Erasmus University of Rotterdam

Master Thesis

Simplicity vs. Complexity: Jump Diffusions in Affine Term Structure Models

Author:
Akshay Ramkisoensing

Supervisors:
Michel van der Wel
Marcin Jaskowski
Second Reader:
Rutger-Jan Lange

A thesis submitted in fulfilment of the requirements for the Master degree of Econometrics and Management Science in Quantitative Finance

at the

Department of Econometrics

October 2016

Declaration of Authorship

I, Akshay Ramkisoensing, declare that this thesis titled, 'Simplicity vs. Complexity: Jump Diffusions in Affine Term Structure Models' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a master degree at this University.
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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: Akshay Ramkisoensing

Date: 18-10-2016

“Education is key.”

- W.K. Bahadoer (Grandfather)

Abstract

Simplicity vs. Complexity: Jump Diffusions in Affine Term Structure Models

by Akshay Ramkisoensing

Affine Jump Term Structure Models (AJTSMs) add a jump diffusion component to Affine Term Structure Models (ATSMs) to model the term structure of interest rates. I investigate whether there is a significant difference in the in- and out-of-sample performance of ATSMs and AJTSMs for riskless interest rates in pre-, mid- and post-crisis periods. I consider the one-, two- and three-factor Vasicek model within the ATSM- and AJTSM-framework and use Quasi-Maximum Likelihood Estimation (QMLE) to estimate the parameters. Firstly, I find that the three-factor AJTSM is unidentified and that imposed restrictions result in an unrealistic economic model. Secondly, the results show that jump diffusion components are empirically justified in the complete sample and pre- and mid-crisis samples. Thirdly, goodness-of-fit measures show that the in-sample fit of one- and two-factor ATSMs and AJTSMs is poor. The three-factor ATSM is superior in fitting the yield curve of the riskless interest rates. Lastly, I establish that ATSMs and AJTSMs perform poorly in out-of-sample VaR and ES estimation for Risk Management purposes.

Keywords: Affine Term Structure Model · Affine Jump Term Structure Model · Jump diffusion · Fisher Information matrix · Quasi-Maximum Likelihood Estimation · Vasicek model · Value-at-Risk · Expected Shortfall

J.E.L. Subject Classifications: C.32 · E.43 · G.12

Acknowledgements

After four months of academic research, I have, finally, come to put the last touch on this thesis. Although personal and intellectual development of a person is never completed, this paper does mark the end of my academic career. It has been a life-changing four-year period with moments of unfamiliarity, development, excellence and, at last, reflection. I realize, upon reflecting on the last four years, that this has not solely been the result of my knowledge and actions. And, therefore, I would like to take a moment to express my gratitude to those who have supported me throughout this period.

First of all, I would like to thank my supervisors, Michel VAN DER WEL and Marcin JASKOWSKI, for their help and comments. Their knowledge has been useful in multiple ways, oftentimes resulting in intellectual discussions on the thesis and related topics. I would like to thank Rutger-Jan LANGE, as well, for his critical comments and discussion as the second reader of my thesis. Moreover, I would like to thank all (assistant-) professors for their challenging lectures and assignments that have developed my analytical skill set and encouraged me to think critically.

Second of all, I would like to thank my supervisors, Dominika BUDNY and Jaroslav KRYSTUL, at APG for providing me their expertise in the field of Risk Management. I am grateful for their support to write a second thesis, in addition to the project at APG, and their insights in the applicability of Econometrics and Stochastic Calculus in practice.

Last, and most important, of all, I would like to express my sincere gratitude to my family and, especially, to my parents, RADJINDER and INGRID RAMKISOENSING. You have raised me and provided me with the necessary tools and opportunities to become the person I am today. And for that I am deeply grateful. This unlimited support has always been present in my family as well. Although they go unnamed in this paper, each and everyone of them knows that they play an important part in my life.

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Abbreviations

AIC	**A**kaike **I**nformation **C**riterion
AJTSM	**A**ffine **J**ump **T**erm **S**tructure **M**odel
ATSM	**A**ffine **T**erm **S**tructure **M**odel
BIC	**B**ayesian **I**nformation **C**riterion
BP	**B**ox-**P**ierce
BM	**B**rownian **M**otion
DGP	**D**ata **G**enerating **P**rocess
ES	**E**xpected **S**hortfall
LIBOR	**L**ondon **I**nter**B**ank **O**ffered **R**ate
LR	**L**ikelihood **R**atio
MAE	**M**ean **A**bsolute **E**rror
MSPE	**M**ean **S**quared **P**rediction **E**rror
ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
QMLE	**Q**uasi-**M**aximum **L**ikelihood **E**stimation
RMSE	**R**oot **M**ean **S**quared **E**rror
SDE	**S**tochastic **D**ifferential **E**quation
SPA	**S**addle**P**oint **A**pproximation
VAR	**V**ector **A**uto-**R**egressive
VaR	**V**alue-**a**t-**R**isk
mds	**m**artingale **d**ifference **s**equence
pdf	**p**robability **d**ensity **f**unction

1. *Introduction*

Affine Term Structure Models (ATSMs) define zero-coupon bond yields as linear functions of state variables, or factors, and are prominently used in both the finance industry and in academics. Research in the field of term structure models for interest rates shows that these ATSMs have desirable properties, which justify the popularity of these models. The key properties, in this regard, are its analytical tractability and empirical flexibility. Despite its empirical flexibility, Lin and Yeh (1999) show that ATSMs are not able to capture perceptible jumps in the time ($\mathbb{P}$) dynamics[1] of the term structure of the interest rates. To implement this discontinuity in the interest rates, Duffie et al. (2000) provide a framework by deriving analytical tractable ATSMs with Poisson distributed jump times. These ATSMs with jump diffusion components are known as Affine Jump Term Structure Models (AJTSMs).

The contribution of this paper is to answer the research question whether there is a significant difference in modeling the term structure of the riskless interest rates with ATSMs and AJTSMs in pre-, mid- and post-crisis periods. I investigate one-, two- and three-factor models for both the ATSMs and AJTSMs. This research focuses on the economic and practical aspect of the main research question. From an economic standpoint, this paper investigates whether the economic justification of an additional jump diffusion component is empirically justified. This, additionally, extends to investigating what the optimal model is, with regard to ATSMs and AJTSMs, to model and fit the term structure of the riskless interest rates in a Vasicek (1977) framework. From a practical standpoint, this paper investigates whether there is a significant difference in the performance of ATSMs and AJTSMs in Risk Management. The latter research question focuses on Value-at-Risk (VaR) and Expected Shortfall (ES).

This paper builds upon the pioneering work of Vasicek (1977), Duffie and Kan (1996),

[1]Henceforth, I will use dynamics and $\mathbb{P}$-dynamics interchangeably. Whenever risk-neutral $\mathbb{Q}$-dynamics are implied, this will be explicitly indicated.

Baz and Das (1996) and Lin and Yeh (2001). Vasicek (1977) proposes to model the term structure of interest rates using an Ornstein-Uhlenbeck process and, subsequently, derives analytical expressions for zero-coupon bond yields[2]. The Vasicek model is classified as an ATSM in Duffie and Kan (1996), who highlight the desirable properties of the ATSMs. Empirical research into the applicability of these models, such as Chan et al. (1992), Mc Manus et al. (1999) and Wu et al. (2011), suggest that a multi-factor model is necessary to capture the complete dynamics of the term structure of the interest rates.

In addition to a multi-factor model, Lin and Yeh (1999) emphasize the essence of jump diffusion components to model the term structure of interest rates. Lin and Yeh (2001) perform an empirical research on the use of a one-, and two-factor Vasicek model with jump diffusion components for the Taiwanese Government Bond market. They perform Quasi-Maximum Likelihood Estimation (QMLE) using approximations of the AJTSM, proposed in Baz and Das (1996). Their results indicate that the two-factor ATSM and AJTSM perform significantly better than the one-factor ATSM and AJTSM in fitting the term structure of interest rates. More importantly, they find significant parameters related to the jump intensity and jump size, indicating the presence of jumps.

In this paper, the Vasicek (1977) model is used to capture the dynamics of the factors in the ATSMs. The AJTSMs are comprised of these ATSMs and additional jump diffusion components, which have Gaussian distributed jump sizes. Analogous to Piazzesi (2003), the parameters of both types of models are estimated by Quasi-Maximum Likelihood Estimation (QMLE). In combination with a Global Search algorithm, the Fisher Information matrix is justifiably used to verify whether all parameters are identified. Subsequently, I perform an elaborate analysis on the economic interpretation of the parameters as well as on the in-sample fit of the models using goodness-of-fit measures. The out-of-sample performance is based on one-week-ahead VaR and ES estimates for the interest rate swaps, constructed by means of Monte Carlo simulation. The backtests for VaR estimates include the conventional Unconditional Coverage, Independence and Conditional Coverage tests. The Saddlepoint Approximation and Box-Pierce tests are used to backtest ES estimates due to their advantageous small sample properties.

I examine the empirical application of ATSMs and AJTSMs on the riskless interest rates. In order to approximate the riskless interest rates and, subsequently, to bootstrap the riskless yield curve, US LIBOR money market deposits and US interest rate swaps

[2]For elaborate mathematical derivations and analyses of both models, I refer the reader to Bolder (2001).

are used. The time series of these securities range from January 6, 2006 to January 1, 2016 to capture the pre-, mid- and post-crisis periods of the financial crisis of 2008. This time period, in particular, includes the effect of monetary policies and interventions of central banks on the financial market.

The empirical results imply several significant differences and similarities between modeling the riskless interest rates with ATSMs and AJTSMs. Firstly, I find that there is an empirical justification of jump diffusion components in the riskless interest rate process, implied by the significance of the jump parameters in the AJTSMs. In economic terms, this result attributes to the presence of jumps in the riskless interest rate process. In the presence of significant jump diffusion components, I, generally, find that volatility estimates of the non-jump process decrease. For the post-crisis period, I find that the riskless interest rate process does not exhibit significant jumps. Secondly, I find that the three-factor ATSM is superior in fitting the entire yield curve for the complete sample and all sub-samples. The goodness-of-fit measures indicate that the one- and two-factor ATSM and AJTSM, generally, fit the yield curve poorly and are misspecified for long-term yields. A comparison of ATSMs and AJTSMs shows that the AJTSM-framework performs marginally better than the ATSM-framework in the case of one-factor models and the ATSM-framework performs substantially better in the case of the two-factor models. Due to identification problems, I am not able to estimate the parameters of the three-factor AJTSM. Lastly, the results show that ATSMs and AJTSMs perform poorly in terms of Value-at-Risk and Expected Shortfall estimation. From a Risk Management perspective, ATSMs and AJTSMs are inadequate for interest rate swaps.

The key novelty of this research is the deviation from and, thereby, the contribution to the current literature in four significant ways. First, the focus of this research is on the difference between ATSMs and AJTSMs with respect to two separate aspects, namely from an economic, in-sample fit, standpoint and a practical, Risk Management, standpoint. To my knowledge, there has neither been an empirical paper on the Risk Management application of ATSMs, nor has there been a comparable study with respect to this application of AJTSMs. Therefore, this is regarded to be the key contribution of this paper. Second, this paper concerns the modeling and fitting of the riskless interest rates, whereas Lin and Yeh (2001) focus on the Taiwanese Government Bond market. So far, an empirical study that compares the performance of ATSMs and AJTSMs for the riskless interest rates has been lacking in the current literature. Third, in addition to the one- and two-factor ATSMs and AJTSMs in Lin and Yeh (2001), this paper derives and investigates the three-factor models in both frameworks. This builds upon the results of Lin and Yeh (2001), that indicate that the in-sample fit of the term structure of interest rates improves by modeling additional factors. Fourth, I deviate from the QMLE procedure of Lin and Yeh (2001), which allows for misspecification of the ATSM and AJTSM models. To this extent, I propose a QMLE procedure along the lines of Piazzesi (2003).

The remainder of this paper adheres to the following structure. Section 2 discusses the models and methodology of this paper. Section 3 analyzes the data. Section 4 reports and analyzes the main results. Section 5 concludes and Section 6 presents the limitations in this research as well as directions for further research.

2. *Models and Methodology*

In this section, I discuss the main models and methodology of this research. The first subsection introduces the Vasicek model and derives its implications for the yield curve. The second subsection proposes the Quasi-Maximum Likelihood Estimation (QMLE) procedure to obtain the parameters of the Vasicek model. Lastly, the third subsection considers measures to evaluate the goodness-of-fit of models and measures to backtest VaR and ES estimates.

2.1 Vasicek model

This subsection is divided in three main parts to emphasize the theoretical background and derivations of ATSMs and AJTSMs. Firstly, I discuss basic concepts and the general idea in term structure modeling. Secondly, I derive the relation between the Vasicek model and the term structure of yields within the ATSM- and AJTSM-framework. Lastly, I provide the exact solutions for the ATSMs and the approximated solutions for the AJTSMs.

2.1.1 Basic concepts and general idea

The single, most important security in Fixed Income is the riskless zero-coupon bond. This bond pays, with certainty, one unit of currency at maturity. The price of a riskless zero-coupon bond, $P(t,T)$, depends on the current time, t, and its maturity, T. By definition, it holds that $P(T,T) = 1$. Given the prices of riskless zero-coupon bonds for all maturities, we determine the term structure of the riskless interest rates. Conventionally, this is done by exploiting the following relation:

$$\begin{aligned} e^{y(t,T)\times(T-t)} &= \frac{1}{P(t,T)} \\ y(t,T)\times(T-t) &= -\ln(P(t,T)) \\ y(t,T) &= -\frac{\ln(P(t,T))}{T-t}, \end{aligned} \tag{2.1}$$

where $y(t,T)$ denotes the yield of a riskless zero-coupon bond at time t with maturity T. The collection of yields for all maturities represents the term structure of the riskless interest rates.

The main theoretical concept in term structure modeling is the instantaneous interest rate, $r(t)$. This theoretical quantity is defined as the yield on a very short bond:

$$r(t) = \lim_{T\to t} y(t,T). \tag{2.2}$$

The general idea is to define a model for the instantaneous interest rate, or short rate. The short rate is assumed to drive the dynamics of the price of the riskless zero-coupon bonds. Therefore, the short rate drives the dynamics of the term structure, as well. As a matter of fact, the dynamics of the entire term structure, in ATSMs and AJTSMs within the Vasicek framework, are completely specified by the short rate. This relation is established by means of no-arbitrage conditions and the market price of risk. The following paragraphs provide the mathematical framework of this general idea and, further, define the dynamics of the short rate, the no-arbitrage conditions and the market price of risk.

2.1.2 Derivations: ATSM

Initially, I assume a single-factor framework for the short rate model. This assumption will be relaxed at the end of this subsection. Moreover, I assume that this single factor is the short rate itself, that the short rate adheres to the Markov-property and that it has the following real world $\mathbb{P}$-dynamics, excluding jumps:

$$dr(t) = \mu(r,t)dt + \sigma(r,t)dW(t), \tag{2.3}$$

where $W(t)$ is the Brownian Motion (BM) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mu(r,t)$ is the drift of the process and $\sigma(r,t)$ is the volatility of the process.

The short rate dynamics in Equation (2.3) imply, by Itô's Lemma, that the price of a riskless zero-coupon bond, $P(r(t), t, T)^1$, has the following $\mathbb{P}$-dynamics:

$$\begin{aligned} dP(t,T) &= P_t dt + P_r dr(t) + \frac{1}{2} P_{rr} dr(t) dr(t) \\ &= \left(P_t + \mu(r(t),t) P_r + \frac{\sigma(r(t),t)^2}{2} P_{rr} \right) dt + \sigma(r(t),t) P_r dW(t), \end{aligned} \tag{2.4}$$

where P_x denotes the first-order derivative of P with respect to x and P_{xx} denotes the second-order derivative of P with respect to x.

Analogous to the Black-Scholes portfolio, that mimics the pay-off of an option, I aim to construct a similar self-financing portfolio. The goal is to construct a self-financing portfolio with no randomness. No-arbitrage conditions imply that this portfolio earns the riskless rate.

To construct this portfolio, I choose two zero-coupon bonds with maturities, T_1 and T_2. Furthermore, I define V to be the value of the self-financing portfolio and w_i to be the weight of bond i in the portfolio. Subsequently, the return on the self-financing portfolio can be described by the following stochastic differential equation (SDE):

$$\frac{dV(t)}{V(t)} = w_1 \frac{dP_1}{P_1} + w_2 \frac{dP_2}{P_2}, \tag{2.5}$$

where P_i denotes the riskless zero-coupon bond price with maturity T_i, that is $P(t, T_i)$, and $w_i \in \mathbb{R}$. Using Equation (2.4), this can be written as:

$$\begin{aligned} \frac{dV(t)}{V(t)} = w_1 & \frac{(P_{1,t} + \mu P_{1,r} + \frac{\sigma^2}{2} P_{1,rr})dt + \sigma P_{1,r} dW(t)}{P_1} \\ & + w_2 \frac{(P_{2,t} + \mu P_{2,r} + \frac{\sigma^2}{2} P_{2,rr})dt + \sigma P_{2,r} dW(t)}{P_2}. \end{aligned} \tag{2.6}$$

In order to simplify this expression, I define:

$$\begin{aligned} \mu_i &= \frac{P_{i,t} + \mu P_{i,r} + \frac{\sigma^2}{2} P_{i,rr}}{P_i}, \\ \sigma_i &= \frac{\sigma P_{i,r}}{P_i}, \end{aligned} \tag{2.7}$$

[1] For notational convenience, I suppress $r(t)$ in $P(r(t), t, T)$ throughout the paper.

which simplifies Equation (2.6) to:

$$\begin{aligned}\frac{dV(t)}{V(t)} &= w_1(\mu_1 dt + \sigma_1 dW(t)) + w_2(\mu_2 dt + \sigma_2 dW(t)) \\ &= (w_1\mu_1 + w_2\mu_2)dt + (w_1\sigma_1 + w_2\sigma_2)dW(t).\end{aligned} \tag{2.8}$$

To obtain a self-financing portfolio without uncertainty, we solve the following system of equations:

$$\begin{aligned}w_1 + w_2 &= 1, \\ w_1\sigma_1 + w_2\sigma_2 &= 0.\end{aligned} \tag{2.9}$$

The solution to this system of equations is:

$$\begin{aligned}w_1 &= \frac{-\sigma_2}{\sigma_1 - \sigma_2}, \\ w_2 &= \frac{\sigma_1}{\sigma_1 - \sigma_2}.\end{aligned} \tag{2.10}$$

Substitution of the solution in Equation (2.8) yields a self-financing portfolio without uncertainty:

$$\begin{aligned}\frac{dV(t)}{V(t)} &= \left(\frac{-\sigma_2}{\sigma_1 - \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2\right)dt + \left(\frac{-\sigma_2}{\sigma_1 - \sigma_2}\sigma_1 + \frac{\sigma_1}{\sigma_1 - \sigma_2}\sigma_2\right)dW(t) \\ &= \left(\frac{-\sigma_2}{\sigma_1 - \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2\right)dt.\end{aligned} \tag{2.11}$$

In order to exclude arbitrage, the drift of the self-financing portfolio should equal the short rate, $r(t)$. Equating Equation (2.11) to the short rate yields:

$$\begin{aligned}\left(\frac{-\sigma_2}{\sigma_1 - \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2\right) &= r(t) \\ -\sigma_2\mu_1 + \sigma_1\mu_2 &= \sigma_1 r(t) - \sigma_2 r(t) \\ \sigma_2(r(t) - \mu_1) &= \sigma_1(r(t) - \mu_2) \\ \frac{\mu_1 - r(t)}{\sigma_1} = \frac{\mu_2 - r(t)}{\sigma_2} &= \xi(t).\end{aligned} \tag{2.12}$$

This expression represents the market price of risk, $\xi(t)$. The market price of risk can be either positive or negative and is often interpreted as the Sharpe ratio of a security. More importantly, it constitutes, in an analytical sense, as an internal consistency relation and is equal for all riskless zero-coupon bonds, as is illustrated by the last line in Equation (2.12). It provides the shift from the real world $\mathbb{P}$-measure to the risk-neutral

$\mathbb{Q}$-measure. The $\mathbb{Q}$-measure is needed to uniquely price riskless zero-coupon bonds and other Fixed Income securities. The key is to rewrite the market price of risk and to find the partial differential equation (PDE) that needs to be satisfied for an arbitrage-free price:

$$\begin{aligned}
\frac{\mu_i - r(t)}{\sigma_i} &= \xi(t) \\
\mu_i - r(t) &= \xi(t)\sigma_i \\
\text{(Substituting (2.7))} \rightarrow \frac{P_{i,t} + \mu P_{i,r} + \frac{\sigma^2}{2} P_{i,rr}}{P_i} - r(t) &= \xi(t)\frac{\sigma P_{i,r}}{P_i} \\
\text{(Equivalent to)} \rightarrow \frac{P_t + \mu P_r + \frac{\sigma^2}{2} P_{rr}}{P} - r(t) &= \xi(t)\frac{\sigma P_r}{P} \\
P_t + \mu P_r + \frac{\sigma^2}{2} P_{rr} - r(t)P &= \xi(t)\sigma P_r \\
\text{(PDE)} \rightarrow P_t + (\mu - \xi(t)\sigma)P_r + \frac{\sigma^2}{2} P_{rr} - r(t)P &= 0.
\end{aligned} \tag{2.13}$$

The solution, $P \equiv P(t, T)$, to this PDE provides the pricing equation for riskless zero-coupon bonds in an arbitrage-free world. Using Equation (2.1), this pricing equation constitutes the link between the term structure of riskless interest rates and the short rate. Duffie and Kan (1996) show that there is a class of models, namely Affine Term Structure Models, that uniquely solves this PDE. Assuming a constant relation between the price of a bond and its maturity and affine functions for μ and σ, the solution of the PDE can be represented by:

$$P(t, T) \equiv P(t, \tau) = e^{A(\tau) - B(\tau)r(t)}, \tag{2.14}$$

where τ is the time to maturity $(T - t)$.

The Vasicek model belongs to the class of ATSMs. In this paper, I assume that the $\mathbb{P}$-dynamics of the short rate are described by the Vasicek model, that is:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t), \tag{2.15}$$

where θ is the long-term mean, κ captures the speed of the mean-reversion and is positive (> 0), σ is the volatility of the short rate and $W(t)$ is a Brownian Motion (BM) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The $\mathbb{Q}$-dynamics of the short rate are given by:

$$dr(t) = \kappa(\tilde{\theta} - r(t))dt + \sigma d\tilde{W}(t), \tag{2.16}$$

where $\tilde{\theta} = \theta - \frac{\xi(t)\sigma}{\kappa}$ incorporates the market price of risk[2] and $\tilde{W}(t)$ is a BM defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Throughout this paper, I assume, for simplicity, that the market price of risk is constant over time, that is $\xi(t) = \xi$.

Using the dynamics of the short rate and the solution form in Equation (2.14), I derive the PDE in the terms of the Vasicek model in Appendix A. This results in:

$$0 = A_t(\tau) - \kappa\tilde{\theta}B(\tau) + \frac{\sigma^2}{2}B^2(\tau) - (1 + B_t(\tau) - \kappa B(\tau))r(t). \tag{2.17}$$

Equation (2.17) can be rewritten as a system of ODEs, namely:

$$\begin{aligned} A_t(\tau) - \kappa\tilde{\theta}B(\tau) + \frac{\sigma^2}{2}B^2(\tau) &= 0, \\ A(0) &= 0, \\ B_t(\tau) - \kappa B(\tau) &= -1, \\ B(0) &= 0. \end{aligned} \tag{2.18}$$

where $A(0) = 0$ and $B(0) = 0$ are inferred by the definition of the price of riskless zero-coupon bonds, $P(T, \tau) = 1$.

In this entire derivation, I have shown that the price of riskless zero-coupon bonds, in the ATSM-framework, is an exponential function of the short rate. Its coefficients, $A(\tau)$ and $B(\tau)$, are functions of maturity, τ, and are solutions to a system of ODEs (2.18). In the ATSM-framework, the Vasicek model has a closed-form solution. That is, the system of ODEs has a unique solution, see (2.27). The following paragraphs show that this can, similarly, be done in the AJTSM-framework. In contrast to the ATSM-framework, approximations are needed to obtain a system of ODEs, that is uniquely solvable, in the AJTSM-framework.

[2]In mathematical terms, this can also be interpreted as the drift resulting from Girsanov's Theorem.

2.1.3 Derivations: AJTSM

In the previous paragraphs, the dynamics of the short rate did not include jumps. I continue this subsection with the addition of a jump diffusion component and derive the pricing equation for riskless zero-coupon bonds in the AJTSM-framework. I follow Baz and Das (1996) and show that the pricing function can, similar to the ATSM derivation, be reduced to a set of ODEs.

In the AJTSM-framework, I assume that the $\mathbb{P}$-dynamics of the short rate follow the Vasicek model with a jump diffusion:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t) + Y dN(t), \tag{2.19}$$

where θ is the long-term mean, κ is the mean-reversion coefficient and is positive (> 0), σ is the volatility of the short rate, $W(t)$ is the Brownian Motion (BM) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Y is the Gaussian distributed jump size, $\mathrm{N}(\alpha, \beta^2)$, and $dN(t)$ is the Poisson process with intensity λdt. The Brownian Motion and the Poisson process are assumed to be independent and jump risk is assumed to be diversifiable. These short rate dynamics imply, by Itô's Lemma, that the price of a riskless zero-coupon bond, $P(t,T)$, has the following $\mathbb{P}$-dynamics:

$$\begin{aligned} dP(t,T) = \left(P_t + \kappa(\theta - r(t))P_r + \frac{\sigma^2 P_{rr}}{2}\right) dt + \sigma P_r dW(t) \\ + \left[P(r+Y,t,T) - P(r,t,T)\right] \end{aligned} \tag{2.20}$$

where P_x and P_{xx} are similarly defined as in Equation (2.4) and $[P(r+Y,t,T) - P(r,t,T)]$ results from Itô's lemma for jump diffusions.

By similar no-arbitrage conditions, Baz and Das (1996) derive the PDE to ensure an arbitrage-free price:

$$\begin{aligned} 0 = P_t + (\kappa(\theta - r(t)) - \xi\sigma)P_r + \frac{\sigma^2 P_{rr}}{2} - r(t)P(t,T) \\ + \lambda E\left[P(r+Y,t,T) - P(r,t,T)\right]. \end{aligned} \tag{2.21}$$

The solution to this PDE provides the price of riskless zero-coupon bonds in the AJTSM-framework. Duffie et al. (2000) show that the solution can be represented in the same form as the solution in the ATSM-framework, namely an exponential function of the short rate (2.14). Using Equation (2.14), the PDE can be rewritten as:

$$\begin{aligned} 0 = P_t + (\kappa(\theta - r(t)) - \xi\sigma)P_r + \frac{\sigma^2 P_{rr}}{2} - r(t)P(t,T) \\ + \lambda P(t,T) E\left[e^{-YB(\tau)} - 1\right]. \end{aligned} \tag{2.22}$$

Baz and Das (1996) approximate the expectation expression in the PDE by a two-term Taylor expansion, that is:

$$E\left[e^{-YB(\tau)} - 1\right] \approx E\left[-YB(\tau) + \frac{Y^2 B(\tau)^2}{2}\right] = -\alpha B(\tau) + \frac{\beta^2 + \alpha^2}{2} B(\tau)^2. \tag{2.23}$$

The approximation is needed to obtain a system of ODEs, that is uniquely solvable. Substitution of Equation (2.23) in the PDE yields:

$$\begin{aligned} 0 = P_t + (\kappa(\theta - r(t)) - \xi\sigma)P_r + \frac{\sigma^2 P_{rr}}{2} - r(t)P(t,T) \\ + \lambda P(t,T)\left[-\alpha B(\tau) + \frac{\beta^2 + \alpha^2}{2} B(\tau)^2\right]. \end{aligned} \tag{2.24}$$

I postpone the implementation of the exponential function (2.14) in the PDE to Appendix A. This derivation results in the following PDE:

$$\begin{aligned} 0 =& A_t(\tau) + \frac{\sigma^2 B^2(\tau)}{2} - \kappa\tilde{\theta} B(\tau) + \lambda\left[-\alpha B(\tau) + \frac{\beta^2 + \alpha^2}{2} B(\tau)^2\right] \\ &+ (-1 - B_t(\tau) + \kappa B(\tau))r(t). \end{aligned} \tag{2.25}$$

Equation (2.25) can be rewritten as a system of ODEs, that is:

$$\begin{aligned} A_t(\tau) - \kappa\tilde{\theta} B(\tau) + \frac{\sigma^2}{2} B^2(\tau) + \lambda\left[-\alpha B(\tau) + \frac{\beta^2 + \alpha^2}{2} B(\tau)^2\right] &= 0, \\ A(0) &= 0, \\ B_t(\tau) - \kappa B(\tau) &= -1, \\ B(0) &= 0. \end{aligned} \tag{2.26}$$

These derivations provide the key insight that the price of riskless zero-coupon bonds are exponential functions of the short rate in, both, the ATSM- and AJTSM-framework. The coefficients, $A(\tau)$ and $B(\tau)$, are functions of maturity, τ, and are solutions to a system of ODEs ((2.18) and (2.26)). In the following paragraphs, I present the solutions to the systems of ODEs and extend the results to a multi-factor framework.

2.1.4 Solutions

The previous paragraphs derive a system of ODEs for both the ATSM- and AJTSM-framework. The solutions to these systems of ODEs provide the coefficients of the pricing function (2.14). I refrain from deriving the solution of these ODEs, as there is a large literature that provides elaborate derivations of these solutions. In this paper, I use the results of Bolder (2001) and Lin and Yeh (2001) in, respectively, the ATSM- and AJTSM-framework.

Bolder (2001) provides the following solutions to the system of ODEs (2.18) in the ATSM-framework:

$$\begin{aligned} B(\tau) &= \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right), \\ A(\tau) &= \left(\theta - \frac{\xi\sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2}\right)(B(\tau) - \tau) - \frac{\sigma^2 B^2(\tau)}{4\kappa}. \end{aligned} \tag{2.27}$$

Due to additional expressions in the approximated system of ODEs in Equation (2.26), the solution is more complex in the AJTSM-framework. Lin and Yeh (2001) provide the following expressions for $A(\tau)$ and $B(\tau)$ in this framework:

$$\begin{aligned} B(\tau) &= \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right), \\ A(\tau) &= \frac{-Ee^{-2\kappa\tau}}{4\kappa^3} + \frac{(\kappa D + E)e^{-\kappa\tau}}{\kappa^3} + \frac{(2\kappa D + E)\tau}{2\kappa^3} - C, \\ &\text{where} \\ C &= \frac{D}{\kappa^2} + \frac{3E}{4\kappa^3}, \\ D &= \xi\sigma - \kappa\theta - \alpha\lambda, \\ E &= \sigma^2 + (\alpha^2 + \beta^2)\lambda, \\ 2\kappa D &+ E < 0. \end{aligned} \tag{2.28}$$

The constraint ensures that the price of riskless zero-coupon bonds converges to zero when maturity increases to infinity. Given the parameters of the short rate dynamics, the price of riskless zero-coupon bonds can be determined by substituting $A(\tau)$ and $B(\tau)$ in:

$$P(t,\tau) = e^{A(\tau) - B(\tau)r(t)}. \tag{2.29}$$

Subsequently, the term structure of the riskless interest rates is computed using the relation in Equation (2.1).

In the beginning of this subsection, I assumed a single-factor model. This assumption implies that all yields are perfectly correlated. However, the Data Analysis in Section 3 shows that this does not hold in practice. In order to capture the empirical yield dynamics, the current literature provides extensions for the ATSM- and AJTSM-framework to incorporate a multi-factor model. In this model, the multiple factors continue to be related to the short rate in the sense that the following relation holds:

$$r(t) = \sum_{i=1}^{n} y_i(t). \tag{2.30}$$

That is, the factors $y_1(t), \ldots, y_n(t)$ drive the dynamics of the short rate, $r(t)$.

The extension of the Vasicek model in a multi-factor framework is one of the model's main advantages. Assuming independence between the factors, the multi-factor Vasicek model in the ATSM-framework is defined as:

$$\begin{aligned} dy_1(t) &= \kappa_1(\theta_1 - y_1(t))dt + \sigma_1 dW_1(t), \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ dy_n(t) &= \kappa_n(\theta_n - y_n(t))dt + \sigma_n dW_n(t), \end{aligned} \tag{2.31}$$

where θ_i is the long-term mean of the i^{th} factor, κ_i is the mean-reversion coefficient of the i^{th} factor, σ_i is the volatility of the i^{th} factor and $W_i(t)$ is the Brownian Motion of the i^{th} factor defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consequently, the multi-factor Vasicek model in the AJTSM-framework is defined as:

$$\begin{aligned} dy_1(t) &= \kappa_1(\theta_1 - y_1(t))dt + \sigma_1 dW_1(t) + Y_1 dN_1(t), \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ dy_n(t) &= \kappa_n(\theta_n - y_n(t))dt + \sigma_n dW_n(t) + Y_n dN_n(t), \end{aligned} \tag{2.32}$$

where θ_i, κ_i, σ_i and $W_i(t)$ are equivalently defined as in Equation (2.31), Y_i is the Gaussian distributed jump size of the i^{th} factor, $\mathrm{N}(\alpha_i, \beta_i^2)$, and $dN_i(t)$ is the Poisson process of the i^{th} factor with intensity $\lambda_i dt$. Lastly, I denote the market price of risk for the i^{th} factor by ξ_i. The overall market price of risk is $\sum_{i=1}^{n} \xi_i$.

By defining the short rate as the sum of the factors, the price of riskless zero-coupon bonds is modified to:

$$P(t, \tau) = e^{A(\tau) - \sum_{i=1}^{n} B_i(\tau) y_i(t)}. \tag{2.33}$$

Similar to the analysis in the previous paragraphs, one can derive the PDEs, the system of ODEs and the solutions for $A(\tau)$ and $B_i(\tau)$ in the ATSM- and AJTSM-framework. I refer the interested reader to Bolder (2001) and Lin and Yeh (2001). Moreover, I use the results of these papers for the multi-factor models in both frameworks, as well. Bolder (2001) provides the following expressions for $A(\tau)$ and $B_i(\tau)$ in the ATSM-framework:

$$\begin{aligned} B_i(\tau) &= \frac{1}{\kappa_i}\left(1 - e^{-\kappa_i \tau}\right), \\ A(\tau) &= \sum_{i=1}^{n} \left(\theta_i - \frac{\xi_i \sigma_i}{\kappa_i} - \frac{\sigma_i^2}{2\kappa_i^2}\right)(B_i(\tau) - \tau) - \frac{\sigma_i^2 B_i^2(\tau)}{4\kappa_i}, \end{aligned} \tag{2.34}$$

while Lin and Yeh (2001) provide expressions for $A(\tau)$ and $B_i(\tau)$ in the AJTSM-framework:

$$\begin{aligned} B_i(\tau) &= \frac{1}{\kappa_i}\left(1 - e^{-\kappa_i \tau}\right), \\ A(\tau) &= \sum_{i=1}^{n} \frac{-E_i e^{-2\kappa_i \tau}}{4\kappa_i^3} + \frac{(\kappa_i D_i + E_i)e^{-\kappa_i \tau}}{\kappa_i^3} + \frac{(2\kappa_i D_i + E_i)\tau}{2\kappa_i^3} - C_i, \\ &\text{where} \\ C_i &= \frac{D_i}{\kappa_i^2} + \frac{3E_i}{4\kappa_i^3}, \\ D_i &= \xi_i \sigma_i - \kappa_i \theta_i - \alpha_i \lambda_i, \\ E_i &= \sigma_i^2 + (\alpha_i^2 + \beta_i^2)\lambda_i, \\ 2\kappa_i D_i &+ E_i < 0, \qquad \text{for } i = 1, \ldots, n. \end{aligned} \tag{2.35}$$

Analogous to the single-factor models, the price of riskless zero-coupon bonds can be determined by substituting $A(\tau)$ and $B_i(\tau)$ in Equation (2.33). The term structure of the riskless interest rates is computed using the relation in Equation (2.1).

2.2 Quasi-Maximum Likelihood Estimation (QMLE)

This subsection establishes the empirical methodology to estimate the Vasicek model in the AJTSM-framework. Using Lin and Yeh (2001), I specify the probability density function (pdf) for the short rate. The pdf is approximated for feasibility purposes. Subsequently, I elaborate on the Quasi-Maximum Likelihood Estimation (QMLE) for the parameters of the Vasicek model, along the lines of Piazzesi (2003). Lastly, I extend the QMLE procedure to a multi-factor framework and contribute to the literature by deriving the three-factor ATSM and AJTSM. The results of this subsection extend to parameter estimation of the Vasicek model without jumps by applying trivial constraints on the jump parameters.

First, I assume the dynamics of the short rate are described by a one-factor AJTSM, that is a Vasicek model with jumps (2.19). This model implies that the short rate adheres to the Markov-property. Therefore, the likelihood function is given by:

$$\mathcal{L}(r_1, \dots, r_T; \Theta) = \prod_{t=1}^{T} f(r_t | r_{t-1}), \tag{2.36}$$

where $\Theta = \{\kappa, \theta, \sigma, \lambda, \alpha, \beta, \xi\}$ denotes the parameter-set. Maximizing the likelihood function, $\mathcal{L}$, with respect to Θ, yields the optimal parameter set $\hat{\Theta}_{MLE}$. Consequently, we need an expression for the conditional pdf of r_t to perform MLE.

In this AJTSM-framework, Lin and Yeh (1999) derive the following expression for r_t given r_{t-1} :

$$r_t | r_{t-1} = e^{-\kappa \Delta t} \left(r_{t-1} + \int_{t-1}^{t} e^{\kappa u} \kappa \theta du + \int_{t-1}^{t} e^{\kappa u} \sigma dW(u) + \sum_{j=N(t-1)}^{N(t)} e^{\kappa \psi_j} Y^j \right), \tag{2.37}$$

where Δt denotes the time between subsequent observations t and $t-1$, ψ_j denotes the time of the j^{th} jump, Y^j is the corresponding jump size and $N(t)$ denotes the counting process of the number of jumps in the interval $[0, t]$. In this paper, I allow for a maximum of one jump per time period. The probability for more than one jump, within a time period, is empirically negligible. The Gaussian distribution of the Brownian Motion and the Poisson distribution of the jump diffusion imply the following pdf for the short rate[3]:

$$\begin{aligned} f(r_t | r_{t-1}) = \sum_{n=0}^{\infty} & \frac{e^{-\lambda \Delta t} (\lambda \Delta t)^n}{n!} \\ & \times \int_{t-1}^{t} \int_{t-1}^{t} \cdots \int_{t-1}^{t} \left(\phi(r_t; r_{t-1}, m, s) \times \frac{1}{(\Delta t)^n} \right) d\psi_1 d\psi_2 \cdots d\psi_n, \end{aligned} \tag{2.38}$$

[3]The derivation of this pdf can be found in Lin and Yeh (1999).

where $\phi(r_t; m, s)$ denotes a Gaussian distribution with mean m and variance s, that is:

$$\begin{aligned} m &= e^{-\kappa\Delta t} r_{t-1} + \theta(1 - e^{-\kappa\Delta t}) + \alpha e^{-\kappa\Delta t} \sum_{j=1}^{n} e^{\kappa\psi_j}, \\ s &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}) + \beta^2 e^{-2\kappa\Delta t} \sum_{j=1}^{n} e^{2\kappa\psi_j}. \end{aligned} \tag{2.39}$$

Equation (2.38) shows that the pdf of the short rate contains multiple integrals. These integrals are a result of the jump diffusion component, but impose a burden on the feasibility of the estimation procedure. To this extent, Lin and Yeh (1999) propose an approximation of the current form of the pdf, namely:

$$f(r_t | r_{t-1}) \approx \sum_{n=0}^{\infty} \frac{e^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!} \times \phi(r_t; r_{t-1}, \hat{m}, \hat{s}), \tag{2.40}$$

where $\phi(r_t; \hat{m}, \hat{s})$ denotes a Gaussian distribution with mean $\hat{m}$ and variance $\hat{s}$, that is:

$$\begin{aligned} \hat{m} &= e^{-\kappa\Delta t} r_{t-1} + \left(\theta + \frac{n}{\kappa\Delta t}\alpha\right)(1 - e^{-\kappa\Delta t}), \\ \hat{s} &= \left(\sigma^2 + \frac{n}{\Delta t}\beta^2\right) \frac{1 - e^{-2\kappa\Delta t}}{2\kappa}. \end{aligned} \tag{2.41}$$

The approximation is based on the assumption that jumps in the riskless interest rate are equally spread over time. This assumption is used to take the expectation of the summation components in m and s in Equation (2.39). I provide the calculations of this procedure in Appendix A. Additionally, Lin and Yeh (1999) show that the approximation of the pdf converges to the true density, $f(r_t | r_{t-1})$, when $\kappa\Delta t \to 0$.

The approximate pdf allows for Maximum Likelihood Estimation of the AJTSM parameter-set, Θ, given $r_1, \ldots, r_T$. It should be noted that the exact pdf in Equations (2.38) and (2.39) is used for the estimation of the ATSM parameter-set. The multiple integrals (and jump components) are eliminated in the absence of jumps and, therefore, the approximation (2.40) is not needed.

In order to use Maximum Likelihood Estimation, $r_1, \ldots, r_T$ are to be observed. However, these variables are not observed in practice and, therefore, I propose to perform a QMLE procedure by expressing $r_1, \ldots, r_T$ in observable variables. This procedure has been popularized by Piazzesi (2003) and solves a system of equations to obtain the unobservable variables. The system of equations results from the pricing equation of the riskless zero-coupon bond:

$$P(t,\tau) = e^{A(\tau)-B(\tau)r(t)}. \tag{2.42}$$

A collection of zero-coupon bonds with m different maturities constitutes the system of equations as follows:

$$\begin{aligned}
\ln\left[P(t,\tau_1)\right] &= A(\tau_1) - B(\tau_1)r(t), \\
\ln\left[P(t,\tau_2)\right] &= A(\tau_2) - B(\tau_2)r(t) + \epsilon_{1,t}, \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
\ln\left[P(t,\tau_m)\right] &= A(\tau_m) - B(\tau_m)r(t) + \epsilon_{m-1,t},
\end{aligned} \tag{2.43}$$

where $\epsilon_t = [\epsilon_{1,t} \ldots \epsilon_{m-1,t}]' \sim N(0,\Omega)$. The bond with the shortest maturity is modeled without measurement error in order to use the first equation to obtain an estimate of the unobservable variables, that is:

$$\hat{r}_t = \frac{-\ln\left[P(t,\tau_1)\right] + A(\tau_1)}{B(\tau_1)}. \tag{2.44}$$

In this paper, I use the three-month riskless zero-coupon bond to obtain $\hat{r}_1, \ldots, \hat{r}_T$ in the single-factor framework. The system of equations is completed with the riskless zero-coupon bond prices of the six-month, one-, two-, three-, four-, five-, six-, seven-, eight-, nine-, 10-, 15-, 20-, 25- and 30-year maturities, which are modeled with measurement errors. I refer the interested reader to Piazzesi (2003) for an elaborate analysis of this estimation procedure and the necessity of measurement errors to break stochastic singularity.

The substitution of $\hat{r}_t$ in the approximate density implies a transformation of the pdf by means of the Jacobian. In the single-factor framework, the Jacobian is the first order derivative of $\hat{r}_t$ with respect to $\ln\left[P(t,\tau_1)\right]$, that is:

$$J = -\frac{1}{B(\tau_1)}. \tag{2.45}$$

The approximate pdf, system of equations and transformation yield the following log-likelihood function for the QMLE procedure:

$$\max_{\Theta} \mathbb{L}(\Theta) = \underbrace{\ln\left(\mathcal{L}(\hat{r}_1, \ldots, \hat{r}_T; \Theta)\right) - T \ln|J|}_{1} \overbrace{- \frac{T}{2} \ln|\Omega| - \frac{1}{2} \sum_{t=1}^{T} \epsilon_t' \Omega^{-1} \epsilon_t}^{2}, \tag{2.46}$$

where the first part maximizes the likelihood of the observed data and the second part minimizes the measurement errors for the bond prices in the system of equations.

This procedure is easily extended to a multi-factor framework. To illustrate this transition, I use the pricing function of the riskless zero-coupon bonds in the multi-factor framework:

$$P(t, \tau) = e^{A(\tau) - \sum_{i=1}^{n} B_i(\tau) y_i(t)}. \tag{2.47}$$

Analogous to the single-factor framework, the system of equations is constructed by a collection of m bonds with different maturities, namely:

$$\begin{aligned} \ln\left[P(t, \tau_1)\right] &= A(\tau_1) - \sum_{i=1}^{n} B_i(\tau_1) y_i(t), \\ \ln\left[P(t, \tau_2)\right] &= A(\tau_2) - \sum_{i=1}^{n} B_i(\tau_2) y_i(t) + \epsilon_{1,t}, \\ \vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\ \ln\left[P(t, \tau_m)\right] &= A(\tau_m) - \sum_{i=1}^{n} B_i(\tau_m) y_i(t) + \epsilon_{m-n,t}, \end{aligned} \tag{2.48}$$

where $\epsilon_t = [\epsilon_{1,t} \ldots \epsilon_{m-n,t}]' \sim N(0, \Omega)$. In the multi-factor framework, the number of bond prices, m, should be greater or equal to the number of factors, n. This ensures that the system of equations can uniquely be solved for the n factors. To capture the short- and long-term dynamics of the short-rate in the two-factor framework, the three-month and 30-year bond prices are modeled without errors to uniquely solve for the two factors. In the three-factor framework, I choose to model the three-month, 10- and 30-year bond prices without errors to uniquely solve for the three factors. The solutions to these systems of equations are derived in Appendix A. The solutions are, subsequently, used to compute the Jacobian.

The log-likelihood function in the multi-factor framework can, thus, be defined as:

$$\max_{\Theta} \mathbb{L}(\Theta) = \sum_{i=1}^{n} \ln\left(\mathcal{L}(\hat{y}_{i,1}, \ldots, \hat{y}_{i,T}; \Theta)\right) - T\ln|J| - \frac{T}{2}\ln|\Omega| - \frac{1}{2}\sum_{t=1}^{T} \epsilon_t' \Omega^{-1} \epsilon_t, \quad (2.49)$$

where $\Theta = \{\kappa_i, \theta_i, \sigma_i, \lambda_i, \alpha_i, \beta_i, \xi_i\}_{i=1}^{n}$ denotes the parameter-set and $\{\hat{y}_{i,1}, \ldots, \hat{y}_{i,T}\}_{i=1}^{n}$ are assumed to be the factors that constitute the short rate.

2.2.1 Fisher Information matrix

In order to obtain the uncertainty with respect to the estimates of QMLE parameters, I compute the observed Fisher Information matrix. The Fisher Information matrix is defined as the negative Hessian of the log-likelihood function, that is:

$$I(\Theta) = -\frac{\partial^2}{\partial\Theta_i \partial\Theta_j} \mathbb{L}(\Theta). \quad (2.50)$$

The observed Fisher Information matrix is obtained by evaluating the Fisher Information matrix at the QMLE estimate, $I(\hat{\Theta}_{QMLE})$. This is used in the specification of the asymptotic distribution of the parameter estimates as follows:

$$\hat{\Theta}_{QMLE} \overset{a}{\sim} N(\Theta, [I(\hat{\Theta}_{QMLE})]^{-1}). \quad (2.51)$$

Thus, the diagonal of the inverse of the observed Fisher Information matrix quantifies the uncertainty in the parameter estimates.

Additionally, I use the Fisher Information for identification purposes. The observed Fisher Information matrix can be interpreted as the amount of information in the data, regarding the parameter set. Piazzesi (2003) shows that the non-invertibility of $I(\hat{\Theta}_{QMLE})$ might indicate possible identification issues. A pre-condition for the use of this measure is to maximize the log-likelihood function for many different trial parameterizations. This is done by means of a Global Search algorithm in MATLAB. The Fisher Information matrix is used as an indication of identification issues. The theoretical results of Dai and Singleton (2000) are used to establish whether the ATSMs and AJTSMs are identified. This is done in the Parameter Estimation section of the Results (Section 4) by observing that ATSMs can be represented in the canonical framework of Dai and Singleton (2000).

2.3 Evaluation measures

In this subsection, I define the evaluation measures to test the goodness-of-fit and the out-of-sample performance of the models. In accordance with the current literature, I propose basic measures to evaluate the goodness-of-fit. The backtesting methodology for the VaR and ES estimates are, however, more involved and I briefly describe these methodologies in this subsection. For a more elaborate description, I refer the reader to Christoffersen (1998), Wong (2008) and Du and Escanciano (2015).

2.3.1 Goodness-of-fit

In this paper, I propose several models in the ATSM- and AJTSM-framework. The parameters of each model are estimated for the complete sample and, separately, for the pre-, mid- and post-crisis sample periods. Each model fits a yield curve at each period, t, in time. I compare the fitted yield curve with the observed yield curve by means of the Mean Squared Prediction Error (MSPE), Mean Absolute Error (MAE), Root Mean Squared Error (RMSE) and the in-sample Adjusted R^2. Additionally, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are computed for each model to compare the relative quality of the models.

2.3.2 Value-at-Risk and Expected Shortfall

In order to test the out-of-sample performance of the models, I compute VaR and ES estimates for the interest rate swaps. These estimates are constructed by simulation and employing the methodology of Hull (2006), proposed in subsection 3.1. First, I use the parameter estimates of the models to simulate M yield curves. Using the simulated yield curves and the relation in Equation (3.1), I determine the interest rate swap rates. Subsequently, I compute their one-week-ahead VaR and ES estimates.

With regard to testing the VaR estimates, I follow Christoffersen (1998) and apply three interrelated tests. These tests are regularly used in the industry to backtest two properties of the VaR estimates, namely their independence and (un)conditional coverage. The following tests examine these properties using Likelihood Ratio (LR) tests:

Correct Unconditional Coverage: Assuming that the independence property holds, the fraction of VaR violations should be equal to the nominal coverage probability, γ. This is, in essence, equivalent to testing: $H_0 = \mathrm{P}[I_{t+1} = 1] = \mathrm{E}[I_{t+1}] = \gamma$, where I_t is an indicator function of the VaR violations. I test this hypothesis by the following LR-test:

$$LR_{UC} = -2\log\left(\frac{(1-\gamma)^{T_0} \times \gamma^{T_1}}{(1-\pi_1)^{T_0} \times \pi_1^{T_1}}\right) \sim \chi^2(1), \tag{2.52}$$

where π_1 is the average of VaR violations, T_1 is the number of VaR violations, T_0 is the number of non-VaR violations and $\chi^2(1)$ is the Chi-squared distribution with one degree of freedom.

Independence: VaR violations should be spread out and not come in clusters. This is equivalent to testing the hypothesis; $H_0 = \mathrm{P}[I_{t+1} = 1|I_t] = \mathrm{P}[I_{t+1} = 1]$. I test this hypothesis with the following LR-test:

$$LR_{IND} = -2\log\left(\frac{(1-\pi_1)^{T_{00}+T_{10}} \times \pi_1^{T_{01}+T_{11}}}{(1-\pi_{01})^{T_{00}} \times \pi_{01}^{T_{01}} \times (1-\pi_{11})^{T_{10}} \times \pi_{11}^{T_{11}}}\right) \sim \chi^2(1), \tag{2.53}$$

where π_1 and $\chi^2(1)$ are defined in Equation (2.52), π_{01} is the average number of non-VaR violations followed by a VaR violation, π_{11} is the average number of VaR violations followed by a VaR violation, T_{00} is the number of non-VaR violations followed by a non-VaR violation, T_{01} is the number of non-VaR violations followed by a VaR violation, T_{10} is the number of VaR violations followed by a non-VaR violation and T_{11} is the number of VaR violations followed by a VaR violation.

Correct Conditional Coverage: Accurate VaR estimates should result in independent VaR violations and correct unconditional coverage. That is, the fraction of VaR violations should be equal to the nominal coverage probability, while they should be spread out over the sample and not come in clusters. The following hypothesis applies to this case: $H_0 = \mathrm{P}[I_{t+1} = 1|I_t] = \mathrm{P}[I_{t+1} = 1] = \gamma$. To test these two properties simultaneously, I use the sum of both LR-tests:

$$LR_{CC} = LR_{UC} + LR_{IND} \sim \chi^2(2), \tag{2.54}$$

where $\chi^2(2)$ is the Chi-squared distribution with two degrees of freedom.

In contrast to backtesting VaR estimates, ES estimates do not have common backtests. In this paper, I choose the Saddlepoint Approximation test and the Box-Pierce test to examine the ES estimates. Wong (2008) and Du and Escanciano (2015) show, by Monte Carlo simulations, that both tests have favorable small sample properties. The following summary describes the main aspects of these tests.

Saddlepoint Approximation Test The Saddlepoint Approximation (SPA) test is analogous to the unconditional coverage test for the VaR estimates. Wong (2008) uses a saddlepoint approximation technique by Lugannani and Rice (1980) to test whether the ES estimates capture the tail risk accurately. He compares the empirically estimated ES_n with the saddlepoint approximated ES_0, implied by the model under H_0. The test statistic is defined as:

$$ES_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \tag{2.55}$$

where n is the number of VaR violations and X_i is the ES estimate corresponding to the i^{th} VaR violation. The derivation of the saddlepoint approximated ES_0 is beyond the scope of this paper. I refer the reader to Proposition 2 and Equation (9), in Wong (2008), for an elaborate explanation of the saddlepoint approximated ES_0 and the corresponding p-value of the test.

Box-Pierce Test The Box-Pierce (BP) test is analogous to the independence test for the VaR estimates. Du and Escanciano (2015) define a cumulative violation process, $H_t(\gamma)$, which should be a martingale difference sequence (mds) after centering. Using this property, Du and Escanciano (2015) derive a conditional backtest for the ES estimates by testing the autocorrelation of the cumulative violation process. This is accomplished by a Box-Pierce test:

$$BP_{ES}(m) = n \sum_{j=1}^{m} \hat{p}_{nj}^2 \sim \chi^2(m), \tag{2.56}$$

where $\hat{p}_{nj}$ is the lag-j autocorrelation of the H_t with n violations and $m = 3$, in accordance with the power tests in Du and Escanciano (2015).

Although the VaR and ES backtests have favorable small sample properties, I emphasize that these tests lack power to provide conclusive results. Therefore, I estimate VaR and ES estimates with $\gamma = \{1\%, 5\%, 10\%\}$. This gives a more complete picture of the out-of-sample performance of the models.

3. *Data Analysis*

In this section, I discuss the data that is used in this research. In the first subsection, I focus on the modification of the data to construct the riskless yield curve. The models and methodology are applied on this riskless yield curve. The second subsection presents evidence for jumps in the riskless interest rates. Lastly, the third subsection provides summary statistics and further analysis of the yield curve for the entire sample as well as for three sub-samples.

3.1 Construction of yield curve

In this paper, the riskless yield curve is modeled by ATSMs and AJTSMs in the Vasicek-framework. In order to obtain the riskless interest rates and, subsequently, the riskless yield curve, US LIBOR money market deposits and US government based securities are used, in accordance with Dai and Singleton (2000). Although these securities are not completely riskfree, they are, generally, regarded to accurately approximate the riskless interest rates. Moreover, Feldhütter and Lando (2008) show that interest rate swap data most accurately approximates the riskless interest rates. Therefore, I use three- and six-month US LIBOR money market deposits and one-, two-, three-, four-, five-, six-, seven-, eight-, nine-, 10-, 15-, 20-, 25- and 30-year US interest rate swaps. The data is obtained from the Bloomberg database[1] for the period covering January 6, 2006 to January 1, 2016.

[1]Tickers for the US LIBOR money market deposits are US0003M INDEX and US0006M INDEX. Tickers for the US interest rate swaps are: USSW1 CMPN CURNCY, USSW2 CMPN CURNCY, USSW3 CMPN CURNCY, USSW4 CMPN CURNCY, USSW5 CMPN CURNCY, USSW6 CMPN CURNCY, USSW7 CMPN CURNCY, USSW8 CMPN CURNCY, USSW9 CMPN CURNCY, USSW10 CMPN CURNCY, USSW15 CMPN CURNCY, USSW20 CMPN CURNCY, USSW25 CMPN CURNCY and USSW30 CMPN CURNCY.

US LIBOR money market deposits have no coupon payment and are, therefore, directly substituted in the yield curve. US interest rate swaps are, however, more complicated to bootstrap zero-coupon yields from. I apply the methodology, proposed in Hull (2006), to bootstrap the term structure of the riskless interest rates. The methodology is based on the concept of an interest rate swap, with maturity T, being equivalent to a coupon bond, with the same maturity T. More elaborately, I provide a modified version of the example in Hull (2006), using linear interpolation of the yield curve:

> Suppose the six- and 12-month zero-coupon yields are 4% and 4.5%, respectively. The two-year interest rate swap is 5%. This is equivalent to a bond, with a principal of \$100 and a semi-annually coupon of 5% per annum, selling at par. Let X denote the two-year zero-coupon yield. By means of linear interpolation of the yield curve, the 18-month zero-coupon yield is $4.5\% + \frac{X-4.5\%}{2}$. Therefore, the following equation should hold[2]:
>
> $$100 = 2.5e^{-4\%\times 0.5} + 2.5e^{-4.5\%\times 1.0} + 2.5e^{-(4.5\%+\frac{X-4.5\%}{2})\times 1.5} + 102.5e^{-X\times 2.0}. \quad (3.1)$$
>
> Solving this equation yields a two-year zero-coupon yield of 4.95%.

Similarly, I apply this methodology to US interest rate swaps of all maturities and, thereby, obtain the entire riskless yield curve.

3.2 Jumps

In order to justify the use of AJTSMs, I analyze whether jumps are, in fact, apparent in the data. For illustrative purposes, Figure 3.1 only plots the weekly yield changes of the three-month and 10-year yields. Both weekly yield changes exhibit relatively infrequent, large spikes. Figure 3.1 shows that the magnitude of multiple spikes exceeds the three standard deviations barrier. These observations can be interpreted as jumps. Jumps are often caused by the arrival of new information that significantly impacts the view of market participants on the future state of the economy. We observe multiple jumps during the mid-crisis period, which can not be captured by ATSMs. This justifies the use of AJTSMs to capture the discontinuity in the riskless interest rates.

[2]Throughout the entire paper, I assume continuous compounding.

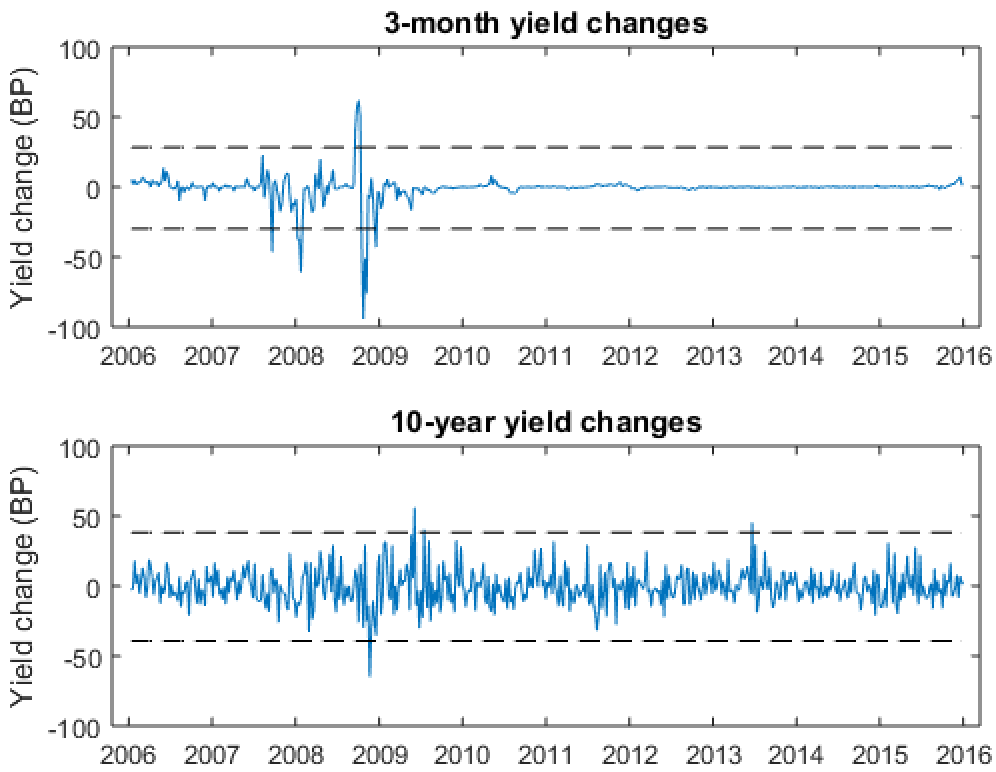


FIGURE 3.1: Weekly Yield Changes

Notes: This figure shows the weekly yield changes (in basis points) of the three-month and 10-year yields from January 6, 2006 to January 1, 2016. The upper and lower dashed lines indicate the upper and lower three standard deviation barriers.

The most evident jump took place at the end of 2008 and is, in fact, related to monetary policy of central banks. In particular, the Federal Reserve decided on December 16, 2008 to lower the interest rate to the range of 0-0.25% and this constituted the significant jump in Figure 3.1. Given the previous definition of jumps, interventions of central banks are interpreted as jumps. This suggests that AJTSMs might outperform ATSMs in an economic environment with many unanticipated interventions of central banks. However, anticipated interventions do not constitute large yield changes. This is apparent in the short-term yield changes after 2009, which are flat due to the zero interest rate policies of central banks.

3.3 Summary statistics and stylized facts

The summary statistics of the weekly yield curve from January 6, 2006 to January 1, 2016 are reported in Table 3.1. In accordance with established stylized facts in the literature, the mean estimates of the yields, in Panel A, show that the yield curve is generally increasing and concave. A graphical representation of this fact can be found in Figure B.1 in Appendix B.

Panel A

Maturity	Mean (%)	SD (%)	Skewness	Kurtosis	ρ_1	ρ_{26}	ρ_{52}
3-month	1.6408	2.0524	1.1222	2.4850	0.9966	0.8482	0.6718
6-month	1.7868	1.9550	1.0779	2.4281	0.9968	0.8512	0.6800
1-year	1.6663	1.9087	1.1574	2.6081	0.9963	0.8457	0.6714
2-year	1.8583	1.7677	1.0885	2.5278	0.9956	0.8379	0.6911
3-year	2.1220	1.6622	0.9601	2.3660	0.9951	0.8301	0.7057
4-year	2.3834	1.5679	0.8431	2.2460	0.9946	0.8206	0.7071
5-year	2.6201	1.4813	0.7507	2.1672	0.9940	0.8096	0.6992
6-year	2.8260	1.4054	0.6755	2.1044	0.9935	0.7979	0.6870
7-year	2.9980	1.3421	0.6150	2.0530	0.9930	0.7863	0.6732
8-year	3.1388	1.2911	0.5682	2.0118	0.9926	0.7755	0.6589
9-year	3.2566	1.2498	0.5284	1.9777	0.9922	0.7657	0.6446
10-year	3.3567	1.2155	0.4947	1.9502	0.9920	0.7568	0.6311
15-year	3.6784	1.1144	0.3726	1.8623	0.9908	0.7265	0.5780
20-year	3.8046	1.0758	0.3287	1.8442	0.9904	0.7082	0.5448
25-year	3.8596	1.0555	0.3057	1.8357	0.9900	0.6949	0.5240
30-year	3.8894	1.0411	0.2900	1.8323	0.9896	0.6874	0.5131

Panel B

Maturity	Mean (%)	SD (%)	Skewness	Kurtosis	ρ_1	ρ_{26}	ρ_{52}
3-month	-0.0077	0.0970	-2.7751	39.413	0.6086	0.0152	0.0528
6-month	-0.0074	0.0783	-2.2179	22.874	0.5428	-0.0007	0.0366
1-year	-0.0076	0.0830	-0.5303	12.241	0.1219	-0.0081	0.0498
2-year	-0.0069	0.1018	0.2188	9.0549	-0.0391	-0.0418	0.0436
3-year	-0.0065	0.1131	0.3860	6.9851	-0.0481	-0.0664	0.0598
4-year	-0.0061	0.1197	0.4090	5.6894	-0.0422	-0.0689	0.0604
5-year	-0.0059	0.1249	0.4163	5.1153	-0.0560	-0.0733	0.0597
6-year	-0.0057	0.1276	0.4170	4.9190	-0.0680	-0.0819	0.0598
7-year	-0.0055	0.1290	0.3808	4.7880	-0.0721	-0.0783	0.0599
8-year	-0.0054	0.1293	0.3410	4.8621	-0.0745	-0.0839	0.0600
9-year	-0.0052	0.1300	0.2902	4.9018	-0.0804	-0.0837	0.0609
10-year	-0.0052	0.1290	0.2467	4.9442	-0.0764	-0.0910	0.0666
15-year	-0.0049	0.1283	0.1018	5.3798	-0.0964	-0.0832	0.0923
20-year	-0.0048	0.1272	0.1087	4.8059	-0.1011	-0.0881	0.0953
25-year	-0.0047	0.1283	-0.1145	5.7621	-0.1064	-0.0886	0.1009
30-year	-0.0047	0.1300	-0.1152	5.8883	-0.1226	-0.0930	0.1038

Table 3.1: Summary Statistics

Notes: This table shows the summary statistics of the weekly yield curve (**Panel A**) and the weekly changes in the yield curve (**Panel B**) from January 6, 2006 to January 1, 2016. The table provides the mean (in %), standard deviation (SD in %), skewness and kurtosis. The one-, 26- and 52-week auto-correlation coefficients are denoted by, respectively, ρ_1, ρ_{26} and ρ_{52}.

Moreover, standard deviation estimates in Panel A of Table 3.1 decrease with increasing maturities. This indicates that the short end of the yield curve is more volatile than its long end. Both the short and long end of the yield curve exhibit high persistence in their dynamics, emphasized by their high auto-correlation coefficients, ρ_1. The cross-correlation among yields with different maturities can be found in Panel A of Table B.2 in Appendix B for the complete sample. This table indicates, as expected, that cross-correlation among yields is high, ranging from 0.8191 to 0.9998.

In contrast to what the established literature suggests, the half- and one-year auto-correlation coefficients, ρ_{26} and ρ_{52}, indicate that the yield dynamics are more persistent at the short end of the yield curve than at the long end. A possible explanation for this anomaly is the aggressive intervention of central banks in recent years. Monetary policy is known to affect the short end of the yield curve more profoundly than the long end. Keeping interest rates low since 2008, arguably, constituted these contradictory results.

Panel B in Table 3.1 reports the summary statistics of the weekly changes. The mean estimates show that all yields, on average, display negative weekly changes, consistent with decreasing interest rates in the past 10 years. This is graphically represented in Figure 3.2 for the three-month, three-year, 10-year and 30-year yields. More interestingly, the standard deviation estimates show reverse patterns in Panel A and B in Table 3.1. Apparently, weekly changes in yields are smaller and more dispersed at the long end of the yield curve than the short end. Additionally, Figure 3.2 shows that the level of the yields diverged, since 2008. This phenomenon might also be explained by the low interest rates, set by the central banks. The skewness and kurtosis estimates in both Panels A and B in Table 3.1 conclude that the yields and the yield changes are not Normally distributed.

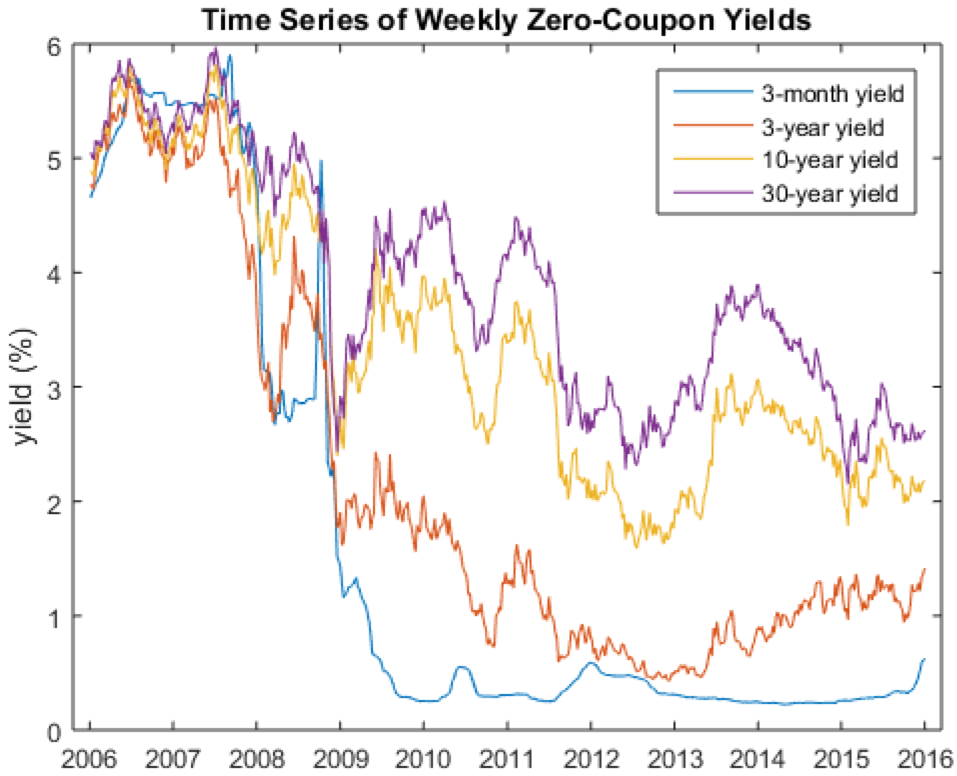


FIGURE 3.2: Time Series of Yields

Notes: This figure plots weekly yields (in %) from January 6, 2006 to January 1, 2016. The yields have three-month (*blue*), three-year (*red*), 10-year (*yellow*) and 30-year (*purple*) maturities.

In this paper, I investigate the performance of ATSMs and AJTSMs in different economic environments. Therefore, the sample is divided into three sub-samples. The first sub-sample ranges from January 6, 2006 to December 28, 2007 and captures the pre-crisis period. The second sub-sample ranges from January 4, 2008 to December 25, 2009 and captures the mid-crisis period. Lastly, the third sub-sample ranges from January 1, 2010 to January 1, 2016 and capture the post-crisis period. For the sake of brevity, only the mean and standard deviation (SD) of the yields are reported in Table 3.2 for each sub-sample. The complete summary statistics and cross-correlation of these sub-samples are reported in Tables B.1 and B.2 in Appendix B.

The mean estimates in Table 3.2 generally indicate that the average yield curve is increasing and concave, with an exception for the pre-crisis period. The pre-crisis period exhibits a flat yield curve. These findings are graphically represented in Figure B.2 in Appendix B. In accordance with previous findings for the complete sample, first-order auto-correlation coefficients in Panels A, B and C of Table B.1 also show that the yield dynamics, in every sub-sample, are highly persistent.

However, the sub-sample statistics additionally present remarkable results, regarding cross-correlations and standard deviations of the yields. Firstly, Panels B and C in Table B.2 show that cross-correlations are, relatively, high in the pre- and mid-crisis periods. The cross-correlation coefficients range from 0.3942 to 0.9998 and 0.4485 to 0.9996 for these periods, respectively. Panel D in Table B.2 indicates, however, that the cross-correlation relation is distorted after the crisis, with coefficients ranging from -0.3597 to 0.9997. Further analysis shows that negative, low cross-correlations are only apparent at the short end of the yield curve, that is for the three-month, six-month and 1-year yields.

Secondly, the SD estimates in Table 3.2 indicate that standard deviation dynamics changed from 2006 to 2016. During the pre-crisis period, the standard deviation had a humped term structure. That is, the standard deviation increased from low to medium maturities and decreased from medium to high maturities. During the mid-crisis period, standard deviation decreased as maturity increased. This is in contrast with the post-crisis period, when standard deviation increased as maturity increased. A comparison of SD estimates across periods shows that the standard deviation spiked in the mid-crisis period, as is expected during recessions.

Monetary policies, affecting the short end of the yield curve, can be interpreted as an explanation for these abnormalities. The data analysis concludes that this research is, partly, focused on the applicability of ATSMs and AJTSMs in unconventional economic environments. In this case, the unconventionality is caused by monetary policy experiments of central banks.

	Pre-Crisis		Mid-Crisis		Post-Crisis	
Maturity	Mean (%)	SD (%)	Mean (%)	SD (%)	Mean (%)	SD (%)
3-month	5.3962	0.2685	1.8514	1.2881	0.3272	0.0983
6-month	5.3432	0.2529	2.1231	1.1183	0.4975	0.1477
1-year	5.2004	0.3420	1.8308	1.0219	0.4412	0.1308
2-year	5.0695	0.3871	2.1903	0.8774	0.6848	0.2329
3-year	5.0576	0.3674	2.6018	0.7906	0.9908	0.3538
4-year	5.0841	0.3453	2.9323	0.7278	1.3072	0.4393
5-year	5.1225	0.3233	3.1833	0.6826	1.6047	0.4943
6-year	5.1597	0.3068	3.3847	0.6579	1.8680	0.5296
7-year	5.1937	0.2936	3.5445	0.6437	2.0897	0.5522
8-year	5.2253	0.2834	3.6676	0.6387	2.2725	0.5673
9-year	5.2549	0.2757	3.7677	0.6362	2.4254	0.5790
10-year	5.2822	0.2701	3.8507	0.6325	2.5553	0.5885
15-year	5.3822	0.2544	4.1141	0.6300	2.9698	0.6195
20-year	5.4279	0.2484	4.1906	0.6586	3.1392	0.6279
25-year	5.4404	0.2466	4.2128	0.6805	3.2189	0.6313
30-year	5.4409	0.2459	4.2273	0.6881	3.2635	0.6306

Table 3.2: Summary Statistics of Sub-Samples (Mean and Standard Deviation)

Notes: This table shows an excerpt of the summary statistics of the weekly yield curve for the pre-crisis, mid-crisis and post-crisis samples in Table B.1. The pre-crisis sample ranges from January 6, 2006 to December 28, 2007, the mid-crisis sample ranges from January 4, 2008 to December 25, 2009 and the post-crisis sample ranges from January 1, 2010 to January 1, 2016. The table provides the mean (in %) and standard deviation (SD in %) estimates of these samples.

4. *Results*

In this section, I present the results of this research. The first subsection provides the parameter estimates of the ATSMs and AJTSMs for the complete sample and, separately, for the pre-, mid- and post-crisis samples. The second subsection provides the goodness-of-fit measures to evaluate the in-sample performance of the models. Lastly, the third subsection reports the results of the backtests for VaR and ES estimates to evaluate the out-of-sample Risk Management performance.

4.1 Parameter estimation

The parameter estimates of ATSMs and AJTSMs for the complete sample and the pre-, mid- and post-crisis samples are presented in, respectively, Tables 4.1 and C.1, C.2 and C.3 in Appendix C. The following paragraphs provide an elaborate analysis of these results and focus on the empirical justification of the jump diffusion component.

Before I analyze the results, I discuss the application of the methodology to the empirical data. Firstly, I estimate the parameters of the ATSMs and AJTSMs by the QMLE procedure in conjunction with a Global Search procedure to avoid local optima. I find that this algorithm is robust in finding a global optimum in most cases.

Secondly, I experience difficulties in approximating the Hessian matrix and, thereby, the Fisher Information matrix by numerical optimization. This problem arises due to the incorporation of constraints on the parameters in the model. A constrained problem includes the constraints by means of Karish-Kuhn-Tucker multipliers, which ultimately affect the Hessian matrix. I optimize the constrained problem and calculate the Hessian matrix of the unconstrained problem, evaluated in the solution of the constrained problem. This results in a non-optimal Hessian matrix and in a negative semi-definite covariance matrix. To avoid negative variances, I report the standard errors from the nearest positive semi-definite covariance matrix, based on the Frobenius-norm.

Thirdly, I find that the one-, two- and three-factor ATSMs are identified, according to the observed Fisher Information matrix. This in accordance with the theoretical framework of Dai and Singleton (2000) as well. The ATSMs, in this paper, adhere to the canonical representation in Dai and Singleton (2000)[1] and are, therefore, classified as admissible ATSMs. These admissible ATSMs are econometrically identified. The AJTSMs are not classified within the theoretical framework of Dai and Singleton (2000), due to the jump diffusion component. The observed Fisher Information matrix, in combination with the Global Search algorithm, show that the one- and two-factor AJTSMs are identified. The three-factor model in the AJTSM-framework, however, has identification issues according to the Hessian matrix. The estimation algorithm reports a near-singular Hessian matrix during the optimization procedure, which indicates that certain parameters are not identified. The final Hessian matrix and the observed Fisher Information matrix, however, have full rank. Given this anomaly, the estimates of the three-factor AJTSM are suspect. I impose restrictions on the parameters to solve the identification problem. However, this results in either the two-factor AJTSM, the three-factor ATSM or an unrealistic economic model. Due to this identification problem and computational feasibility, I only include the three-factor models in the parameter estimation and goodness-of-fit analysis and exclude these models in the VaR and ES analysis.

Fourthly, given the identification problem, the robustness property of the Global Search algorithm does not apply to the three-factor AJTSMs. By definition of an unidentified model, I find that the estimated parameters are highly dependent on the initialization of the QMLE procedure. Remarkably, initialization of the QMLE procedure with the parameter estimates of the ATSMs result in sound economic parameter estimates of the AJTSMs[2]. Although this procedure is practical, it is not econometrically sound. I report the results of the three-factor AJTSMs to portray its anomalous behaviour and to, possibly, stimulate further research.

[1]ATSMs within the Vasicek framework comply with the constraints in Definition 1 of Dai and Singleton (2000).

[2]This holds for parameter estimation in the sub-samples and is not observed for parameter estimation in the complete sample.

4.1.1 Complete sample analysis

Panel A in Table 4.1 shows that all parameters of the one-, two- and three-factor models in the ATSM-framework are significantly different from zero on a 95% confidence level, except for the long-term mean (θ). Equivalently, all parameters of the one-, two- and three-factor models but five, in the AJTSM-framework, are significantly different from zero on a 95% confidence level in Panel B in Table 4.1. The five insignificant parameters include the long-term mean as well.

The mean-reversion coefficients (κ) determine the speed of the process to return to its long-term mean. For interpretational purposes, I modify this coefficient to the mean half life of the process, $-\frac{\ln(0.5)}{\kappa}$. This expression is interpreted as the expected time for the process to return halfway to its long-term mean, according to Lin and Yeh (2001). In the ATSM-framework, Panel A in Table 4.1 shows that the first factor has mean half lives of 4.5, 1.5 and 5.1 years in, respectively, the one-, two- and three-factor model. The third factor has a mean half life of 1.3 years in the three-factor model. The first and third factor, therefore, seem to exhibit some periodicity through their mean-reversion coefficients. In contrast, the mean half life of the second factor in the two- and three-factor model is relatively high, namely 69 years in both models. The mean-reversion estimates in the AJTSM-framework, in Panel B in Table 4.1, exhibit a similar pattern, with the exception of the three-factor model. The first factor has mean half lives of 4.5 and 2.0 years in the one- and two-factor model, but a mean half life of 21 years in the three-factor model. The second factor has a mean half life of 24 years in the two-factor model. In the three-factor model, however, the second and the third factor exhibit significant mean-reversion with mean half lives of, respectively, 2.2 and 0.4 years.

As previously stated, the long-term mean estimates (θ) in the ATSM-framework are not significantly different from zero on 95% confidence level. The insignificance of these estimates are, understandably, a result of the zero-interest rate policies of central banks, which covers a significant part of the complete sample. However, several factors in the AJTSMs have significantly large long-term mean estimates. The analysis of the jump component, partly, explains these results.

Panel A: ATSM

Θ	1-Factor model		2-Factor model				3-Factor model					
	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.155	0.009 (0.00)	0.455	0.014 (0.00)	0.010	0.002 (0.00)	0.136	0.002 (0.00)	0.010	0.002 (0.00)	0.539	0.030 (0.00)
θ	0.000	0.008 (1.00)	0.000	0.735 (1.00)	0.000	0.741 (1.00)	0.000	0.090 (1.00)	0.000	0.002 (1.00)	0.000	0.210 (1.00)
σ	0.007	0.001 (0.00)	0.014	0.004 (0.00)	0.011	0.002 (0.00)	0.075	0.003 (0.00)	0.011	0.005 (0.03)	0.020	0.010 (0.05)
ξ	-1.069	0.027 (0.00)	-0.508	0.118 (0.00)	-0.190	0.011 (0.00)	-0.228	0.032 (0.00)	-0.345	0.012 (0.00)	-1.256	0.401 (0.00)

Panel B: AJTSM

Θ	1-Factor model		2-Factor model				3-Factor model					
	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.153	0.010 (0.00)	0.351	0.006 (0.00)	0.029	0.001 (0.00)	0.033	0.000 (0.00)	0.312	0.002 (0.00)	1.789	0.017 (0.00)
θ	0.000	0.003 (1.00)	0.021	0.007 (0.00)	0.023	0.003 (0.00)	0.015	0.008 (0.05)	0.440	0.009 (0.00)	0.327	0.038 (0.00)
σ	0.001	0.000 (0.00)	0.021	0.001 (0.00)	0.011	0.000 (0.00)	0.019	0.000 (0.00)	0.007	0.001 (0.00)	0.139	0.019 (0.00)
λ	7.663	0.087 (0.00)	0.000	0.001 (1.00)	0.655	0.004 (0.00)	0.277	0.014 (0.00)	29.06	0.024 (0.00)	2.604	0.042 (0.00)
α	-0.002	0.000 (0.00)	0.011	0.314 (0.97)	0.001	0.000 (0.00)	0.036	0.001 (0.00)	-0.003	0.000 (0.00)	-0.129	0.013 (0.00)
β	0.003	0.002 (0.15)	0.904	0.147 (0.00)	0.002	0.001 (0.00)	0.004	0.001 (0.00)	0.004	0.001 (0.00)	0.046	0.014 (0.00)
ξ	-1.996	0.045 (0.00)	0.328	0.027 (0.00)	-0.067	0.003 (0.00)	-0.033	0.000 (0.00)	-0.582	0.000 (0.00)	-0.484	0.000 (0.00)

TABLE 4.1: Parameter Estimates Complete Sample

Notes: This table reports the parameter estimates, based on the QMLE procedure, in the ATSM-framework (**Panel A**) and in the AJTSM-framework (**Panel B**) using the weekly yield curve from January 6, 2006 to January 1, 2016 (522 observations). The table provides the parameter estimates of κ, θ, σ and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the ATSM-framework (**Panel A**). Additionally, the table provides the parameter estimates of κ, θ, σ, λ, α, β and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the AJTSM-framework (**Panel B**). In the models, I assume that the dynamics of the factors are described by a Vasicek model and the market price of risk of each factor is constant. The standard errors are based on the nearest symmetric-positive definite covariance matrix, derived from the unconstrained Hessian matrix. In the AJTSM-framework, I assume that jump risk is diversifiable and the Brownian Motion and Poisson process are independent as well.

Panel B in Table 4.1 shows that all jump intensity parameters (λ) are significantly different from zero on a 95% confidence level, with the exception of the first factor in the two-factor AJTSM. Factors, that exhibit significant jump intensities, generally have significant parameters for the distribution of the jump size, that is α and β. This indicates that there is empirical evidence for the presence of jumps in the riskless interest rate. Moreover, we identify three factors with particular importance for the jump diffusion, namely the first factor in the one-factor model and the second and third factor in the three-factor model. These factors exhibit, on average, 7.7, 29 and 2.6 jumps per year, respectively. The size of these jumps follow a Gaussian distribution with a negative mean. In the case of the three-factor AJTSM, large negative jumps provide a possible explanation for the positive long-term mean estimates. The volatility of the jump process is small and, generally, lower than the volatility estimates.

A comparison of the volatility estimates (σ) in Table 4.1 shows that the volatility of the non-jump processes decreases in the presence of a significant jump diffusion component. I reason that this advocates the presence of jumps in the riskless interest rates. That is, observations with jumps increase the volatility of a process as the model aims to capture these aberrant observations within its framework. By capturing the jumps in the aberrant observations with a jump diffusion process, the volatility estimate should decrease substantially. This result is observed for the factors with significant jumps. The third factor in the three-factor AJTSM is an exception to this general result as it has an unreasonably high volatility estimate. Most factors have volatility estimates of the same order as standard deviations of the yields in Panel A in Table 3.1 in the Data Analysis of Section 3. The irregularly high volatility estimate of the third factor, possibly, indicates the unreliability of the parameter estimates in the three-factor AJTSM.

Lastly, Table 4.1 shows that most factors have negative market prices of risk (ξ). Essentially, negative market prices of risk indicate negative Sharpe ratios, which would make these securities unattractive to investors. The market, however, infers that investors are willing to take on the risk in exchange for a default-free security. In light of the recession, this assessment is justified. However, I emphasize that this relation might be distorted in recent years by (short-term) monetary policies.

4.1.2 Sub-sample analysis

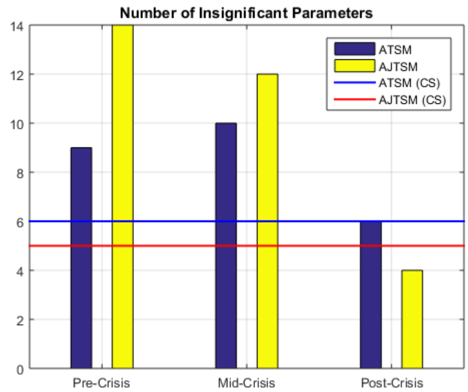
In order to investigate whether there is an empirical justification for the jump diffusion component in different economic environments, I compare the ATSMs and AJTSMs for different sub-samples. The subsequent paragraphs analyze the parameter estimates of the ATSMs and AJTSMs, that are specifically different in the pre-, mid- and post-crisis samples. To this extent, Figure 4.1 is used to graphically present the variations in the parameter estimates over time. The complete statistics of the parameter estimation of the sub-samples is reported in Tables C.1, C.2 and C.3 in Appendix C.

Figure (A) in 4.1 presents the number of insignificant parameters in the ATSMs and AJTSMs for the pre-, mid- and post-crisis samples as well as for the complete sample. The pre- and mid-crisis samples are substantially smaller (104 observations) than the post-crisis and complete sample (respectively, 314 and 522 observations). This affects the parameter estimates in the sense that the estimates are less accurate and have large standard errors. Figure (A) in 4.1 reflects this result. The figure shows that, on a 95% confidence level, the ATSMs and AJTSMs in the pre-crisis sample have, respectively, nine and 14 insignificant parameters. Equivalently, the ATSMs and AJTSMs in the mid-crisis sample have, respectively, 10 and 12 parameters that are not significantly different from zero. This is in contrast with the parameter estimates of the post-crisis and complete samples, which have 4 to 6 insignificant parameters. Tables C.1, C.2 and C.3 show that the insignificance is largely restricted to the long-term mean estimates in the ATSM-framework, while varying long-term mean estimates and jump parameters are insignificant in the AJTSM-framework.

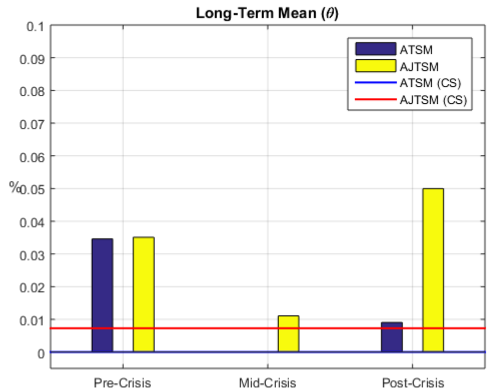
Figure (B) in 4.1 provides the average long-term mean estimates of the ATSMs and AJTSMs for the complete, pre-, mid- and post-crisis samples. In contrast to the complete sample results, Figure (B) in 4.1 shows that the average long-term mean estimates of the pre-crisis sample are significantly different from zero. The average long-term mean estimates range between 3%-4% for, both, ATSMs and AJTSMs. Table C.1 adheres to this result by reporting that all models, for the pre-crisis sample, have one factor with a significant positive long-term mean between the range of 5%-6%. This result is expected for the pre-crisis sample since Panel A in Table B.1 (Appendix B) shows a three-month interest rate of 5.4% during this period. Additionally, Panel B and C in Table B.1 show a decrease of the short-term interest rates over time. This is accurately captured by the ATSMs as Figure (B) in 4.1 illustrates the decrease in their average long-term mean estimates for the mid- and post-crisis periods. These results are more in accordance with the complete sample results and reflect the fall of the riskless interest rates during the crisis in 2008. We do, however, observe positive average long-term mean estimates for the AJTSMs during the mid- and post-crisis periods. Panel B in Table C.2 indicates that the long-term mean estimates are compensated by significant negative jumps during the mid-crisis period. This result does not hold during the pre-crisis period. This anomaly can be explained by the long-term mean estimates of the three-factor AJTSM, which have a large impact on the average long-term mean estimate, but are suspect.

In order to establish the empirical justification of jump diffusion components, I analyze the significance of the jump intensities (λ). Panel B in Tables C.1 and C.2 show that all, but one, jump intensities are significantly different from zero in the pre- and mid-crisis samples. Excluding the three-factor AJTSM due to identification issues, the jump intensities in the post-crisis sample are insignificant, according to Panel B in Table C.3. The jump intensities are modified in Figure (C) in 4.1, to graphically indicate the average number of jumps per year in each AJTSM. The figure shows that the average number of jumps in each AJTSM is large during the pre- and mid-crisis periods and small during the post-crisis period. These results attribute to the presence of jumps in the riskless interest rates during the pre- and mid-crisis periods and the absence of jumps during the post-crisis period. Figure 3.1 in the Data Analysis of Section 3 provides the justification for this observation. The figure shows that, while there are a significant number of jumps from 2006 to 2009, there are no observable jumps in the three-month yield changes during the post-crisis sample. Effectively, this result suggests that AJTSMs are not applicable in an economic environment without the presence of jumps. In the case of significant jumps, we observe that the mean jump size, generally, is non-positive. Moreover, we observe that volatility estimates of the non-jump process decrease in the presence of these jumps. This is in line with the complete sample results, but does not hold during the mid-crisis period as crises are regarded to be extremely volatile.

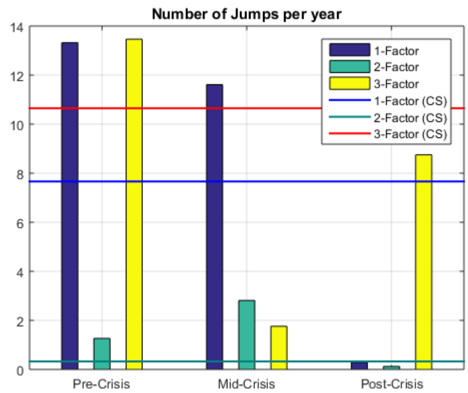
Lastly, Figure (D) in Table 4.1 plots the average market price of risk of the ATSMs and AJTSMs for the complete, pre-, mid- and post-crisis samples. The figure shows that that the average market prices of risk are positive, during the pre-crisis period, and negative during the mid- and post-crisis periods. Analogous to previous results, these findings emphasize the contrast of the pre-crisis results and the congruence of the midand pre-crisis results with the complete sample results. There are several interpretations of these discrepancies. An economic reason for the positive average market prices of risk is that the market did not highly demand default-free securities in the pre-crisis period (2006-2007). Investors demanded a return for the risk on these securities. Understandably, the market sentiment changed during and after the financial crisis of 2008 with regard to the default-free securities. The negative average market prices of risk in the mid-crisis period (2008-2009) demonstrate this change in market sentiment during the financial crisis of 2008. Investors fled to safe securities and the market's demand for default-free securities rose. The continuation of negative market prices of risk during the post-crisis period (2010-2016) emphasizes that the market sentiment has not shifted back in the recovery period after the financial crisis to pre-crisis levels.



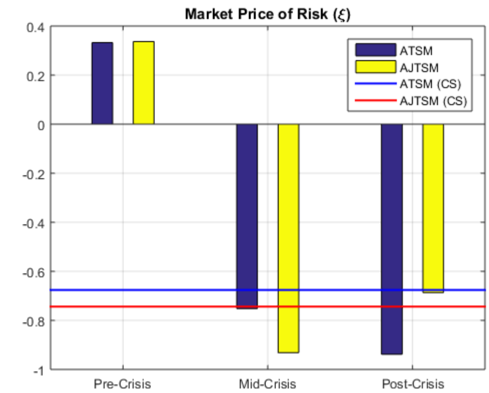
(A)



(B)



(C)



(D)

FIGURE 4.1: Key Results of Sub-Sample Analysis

Notes: These figures provide key results of the sub-sample parameter estimates in Tables C.1, C.2 and C.3 in Appendix C. **(A)** plots the number of insignificant parameters of all ATSMs (*blue*) and AJTSMs (*yellow*) for each sub-sample in a bar chart. The lines represent the number of insignificant parameters for the ATSM (*blue*) and AJTSM (*red*) in the complete sample. ATSMs have a total of 24 parameters in each sample, while AJTSMs have 42. **(B)** plots the average long-term mean estimates (θ) of the ATSMs (*blue*) and AJTSMs (*yellow*) for each sub-sample in a bar chart. The lines represent the average long-term mean estimates for the ATSM (*blue*) and AJTSM (*red*) in the complete sample. **(C)** lots the average number of jumps per year for the one- (*blue*), two- (*green*) and three-factor (*yellow*) AJTSM for each sub-sample in a bar chart. The lines represent the number of jumps per year for the one- (*blue*), two- (*green*) and three-factor (*red*) AJTSM in the complete sample. **(D)** plots the average market price of risk (ξ) of the ATSMs (*blue*) and AJTSMs (*yellow*) for each sub-sample in a bar chart. The lines represent the average market price of risk for the ATSM (*blue*) and AJTSM (*red*) in the complete sample.

4.2 Goodness-of-fit

In this subsection, I evaluate the performance of the ATSMs and AJTSMs in terms of their ability to fit the term structure of the riskless interest rates. Analogous to the previous subsection, I perform a complete sample analysis and focus on discrepancies with the complete sample results in a separate sub-sample analysis. Table 4.2 reports the AIC and BIC values of all models for each sample and Table 4.3 provides the goodness-of-fit measures[3] for the complete sample. Due to contrasting results, the goodness-of-fit measures for the pre-crisis sample are presented in Table 4.4 as well. For sake of brevity, I postpone the results for the mid- and post-crisis samples to Tables C.4 and C.5 in Appendix C.

4.2.1 Complete sample analysis

Firstly, Panel A in Table 4.3 shows that the performance of the one-factor ATSM and AJTSM, in terms of goodness-of-fit, is similar. Both one-factor models exhibit increasing MSPEs, RMSEs and MAEs and decreasing R_A^2's for yields with longer maturities. In particular, the negative R_A^2's assert that the model is misspecified for these yields. By definition of the R_A^2, the results indicate that one would obtain a better fit by using the average of the yield itself. Emphasizing the disparity in the fit of the yield curve, the MAE[4] ranges from 2.505 to 43.89 basis points for the one-factor ATSM, while it ranges from 2.511 to 43.75 basis points for the one-factor AJTSM. This pattern of increasing difficulty to fit long-term yields results from two separate issues. First, the restriction of the model to one factor only enables the model to capture the level of the short rate. I choose to approximate this factor by modeling the three-month yield without error. Consequently, the short-term yields are fitted better than long-term yields. Second, the one-factor Vasicek model is known to capture $\mathbb{P}$-dynamics much better than the $\mathbb{Q}$-dynamics (see Bolder (2001)). Effectively, the increasing MAEs portray this empirical fact. The market price of risk of one factor is not able to capture the internal consistency relation across the yield curve in the Vasicek framework. To improve this characteristic within the framework, factors are added. The results of multi-factor models are discussed

[3] The goodness-of-fit measures for the 3-month, 30-year and 10-year yield are not provided, because I model these yields without error for, respectively, the one-, two- and three-factor model.

[4] I put more emphasis on the MAE as this measure uses equal weighting of the errors and portrays a fairer impression of the average error than the MSPE and RMSE.

in the two- and three-factor analysis.

Although the performance of both one-factor models is similar, Panel A in Table 4.3 indicates a marginal difference in favor of the one-factor AJTSM. This is confirmed by the AIC and BIC values in Table 4.2. However, all goodness-of-fit measures show that the one-factor model, in both frameworks, is not able to fit the entire yield curve well.

	ATSM						AJTSM					
	1-Factor model		2-Factor model		3-Factor model		1-Factor model		2-Factor model		3-Factor model	
Sample	AIC	BIC	AIC	BIC	AIC	BIC	AIC	BIC	AIC	BIC	AIC	BIC
Complete	-49334	-49317	-68357	-68323	-79773	-79722	-50372	-50342	-63736	-63676	52705	52614
Pre-Crisis	-11470	-11459	-15995	-15974	-19003	-18971	-11547	-11528	-14414	-14377	-18354	-18298
Mid-Crisis	-9688.4	-9677.4	-13024	-13003	-16050	-16018	-9798.3	-9779.8	-12066	-12029	-15582	-15527
Post-Crisis	-18557	-18572	-28431	-28382	-30212	-29019	-19439	-19413	-27194	-27142	-27769	-27690

Table 4.2: AIC and BIC values of ATSMs and AJTSMs

Notes: This table reports statistics concerning the relative fit of the one-, two- and three-factor models in the ATSM- and AJTSM-framework. The table provides the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for the complete sample and the pre-, mid- and post-crisis samples.

Secondly, Panel B in Table 4.3 shows a significant improvement in the performance of the two-factor model in, both, the ATSM- and AJTSM-framework in comparison with the one-factor model. There is a pronounced decrease in the MSPEs, RMSEs and MAEs of both two-factor models. The MAE ranges from 2.374 to 12.52 basis points for the two-factor ATSM and from 2.426 to 17.32 basis points for the two-factor AJTSM. Although both models are performing as well for short-term yields as the one-factor model, the magnitude of the MSPEs, RMSEs and MAEs of long-term yields is greatly diminished for the two-factor models. Moreover, the error measures do not increase with maturity. Instead, their evolution can be described as a concave function of maturity. These improvements in the results are attributable to the fact that the two-factor models use the three-month and 30-year yields to approximate the factors. The model is able to capture the short- and long-term dynamics, which can be transformed to and interpreted as the level and the slope of the short rate.

Although an additional factor improves in-sample fitting performance of both models, the negative R_A^2's still indicate that the model is misspecified for a majority of the yields. This holds for the two-factor AJTSM and, to a lesser extent, for the two-factor ATSM. In contrast to the one-factor models, the two-factor models exhibit a clear difference in their performances. The two-factor ATSM outperforms the two-factor AJTSM and, as is shown in Table 4.2, the AIC and BIC favor the two-factor ATSM over the one-factor ATSM and one- and two-factor AJTMS.

	Panel A: 1-Factor model							
	MSPE		RMSE		MAE		R_A^2	
τ	ATSM	AJTSM	ATSM	AJTSM	ATSM	AJTSM	ATSM	AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.007	0.007	0.085	0.085	**2.505**	2.511	0.998	0.998
1-year	0.077	0.077	0.277	0.277	4.354	**4.345**	0.979	0.979
2-year	0.621	**0.620**	0.788	0.788	7.854	**7.844**	**0.801**	0.799
3-year	**1.906**	1.907	1.381	1.381	10.49	10.49	**0.310**	0.300
4-year	**4.132**	4.145	**2.033**	2.036	**12.62**	12.63	**-0.681**	-0.709
5-year	**7.352**	7.391	**2.712**	2.719	**14.50**	14.53	**-2.351**	-2.414
6-year	**11.57**	11.65	**3.402**	3.413	**16.28**	16.31	**-4.859**	-4.978
7-year	**16.68**	16.80	**4.085**	4.099	**17.89**	17.93	**-8.262**	-8.454
8-year	**22.39**	22.53	**4.732**	4.747	**19.34**	19.36	**-12.43**	-12.70
9-year	**28.78**	28.93	**5.365**	5.378	**20.68**	20.69	**-17.42**	-17.77
10-year	**35.82**	35.95	**5.985**	5.996	21.98	**21.96**	**-23.25**	-23.66
15-year	84.23	**83.74**	9.178	**9.151**	28.00	**27.91**	**-66.83**	-67.35
20-year	166.5	**164.6**	12.90	**12.83**	33.47	**33.34**	**-142.9**	-143.2
25-year	309.7	**306.5**	17.60	**17.51**	38.76	**38.62**	**-277.0**	-277.9
30-year	525.8	**522.1**	22.93	**22.85**	43.89	**43.75**	**-484.1**	-487.3

	Panel B: 2-Factor model							
	MSPE		RMSE		MAE		R_A^2	
τ	ATSM	AJTSM	ATSM	AJTSM	ATSM	AJTSM	ATSM	AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	**0.006**	0.007	**0.078**	0.081	**2.374**	2.426	0.998	0.998
1-year	0.099	0.099	**0.314**	0.315	4.950	**4.881**	**0.973**	0.972
2-year	**0.736**	0.750	**0.858**	0.866	**8.323**	8.365	**0.764**	0.753
3-year	**1.685**	1.749	**1.298**	1.323	**10.30**	10.55	**0.390**	0.349
4-year	**2.491**	2.700	**1.578**	1.643	**11.47**	11.94	**-0.013**	-0.129
5-year	**2.914**	3.418	**1.707**	1.849	**12.01**	12.69	**-0.328**	-0.601
6-year	**3.087**	4.045	**1.757**	2.011	**12.19**	13.14	**-0.563**	-1.104
7-year	**3.233**	4.781	**1.798**	2.186	**12.23**	13.54	**-0.795**	-1.727
8-year	**3.386**	5.604	**1.840**	2.367	**12.24**	13.93	**-1.031**	-2.455
9-year	**3.555**	6.503	**1.885**	2.550	**12.26**	14.42	**-1.276**	-3.278
10-year	**3.729**	7.465	**1.931**	2.732	**12.28**	14.98	**-1.524**	-4.192
15-year	**3.968**	11.28	**1.992**	3.359	**12.52**	17.32	**-2.195**	-8.336
20-year	**1.490**	7.833	**1.221**	2.799	**9.976**	15.87	**-0.288**	-5.955
25-year	**0.606**	2.441	**0.779**	1.562	**7.721**	11.74	**0.456**	-1.251
30-year	-	-	-	-	-	-	-	-

	Panel C: 3-Factor model							
	MSPE		RMSE		MAE		R_A^2	
τ	ATSM	AJTSM	ATSM	AJTSM	ATSM	AJTSM	ATSM	AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	**0.008**	0.965	**0.089**	0.982	**2.605**	9.888	**0.998**	0.737
1-year	**0.052**	17.34	**0.228**	4.164	**4.092**	20.34	**0.986**	-3.959
2-year	**0.287**	100.0	**0.535**	10.00	**6.591**	31.52	**0.908**	-32.36
3-year	**0.565**	184.9	**0.752**	13.60	**7.984**	36.76	**0.795**	-68.74
4-year	**0.739**	222.2	**0.860**	14.90	**8.670**	38.49	**0.699**	-93.16
5-year	**0.709**	206.2	**0.842**	14.36	**8.625**	37.79	**0.677**	-96.93
6-year	**0.531**	155.8	**0.729**	12.48	**8.040**	35.24	**0.731**	81.21
7-year	**0.323**	95.49	**0.568**	9.772	**7.102**	31.18	**0.821**	-54.24
8-year	**0.148**	43.86	**0.384**	6.623	**5.801**	25.67	**0.911**	-26.42
9-year	**0.040**	10.90	**0.199**	3.301	**4.109**	18.12	**0.975**	-6.269
10-year	-	-	-	-	-	-	-	-
15-year	**0.562**	156.0	**0.750**	12.49	**8.132**	35.24	**0.547**	-129.9
20-year	**0.635**	232.0	**0.797**	15.23	**8.230**	38.92	**0.452**	-207.9
25-year	**0.299**	102.5	**0.547**	10.12	**6.703**	31.72	**0.732**	-94.84
30-year	-	-	-	-	-	-	-	-

Table 4.3: Goodness-of-Fit Measures of Complete Sample

Notes: This table reports and compares the goodness-of-fit measures for ATSMs and AJTSMs for the yield curve from January 6, 2006 to January 1, 2016. **Panel A** presents the results of the one-factor ATSM and AJTSM, **Panel B** presents the results of the two-factor ATSM and AJTSM, and **Panel C** presents the results of the three-factor ATSM and AJTSM. The MSPEs are denoted in squared basis points, the RMSEs and MAEs are denoted in basis points, R_A^2 denotes the adjusted R^2 and τ denotes the maturity. The bold numbers show the best model (ATSM or AJTSM) with respect to the goodness-of-fit measure.

Thirdly, Panel C in Table 4.3 demonstrates the effect of the parameter estimation problems in the three-factor model within the AJTSM-framework. The MSPEs, RMSEs, MAEs and R_A^2's of the three-factor AJTSM are worse than all other models in both frameworks. The negative performance is recognized by the AIC and BIC values in Table 4.2 as well. In contrast to the three-factor AJTSM, the three-factor ATSM performs particularly well. The MAE ranges from 2.605 to 8.670 basis points and shows that the three-factor ATSM performs slightly worse for the six-month yield than the one- and two-factor models. This, however, is compensated by low MSPEs, RMSEs and MAEs for all remaining yields. As a matter of fact, the three-factor ATSM is the best model to fit the yield curve for the complete sample, according to Table 4.2. Essentially, this result shows that increasing the number of factors increases the fit of the entire yield curve. The positive R_A^2's, in Panel C in Table 4.3, confirm this statement. Multiple factors are able to capture the market price of risk and, thereby, increase the model's ability to capture the internal consistency relation and, consequently, the $\mathbb{Q}$-dynamics. These results might be attributed to the incorporation of the short-term, mid-term and long-term dynamics in the three-factor model by modeling, respectively, the three-month, 10-year and 30-year yields without error. These dynamics are transformed to and interpreted as the level, slope and curvature of the short rate.

4.2.2 Sub-sample analysis

In the following paragraphs, I analyze the results of the pre-, mid- and post-crisis samples. I focus on contrasting results for different economic environments and discrepancies with the complete sample analysis. The results of the pre-crisis sample are reported in Table 4.4. The results of the mid- and post-crisis samples are reported in Tables C.4 and C.5 in Appendix C.

Panel A in Table 4.4 shows that the results of the pre-crisis sample for the one-factor ATSM and AJTSM are comparable to the complete sample results. Both one-factor models perform poorly in terms of the goodness-of-fit measures and the one-factor AJTSM is marginally superior to the one-factor ATSM. This is confirmed by the AIC and BIC values in Table 4.2. More importantly, the MSPEs, RMSEs and MAEs show increasing difficulty to fit long-term yields and the negative R_A^2's indicate that both models are misspecified for the long-term yields.

In contrast to the complete sample results, Table 4.4 shows that the magnitude of the poor performance of the one-factor models is much smaller in the pre-crisis period. The MAE ranges from 2.157 to 25.90 basis points for the one-factor ATSM and from 2.141 to 26.02 basis points for the one-factor AJTSM. This is a sharp decrease in comparison with the MAEs of the complete sample results. A similar pattern holds for the MSPEs, RMSEs and R_A^2's of the one-factor model.

The two-factor models in Panel B in Table 4.4 improve upon the one-factor models in accordance with the results of the complete sample. However, the performance of the two-factor models is comparable to the performance of the three-factor ATSM in the complete sample results. Both two-factor models have remarkably low MSPEs, RMSEs and MAEs and the R_A^2's indicate that a large part of the variance of the yields is explained. Apparently, the Vasicek framework is more applicable in the pre-crisis period.

Panel B in Table 4.4 shows that the two-factor ATSM and AJTSM have comparable results for the six-month and one-year yields. The models diverge in performance when the maturity increases. The two-factor ATSM outperforms the two-factor AJTSM in terms of goodness-of-fit. This is also deduced from the results in Table 4.2 and observed in the complete sample results.

Panel C in Table 4.4 provides the results for the three-factor models. As mentioned in the Parameter Estimation subsection, the results of the three-factor AJTSM are provided to portray its anomalous behaviour. Despite the fact that the model is unidentified, the results indicate that the performance of the three-factor AJTSM is comparable to the three-factor ATSM. The three-factor ATSM, however, is identified and performs particularly well. The MAE of this model ranges from 1.629 to 5.138 basis points, which is significantly lower than the MAEs in the complete sample results. Additionally, the MSPEs, RMSEs and R_A^2's indicate that the three-factor ATSM is superior in fitting the yield curve during the pre-crisis period. Analogous to the complete sample, the AIC and BIC values in Table 4.2 confirm its superiority over the one- and two-factor models.

Panel A: 1-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R_A^2 ATSM	R_A^2 AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.004	0.004	0.061	**0.060**	2.157	**2.141**	0.999	0.999
1-year	0.113	**0.111**	0.336	**0.333**	4.751	**4.691**	**0.969**	0.967
2-year	0.805	**0.788**	0.897	**0.888**	8.028	**7.949**	**0.742**	0.729
3-year	1.698	**1.664**	1.303	**1.290**	9.860	**9.780**	**0.385**	0.354
4-year	2.504	**2.462**	1.582	**1.569**	11.03	**10.98**	**-0.018**	-0.074
5-year	3.107	**3.071**	1.763	**1.752**	11.84	**11.79**	**-0.416**	-0.502
6-year	3.635	**3.617**	1.907	**1.902**	12.44	**12.39**	**-0.840**	-0.965
7-year	**4.146**	4.155	**2.036**	2.038	12.93	**12.90**	**-1.302**	-1.474
8-year	**4.706**	4.748	**2.169**	2.179	13.37	**13.36**	**-1.823**	-2.056
9-year	**5.392**	5.473	**2.322**	2.339	**13.81**	13.84	**-2.452**	-2.759
10-year	**6.293**	6.417	**2.509**	2.533	**14.28**	14.33	**-3.259**	-3.660
15-year	**15.06**	15.44	**3.881**	3.930	**17.41**	17.53	**-11.13**	-12.34
20-year	**31.08**	31.73	**5.575**	5.633	**20.76**	20.89	**-25.85**	-28.41
25-year	**51.49**	52.40	**7.176**	7.238	**23.55**	23.68	**-45.22**	-49.46
30-year	**75.05**	76.18	**8.663**	8.728	**25.90**	26.02	**-68.24**	-74.41

Panel B: 2-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R_A^2 ATSM	R_A^2 AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.002	0.002	0.049	0.049	2.058	**1.974**	0.999	0.999
1-year	**0.065**	0.078	**0.255**	0.279	4.424	**4.312**	**0.982**	0.975
2-year	**0.035**	0.487	**0.188**	0.698	**6.328**	7.053	**0.887**	0.820
3-year	**0.622**	0.935	**0.788**	0.967	**7.252**	8.567	**0.775**	0.608
4-year	**0.802**	1.268	**0.895**	1.126	**7.820**	9.372	**0.674**	0.403
5-year	**0.858**	1.419	**0.926**	1.191	**8.092**	9.738	**0.609**	0.252
6-year	**0.851**	1.460	**0.923**	1.208	**8.216**	9.872	**0.569**	0.145
7-year	**0.792**	1.414	**0.890**	1.189	**8.172**	9.866	**0.561**	0.092
8-year	**0.705**	1.313	**0.840**	1.146	**8.028**	9.721	**0.577**	0.088
9-year	**0.605**	1.175	**0.778**	1.084	**7.812**	9.483	**0.613**	0.129
10-year	**0.525**	1.052	**0.725**	1.025	**7.628**	9.240	**0.644**	0.176
15-year	**0.271**	0.833	**0.521**	0.913	**6.464**	8.407	**0.781**	0.223
20-year	**0.104**	0.869	**0.322**	0.932	**4.923**	8.766	**0.910**	0.131
25-year	**0.043**	0.374	**0.206**	0.612	**4.050**	7.168	**0.962**	0.611
30-year	-	-	-	-	-	-	-	-

Panel C: 3-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R_A^2 ATSM	R_A^2 AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.002	**0.001**	0.043	**0.032**	1.868	**1.629**	1.000	1.000
1-year	**0.021**	0.023	**0.146**	0.150	3.626	**3.449**	**0.994**	0.992
2-year	**0.059**	0.092	**0.244**	0.302	**4.443**	4.778	**0.981**	0.963
3-year	**0.069**	0.123	**0.262**	0.351	**4.488**	5.093	**0.975**	0.944
4-year	**0.067**	0.127	**0.260**	0.357	**4.408**	5.138	**0.973**	0.935
5-year	**0.052**	0.102	**0.228**	0.319	**4.113**	4.894	**0.976**	0.942
6-year	**0.038**	0.073	**0.195**	0.271	**3.784**	4.522	**0.981**	0.953
7-year	**0.022**	0.042	**0.148**	0.205	**3.202**	3.902	**0.988**	0.971
8-year	**0.010**	0.019	**0.100**	0.136	**2.587**	3.172	**0.994**	0.986
9-year	**0.003**	0.005	**0.051**	0.067	**1.851**	2.245	**0.998**	0.996
10-year	-	-	-	-	-	-	-	-
15-year	**0.036**	0.069	**0.191**	0.263	**3.677**	4.576	**0.971**	0.930
20-year	**0.049**	0.118	**0.221**	0.344	**4.070**	5.233	**0.958**	0.872
25-year	**0.045**	0.065	**0.213**	0.254	**4.233**	4.475	**0.959**	0.927
30-year	-	-	-	-	-	-	-	-

Table 4.4: Goodness-of-Fit Measures of Pre-Crisis Sample

Notes: This table reports and compares the goodness-of-fit measures for ATSMs and AJTSMs for the yield curve from January 6, 2006 to December 28, 2007. **Panel A** presents the results of the one-factor ATSM and AJTSM, **Panel B** presents the results of the two-factor ATSM and AJTSM, and **Panel C** presents the results of the three-factor ATSM and AJTSM. The MSPEs are denoted in squared basis points, the RMSEs and MAEs are denoted in basis points, R_A^2 denotes the adjusted R^2 and τ denotes the maturity. The bold numbers show the best model (ATSM or AJTSM) with respect to the goodness-of-fit measure.

The results of the mid- and post-crisis samples are more aligned with the complete sample results than with the pre-crisis sample results. Panel A in Tables C.4 and C.5 show that the one-factor models are misspecified due to largely negative R_A^2's, in accordance with the complete and pre-crisis sample results. However, the magnitude of the R_A^2's is comparable with the complete sample results. Moreover, the MAE ranges from 2.128 to 41.51 basis points in the ATSM-framework and from 2.142 to 41.49 basis points in the AJTSM-framework for, both, the mid- and post-crisis samples. This is similar to the performance of the one-factor models in the complete sample. Additionally, Table 4.2 shows that the one-factor AJTSM is preferred over the one-factor ATSM in the mid- and post-crisis samples.

An identical pattern is observed for the two-factor models. The two-factor models are not misspecified in the pre-crisis sample and, correspondingly, have low MSPEs, RMSEs and MAEs. Although the R_A^2's of the two-factor models in the mid- and post-crisis samples indicate an improvement over the one-factor models, the models remain misspecified. The corresponding MSPEs, RMSEs and MAEs are comparable to the observed measures in the complete sample results. Furthermore, the AIC and BIC values in Table 4.2 show that the two-factor ATSMs are preferred over the two-factor AJTSMs in the mid- and post-crisis samples. This is equivalent to the complete and pre-crisis sample results.

The results of the three-factor AJTSM in the mid- and post-crisis sample is presented to portray the anomalous behaviour of the unidentified model. Similar to the pre-crisis sample results, the three-factor ATSM results are comparable to the results of this erroneous model. However, the three-factor ATSM is identified and performs particularly well across the yield curve. The MSPEs, RMSEs and MAEs are relatively low for, both, the mid- and post-crisis samples. Moreover, the R_A^2's in Panel C in Tables C.4 and C.5 show that the model is not misspecified for the yield curve. Table 4.2 shows that the three-factor ATSM outperforms the one- and two-factor ATSMs and AJTSMs. More importantly, the three-factor ATSM is superior in fitting the entire yield curve in the mid- and post-crisis periods.

The complete and sub-sample analysis show that the three-factor ATSM is superior in fitting the yield curve in the pre-, mid- and post-crisis periods. The analysis indicates that the one-factor models are inadequate in fitting the entire yield curve in the pre-, mid- and post-crisis periods. This holds for the two-factor models in the midand post-crisis periods as well. The two-factor model in the pre-crisis period, however, is able to fit the entire yield curve. For this pre-crisis sample, the two-factor ATSM consistently outperforms the two-factor AJTSM. Moreover, the goodness-of-fit measures indicate that the pre-crisis period is fitted with the smallest average error in comparison with the other periods. This indicates that the Vasicek model, either with or without jump diffusions, is more applicable in the pre-crisis period than the mid- and post-crisis periods. A possible explanation for these results is the monetary policy of central banks since the financial crisis of 2008. These interventions distort the internal consistency relation across the yield curve, which is captured by the market price of risk. The results show that multiple factors are needed to capture this relation and fit the entire yield curve appropriately.

I emphasize that the analysis of the results in this subsection are based on in-sample goodness-of-fit statistics. The Vasicek model is known to capture the $\mathbb{P}$-dynamics better than the $\mathbb{Q}$-dynamics. The results show that increasing the number of factors increases the ability to capture the $\mathbb{Q}$-dynamics. However, this is restricted to the in-sample performance of the models. The out-of-sample performance exposes the inferiority of the Vasicek framework. This is discussed in the Value-at-Risk and Expected Shortfall subsection.

4.3 Value-at-Risk and Expected Shortfall

In this subsection, I analyze the results of the backtests on the one-week-ahead VaR and ES estimates from January 4, 2008 to January 1, 2016. The analysis provides a comparison of the performance of the ATSMs and AJTSMs from a Risk Management perspective. The VaR and ES estimates are constructed by Monte Carlo simulation of the one- and two-factor models. The parameters of these models are estimated using an expanding window. I provide the results of the Correct Unconditional Coverage, Independence and Correct Conditional Coverage tests in Tables 4.5 and 4.6. The results of the Saddlepoint Approximation and Box-Pierce tests are presented in Table 4.7. Figures C.1 and C.2, in Appendix C, provide graphical representations of the one-week-ahead $VaR_{1\%}$ and $ES_{1\%}$ estimates for the one- and 30-year interest rate swaps.

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Panel A and B in Table 4.5 show that the Correct Unconditional Coverage, Independence and Correct Conditional Coverage tests significantly reject their corresponding hypotheses for the one-factor ATSM and AJTSM. That is, the Correct Unconditional Coverage test rejects that the fraction of VaR violations is equal to the nominal coverage probability and the Independence test rejects that the VaR violations are independent. Consequently, the Correct Conditional Coverage tests rejects that the VaR estimates exhibit both properties. Essentially, the results show that the VaR estimates of the one-factor ATSM and AJTSM are inadequate for Risk Management purposes.

Further analysis shows that the one-factor models are able to capture the $\mathbb{P}$-dynamics of the one-year interest rate swap, according to Figures (A) and (C) in C.1 in Appendix C. However, the level of the VaR estimates is not conservative enough, resulting in a large number of violations and the rejection of the Correct Unconditional Coverage. Figures (A) and (C) in C.1 also show that the VaR violations occur in groups rather than separately, rejecting the Independence property of the VaR estimates. Additionally, the level of the $VaR_{1\%}$ estimates in the figures indicate that the one-factor AJTSM describes a wider distribution for the one-year interest rate swap rate[5]. Consequently, Panel A and B in Table 4.5 show that the number of violations is larger for $VaR_{1\%}$ estimates of the one-factor ATSM than for $VaR_{1\%}$ estimates of the one-factor AJTSM. The opposite is true for the $VaR_{5\%}$ and $VaR_{10\%}$ estimates. This indicates that the relative performance of the one-factor ATSM is worse than the performance of the one-factor AJTSM for the $VaR_{1\%}$ estimates, but better for the $VaR_{5\%}$ and $VaR_{10\%}$ estimates.

[5]That is, the one-year interest rate swap rate can take a wider range of values in the one-factor AJTSM than in the one-factor ATSM.

Panel A: ATSM

	$VaR_{1\%}$				$VaR_{5\%}$				$VaR_{10\%}$			
τ	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}
1-year	123	632.3 (0.00)	221.3 (0.00)	853.6 (0.00)	260	1020 (0.00)	354.7 (0.00)	1374 (0.00)	291	853.5 (0.00)	296.7 (0.00)	1150 (0.00)
2-year	242	1663 (0.00)	466.3 (0.00)	2130 (0.00)	263	1040 (0.00)	384.9 (0.00)	1425 (0.00)	276	765.2 (0.00)	348.0 (0.00)	1113 (0.00)
3-year	211	1368 (0.00)	357.3 (0.00)	1725 (0.00)	230	822.1 (0.00)	396.1 (0.00)	1218 (0.00)	237	557.6 (0.00)	393.0 (0.00)	950.6 (0.00)
4-year	171	1014 (0.00)	398.5 (0.00)	1413 (0.00)	186	563.8 (0.00)	373.4 (0.00)	937.2 (0.00)	199	384.1 (0.00)	327.4 (0.00)	711.4 (0.00)
5-year	151	849.3 (0.00)	392.6 (0.00)	1242 (0.00)	156	409.3 (0.00)	385.9 (0.00)	795.2 (0.00)	167	259.5 (0.00)	362.3 (0.00)	621.7 (0.00)
6-year	143	785.5 (0.00)	349.2 (0.00)	1135 (0.00)	153	394.8 (0.00)	371.5 (0.00)	766.3 (0.00)	158	228.1 (0.00)	365.4 (0.00)	593.5 (0.00)
7-year	141	769.8 (0.00)	357.7 (0.00)	1128 (0.00)	146	361.8 (0.00)	368.8 (0.00)	730.5 (0.00)	153	211.3 (0.00)	388.2 (0.00)	599.5 (0.00)
8-year	135	723.2 (0.00)	367.4 (0.00)	1091 (0.00)	144	352.5 (0.00)	377.9 (0.00)	730.4 (0.00)	146	188.8 (0.00)	380.3 (0.00)	569.1 (0.00)
9-year	134	715.5 (0.00)	366.0 (0.00)	1082 (0.00)	141	338.8 (0.00)	392.9 (0.00)	731.7 (0.00)	144	182.5 (0.00)	409.3 (0.00)	591.8 (0.00)
10-year	134	715.5 (0.00)	383.3 (0.00)	1099 (0.00)	137	320.8 (0.00)	375.4 (0.00)	696.2 (0.00)	141	173.3 (0.00)	380.7 (0.00)	554.0 (0.00)
15-year	108	523.3 (0.00)	294.0 (0.00)	817.3 (0.00)	114	224.4 (0.00)	327.0 (0.00)	551.4 (0.00)	116	104.1 (0.00)	319.6 (0.00)	423.7 (0.00)
20-year	95	433.4 (0.00)	277.1 (0.00)	710.5 (0.00)	99	168.2 (0.00)	296.8 (0.00)	465.1 (0.00)	101	69.70 (0.00)	290.3 (0.00)	359.9 (0.00)
25-year	79	329.2 (0.00)	205.3 (0.00)	534.4 (0.00)	82	111.9 (0.00)	212.9 (0.00)	324.8 (0.00)	85	39.40 (0.00)	233.2 (0.00)	272.6 (0.00)
30-year	48	151.5 (0.00)	112.7 (0.00)	264.2 (0.00)	50	31.20 (0.00)	119.5 (0.00)	150.7 (0.00)	53	3.121 (0.08)	113.7 (0.00)	116.8 (0.00)

Panel B: AJTSM

	$VaR_{1\%}$				$VaR_{5\%}$				$VaR_{10\%}$			
τ	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}
1-year	30	68.26 (0.00)	105.7 (0.00)	174.0 (0.00)	326	1522 (0.00)	188.5 (0.00)	1710 (0.00)	374	1450 (0.00)	100.9 (0.00)	1551 (0.00)
2-year	192	1196 (0.00)	298.9 (0.00)	1495 (0.00)	286	1205 (0.00)	304.0 (0.00)	1509 (0.00)	322	1052 (0.00)	249.3 (0.00)	1302 (0.00)
3-year	174	1040 (0.00)	377.9 (0.00)	1417 (0.00)	248	938.5 (0.00)	364.5 (0.00)	1303 (0.00)	271	736.9 (0.00)	323.0 (0.00)	1060 (0.00)
4-year	151	849.3 (0.00)	369.4 (0.00)	1218 (0.00)	207	682.5 (0.00)	319.0 (0.00)	1001 (0.00)	232	533.2 (0.00)	288.3 (0.00)	821.5 (0.00)
5-year	139	754.2 (0.00)	402.9 (0.00)	1157 (0.00)	165	453.7 (0.00)	340.2 (0.00)	793.9 (0.00)	187	335.0 (0.00)	338.1 (0.00)	673.1 (0.00)
6-year	128	669.7 (0.00)	413.7 (0.00)	1083 (0.00)	159	423.9 (0.00)	366.3 (0.00)	790.2 (0.00)	172	277.6 (0.00)	381.7 (0.00)	659.4 (0.00)
7-year	124	639.7 (0.00)	393.5 (0.00)	1033 (0.00)	154	399.6 (0.00)	389.2 (0.00)	788.8 (0.00)	161	238.4 (0.00)	384.3 (0.00)	622.7 (0.00)
8-year	116	580.7 (0.00)	385.0 (0.00)	967.7 (0.00)	146	361.8 (0.00)	380.3 (0.00)	742.1 (0.00)	156	221.3 (0.00)	368.4 (0.00)	589.8 (0.00)
9-year	116	580.7 (0.00)	348.9 (0.00)	929.6 (0.00)	146	361.8 (0.00)	386.9 (0.00)	748.6 (0.00)	148	195.1 (0.00)	377.5 (0.00)	572.6 (0.00)
10-year	109	530.4 (0.00)	312.9 (0.00)	843.3 (0.00)	141	338.8 (0.00)	369.0 (0.00)	707.8 (0.00)	145	185.6 (0.00)	374.0 (0.00)	559.6 (0.00)
15-year	94	426.7 (0.00)	250.7 (0.00)	677.4 (0.00)	118	240.3 (0.00)	323.1 (0.00)	563.4 (0.00)	123	122.1 (0.00)	310.4 (0.00)	432.5 (0.00)
20-year	87	380.4 (0.00)	207.2 (0.00)	587.6 (0.00)	103	182.7 (0.00)	316.7 (0.00)	499.3 (0.00)	105	78.29 (0.00)	309.5 (0.00)	387.8 (0.00)
25-year	61	221.6 (0.00)	154.7 (0.00)	376.3 (0.00)	84	118.1 (0.00)	227.2 (0.00)	345.3 (0.00)	88	44.54 (0.00)	250.4 (0.00)	294.9 (0.00)
30-year	42	121.7 (0.00)	124.8 (0.00)	246.6 (0.00)	53	37.11 (0.00)	113.7 (0.00)	150.8 (0.00)	62	9.594 (0.00)	118.8 (0.00)	128.3 (0.00)

Table 4.5: Value-at-Risk Backtests of One-Factor Models

Notes: This table reports the results of the Value-at-Risk (VaR) backtests for the one-factor model in the ATSM-framework (**Panel A**) and in the AJTSM-framework (**Panel B**). The VaR estimates are constructed for the interest rate swaps from January 4, 2008 to January 1, 2016 (418 observations) by means of Monte Carlo simulation. These VaR estimates have three different nominal coverage probabilities, namely $\gamma = \{1\%, 5\%, 10\%\}$. This table provides the number of VaR violations (T_1) and the likelihood-ratio test statistics of the Correct Unconditional Coverage (LR_{UC}), Independence (LR_{IND}) and Correct Conditional Coverage (LR_{CC}) tests. The corresponding p-values are reported in brackets.

Using Panel A and B in Table 4.5 and Figures (B) and (D) in C.1, the analysis of the rejected Correct Unconditional Coverage and Independence properties and the difference between the one-factor ATSM and AJTSM can be extrapolated to the 30-year interest rate swap as well. However, an analysis of the performance of the one-factor models across the yield curve is more insightful. Figures (B) and (D) in C.1 show that both models are unable to capture the $\mathbb{P}$-dynamics of the 30-year interest rate swap. This is related to the fact that one-factor models are not able to capture the $\mathbb{Q}$-dynamics with the market price of risk of a single factor. Similarly, in the previous subsection, the goodness-of-fit measures show that the one-factor models perform poorly and are misspecified for long-term yields. Figures (B) and (D) in C.1 adhere to these results. They indicate that the dynamics of the 30-year interest rate swap are represented by, approximately, a straight line as the $\mathbb{Q}$-dynamics are not captured. According to the one-factor models, it is unclear how the long-term yields are related to the short rate and evolve over time. Effectively, the VaR and ES estimates are more static and conservative as the maturity of the interest rate swap increases, resulting in a decrease in the number of violations. The number of violations, however, remain substantial and result in the inadequacy of both one-factor models for Value-at-Risk estimation.

Table 4.6 presents the results of the two-factor ATSM and AJTSM. Similar to the results of the one-factor models, Correct Unconditional Coverage and Independence are rejected for the one- to 10-year interest rate swap VaR estimates of the two-factor ATSM and for the one- to five-year interest rate swap VaR estimates of the two-factor AJTSM. The six- and seven-year interest rate swap VaR estimates of the two-factor AJTSM exhibit Correct Unconditional Coverage, Independence and Correct Conditional Coverage in several instances. However, the results suggest that this is most likely a coincidence due to the decrease in VaR violations as the maturity increases. The large number of rejected hypotheses attest to this assumption. For the remaining VaR estimates of the interest rate swaps, there are no VaR violations. These results imply that the two-factor ATSM and AJTSM are inadequate for Risk Management purposes as well.

A comparison of the two-factor ATSM and AJTSM shows that, in terms of VaR violations, the two-factor AJTSM outperforms the two-factor ATSM. Moreover, a comparison of the one- and two-factor models shows that the two-factor ATSM performs worse than the one-factor ATSM and that the two-factor AJTSM is superior to all one- and two-factor models. However, the overall performance of the one- and two-factor models is poor and not useful for Value-at-Risk in Risk Management.

To gain further insight in the absence of VaR violations for a number of long-term yields, I analyze Figures C.2 in Appendix C. In contrast to the one-factor models, the figures show that the two-factor models are able to capture the $\mathbb{P}$-dynamics of the one- and 30-year interest rate swaps to a certain degree. This, indirectly, implies that the two-factor models are able to capture the $\mathbb{Q}$-dynamics to a larger extent than the one-factor models. However, Figures (B) and (D) in C.2 indicate that the VaR estimates are very conservative for the 30-year interest rate swap. Apparently, the two-factor models do not capture the $\mathbb{Q}$-dynamics to such an extent that the VaR estimates are accurate. This results in zero VaR violations for long-term yields and inadequacy of ATSMs and AJTSMs in Value-at-Risk estimation.

Panel A: ATSM

τ	$VaR_{1\%}$				$VaR_{5\%}$				$VaR_{10\%}$			
	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}
1-year	221	1461 (0.00)	356.1 (0.00)	1817 (0.00)	329	1547 (0.00)	296.3 (0.00)	1843 (0.00)	351	1262 (0.00)	246.7 (0.00)	1509 (0.00)
2-year	345	2791 (0.00)	290.1 (0.00)	3081 (0.00)	361	1835 (0.00)	251.1 (0.00)	2086 (0.00)	380	1503 (0.00)	141.1 (0.00)	1644 (0.00)
3-year	341	2742 (0.00)	276.4 (0.00)	3019 (0.00)	362	1845 (0.00)	234.4 (0.00)	2079 (0.00)	378	1485 (0.00)	119.7 (0.00)	1605 (0.00)
4-year	330	2610 (0.00)	250.4 (0.00)	2861 (0.00)	356	1788 (0.00)	219.3 (0.00)	2007 (0.00)	372	1433 (0.00)	108.5 (0.00)	1541 (0.00)
5-year	311	2391 (0.00)	336.7 (0.00)	2727 (0.00)	339	1633 (0.00)	248.8 (0.00)	1882 (0.00)	354	1285 (0.00)	166.6 (0.00)	1452 (0.00)
6-year	281	2062 (0.00)	301.2 (0.00)	2363 (0.00)	312	1406 (0.00)	290.1 (0.00)	1696 (0.00)	328	1094 (0.00)	245.3 (0.00)	1339 (0.00)
7-year	206	1322 (0.00)	301.5 (0.00)	1623 (0.00)	252	965.2 (0.00)	303.1 (0.00)	1268 (0.00)	285	817.6 (0.00)	268.5 (0.00)	1086 (0.00)
8-year	137	738.6 (0.00)	341.6 (0.00)	1080 (0.00)	171	484.3 (0.00)	315.7 (0.00)	800.0 (0.00)	199	384.1 (0.00)	318.2 (0.00)	702.3 (0.00)
9-year	89	393.5 (0.00)	315.7 (0.00)	709.1 (0.00)	112	216.6 (0.00)	301.7 (0.00)	518.2 (0.00)	133	149.6 (0.00)	358.1 (0.00)	507.8 (0.00)
10-year	48	151.5 (0.00)	256.3 (0.00)	407.8 (0.00)	78	99.92 (0.00)	299.3 (0.00)	399.2 (0.00)	85	39.41 (0.00)	318.3 (0.00)	357.7 (0.00)
15-year	0	-	-	-	0	-	-	-	0	-	-	-
20-year	0	-	-	-	0	-	-	-	0	-	-	-
25-year	0	-	-	-	0	-	-	-	0	-	-	-
30-year	0	-	-	-	0	-	-	-	0	-	-	-

Panel B: AJTSM

τ	$VaR_{1\%}$				$VaR_{5\%}$				$VaR_{10\%}$			
	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}	T_1	LR_{UC}	LR_{IND}	LR_{CC}
1-year	27	56.40 (0.00)	20.54 (0.00)	76.39 (0.00)	119	244.3 (0.00)	172.3 (0.00)	416.5 (0.00)	179	303.9 (0.00)	238.1 (0.00)	542.1 (0.00)
2-year	120	610.1 (0.00)	125.5 (0.00)	735.6 (0.00)	215	729.9 (0.00)	245.1 (0.00)	975.4 (0.00)	254	644.3 (0.00)	236.3 (0.00)	880.6 (0.00)
3-year	118	595.3 (0.00)	107.2 (0.00)	702.6 (0.00)	210	700.1 (0.00)	260.5 (0.00)	960.6 (0.00)	244	592.6 (0.00)	336.6 (0.00)	929.2 (0.00)
4-year	86	373.9 (0.00)	76.54 (0.00)	450.4 (0.00)	170	479.2 (0.00)	240.4 (0.00)	719.6 (0.00)	204	405.3 (0.00)	309.9 (0.00)	715.2 (0.00)
5-year	36	93.95 (0.00)	24.47 (0.00)	118.3 (0.00)	92	144.1 (0.00)	119.9 (0.00)	263.9 (0.00)	127	132.8 (0.00)	183.6 (0.00)	316.4 (0.00)
6-year	5	0.200 (0.70)	4.100 (0.04)	4.300 (0.12)	32	5.401 (0.02)	26.23 (0.00)	31.69 (0.00)	48	1.211 (0.32)	38.83 (0.00)	39.82 (0.00)
7-year	0	-	-	-	2	29.30 (0.00)	-	-	8	44.16 (0.00)	2.200 (0.13)	46.39 (0.00)
8-year	0	-	-	-	0	-	-	-	0	-	-	-
9-year	0	-	-	-	0	-	-	-	0	-	-	-
10-year	0	-	-	-	0	-	-	-	0	-	-	-
15-year	0	-	-	-	0	-	-	-	0	-	-	-
20-year	0	-	-	-	0	-	-	-	0	-	-	-
25-year	0	-	-	-	0	-	-	-	0	-	-	-
30-year	0	-	-	-	0	-	-	-	0	-	-	-

Table 4.6: Value-at-Risk Backtests of Two-Factor Models

Notes: This table reports the results of the Value-at-Risk (VaR) backtests for the two-factor model in the ATSM-framework (**Panel A**) and in the AJTSM-framework (**Panel B**). The VaR estimates are constructed for the interest rate swaps from January 4, 2008 to January 1, 2016 (418 observations) by means of Monte Carlo simulation. These VaR estimates have three different nominal coverage probabilities, namely $\gamma = \{1\%, 5\%, 10\%\}$. This table provides the number of VaR violations (T_1) and the likelihood-ratio test statistics of the Correct Unconditional Coverage (LR_{UC}), Independence (LR_{IND}) and Correct Conditional Coverage (LR_{CC}) tests. The corresponding p-values are reported in brackets. The test statistics, and their corresponding p-values, can not be computed for VaR estimates with zero VaR violations.

Expected Shortfall

The analysis of the ES estimates in Figures C.1 and C.2 demonstrate that the ES estimates follow the same dynamics as their corresponding VaR estimates. That is, the ES estimates of one-factor models are able to capture the $\mathbb{P}$-dynamics of short-term yields, but not of the long-term yields. The ES estimates of two-factor models are able to capture the $\mathbb{P}$-dynamics of short-term yields and the long-term yields, to a larger extent than the ES estimates of the one-factor models. As mentioned in the previous analysis of the VaR estimates, this result is related to the inability of the models to capture the $\mathbb{Q}$-dynamics of the yield curve. More importantly, Figures C.1 and C.2 graphically confirm that the one- and two-factor ATSMs and AJTSMs are inadequate for Risk Management purposes with respect to Expected Shortfall estimation.

Panel A in Table 4.7 provides quantitative results to confirm this inadequacy by rejecting the correct unconditional coverage and independence properties of the ES estimates of the one-factor models. The table shows that the Saddlepoint Approximation tests reject the correct unconditional coverage for all one-factor models as all p-values are zero. Similarly, the Box-Pierce tests reject that there is no autocorrelation in the cumulative violation process of the ES estimates. The results indicate that there is no difference in the poor performance of the one-factor ATSM and AJTSM.

Similar to the VaR backtests for the two-factor models, Panel B in Table 4.7 shows that a large number of tests can not be computed due to scarcity of VaR violations. The feasible tests indicate that there is a distinction between the performance of the two-factor ATSM and AJTSM. The results of the two-factor ATSM show that all, but two, Saddlepoint Approximation and Box-Pierce tests return zero p-values. Therefore, the ES estimates of the two-factor ATSM do not exhibit correct unconditional coverage or independence. This reflects the poor performance of the two-factor ATSM VaR estimates. The results of the two-factor AJTSM, however, indicate a diverse performance with respect to correct unconditional coverage and independence. The Box-Pierce tests show that the $ES_{1\%}$ estimates of the two-factor AJTSM exhibit independence. This holds, to a lesser extent, for the $ES_{5\%}$ estimates as well. The Saddlepoint Approximation tests reject the correct conditional coverage property for the majority of ES estimates of the two-factor AJTSM. Exceptions occur for several cases with a small number of VaR violations.

Panel A: One-Factor Models

	1-Factor ATSM						1-Factor AJTSM					
	$ES_{1\%}$		$ES_{5\%}$		$ES_{10\%}$		$ES_{1\%}$		$ES_{5\%}$		$ES_{10\%}$	
τ	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}
1-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
15-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30-year	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Panel B: Two-Factor Models

	2-Factor ATSM						2-Factor AJTSM					
	$ES_{1\%}$		$ES_{5\%}$		$ES_{10\%}$		$ES_{1\%}$		$ES_{5\%}$		$ES_{10\%}$	
τ	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}	p_{BP}	p_{SPA}
1-year	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.01	0.00	0.00	0.00
2-year	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.00	1.00	0.00	0.00	0.00
3-year	0.00	0.00	0.00	0.00	0.00	0.00	0.99	0.00	0.94	0.00	0.00	0.00
4-year	0.00	0.00	0.00	0.00	0.00	0.00	0.99	0.00	0.10	0.00	0.00	0.00
5-year	0.00	0.00	0.00	0.00	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.04
6-year	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.01	0.00	0.96
7-year	0.00	0.00	0.00	0.00	0.00	0.00	-	-	0.00	0.12	0.00	1.00
8-year	0.00	0.00	0.00	0.00	0.00	0.00	-	-	-	-	-	-
9-year	0.00	0.00	0.00	0.91	0.00	0.00	-	-	-	-	-	-
10-year	0.00	0.00	0.00	0.85	0.00	0.00	-	-	-	-	-	-
15-year	-	-	-	-	-	-	-	-	-	-	-	-
20-year	-	-	-	-	-	-	-	-	-	-	-	-
25-year	-	-	-	-	-	-	-	-	-	-	-	-
30-year	-	-	-	-	-	-	-	-	-	-	-	-

TABLE 4.7: Expected Shortfall Backtests of One- and Two-Factor Models

Notes: This table reports the results of the Expected Shortfall (ES) backtests for the one-factor (**Panel A**) and two-factor (**Panel B**) model in, both, the ATSM-framework and the AJTSM-framework. The ES estimates are constructed for the interest rate swaps from January 4, 2008 to January 1, 2016 (418 observations) by means of Monte Carlo simulation. These ES estimates have three different nominal coverage probabilities, namely $\gamma = \{1\%, 5\%, 10\%\}$. This table provides the p-values of the Saddlepoint Approximation test (p_{SPA}) and the Box-Pierce test (p_{BP}). The test statistics, and their corresponding p-values, can not be computed for ES estimates with zero VaR violations.

These results indicate that both two-factor models are inadequate for Expected Shortfall estimation in Risk Management. Although the performance of the two-factor ATSM is worse than the two-factor AJTSM, the results refute that either the ATSMs or AJTSMs are able to accurately estimate Expected Shortfall values.

5. *Conclusion*

This paper considers whether there is a significant difference between Affine Term Structure Models (ATSMs) and Affine Jump Term Structure Models (AJTSMs) with respect to their in-sample and out-of-sample performance for the riskless interest rates. In this research, I derive and propose to use one-, two- and three-factor ATSMs and AJTSMs within the Vasicek framework for the riskless interest rates. Subsequently, I test whether the jump diffusion component is empirically justified in each AJTSM. I propose goodness-of-fit measures to compare the in-sample performance of the ATSMs and AJTSMs. Lastly, I compute and backtest Value-at-Risk (VaR) and Expected Shortfall (ES) estimates of interest rate swaps to compare the out-of-sample performance of the ATSMs and AJTSMs.

Firstly, I find that the significance of the jump parameters in the AJTSMs indicates the empirical justification of jump diffusion components in the riskless interest rate process. In economic terms, this result attributes to the presence of jumps in the riskless interest rate process. I find significant jump diffusion components in the parameter estimation for the complete sample and, more specifically, for the pre- and mid-crisis samples. For these samples, each model in the AJTSM-framework incorporates, at least, one factor with a significant jump diffusion component. In the presence of significant jump diffusion components, I find that volatility estimates decrease, except during the mid-crisis period. In contrast to the complete sample and the pre- and mid-crisis samples, I find that the post-crisis period does not exhibit significant jump parameters. That is, there is no presence of jumps in the riskless interest rate process after the financial crisis of 2008.

Secondly, I find that, from an economic perspective, the three-factor ATSM is superior in fitting the entire yield curve for the complete sample and all sub-samples. The goodness-of-fit measures indicate that the one-factor ATSM and AJTSM fit the yield curve poorly and are misspecified for long-term yields. Although the two-factor ATSM and AJTSM exhibit an improvement in their performance over the one-factor models, both two-factor models are misspecified for long-term yields as well. The two-factor models for the pre-crisis sample are exceptions to this result. Ultimately, the three-factor ATSM fits the entire yield curve well and the R_A^2's indicate that the model is not misspecified. A comparison of ATSMs and AJTSMs shows that the AJTSM-framework performs marginally better than the ATSM-framework in the case of one-factor models. However, the ATSM-framework performs substantially better in the case of the two-factor models. This result is observed for the complete sample and all sub-samples. Due to identification problems, I am not able to estimate the parameters of the three-factor AJTSM.

Thirdly, the results in this paper show that ATSMs and AJTSMs perform poorly in terms of Value-at-Risk and Expected Shortfall estimation. From a Risk Management perspective, ATSMs and AJTSMs are inadequate for interest rate swaps. I find that VaR and ES estimates of one-factor ATSMs and AJTSMs are inaccurate for short-term interest rate swaps and are not able to capture the dynamics of the long-term interest rate swaps. Although two-factor ATSMs and AJTSMs improve upon the one-factor ATSMs and AJTSMs in terms of VaR and ES estimates, Correct Unconditional Coverage, Independence, Correct Conditional Coverage, Saddlepoint Approximation and Box-Pierce tests reject the application of these two-factor models in Risk Management as well. There is, however, a significant difference in the performance of the two-factor models in favor of the two-factor AJTSM.

In conclusion, this paper confirms that there is empirical justification for a jump diffusion component in the riskless interest rate process. The addition of this jump diffusion component in Affine Term Structure Models does not result in a significant improvement in the in-sample performance. That is, the three-factor ATSM is superior in fitting the entire yield curve in-sample. Lastly, the results in this paper establish that ATSMs and AJTSMs are not applicable in out-of-sample VaR and ES estimation in Risk Management.

6. *Limitations and Further Research*

In this research, I encountered certain limitations with respect to the data, models and parameter estimation. Firstly, I emphasize that the data for the pre- and mid-crisis samples contains 104 observations for each period. This amount is too small for accurate parameter estimation and is reflected in the insignificance of parameters in these samples. Secondly, I encountered long computation times for the parameter estimation of the three-factor ATSM. In order to compute VaR and ES estimates, the parameter estimation procedure reiterates 418 times, which is not feasible within the time-frame of this research.

This paper is first in its use of AJTSMs on the riskless interest rates and in its application of ATSMs and AJTSMs in Risk Management. Therefore, there are several directions for further research to explore these new fields of application. The first direction for further research is to investigate whether the outperformance of the two-factor AJTSM by the two-factor ATSM, in the presence of jumps, is due to the approximation of the AJTSM. This can be done by means of a simulation study. One would expect that each model performs best for its own Data Generating Process (DGP). In case the ATSMs outperform AJTSMs for DGPs, that are based on AJTSMs, the approximation of the AJTSM is not accurate. The second direction is to perform a similar research with ATSMs and AJTSMs in a different framework than the Vasicek-framework. In order to improve upon capturing the $\mathbb{Q}$-dynamics, one might investigate the Hull-White or Heath-Jarrow-Morton framework in this respect. More interestingly, the assumption of diversifiable jump risk could be relaxed, using Baz and Das (1996), to increase the flexibility in the $\mathbb{Q}$-dynamics. Similarly, the Cox-Ingersoll-Ross framework would be attractive in order to provide a more realistic model for the $\mathbb{P}$-dynamics of the riskless interest rates. The third direction for further research is to investigate the anomalous behaviour of the QMLE procedure for parameter estimation of the three-factor AJTSM. The fourth direction for further research would be to compare the VaR and ES estimates of ATSMs and AJTSMs to VaR and ES estimates of a vector autoregressive (VAR) model. The VAR model is statistically optimal and, therefore, is a credible benchmark for the VaR and ES estimates. Besides VaR and ES estimation, which focus on the variance of the interest rate swap forecasts, the out-of-sample performance could be based on the RMSE and MAE, which concern the mean of the interest rate swap forecasts.

A. *Models and Methodology Appendix*

A.1 PDE derivation for ATSMs

In this subsection, I derive the ODEs for $A(\tau)$ and $B(\tau)$ in the pricing equation:

$$P(t,\tau) = e^{A(\tau)-B(\tau)r(t)}. \tag{A.1}$$

Firstly, I derive the partial derivatives of $P(t,\tau)$:

$$\begin{aligned} P_t &= (A_t(\tau) - B_t(\tau)r(t))e^{A(\tau)-B(\tau)r(t)} \\ &= (A_t(\tau) - B_t(\tau)r(t))P(t,\tau), \\ P_r &= -B(\tau)e^{A(\tau)-B(\tau)r(t)} \\ &= -B(\tau)P(t,\tau), \\ P_{rr} &= B^2(\tau)e^{A(\tau)-B(\tau)r(t)} \\ &= B^2(\tau)P(t,\tau). \end{aligned} \tag{A.2}$$

Substitution of these derivatives in the PDE yields:

$$\begin{aligned} 0 &= P_t + (\mu - \xi(t)\sigma)P_r + \frac{\sigma^2}{2}P_{rr} - r(t)P(t,\tau) \\ 0 &= (A_t(\tau) - B_t(\tau)r(t))P(t,\tau) - (\mu - \xi(t)\sigma)B(\tau)P(t,\tau) + \frac{\sigma^2}{2}B^2(\tau)P(t,\tau) \\ &\quad - r(t)P(t,\tau) \\ 0 &= (A_t(\tau) - B_t(\tau)r(t)) - (\mu - \xi(t)\sigma)B(\tau) + \frac{\sigma^2}{2}B^2(\tau) - r(t) \\ 0 &= A_t(\tau) - (1 + B_t(\tau))r(t) - (\mu - \xi(t)\sigma)B(\tau) + \frac{\sigma^2}{2}B^2(\tau). \end{aligned} \tag{A.3}$$

The Vasicek model is defined by:

$$\begin{aligned} \mu &\equiv \kappa\left(\theta - r(t)\right), \\ \sigma &\equiv \sigma. \end{aligned} \tag{A.4}$$

Using Equation (A.4), the PDE simplifies to:

$$\begin{aligned} 0 &= A_t(\tau) - (1 + B_t(\tau))r(t) - (\kappa\left(\theta - r(t)\right) - \xi(t)\sigma)B(\tau) + \frac{\sigma^2}{2}B^2(\tau) \\ 0 &= A_t(\tau) - (1 + B_t(\tau))r(t) - \kappa\left(\tilde{\theta} - r(t)\right)B(\tau) + \frac{\sigma^2}{2}B^2(\tau) \\ 0 &= A_t(\tau) - \kappa\tilde{\theta}B(\tau) + \frac{\sigma^2}{2}B^2(\tau) - (1 + B_t(\tau) - \kappa B(\tau))r(t). \end{aligned} \tag{A.5}$$

A.2 PDE derivation for AJTSMs

For notational convenience, I transform the PDE in Equation (2.24) from coefficients in the $\mathbb{P}$-measure to coefficients in the $\mathbb{Q}$-measure. This is done by substituting $\tilde{\theta} = \theta - \frac{\xi(t)\sigma}{\kappa}$. The PDE simplifies to:

$$\begin{aligned} 0 = P_t + \kappa(\tilde{\theta} - r(t))P_r + &\frac{\sigma^2 P_{rr}}{2} - r(t)P(t,\tau) \\ &+ \lambda P(t,\tau)\left[-\alpha B(\tau) + \frac{\beta^2 + \alpha^2}{2}B(\tau)^2\right]. \end{aligned} \tag{A.6}$$

I restate the partial derivatives of the pricing function, $P(t,\tau)$, for convenience:

$$\begin{aligned} P_t &= (A_t(\tau) - B_t(\tau)r(t))e^{A(\tau) - B(\tau)r(t)} \\ &= (A_t(\tau) - B_t(\tau)r(t))P(t,\tau), \\ P_r &= -B(\tau)e^{A(\tau) - B(\tau)r(t)} \\ &= -B(\tau)P(t,\tau), \\ P_{rr} &= B^2(\tau)e^{A(\tau) - B(\tau)r(t)} \\ &= B^2(\tau)P(t,\tau), \end{aligned} \tag{A.7}$$

and substitution in Equation (A.6) yields:

$$\begin{aligned}
0 =&(A_t(\tau) - B_t(\tau)r(t))P(t,\tau) - \kappa(\tilde{\theta} - r(t))B(\tau)P(t,\tau) \\
&+ \frac{\sigma^2 B^2(\tau)P(t,\tau)}{2} - r(t)P(t,\tau) + \lambda P(t,\tau)\left[-\alpha B(\tau) + \frac{\beta^2+\alpha^2}{2}B(\tau)^2\right], \\
0 =&A_t(\tau) - B_t(\tau)r(t) - \kappa(\tilde{\theta} - r(t))B(\tau) + \frac{\sigma^2 B^2(\tau)}{2} - r(t) \\
&+ \lambda\left[-\alpha B(\tau) + \frac{\beta^2+\alpha^2}{2}B(\tau)^2\right], \\
0 =&A_t(\tau) + \frac{\sigma^2 B^2(\tau)}{2} - \kappa\tilde{\theta}B(\tau) + \lambda\left[-\alpha B(\tau) + \frac{\beta^2+\alpha^2}{2}B(\tau)^2\right] \\
&+ (-1 - B_t(\tau) + \kappa B(\tau))r(t).
\end{aligned} \tag{A.8}$$

A.3 Approximation of $f(r_t|r_{t-1})$

The approximation of $f(r_t|r_{t-1})$ is based on taking the expectation of $\sum_{j=1}^n e^{\kappa\psi_j}$ and $\sum_{j=1}^n e^{2\kappa\psi_j}$. Assuming that jumps in the riskless interest rate are equally spread over time, Lin and Yeh (1999) show:

$$E\left[\sum_{j=1}^n e^{\kappa\psi_j}\right] = \sum_{j=1}^n E\left[e^{\kappa\psi_j}\right] = nE\left[e^{\kappa\psi}\right] = n\int_{t-1}^t e^{\kappa\psi}\frac{1}{\Delta t}\psi = \frac{n}{\kappa\Delta t}\left(e^{\kappa\Delta t} - 1\right). \tag{A.9}$$

Equivalently,

$$\begin{aligned}
E\left[\sum_{j=1}^n e^{2\kappa\psi_j}\right] &= \sum_{j=1}^n E\left[e^{2\kappa\psi_j}\right] = nE\left[e^{2\kappa\psi}\right] = n\int_{t-1}^t e^{2\kappa\psi}\frac{1}{\Delta t}\psi \\
&= \frac{n}{2\kappa\Delta t}\left(e^{2\kappa\Delta t} - 1\right).
\end{aligned} \tag{A.10}$$

Substituting Equations (A.9) and (A.10) in the conditional pdf (2.38) results in the approximation of the conditional pdf (2.40).

A.4 Multi-factor framework: System of equations

The goal of solving the system of equations is to obtain an expression for the n factors. In this subsection, I derive these expressions for the two- and three- factor model. To this extent, I use the equations that are modeled without measurement errors. For the two-factor model, I consider the following system of equations:

$$\begin{aligned} x_1 &= A(\tau_1) - B_1(\tau_1)y_1 - B_2(\tau_1)y_2, \\ x_2 &= A(\tau_2) - B_1(\tau_2)y_1 - B_2(\tau_2)y_2, \end{aligned} \tag{A.11}$$

where $x_i = \ln\left[P(t, \tau_i)\right]$ and $y_j = y_j(t)$ for brevity. Solving for y_1 and y_2, yields:

$$\begin{aligned} \text{(Express } y_1 \text{ in } y_2\text{)} \;\rightarrow y_1 &= \frac{A(\tau_1) - x_1 - B_2(\tau_1)y_2}{B_1(\tau_1)}, \\ \text{(Substitution)} \;\rightarrow x_2 &= A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1 - B_2(\tau_1)y_2}{B_1(\tau_1)} - B_2(\tau_2)y_2, \\ x_2 &= A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} \\ &\quad + y_2\left(\frac{B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_1)} - B_2(\tau_2)\right), \\ \text{(Solve for } y_2\text{)} \;\rightarrow \hat{y}_2 &= \frac{A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} - x_2}{B_2(\tau_2) - \frac{B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_1)}}, \\ \hat{y}_1 &= \frac{A(\tau_1) - x_1 - B_2(\tau_1)\hat{y}_2}{B_1(\tau_1)}. \end{aligned} \tag{A.12}$$

For the three-factor model, I consider the following system of equations:

$$\begin{aligned} x_1 &= A(\tau_1) - B_1(\tau_1)y_1 - B_2(\tau_1)y_2 - B_3(\tau_1)y_3, \\ x_2 &= A(\tau_2) - B_1(\tau_2)y_1 - B_2(\tau_2)y_2 - B_3(\tau_2)y_3, \\ x_3 &= A(\tau_3) - B_1(\tau_3)y_1 - B_2(\tau_3)y_2 - B_3(\tau_3)y_3, \end{aligned} \tag{A.13}$$

where x_i and y_j are similarly defined.

Solving for y_1, y_2 and y_3, I first express y_1 and y_2 in terms of y_3:

$$\begin{aligned}
y_1 &= \frac{A(\tau_1) - x_1 - B_2(\tau_1)y_2 - B_3(\tau_1)y_3}{B_1(\tau_1)}, \\
x_2 &= A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1 - B_2(\tau_1)y_2 - B_3(\tau_1)y_3}{B_1(\tau_1)} \\
&\quad - B_2(\tau_2)y_2 - B_3(\tau_2)y_3, \\
x_2 &= A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} + y_2\left(\frac{B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_1)} - B_2(\tau_2)\right) \\
&\quad + y_3\left(\frac{B_1(\tau_2)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_2)\right), \\
y_2 &= \left[A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} - x_2 + y_3\left(\frac{B_1(\tau_2)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_2)\right)\right] \\
&\quad \times \left(B_2(\tau_2) - \frac{B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_1)}\right)^{-1}.
\end{aligned} \tag{A.14}$$

Substituting y_1 and y_2 in x_3, yields:

$$\begin{aligned}
x_3 = A(\tau_3) - B_1(\tau_3)&\frac{A(\tau_1) - x_1 - B_2(\tau_1)y_2 - B_3(\tau_1)y_3}{B_1(\tau_1)} - B_2(\tau_3) \\
&\times \left[A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} - x_2 + y_3\left(\frac{B_1(\tau_2)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_2)\right)\right] \\
&\times \left(B_2(\tau_2) - \frac{B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_1)}\right)^{-1} - B_3(\tau_3)y_3.
\end{aligned} \tag{A.15}$$

Rearranging y_3, generates:

$$\begin{aligned}
x_3 &= A(\tau_3) - B_1(\tau_3)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} + \left[A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1) - x_1}{B_1(\tau_1)} - x_2\right] \\
&\times \left(\left(\frac{B_1(\tau_2)B_2(\tau_1)}{B_2(\tau_3)B_1(\tau_1)} - \frac{B_2(\tau_2)}{B_2(\tau_3)}\right)^{-1} + \left(\frac{B_1(\tau_1)B_2(\tau_2) - B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_3)B_2(\tau_1)}\right)^{-1}\right) \\
&+ y_3\left[\frac{B_1(\tau_3)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_3) + \left(\frac{B_1(\tau_2)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_2)\right) \times \right. \\
&\left.\left(\left(\frac{B_1(\tau_2)B_2(\tau_1)}{B_2(\tau_3)B_1(\tau_1)} - \frac{B_2(\tau_2)}{B_2(\tau_3)}\right)^{-1} + \left(\frac{B_1(\tau_1)B_2(\tau_2) - B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_3)B_2(\tau_1)}\right)^{-1}\right)\right].
\end{aligned} \tag{A.16}$$

Subsequently, y_1, y_2 and y_3 can be expressed as:

$$\begin{aligned}
\hat{y}_3 &= A(\tau_3) - x_3 - B_1(\tau_3)\frac{A(\tau_1)-x_1}{B_1(\tau_1)} + \left[A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1)-x_1}{B_1(\tau_1)} - x_2\right] \\
&\times \left(\left(\frac{B_1(\tau_2)B_2(\tau_1)}{B_2(\tau_3)B_1(\tau_1)} - \frac{B_2(\tau_2)}{B_2(\tau_3)}\right)^{-1} + \left(\frac{B_1(\tau_1)B_2(\tau_2)-B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_3)B_2(\tau_1)}\right)^{-1}\right) \\
&\times \left[B_3(\tau_3) - \frac{B_1(\tau_3)B_3(\tau_1)}{B_1(\tau_1)} - \left(\frac{B_1(\tau_2)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_2)\right) \times \right. \\
&\left.\left(\left(\frac{B_1(\tau_2)B_2(\tau_1)}{B_2(\tau_3)B_1(\tau_1)} - \frac{B_2(\tau_2)}{B_2(\tau_3)}\right)^{-1} + \left(\frac{B_1(\tau_1)B_2(\tau_2)-B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_3)B_2(\tau_1)}\right)^{-1}\right)\right]^{-1} \qquad \text{(A.17)} \\
\hat{y}_2 &= \left[A(\tau_2) - B_1(\tau_2)\frac{A(\tau_1)-x_1}{B_1(\tau_1)} - x_2 + \hat{y}_3\left(\frac{B_1(\tau_2)B_3(\tau_1)}{B_1(\tau_1)} - B_3(\tau_2)\right)\right] \\
&\times \left(B_2(\tau_2) - \frac{B_1(\tau_2)B_2(\tau_1)}{B_1(\tau_1)}\right)^{-1} \\
\hat{y}_1 &= \frac{A(\tau_1) - x_1 - B_2(\tau_1)\hat{y}_2 - B_3(\tau_1)\hat{y}_3}{B_1(\tau_1)}.
\end{aligned}$$

A.5 Multi-factor framework: Jacobian

Using the solutions of the factors, I calculate the corresponding Jacobian for the two- and three-factor models. The Jacobian is defined as:

$$J = \begin{bmatrix} \frac{\hat{y}_1}{x_1} & \cdots & \frac{\hat{y}_1}{x_m} \\ \vdots & \ddots & \vdots \\ \frac{\hat{y}_n}{x_1} & \cdots & \frac{\hat{y}_n}{x_m} \end{bmatrix}, \qquad \text{(A.18)}$$

where y_i denotes the i^{th} factor and $x_i = \ln[P(t,\tau_i)]$. This results in:

$$J_2 = \begin{bmatrix} \frac{\hat{y}_1}{x_1} & \frac{\hat{y}_1}{x_2} \\ \frac{\hat{y}_2}{x_1} & \frac{\hat{y}_2}{x_2} \end{bmatrix} \text{ and } J_3 = \begin{bmatrix} \frac{\hat{y}_1}{x_1} & \frac{\hat{y}_1}{x_2} & \frac{\hat{y}_1}{x_3} \\ \frac{\hat{y}_2}{x_1} & \frac{\hat{y}_2}{x_2} & \frac{\hat{y}_2}{x_3} \\ \frac{\hat{y}_3}{x_1} & \frac{\hat{y}_3}{x_2} & \frac{\hat{y}_3}{x_3} \end{bmatrix}, \qquad \text{(A.19)}$$

for, respectively, the two- and three-factor models. Due to the linear relation between $\hat{y}_j$ and x_i, the first-order partial derivatives are easily derived from the results in the previous subsection.

B. *Data Analysis Appendix*

B.1 Average yield curves

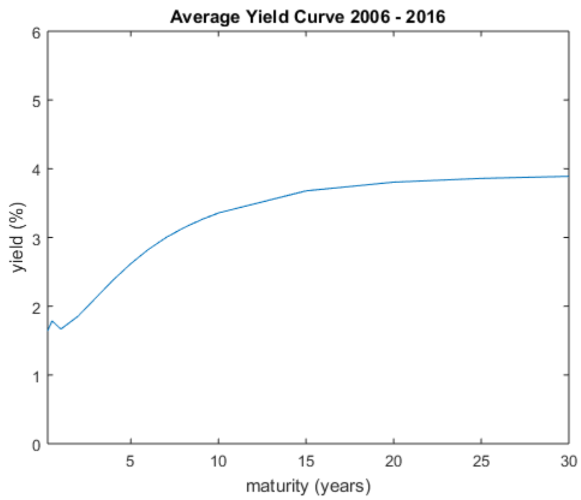


FIGURE B.1: Average Yield Curve

Notes: This figure shows the average yield curve (in %) of the riskless interest rates. The yield curve is based on the complete sample, from January 6, 2006 to January 1, 2016, and is presented until a maturity of 30 years. The yield curve is bootstrapped from US LIBOR money market deposits and US interest rate swaps.

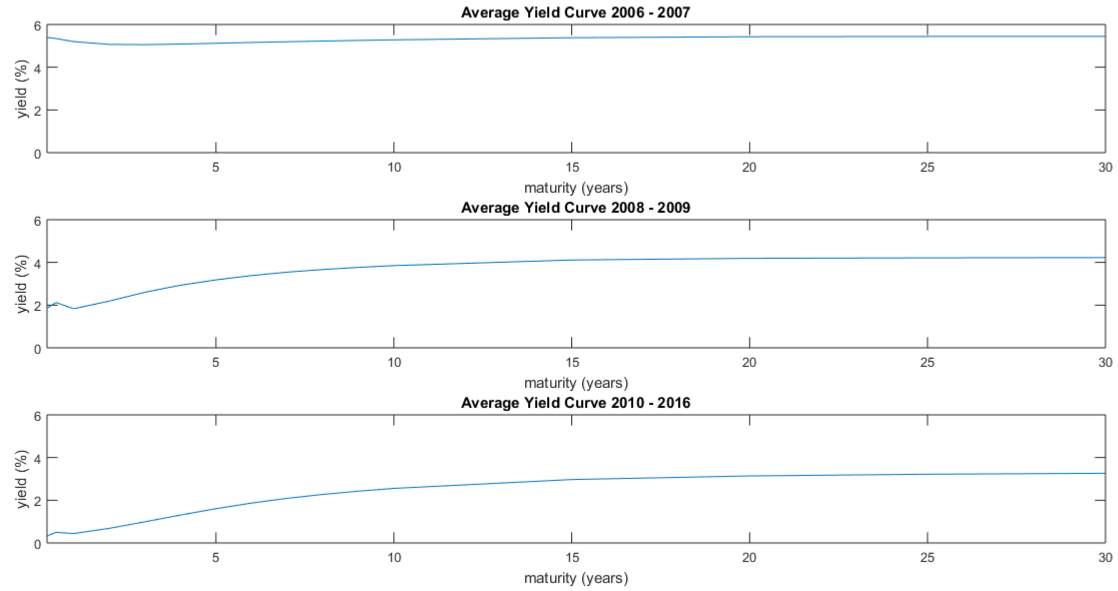


FIGURE B.2: Average Yield Curve of Sub-Samples

Notes: This figure shows the average yield curve (in %) of the riskless interest rates for (from top to bottom) the pre-crisis, mid-crisis and post-crisis samples. The pre-crisis sample ranges from January 6, 2006 to December 28, 2007, the mid-crisis sample ranges from January 4, 2008 to December 25, 2009 and the post-crisis sample ranges from January 1, 2010 to January 1, 2016. The yield curve is presented until a maturity of 30 years and is bootstrapped from US LIBOR money market deposits and US interest rate swaps.

B.2 Summary Statistics

Panel A: Pre-Crisis Sample

Maturity	Mean (%)	SD (%)	Skewness	Kurtosis	ρ_1	ρ_{26}	ρ_{52}
3-month	5.3962	0.2685	-0.9759	3.3675	0.9054	-0.1881	-0.0296
6-month	5.3432	0.2529	-0.9879	3.0846	0.9086	-0.2310	-0.0034
1-year	5.2004	0.3420	-1.3615	4.3649	0.9162	-0.1241	-0.0385
2-year	5.0695	0.3871	-1.3178	4.6107	0.9133	-0.1280	0.0637
3-year	5.0576	0.3674	-1.0798	4.2515	0.9118	-0.1679	0.1318
4-year	5.0841	0.3453	-0.8485	3.8558	0.9088	-0.2017	0.1775
5-year	5.1225	0.3233	-0.6208	3.4785	0.9060	-0.2400	0.2179
6-year	5.1597	0.3068	-0.4283	3.1566	0.9047	-0.2722	0.2478
7-year	5.1937	0.2936	-0.2623	2.9057	0.9037	-0.3046	0.2753
8-year	5.2253	0.2834	-0.1312	2.7161	0.9029	-0.3313	0.2962
9-year	5.2549	0.2757	-0.0305	2.5789	0.9022	-0.3528	0.3116
10-year	5.2822	0.2701	0.0439	2.4839	0.9021	-0.3689	0.3225
15-year	5.3822	0.2544	0.1729	2.2854	0.9010	-0.4007	0.3451
20-year	5.4279	0.2484	0.2021	2.2306	0.9007	-0.4091	0.3480
25-year	5.4404	0.2466	0.2138	2.2289	0.8997	-0.4083	0.3480
30-year	5.4409	0.2459	0.2195	2.2379	0.9002	-0.4068	0.3459

Panel B: Mid-Crisis Sample

Maturity	Mean (%)	SD (%)	Skewness	Kurtosis	ρ_1	ρ_{26}	ρ_{52}
3-month	1.8514	1.2881	0.3359	2.0194	0.9552	0.2720	-0.3688
6-month	2.1231	1.1183	0.0727	1.8577	0.9563	0.2627	-0.3695
1-year	1.8308	1.0219	0.1709	1.4489	0.9627	0.2626	-0.3864
2-year	2.1903	0.8774	0.3874	1.5833	0.9643	0.1672	-0.3494
3-year	2.6018	0.7906	0.4661	1.7185	0.9627	0.0589	-0.2943
4-year	2.9323	0.7278	0.4019	1.7708	0.9598	-0.0261	-0.2480
5-year	3.1833	0.6826	0.2654	1.8187	0.9557	-0.0905	-0.2141
6-year	3.3847	0.6579	0.1065	1.8750	0.9524	-0.1354	-0.1956
7-year	3.5445	0.6437	-0.0395	1.9582	0.9498	-0.1655	-0.1853
8-year	3.6676	0.6387	-0.1554	2.0245	0.9488	-0.1811	-0.1842
9-year	3.7677	0.6362	-0.2547	2.0955	0.9476	-0.1897	-0.1891
10-year	3.8507	0.6325	-0.3279	2.1658	0.9487	-0.1971	-0.1925
15-year	4.1141	0.6300	-0.5252	2.3675	0.9480	-0.2057	-0.2169
20-year	4.1906	0.6586	-0.5819	2.4469	0.9526	-0.2010	-0.2393
25-year	4.2128	0.6805	-0.6132	2.4664	0.9548	-0.2072	-0.2450
30-year	4.2273	0.6881	-0.6562	2.5507	0.9529	-0.2146	-0.2455

Panel C: Post-Crisis Sample

Maturity	Mean (%)	SD (%)	Skewness	Kurtosis	ρ_1	ρ_{26}	ρ_{52}
3-month	0.3272	0.0983	1.2314	3.3658	0.9766	0.2538	-0.1247
6-month	0.4975	0.1477	0.8075	2.4022	0.9850	0.3851	-0.0732
1-year	0.4412	0.1308	0.8679	3.5359	0.9411	0.3492	-0.0134
2-year	0.6848	0.2329	0.6367	2.6718	0.9451	0.3776	0.2909
3-year	0.9908	0.3538	0.4926	2.7354	0.9585	0.3595	0.3439
4-year	1.3072	0.4393	0.4062	2.8689	0.9634	0.3485	0.3102
5-year	1.6047	0.4943	0.3987	2.8895	0.9663	0.3449	0.2605
6-year	1.8680	0.5296	0.4315	2.8268	0.9684	0.3443	0.2171
7-year	2.0897	0.5522	0.4663	2.7206	0.9698	0.3441	0.1819
8-year	2.2725	0.5673	0.4958	2.5980	0.9711	0.3446	0.1544
9-year	2.4254	0.5790	0.5123	2.4867	0.9721	0.3469	0.1322
10-year	2.5553	0.5885	0.5256	2.4000	0.9725	0.3475	0.1157
15-year	2.9698	0.6195	0.5131	2.1756	0.9748	0.3611	0.0773
20-year	3.1392	0.6279	0.4541	2.0682	0.9761	0.3727	0.0579
25-year	3.2189	0.6313	0.4211	2.0172	0.9764	0.3783	0.0491
30-year	3.2635	0.6306	0.4034	1.9996	0.9769	0.3817	0.0456

Table B.1: Summary Statistics of Sub-Samples

Notes: This table shows the summary statistics of the weekly yield curve for the pre-crisis (**Panel A**), mid-crisis (**Panel B**) and post-crisis (**Panel C**) samples. The pre-crisis sample ranges from January 6, 2006 to December 28, 2007, the mid-crisis sample ranges from January 4, 2008 to December 25, 2009 and the post-crisis sample ranges from January 1, 2010 to January 1, 2016. The table provides the mean (in %), standard deviation (SD in %), skewness and kurtosis. The one-, 26- and 52-week auto-correlation coefficients are denoted by, respectively, ρ_1, ρ_{26} and ρ_{52}.

B.3 Cross-correlations

Panel A: Complete Sample

Maturity	3-month	6-month	1 year	2 year	3 year	4 year	5 year	6 year	7 year	8 year	9 year	10 year	15 year	20 year	25 year	30 year
3-month	1.0000															
6-month	0.9976	1.0000														
1 year	0.9930	0.9950	1.0000													
2 year	0.9811	0.9855	0.9943	1.0000												
3 year	0.9645	0.9699	0.9799	0.9951	1.0000											
4 year	0.9474	0.9529	0.9637	0.9849	0.9970	1.0000										
5 year	0.9321	0.9374	0.9488	0.9736	0.9905	0.9980	1.0000									
6 year	0.9187	0.9235	0.9353	0.9621	0.9823	0.9933	0.9986	1.0000								
7 year	0.9069	0.9113	0.9233	0.9512	0.9737	0.9873	0.9952	0.9990	1.0000							
8 year	0.8972	0.9010	0.9131	0.9415	0.9654	0.9810	0.9908	0.9966	0.9993	1.0000						
9 year	0.8884	0.8917	0.9038	0.9325	0.9574	0.9744	0.9859	0.9933	0.9975	0.9994	1.0000					
10 year	0.8803	0.8832	0.8954	0.9242	0.9498	0.9680	0.9808	0.9896	0.9950	0.9980	0.9996	1.0000				
15 year	0.8487	0.8501	0.8629	0.8914	0.9187	0.9403	0.9574	0.9704	0.9799	0.9865	0.9913	0.9947	1.0000			
20 year	0.8346	0.8345	0.8484	0.8761	0.9033	0.9257	0.9442	0.9588	0.9698	0.9779	0.9841	0.9888	0.9987	1.0000		
25 year	0.8259	0.8249	0.8397	0.8668	0.8939	0.9165	0.9357	0.9512	0.9630	0.9720	0.9789	0.9842	0.9968	0.9995	1.0000	
30 year	0.8204	0.8191	0.8342	0.8611	0.8880	0.9109	0.9305	0.9464	0.9587	0.9681	0.9755	0.9813	0.9953	0.9989	0.9998	1.0000

Panel B: Pre-Crisis Sample

Maturity	3-month	6-month	1 year	2 year	3 year	4 year	5 year	6 year	7 year	8 year	9 year	10 year	15 year	20 year	25 year	30 year
3-month	1.0000															
6-month	0.8983	1.0000														
1 year	0.5358	0.8205	1.0000													
2 year	0.4120	0.7334	0.9769	1.0000												
3 year	0.3942	0.7138	0.9521	0.9930	1.0000											
4 year	0.3945	0.7062	0.9287	0.9790	0.9959	1.0000										
5 year	0.4017	0.7024	0.9028	0.9591	0.9848	0.9963	1.0000									
6 year	0.4159	0.7033	0.8772	0.9366	0.9694	0.9872	0.9971	1.0000								
7 year	0.4283	0.7010	0.8481	0.9100	0.9493	0.9731	0.9890	0.9973	1.0000							
8 year	0.4409	0.6992	0.8197	0.8827	0.9274	0.9562	0.9772	0.9904	0.9978	1.0000						
9 year	0.4520	0.6960	0.7916	0.8550	0.9042	0.9373	0.9629	0.9804	0.9921	0.9982	1.0000					
10 year	0.4591	0.6916	0.7671	0.8305	0.8832	0.9197	0.9488	0.9698	0.9849	0.9941	0.9988	1.0000				
15 year	0.4907	0.6814	0.6882	0.7464	0.8071	0.8525	0.8915	0.9224	0.9475	0.9661	0.9796	0.9880	1.0000			
20 year	0.4991	0.6695	0.6468	0.7027	0.7667	0.8157	0.8589	0.8939	0.9233	0.9460	0.9633	0.9749	0.9975	1.0000		
25 year	0.5019	0.6638	0.6308	0.6856	0.7506	0.8009	0.8454	0.8819	0.9128	0.9369	0.9556	0.9684	0.9951	0.9995	1.0000	
30 year	0.5025	0.6599	0.6212	0.6758	0.7415	0.7926	0.8379	0.8753	0.9070	0.9319	0.9514	0.9649	0.9936	0.9990	0.9998	1.0000

Panel C: Mid-Crisis Sample

Maturity	3-month	6-month	1 year	2 year	3 year	4 year	5 year	6 year	7 year	8 year	9 year	10 year	15 year	20 year	25 year	30 year
3-month	1.0000															
6-month	0.9883	1.0000														
1 year	0.9550	0.9690	1.0000													
2 year	0.8900	0.9088	0.9782	1.0000												
3 year	0.8384	0.8552	0.9407	0.9888	1.0000											
4 year	0.7956	0.8076	0.9018	0.9656	0.9930	1.0000										
5 year	0.7536	0.7599	0.8615	0.9363	0.9760	0.9947	1.0000									
6 year	0.7192	0.7188	0.8246	0.9061	0.9547	0.9825	0.9963	1.0000								
7 year	0.6878	0.6820	0.7920	0.8783	0.9333	0.9677	0.9882	0.9976	1.0000							
8 year	0.6663	0.6555	0.7677	0.8563	0.9151	0.9539	0.9790	0.9927	0.9986	1.0000						
9 year	0.6463	0.6311	0.7460	0.8363	0.8978	0.9401	0.9689	0.9861	0.9951	0.9988	1.0000					
10 year	0.6284	0.6102	0.7282	0.8196	0.8830	0.9275	0.9592	0.9789	0.9904	0.9959	0.9989	1.0000				
15 year	0.5537	0.5242	0.6559	0.7527	0.8203	0.8706	0.9105	0.9381	0.9576	0.9695	0.9792	0.9864	1.0000			
20 year	0.5235	0.4889	0.6272	0.7247	0.7920	0.8429	0.8849	0.9148	0.9368	0.9509	0.9630	0.9727	0.9971	1.0000		
25 year	0.5031	0.4659	0.6071	0.7060	0.7742	0.8259	0.8690	0.9003	0.9237	0.9391	0.9525	0.9635	0.9936	0.9988	1.0000	
30 year	0.4865	0.4485	0.5916	0.6919	0.7612	0.8141	0.8586	0.8910	0.9157	0.9318	0.9461	0.9579	0.9912	0.9978	0.9996	1.0000

Panel D: Post-Crisis Sample

Maturity	3-month	6-month	1 year	2 year	3 year	4 year	5 year	6 year	7 year	8 year	9 year	10 year	15 year	20 year	25 year	30 year
3-month	1.0000															
6-month	0.9751	1.0000														
1 year	0.7233	0.6810	1.0000													
2 year	0.1350	0.0797	0.7088	1.0000												
3 year	-0.1244	-0.1917	0.4658	0.9438	1.0000											
4 year	-0.2211	-0.2945	0.3447	0.8691	0.9806	1.0000										
5 year	-0.2611	-0.3353	0.2719	0.7984	0.9410	0.9883	1.0000									
6 year	-0.2775	-0.3496	0.2203	0.7339	0.8952	0.9627	0.9925	1.0000								
7 year	-0.2855	-0.3543	0.1787	0.6763	0.8500	0.9322	0.9760	0.9952	1.0000							
8 year	-0.2918	-0.3569	0.1430	0.6260	0.8083	0.9016	0.9561	0.9846	0.9969	1.0000						
9 year	-0.2962	-0.3577	0.1130	0.5824	0.7710	0.8728	0.9356	0.9715	0.9899	0.9979	1.0000					
10 year	-0.2996	-0.3575	0.0867	0.5441	0.7375	0.8460	0.9155	0.9574	0.9808	0.9930	0.9985	1.0000				
15 year	-0.3026	-0.3480	0.0093	0.4255	0.6288	0.7540	0.8414	0.8996	0.9371	0.9613	0.9767	0.9868	1.0000			
20 year	-0.3117	-0.3526	-0.0274	0.3773	0.5853	0.7163	0.8094	0.8730	0.9152	0.9434	0.9624	0.9755	0.9981	1.0000		
25 year	-0.3143	-0.3526	-0.0443	0.3532	0.5629	0.6964	0.7923	0.8585	0.9030	0.9332	0.9539	0.9685	0.9957	0.9995	1.0000	
30 year	-0.3143	-0.3507	-0.0540	0.3376	0.5475	0.6823	0.7798	0.8478	0.8938	0.9254	0.9473	0.9629	0.9935	0.9986	0.9997	1.0000

Table B.2: Cross-Correlations

Notes: This table reports the cross-correlations of the yield curve for the complete (**Panel A**), pre-crisis (**Panel B**), mid-crisis (**Panel C**) and post-crisis (**Panel D**) samples. The complete sample ranges from January 6, 2006 to January 1, 2016, the pre-crisis sample ranges from January 6, 2006 to December 28, 2007, the mid-crisis sample ranges from January 4, 2008 to December 25, 2009 and the post-crisis sample ranges from January 1, 2010 to January 1, 2016. The lower triangle of the table provides the cross-correlations of the yields.

C. *Results Appendix*

C.1 Parameter Estimates

The parameter estimates of the pre-, mid- and post-crisis samples are reported on the following pages.

Panel A: ATSM

Θ	1-Factor model		2-Factor model				3-Factor model					
	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.609	0.093 (0.00)	2.034	0.113 (0.00)	0.010	0.004 (0.01)	0.010	0.002 (0.00)	0.423	0.032 (0.00)	1.684	0.080 (0.00)
θ	0.056	0.010 (0.00)	0.002	0.251 (0.99)	0.055	0.025 (0.03)	0.000	0.128 (1.00)	0.058	0.017 (0.00)	0.000	0.201 (1.00)
σ	0.006	0.005 (0.21)	0.013	0.009 (0.18)	0.007	0.002 (0.00)	0.006	0.003 (0.06)	0.014	0.010 (0.16)	0.019	0.013 (0.14)
ξ	0.332	0.089 (0.00)	1.005	0.272 (0.00)	-0.112	0.010 (0.00)	0.002	0.024 (0.94)	-0.871	0.252 (0.00)	1.530	0.252 (0.00)

Panel B: AJTSM

Θ	1-Factor model		2-Factor model				3-Factor model					
	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.751	0.040 (0.00)	1.656	0.063 (0.00)	0.035	0.003 (0.00)	0.024	0.001 (0.00)	0.293	0.005 (0.00)	2.207	0.018 (0.00)
θ	0.049	0.010 (0.00)	0.000	0.014 (0.99)	0.040	0.001 (0.00)	0.005	0.002 (0.05)	0.104	0.005 (0.00)	0.000	0.030 (1.00)
σ	0.002	0.001 (0.28)	0.023	0.019 (0.24)	0.005	0.000 (0.00)	0.007	0.000 (0.00)	0.004	0.001 (0.00)	0.016	0.016 (0.32)
λ	13.32	0.230 (0.00)	0.013	0.055 (0.81)	2.538	0.022 (0.00)	0.034	0.002 (0.00)	40.27	0.051 (0.00)	0.088	0.036 (0.02)
α	0.000	0.000 (0.24)	0.075	0.057 (0.19)	0.001	0.000 (0.00)	0.000	0.000 (1.00)	-0.001	0.000 (0.00)	0.509	0.089 (0.00)
β	0.001	0.005 (0.78)	0.005	0.118 (0.97)	0.001	0.000 (0.00)	0.002	0.002 (0.28)	0.002	0.003 (0.54)	0.993	0.087 (0.00)
ξ	1.445	0.094 (0.00)	-0.491	0.164 (0.00)	-0.211	0.005 (0.00)	-0.127	0.004 (0.00)	-0.109	0.019 (0.00)	-0.015	0.358 (0.00)

Table C.1: Parameter Estimates Pre-Crisis Sample

Notes: This table reports the parameter estimates, based on the QMLE procedure, in the ATSM-framework (**Panel A**) and in the AJTSM-framework (**Panel B**) using the pre-crisis weekly yield curve from January 6, 2006 to December 28, 2007 (104 observations). The table provides the parameter estimates of κ, θ, σ and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the ATSM-framework (**Panel A**). Additionally, the table provides the parameter estimates of κ, θ, σ, λ, α, β and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the AJTSM-framework (**Panel B**). In the models, I assume that the dynamics of the factors are described by a Vasicek model and the market price of risk of each factor is constant. The standard errors are based on the nearest symmetric-positive definite covariance matrix, derived from the unconstrained Hessian matrix. In the AJTSM-framework, I assume that jump risk is diversifiable and the Brownian Motion and Poisson process are independent as well.

Panel A: ATSM

	1-Factor model		2-Factor model				3-Factor model					
Θ	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.330	0.032 (0.00)	0.545	0.032 (0.00)	0.010	0.010 (0.33)	0.107	0.004 (0.00)	0.010	0.006 (0.10)	1.134	0.074 (0.00)
θ	0.000	0.017 (1.00)	0.000	0.251 (1.00)	0.000	0.251 (1.00)	0.000	0.226 (1.00)	0.000	0.403 (1.00)	0.000	0.368 (1.00)
σ	0.015	0.002 (0.00)	0.022	0.005 (0.00)	0.010	0.001 (0.00)	0.058	0.007 (0.00)	0.017	0.004 (0.00)	0.025	0.013 (0.07)
ξ	-1.053	0.061 (0.00)	-1.504	0.118 (0.00)	-0.127	0.037 (0.00)	-0.024	0.091 (0.79)	-0.504	0.034 (0.00)	-0.633	0.265 (0.02)

Panel B: AJTSM

	1-Factor model		2-Factor model				3-Factor model					
Θ	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.330	0.031 (0.00)	0.032	0.000 (0.00)	0.433	0.013 (0.00)	0.022	0.001 (0.00)	0.092	0.003 (0.00)	1.050	0.031 (0.00)
θ	0.000	0.007 (1.00)	0.000	0.005 (1.00)	0.051	0.017 (0.00)	0.023	0.007 (0.00)	0.016	0.021 (0.44)	0.010	0.015 (0.53)
σ	0.003	0.001 (0.00)	0.011	0.000 (0.00)	0.024	0.006 (0.00)	0.011	0.000 (0.00)	0.041	0.002 (0.00)	0.018	0.012 (0.15)
λ	11.61	0.236 (0.00)	0.037	0.002 (0.00)	5.583	0.062 (0.00)	0.181	0.007 (0.00)	1.982	0.015 (0.00)	3.113	0.205 (0.00)
α	0.000	0.000 (0.32)	-0.029	0.001 (0.00)	-0.007	0.000 (0.00)	-0.007	0.001 (0.00)	0.000	0.001 (0.70)	-0.003	0.008 (0.68)
β	0.004	0.006 (0.56)	0.018	0.002 (0.00)	0.000	0.012 (1.00)	0.004	0.002 (0.02)	0.016	0.003 (0.00)	0.007	0.014 (0.64)
ξ	-2.000	0.100 (0.00)	-0.336	0.006 (0.00)	-0.848	0.082 (0.00)	-0.119	0.008 (0.00)	-0.241	0.025 (0.00)	-0.250	0.014 (0.97)

Table C.2: Parameter Estimates Mid-Crisis Sample

Notes: This table reports the parameter estimates, based on the QMLE procedure, in the ATSM-framework (**Panel A**) and in the AJTSM-framework (**Panel B**) using the mid-crisis weekly yield curve from January 4, 2008 to December 25, 2009 (104 observations). The table provides the parameter estimates of κ, θ, σ and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the ATSM-framework (**Panel A**). Additionally, the table provides the parameter estimates of κ, θ, σ, λ, α, β and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the AJTSM-framework (**Panel B**). In the models, I assume that the dynamics of the factors are described by a Vasicek model and the market price of risk of each factor is constant. The standard errors are based on the nearest symmetric-positive definite covariance matrix, derived from the unconstrained Hessian matrix. In the AJTSM-framework, I assume that jump risk is diversifiable and the Brownian Motion and Poisson process are independent as well.

Panel A: ATSM

	1-Factor model		2-Factor model				3-Factor model					
Θ	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.165	0.013 (0.00)	0.269	0.014 (0.00)	0.015	0.003 (0.00)	0.201	0.007 (0.00)	0.022	0.003 (0.00)	0.877	0.032 (0.00)
θ	0.029	0.006 (0.00)	0.000	0.901 (1.00)	0.000	0.190 (1.00)	0.000	0.144 (1.00)	0.058	1.069 (0.96)	0.147	0.154 (0.34)
σ	0.001	0.000 (0.00)	0.011	0.004 (0.01)	0.019	0.001 (0.00)	0.084	0.005 (0.00)	0.020	0.001 (0.00)	0.016	0.003 (0.00)
ξ	-2.000	0.055 (0.00)	-0.135	0.075 (0.07)	-0.176	0.021 (0.00)	-0.171	0.021 (0.00)	0.192	0.016 (0.00)	-2.000	0.069 (0.00)

Panel B: AJTSM

	1-Factor model		2-Factor model				3-Factor model					
Θ	1^{st}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	1^{st}-F	SE (p-value)	2^{nd}-F	SE (p-value)	3^{rd}-F	SE (p-value)
κ	0.045	0.002 (0.00)	0.035	0.000 (0.00)	0.192	0.007 (0.00)	0.020	0.000 (0.00)	0.314	0.002 (0.00)	1.546	0.010 (0.00)
θ	0.000	0.006 (0.99)	0.028	0.003 (0.00)	0.012	0.005 (0.02)	0.036	0.002 (0.00)	0.129	0.008 (0.00)	0.230	0.017 (0.00)
σ	0.001	0.000 (0.00)	0.009	0.000 (0.00)	0.016	0.002 (0.00)	0.011	0.000 (0.00)	0.016	0.001 (0.00)	0.143	0.006 (0.00)
λ	0.284	0.008 (0.00)	0.240	0.003 (0.00)	0.002	0.002 (0.43)	0.313	0.002 (0.00)	23.89	0.022 (0.00)	2.052	0.032 (0.00)
α	0.019	0.002 (0.00)	0.000	0.000 (0.24)	-0.029	0.020 (0.16)	0.008	0.000 (0.00)	-0.001	0.000 (0.00)	-0.181	0.003 (0.00)
β	0.040	0.002 (0.00)	0.006	0.001 (0.00)	0.100	0.028 (0.00)	0.004	0.000 (0.00)	0.021	0.002 (0.00)	0.102	0.025 (0.00)
ξ	-1.170	0.011 (0.00)	-0.089	0.004 (0.00)	-0.096	0.013 (0.00)	-1.253	0.003 (0.00)	-0.011	0.283 (0.00)	-1.131	0.042 (0.00)

Table C.3: Parameter Estimates Post-Crisis Sample

Notes: This table reports the parameter estimates, based on the QMLE procedure, in the ATSM-framework (**Panel A**) and in the AJTSM-framework (**Panel B**) using the post-crisis weekly yield curve from January 1, 2010 to January 1, 2016 (318 observations). The table provides the parameter estimates of κ, θ, σ and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the ATSM-framework (**Panel A**). Additionally, the table provides the parameter estimates of κ, θ, σ, λ, α, β and ξ, their corresponding standard errors and p-values for the one-, two- and three-factor model in the AJTSM-framework (**Panel B**). In the models, I assume that the dynamics of the factors are described by a Vasicek model and the market price of risk of each factor is constant. The standard errors are based on the nearest symmetric-positive definite covariance matrix, derived from the unconstrained Hessian matrix. In the AJTSM-framework, I assume that jump risk is diversifiable and the Brownian Motion and Poisson process are independent as well.

C.2 Mid- and post-crisis goodness-of-fit statistics

The goodness-of-fit measures of the mid- and post-crisis samples are reported on the following pages.

Panel A: 1-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R^2_A ATSM	R^2_A AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.018	0.018	0.134	0.134	**3.440**	3.443	0.995	0.995
1-year	0.232	**0.231**	0.481	**0.480**	5.965	**5.954**	**0.936**	0.932
2-year	1.212	**1.206**	1.101	**1.098**	9.054	**9.042**	0.612	**0.586**
3-year	2.300	**2.291**	1.516	**1.514**	11.05	11.05	0.168	**0.110**
4-year	3.638	**3.632**	1.907	**1.906**	12.76	12.76	**-0.480**	-0.585
5-year	5.503	**5.502**	2.346	2.346	**14.26**	14.27	**-1.508**	-1.690
6-year	**7.918**	7.925	**2.814**	2.815	**15.64**	15.65	**-3.009**	-3.305
7-year	**11.08**	11.09	**3.328**	3.330	**16.96**	16.97	**-5.150**	-5.606
8-year	**14.87**	14.88	**3.856**	3.858	18.17	18.17	**-7.920**	-8.580
9-year	**19.49**	19.51	**4.415**	4.417	**19.35**	19.36	**-11.48**	-12.40
10-year	**24.77**	24.78	**4.977**	4.978	**20.54**	20.55	**-15.77**	-17.00
15-year	64.84	**64.82**	8.053	**8.051**	26.13	26.13	**-51.22**	-55.00
20-year	133.2	**133.1**	11.54	11.54	30.71	**30.70**	**-114.1**	-122.4
25-year	234.5	234.5	15.31	15.31	34.74	**34.72**	**-209.5**	-224.8
30-year	**361.8**	362.0	**19.02**	19.03	38.22	**38.20**	**-332.8**	-357.3

Panel B: 2-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R^2_A ATSM	R^2_A AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	**0.017**	0.019	**0.132**	0.139	**3.365**	3.453	**0.995**	0.994
1-year	0.229	**0.225**	0.478	**0.475**	6.162	**5.973**	**0.937**	0.928
2-year	**1.034**	1.056	**1.017**	1.028	8.796	**8.389**	**0.669**	0.609
3-year	**1.476**	1.703	**1.215**	1.305	**9.020**	9.121	**0.466**	0.287
4-year	**1.717**	2.373	**1.310**	1.541	**9.316**	10.43	**0.302**	-0.117
5-year	**2.051**	3.312	**1.432**	1.820	**10.19**	11.84	**0.065**	-0.747
6-year	**2.492**	4.414	**1.578**	2.101	**11.13**	13.04	**-0.261**	-1.586
7-year	**3.095**	5.679	**1.759**	2.383	**12.00**	14.06	**-0.718**	-2.649
8-year	**3.706**	6.872	**1.925**	2.622	**12.63**	14.81	**-1.223**	-3.771
9-year	**4.222**	7.914	**2.055**	2.813	**13.07**	15.40	**-1.703**	-4.863
10-year	**4.671**	8.889	**2.161**	2.981	**13.41**	15.89	**-2.162**	-5.963
15-year	**5.298**	11.23	**2.302**	3.351	**13.69**	16.87	**-3.266**	-9.466
20-year	**1.945**	6.315	**1.395**	2.513	**10.65**	14.33	**-0.680**	-5.314
25-year	**0.766**	1.796	**0.875**	1.340	**8.622**	10.41	**0.312**	-0.865
30-year	-	-	-	-	-	-	-	-

Panel C: 3-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R^2_A ATSM	R^2_A AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.020	**0.018**	0.141	**0.135**	3.418	**3.341**	**0.995**	0.994
1-year	**0.107**	0.124	**0.327**	0.351	**5.047**	5.271	**0.971**	0.957
2-year	**0.393**	0.447	**0.627**	0.669	**7.254**	7.557	**0.874**	0.820
3-year	**0.474**	0.510	**0.688**	0.714	**7.696**	7.856	**0.829**	0.768
4-year	**0.431**	0.449	**0.657**	0.670	**7.367**	7.417	**0.825**	0.771
5-year	**0.333**	0.353	**0.577**	0.594	6.753	**6.752**	**0.848**	0.798
6-year	**0.215**	0.243	**0.464**	0.493	**5.959**	5.986	**0.891**	0.845
7-year	**0.130**	0.160	**0.361**	0.400	**5.197**	5.361	**0.928**	0.888
8-year	**0.068**	0.086	**0.261**	0.293	**4.332**	4.510	**0.959**	0.935
9-year	**0.028**	0.033	**0.167**	0.182	**3.275**	3.418	**0.982**	0.973
10-year	-	-	-	-	-	-	-	-
15-year	0.491	**0.462**	0.701	**0.680**	7.836	**7.641**	**0.604**	0.532
20-year	**0.470**	0.550	**0.686**	0.741	**7.518**	7.861	**0.594**	0.403
25-year	**0.263**	0.270	**0.513**	0.519	**6.252**	6.402	**0.764**	0.696
30-year	-	-	-	-	-	-	-	-

Table C.4: Goodness-of-Fit Measures of Mid-Crisis Sample

Notes: This table reports and compares the goodness-of-fit measures for ATSMs and AJTSMs for the yield curve from January 4, 2008 to December 25, 2009. **Panel A** presents the results of the one-factor ATSM and AJTSM, **Panel B** presents the results of the two-factor ATSM and AJTSM, and **Panel C** presents the results of the three-factor ATSM and AJTSM. The MSPEs are denoted in squared basis points, the RMSEs and MAEs are denoted in basis points, R^2_A denotes the adjusted R^2 and τ denotes the maturity. The bold numbers show the best model (ATSM or AJTSM) with respect to the goodness-of-fit measure.

Panel A: 1-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R_A^2 ATSM	R_A^2 AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.003	0.003	**0.053**	0.054	**2.128**	2.142	0.999	0.999
1-year	0.022	0.022	0.150	**0.147**	3.603	**3.569**	0.994	0.994
2-year	0.334	**0.327**	0.578	**0.572**	6.952	**6.913**	0.893	0.893
3-year	1.396	**1.384**	1.181	**1.177**	9.735	**9.715**	**0.495**	0.488
4-year	3.507	**3.506**	1.873	**1.872**	**12.16**	12.17	**-0.426**	-0.459
5-year	**6.883**	6.910	**2.624**	2.629	**14.50**	14.52	**-2.137**	-2.221
6-year	**11.66**	11.71	**3.414**	3.422	**16.66**	16.69	**-4.902**	-5.063
7-year	**17.67**	17.72	**4.204**	4.210	**18.52**	18.54	**-8.810**	-9.062
8-year	**24.64**	24.65	**4.964**	4.965	**20.12**	20.13	**-13.78**	-14.13
9-year	32.49	**32.42**	5.700	**5.694**	**21.59**	21.60	**-19.80**	-20.23
10-year	41.12	**40.94**	6.412	**6.399**	22.96	22.96	**-26.83**	-27.35
15-year	95.53	**95.03**	9.774	**9.748**	**28.81**	28.86	**-75.93**	-77.28
20-year	**168.3**	169.2	**12.97**	13.01	**33.66**	33.74	**-144.4**	-148.6
25-year	**267.1**	269.0	**16.34**	16.40	**37.89**	37.96	**-238.7**	-246.0
30-year	384.6	**383.8**	19.61	**19.59**	41.51	**41.49**	**-353.8**	-361.2

Panel B: 2-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R_A^2 ATSM	R_A^2 AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.003	0.003	**0.051**	0.052	**1.946**	1.957	0.999	0.999
1-year	0.041	**0.040**	0.203	**0.199**	4.242	**4.201**	0.989	0.989
2-year	0.485	**0.466**	0.696	**0.683**	7.855	**7.786**	**0.845**	0.844
3-year	1.409	**1.356**	1.187	**1.164**	10.22	**10.14**	**0.490**	0.486
4-year	2.331	**2.247**	1.527	**1.499**	11.61	**11.53**	**0.052**	0.043
5-year	**2.886**	2.796	1.699	**1.672**	12.19	**12.12**	**-0.315**	-0.334
6-year	3.115	**3.054**	1.765	**1.747**	12.38	**12.33**	**-0.577**	-0.618
7-year	3.183	**3.171**	1.784	**1.781**	12.40	12.40	**-0.767**	-0.843
8-year	**3.153**	3.193	**1.776**	1.787	**12.34**	12.36	**-0.892**	-1.005
9-year	**3.059**	3.144	**1.749**	1.773	**12.20**	12.27	**-0.959**	-1.107
10-year	**2.900**	3.020	**1.703**	1.738	**12.02**	12.11	**-0.963**	-1.140
15-year	**1.888**	1.984	**1.374**	1.409	**10.58**	10.68	**-0.520**	-0.673
20-year	**0.732**	0.762	**0.856**	0.873	**8.161**	8.328	**0.367**	0.311
25-year	**0.368**	0.450	**0.606**	0.671	**7.167**	7.516	**0.670**	0.577
30-year	-	-	-	-	-	-	-	-

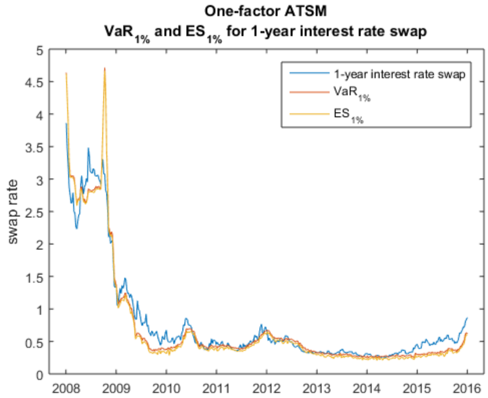
Panel C: 3-Factor model

τ	MSPE ATSM	MSPE AJTSM	RMSE ATSM	RMSE AJTSM	MAE ATSM	MAE AJTSM	R_A^2 ATSM	R_A^2 AJTSM
3-month	-	-	-	-	-	-	-	-
6-month	0.009	0.009	0.096	0.096	2.720	2.720	**0.998**	0.997
1-year	0.048	0.048	0.219	0.219	4.310	4.310	**0.987**	0.986
2-year	0.200	0.200	0.447	0.447	6.130	6.130	**0.936**	0.931
3-year	**0.255**	0.256	0.505	0.505	6.575	6.575	**0.908**	0.901
4-year	0.279	0.279	0.528	0.528	6.617	6.617	**0.886**	0.878
5-year	0.257	0.257	0.507	0.507	6.341	6.341	**0.883**	0.874
6-year	0.180	0.180	0.425	0.425	5.837	5.837	**0.909**	0.902
7-year	0.099	0.099	0.315	0.315	5.104	5.104	**0.945**	0.941
8-year	0.044	0.044	0.211	0.211	4.180	4.180	**0.973**	0.971
9-year	0.013	0.013	0.112	0.112	2.999	2.999	**0.992**	0.991
10-year	-	-	-	-	-	-	-	-
15-year	0.130	0.130	0.360	0.360	5.205	5.205	**0.895**	0.888
20-year	0.394	0.394	0.628	0.628	7.149	7.149	**0.659**	0.635
25-year	0.778	0.778	0.882	0.882	9.028	9.028	**0.302**	0.251
30-year	-	-	-	-	-	-	-	-

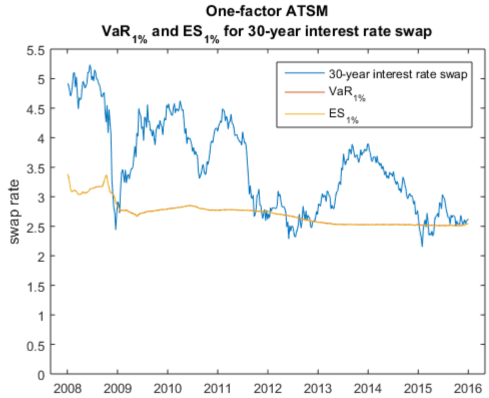
Table C.5: Goodness-of-Fit Measures of Post-Crisis Sample

Notes: This table reports and compares the goodness-of-fit measures for ATSMs and AJTSMs for the yield curve from January 1, 2010 to January 1, 2016. **Panel A** presents the results of the one-factor ATSM and AJTSM, **Panel B** presents the results of the two-factor ATSM and AJTSM, and **Panel C** presents the results of the three-factor ATSM and AJTSM. The MSPEs are denoted in squared basis points, the RMSEs and MAEs are denoted in basis points, R_A^2 denotes the adjusted R^2 and τ denotes the maturity. The bold numbers show the best model (ATSM or AJTSM) with respect to the goodness-of-fit measure.

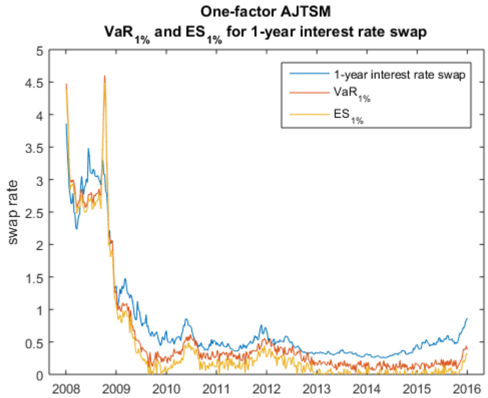
C.3 Value-at-Risk and Expected Shortfall Estimates



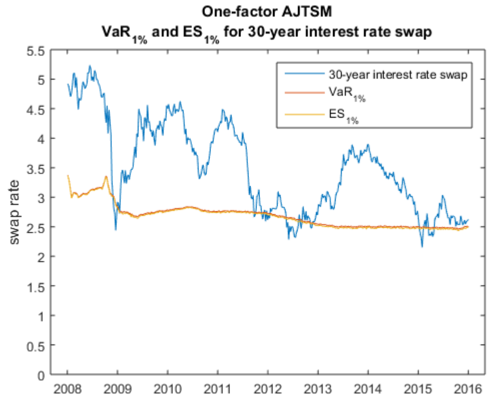
(A)



(B)



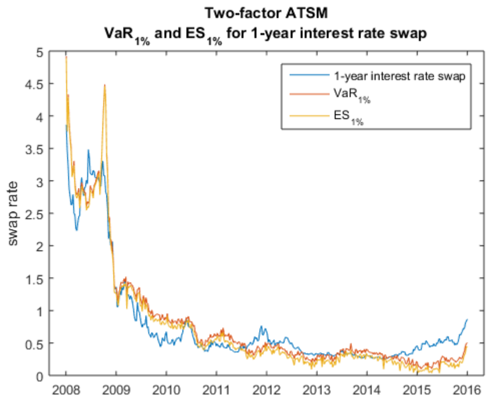
(C)



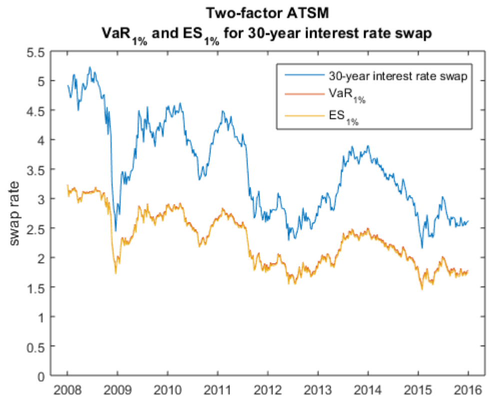
(D)

FIGURE C.1: VaR and ES Estimates of One-Factor Models

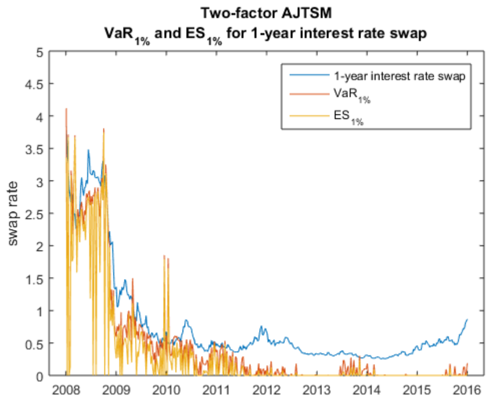
Notes: These figures plot the one- and 30-year interest rate swaps and their one-week-ahead VaR and ES estimates from January 4, 2008 to January 1, 2016. The VaR and ES estimates have a nominal coverage probability of $\gamma = 1\%$ and are constructed by the one-factor models. **(A)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the one-year interest rate swap, based on the one-factor ATSM, **(B)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the 30-year interest rate swap, based on the one-factor ATSM, **(C)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the one-year interest rate swap, based on the one-factor AJTSM, and **(D)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the 30-year interest rate swap, based on the one-factor AJTSM.



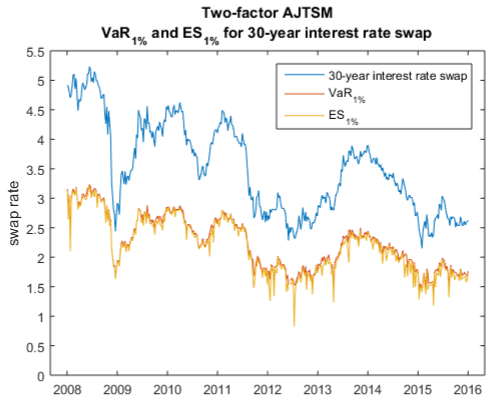
(A)



(B)



(C)



(D)

FIGURE C.2: VaR and ES Estimates of Two-Factor Models

Notes: These figures plot the one- and 30-year interest rate swaps and their one-week-ahead VaR and ES estimates from January 4, 2008 to January 1, 2016. The VaR and ES estimates have a nominal coverage probability of $\gamma = 1\%$ and are constructed by the two-factor models. **(A)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the one-year interest rate swap, based on the two-factor ATSM, **(B)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the 30-year interest rate swap, based on the two-factor ATSM, **(C)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the one-year interest rate swap, based on the two-factor AJTSM, and **(D)** plots the $VaR_{1\%}$ and $ES_{1\%}$ of the 30-year interest rate swap, based on the two-factor AJTSM.

Bibliography

Baz, J. and Das, S. R. (1996). Analytical approximations of the term structure for jump-diffusion processes: a numerical analysis. *The Journal of Fixed Income*, 6(1):78–86.

Bolder, D. (2001). Affine term-structure models: Theory and implementation. Staff Working Papers 01-15, Bank of Canada.

Chan, K. C., Karolyi, G. A., Longstaff, F. A., and Sanders, A. B. (1992). An empirical comparison of alternative models of the short-term interest rate. *The journal of finance*, 47(3):1209–1227.

Christoffersen, P. F. (1998). Evaluating interval forecasts. *International Economic Review*, 39(4):841–862.

Dai, Q. and Singleton, K. (2000). Specification analysis of affine term structure models. *The Journal of Finance*, 55(5):1943–1978.

Du, Z. and Escanciano, J. (2015). Backtesting expected shortfall: Accounting for tail risk (working paper no: 2015-001).

Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6(4):379–406.

Duffie, D., Pan, J., and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376.

Feldhütter, P. and Lando, D. (2008). Decomposing swap spreads. *Journal of Financial Economics*, 88(2):375–405.

Hull, J. C. (2006). *Options, futures, and other derivatives.* Pearson Education India.

Lin, B.-H. and Yeh, S.-K. (1999). Jump-diffusion interest rate process: An empirical examination. *Journal of Business Finance & Accounting*, 26(7-8):967–995.

Lin, B.-H. and Yeh, S.-K. (2001). Estimation for factor models of term structure of interest rates with jumps: the case of the taiwanese government bond market. *Journal of International Financial Markets, Institutions and Money*, 11(2):167–197.

Lugannani, R. and Rice, S. (1980). Saddle point approximation for the distribution of the sum of independent random variables. *Advances in applied probability*, pages 475–490.

Mc Manus, D., Watt, D., et al. (1999). Estimating one-factor models of short-term interest rates. Technical report, Bank of Canada.

Piazzesi, M. (2003). Affine term structure models. *Handbook of financial econometrics*, 1:691–766.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188.

Wong, W. (2008). Backtesting trading risk of commercial banks using expected shortfall. *Journal of Banking & Finance*, 32(7):1404–1415.

Wu, H., Chen, P., Yan, W., and Hu, G. (2011). Empirical research of two-factor vasicek model based on copulas. In *Business Management and Electronic Information (BMEI), 2011 International Conference on*, volume 2, pages 533–536. IEEE.