Erasmus University Rotterdam

Thesis

Bicycle Allocation Methods Within Public Bicycle-Sharing Systems

Author:
Emma Brocken, 373223

Supervisors:
Dr. Y. Zhang
Dr. A. F. Gabor

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Abstract

This paper is a study of the paper of J. Shu et al.[2], which discusses the development of operational research models that simplify decision making in the management and design of public bicycle-sharing systems.. To begin with, the results of a simple example from the paper of J. Shu et al.[2] will be reproduced. This is a three-station example, for which the time-average level of bicycles that are traveling in between stations is calculated both by implementing a theoretical analysis, as well as running a simulation. The results of the two different techniques will be compared, and turn out to be quite similar. To obtain better insight in the results, the time-average amount of bicycles at each station and the percentage of customers that is lost is calculated. The simple example is extended by taking into account arbitrary travel times, as a result of which the theoretical model and the simulation need to be adapted. In the second part of the paper, two larger and more complex problems are discussed. In this way, we can test if the simulation model also works for problems other than the simple example. The results of the simulation and the theoretical model for the 4-node example with steady travel rates and travel time excluded, show on average a 33.8% difference in the average amount of bicycles moving in between stations for all combination of stations, and the 6-node example shows a 10.3% difference. But we cannot jump to the conclusion that the theoretical model is not accurate, moreover we will see that the ratios of the bicycles leaving from station i traveling to station j divided by total bicycles number of leaving from station i are the same for the simulation and the deterministic model for all stations.

Contents

1 Introduction

1.1 Problem Statement

With the continuously increasing world population, governments have great concerns about carbon emissions, traffic congestion and lack of oil stocks. Looking for alternative ways of traveling is necessary. One of these alternatives is the use of a public bicycle-sharing system (BSS), this is a "self-service, one-way-capable, public place for bike rental, which has network characteristics" [1], and is most applicable for short distances. Bike-share began in Europe in 1965 but is evolving rapidly. Nowadays we can find bicycle-sharing systems all around the world. There are many advantages for the use of bicycle sharing systems, which include lower greenhouse gas emission, increased transit use, improved public health and decreased personal vehicle trips [2]. The paper of J. Shu et al. [2] develops operational research models that simplify the decision making process in management and design of public bicycle-sharing systems. A model is developed using proportionality constraints in order to estimate the movement of bicycles in the networks and the amount of trips that are supported, when the initial endowment of bikes at each station is specified. //
Another problem the paper explains is how the flow of bicycles and the utilization rate is affected by the redistribution and stationing of bicycles. In my research, I will first reproduce the first part of the results of this paper by building the same simulation and theoretical model, after that I will elaborate on a simple example discussed in the paper by adding some features. These features consist of including arbitrary travel times in the simulation and adapting the simulation such that it can handle travel rates that vary over time, whereby the results of the adapted simulation will be compared with a theoretical model for time-varying demand. Finally, two larger examples will be discussed, and the outcome of the time-average amount of bicycles traveling in between stations for the simulation with steady demand (travel time not taken into account) and the deterministic model for the steady state are compared.

1.2 Research Questions

Considering the topics discussed in the article of J. Shu et al. [2], and the possible extensions that could be made, I formulated the following research questions:

- Looking at the 3-node example in the paper, is it possible, for both the simulation and the deterministic model, to reproduce the average number of bicycles present per station?
- Are the results of the simulation and the deterministic model comparable for a larger example?
- Can we take travel time into account? If so, do the results change and how?
- What are the consequences of time-varying demand (for both the simulation and the deterministic model)?

1.3 Structure of the paper

In the next section (Section 2) the different methods that are used will be discussed. To begin with, two theoretical models will be discussed. The first model is the one for time-varying demand, the second one is the model for the case when the system is in equilibrium. Next, in Section 2.2, the two different ways in which the simulation is made will be explained and the simulation will be validated. After this, in Section 3, the obtained results from applying these methods will be presented. This section consists of four subsections, in which respectively the time-average number of bicycles at each station, the percentage of lost customers, the time average amount of bicycles traveling between stations and larger networks will be discussed, The results will be used to base the conclusions upon Section 4. Finally, some ideas for future research are discussed.

2 Methods

In this section the LP formulation of two theoretical models will be discussed. These models determine the expected amount of customers traveling from station i to j either in steady state or per time unit (in case of non-stationary travel rates). The deterministic models are solved using the CPLEX LP solver. After the theoretical part, two different simulations are discussed, the first one is the simulation as discussed in the paper of J.Shu et al. [2], the second one takes into account travel time as extra variable. The simulations are made (and solved) in Matlab. The formula that is used to calculate the gap between the deterministic model and the simulation in percentages is:

$$\frac{\textit{Output of the deterministic model - output of the simulation}}{\textit{Output of the simulation}} \tag{1}$$

2.1 The Deterministic Model

The goals of the models discussed in this section are to analyze and make an approximation of the amount of trips that are supported by a BSS, when the initial endowment of bikes at each station is specified. To do so, I will use the formulations as in J. Shu et al. [2]. An LP formulation for the time-varying model and the steady state model are the two topics from the paper that will be discussed in this section. The goal of the time-varying model is to analyze and make an approximation of the amount of trips that are supported by a BSS, when the initial endowment of bikes at each station is specified. The steady-state model does the same, but then for the case in which the system is already in equilibrium.

Before I move on to the set of equations that form the time-varying model, the notation and the underlying formulas need to be clarified. This will done by beginning with the initialization and step by step adding more variables until we reach the final model.

The set of stations that we work with is notated as φ. As described in the introduction the amount of people who arrive at station i in time period t and want to travel to station j, is Poisson distributed with stationary rate r_{ij}. Because there is only a limited amount of bicycles available, it will happen that not all customers are able to get a bicycle. I used a “first-come, first-served” service basis, i.e. passengers are attended to in the order they arrive. The customers who did not get a bike are the so-called lost customers. The total amount of customers who want a bike and want to travel from station i to station j during time period t is equal to D_{ij}(t), likewise the amount of people that (have plans to) leave from station i in time period t is equal to D_i(t). The assumption is made that all bicycles have reached their destination before the next time period, which means that every trip can be completed within a *single time period.*

The amount of bikes at the beginning of time period t for each station i in the set φ is equal to x_i(t). Therefore, the amount of trips that leave from station i in time period t is limited by either the amount of customers that arrive or the amount of available bicycles and is thus equal to $min(x_i(t), D_i(t))$. The next step is to determine the amount of people that leave from station i and travel to station j in time period t. To do so we introduce p, with $0 < p < 1$, now let D_i(t) be the number of tagged customers, where a customer that arrives at station i is tagged with probability p. η_i(p) is a Bernoulli random variable for passenger i, it is equal to 1 with probability p and equal to 0 with probability 1-p. Let $\{\eta_i(\text{p})\}$ be the sequence of identical and independent Bernoulli variables with mean p. If we add the Bernoulli variables for all passengers that want to leave from station i we obtain the number of tagged passengers:

$$D_i(t)[p] = \sum_{k=1}^{D_i(t)} \eta_k(p) \tag{2}$$

Because D_i(t) is Poisson distributed with rate $\sum_{j\neq i} r_{ij}$, by applying the Poisson thinning lemma we can conclude that D_i(t)[p] is also Poisson distributed, with rate p x $\sum_{j\neq i} r_{ij}$. If we now take $p_{ij}(t) \equiv$

$r_{ij}(t)/\sum_{k:k\neq i} r_{ik}(t)$ then the amount of people who want to travel from station i to station j is Poisson distributed:

$$D_{ij}(t) \sim D_i(t)[p_{ij}(t)] \tag{3}$$

Taking into account the maximum number of available bicycles at station i ($x_i(t)$) we get:

$$min(x_i(t), D_i(t))[p] = \sum_{k=1}^{min(x_i(t), D_i(t))} \eta_k(p) \tag{4}$$

The amount of customers who travel from station i to station j during time period t, is dependent on the order of arriving customers at station i. The customers who come later might not be able to get a bike and travel to their destination. The amount of traveling customers that travel to station j has the following distribution:
$min(D_i(t), x_i(t))[p_{ij}(t)]$
The number of bicycles at station i at the end of time period t is given by the initial amount of bicycles subtracted by the total amount of departures and added by the total amount of arrivals:

$$\begin{align} x_i(t+1) &= x_i(t) - min(D_i(t), x_i(t)) + \sum_{j:j\neq i} (min(D_j(t), x_j(t))[p_{ji}(t)]) \tag{5}\\ &= x_i(t) - \sum_{j:j\neq i} (min(D_i(t), x_i(t))[p_{ij}(t)]) + \sum_{j:j\neq i} (min(D_j(t), x_j(t))[p_{ji}(t)]) \tag{6}\end{align}$$

For every station i, the variable $y_i(t)$ is equal to the expected number of bicycles at station i at the beginning of time period t (E(x_i(t)), and we obtain:

$$\begin{cases} y_{ij}(t) = E(min(D_i(t), x_i(t))[p_{ij}(t)]) \\ y_{ii}(t) = y_i(t) - \sum_{j:j\neq i} y_{ij}(t) \end{cases}$$

By this definition $y_{ij}(t)$ stands for the expected amount of customers traveling from station i to j in time period t and $y_{ii}(t)$ stands for the number of bicycles that is not used and stays at station i during time period t.

Now I will describe some structural properties of $y_{ij}(t)$, which will be used to formulate the LP model.
Lemma 1. $y_{ij}(t)/y_{il}(t) = r_{ij}(t)/r_{il}(t)$
Proof: I have shown before that

$$\begin{align} y_{ij}(t) &= E(min(D_i(t), x_i(t))[p_{ij}(t)]) \tag{7}\\ &= E\Bigg(\sum_{k=1}^{min(D_i(t), x_i(t))} \eta_k(p_{ij}(t))\Bigg) \tag{8}\end{align}$$

In the same way:

$$\begin{align} y_{il}(t) &= E(min(D_i(t), x_i(t))[p_{il}(t)]) \tag{9}\\ &= E\Bigg(\sum_{k=1}^{min(D_i(t), x_i(t))} \eta_k(p_{il}(t))\Bigg) \tag{10}\end{align}$$

Conditional on $min(x_i(t), D_i(t))$, using the law of total expectation (E[E[X| H]]=E[X]) and making use of the fact that:

$E(\eta_k(p_{ij}(t))) = p_{ij}(t), \qquad \text{and} \qquad E(\eta_k(p_{il}(t))) = p_{il}(t)$

We obtain the following equations for $y_{ij}(t)$ and $y_{il}(t)$:

$$y_{ij}(t) = E\Big[E(min(D_i(t), x_i(t))[p_{ij}(t)]) \mid min(D_i(t), x_i(t)\Big] \tag{11}$$

$$= E\Bigg(\sum_{k=1}^{min(D_i(t),x_i(t))} E[\eta_k(p_{ij}(t))]\Bigg) \tag{12}$$

$$= E\Bigg(\sum_{k=1}^{min(D_i(t),x_i(t))} p_{ij}(t)\Bigg) \sim r_{ij}(t) \tag{13}$$

In the same way:

$$y_{il}(t) = E\Big[E(min(D_i(t), x_i(t))[p_{il}(t)]) \mid min(D_i(t), x_i(t)\Big] \tag{14}$$

$$= E\Bigg(\sum_{k=1}^{min(D_i(t),x_i(t))} E[\eta_k(p_{il}(t))]\Bigg) \tag{15}$$

$$= E\Bigg(\sum_{k=1}^{min(D_i(t),x_i(t))} p_{il}(t)\Bigg) \sim r_{il}(t) \tag{16}$$

This gives us:

$$y_{ij}(t)/y_{il}(t) = r_{ij}(t)/r_{il}(t) \tag{17}$$

Lemma 2. $y_{ij}(t) \leq r_{ij}(t)$
Proof: This follows from

$$y_{ij}(t) = E(min(D_i(t), x_i(t))[p_{ij}(t)]) \leq E(D_i(t)[p_{ij}(t)]) \tag{18}$$

Since $D_i(t)[p_{ij}(t)]$ is Poisson distributed with rate r_{ij}(t) it follows that

$$y_{ij}(t) = E(min(D_i(t), x_i(t))[p_{ij}(t)]) \leq r_{ij}(t) \tag{19}$$

Lemma 3. $y_i(t+1) = y_i(t) - \sum_{j:j\neq i} y_{ij}(t) + \sum_{j:j\neq i} y_{ji}(t)$
Proof: Formulas 4 and 5 show that, adding the net flow of bikes into station i during the time period to the amount of bicycles at the start of period t, equals the amount of bikes at the start of the next time period (t+1). The same counts for the expectation of these values. That is, if you add the net flow of bikes into station i in time period to the expected amount of bicycles at the start of period t, it equals the expected amount of bikes at the start of time period t+1.

2.1.1 The Time-Varying Model

Our goal is to analyze and make an approximation of the amount of trips that are supported by a BSS, when the initial endowment of bikes and the process with which customers arrive is given. To do this an LP formulation is introduced described by Formulas (20) to (25). In this model, y_{ij}(t) is the only decision variable, because we are interested in the number of trips between stations. Formulas 21, 23 and 24 are explained by respectively Lemma 3,1 and 2 .The objective function (20) maximizes the expected number of bicycles traveling from station i to station j during time period t (y_{ij}(t)) over all possible stations and time periods, so the total amounts of bikes used over the whole planning period is maximized.

$$Z^* = \max \sum_{t=0}^{N} \sum_{i\in\varphi} \sum_{j:j\neq i} y_{ij}(t) \tag{20}$$

subject to

$$y_i(t+1) = y_i(t) - \sum_{j:j\neq i} y_{ij}(t) + \sum_{j:j\neq i} y_{ji}(t),\ \textit{for all i,t} \tag{21}$$

$$y_i(t) = y_{ii}(t) + \sum_{j:j\neq i} y_{ij}(t),\ \textit{for all i,t} \tag{22}$$

$$\frac{y_{ij}(t)}{y_{il}(t)} = \frac{r_{ij}(t)}{r_{il}(t)},\ \textit{for all i,j,l,t} \tag{23}$$

$$y_i(0) = x_i(0),\ \textit{for all } i \tag{24}$$

$$0 \leq y_{ij}(t) \leq r_{ij}(t),\ \textit{for all } t, i \neq j \tag{25}$$

The second constraint says that, for every station i and every time period t, the total amount of bicycles in the current period (y_i(t)) is equal to the number of bicycles that do not leave the station (y_{ii}(t)) added by the total amount of bicycles that leave the station. The reason for this constraint is to make sure that all the bicycles 'do something', they either stay in the station or leave.

The expectation of the initial allotment of bicycles ($y_i(0)$) is of course equal to the initial allotment of bicycles at station i ($x_i(0)$), because this is a fixed number. This is stated in constraint (24). Given $x_i(0)$, the expected amount of trips that are supported in the bicycle sharing system on each link gives feasible results to the LP.

After solving the LP model, the average level of bicycles moving between stations i and j in time t is given by y_{ij}(t) for every i,j: j$\neq$ i.

Theorem 1. *Z^* is an upper bound to the amount of bicycle trips that are expected in the system when the initial endowment of bicycles to station i is equal to x_i(0).*

This upper bound can be explained by the fact that for the LP formulation the travel rates for all time units are know. So if from the travel rates it appears that a bicycle will get stuck at a station because no people arrive at this station anymore who want to take this bike, then the deterministic model does not serve the customer who wants to take this bike to the station were it gets stuck, and the number of bicycle will be lower than the expectation. But in practice we won't always no this bike will get stuck so we do rent out the bicycle. Therefore the solution of the LP formulation denotes an upper bound to the expected number of bicycles trips in the system.

2.1.2 Model for the equilibrium state (in Time-Invariant System)

The LP described by formulas (27) to (31) is the first step to characterizing the amount of bikes at each station of the bicycle-sharing network, when the system in equilibrium state. When the system is in equilibrium and t$\rightarrow \infty$, y_i(t+1)=y_i. Let

$$y_{ij} = \lim_{t\rightarrow\infty} y_{ij}(t) \tag{26}$$

y^*_{ij} is the optimal solution to the LP described by formulas (27) to (31):

$$Z^*(\infty) = \max \sum_{i,j\in\varphi:j\neq i} y_{ij} \tag{27}$$

subject to

$$\sum_{j:j\neq i} y_{ij} = \sum_{j:j\neq i} y_{ji},\ \textit{for all } i \tag{28}$$

$$\frac{y_{ij}}{y_{il}} = \frac{r_{ij}}{r_{il}},\ \textit{for all i,j,l} \tag{29}$$

$$0 \leq y_{ij} \leq r_{ij},\ \textit{for all i,j}, i \neq j \tag{30}$$

$$\sum_{i} \left(y_{ii} + \sum_{j:j\neq i} y_{ij} \right) = N \tag{31}$$

In this LP, N equals the total number of bikes in the system. After implementing this we obtain y^*_{ij}, the optimal solution of the LP problem. We now take several steps, following the procedure in J.Shu et al. [2]. It can be seen that i* exist such that $y^*_{i^*j}$=r_{i^*j} for all j $\neq$ i. If this was not the case, we could improve the objective value by scaling the solution. Nodes for which $y^*_{i^*j}$=r_{i^*j} are called sink stations. Besides, if an i exist for which $y^*_{ii} < r_{ij}$, then, by shifting y_{ii} to station i, we could modify the solution without affecting the feasibility (and quality) of the solution.

$y_{ii} \leftarrow 0, \qquad\qquad y_{i^*i^*} \leftarrow y_{i^*i^*}$+$y^*_{ii}$

Nodes for which $y*_{ij} < r_{ij}$ tare called transient stations. If a station i is transient, we can assume that y_{ii}=0, without loss of generality. When I obtain results for the steady state model by implementing the set of equations denoted in formulas (27) to (31), I will apply the method described above after.

2.2 Simulation

2.2.1 Simulation

For the reproduction part of this paper I built the same simulation as in the paper of J. Shu et al. [2]. This simulation does not take into account travel time. Instead, it is assumed that all bicycle rides are completed within the time period. For this reason, bikes can only be used once per time period. With this information, the following steps should be taken in order to implement the simulation:

- The input of the simulation model is the initial allotment of bicycles at t=0 and the travel rates which are Poisson distributed.
- In each simulation, the amount of customers that arrives at station i is Poisson distributed. We can calculate the inter-arrival time as $1/r_{ij}$.
- Bicycles are used on a “first-come, first-served” service basis as mentioned before.

To gain more insight in how the simulation is built I made the block-schedule shown in Figure 1. It shortly describes all the steps taken in the simulation. All the results in this paper are the average over 2000 simulations, this means that we have to go through this block-schedule 2000 times, in the last step the overall average is saved.

To calculate the time-average level of bicycles in the simulation I assumed that when a bike leaves a station, it immediately arrives at the station of destination. In this way, the time-average level of bicycles over all stations adds up to ten, just like in the paper of J. Shu et al. [2].

2.2.2 Extended simulation

To make the simulation of J. Shu et al. [2] more realistic, I included travel time. The extended simulation is made such that it can operate with every chosen travel time (including randomly distributed travel times).

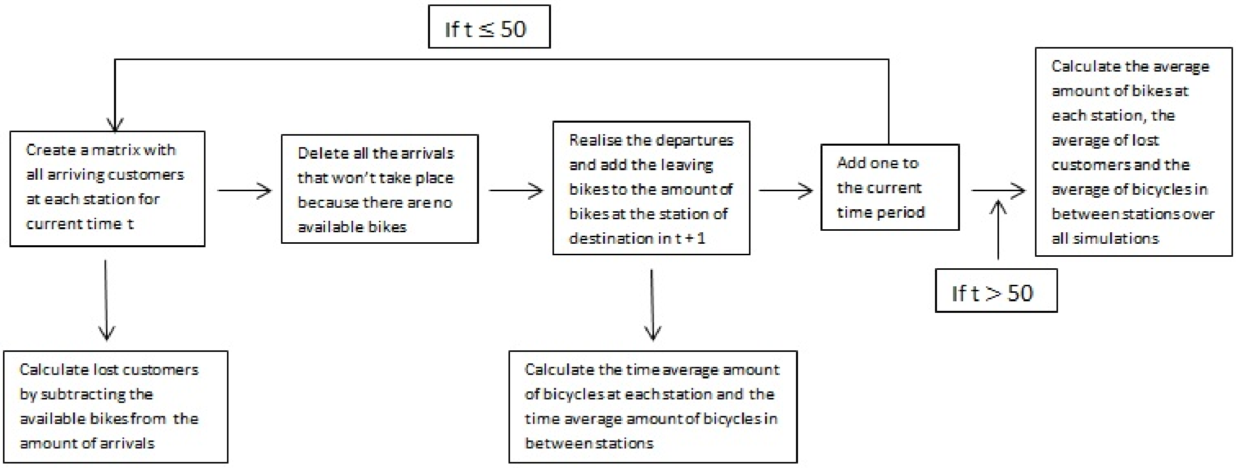

Figure 1: Block schedule of the simulation without travel time

In this report however, I work with stable travel-times. Additionally, the simulation also takes as input the initial allocation of bicycles and the travel rates. This means that it can also handle problems with more stations and/or different travel rates between stations. The simulation is too complex to describe in detail here, but the pseudocode of the extended simulation can be found in the Appendix. A broad overview is shown in the block schedule in Figure 2.

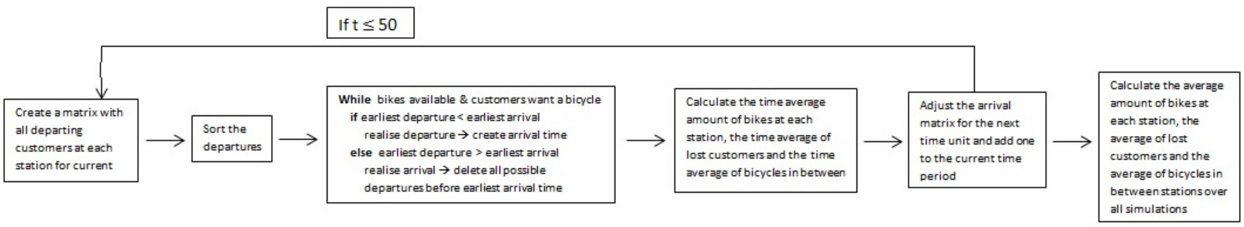

Figure 2: Block schedule of the simulation with travel time

Instead of assuming that bikes can only be used ones per time period, I now make the more realistic assumption that bikes can be used again as soon as they arrive at the station of destination. Again the time-average level of bicycles at each station, the amount of lost customers and the average amount of bicycles moving in between stations, for all stations is calculated and the outcomes are the average over 2000 simulations.

2.2.3 Validation

To check whether the simulation is trustworthy, I looked at two things. First of all, I checked for the simulation with steady demand, the simulation with time-varying travel rates and the simulation with travel time included (tr=0) and steady demand, if the sum of the average number of bicycles at each station adds up to ten. We saw in Section 2.2.1 that this has to be the case in order for the simulation to be correct. For the simulation with travel time the travel time needs to be set to zero in order for the sum of the average to be equal to ten, with any other travel time there are bicycles in between station so the average amount of bicycles at each station does not sum up to ten. To check this property I included a test in my simulation and the result was positive, the sum of the average amount of bicycles at each station indeed added up to ten. This was the case for all the examples (except for the ones with a travel time larger than one) given later, because the total number of bicycles in the system is 10 for all cases.

The second part of the validation was checking the following:

$$Total\ \ customers = lost\ \ customers + served\ \ customers \tag{32}$$

I checked this for all the simulations I runned. The equation has to hold because all customers that arrive either get a bike, in which case they are a served customer, or not, in which case they are a lost customer. Instead of calculating the percentage of lost customers (which is used in Section Results) for this end I calculated the amount of lost customers. Calculating the total customers and served customers was a matter of determining the total number of arrivals from the arrival matrix respectively the total number of departures from the departure matrix. It turned out that the equation above indeed was satisfied. After doing this validation check we are more secure about that the reliability of the results.

3 Results

By applying the methods described in the previous section, I obtained the results described below. For the models described in Section 2, I calculated the time-average level of bicycles and the amount of lost customers (which is the amount of customers that do not get a bike while they do want one) by means of simulation. The results in this section are the results of taking the average over 2000 simulations. I compared the simulation and the deterministic model by looking at the difference between the average amount of bicycles traveling between stations. The average amount of bicycles moving between stations is obtained from implementing and solving the LP formulation for the steady state model using the CPLEX LP solver. The y_{ij} that are obtained are equal to the expected number of customers that travel from station i to station j. Additionally, I am interested in how to redistribute bicycles. Redistribution possibly makes the system more efficient which will lead to a decrease in the percentage of lost customers.

3.1 Time-average number of bicycles

3.1.1 Stationary travel rates

In the first part of the paper of J. Shu et al. [2] a simple numerical example is discussed. This example is shown in figure 3. The numbers besides the arcs stand for the rates r_{ij}. For this example, the time-average level of bicycles at each station and the total number of lost costumers determined by simulation is plotted over 50 time periods. Analyzing results after more than 50 time periods is not necessary, because the results will show that after 50 time periods the steady state is already reached.

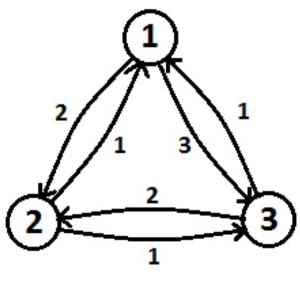

Figure 3: Example with three stations

The simulation they use in the paper has a stationary travel rate and does not take into account travel time, instead they assume that all trips can be completed within the time period. Holding on to these same assumptions, I tried to reproduce Table 2 of J. Shu et al. [2], the results are shown in Table 1. For every number I obtained, I displayed the difference with the paper of J.Shu et al. [2] calculated as:

$$\frac{\textit{Result of J. Shu et al.[2] - obtained result}}{\textit{obtained result}} \tag{33}$$

These differences (in percentages) are displayed in the column next to the obtained results. The last row of Table 1 shows the average difference in the average number of bicycles at each station over all stations for t=10 and t=50.

Table 1: Avg no. of bicycles for the equilibrium model

	t=0	t=10	Diff. for t=10	t=50	Diff. for t=50
Avg no. of bicycles at station 1	5	1,718	-3.6 %	1,484	11.6 %
Avg no. of bicycles at station 2	2	5,096	8.0 %	5,608	-1.8 %
Avg no. of bicycles at station 3	3	3,186	-11.0 %	2,907	-2.4 %
Avg diff. with J.Shu et al. [2]		7.5 %		5.3%	

For t=50, there is on average only a 5.3% difference with the results in the paper of J.Shu et al. [2]. In the paper it is not explained how they calculated the average, but after some experimenting I found out that the difference in results is the least, when it is assumed that when a bike leaves a station, it immediately arrives at the station of destination, even though it can only be used again in the next time period. This is therefore the method I used to obtain the time averages shown in Table 1.

3.1.2 Non-stationary travel rates

The next results are the ones for the model with time-varying demand after a warming-up of 100 time periods, this means that I ran the simulation for 150 time periods, but only started saving the results after 100 time periods. I did this to make sure the system is in steady state when I start measuring results. Furthermore, the same assumptions are made as in the steady state model, but now the travel rates change over time. It is very useful to investigate the results for non-stationary travel rates because the results will be closer to the results of real data (in which travel rates are never stable), also we might get ideas about how to redistribute bicycles in the most profitable way. The travel rates are chosen in such a way that they mimic a real day. In the morning, during rush hour, (time units 1-10) the travel rates are 1.5 times as high as originally in the 3-node example, in the afternoon (time-units 11-20) the travel rates are identical to the 3-node example, and in the evening (time-units 21-30) it is rush hour again, and the travel rates are again 1.5 times as high. During the night (time-units 31-50) it is normally not that busy, so the travel-rates are halved. This means that on average, over the 50 time units, the travel rates are the same as in the 3-node example. The time average is calculated per ten time units, because of the changing travel rates per 10 time units, so every ten time units the time-average is reset to zero. The results of the simulation with non-stationary travel rates and not taking into account travel time, are shown in Figure 4.

In Figure 4 it is very clear that we are dealing with time-varying demand since we see a change in direction of the three graphs every ten time units. Also we can see that the average amount of bicycles at station 2 tends to be higher for low travel rates than for high travel rates, while for station 1 and 3 the average amount of bicycles at the station tends to be lower for low travel rates. This can be explained by the fact that the rate of people who want to travel to station 2 is higher than the rate of people who want to leave from station 2. When the travel rates are low, it takes longer for a bicycle to leave station 2 again, on average only one bike per time unit is leaving from station two. While when a bike arrives at either station 1 or 3, it leaves relatively quicker. This is why the average amount of bicycles at station 2 is higher when travel rates are low. The periods with low travel rates are a good opportunity to redistribute the bicycles, because a big part of the bicycles are at station 2, where they are not needed (not many customers arrive because of the low travel rates), so they can easily be redistributed, without loss of customers.

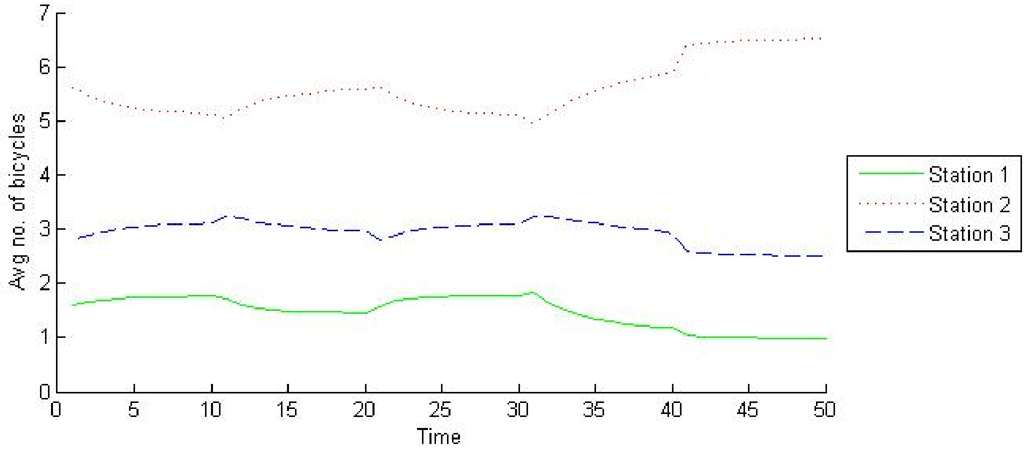

Figure 4: Time-average amount of bicycles for time-varying demand

If we would calculate the time-average amount of bicycles per station over the full time period instead of per 10 time units, the results are somewhat different. The difference between the time-average over 50 time periods after a warming-up of 100 time periods, for the simulation with stationary travel rates and the one with non-stationary travel rates (both not taking into account travel time) are calculated as

$$\frac{\textit{Result for stationary travel rates - result for non-stationary travel rates}}{\textit{Result for non-stationary travel rates}} \tag{34}$$

The results are shown in Table 2.

Table 2: Time average no. of bicycles for stationary and non-stationary travel rates at t=50

	Stationary travel rates	Non-stationary travel rates	Difference
Avg no. of bicycles at station 1	1.5330	1.4475	5.91 %
Avg no. of bicycles at station 2	5.6204	5.7342	-1.98 %
Avg no. of bicycles at station 2	2.8466	2.8183	1.00 %

We can see in Table 2 that after 50 time periods there is on average only a 2.96 % difference between the results of the steady state model and the results of the model with time-varying demand. So in this case the time-average amount of bicycles over all time units is the same for both models, as well as the average of travel rates (for time-varying travel rates) over all time units is the same as the fixed travel rate. If we want to find out if it is in general the case that when the average of travel rates (for time-varying travel rates) over all time units is the same as the fixed travel rate, the time-average amount of bicycles over all time units will be the same for both models (so in this case we would not have to take into time-varying demand to calculate the time-average) we would have to do more research. To do this research we would have to let the initial travel rates vary according to a uniform distribution (for example with rates U $\sim$ (initial rate -10 % rate, initial rate +10 % rate). The ratios for the different times of the day have to stay the same, because we want the average of travel rates (for time-varying travel rates) over all time units to be the same as the fixed travel rate. If we would calculate the time-average amount of bicycles over all time units for both models for a large number of cases with uniformly distributed travel rates, we could check if it is indeed the case that they are (approximately) equal to each other, which would mean that the statement holds.

3.1.3 Stationary travel rates, travel time included

Finally, in Table 3 the average number of bicycles at each station is displayed when travel time is included in the simulation and the travel rates are stable. We can see that there is no warming-up, because of the wish to compare the results with the results in the the paper of J.Shu et al. [2]. Now the assumption is made that when a bicycle arrives at the station of destination, it can immediately be rented out again. It can be seen in Table 3 that with a travel time equal to 0, the average amount of bicycles at station 2 for the model with stationary travel rates that takes into account travel time is higher than for the model with stationary rates but not taking into account travel time (these results can be found in Table 1). This difference is caused by the fact that bicycles can leave again straight away (and station 2 has the highest arrival rate) in the model that includes travel time. The simulation for which this is the case gives more reliable results for a real life situation.

Table 3: Avg no. of bicycles for the model with travel time (tr=0) included (steady demands)

	t=0	t=10	t=50
Avg no. of bicycles at station 1	5	1.002	0.596
Avg no. of bicycles at station 2	2	5.998	7.292
Avg no. of bicycles at station 3	3	3.000	2.113

In Table 4 the average amount of bicycles that are on the way is shown when applying different travel times. It can be seen that when the travel time is twice as high, there also almost twice as many bicycles on the way on average. This is not what I expected, because the bicycles can be used again as soon as they arrive. So for example if the travel time is 0.25 the bike can be used four times in one time period, while for a travel time equal to one it can only be used once. For this reason I expected that the numbers would be rather close together. Why this is not the case can be explained by the fact that the travel rates for the 3-node example are low. The numbers are not closer because there are just not enough customers that want to take a bicycle (so if the travel time is 0.25, there are not four people in one time period who want to use the bike). When we multiply the travel rates by three, the average amount of bicycles on the way for tr=0.5 becomes 5.4593, and for tr=2 it becomes 8.1521. So if the renter would pay per time unit, the travel time is a particularly important factor in determining the total revenues if the travel rates are low.

Table 4: Average amount of bicycles on the way for different travel times and computation times

	t=10	t=50	Computation Time
Travel time equal to 0.25	1,5799	1,4325	302 seconds
Travel time equal to 0.5	2,9927	2,7913	311 seconds
Travel time equal to 1	5,0744	4,9809	297 seconds
Travel time equal to 2	7,0105	7.1420	251 seconds

3.2 Percentage of Lost Customers

In Figure 5 the percentage of people who arrived at a station but were not able to get a bike is shown for each station and every time unit until t=50 for the 3-node example as displayed in Figure 3. This figure is interesting, because it gives us more information about the efficiency of the system (if there are enough bicycles in the system for example) and the opportunities for redistribution. Again a warming-up of 100 time periods is used and for the simulation with travel time, the travel time is taken equal to 1. For this example it can be seen that for every station the number of lost customers for the steady state model is approximately equal to that of the model where travel time is included. The expectation was that the amount of lost customers for the model with travel time would be higher, because every hired bicycle is gone for a full time period, so there are less bicycles available at the stations. The outcome can be explained by the assumption made in the steady state model that bicycles can only leave in the next time unit again, because of which a lot of customers don't get a bike while there actually may be one available.

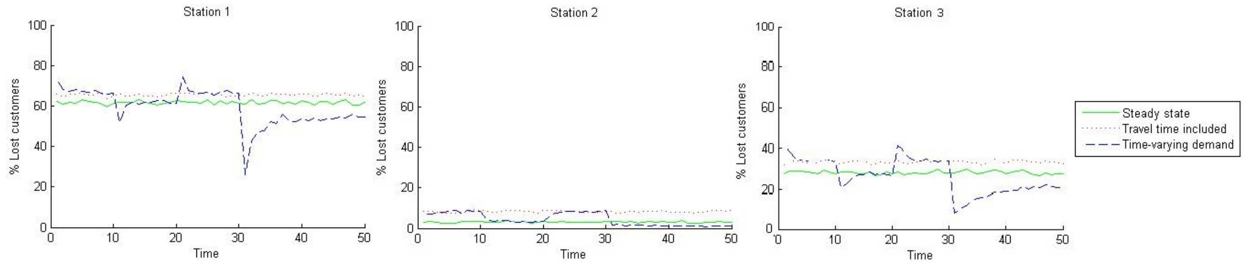

Figure 5: Percentage of lost customers per time unit

Figure 5 tells us a lot about possibilities to redistribute bicycles. At station 1 the percentage of people who cannot get a bicycle, while they do want one, is very high compared to the other stations. This means that it could be profitable to redistribute the bicycles from station 2 to station 1 after every x time units. Hereby, x would have to be determined by including the redistribution in the simulation and using a trial-and-error method. To do this, the redistribution costs and the bicycle hiring costs would have to be known.

3.3 Average amount of bicycles moving between stations

3.3.1 Stationary travel rates

In Tables 5 and 7 the average amount of bicycles that are moving between stations is displayed for all combinations of stations. This is done for the simulation and the deterministic model with stationary travel rates and which do not take into account travel time. A warming-up of 100 time periods is used. The reason why we look at these tables is because the results of the simulation and deterministic model cannot be compared by looking at time-average number of bicycles at each station, due to the fact that there are many ways in which the solution can be scaled. Table 5 shows the results of the simulation after 50 time units after the warming-up.

	Station 1	Station 2	t=50
Station 1	-	0.6604	0.9920
Station 2	0.9400	-	0.9374
Station 3	0.6469	1.2861	-

Table 5: Steady state: simulation

	Station 1	Station 2	Station 3
Station 1	0	0.94%	0.80%
Station 2	6.00%	0	6.26%
Station 3	2.97%	3.54%	0

Table 6: Steady state: diff. between deterministic model and simulation

	Station 1	Station 2	Station 3
Station 1	4.3333	0.6667	1
Station 2	1	0	1
Station 3	0.6667	1.3333	0

Table 7: Steady state: deterministic model

	Station 1	Station 2	Station 3
Station 1	0	0.6667	1
Station 2	1	4.3333	1
Station 3	0.6667	1.3333	0

Table 8: Steady state: deterministic model after improvement

The difference between the average amount of bicycles in between stations for the simulation and the deterministic model is calculated as:

$$\frac{\textit{Results of the deterministic model - results of the simulation}}{\textit{Results of the simulation}} \tag{35}$$

The average difference between the two models is equal to 3.6 %. Because the results of the simulation and the deterministic model are very alike, we can assume that the deterministic produces trustworthy results for the 3-node example. With this information, we can start making extensions for the 3-node example. Table 8 shows the results of executing the steps described in 2.1.2 after obtaining the solution for the LP model. Station 2 is the sink station, the outcomes of the LP are 1 for both station 1 and 3, which are equal to the travel rates. Station 3 is a transient station, both y_{31} and y_{32} are greater than zero but smaller than their travel rate. Therefore, we can modify the solution by shifting y_{11}^* to station 2 without affecting the feasibility and quantity of the solution.

3.3.2 Non-stationary travel rates

Comparing the results of the simulation and deterministic model for time-varying demand is more complicated. Again a warming-up of 100 periods is included and the simulation and deterministic model do not take into account travel time. Because the demand changes every ten time units, I calculated the time average number of bicycles that are traveling in between stations per ten time units. The results for each pair of stations is shown in Figure 6.

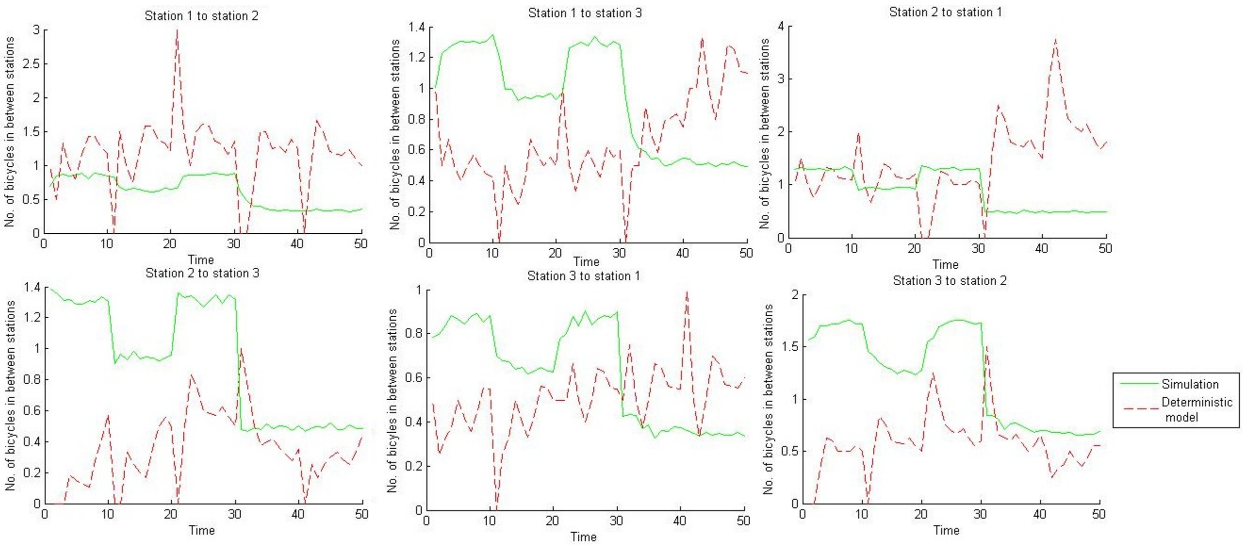

Figure 6: Time-average amount of passengers traveling between stations per time unit

It can be seen in the figure that the time-average amount of bicycles in between stations does not stabilize for the deterministic model. That is, it does not reach a steady state. Obviously, for a model with time-varying demand, the simulation is a more accurate way to estimate the amount of bicycles moving between stations. But still it could be preferable to use a deterministic model, because the simulation takes a long time to execute, we will see this also in Section 3.4. Again, it is easy to see the time-varying travel rates. What is remarkable is that the change in the amount of bicycles moving between station i and j is not proportional to the changing travel rates r_{ij}(t), this can be clarified by the fact that there are not enough bicycles available. By taking $min(x_i(t), D_i(t))$ it will often be $x_i(t)$ that outlines the amount of bikes that leaves from station i.

3.4 Larger networks

In Figures 7 and 8 two larger networks are shown. The travel rates can be obtained from the figures, the initial allocation of bicycles stays the same as in the 3-node example, 5 bicycles at station 1, 2 bicycles at station 2 and 3 bicycles at station 3, the added stations begin with 0 bicycles. There are two reasons why I have a look at the two larger networks shown in Figures 7 and 8. The first reason is that by obtaining the average amount of bicycles in between stations for the simulation and the deterministic model we can check for correctness of the deterministic model. These examples are more complex, and the goal is to find out if the deterministic model still gives an accurate outcome. To do this we calculate the difference between the outcomes in the same way as in Formula 35. Hereby the deterministic is the set of equations for the steady state, and the simulation is the simulation without travel time with steady demands and with a warming-up of 100 time periods. The second reason is to find out if adding an extra station which functions as a redistribution station is lucrative, hereby we only look at the example with four stations, whereby the fourth station is added with the purpose of redistribution. But first we will compare the running times of the different examples for the simulation and the deterministic model.

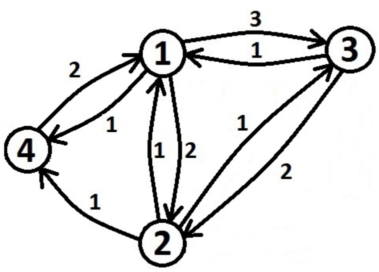

Figure 7: Four station example

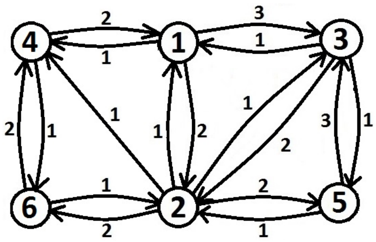

Figure 8: Six station example

The running times in seconds for the simulation and the deterministic model are displayed in Table 9. The running times for the simulation are including the time needed for the warming-up. We can see that the running times for the simulation are a lot larger that those for the steady-state model. While for the steady-state model the running times are only a bit larger than one second and almost don't increase if the amount of stations in the example increases, contrarily the running times for the simulation are all larger than 5 minutes and increase heavily when the amount of stations increases. Therefore, it would be convenient if the deterministic model gives reliable results and a lot of time could be saved.

Table 9: Running Times (in seconds)

	Simulation	Deterministic model	Accuracy
3-node example	357.67	1.24	3.6%
4-node example	478.18	1.26	33.8%
6-node example	874.30	1.29	10.3%

The time-average amounts of bicycles in between stations for the stationary simulation without travel time and the steady-state deterministic model are shown in table 10 and 11. The difference between the outcomes of the simulation and the deterministic model is much greater for the 4-node example than for the 3-node example. The average difference between the outcomes 33.8%. This is calculated as taking the average over the differences between each pair of stations. The conclusion could be made that the results are not reliable, but on the other hand we can see that the relations between the numbers show great similarities. Namely if we compare the results of $a_{ij}/\sum_j a_{ij}$ for all i and j, whereby a_{ij} is the average number of bicycles between station i and j there is only a 0.03% difference. We can conclude that when we look at the distribution of bicycles between stations the simulation is similar to the deterministic model, and thus gives realistic results for this 4-node example.

To check if the increasing difference had to do with the increasing amount of stations I also compared the outcomes of the simulation and the deterministic model for the 6-node example shown in figure 8. The difference, calculated as taking the average over the differences between each pair of stations, is now only 10.3%. If we compare the results of $a_{ij}/\sum_j a_{ij}$ for all i and j again, we obtain a 5.0% difference. We know now that the accuracy does not necessarily decreases with the increase of the amount of stations. However,

Table 10: Avg no. of bicycles in between stations, determined by simulation

	Station 1	Station 2	Station 3	Station 4
Station 1	-	0.8070	1.2085	0.4015
Station 2	0.6880	-	0.6880	0.6915
Station 3	0.6348	1.2613	-	0
Station 4	1.0934	0	0	-

Table 11: Avg no. of bicycles in between stations, determined by a deterministic model

	Station 1	Station 2	Station 3	Station 4
Station 1	-	1.0770	1.6154	0.5385
Station 2	0.9231	-	0.9231	0.9231
Station 3	0.846	1.6923	-	0
Station 4	1.0646	0	0	-

the large difference between the results for the simulation and the deterministic model for the 4-node example are hard to explain. One possible explanation could be the fact that bicycles travel from station 2 to station 4, but do no travel the other way around, maybe this is handled differently by the simulation than by the deterministic model.

To look at the difference between the percentage of lost customers for the simulation with stationary travel rates (without travel time) and the steady-state model for the 4-node example, I plotted the sum of lost customers over station 1, 2 and 3 for every time unit. The comparison is useful because it gives us information about if the fourth station could be used as a redistribution station. The results that I obtained after a warming-up of 100 time periods are shown in figure 9.

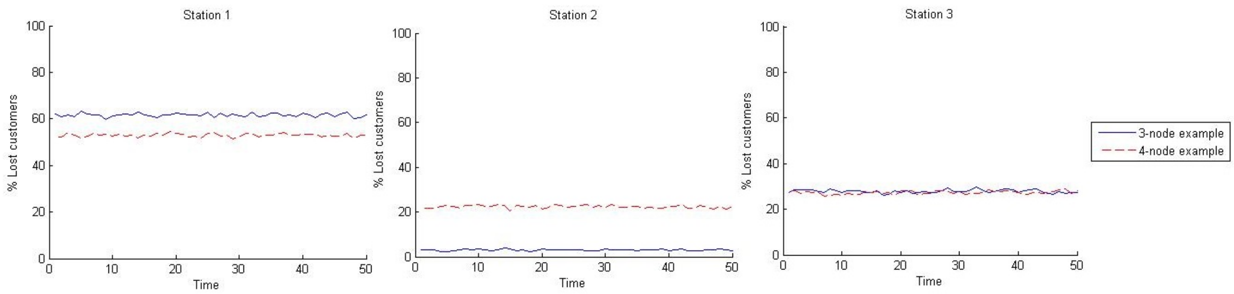

Figure 9: Percentage of total customers that is lost per time unit

If we have a look at Figure 9 we can see that the percentage of lost customers per time unit stays approximately the same for station 3, the average percentage of lost customers decreases from 61.67% (avg. of 3.05 lost customers per time unit) to 52.99%(avg. of 3.17 lost customers per time unit), but increases from 3.05%(avg. of 0.06 lost customers per time unit) to 22.36%(avg. of 0.67 lost customers per time unit). So the total amount of lost customers per time unit increases, but not proportionally to the two extra customers that want to leave from station 1 and 2. However, additionally on average 32.0% of the customers who arrive at station 4 and want to travel to station 1 also can't get a bike. It can be concluded that if the goal of the manager is to minimize the percentage of lost customers adding a redistribution station is not profitable, but if the manager wants to maximize the bicycle utilization rate than it is profitable to add a fourth station in this example. .

Table 12: Avg no. of bicycles in between stations, determined by simulation

	Station 1	Station 2	Station 3	Station 4	Station 5	Station 6
Station 1	-	0.5443	0.8165	0.2702	0	0
Station 2	0.3066	-	0.3028	0.3056	0.6081	0.6027
Station 3	0.4852	0.9700	-	0	0.4856	0
Station 4	0.8394	0	0	-	0	0.4196
Station 5	0	0.2712	0.8207	0	-	0
Station 6	0	0.3397	0	0.6829		-

Table 13: Avg no. of bicycles in between stations, determined by a deterministic model

	Station 1	Station 2	Station 3	Station 4	Station 5	Station 6
Station 1	-	0.5983	0.8974	0.2991	0	0
Station 2	0.3349	-	0.3349	0.3349	0.6698	0.6698
Station 3	0.5338	1.0675	-	0	0.5338	0
Station 4	0.9262	0	0	-	0	0.4196
Station 5	0	0.3009	0.9027	0	-	0
Station 6	0	0.3776	0	0.7553	0	-

4 Conclusion

After describing the methods that were used in the paper of J.Shu et al. [2] and applying them, I obtained a number of interesting results.

First of all, reproducing the time-average number of bicycles for the 3-node example shown in figure 3 turned out not to be entirely possible for the deterministic model. This was due to the fact that the excessive amount of bicycles (the total amount of bicycles that don't depart from a station in a time period) can be distributed to any of the y_{ii}, the way in which this is done does not affect the optimality of the solution. This means that it could be done as in the example of the paper, but it could as well be done in any other way. For the simulation however, we were able to reproduce the results. The difference between these results and the results of the paper of J.Shu et al. [2] turned out to be only 2.96 %.

Next we looked at the time-average amount of bicycles at each station for time-varying demand by using simulation. I saw that the periods with low travel rates are a good opportunity for redistribution. Not many customers arrive at station 2 because of the low travel rates, but there are a lot of bicycles present, which we can redistribute to station 1 and/or station 3. Another part of the results also taught us about possibilities of redistribution, namely the results for the percentage of lost customers. With the graphs of the percentage of lost customers we can easily see at which stations there is a shortage of bicycles, and at which stations there is a surplus of bicycles. This gives us information about how to redistribute.

When I compared the average amount of bicycles moving in between stations, I found that there was on average over all pairs of stations only a 3.6% difference between the simulation and the deterministic model with stationary travel rates (and without travel time). Because of this we can assume that the theoretical model produces trustworthy results for the 3-node example. However, when I compared the results of the simulation and the deterministic model for the 4-node example I found a 33.8% difference. We could jump to the conclusion that for this case the results of the simulation are not reliable, but what we could see on the other hand, was that the proportion of bicycles that travel from station i to station j divided by the total amount of bicycles leaving from station i only showed a 0.03% difference on average. For the 6-node the average difference for these proportions was 5.0% and the difference between the initial outcomes of the simulation and the deterministic model(as described before) was 10.3% on average for all pairs of stations. Therefore, we could conclude that the accuracy does not necessarily decrease with the increase of the number of stations.

At last, I had a look at the percentage of lost customer for the 4-node example. The fourth station was placed such that it could possibly function as a redistribution station. We saw that the percentage of lost customers over the first three stations increases when the fourth station is added, but that it depends on the wants of a manager if adding the fourth station is profitable.

5 Future Research

This paper adds some features to the simulation for the 3-node example as described of the paper of J.Shu et al. [2]. It also describes and shows results for the first two theoretical models that are introduced in the paper of J.Shu et al. [2], but still there are a lot of opportunities for future research, some of them will be discussed below.

First of all, it is assumed that travel time is fixed over time. In reality, people bike at different speeds, so the travel time might have a random distribution. It can as well be that people bike faster in the morning (because they are in a rush to get to their job) than in the evening, this would also have to be taken into account. The simulation is already made in such a way that it can take into account unstable travel times, but the deterministic model is not yet, hence it could be extended.

Secondly, the initial allotment of bicycles can be improved. I now took the initial allotment of bikes as given, but in reality we would have to determine the optimal initial allotment. This could be done by for example minimizing the total amount of lost customers.

Perhaps the most interesting extension that could be made is including the possibility of redistribution. One could make an algorithm do determine the most profitable redistributions. It needs to be decided how frequently to do it, how many bikes to transfer and which stations to transfer to. To do this, information about the redistribution costs, fixed starting costs for the passengers and the kilometer/minute tariff would be needed. In combination with applying the optimal initial allotment, this could significantly improve the efficiency of the system.

After accomplishing all the extensions above, it would be interesting to look at a real life situation. By applying either the simulation or the deterministic approach one could determine the optimal initial allotment and redistribution of bicycles. By doing so, public bicycle-sharing can be made more beneficial.

References

[1] OBIS project (2011) optimising bike sharing in european cities-a handbook. `http://www.eltis.org/docs/tools/Obis_Handbook.pdf`.

[2] J. Shu, M. Chou, Q. Liu, C.-P. Teo, and I.-L. Wang. Models for effective deployment and redistribution of bicycles within public bicycle-sharing systems. *Operations Research*, 61(6):1346–1359, 2013.

Appendix

Pseudocode Simulation

Main function

Input variables:

- Allocation: the initial allotment of bikes over the station
- Travel rates: the mean r_{ij} of the Poisson distribution between station i and j, for every i and j
- Time: the amount of time periods for which the simulation has to run
- Simulations: the total number of simulations over which the average should be taken

Output arguments:

- Time average level: the time average amount of bicycles at each station
- Average on way: the average amount of bicycles in between stations

1. Determine the amount of stations from the size of 'allocation'
2. Initialize the time average level as a vector of (t-1) by 1
3. For every simulation
   (a) initialize
      - the matrix with the amount of bikes (equal to allocation for t=0, and zero otherwise)
      - the arrival matrix as infinity
      - the average amount of bicycles for each station per time unit as zero
      - the spot in the arrival matrix to enter new arrivals as ones for each station

   (b) For every time unit
      i. Initialize:
         - the departure matrix as infinity
         - the spot in the departure matrix to enter new departures as ones for each station

      ii. For each station i
         A. Create a vector with travel rates
         B. Create a vector with the stations that is travelled to from station i
         C. Create a matrix with all the departures, by running the departuresStation function, and adding the departures for station i to the matrix

      iii. Create a matrix with the realised the departures with the Realise_departures function
      iv. Calculate the time average for this time unit with the Calc_Time_average function
      v. Calculate the time average for all time units till so far
      vi. Adjust the arrival matrix for the next time unit by running the Make_arrival_matrix function

   (c) Calculate the time average level over all simulations till so far
4. Calculate the average amount of bicycles on the way

departuresStation function

Input variables:

- ExpMean: a vector with the Poisson rates for each station that is travelled to
- Station: a vector with the number of each station that is travelled to

Output variable:

- Departures: a matrix with all the departures from the station, and the number of the station to which is travelled

1. Initialize a temporarily departures matrix as an empty matrix
2. For every station that is travelled to
   (a) Initialize
      - The current time equal to zero
      - An exponentially distributed random variable with the mean of the current station

(b) **while** time is smaller than one
Set the current time equal to the current time plus the random variable
Add the current time to the arrival vector
Create a new exponentially distributed random variable with the mean of the current station
end
(c) Add a row next to the arrivals with the number of the current station
(d) Add the arrival matrix for the current station to the total departure matrix
3. Sort the departure matrix chronologically

Realise_departures function

Input variables:

- Total_departures: a matrix with all the departures from each station, and the number of the station to which is travelled
- Total_arrivals: a matrix with all the departures till so far (new arrivals will be added later)
- AmountBikes: the matrix with the amount of bikes at each station for each time unit
- Spot: a vector with for each station the spot in the matrix where a new arrival can be entered
- Spot_dep: a vector with for each station the spot in the matrix where a new departure can be entered
- Amount_stations: a matrix with the amount of bikes at each station for the beginning of each time unit
- Travel_time: the travel time between the stations (equal for every station)
- T: the current time period

Output arguments:

- Realised_departures: a matrix with the realised departures for each station, and to which station was traveled
- Total_arrivals: a matrix with all the departures till so far

1. Initalize:
   - Realised_departures as a matrix with zeros
   - Delete_arrivals as a temporary matrix equal to Total_arrivals
   - Original_departures as a temporary matrix equal to Total_departures
   - Original_amount as a vector with the amount of bikes at each station at the beginning of the time period
   - counter (counts the amount of departures that take place) as a vector with ones

2. **if** all bicycles are on the way in the beginning of the time period
   Determine the arrival with the minimum arrival time
   Delete the arrival in the delete_arrival matrix
   Add one to the original amount of bikes for the specific station
   Delete all the departures that take place earlier than the minimum arrival time (these departures won't take place)
   end

3. **while** there are bicycles available

   (a) check at which station(s) there are bikes available
   (b) Make a matrix Delete_departures with only the departures for the stations with available bikes
   (c) **if** there are still departures that could take place
      i. Determine the departure with the minimum arrival time
      ii. Determine the arrival with the minimum arrival time
      iii. **if** the departure is earlier than the arrival
         - Add the departure in the Realised_departure matrix

- Add the arrival time (=departure time + travel time) to the total_arrival matrix
- Add the arrival time to the delete_arrival matrix
- delete the departure from the Originial_departure matrix
- Update the counter the counter, spot and spot_dep vectors

iv. **else**
- Delete the arrival in the delete_arrival matrix
- Delete all the departures that take place earlier than the minimum arrival time (these departures won't take place)
- Add one to the original amount of bikes for the specific station

v. **end**

vi. **if** all bicycles are on the way after a departure took place Determine the arrival with the minimum arrival time
Delete the arrival in the delete_arrival matrix
Add one to the original amount of bikes for the specific station
Delete all the departures that take place earlier than the minimum arrival time (these departures won't take place)
if there are no more arrivals that have to take place
Set the original amount equal to zeros **end**

(d) **elseif** there are no more departures left, but there are still arrivals

i. Determine the arrival with the minimum arrival time
ii. Delete the arrival in the delete_arrival matrix
iii. Delete all the departures that take place earlier than the minimum arrival time (these departures won't take place)
iv. Add one to the original amount of bikes for the specific station

(e) **elseif** There are no more departures nor arrivals that have to take place
- set the original amount equal to zeros

Calc_Time_Average function

Input variables:
- Total_departures: a matrix with all the departures from the station, and the number of the station to which is travelled
- Total_arrivals: a matrix with all the departures till so far (new arrivals will be added later)
- AmountBikes: the matrix with the amount of bikes at each station for eacht time unit
- T: the current time period

Output arguments:
- Time_average: a matrix with the time average of bycicles (till so far) for each station for all time units
- AmountBikes: the matrix with the amount of bikes at each station for eacht time unit, updated for time t+1

Initialize:
- the amount of stations
- the time average matrix as zeros

for each station

1. Add the arrivals and departures of the specific station together in a matrix, with on the second row, a -1 in case of a departure, and a +1 in case of an arrival
2. Sort the matrix chronologically

3. **if** there are departures and arrivals for this station
   calculate the time average for the current time unit
   else
   set the time average equal to the amount of bikes at the beginning of the time period
   end

4. Calculate the amount of bikes for t+1 by taking the amount of bikes for t, and adding all the arrived bikes and subtracting all the departed bikes

end

Make_arrival_matrix function

Input variables:

- Total_arrivals: a matrix with all the departures till so far

Output arguments:

- Total_arrivals: the updated matrix with all the departures till so far

1. initialize the total amount of stations
2. Subtract one from the total arrival matrix (because it needs to be adapted to the next time period)
3. **for** each station
   Take the vector with arrivals for that specific station
   Delete the arrivals with an arrival time below one
   Fill the remaining space with a vector with infinity
   Replace the original arrival vector in the matrix with the new vector
   end