their other side, implying a distance of $2v$ to their next neighbor, who then receives v itself, unless it is also copied. This thus extends to the entire spectrum, with the exceptions of the positions at 0 and 1. The lemma follows.

Lemma 6d. Copying a pawn in an area is only compatible with a victors' share that is at most one quarter of the size of the area.

Proof: Suppose a new master plan requires that only one new copied position is to position in an area. These two pawns then combined receive half the distance between their two neighbors (similar to Lemma 3), which implies that both receive one quarter of the size of the area. As adding pawns cannot increase a victors' share (by Lemma 5), The lemma follows.

Note that integrating all sub lemmas from Lemma 6 implies that copying a position requires the copying of multiple positions, as well as a victors' share that is likely to be at most one quarter of the size of an area. In the remainder of the paper however, areas are generally dealt with using two pawns (and including one or more neighbors), which results in a victors' share that is larger than one quarter of the size of the area. As a larger victors' share requires either the same amount of pawns to cover areas, or fewer (by Lemma 5), it is generally safe to say that a larger victors' share is preferable to a smaller one. Therefore, in the proofs that follow, copying positions in order to interfere with areas or pawns is not always mentioned, as it should then be clear that it leads to inferior situations, compared to the master or counter plan at hand.

Lemma 7. If a new victors' share is smaller than an older victors' share ($v' < v$), then all (excluded) parties that were eligible for the older victors' share must be interfered with.

Proof: Suppose the new master or counter plan is completed, without interfering with at least one excluded party that was eligible for the old victors' share. This implies that the excluded party receives at least the old victors' share in terms of votes, as it has the same neighbors, or even more distant neighbors, compared to the old master plan. This would then make this excluded party superior to the new master plan. Consequently, such parties must be interfered with, and the lemma follows.

Note that if all the excluded parties that were eligible for the older victors' share were positioned on unique locations, that new entrants can at most interfere with two of those parties at the same time, by positioning between them. Even if this would provide exactly enough votes for all the interfering pawns, it would still require at least half the number of excluded parties that were previously eligible for the old victors' share, as new potential entrants.

4 Analysis

This section shows why all two party setups, initiated by P_1, are unstable under the assumptions of the model. It first introduces the structure in which the proofs will be constructed, which is mentioned under the general remarks. After that, the proofs start with the simple two party equilibria that include P_1, followed by Osborne's conjecture.

4.1 General remarks

The following section consists of the proofs for the claims made in the introduction. As these proofs are rather technical and sometimes repetitive, this paper will use the following setup to describe the proofs. The following sections will describe the incentives for pawns to play nice (or to deviate), using backward induction. For all pawns, the potentials for deviation will be described, ordered by the number of pawns that are required. For instance, a pawn may first consider to deviate on its own, which will then be described under **(1)**.Whenever **(1)** is written at the beginning of a line, the following is implied: 'Suppose pawn x tries to deviate on its own'. If the pawn uses one other pawn to construct its own master plan, it will be described under **(2)**. Whenever **(2)** is used at the start of a line, this implies: 'Suppose pawn x tries to deviate using a master plan consisting of two pawns'. This goes on until the number of pawns is equal to, or higher than the number of pawns involved in playing nice (e.g. **(->6)**), or violates other constraints. Once the number of pawns required for deviation is at least as high as the number of pawns involved in playing nice, then there is no incentive to deviate, as playing nice results in a chance of victory that is at least as large. This is therefore the last number that is to be considered. Furthermore, it may be the case that some possibilities are covered simultaneously. For instance, it may be the case that the deviation options with two, three and four pawns are sufficiently similar, such that they can be covered at once. In those cases, the line starts with **(2-4)**.

In some cases, it is possible for a pawn to deviate with a new master plan that uses less pawns than playing nice. However, in all of these cases, the remaining (potential) entrants can come up with a counter plan, that comes at the expense of the new master plan. In these cases, one is always to assume that the counter plan uses the latest possible parties. For example, if there is a deviating master plan that ends with P_4, and the counter plan requires seven new parties, then

P_6-P_{12} will participate in this counter plan, whereas P_5 does not enter[6]. If the counter plan is self sustaining, then there is no further debate about what happens next, as there are no more potential entrants to be considered.

Furthermore, the proofs for each pawn will be numbered as Lemma $w.x - z$, or $w.x.y - z$. Where w represents the section, x represents the party that initiated the master plan (mostly P_2), y represents the initiator of a counter plan, and z represents the pawn that is considering to deviate within that respective master or counter plan. For example, Lemma 3.2-7 means that it is in section 3, where P_2 initiated a master plan, and that P_7 is considering whether or not to deviate.

Finally, for all pawns in each master plan, it will be described why deviation is not profitable. This will be shown by eliminating all potential options for deviation, in order of the size of the corresponding victors' share. This means that first the options will be discussed that results in the largest share of votes for the deviator, followed by the smaller ones, until the required victors' share is so small, that it requires more pawns to complete it compared to playing nice.

4.2 Simple Examples

4.2.1 Uncommon two party master plans of P_1 (Position $P_1 \neq 0.5$)

Lemma 3.1-z[7] P_1.

Any two party plan from P_1 where P_1 does not position at 0.5 is unstable.

Suppose P_1 positions somewhere along the distribution, which is not one half, with the intention of a two party master plan. The party that is then to finish this master plan, can always deviate marginally towards the middle. This then requires the same number of parties to form a counter plan, but provides the second party with strictly more votes than P_1. Therefore, P_1 did not have an incentive to enter here. The lemma follows.

4.2.2 P_1 positions at 0.5, another party (not P_{12}) copies this position

Note that if the second party or pawn does not copy this position, that then P_1 has strictly more votes, making the second party inferior.

Lemma 3.1-z. (z is one of P_2 - P_{11})

Any two party plan from P_1, with two parties at 0.5 where the second party is not P_{12} (P_n), is unstable.

Suppose any one of the parties from P_2 - P_{14} copies P_1. P_{12} can then win the election by positioning slightly away from the middle. This provides almost half the votes to P_{12}, whereas P_1 and the other party have to share a little over half equally, making them inferior. The lemma follows.

[6]Any successful new master plan would prevent further entry. Consequenly, assuming that a party does not enter between the ending of the (new) master plan and the beginning of the counter plan is thus consistent with both plans, and the assumption is thus harmless.

[7]Where z represents the second party in P_1's setup.

4.3 Counterproof Osborne conjecture (P_1 positions at the median voter, P_{12} (P_n) copies)

This section shows that after P_1 has positioned, P_2 has an incentive to construct a master plan, which, given the position of P_1, is a sub-game perfect Nash equilibrium.

The master plan of P_2 uses a victors' share of 0.15, and the following (corresponding) positions:

$-P_2 = 0.1$
$-P_6 = 0.2$
$-P_4 = 0.4$
$-P_1 = 0.5$
$-P_5 = 0.6$
$-P_7 = 0.8$
$-P_3 = 0.9$

The positions of the pawns will be justified using backward induction. Note that P_1 receives 0.1 votes, which is inferior to 0.15.

4.3.1 The positioning of P_7

Lemma 3.2-7.

P_7 cannot deviate profitably from P_2's master plan.

(1). Suppose P_7 deviates within the area it was supposed to position. This yields an unchanged share of votes for P_7, as it still receives half the distance between its neighbors. One of the neighbors, on the other hand, will end up with a larger share of votes, as P_7 is positioned further away from it. As both neighbors were to receive a victors' share each, now either of the neighbors will receive strictly more than the old victors' share. As P_7's votes are unchanged, this is inferior from the perspective of P_7.

Deviation outside of the area implies ignoring the largest share of votes, and will be inferior to either P_3 or P_5.

(2->6). Suppose P_7 creates a new master plan. This then still requires interference with the area between P_5 and P_3, as it is by far the largest area. Using one pawn in this area does not work due to the reasoning above, and using two pawns violates Lemma 1, unless at least one neighbor is included. Consequently, at least three pawns are required to interfere with the area between P_5 and P_3. As there are now more pawns between P_5 and P_3 compared to the initial master plan, the corresponding victors' share is strictly smaller than the initial master plan (by Lemma 5). By Lemma 7, P_2, P_4 and P_6 now also require interference. Interfering with P_4 and P_6 may be done with one pawn between them, which requires a victors' share of 0.1 However, the pawn interfering with P_2 may never claim exactly 0.1, as it can receive at most slightly less than 0.1 by positioning a small distance to the left of P_2. Consequently, at least two pawns are required to interfere with P_4 and P_6. This then requires at least three new pawns on the left side of the spectrum, as well as at least three pawns on the right side of the spectrum, for at least six pawns in total. The corresponding chance of victory is thus at best on in six. As the chance of victory in case of playing nice is exactly one in six, P_7 does not have an incentive to deviate in such a manner. Combining all the above, the lemma follows.

4.3.2 The positioning of P_6

Lemma 3.2-6.

P_6 cannot deviate profitably from P_2's master plan.

(1). Suppose P_6 deviates within the area it was supposed to position. This then faces similar problems as P_7 under (1).

(2->6). Interfering with more than one pawn between P_2 and P_4 as well as more than one pawn between P_5 and P_3 faces similar problems as P_7 under (2->6), as the situation between P_5 and P_3 is the mirror image of the one between P_2 and P_4. Consequently, there is no incentive to deviate and the lemma follows.

4.3.3 The positioning of P_5

Lemma 3.2-5.

P_5 cannot deviate profitably from P_2's master plan.

(1). Suppose P_5 deviates between P_1 and P_3. This results in a share of votes equal to 0.2 (half of the distance between P_1 and P_3). Without further entry, P_2 would then receive 0.25 votes, which is superior. Consequently, P_5 must interfere with both the area between P_1 and P_3, as well as with P_2.

(2->6). Suppose P_5 creates a new master plan. This requires at least three pawns to cover the area between P_1 and P_3. This is either with two new pawns and by including P_1 (including P_3 implies playing nice, and using two pawns alone violates Lemma 1), or with three new pawns. At most (in the case of including P_1), this requires a victors' share that is equal to 0.125 (by Lemma 3). This then requires interference with P_2 and P_4 (by Lemma 7 (as $v' < v$; $0.125 < 0.15$)). If P_5 were to use one pawn between P_2 and P_4, then this pawn would receive 0.15 votes (half the distance between P_2 and P_4), which is not compatible with 0.125. Using two pawns between P_2 and P_4 violates Lemma 1, and using more than two pawns implies using at least six pawns in total. This then results in a chance of victory of at best one in six. As the chance of victory in case of playing nice is exactly one in six, P_5 does not have an incentive to deviate in this manner. Combining all the above, the lemma follows.

4.3.4 The positioning of P_4

Lemma 3.2-4.

P_4 cannot deviate profitably from P_2's master plan.

(1). P_4 has to consider both the area between P_1 and P_2, as well as the area between P_1 and P_3 (as both are symmetric, and sufficiently large). However, it is not possible to do so with only one pawn, as copying P_1 violates Lemma 6c.

(2). If P_4 is to win with a new master plan using two pawns, then this can only be done by locating one pawn in the area between P_2 and P_1, and one pawn in the area between P_1 and P_3. With one pawn in either area, both receive 0.2 votes each, and 0.4 combined. The remaining 0.6 is thus to be distributed over P_1, P_2 and P_3, implying that they all must receive 0.2 each, or else at least one of them is superior to P_4 and P_5. This is only possible when P_4 and P_5 position at 0.3 and 0.7. As P_2 and P_3 must be included, this may thus at best provide a more efficient way of showing that P_1 cannot initiate a two party master plan when there are at least twelve parties in total. This new master plan from P_4 however, may easily be countered by the five last potential entrants (P_8-P_{12}), as they may copy the five positions, which then shares victory with ten parties in total. As this is inferior to playing nice, P_4 either has no incentive to deviate in such a manner, or it still disproves P_1's two party setup.

(3->6). If P_4 is to use two pawns between P_2 and P_1, then also two pawns between P_1 and P_3 are required. However, this violates Lemma 1, unless (at least) either both P_2 and P_3 are included, or P_1 is included. Including P_2 and P_3 implies playing nice, making including P_1 the only potentially profitable option. Including P_1 implies that there are then five pawns between P_2 and P_3, and the corresponding victors' share (by Lemma 3) is approximately 0.133. Combined they capture approximately 0.667 votes. As the remaining 0.333 votes are split equally among P_2 and P_3, they will both become superior with approximately 0.167 votes each (by Lemma 2).

(6->6). Any new master plan that uses six or more pawns results in a chance of victory of at most one in six. As the chance of victory in case of playing nice is also one in six, there is no incentive to deviate with such a master plan. The lemma follows.

4.3.5 The positioning of P_3

Lemma 3.2-3.

P_3 cannot deviate profitably from P_2's master plan.

(1). Any position to the left of 0.7 can be countered by P_{12} positioning at 0.7. This then provides P_{12} with a little more than 0.3 votes, which is superior to P_1 , P_2 and P_3 (who receive at most 0.3, 0.3 and 0.2 respectively).

For the position of 0.7 exactly, there is room for entry at 0.9 and 0.3 (in that order). This is then similar to the new master plan from P_4 under Lemma 3.2-4 (2). Consequently, the counter plan is also the same.

For any position between 0.7 to 0.8 (0.7 and 0.8 not included), the remaining parties can come up with a counter plan using a victors' share of $0.1 - 0.125m$, where m depends on the exact position of P_3. The counter plan is described and justified in appendix A1. After the completion of this counter plan, P_3 has a chance of victory of one in ten, which is inferior to playing nice.

For the position of 0.8 exactly, the initial master plan from P_2 may be completed, and P_3 may even be excluded itself as P_5 (the pawn intended to locate at 0.6 in P_2's master plan) may then position itself marginally towards 0.8.

Any position to the right of 0.8 is inferior to P_1 in terms of votes[8].

[8]For any such a position P_3 receives at most slightly less than 0.35 votes (everything to the

(2). Using two pawns for a new master plan either implies one pawn between P_2 and P_1 (and one pawn to the right of P_1), or two pawns to the right of P_1. In the first case, the pawn between P_2 and P_1 receives 0.2 votes. The pawn to the right of P_1 however, receives at least 0.25 votes (by positioning at 1), which is thus incompatible. With two pawns to the right of P_1, both pawns receive at most slightly less than 0.25 each. However, P_2 then receives 0.3, which is superior.

(3). The next attempt is thus to use two pawns to the right of P_1, and one between P_1 and P_2. This will result in a victors' share of 0.2 (half the distance between P_1 and P_2 (by Lemma 3)). Furthermore, this requires fixed positions for the two pawns to the right of P_1. These positions are 0.7 and 0.9, as otherwise these parties do not receive 0.2 votes each. The pawn between P_2 and P_1 must position itself at 0.3 exactly, as otherwise P_2 or P_1 receives more than 0.2 in total. This situation is now the same as the one under Lemma 3.2-4 (2). Consequently, also here the last five potential entrants may copy all positions, resulting in a chance of victory of one in ten. This thus also results in an inferior chance of victory compared to playing nice.

(4). As one pawn between P_1 and P_2 does not work, at least two pawns will be required. However, as two pawns alone violates Lemma 1, this requires including either P_1 or P_2. Including either one implies that only one[9] pawn remains to cover the edge to the right of P_1. This pawn will then receive at least 0.25 votes (by positioning at 1), whereas the pawns between P_1 and P_2 receive at most slightly less than 0.2 each (by Lemma 2), which is incompatible.

(5). Using five pawns in total implies that there may be two pawns to the right of P_1, as well as two pawns between P_2 and P_1, and P_1 or P_2 may be included. If P_2 is included however, the required victors' share is equal to 0.15 (and the situation on the left side of the spectrum is equal to the initial master plan from P_2). This then requires three pawns to the right of P_1, for six in total. Using two pawns between P_2 and P_1 while including P_1 is possible, but only with the following positions: 0.15, 0.45, 0.5 (P_1), 0.8, and 0.85. These positions must be as mentioned, due to the fixed position of P_1. The new victors' share is then 0.175. This new master plan, however, can be countered by the remaining parties, as will be described in appendix A2.

(6->6). Any new master plan that uses six or more pawns results in a chance of victory of at most one in six. As the chance of victory in case of playing nice is also one in six, there is no incentive to deviate with such a master plan.

Combining all the above, the lemma follows.

4.3.6 The positioning of P_2

Lemma 3.2-2.

P_2 has an incentive to enter, whenever P_1 positions at 0.5.

right of its position, plus half the distance between P_1 and P_3). P_1 receives at least slightly more than 0.35 votes (half the distance between P_2 and P_3).

[9](4 - 2 (between P_1 and P_2) -1 (either P_1 or P_2) =1)

If P_2 positions itself at 0.1, after P_1 has positioned itself at 0.5, P_3-P_7 will not deviate from the master plan of P_2 as long as there is no counter plan from P_8-P_{12}. Lemmas 3.2-3 till 3.2-7 show that P_3-P_7 cannot deviate profitably from P_2's master plan. This Lemma (3.2-2) will thus be concerned with the counter plan opportunities of P_8-P_{12}.

(1). Once P_7 has completed the master plan of P_2, all parties from P_2-P_7 are to receive 0.15 each, if they are not interfered with. If one party after P_7 is to locate itself anywhere, it can get at most 0.1 (between P_6 and P_4, or between P_5 and P_7). Furthermore, it can at most interfere with two parties. As at least four parties from the master plan of P_2 remain uninterfered with, this solo attempt is inferior (by Lemma 7).

(2->12). Any form of entrance thus requires interference with all pawns from P_2's master plan. Any new pawn interfering with P_2 however, can at most receive slightly less than 0.1 (by positioning to the left of it). However, as P_1 receives 0.1 exactly, it will also require interference. Any pawn interfering with P_1 may receive 0.05 votes, regardless of its exact position. If under this new victors' share no excluded party is to be superior, then at least 20 parties are required who receive 0.05 each or less. As there are only twelve parties however, this is not feasible.

Combining all the above, the lemma follows.

5 Implications and generality of the results

This paper has shown that Osborne's conjecture[10] does not hold when there are at least twelve parties. For the given model, P_1 cannot profitably position itself in the middle with the intention of a two party master plan, as P_2 has an incentive to enter with the intent of a six party master plan. It remains to be proven what the new equilibrium is when there are twelve or more parties that consider entering. Although the proof uses exactly twelve parties, the results an easily be extended by replacing P_2 by P_{n-10}, P_3 by P_{n-9}, and so on until $P_{12} = P_n$. All parties between P_1 and P_{n-10} can be presumed to refrain from entering w.l.o.g., as the conjecture already presumed their absence[11]. Since there is no clear new equilibrium, it is not yet possible to determine whether this model argues in favor or against Duverger's law. What it does show, is that there is some potential for mulch-party setups, if the number of potential entrants is sufficiently large. An interesting followup question might be whether or not that is exogenous, or endogenous. For this paper has used twelve potential entrants, but only seven parties actually enter the arena. One might thus argue, that in a repeated setting, only six or seven parties would remain, which would then lead to a different outcome (plausibly the outcome of the Osborne conjecture).

[10]The conjecture was that P_1 and P_n would position at the median voter, and that all other parties would refrain from entering.

[11]This means that these intermediary parties either do not enter, and allow for the proof of this paper, or they do have some incentive to enter, but then that would still disprove the conjecture. Thus, as long as there is at least one setup that disproves the conjecture, one may conclude that the conjecture is wrong for all cases with at least twelve parties.

Regarding Duverger's law in general, it appears that the assumptions are critical in determination of whether or not there will be exactly two parties, en where they will position. In general, it appears to be the case that plurality rule leads to a two party system (especially with strategic voting), but some exceptions still remain, both in the models as well as in reality. Since the models do not (always) appear to result in the same findings as in reality, it might be wise to reconsider the added value of creating all these theoretical models, and their underlying assumptions.

Future research might thus be concerned with both solving the model, as well as making a meta-analysis regarding the assumptions, their empirical value, and their effect on Duvergers' law.

References

K. Benoit. Duverger's Law and the Study of Electoral Systems. French Politics, Vol 4 (2006) pp 69-83.

A. Blais and R.K. Carty. The Psychological Impact of Electoral Laws: Measuring Duverger's Elusive Factor. British Journal of Political Science, Volume 21, No. 01 (January 1991) pp 79 – 93.

S. Callander. and C.H. Wilson. Turnout, Polarization and Duverger's Law. Journal of Politics, Vol. 69, No. 4, (Nov 2007), pp 1047–1056.

Electoral Competition with Entry. Social Science Working Paper 1083 (December 1999).

E. Clough. Strategic Voting under Conditions of Uncertainty: A Re-Evaluation of Duverger's Law. British Journal of Political Science, Vol. 37, No. 2 (Apr., 2007), pp. 313-332.

J.M Colomer. It's Parties That Choose Electoral Systems (or, Duverger's Laws Upside Down). Political Studies, Volume 53 (2005), pp 1-21.

R.D. Congleton and B. Steunenberg. Voter discernment and candidate entry in pluralitarian election. Public Choice, Vol. 95, (1998), pp 287–305.

R. Diwakar. Duverger's Lawn and the Size of the Indian Party System. Party Politics, Vol. 13, No. 5 (2007), pp 539-561.

W.M. Dobell. Updating Duverger's Law. Canadian Journal of Political Science / Revue canadienne de science politique, Vol. 19, No. 3 (Sep., 1986), pp. 585-595.

J.W. Endersby and K.B. Shaw. Strategic Voting in Plurality Elections: A Simulation of Duverger's Law. Political Science, (April 2009) pp 393-399.

T.J. Feddersen. A Voting Model Implying Duverger's Law and Positive Turnout. American Journal of Political Science, Vol. 36, No. 4 (Nov., 1992), pp. 938-962.

T. J. Fedderson et al. Rational Voting and Candidate Entry under Plurality Rule. American Journal of Political Science, Vol. 34, No. 4 (Nov., 1990), pp 1005-1016.

M. Fey. Stability and Coordination in Duverger's Law: A Formal Model of Preelection Polls andStrategic Voting. The American Political Science Review, Vol. 91, No. 1 (Mar., 1997), pp. 135-147.

R. Forsythe et al. An experiment on coordination in multi-candidate elections: The importance of polls and election histories. Soc Choice Welfare, Vol 10. (1993) pp 223-247.

T. Fujiwara, A Regression Discontinuity Test of Strategic Voting and Duverger's Law. Quarterly Journal of Political Science, Vol. 6, (2011), pp 197–233.

B.J. Gaines. Duverger's Lawn and the Meaning of Canadian Exceptionalism. University of Illinois at Urbana–Champaign. Comparative Political Studies, Vol. 32, No. 7 (October 1999), pp 835-861.

B.J. Gaines. Where to Count Parties. Electoral Studies, Vol. 16, No. l (1997), pp. 49-58.

B. Grofman et al. Duverger's Law of Plurality Voting; The logic of Party competition in Canada, India, the United Kingdom and the United States (2009) (book).

J. Grosser and T.R. Palfrey. Candidate Entry and Political Polarization: An Antimedian Voter Theorem. American Journal of Political Science, Vol. 58, No. 1, (January 2014), pp 127–143.

B.D. Humes. Multi-party competition with exit: A comment on Duverger's Law. Public Choice, Vol. 64, (1990), pp 229-238.

F.E. Lehoucq. Institutional Change and Political Conflict: Evaluating Alternative Explanations of Electoral Reform in Costa Rica. Electoral Studies, Vol. 14. No. I. (1995), pp 23-45.

M.J. Osborne. Entry-Deterring Policy Differentiation by Electoral Candidates. Mathematical Social Sciences, Vol. 40, (2000), 41-62.

M.J. Osborne. Spatial models of political competition under plurality rule: a survey of some explanations of the number of candidates and the positions they take. The Canadian Journal of Economics / Revue canadienne d'Economique, Vol. 28, No. 2 (May, 1995), pp 261-301.

S.R. Reed. Duverger's Lawn is Working in Italy. Comparative Political Studies, Vol. 34, No. 3 (April 2001), pp 312-327.

T. Rietz. Three-way Experimental Election Results: Strategic Voting, Coordinated Outcomes and Duverger's Law. Working Draft for eventual publication in The Handbook of Experimental Economics Results (March 2003).

W.H. Riker. The Number of Political Parties: A Reexamination of Duverger's Law. Comparative Politics, Vol. 9, No. 1 (Oct., 1976), pp. 93-106.

H.A. Scarrow. Duverger's Law, Fusion, and the Decline of American "Third" Parties. The Western Political Quarterly, Vol. 39, No. 4 (Dec., 1986), pp 634-647.

R. Taagepera and B. Grofman. Rethinking Duverger's Law: Predicting the Effective Number of Parties in Plurality and PR Systems - Parties Minus Issues Equals One*. European Journal of Political Research, Vol 13 (1985) pp 341-352

Appendix

The appendix contains the proofs for the counter plans against deviations of P_3 in the master plan from P_2. These proofs are located here to alleviate the technicality of the main part of the paper.

A1. Counter-proof P_3 soloing (as deviation to the master plan of P_2).

P_3 may attempt to deviate on its own at any location between 0.7 and 0.8.For all those locations however, the remaining parties (P_4-P_{12}) can come up with a counter plan using a victors' share of $0.1 - 0.125m$, where m depends on the exact location of P_3.

The setup then looks as follows:
$x1 = 0.1 - 0.25m$
$-P_2 = 0.1$
$x2 = 0.1 + 0.5m$
$x3 = 0.3 - 0.25m$
$x4 = 0.3 + 0.25m$
$x5 = 0.5 - 0.5m$
$-P_1 = 0.5$
$x6 = 0.5 + 1.25m$
$x7 = 0.7 - 0.25m$
$-P_3 = 0.7 + m$
$x8 = 0.9 - 0.5m$
$x9 = 0.9 + 0.75m$

Where $[0 < m < 0.1]$[12], and where x1-x9 represent the positions of the parties (P_5-P_{13}). Note that the numbering is strictly left to right, and is not related to the ordering of positioning. With these positions, the ten pawns receive $0.1 - 0.125m$ votes each. Even for the most extreme values of m, this is larger than 0.0875. P_1 receives at most $0.875m$, which is slightly less than 0.0875. P_2 receives at most $0.375m$, which is at most slightly less than 0.0375.

The ordering of the positions is (grouped) as follows:
[1] - x9, x3, x6
[2] - x2, x1, x5,
[3] - x4, x7, x8.

The justification of the order of positioning will be performed using backward induction.

The positioning of group [3] (x4, x7, x8).

Lemma A.3.7-[3] Group [3] cannot deviate profitably from the counter plan.

(1->10). The most any deviator can receive is an initial victors' share, v, by positioning around any of the locations from the members of this group.

[12] Note that all pawns remain between their respective neighbors (they do not copy or jump over them for any value of m within its range).

However, this either implies playing nice, or giving one of either neighbors more votes compared to the counter plan. As these neighbors received v in the counter plan, they become superior upon deviation. Interfering with more pawns per area results in a smaller victors' share (by Lemma 5), which then requires interference with all areas that the initial counter plan interfered with, as well as at least as many pawns in those areas (by lemmas 5 and 7). This requires more potential entrants compared to the initial counter plan. However, as the last pawn to enter in the initial counter plan is P_{12}, this is not feasible. The lemma follows.

The positioning of group [2] (x2, x1, x5).

Lemma A.3.4-[2] Group [2] cannot deviate profitably from the counter plan.

(1->10). The most any deviator can receive by deviating is $0.1+0.125m$, by positioning between x3 and P_1. However, x6 then receives $0.1 + 0.5m$, which is superior. Interfering with x6 faces the same problems as group [3]. The lemma follows.

The positioning of group [1] (x9, x3, x6).

For this specific section, an additional general lemma will be introduced, namely Lemma 8. The situation depicted in the lemma is applicable for most situations throughout Lemma 6.3.4-[1], and is therefore captured in a separate lemma.

Lemma 8. It is impossible to have a required victors' share that is strictly between 0.1 and 0.15, when the only parties on the left side of the spectrum are P_2 (at 0.1) and P_1 (at 0.5).

Suppose that a victors' share between 0.1 and 0.15 is required to construct a master plan on the right side of the spectrum. This share then prevents any pawns from entering to the left of P_2, as any pawn there could then receive at most slightly less than 0.1 votes. Furthermore, exactly two parties must be positioned between P_2 and P_1 (by Lemma 2). This then results in two excess shares that are larger than 0.05, but smaller than 0.1 (by Lemma 2). As a consequence, P_2 receives 0.1 on its left side, as well as a share between 0.05 and 0.1 votes on its right side, for a combined share of votes between 0.15 and 0.2. As the required victors' share was between 0.1 and 0.15, this is inferior to P_2. If any pawn were to copy P_2 (by positioning at 0.1), then the share of P_2 would have to be shared by at least two parties. In the absence of any other parties around it, this implies a share of exactly 0.15, which is incompatible the required victors' share (as that lies between 0.1 and 0.15). If there are two parties between P_2 and P_1, then (by Lemma 2) those parties provide an excess share between 0.05 and 0.1 to P_2. If P_2 has to share its left flank, as well as those excess votes, then both P_2 and the copying party receive at most slightly less than 0.1 each, which is incompatible with the required victors' share. Combining all the above, the lemma follows.

Corollary 1. It is impossible to have a required victors' share between 0.1 and 0.15 when the only parties on the left side of the spectrum are P_2 (at 0.1), P_5 (at $0.3 - 0.25m$)[13] and P_1 (at 0.5).

Similar to the situation in Lemma 8, no pawn may position itself to the left of P_2. Furthermore, as the distance between P_2 and P_5 is less than 0.2, any pawn positioning here receives at most slightly less than 0.1. Therefore, P_2 will receive slightly less than 0.2 combined, which is superior to the required victors' share. Any pawn trying to copy P_2 receives at most slightly less than 0.1, which is incompatible with the required victors' share.

Lemma A.3.4-[1] Group [1] cannot deviate profitably from the counter plan. Note that x9 initiates the master plan. Therefore, as long as all other pawns play nice, and there is no threat of further entry, x9 has an incentive to initiate the counter plan. As the last pawn to enter is P_{12}, there are no further potential entrants, and there is no risk of another counter plan. Furthermore, by lemmas A.3.4-[2] and -[3], pawns x1, x2, x4, x5, x7 and x8 play nice. The remainder of this lemma is thus concerned with x3 and x6. Note that the ordering of the positioning of x9, x3 and x6 is strictly as mentioned. This is different from the other groups, where the exact ordering does not really matter, as the situations are sufficiently similar.

x3 (1-4). The most x3 can receive by deviating is 0.2, by positioning between P_2 and P_1. However, P_3 then receives $0.2 + 0.375m$, which is superior. Interfering with P_3 results in at most $0.1 + 0.5m$ (max slightly less than 0.15), by positioning between P_1 and P_3. By Lemma 8, such deviation is inferior.

(5->10). Using two pawns between P_1 and P_3 violates Lemma 1, unless either P_3 or P_1 is included. Including P_3 may be combined with zero or one pawn between P_3 and x9, and including P_1 may be combined with two or three pawns between P_2 and P_1. Using more or less pawns in the areas P_3-x9 or P_2-P_1 is not feasible, as then the corresponding theoretical victors' shares are incompatible. The sizes of theoretical victors' shares follow from either Lemma 2 or Lemma 3. The situations with two pawns between P_1 and P_3 will now be described in the order in which they were mentioned above.

(Including P_3, zero pawns P_3-x9). By Lemma 3, this setup results in a required victors' share of one quarter of the distance between P_1 and x9, or $0.1 + 0.1875m$. By Lemma 8, this is inferior.

(Including P_3, one pawn P_3-x9). This setup implies playing nice, as the required victors' share is v (by Lemma 3).

(Including P_1, two pawns P_2-P_1). By Lemma 3, this setup results in a victors' share that is equal to one sixth of the distance between P_2 and P_3, or $0.1 + 0.167m$. By Lemma 8, such deviation is inferior.

(Including P_1, three pawns P_2-P_1). By Lemma 3, this setup results in a required victors' share of one quarter of the distance between P_2 and P_1, or 0.1. This however, is inferior to x9, who receives $0.2 - 0.875m$ votes (> 0.1125).

[13]**With** $0 < m < 0.1$

Furthermore, any interference with x9 results in a share of votes of at most $0.1 - 0.125m$, which is incompatible with 0.1.

Using more pawns in any of the mentioned areas results in a victors' share that is smaller than the initial victors' share (by Lemma 5), which in turn requires at least as many pawns in all areas, which then requires more potential entrants compared to the initial counter plan, which is not feasible.

x6 (1->10). Similar to x3, x6 must interfere with P_3. Then, similar to x3, this results in a required victors' share that is either larger than 0.1 (max $0.1 + 0.5m$, < 0.15), or a victors' share that is smaller than the initial victors' share. For all the cases where it is larger than 0.1, it is inferior by Lemma 8. For all the cases with a smaller new victors' share ($v' < v$), interference with all areas is required (by Lemma 7). Furthermore, as more pawns are then required around P_3 compared to the initial counter plan, the total number of potential entrants is increased. However, as the last pawn from the initial counter plan was P_{12}, this is not feasible.

Combining all the above, the lemma follows.

A2. Counter-proof new master plan P_3 with five pawns (including P_1).

The remaining parties (P_7-P_{12}) have an incentive to enter using a new victors' share of 0.0875. With this counter plan they include P_1, P_3, P_4, P_5 and P_6, resulting in eleven pawns in total. Note that this counter plan still prevents P_3 from deviating in such a manner, as playing nice would result in a chance of victory of one in six.

The setup then looks as follows:

$x1 = 0.075$
$-P_2 = 0.1$
$-P_4 = 0.15$
$x2 = 0.275$
$x3 = 0.325$
$-P_6 = 0.45$
$-P_1 = 0.5$
$x4 = 0.625$
$x5 = 0.675$
$-P_5 = 0.8$
$-P_3 = 0.85$
$x6 = 0.975$

Where x1-x6 represent the positions of the parties P_7-P_{12}. Note that the numbering is strictly left to right, and is not related to the ordering of positioning. With these positions, the eleven pawns get 0.0875 votes each. The ordering of the positions is (grouped) as follows:

[1] - x2, x4,
[2] - x1, x6,
[3] - x3, x5.

The justification of the order of positioning will be shown using backward induction.

The positioning of group [3] (x3, x5).

Lemma A.3.7-[3].

Group [3] cannot deviate profitably from the counter plan.

(1->11). The most either pawn can receive by deviating is 0.0875, between x2 and P_6, or between x4 and P_5. However, this either implies playing nice, or results in more votes for either neighbor by deviating slightly from the intended position. As all neighbors receive a victors' share under the counter plan, any amount more will make them superior. Consequently, it is not feasible to deviate with one pawn in the areas where x3 and x5 are meant to position. Using more than one pawn implies a smaller victors' share (by Lemma 5), which in turn implies interference with all pawns from the counter plan (by Lemma 7). This in turn requires at least as many pawns in all areas compared to the extended master plan (by Lemma 5). Combined with the additional entrants that are required in the areas of x3 and x5, this thus requires more potential entrants than the counter plan. However, as the last pawn from the extended master plan is P_{12}, there are no more potential entrants to consider, implying that such deviation is not feasible. Consequently, the lemma follows.

The positioning of group [2] (x1, x6).

Lemma A.3.7-[2] Group [2] cannot deviate profitably from the extended master plan.

(1->11). The most any pawn can receive by deviating, is slightly less than 0.15, by positioning to the right of P_3. However, this is inferior to x2 and x4, who receive exactly 0.15 each. Interfering with x2 and x4 is similar to group [3], which is thus not feasible. The lemma follows.

The positioning of group [1] (x2, x4).

Lemma A.3.7-[1] Group [1] cannot deviate profitably from the extended master plan.

Note that x2 initiates the counter plan. Therefore, as long as all other pawns play nice, and there is no threat of entry, x2 has an incentive to enter. Lemmas A.3.7-[2] and -[3] show that x1, x3, x5 and x6 have no incentive to deviate, and this lemma will show that also x4 has no incentive to deviate. Furthermore, as the last pawn (x5) is P_{12}, there are no additional potential entrants, and there is no threat of entry.

x4 (1->11). The most x4 can receive by deviating is 0.15, by positioning between P_1 and P_5. However, P_3 then receives 0.175, which is superior. Interfering with P_3 results in at most slightly less than 0.15, by positioning to the right of it. In that case, x2 receives exactly 0.15, which is superior. Interfering with x2 is inferior for similar reasons as group [3].

Combining all the above, the lemma follows.