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[The analysis of competition through the multi-unit auction with the unit demand and the symmetric private information]

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Abstract

The Multi-unit all-pay auction can be used as an analytical tool of the contest. This paper investigates the competitive behavior during the contest through the symmetric multi-unit auction with the private information and unit demands of bidders. Allowing lags between adjacently ranked prizes, the equilibrium bidding strategy shows that the lag size has a negative effect and the valuation level has a positive effect on the equilibrium bidding strategy and the expected revenue. Additionally, the paper calls for the distribution of prizes by confirming that the increasing lag size might reduce the expected revenue more than the reduced valuations on prizes. The example results that the revenue generated from the single-unit auction might be higher than that of the multi-unit auction.

I. Introduction

An auction is the tool for allocating resources when the seller does not have sufficient information of potential buyers. Regarding the information, the value of the product and the highest valuing buyer are included. Utilizing various types of auctions, the auctioneer can proceed to the transaction with the highest valuing person. As a consequence, some auction schemes are expected to bring higher revenue to the seller compared to the revenue from mere trade in the market when it is designed properly. Those well-defined auctions are also conducive to attaining social efficiency by inducing the Pareto Improvements between the auctioneer and bidders. Following those advantages, numerous practical examples exist with different auction styles. The most well-known case is where the government tries to grant the permission on natural monopoly or the exclusive right to provide public goods. Moreover, recent popularity in Internet auction platforms such as E-bay or Alibaba shows that auction has become an essential way of trade in our daily life. With the pervasiveness of auction, multitudinous theorems from various literatures succeeded to explain the bidding strategy in different auction schemes.

Among various auction types of the auction, the studies have been divided into two types according to whether bids are disclosed or not before the result of the auction is announced. The first case is called as the open-bid auction, and latter is called the sealed-bid auction. Open-bid auction was clearly and thoroughly examined since the dominating strategy is apparent, which is sustaining bids until one's valuation is reached. In contrast, the sealed bid auction cannot be analyzed intuitively due to the uncertainty. Vickery (1961), nevertheless, started the attempt to make a formal theory about the sealed-bid auction. In his paper, the Nash Equilibrium on the winner-pay auction with private information is suggested. With additional studies done in Clarke (1971) and Groves (1973), the general methodology of auction with incomplete information has become formalized over the years.

Another important baseline theory of the auction is the revenue equivalence theorem suggested in Myerson (1981). According to Myerson, the expected revenue oriented by various auction methods should be the same when several assumptions are satisfied. The first assumption is about the game setting where all bidders are risk-neutral and the valuation of the prize should be drawn randomly from the same distribution function. Secondly, the prize should always go to the bidder with the highest bid. Also, revenue equivalence can be attained with the revelation principle, which holds the equilibrium strategy whereby each player follows strictly increasing bidding function according to their valuations. The last condition is that any bidder of the lowest valuing type expects zero utility from the auction, therefore bids nothing.

Succeeding those rudimentary frameworks, analyses of various sealed-bid auction schemes have been conducted. The first particular case is the single-unit auction ranging from the winner-pay auction to the all-pay auction. The beauty of the single-unit auction is that it is straightforward to formulate the bidding strategy from the schematized payoff function. On the other hand, diverse concerns such as different valuation of each prize or maximum units of attainable items are left while explaining for the multi-unit auction.

Even with those controversies, multi-unit auction illustrates some circumstances which cannot be explained by single-unit auction, i.e. dividing the share, or attracting low-valued bidders. The Treasury auction is the most well-known example, where the size of the whole Treasury is excessively enormous in that no individual economic unit can bid at the appropriate level. Following those necessities, to explain one aspect of those complexities, several models of multi-unit auction have been scrutinized both theoretically and empirically. Harris and Raviv (1981) examines the auction under the situation in which each bidder follows non-uniform value distribution about multiple prizes. Maskin and Riley (1989) further generalizes the bidding strategy when bidders compete for multiple objects.

On top of those findings, experimental literatures tried to validate whether theories predict an individual's behavior correctly. Since theorems are clearly formulated and individual's behavior is easier to be observed, the auction often invites experimental economists. Kagel (1995) compared different types of auctions extensively with complete information. Several experimental studies such as Cox *et al.* (1984, 1985) examines the *pay-as-bid* multi-unit auction with incomplete information and checked the robustness of theoretical assumptions with experiments. However, above analyses focuses on the winner-pay auction which only considers the cost incurred by winners.

Another feature of real life which can be explained by the auction theory is the all-pay auction. All-pay auction assumes that the cost occurred during the competition is irreversible; therefore, all participants ought to pay their bids regardless of winning the prize. Even though participants in the all-pay auction bid more passively than bids in other auctions, all-pay auction is the best tool to analyze the situation when the bidding is implicit such as political lobbying. Another possibility of explanation from the theory of all-pay auction is an application to real life instances such as R&D, lobbying, and contest exhibit. In the R&D case, for example, several firms compete for the innovation e.g. technological advance, and the firm who achieves the innovation earlier relishes an advantage e.g. patent. Also, the more investment they make the more possibility to arrive at the innovation earlier than its competitors.[1]

Furthermore, not all players compete to get the single-unit prize. The most well-known case is the exhibit contest. The participants of this contest decide whether to participate or not, and make an effort to win the prize without knowing their competitors' exact abilities or valuations of the prize. Also, each participant's endeavor used in the contest cannot be reimbursed unless they win the prize. After all submissions having been finished, endeavors will be converted into the benefit to the contest holder in a tangible form. The contest designer then grants discriminative remunerations to winners from the first place. The goal of the contest designer is therefore to set the optimal remuneration scheme which maximizes the aggregate effort level of participants.

Following this practical relevance with the all-pay auction model, several studies are done on the all-pay auction with different information settings. The first group of research about the topic is assessed by the Holt (1979) and Holt and Sherman (1982). In those papers, they established the theoretical groundwork explaining the all-pay auction. Following that, Barut and Kovenock (1998) proved the existence of equilibrium solution on multiunit all-pay auction with complete information. Davis and Reilly (1998) and Gneezy and Smorodinsky (1999) conducted an experiment on the all-pay auction with complete information about others and discovered that people tend to bid more aggressively than the risk neutral Nash Equilibrium strategy.

Even though less attention is made due to the complexity of the model, similar results are found with incomplete information. Glazer and Hassin (1988) used a tool of the multi-unit all-pay auction to analyze the contest and showed that the design of the all-pay auction generally follows the aim of the contest. Amann and Leininger (1996) reaffirmed the participants' overbidding tendency compared to the risk-neutral Bayesian Equilibrium under the asymmetric incomplete information.

The complexity remains, however, because those studies failed to account for two important aspects. One is that the bidders may have a different ranking of valuation on prizes. The ranking can be validated by various criterions such as the timing of achieving the prize or

[1] A doubt may arise as the R&D contest is sometimes analyzed with the second-highest winning bid auction. The doubt entails that although the highest bidder wins the advantage, the winner may not have to make any further endeavor after the winner being assigned. The dynamic concern rebuts this criticism since, even after the winner is already assigned, firms still have to compete for other possible innovations. In this regard, the competition resembles more to the multiple-unit all-pay auction than the single unit auction.

different shares among winners of the overall prize. The other failure is ignoring the difference between the *pay-as-bid* (or “discriminatory-price”) auction and the *uniform-price* auction. Researches done in the above by Glazer and Amann are about the *uniform-price* auction. The real life proxies of the *uniform-price* auction are Treasury auctions or the spectrum market auction. On the contrary, *pay-as-bid* auction is done in a different payment setting where each bidder reports only one bid throughout the auction.

The comparison between *pay-as-bid* and *uniform-price* auction has brought many controversies over the years. Beck and Zender (1993) showed that revenues are higher from the *pay-as- bid* auction with particular equilibrium strategy. Also, Ausubel et al. (2014) generated a situation where the *pay-as-bid* auction is better than the *uniform-price* auction in terms of the size of the revenue and efficiency, given the symmetric valuation among bidders. However, the general superiority between *uniform-price* auctions and *pay-as-bid* auctions is not clear as shown by Ausbel and Cramton (2002).

Following that, the major concern of the contest designer occurs on formulating the optimal auction scheme to maximize the revenue. To be more precise, the contest designer can consider various choice variables such as the auction rule, the number of prizes and different lags between each prize. The difficulty of the mechanism design arises because there is no general rule to judge among different auction schemes. Therefore, it seems futile to discuss superiority of either one of the *pay-as-bid* or the *uniform-bid* auction scheme in terms of the revenue.

Instead, each of two different auctions has its own applications. There are numerous real life cases which can be only explained by each model and there is no exception for the *pay-as-bid* auction. For instance, various kinds of contests are the epitome which is only explained by the *pay-as-bid* auction. In this case, each bidder can attain one prize at maximum and the ranking of the prize is the same and unambiguous for each player, i.e. being the first is strictly better than being the second. The most familiar case is an athletic competition such as the 100m running race. Even though one athlete performs extraordinarily during the game, she or he only gains one record thus only one prize would be conferred. On top of that, even when records differ minimally even less than 0.01 seconds, the remuneration for each player such as reputations or annual income differs significantly according to the ranking. With this setting, the committee of the contest is interested in finding the optimal level of remuneration scheme to maximize each player’s performance.

Cohen and Sela (2008) theoretically shows that the multi-unit *pay-as-bid* all-pay auction sometimes brings higher revenue than the single-unit all-pay auction under asymmetric valuation of the same ranking of prizes. However, Cohen and Sela (2008) found that the only ambiguous rule exists due to complexities brought by the asymmetric information about valuations. This paper instead assumes the symmetric valuation for all players to untangle those complexities. Furthermore, we introduce the lag between each succeeding prizes to satisfy for the symmetry of valuations to hold for all prizes. Following that, we investigate two models. The first model is constructed to do the analysis from the constrained situation where the same amount of lag size. The second model lexes the assumption such that heterogeneous size of lags between prizes is allowed.

Therefore, the model considers the *pay-as-bid* auction with *N* players competing for *m* units of prizes. The winner reports the bid simultaneously and each player's valuation is the private information. We assume that each player bids unilaterally for the contest which is done only once and can attain maximum one prize. The first prize which is the most valuable goes to the highest bidder, and the second goes to the next highest, and so on. About each bidder's behavior, we assume the risk-neutral behavior of each bidder. The game, therefore, is a Bayesian Game with the information of prizes and their probabilities of winning.

The analysis starts from identifying the equilibrium bidding strategy, followed by the expected revenue. To show the clearest result as possible, this paper mainly focuses on the effect of valuation of prizes and lags on the revenue, isolating other issues such as the number of prizes or participation. By the same token, the equilibrium discussed here follows the revelation principle from which all bidders follow the same strategy profile. The result shows that biddings in equilibrium follow the natural conjecture that higher valuation and lower level of lags are linked with the higher bids. This means that players bid more fiercely when they expect higher gain from winning either the top prize or depreciated one. As a matter of the revenue, the same relationship is found with variables and the expected revenue as for the equilibrium strategy. The notable finding here is that the dissipation in the revenue is always expected to be higher than the summation of all lags.

Another interest can be a comparison between multiunit *pay-as-bid* and single-unit all-pay auction. Many emoluments for competitors such as athletics tend to follow 'the winner takes it all' structure. Though it is the outcome of the competition each other to get more valuable prizes, the efficiency of concentrating prizes only to one winner is not thoroughly analyzed. The numerical example discusses this intuition to show how the revenue would change between single-unit and multi-unit auctions with the same amount of total valuations.

Section II provides the structure of the model. Given the structure, Section III optimizes the payoff function and provides the equilibrium bidding strategy. The first and the third proposition in Section III show that the originally complex first order condition can be simplified after introducing the lag. Remaining two propositions tells about the revenue under the equilibrium strategy. Section IV simulates the auction between the single-unit and the multi-unit auction, and results that the expected revenue from the single-unit all-pay auction is higher than the revenue from the two-unit *pay-as-bid* auction. Section V concludes the paper.

II. Models

The model considers the *pay-as-bid* auction with N players competing for m units of prizes. The winner reports the bid simultaneously and each player's valuation is the private information. We assume that each player bids unilaterally for the contest which is done only once, and in which each bidder can attain maximum one prize. The first prize which is the most valuable goes to the highest bidder, and the second goes to the next highest, and so on. Without the loss of generality, the number of prizes is smaller or the same as the number of bidders, $m < N$[2].

To recognize differences on valuations in prizes, the lag concept is adopted to the model. The first model is built with the same lag size, and the second model relieves the assumption so that each lag can have different values. However, lags should be jointly smaller than the value of the first prize. This is the condition to guarantee to have a nonnegative valuation for any prize. We also assume the risk-neutral behavior of each bidder. Therefore, with the information of prizes and their probabilities of winning, the game is a Bayesian Game. That means the expected valuation of the contest is the linear summation of prizes multiplied by its probability of winning.

Due to the symmetry of valuations, each bidder expects that the payoff of the top prize follows the same differentiable distribution function, that is, $x_i \sim F[\underline{v}, \overline{v}]$. The nature draws the exact value of x_i randomly from the given distribution function. Furthermore, given the assumption of the private information, the valuation of each bidder is known only to the

[2] The maximum number of prizes is the number of participants since each bidder can maximally attain one prize. Also, when the number of prizes is the same with the number of bidders, no endeavor for each bidder is needed to win the prize. This is because even though the bidder pays nothing to the auctioneer, the prize would be granted, thus the valuation of the least valued product can be viewed as a minimum guarantee. Considering that the bidders increase their bid to increase more valued prizes, the minimum guarantee does not affect the bidding strategy after the participation is decided.

bidder herself. At the final stage, the valued product is only given to the bidder who wins the pertaining prize.

Assuming that x_i^k denotes the valuation of the *k-th* most valuable prize for player *i*, the difference between x_i^{k+1} and x_i^k to be *l*, which is strictly positive. Furthermore, the valuation of any prize should not be negative in the first model. If it is the case, some players might not want to win the negatively valued prizes rather than to gain nothing. Following that, they bid zero amounts of bids (or negative bids if possible) and also affect others' bidding strategy. Following mathematical formula prevents this situation from having the abovementioned problem;

$$\forall i \in \{1,2,\cdots,N\}, x_i^k = x_i - (k-1)l \;\; s.t. \;\underline{v} > (m-1)l$$

After that, the second model lexes the assumption of the constant lag, which is represented into

$$\forall i \in \{1,\cdots,N\}, x_i^1 = x_i, and\; x_i^k = x_i - \sum_{t=1}^{k-1} l_t \;\; s.t. \underline{v} > \sum_{t=1}^{m-1} l_t$$

The payoff function of each x_i type of individual who adopts the type *y* bidder's strategy is defined as the linear combination of the expected benefit of the pertaining prize and its probability of winning, deduced by the irreversible cost of bidding. Hence, the following equation is the payoff function of each bidder:

$$\pi_i\big(x_i, y, b_{-i}(x)\big) = \sum_{k=1}^{m}\big(x_i^k \cdot Pr(\theta_k^{N-1} < y < \theta_{k-1}^{N-1})\big) - b(y) \;\; s.t. \theta_0^{N-1} = \overline{v}$$

Terms represented by the series of the payoff function illustrate the summation of expected benefit of being the *k-th* winner. The possibility of the tie is not computed in the payoff function since the probability of being at a point is zero from given continuous probability distribution. To find the probability of being the *k-th* highest bidder, the knowledge regarding the order statistics is used, where θ_j^{N-1} symbolizes the valuation of the *j-th* highest bidder among other bidders[3]. Also, the payoff function satisfies the probabilistic approach since each event is mutually exclusive. However, the general solution of this payoff

[3] The exact value of each probability is following: Given the same distribution function for every players,

$$x_i \sim F[\underline{v}, \overline{v}], \Pr(\theta_k^{N-1} < y < \theta_{k-1}^{N-1}) = \binom{N-1}{k-1}\big(F(x)\big)^{N-k}\big(1 - F(x)\big)^{k-1}$$

s. t F(x) is the cumulative probability distribution function to x.

function gives the equilibrium strategy only to multi-unit auctions since lag should not be accounted in the single-unit auction.

The second model is offered if lags are not the same between succeeding prizes. The valuations, in this case, are slightly modified since the series adds the different value of the *k-th* prize.

$$\pi_i\big(x_i, y, b_{-i}(x)\big) = \sum_{k=1}^{m}\Big(\big(x_i - \sum_{j=0}^{k-1} l_j\big)\cdot \Pr(\theta_k^{N-1} < y < \theta_{k-1}^{N-1})\Big) - b(y)$$

$$s.t.\, l_0 = 0, and\ \theta_0^{N-1} = 0$$

III. Equilibrium

(1) The equilibrium with the same size of lags

To predict the outcome of the model, we first need to figure out the equilibrium strategy. By taking a derivative of the strategic variable y, we can find a point where the payoff is maximized given other people's strategies. The equilibrium strategy discussed in this paper is attained keeping the revelation principle. The revelation principle states that the equilibrium is under the condition where every player adopts the same monotonic bidding function according to their valuations. As the bidding function is assumed to be strictly increasing, at the suggested equilibrium, the highly valuing type of player has the higher possibility of winning the better prize and every player become a candidly reveals her true preference level by her bid. Afterwards, we need to check whether being truthful is the best response tipping upon other bidders' strategy of revealing true preferences. Proposition 1 suggests the equilibrium strategy from the model (1) followed by the mathematical proof:

Proposition 1 *From the multi-unit all-pay auction with N players and m prizes with the uniform demand, the equilibrium bidding strategy is the increasing function of valuation the top prize, and is not monotonic function of the constant lag, each multiplied by marginal probability of winning the prize. The exact bidding function is stated below:*

$$b(x) = \int_{\underline{v}}^{x}\left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - l \cdot \sum_{k=2}^{m}(k-1)\cdot f_k^{N-1}(x_i)\right) dx_i$$

$$s.t.\ m \geq 2$$

(See the Appendix A for the proof)

Counting on the proposition 1, the equilibrium strategy can be explained by two major effects denoted in each term. The first effect is from expected benefits of getting the prize for when the lag size is zero. This is strictly increasing function of *x*, meaning that the more bidder values the prize the higher bids would be made. The second term is about the effect of the lag on bidding strategy. Here, the lag is multiplied by the probability of winning that prize and the number of duplication of the lag. The marginal effect of the lag on the bidding is not straightforward since, for some higher type of bidders, the marginal effect of the lag is sometimes negative.

The simplest case of the All-Pay auction is definitely a single-unit auction. In such case, the value of m should be 1 with all terms related to the lags should not be considered. Therefore, bidding function in the single-unit auction contains only the first term in the integral, and the model equilibrium does not conflict with the equilibrium bidding strategy in the standard single-unit all-pay auction. However, it is hard to compare between the single-unit and multi-unit auctions due to complexities incurred while formulating the fair total prizes between two auctions. We discuss the issue in the next section by simulating with simple distribution and numeric values.

The equilibrium strategy shows that the bidding strategy is monotonically increasing function of *x* but not of *l*. Since the marginal bidding function is always positive, the increment of *x* widens the interval of integration, thus the amount of bid is the increasing function of the valuation. In contrast, the coefficients multiplied by each lags are dependent upon the valuation of the prize. Rather, it is possible to check how the value of lag affects the bidding strategy by observing two types of bidders. The first group is of bidders who have high value in prizes. Having high value in prize implies that there is a high possibility of winning the prize, given the premise of the revelation principle. Thus, the highly valuing bidders care more about winning the specific prize they levy high value in, rather than any other prizes. On the other hand, bidders who have low valuations are more interested in the valuation of the low valued prize since it is realistic to get it. If the lag increases, the depreciation of the value of the least valued prize increase, so winning at the last place becomes less attractive.

The revenue is the other focus of this model. The model is concerned only with the *pay-as-bid* auction. In this regard, the number of prizes, the size of lag and the total valuation of prizes, the number of bidders are only possible effects on the revenue. The second proposition illustrates the expected revenue from the equilibrium strategy defined on the proposition 1.

Proposition 2 *The expected revenue from the equilibrium strategy with the constant lag is increasing function of* x_i *but the decreasing function of lags. Also, the depreciation of the expected revenue due to the lags exceeds sum of all depreciations from the first prize to all lagged prizes:*

$$R = N \cdot \left(\int_0^1 \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) \right) - l \cdot \frac{(m-1) \cdot m \cdot (m+1)}{3}$$

Proposition 2 states two impacts: the expected revenue changes through the valuation of the top prize, and the size of lag. The first effect is the positive relationship between the valuation of the top prize and the expected revenue. This is in line with common sense that people would compete more fiercely if they expect for more precious prizes. The other tendency is the negative effect of the lag on the expected revenue. As bigger lag size reduces the attractiveness of prizes from the second, bidders would bid less fiercely to get prizes when *l* increases. In addition, the size of the reduced revenue is always higher than the reduced size of lags on the prize scheme whenever m is higher than 1/2.[4] Considering that we are testing the multi-unit auction, the minimum value of m is 2, and therefore the reduction of expected revenue always surpasses the reduction of remunerations from the lag. From the seller's point of view, equalizing prizes is revenue maximizing.

(2) The model with different size of lags

We can also formulate another general equilibrium bidding strategy from the payoff function in the second model using the similar methodology. With the small modification of mathematical derivations, the similar form of equilibrium strategy can be found as follows:

Proposition 3 *The equilibrium bidding strategy is the increasing function of the valuation on the top prize, and the decreasing function of each marginal lags multiplied by its marginal probability of containing prize. The exact bidding function is stated below:*

$$b(x) = \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - \sum_{k=2}^{m} (\sum_{j=1}^{k-1} l_j) \cdot f_k^{N-1}(x_i) \right) dx_i$$

(See the Appendix C for the proof)

[4] The auctioneer will grant $\frac{(m-1)m}{2} \cdot l$ less to winners in the first model than the case when all prizes are valued by x_i. Therefore, the claim is verified by subtracting $l \cdot \frac{(m-1) \cdot m \cdot (m+1)}{3}$ to $\frac{(m-1)m}{2} \cdot l$.

Even though the similar structure of the bidding function is made between two propositions, one of two effects in proposition 1 is different in proposition 3. The first terms in equilibrium strategy in both models are the same since all factors other than the lag remained to be the same between two models. However, the second series within the integration is changed since lag structure between two models is different. With the same probability of winning for each ranking of prize followed by the revelation principle, the second model instead looks for the marginal effect of the lag by each.

The major difference in two model equilibria are that the marginal effect of the lag on biddings is ambiguous in the first model, but strictly negative in the second model. This seemingly contradictory phenomenon is attributed to that, in a risk-neutral setting, the marginal increase in the *k-th* lag only decreases the marginal benefit of achieving one step less valuable prize, that is, achieving the *k-th* prize instead of the *(k+1)-th prize*. In this regard, all bidders face the lower expected benefit from their efforts, leading the negative shock to the bidding function.

In contrast, the overall increase of the lag spurs the advantage to be the one step higher winner of the auction for bidders who have higher valuations. Given that they conject relatively high probabilities of winning at least the minimal prize, their concerns are more likely to be the ranking of the winning prize. Therefore, they suppose the valuation of the least valued prize as a guaranteed value with high certainty, and lags become the marginal value from the competition between winners. This situation brings a new subgame of multi-unit auction among winners, and lags are the new valuation of the competition. As a result, the unequal distribution of prizes brings fiercer competition among highly valued bidders. Nevertheless, this behavior only pertains to highly valuing bidder. In this regard the overall effect of the lag on the bidding strategy is ambiguous.

Even with this controversy between equilibrium strategies in two models, the revenue in the second model follows the same structure as in the first model. Proposition 4 reassures that two effects found in proposition 2. Here, in the second model, the effect of the lag on the expected revenue does also surpass all differences of prizes from the most valued one.

Proposition 4 *The expected revenue from the equilibrium strategy with non-constant lags is increasing for the valuation of the first prize but is decreasing for the size of lags. Also, the reduced amount of the expected revenue due to the lags exceeds sum of all depreciations as in the proposition 2.*

$$R = N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) \right) dx_i \, dF(x) \right) - \sum_{k=2}^{m} \left((\sum_{j=1}^{k-1} l_j) \cdot k \right)$$

(See Appendix D for the proof)

The second model is the generalized version of the first model. Proposition 4 assures that the lag affects the expected revenue, but effects of each lag are with different size. l_1 affects the valuation of all prizes except the first one, thus alters the expected revenue mostly and l_m alters minimally from the opposite reason. Therefore, even the total lag size is assumed to be the same, the expected revenue can be modified by interchanging lag sizes. For this reason, in multi-unit *pay-as-bid* auction, the auction designer should care not only the valuation of the first prize but also the distribution of different level among them. Ideally, her revenue is expected to be maximized when the prize is equally distributed in this model setting.

However, even after above propositions are adduced, the actual policy implication remains as a debate. This is attributed to the possibility that not all bidders' efforts do not benefit the seller. For instance, in the contest exhibit case, not all ideas or suggestions contributes to the payoff the event holder. In such cases, since the size of the lag differently affects the bidding strategy to each type of bidders, expected valuations of winners affect the expected payoff of the contest designer. In such a case, the auctioneer also has to know the exact information about how much winners value when designing the optimal lag size.

Summing up, the finding of this section is that in multi-unit all-pay auction, the lag is a relevant factor of bidders' behavior. The main conclusion from the proposition 1 and 3 is that the valuation and the depreciation affect the bidding strategy differently, but only the valuation is relevant to the total expected revenue. Next section analyzes two different auction schemes between single-unit and multi-unit auction with simple numbers and the distribution in a fair environment.

IV. Numerical Example

In this section, we do the simple numeric example and discuss two remaining issues. The first thing is whether the revenue generated is the same between the single-unit and the multi-unit auction. The second aspect is whether the value of lags affects the bidding strategy or the total revenue when the total value of prizes is fixed. In this regard, the simple example suggested here tries to explain how each factor affects the outcome. Therefore, this part analyzes the revenue with examples between the single-unit (*m=1*) and the two-unit auction (*m=2*) with the fixed number of *N* as well as the total value. The analysis starts with the comparison of the revenue between auctions with one prize and two prizes. The first model assumes that four bidders compete for one prize. The second model assumes the same number of bidders with two prizes.

For the fair comparison, the total value of prizes in two auctions is assumed to be the same. The valuation for the highest prize is defined as x_i in the single-unit auction model and x_j^1 in the two-unit auction model, where $x_j^1 = \frac{1}{2}(x_i + l)$ and $x_j^2 = x_j^1 - l \ s.t. l < x_j^1$. We hereby assume the valuation in the single-unit model to be uniformly distributed from 1 to 2. The beauty of this uniform distribution is that the cumulative distribution function is equal to the independent variable subtracted by 1, and the density function is constant for 1 within the support.

The model solution showed that the total expected value from the single unit auction is $\frac{8}{5}$ and $\frac{7}{5}$.from the multiunit auction (See the Appendix F for the whole arithmetic). Therefore, the result shows that the single-unit auction generates higher revenue than the multi-unit auction. The explanation can be articulated by comparing two classes of the bidders. The first group is with the bidders who have a valuation of x_i more than$\frac{3}{2}$. Since only one winner can attain the prize in the single-unit auction, they have to be more aggressive to get the prize in that case. In contrast, remaining types of bidders do not have sufficient incentives to bid fiercely in the single-unit auction since they have the smaller possibility of winning even though they become more aggressive. From aggregation, the result of example shows that the single-unit all-pay auction generates the higher revenue than some multi-unit auctions. Therefore, the result shows that the revenue is not always equivalent when the number of prizes is different.

The size of the lags completely averages out while summing the expected revenue in this example. However, it does not mean that the lag size does not affect the revenue at all. From the model solution, the expected revenue should decrease by *-2l* from mere existence from the lag. However, the increment of the valuation on the top prize by $\frac{1}{2}l$ in the model offsets the decrement of expected revenue mentioned above. Therefore, two effects found in the former section persist in the example, but the effect of lags on the revenue is averaged out when the total valuation on all prizes are fixed in this example.

V. Conclusion

The study of the multi-unit auction is involved with multiple considerations and therefore numerous impacts are related each other to the outcome. Accordingly, it is hard to define the general rule and even comparative statics of changing one variable concludes ambiguously since most effects are related to the size of other fixed variables. Nevertheless, proper idealization of the model into the simple form allows explaining tendencies caused by each choice variable. Among them, we investigated effects of the valuation and the lag on the revenue under *pay-as-bid* multi-unit auction under symmetric private information.

Following that, the main purposes of this paper are twofold; one is to check the relevance between the distribution of the valuation on the prize and the revenue, and the other is the equilibrium strategy and the expected revenue while lags are present. Firstly, the paper reaffirms previous findings that the effect of valuation on bidding strategy is unambiguous as higher reward incentivizes bidders to be more aggressive. Unlike other previous literatures, this paper added distributional issues and concludes that the effect of the lag on biddings is not straightforward. Proposition 1 shows that, when the value of the low ranked prize is depreciated by the same size of the lag, the size of lag affected bidding behaviors differently according to the type, whereas the causality of different size of the lags on the equilibrium bidding strategy is unambiguously positive according to the proposition 3. In connection with the revenue and the lag size, both proposition 2 and 4 show that the lag size negatively affects the revenue, even more than the overall sum of depreciations of remunerations.

Another possible variation of the model can be the number of prizes itself, as done in Cohen and Sela (2008), which compares the expected revenue between the multiunit pay-as-bid auction and the single-unit all-pay auction with asymmetric information. We do the similar analysis with the symmetric information which finds that single-unit auction generated higher revenue when bidders have a uniform distribution of valuation. However, our example shows the opposite dominance, thus more detailed analysis considering the valuation distribution function needs to be attained.

The remaining concern is about the revenue when all costs incurred from bidders do not benefit the seller. In this case, suggested propositions cannot be used to design the optimal auction scheme since totally different mechanism might be needed. To solve the question, it requires the information with respect to the expected valuation of only a few minded numbers of highest bidders and we keep this issue for the future research.

Appendix

Appendix A The proof of proposition 1

Proof

Since the equilibrium bids always follow the strategy of revealing her true valuation, the first derivative should be zero where $y = x_i$. From the profit function suggested by the first model, the first order condition shows a candidate of the equilibrium strategy.

$$by\ F.O.C., \frac{\partial \pi_i(x_i, y, b_{-i}(x))}{\partial y} = x_i \cdot \sum_{k=1}^{m} \frac{\partial F_k^{N-1}(y)}{\partial y} - l \cdot \sum_{k=2}^{m} (k-1) \cdot \frac{\partial F_k^{N-1}(y)}{\partial y} - b^{'}(y)$$

$$= x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - l \cdot \sum_{k=2}^{m} (k-1) \cdot f_k^{N-1}(x_i) - b'(x_i) = 0$$

$F_k^{N-1}(y)$ denotes the probability that type y player has the k-th highest valuation among N players. Additionally, $f_k^{N-1}(y)$ is assumed to be the differentiated value of $F_k^{N-1}(y)$ in terms of y. By isolating the differentiated bidding function, above equation can be simplified. Firstly, by moving all other terms except $b'(x_i)$, it becomes

$$\mathrm{b}'(\mathrm{x_i}) = x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - l \cdot \sum_{k=2}^{m} (k-1) \cdot f_k^{N-1}(x_i)$$

The equilibrium strategy can be attained as suggested in proposition 1 by integrating both sides from the lowest support to the realized value x, and proving that the bids at the lowest support are 0 as described in the model part.

To check whether the bidding function is indeed an equilibrium strategy, the expected payoff from revealing its true valuation through the bidding function should be higher than feigning as having any other valuations. By comparing the payoff from choosing b(x) and the payoff from choosing b(y) where y is an arbitrary number other than x within the support, the revelation principle can be confirmed. The test can be done by first finding the expected payoff for each strategy.

$$b(y) = \int_{\underline{v}}^{y} \sum_{k=1}^{m} z \cdot f_k^{N-1}(z) - l \cdot \sum_{k=2}^{m-2} (k-1) f_k^{N-1}(z) dz$$

Then, the expected profit of feigning as y type is

$$\pi(x, y) = \int_{\underline{v}}^{y} \sum_{k=1}^{m} (x_i - (k-1)l) \cdot f_k^{N-1}(z)\, dz - b(y)$$

$$= \int_{\underline{v}}^{y} \sum_{k=1}^{m} (x_i - (k-1)l) \cdot f_k^{N-1}(z) - z \cdot \sum_{k=1}^{m} f_k^{N-1}(z) + l \cdot \sum_{k=2}^{m} (k-1) f_k^{N-1}(z) dz$$

$$= \int_{\underline{v}}^{y} (x_i - z) \sum_{k=1}^{m} f_k^{N-1}(z)\, dz$$

After that, the dominance of bidding strategy from revealing her true type can be confirmed as suggested in the following equation.

$$\forall y \in [\underline{v}, \overline{v}], \pi(x,x) - \pi(x,y) = \int_{y}^{x} (x_i - z) \sum_{k=1}^{m} f_k^{N-1}(z)\, dz \geq 0$$

The strict positivity of the equation can be verified by assuming two possible cases. If $x \geq y$, the integration is done for positive values on the positive interval. Contrarily, if $x \leq y$, the integration adds negative value for negative interval if $x \leq y$. Therefore, the equation is always positive leading to the conclusion that being truthful is the best strategy one can take.

The next step is to check whether the bidding function is strictly increasing function of x. This can be proved by the fact that the marginal bidding function is always positive within the support. By changing the structure of the equilibrium bidding strategy, the property can be easily attained. The following property is used to reconstruct the equation into the following testable form.

$$l \cdot \sum_{k=2}^{m} (k-1) \cdot f_k^{N-1}(x_i)$$

$$= l \cdot (N-1) \left(\{F(x_i)\}^{N-2} + (m-1) \cdot \binom{N-2}{m-2} \cdot \{F(x_i)\}^{N-m} \{1 - F(x_i)\}^{m-2} - \sum_{k=2}^{m-2} (F_k^{N-2}(x_i)) \right)$$

$$= l \cdot \left(\sum_{k=1}^{m} f_k^{N-1}(x_i) - (N-1) \cdot \sum_{k=2}^{m-2} (F_k^{N-2}(x_i)) \right)$$

Therefore, the equilibrium bidding strategy can be modified as the summation of two positive terms as in the following equation as shown below. The first term is positive since the lag is assumed to be smaller than the valuation of the first winning prize and the summation of all marginal possibility is also higher than zero, as proven by Appendix E. For the second term, every component is by structure positive, including the series of summing all probabilities of winning. Thus, the monotonic positivity of bidding function is verified.

$$b'(x) = (x_i - l) \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) + l(N-1) \cdot \sum_{k=2}^{m-2} F_k^{N-2}(x_i)$$

Appendix B The expected revenue of the first model

Proof

The revenue is calculated from integrating the bidding function from the lowest value of *x* to the highest value multiplied by the probability of having the specific value *x*. The symmetric valuation structure for all N players requires multiplying N to the expected revenue of one individual. To look at the effect of valuation of the first prize and the lag on the revenue separately, we divide the equilibrium bidding function into two different terms, each by the valuation and the lag.

$$N \cdot E[b(x)] = N \cdot \int_{\underline{v}}^{\overline{v}} f(x) \cdot b(x) dx$$

$$= N \cdot \int_{\underline{v}}^{\overline{v}} f(x) \cdot \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - l \cdot \sum_{k=1}^{m} (k-1) \cdot f_k^{N-1}(x_i)\, dx_i \right) dx$$

$$= N \cdot \int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - l \cdot \sum_{k=1}^{m} (k-1) \cdot f_k^{N-1}(x_i)\, dx_i \right) dF(x)$$

$$= N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) - \int_{0}^{1} \int_{\underline{v}}^{x} l \cdot \sum_{k=1}^{m} (k-1) \cdot f_k^{N-1}(x_i)\, dx_i\, dF(x) \right)$$

$$= N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) - l \cdot \sum_{k=1}^{m} (k-1) \cdot \int_{0}^{1} \int_{\underline{v}}^{x} f_k^{N-1}(x_i)\, dx_i dF(x) \right)$$

$$= N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) - l \cdot \sum_{k=1}^{m} (k-1) \cdot \int_{0}^{1} F_k^{N-1}(x) dF(x) \right)$$

After that, the each effect is reduced through the integration procedure. Especially, the integration of $F_k^{N-1}(x)$ in term of *F(x)* from 0 to 1 can be calculated using the integration by parts[5]. Repeating the integration by parts reduces the integration into the arithmetic sum of m as in the proposition 2.

$$= N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) - l \cdot \sum_{k=1}^{m} (k-1) \cdot \binom{N-1}{k-1} \cdot \frac{k!\,(N-k)!}{(N)!} \right)$$

[5] The formulae for the method of integration by parts is the following;
$\int F_k^{N-1}(x) dF(x) = \binom{N-1}{k-1} \int \big(F(x)\big)^{N-k} \big(1-F(x)\big)^{k-1} dF(x) = \binom{N-1}{k-1} \frac{(N-k)}{k} \int \big(F(x)\big)^{N-k} \big(1-F(x)\big)^{k-1} dF(x)$

$$= N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) - l \cdot \sum_{k=1}^{m} \frac{k(k-1)}{N} \right)$$

$$= N \cdot \left(\int_{0}^{1} \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i \right) dF(x) \right) - l \cdot \sum_{k=1}^{m} k(k-1)$$

By calculating the arithmetic sum of the equation, the expected revenue in proposition 2 can be attained.

Appendix C The proof of proposition 3

Proof

Deriving the equilibrium in the second model follows the same procedure as in the Appendix A. Firstly, the revelation principle holds when the first derivative of the payoff function is zero at $y = x_i$. Therefore, the first order condition can be suggested as following:

$$by\ F.O.C., \frac{\partial \pi_i\big(x_i, y, b_{-i}(x)\big)}{\partial y}$$

$$= x_i \cdot \sum_{k=1}^{m} \frac{\partial F_k^{N-1}(y)}{\partial y} - \sum_{k=2}^{m} \left((\sum_{j=1}^{k-1} l_j) \cdot \frac{\partial F_k^{N-1}(y)}{\partial y} \right) - b'(y)$$

$$= x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - \sum_{k=2}^{m} \left((\sum_{j=1}^{k-1} l_j) \cdot f_k^{N-1}(x_i) \right) - b'(x_i) = 0$$

$$\therefore \mathrm{b}'(x_i) = x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - \sum_{k=2}^{m} \left((\sum_{j=1}^{k-1} l_j) \cdot f_k^{N-1}(x_i) \right)$$

Therefore, the equilibrium strategy as in Proposition 2 can be attained by integrating both sides from the lowest bound to x.

Secondly, we need to check whether the suggested strategy is essentially equilibrium. To do so, the equilibrium strategy needs to be modified into

$$b(y) = \int_{\underline{v}}^{y} \sum_{k=1}^{m} z \cdot f_k^{N-1}(z) - \sum_{k=2}^{m} (\sum_{j=1}^{k-1} l_j) \cdot f_k^{N-1}(z) dz$$

[6]

Then, the expected profit of feigning as y type is

$$\pi(x, y) = \int_{\underline{v}}^{y} \sum_{k=1}^{m} \left(x_i - \sum_{j=1}^{k-1} l_j \right) \cdot f_k^{N-1}(z)\, dz - b(y)$$

$$= \int_{\underline{v}}^{y} \sum_{k=1}^{m} \left(x_i - \sum_{j=1}^{k-1} l_j \right) \cdot f_k^{N-1}(z) - z \cdot \sum_{k=1}^{m} f_k^{N-1}(z) + \sum_{k=2}^{m} (\sum_{j=1}^{k-1} l_j) \cdot f_k^{N-1}(z) dz$$

$$= \int_{\underline{v}}^{y} (x_i - z) \cdot f_1^{N-1} + \sum_{k=2}^{m} \left((x_i - \sum_{j=1}^{k-1} l_j - z) \cdot f_k^{N-1}(z) \right) + \sum_{k=2}^{m} \left((\sum_{j=1}^{k-1} l_j) f_k^{N-1}(z) \right) dz$$

$$= \int_{\underline{v}}^{y} \left(\sum_{k=1}^{m} (x_i - z) \cdot f_k^{N-1}(z) \right) dz$$

Therefore, the expected marginal profit from deviating from equilibrium strategy becomes

$$\forall y \in [\underline{v}, \overline{v}], \pi(x, x) - \pi(x, y) = \int_{y}^{x} (x_i - z) \sum_{k=1}^{m} f_k^{N-1}(z)\, dz \geq 0$$

As a result, being truthful is also the best strategy one can take.

To show that the equation is the monotonically increasing function of x, the original payoff function starts from the modified version. Instead of letting all different lags be on the payoff function, let the *i-th* player's valuation on *k-th* prize to be x_i^k. Then the modified payoff function becomes

$$\frac{\partial \pi_i}{\partial y} \left(x_i, x_i, b_{-i}(x_{-i}) \right) = \sum_{k=1}^{m} x_i^k \cdot f_k^{N-1}(y) - \mathrm{b}'(y) = 0$$

Note that, $for\ k \geq 2,\ f_k^{N-1}(y) = (N-1) \cdot \left(F_k^{N-2}(y) - F_{k-1}^{N-2}(y) \right)$.

[6] From the initial condition (1), the first order condition becomes

$$\sum_{k=1}^{m} x_i f_k^{N-1}(y) - l \sum_{k=2}^{m} (k-1) f_k^{N-1}(y) - b'(y) = 0$$

where $f_i^N(y)$ denotes the differentiated order function.

$$\sum_{k=1}^{m} x_i^k \cdot f_k^{N-1}(x_i)$$

$$= (N-1)\left(x_i^1\left(F_1^{N-2}(x_i)\right) + x_i^2\left(F_2^{N-2}(x_i) - F_1^{N-2}(x_i)\right) + \cdots + x_i^m\left(F_m^{N-2}(x_i) - F_{m-1}^{N-2}(x_i)\right)\right)$$

$$= (N-1)\left(F_1^{N-2}(x_i)(x_i^1 - x_i^2) + \cdots + F_{m-1}^{N-2}(x_i)(x_i^{m-1} - x_i^m) + x_i^m \cdot F_m^{N-2}(x_i)\right)$$

$$= (N-1)\left(\sum_{k=1}^{m-1} F_k^{N-2}(x_i) \cdot l_k + x_i^m \cdot F_m^{N-2}(x_i)\right)$$

$$\therefore \mathrm{b}'(x_i) = (N-1) \cdot \left(\sum_{k=1}^{m-1} l_k \cdot F_k^{N-2}(x_i) + \left(x_i^k\right) \cdot F_m^{N-2}(x_i)\right)$$

Since $\mathrm{b}'\left(x_j\right)$ is positive for any value of x_j within the interval, the bidding function is strictly increasing.

Appendix D The expected revenue of the second model

Proof

The calculations to find the revenue in the second model follow the same procedure as in the first model with a slight difference in notations. Therefore, following equations show steps to find the expected revenue using the equilibrium strategy in proposition 3.

$$N \cdot E[b(x)] = N \cdot \int_{\underline{v}}^{\overline{v}} f(x) \cdot b(x)dx$$

$$= N \cdot \int_{\underline{v}}^{\overline{v}} f(x) \cdot \left(\int_{\underline{v}}^{x} \sum_{k=1}^{m} x_i^k \cdot f_k^{N-1}(x_i)\, dx_i\right) dx$$

$$= N \cdot \int_0^1 \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i) - \sum_{k=2}^{m}(\sum_{j=1}^{k-1} l_j) \cdot f_k^{N-1}(x_i)\, dx_i\right) dF(x)$$

$$= N \cdot \left(\int_0^1 \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i\right) dF(x) - \int_0^1 \int_{\underline{v}}^{x} \left(\sum_{k=2}^{m}\left(\sum_{j=1}^{k-1} l_j\right) \cdot f_k^{N-1}(x_i)\, dx_i\right) dF(x)\right)$$

$$= N \cdot \left(\int_0^1 \int_{\underline{v}}^{x} \left(x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i\right) dF(x) - \sum_{k=2}^{m}(\sum_{j=1}^{k-1} l_j) \cdot \int_0^1 F_k^{N-1}(x) dF(x)\right)$$

$$= N \cdot \left(\int_0^1 \int_{\underline{v}}^{x} [x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i\, dF(x) - \sum_{k=2}^{m} (\sum_{j=1}^{k-1} l_j) \cdot \frac{k}{N} \right)$$

$$= N \cdot \left(\int_0^1 \int_{\underline{v}}^{x} [x_i \cdot \sum_{k=1}^{m} f_k^{N-1}(x_i)\, dx_i\, dF(x) \right) - \sum_{k=2}^{m} (\sum_{j=1}^{k-1} l_j) \cdot k$$

Appendix E The positivity of $\sum_{k=1}^{m} f_k^{N-1}(z)$

Proof

$$\sum_{k=1}^{m} f_k^{N-1}(z) = f(z)^7 \Bigg((N-1)\big(F(z)\big)^{N-2} + \left((N-1)(N-2)\big(F(z)\big)^{N-3}\big(1-F(z)\big) - (N-1)\{F(z)\}^{N-2} \right) + \cdots + \left(\frac{(N-1)\cdots(N-k)}{(k-1)!}(N-k-1)\big(F(z)\big)^{N-k-1}\big(1-F(z)\big)^{k-1} - \frac{(N-1)\cdots(N-k)}{(k-2)!}\big(F(z)\big)^{N-k}\big(1-F(z)\big)^{k-2} \right) \Bigg)$$

$$= f(z) \left((N-1)\{F(z)\}^{N-2} + \frac{(N-1)\cdots(N-k)}{(k-1)!}(N-k-1)\{F(z)\}^{N-k-1}\big(1-F(z)\big)^{k-1} \right) \geq 0$$

Appendix F The solution of the numerical Example

1. Single Unit Auction

$\pi_i = x_i\big(F(y)\big)^3 - b(y) = x_i(y-1)^3 - b(y)$

Due to the revelation principle, the optimization of y should be at x_i

$by\ F.O.C., \frac{\partial \pi_i}{\partial y} = 3x_i(x_i-1)^2 - b'(x_i) = 0$

$\therefore 3x_i(x_i-1)^2 - b'(x_i) = 0$

$$b(x) = \int_1^x 3x_i(x_i-1)^2 dx_i = \frac{3}{4}x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{4}$$

[7] *F(z)* denotes the cumulative probability distribution function until z Also, *f(z)* denotes the differentiated function of *F(z)*.

Therefore, the expected revenue is

$$R = NE[b(x)] = 4 \cdot \int_1^2 b(x)dx = 4\int_1^2 \left(\frac{3}{4}x^4 - 2\text{x}^3 + \frac{3}{2}x^2 - \frac{1}{4}\right)dx = \frac{8}{5}$$

2. Two-Unit Auction

$$\pi_i = \frac{1}{2}x_i\left(F_1^3(y) + F_2^3(y)\right) + \frac{1}{2}l\left(F_1^3(y) - F_2^3(y)\right) - b(y)$$

Due to the revelation principle, the first order condition is satisfied where $y = x_i$

$$by\ F.O.C., \frac{\partial \pi_i}{\partial y} = \frac{1}{2}x_i\left(f_1^3(x_i) + f_2^3(x_i)\right) + \frac{1}{2}l\left(f_1^3(x_i) - f_2^3(x_i)\right) - b'(x_i) = 0$$

$$\therefore \text{b}'(x_i) = 3x_i(x_i - 1)(2 - x_i) + 3l(x_i - 1)(2x_i - 3)$$

$$b(x) = 3\int_1^x x_i(x_i - 1)(2 - x_i) + l(x_i - 1)(2x_i - 3)dx_i$$

$$= -\frac{3}{4}x^4 + 3x^3 - 3x^2 + \frac{3}{4} + l(2x^3 - \frac{15}{2}x^2 + 9x - \frac{7}{2})$$

Therefore, the expected revenue is

$$R = N \cdot E[b(x)] = 4\int_1^2 b(x)dx$$

$$= 4\int_1^2 \left(-\frac{3}{4}x^4 + 3x^3 - 3x^2 + \frac{3}{4} + l\left(2x^3 - \frac{15}{2}x^2 + 9x - \frac{7}{2}\right)\right)dx$$

$$= \frac{7}{5}$$

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