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Pension De-Risking
A Partial Buy-Out Solution

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ERASMUS UNIVERSITEIT ROTTERDAM
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Willis Towers Watson

Abstract

The partial buy-out or carve-out is a new pension fund de-risking solution proposed by Willis Towers Watson. In the partial buy-out only part of the pension fund liabilities are transferred to an insurer. The pensioners are generally more risk averse and have a shorter investment horizon than active participants. The pension fund board is obliged to conduct the fund policy decisions in interest of all participants, making the policy generally too risky for the pensioners and not risky enough for the active participants. In this thesis I establish with an ALM study whether transferring the entitlements of the pensioners to an insurer through a partial buy-out can benefit all participants of the fund. The pensioners can benefit from ensured entitlements and possibly indexations, whereas the active participants can benefit from the pension fund policy being more aligned to their preferences.

Keywords: Buy-Out, Partial Buy-Out, Carve-Out, Penion Fund, ALM, Pension De-Risking

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1 Introduction

In recent years the funding statuses of pension funds with defined benefit schemes have increasingly come under pressure. Due to declining interest rates and continuous unanticipated improvements in the life expectancy[1], the liabilities have increased in value, whereas return on investments was not sufficient enough to compensate the increase in liabilities (Lin et al. (2015), Biffis and Blake (2013)). Adding to that, new Dutch and European regulations for financial institutions have been introduced. Dutch pension funds must now comply with more stringent rules for the valuation of the liabilities and the indexation policy. Risks must be monitored more strictly and larger capital buffers are required, thereby making the pension funds risk management more complicated and capital intense.

The resulting higher operating costs and pension funding deficits have adverse effects for both the participants of the fund, as for the sponsoring company. For the participants indexation is very unlikely in the upcoming years and even reductions of the entitlements might be necessary for pension funds to ensure their obligations. The sponsoring companies of pension funds suffer higher contributions. This increase in contributions leads to decreased investments from the sponsoring company (Rauh (2006)), potentially resulting in decreased share value.

With this combination of market circumstances and stricter regulatory requirements, pension funds have increasing incentives to reduce the risk of their liabilities. The three most common pension de-risking solutions are: a pension buy-out, a pension buy-in and longevity hedge strategies (Coughlan et al. (2013)). A new strategy in this field is the partial buy-out, which will be named carve-out in this thesis[2]. An important requirement for a carve-out to be an attractive solution is that it must be beneficial for all stakeholders of the fund. The aim of this thesis is to examine whether a carve-out can be beneficial for both the remaining as the transferred participants of the fund.

A buy-out transfers the pension obligations and assets to an insurer, where the transferred liabilities are no longer the fund's obligations. The buy-out is a bulk annuity contract for the participants of the fund, ensuring a fixed pension income with possibly indexation included. A buy-out therefore eliminates all possible risks involved with both the liabilities and assets for both the pension fund and the sponsoring company. A carve-out is a buy-out where only part of the fund's liabilities are transferred. The pension buy-in is similar to a buy-out. A buy-in transfers the risk from the fund to an insurer by paying a premium in exchange for a bulk annuity that matches the fund's future obligations. With a buy-in the liabilities remain on the balance sheet of the pension fund. However, these liabilities are perfectly matched by the annuity contract on the asset side of the balance sheet. In case of a buy-in, the risk of default of the counter-party arises for the pension fund, whereas with a buy-out the participants bear this risk. This makes a buy-in generally cheaper for the pension fund. Lin et al. (2016) give a clear overview of the differences between buy-ins and buy-outs and their implications for pricing. Similar to the carve-out, also a carve-in or partial buy-in can be considered as possible de-risking solution. In this thesis the counter-party risk is assumed to be zero, the results can thus also

[1]See Cox et al. (2013)

[2]Carve-out is a term introduced by Willis Towers Watson. The term more specifically refers to a partial buy-out where only the pensioners are transferred from the pension fund.

be used to asses the attractiveness for the participants of a carve-in under this assumption.

A buy-out is a relatively expensive de-risking solution, however, an important advantage over the other solutions is that it mitigates all risks for the fund and sponsoring company associated with the assets and obligations (Blake et al. (2008), Bertocchi et al. (2010)). The advantage of a pensioners carve-out compared to a buy-out is that buy-outs become cheaper when pension funds have a lower duration. European insurers are required to hold capital buffers for the risks on their balance sheets. This European regulatory framework is known as Solvency II. With a lower duration the amount of interest rate risk is relatively smaller, which makes the Solvency requirement for the pensioners obligations relatively lower. Furthermore, to buy full indexation for the pensioners a lower funding ratio is required than the fund itself needs to be allowed to pay full indexation following the Dutch pension regulations.

In the decision for a de-risking strategy the board of a pension fund must take the interests of all stakeholders of the fund into account. The interests of the participants and the sponsoring company can be different, but also the interests and risk preferences of young and old participants are quite different. The sponsoring company mostly wants to reduce its pension risk as cost efficient as possible. Lin et al. (2015) show that buyouts create more value than longevity hedges in the enterprise risk management framework. The reason being that buy-outs provide more freedom for a firm to engage in riskier projects with high expected returns.

The participants desire an indexed pension, where the risk a participant is willing to take to achieve this depends on his risk aversion. Generally older participants are more risk averse than younger participants (Campbell and Viceira (2002)). A buy-out is therefore more attractive to older participants, as this assures the pension payments for the participants. For younger participants the combination of a longer investment horizon and lower risk aversion can make a buy-out suboptimal. By taking more risk, a higher indexation can be pursued. A partial buy-out or carve-out, where only the pensioner's assets and liabilities are transferred to an insurer, can therefore be an attractive solution for both young and old participants. The pensioners gain by eliminating the risk of pension reductions in the short term. Whereas the remaining participants gain by being able to adjust the fund's investment policy to their risk preferences and thereby increase the probability of indexation in the future.

In this thesis I analyse whether a carve-out is interesting from the participants perspective and how the assets can best be distributed between the active participants and pensioners. For this purpose, I analyse the development of the pension fund with and without a carve-out by means of an Asset Liability Management (ALM) study, where the participants are assumed to derive utility from their benefits. ALM models are often used to analyse the impact of policy decisions on the dynamics of pension funds, e.g. Boender (1997) and Dert (1995). ALM models provide good insights of the effects of policy decisions on both the asset and liability side of the balance sheet. In the ALM literature various approaches have been proposed. To analyse the impact of a carve-out I follow Hoevenaars and E. Ponds (2007) by using a value-based approach. In this type of ALM model the financial market is modelled under a no-arbitrage assumption consistent with asset pricing theory. Next to the general insights that the standard ALM approach can offer, a value-based approach also provides insights in the market value of embedded options in the pension contract. Hoevenaars and Ponds (2008b) use this approach to gain insights in the inter-generational value transfers caused by policy decisions of the fund management. An important management decision that is involved in a carve-out is how to divide the assets. The management of the fund should strive to divide the assets in a fair manner, where all participants benefit equally. In the carve-out setting fair can be considered as a distribution of assets that results in as small as possible value-transfers. A value-based ALM study enables these value transfers to be analysed. I compare several methods to split the assets based on value neutrality and the resulting utility distribution. As more intuitive assets distribution rules I split the assets based on the nominal, real and regulatory funding ratio and the expected indexation. To judge the intuitive methods in value neutrality I also split the assets based on the no-arbitrage value of the entitlements. And to determine whether a win-win situation is possible I split the assets based on indifference for the pensioners.

The pension fund is assumed to have a defined benefit pension scheme with conditional indexation. This is the most common defined benefit scheme in the Netherlands[3]. Defined benefit schemes ensure the participants a certain level of benefits, where the premium can be adjusted to finance these obligations. In this type of scheme the risks associated with the fund are borne by the fund and the sponsoring company. Conditional on the financial position of the fund, the management can decide to grant indexation, the compensation for the devaluation of entitlements caused by inflation. The success of the carve-out is determined by the utility derived by the participants from the pension payments. The participants are assumed to have Constant Relative Risk Aversion (CRRA) preferences. First I analyse the carve-out under a homogeneous risk aversion assumption, thereafter I assume heterogeneity in the risk aversion of active participants and pensioners.

I find that under the assumption of homogeneous risk aversion no mutually beneficial carve-out is possible for the stylised pension funds I define. A carve-out increases the duration of the pension fund, which results in an increased volatility of the regulatory funding ratio. This increased risk can not be hedged efficiently as the value of the liabilities is determined on a fictional interest rate, namely the Ultimate Forward rate. A carve-out is more attractive for the participants if the fund has a higher funding ratio, this leads to higher funding benefits for the fund as the pensioners require less assets than the funding ratio attributes to their entitlements. A carve-out is also more attractive for participants of a fund with a relatively low duration. The value-based asset distribution generally leads to a well-balanced distribution in terms of utility. Of the more intuitive distribution rules the expected indexation distribution comes closest to this balance for higher funding ratios. For lower funding ratios this leads to a too large proportion of the assets being attributed to the pensioners. In this case the nominal asset distribution is closer to being value and utility neutral. With heterogeneous risk aversion a mutually beneficial carve-out is possible for funds with a relatively low funding ratio (100%) and for the combination of a high funding ratio (130%) and short duration.

[3]Figure 4 in Appendix A.2 shows the distribution of the most common pension schemes in the Netherlands.

2 Data

To perform a pension fund ALM study multiple types of data are required. For the estimation of the financial market model historical equity, inflation and interest rate data is used. All time series data is collected in a monthly frequency. To model equity returns, the MSCI World Total Return Index is used with the corresponding dividend yield. As bond data I use the German government bond constant maturity rates for maturities 5 years, 10 years and 30 years. Both the MSCI World Index with corresponding dividend yield as the German bond rates are from Bloomberg. The 1-month Euribor rate will be used to denote the risk-free rate in the model. As measure of the price inflation I use the European HICP seasonally adjusted. Both the Euribor rate and HICP inflation are available at the Statistical Data Warehouse of the European Central Bank[4]. The above mentioned data is collected for a period ranging from January 1997 until April 2016. Table 1 gives some summary statistics. The average inflation rate, π, in the sample period is lower than the target inflation rate of the ECB. The kurtosis of the inflation rate is larger than 3, indicating that the inflation rate is not normally distributed. The average total net return on the MSCI World Index is equal to 5.4%. The MSCI index returns are negatively skewed and have a kurtosis of roughly 5. Extreme returns are thus more likely compared to normally distributed returns and large negative returns are more likely than large positive returns. The average interest rates are increasing with the maturity, while the volatility decreases for larger maturities.

Table 1: Summary Statistics of the Financial Data

Descriptive statistics of the financial data used in this study. π denotes the European HICP inflation rate, xs denotes the returns on the MSCI World Total Net Return Index, with dy the corresponding dividend yield, $r_{euribor}$ is the 1-month Euribor rate and r_i denotes the i-year German interest rate.

	π	xs	dy	$r_{euribor}$	r_5	r_{10}	r_{30}
Average	1.64%	5.40%	2.23%	2.27%	2.82%	3.43%	4.04%
Std. Dev.	0.58%	15.97%	0.58%	1.65%	1.64%	1.50%	1.42%
Sharpe Ratio	-	0.34	-	-	-	-	-
Skewness	-0.08	-0.90	0.79	-0.05	-0.47	-0.58	-0.51
Kurtosis	3.95	4.99	4.65	1.58	1.93	2.33	2.45

To analyse the development of a pension fund with and without a carve-out, data on the initial composition and development of the fund is required. For the initial composition I make use of fictional self generated data, based on pension fund statistics in the Netherlands and the expert opinion of Willis Towers Watson. Entrance probabilities of new participants are provided by Willis Towers Watson. Furthermore, data on mortality is needed to model the composition of the fund over time. For this purpose I use the mortality rates provided by the dutch actuarial institute[5] for 2016, known as the 'AG sterftetafel 2016'. This file contains both current mortality rates as future expected mortality rates split by age and gender[6]. The composition and development of the fund will be discussed in more detail in Section 4.

The wages used to calculate the pension accruals for the participants is based on

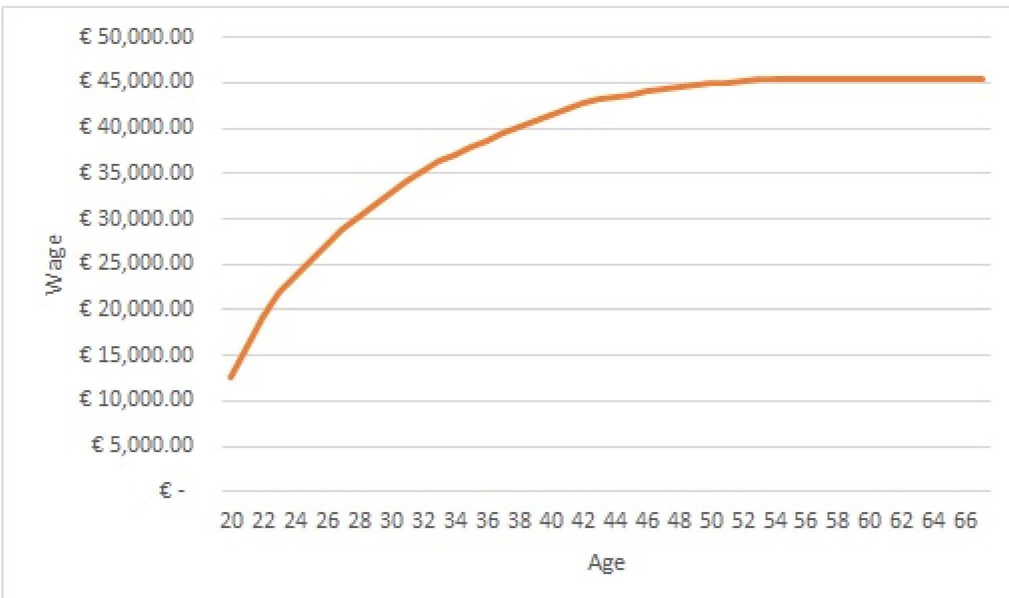
[4]Available at: `http://sdw.ecb.europa.eu/`

[5]Het Actuarieel Genootschap (AG).

[6]Avalaible at: `www.ag-ai.nl`

average wage data per age group in the Netherlands. This data can be collected from the website of the dutch statistics agency, the CBS[7]. The raw wage data can be found in Table 16 in Appendix A. To obtain wages for all ages, I construct a smoothed curve from the raw data. The wages are assumed to remain constant after the age of 50, because the raw data shows some irregularities after that age. Nevertheless, this is a realistic assumption, because productivity is also likely to remain quite constant after this age. Additionally I assume the wages per age group to hold exactly for the average age of that particular group. A curve is obtained by linear interpolation between those points. The resulting curve is shown in Figure 1. In the ALM study I assume wages to grow with the inflation rate over time plus an additional 0.5% wage inflation, thereby following the prescribed approach of the Dutch National Bank (DNB).

Figure 1: Wages Per Age



[7]Centraal Bureau voor de Statistiek. The data set is available at: `http://statline.cbs.nl/Statweb/`

3 Financial Market Model

This section describes the models I use to simulate the dynamics of the financial market. For this purpose I use two separate models. For inflation, equity and dividends I use a Markov Switching VAR model, which is further elaborated in Section 3.1. In the ALM literature VAR models are frequently used to model the dependency in the financial markets, examples can be found in Boender et al. (2007) and Hoevenaars (2008). However, financial time series often show excess skewnnes and kurtosis, heteroskedasticity and time varying correlations, which is not captured by linear VAR models. Ang and Timmermann (2012) argue that regime switching models can successfully capture these characteristics, making these type of models better suited for this purpose. For the interest rates I use a latent factor affine term structure model, which is elaborated in Section 3.2.

Ideally the interest rates and other financial variables would be described with one model, but including interest rates to the regime switching VAR model results in non-stationary interest rates. Therefore, I choose to model interest rates separately, thereby assuming they are independent from inflation, equity and dividends. For interest rates and excess equity returns this assumption will not have a large impact on the results. The correlations between equity returns and interest rates are not very large. Furthermore, the total equity return exist of the risk-free 1-month rate plus the excess return, which will result in small correlations between interest rates and total equity returns. The assumption that inflation and interest rates are uncorrelated is less realistic. In practice periods of low interest rates are often accompanied by low inflation rates and vice versa.

In the simulated scenarios low (high) interests rates with high (low) inflation will be more probable than it would be in reality. For the pension fund in the ALM model this will mean that in times of low interest rates with high inflation, the interest rates will be too low to compensate for inflation. This makes indexation of the entitlements in such scenarios less likely. However, in times of high interest rates with low inflation, the fund can more easily grant indexation. This will slightly impact the volatility of the real pension results of the participants. The assumption will not have a large influence on the carve-out results, as the assumption applies to both the fund with and without a carve-out. The pensioner might benefit slightly more from a carve-out with the independence assumption, because the indexation included in the carve-out does not depend on asset returns and interest rates. The effect, however, will be small.

3.1 Markov Switching Model

3.1.1 Model Specification

To find the best suiting model for inflation, equity and dividends I consider multiple specifications of reduced form Vector Auto-Regressive (VAR) models with regime switches. In this setting the model parameters depend on the current regime, which is unobserved and equal for each variable. Modelling the regimes to be equal across all variables strongly reduces the dimensionality of the model, while still being able to capture the non-linear dynamics of the joint distribution, if the underlying regimes are strongly correlated[8]. The unobserved states are assumed to follow an ergodic time-homogeneous first order Markov process with a finite state space. This type of model was first introduced by Hamilton (1989). A Markov Switching VAR (MSVAR) model in its most general form with M states and p lags is given by

$$x_t = \nu_{s_t} + \Phi_{1,s_t} x_{t-1} + ... + \Phi_{p,s_t} x_{t-p} + \Sigma_{s_t}^{\frac{1}{2}} u_t,$$
$$p_{ij} = P(s_t = j | s_{t-1} = i), \quad \sum_{j=1}^{M} p_{ij} = 1, \tag{1}$$

where $u_t \sim \mathcal{N}(0, I)$ with I the identity matrix and s_t denotes the state at time t. Furthermore, $\Sigma_{s_t}^{\frac{1}{2}}$ is the lower triangular Cholesky decomposition of the covariance matrix of the innovations of x_t in state s_t. In this formulation the intercept ν_{s_t}, auto-regressive matrices Φ_{i,s_t} and covariance matrix $\Sigma_{s_t}^{\frac{1}{2}}$ all switch states. The transition probability from state i to state j is given by p_{ij}. In this model x_t consists of the European HICP inflation rate (π_t), the MSCI World index return in excess of the 1-month Euribor (xs_t) and the corresponding dividend yield (dy_t).

In practice other specifications can be formulated by restricting the shifting parameters to a part of the parameters. Following Krolzig (1997) I consider models where combinations of the intercept, the auto-regressive parameters and the covariance matrix can switch states. To distinguish the different models I use the following notation:

I	Markov-switching *intercept* term,
A	Markov-switching *auto-regressive* term,
H	Markov-switching *heteroskedasticity.*

With this formulation the notation for the most general MSVAR model as in Equation 1 is given by MSIAH. Given this notation the combination of models that can be formulated is shown in Table 2. The models I consider will only have switching intercepts and heteroskedasticity. In the table these models are marked with a red box. Allowing the auto-regressive parameters to switch will result in the loss of a closed form solution for the affine term structure model[9]. The closed form solution for the yield curve allows the model to be tractable and to be calculated much faster. With this restriction however, the model is still much richer than a linear VAR model. Regimes in financial variables are often determined by changes in the levels, variances and cross correlations of the series. The models I consider are able to capture these aspects through the switching intercept and switching covariance of the innovation terms.

Following Guidolin and Timmermann (2006) I search for the best model specification by considering a large set of models. In this search I consider models with lags varying from

[8]Appendix C shows the smoothed regime probabilities of the multivariate model and the univariate counterparts. The univariate regimes show moderate to strong correlations with each other. All univariate regimes show a correlation of at least 0.57 with the multivariate model regimes.

[9]This statement will be proven in Appendix E, where the derivation of the term structure equations is given.

Table 2: Type of MSVAR models

Types of MSVAR model specifications for different parameter restrictions. The models highlighted in red are the models included in this analysis.

		ν varying	ν invariant
Φ invariant	Σ invariant	MSI-VAR	linear VAR
	Σ varying	MSIH-VAR	MSH-VAR
Φ varying	Σ invariant	MSIA-VAR	MSA-VAR
	Σ varying	MSIAH-VAR	MSAH-VAR

0 to 4 and the number of states varying between 1 and 4. The number of states is limited at 4 because of the extremely large number of parameters for higher order specifications. This leads to identification issues when estimating these models. I compare the models based on the Akaike, Bayesian and Hannan Quinn information criteria. These criteria offer an indication of the goodness of fit of the models, corrected for the complexity of the models. For each criteria the models are ranked, whereafter I chose the model with lowest the total sum of the ranks of each individual criteria. Each criterion has its own strengths and weaknesses, by combining the information of multiple criteria the model selection is more robust.

The models are estimated using the Expectation Maximisation algorithm. This algorithm and the estimation steps involved are discussed extensively by Krolzig (1997). To ensure realistic averages for inflation and equity returns in each regime I adjust the intercepts of the model. To approximate the regime average, I calculate a weighted sample average, where the weights are the smoothed regime probabilities for each observation. The intercepts of the model for inflation and equity are adjusted, such that the unconditional VAR expectations of each regime match this weighted sample average. With these adjustments the unconditional model expectations closely resemble, but do not exactly match, the sample averages[10]. The model comparison results can be found in Appendix B. The chosen model is a MSIH(2,1) model, denoting a 2 regime MSVAR model with switching intercept and covariance matrix and 1 lag term.

3.1.2 Estimation Results

Table 3 contains the estimation results for the MSIH(2,1)-model and Figure 2 shows the in sample estimates of the smoothed state probabilities. The model parameters are estimated on data with a monthly frequency. For the inflation equation the lagged parameters for equity and dividend are restricted to zero. This restriction allows equity to be priced more accurately, without losing closed form solutions for yields. Section 3.2 shows this in more detail. This restriction, however, does not influence the fit of the model drastically. The linear VAR model in Appendix B.1 shows that the autoregressive parameters for equity and dividend in the inflation equation do not differ significantly from 0 with a 1% significance level.

The parameter estimates reveal that regime 1 is a low volatility state for all three

[10]Karalis (2014) and Cavicchioli (2017) provide closed from solutions for the first four moments of MSVAR models.

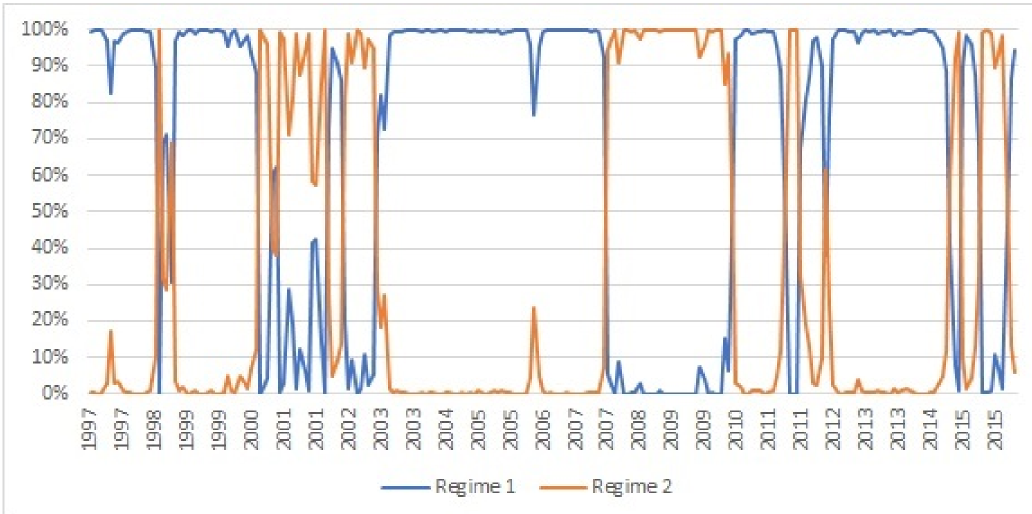
variables. Equity returns in regime 1 are about 1% higher on an annual basis compared to regime 2, with about half the monthly volatility. Although the difference in average return is small, combined with the lower volatility regime 1 has similar characteristics as a bull market regime. Regime 2 is a high volatility regime with relatively lower returns on equity, which is often characterised as bear market regime. All stock market crashes included in the sample, such as the ruble crisis and the 2008 financial crisis, have a smoothed regime 2 probability higher than 50%. The correlation of regime 2 with the OECD Euro Area recession indicator is equal to 0.39, suggesting that the bear market regime does coincide with official recession periods. From the transition probabilities it becomes clear that regime 1 is much more persistent than regime 2. The expected regime 1 duration is about 13.8 months, where for regime 2 this is only 6.5 months. In both regimes a negative correlation between the shocks on equity and dividend is present. This, combined with a positive lagged dividend yield parameter for equity returns, indicates that in both regimes mean-reversion in stock returns is present, making stocks a safer asset for long term investors (Campbell and Viceira (2002)).

Table 3: MSVAR Model Parameter Estimates

Parameter estimates for the MSIH(2,1) model for inflation, equity returns and dividends. The regime expectations denote the unconditional VAR expectation, given the VAR parameters for each regime.

	π_t	xs_t	dy_t
Expectation			
Regime 1 (Bull)	0.140%	0.294%	0.980%
Regime 2 (Bear)	0.131%	0.200%	6.343%
Intercept			
Regime 1 (Bull)	9.06E-04	2.22E-03	-6.69E-05
Regime 2 (Bear)	8.46E-04	-3.59E-02	3.75E-04
Autoregressive param.			
π_{t-1}	3.53E-01	-4.11E+00	1.03E-01
xs_{t-1}	0.00E+00	-8.29E-02	4.74E-04
dy_{t-1}	0.00E+00	6.86E-01	9.92E-01
Correlation / Std. Dev.			
Regime 1 (Bull)			
π_t	0.0012		
xs_t	0.1241	0.0297	
dy_t	-0.0314	-0.8825	0.0006
Regime 2 (Bear)			
π_t	0.0021		
xs_t	0.0056	0.0644	
dy_t	0.0202	-0.9419	0.0018
Transition Probabilities	*Regime 1*		*Regime 2*
Regime 1 (Bull)	0.928		0.072
Regime 2 (Bear)	0.154		0.846
Ergodic Prob.	0.681		0.319
Avr. Regime Duration (Months)	13.8		6.5

Figure 2: Smoothed Regime Probabilities
The estimated smoothed regime probabilities denoted by $P(s_t = j|I_T)$. These are the inferred probabilities for specific time t to belong to state j given the full sample information I_T.



3.2 Arbitrage Free Modelling

3.2.1 Interest Rate Factor Model

Due to non-stationarity in interest rates when incorporated in the MSVAR model, I model the interest rates separately. To be able to generate arbitrage free interest rate scenarios I make use of the Gaussian class of affine term structure models. Early examples of these type of models can be found in Vasicek (1977), Duffie and Kan (1996) and Dai and Singleton (2002). Following this literature, I assume that the nominal term structure can be described by a number of N_f factors f_t, whose dynamics are given by

$$f_{t+1} = \nu + \Phi f_t + \Sigma^{\frac{1}{2}} u_t, \tag{2}$$

where $u_t \sim \mathcal{N}(0, I_{N_f})$ and $\Sigma^{\frac{1}{2}}$ is the lower triangular Cholesky decomposition of the covariance matrix.

In my application I assume the factors to be latent and only observable through their implications for the observed yields, thereby following Dai and Singleton (2000) and Duffee (2002). To achieve identification the factors are assumed to be orthogonal. Additionally I assume the yield dynamics to be captured by a total of three factors, which is often used in the literature (e.g. Nelson and Siegel (1987)). This allows more realistic shapes compared to some more famous one-factor models of Vasicek (1977), Hull and White (1990) and Cox et al. (1990). Litterman and Scheinkman (1991) show with a principal component analysis that for a latent factor model three-factors is generally enough to describe most of the variation in yields. They also showed that the resulting factors can be interpreted as level, slope and curvature.

I do not incorporate macro factors into the model, as is done by for example Ang et al. (2007) and Ang et al. (2006). Ang and Piazzesi (2003) show that macro factors primarily explain movements at the short end and middle of the yield curve while unobservable factors still account for most of the movement at the long end of the yield curve. The long end of the curve has a much greater impact on the financial position of pension funds, due to the high duration of the liabilities and the long term bonds used to invest in. Therefore modelling the long end of the curve accurately is of higher priority in this setting. The interest rate factor model is estimated with the Chi Square estimation procedure as discussed by Hamilton and Wu (2012).

3.2.2 Combined Model

The MSVAR model from Section 3.1 and the interest rate factor model of the previous section combined compose the full factor dynamics of the financial market. The following section will extend these factor dynamics with the no-arbitrage assumption needed to price the assets in the market. The factors incorporated in the MSVAR model will be denoted by x_t and the interest factors by f_t. The notation for the combination of all factors will be X_t in the following sections:

$$\begin{gathered} f_t = \begin{bmatrix} f_{1,t} & f_{2,t} & f_{3,t} \end{bmatrix}', \quad x_t = \begin{bmatrix} \pi_t & xs_t & dy_t \end{bmatrix}', \\ X_t = \begin{bmatrix} f_t' & x_t' \end{bmatrix}'. \end{gathered} \tag{3}$$

The combined model is then formulated as

$$\begin{bmatrix} f_t \\ x_t \end{bmatrix} = \begin{bmatrix} \nu_f \\ \nu_{x,s_t} \end{bmatrix} + \begin{bmatrix} \Phi_f & 0 \\ 0 & \Phi_x \end{bmatrix} \begin{bmatrix} f_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \Sigma_f^{\frac{1}{2}} & 0 \\ 0 & \Sigma_{x,s_t}^{\frac{1}{2}} \end{bmatrix} u_t, \tag{4}$$

in terms of the individual model parameters given in the previous sections. The notation

$$X_t = \nu_{s_t} + \Phi X_{t-1} + \Sigma_{s_t}^{\frac{1}{2}} u_t, \tag{5}$$

will be used as a more general notation for the total model. These notations will be used in the derivations in the following sections, where the term structure equations are determined under the no-arbitrage assumption.

3.2.3 Pricing Kernel and Affine Term Structure

For the model to be arbitrage free, the prices of all assets in the model must be a function of the state variables. This includes both the state variables in the MSVAR model of Equation 1 as the interest rate factors in Equation 2. According to asset pricing theory (Cochrane (2009)) the price P_t of any asset at time t with pay-off Y_{t+1} within the model can be calculated by means of the pricing kernel M_{t+1} by

$$P_t = E_t \left[M_{t_1} Y_{t+1} \right]. \tag{6}$$

Following Cochrane and Piazzesi (2005) the pricing kernel can be described as a function of the state variables by

$$-m_{t+1} = \delta_0 + \delta_1' X_t + \frac{1}{2} \lambda_t' \lambda_t + \lambda_t' u_{t+1}, \tag{7}$$

where $-m_{t+1} = -\log(M_{t+1})$. In this formulation the risk free rate, R_t^f, is equal to

$$R_t^f = \frac{1}{E_t[M_{t+1}]}. \tag{8}$$

The log of the risk free rate or continuously compounded rate, r_t^f, is then equal to

$$r_t^f = \log R_t^f = \delta_0 + \delta_1' X_t. \tag{9}$$

It follows that the short rate is an affine function of the state variables. The short rate is a function of the interest rate factors from Equation 2 and the parameters δ_0 and δ_1 will follow from the estimation procedure for the affine term structure model, discussed in Section 3.2.4. The elements in δ_1 concerning the factors from the MSVAR model are all equal to zero. The short rate r_t can thus be expressed as

$$r_t = \delta_0 + \delta_1' X_t = \delta_0 + \delta_{1,f}' f_t. \tag{10}$$

In Equation 7 the vector λ_t contains the market prices of risk at time t for each of the state variables. The risk premium at time t is then given by $\Sigma_{s_t}^{\frac{1}{2}} \lambda_t$. The market prices of risk are assumed to be time varying and are an affine function of the state variables

$$\begin{bmatrix} \lambda_{f,t} \\ \lambda_{x,t}^{s_t} \end{bmatrix} = \begin{bmatrix} \lambda_{0,f} \\ \lambda_{0,x}^{s_t} \end{bmatrix} + \begin{bmatrix} \Lambda_{1,f} & 0 \\ 0 & \Lambda_{1,x}^{s_t} \end{bmatrix} X_t. \tag{11}$$

In this formulation the market price of risk for the variables in x_t differ by regime[11]. Following Hoevenaars (2008) I assume that the risk premium $\Sigma_{s_t}^{\frac{1}{2}} \lambda_t$ is zero for dividends, because it is a non-tradeable asset. The risk premium on inflation is also restricted to zero. This premium is nearly impossible to estimate without data on real yields. Ang et al. (2008) also argue that models with non-zero inflation risk premium tend to result in lower and more implausible real rates than with this restriction. The parameters concerning the price of equity risk are fixed to the value that assures that the asset pricing equation in Equation 6 holds. This equation holds if the discounted stock price under the risk-neutral measure Q is driftless. For the model in Equation 5 the risk-neutral parameters are given by

$$\begin{aligned} \nu_{s_t}^Q &= \nu_{s_t} - \Sigma_{s_t}^{\frac{1}{2}} \lambda_{0,s_t}, \\ \Phi^Q &= \Phi - \Sigma_{s_t}^{\frac{1}{2}} \Lambda_{1,s_t}. \end{aligned} \tag{12}$$

To make the discounted stock price driftless, Φ_{xs}^Q, the row considering the equity returns in Φ^Q, must be equal to zero. Therefore I set the same row in $\Sigma_{s_t}^{\frac{1}{2}} \Lambda_{1,s_t}$ equal to Φ_{xs} for all s_t. Furthermore, $\nu_{s_t}^Q$ must be equal to the convexity adjustment resulting from the regime switching lognormal distribution, which is different for each state. The prices of risk for the bond factors f_t will follow from the Chi Square estimation procedure.

[11] Ang et al. (2008) introduce a regime switching interest rate model. The way the price of risk is regime dependent in my model, is based on their specification.

Under the above assumptions on the short rate and pricing kernel dynamics, the price of a n-period nominal zero coupon bond can be expressed as an exponentially affine function of the state variables

$$P_t^n = \exp\left(A_n + B_n' X_t\right). \tag{13}$$

The parameters A_n and B_n are not regime dependent due to the fact that nominal bond prices only depend on the bond factors f_t. The terms A_n and B_n are calculated recursively by

$$\begin{aligned} A_n &= -\delta_0 + A_{n-1} + B_{n-1,f}'(\nu_f - \Sigma_f^{\frac{1}{2}}\lambda_{0,f}) + \frac{1}{2}B_{n-1,f}'\Sigma_f B_{n-1,f} \\ B_n &= \begin{bmatrix} B_{n,f} \\ B_{n,x} \end{bmatrix} = \begin{bmatrix} -\delta_{1,f} + (\Phi_f - \Sigma_f^{\frac{1}{2}}\Lambda_{1,f})' B_{n-1,f} \\ 0 \end{bmatrix} \end{aligned} \tag{14}$$

The recursion can be initiated for $n = 1$ with the values of

$$\begin{aligned} A_1 &= -\delta_0, \\ B_1 &= \delta_1. \end{aligned} \tag{15}$$

Following this notation the yield on a n-period bond at time t, y_t^n can be written as

$$y_t^n = -\frac{A_n}{n} - \frac{B_n'}{n} X_t = a_n + b_n' X_t. \tag{16}$$

This expression of the yield is used to derive the bond factor measurement equation in the next section.

The real pricing kernel can be formulated as $\widehat{M}_{t+1} = M_{t+1}P_{t+1}/P_t$[12] with P_t the price level at time t. The real pricing kernel thus equals

$$\widehat{M}_{t+1} = M_{t+1}\exp(\pi_{t+1}) = \exp\left(-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t' u_{t+1} + e_\pi' X_{t+1}\right), \tag{17}$$

where e_π is a vector of zeros and an one on the index of the inflation rate π_{t+1}. Applying the real pricing kernel to a zero coupon bond with maturity n and assuming that the real bond prices are exponentially affine in the state variables, the real bond prices are given by

$$\widehat{P}_t^n(i) = \exp\left(\widehat{A}_n(i) + \widehat{B}_n' X_t\right). \tag{18}$$

The coefficient $\widehat{A}_n$ is a scalar that depends on the current regime i and $\widehat{B}_n$ is $N \times 1$ vector with N the total number of state variables. In this model the coefficients $\widehat{A}_n$ and $\widehat{B}_n$ are recursively given by

$$\begin{aligned} \widehat{A}_{n+1}(i) = -\delta_0 - \widehat{B}_{n,f}'\Sigma_f^{\frac{1}{2}}\lambda_{0,f} + \log\Bigg(\sum_j p_{ij}\exp\Bigg(&\widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right)\nu(j) + \\ &\frac{1}{2}\left(\widehat{B}_n' + e_\pi'\right)\Sigma(j)\left(\widehat{B}_n' + e_\pi'\right)'\Bigg)\Bigg), \\ \widehat{B}_{n+1} = -\delta_1 + \Phi'\left(\widehat{B}_n + e_\pi\right) - e_f'\Lambda_{1,f}'\Sigma_f^{\frac{1}{2}}\widehat{B}_{n,f}.& \end{aligned} \tag{19}$$

[12]See Ang et al. (2008)

In this formulation the matrix e_f transforms the 3×1 vector $\Lambda'_{1,f}\Sigma_f^{\frac{1}{2}}\widehat{B}_{n,f}$ into a 6×1 vector equal to $[\Lambda'_{1,f}\Sigma_f^{\frac{1}{2}}\widehat{B}_{n,f} \quad \mathbf{0}]'$. The vector e_π is a unit vectors with an one on the index of inflation in the state variable vector X_t. The starting values for this recursion are equal to

$$\begin{aligned} \widehat{A}_1(i) &= -\delta_0 + \log \sum_j p_{ij} \exp\left(e'_\pi \nu(j) + \frac{1}{2} e'_\pi \Sigma(j) e_\pi\right), \\ \widehat{B}_1 &= -\delta_1 + \Phi' e_\pi. \end{aligned} \tag{20}$$

The real term structure can be constructed using the following expression for the real yield on a n-period bond:

$$\widehat{y}_t = -\frac{\widehat{A}_n(i)}{n} - \frac{\widehat{B}'_n}{n} X_t = \widehat{a}_n(i) - \widehat{b}'_n X_t. \tag{21}$$

The proof of these recursive formulas can be found in Appendix E.

3.2.4 Minimum Chi Square Estimation

The bond pricing equation given in Equation 16 cannot hold exactly for every maturity. In the latent factor case, this equation can hold exactly for at most N_f maturities (Ang and Piazzesi (2003), Chen and Scott (1993)). If N_m is the total number of maturities that I want to fit my model to, then only N_f maturities can be modelled without error and the remaining $N_e = N_m - N_f$ maturities are thus assumed to be subject to measurement error. If $Y_{1,t}$ is vector containing the yields without error and $Y_{2,t}$ the yields with error, for the factor model in Equation 2 the measurement specification then is[13]

$$\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} f_t + \begin{bmatrix} 0 \\ \Sigma_e \end{bmatrix} u_{e,t}, \tag{22}$$

where A_i and B_i contain the yield parameters a_n and b_n from Equation 16 for the chosen maturities. The measurement errors are assumed to be i.i.d. $u_{e,t} \sim \mathcal{N}(0, I_{N_e})$ and Σ_e is taken to be diagonal. This model is essentially a form of a restricted vector auto-regression. Mapping the variables of the affine term structure model to the reduced form variables gives insight in the identifiability of the model. The model is unidentified if two different parameter values for the structural parameters imply the same reduced-form parameters.

To achieve identification I follow Hamilton and Wu (2012) by applying the restrictions $\Sigma_f = I_{N_f}$, $\delta_{1,f} \geq 0$, $\nu_f = 0$ and Φ_f^Q is lower triangular[14]. With these normalisation conditions applied, the reduced form of Equation 22 and the parameter mappings are

[13] See Hamilton and Wu (2012).

[14] $\Phi_f^Q = \Phi_f - \Sigma_f^{\frac{1}{2}}\Lambda_{1,f}$.

given by

$$
\begin{aligned}
&Y_{1,t} = A_1^* + \phi_{11}^* Y_{1,t-1} + u_{1,t}^*,\\
&A_1^* = A_1 - B_1 \Phi_f B_1^{-1} A_1,\\
&\phi_{11}^* = B_1 \Phi_f B_1^{-1},\\
\\
&Y_{2,t} = A_2^* + \phi_{21}^* Y_{1,t} + u_{2,t}^*,\\
&A_2^* = A_2 - B_2 B_1^{-1} A_1,\\
&\phi_{21}^* = B_2 B_1^{-1},\\
&u_{1,t}^* \sim \mathcal{N}(0, \Omega_1^*),\\
&\Omega_1^* = B_1 B1',\\
&u_{2,t}^* \sim \mathcal{N}(0, \Omega_2^*),\\
&\Omega_2^* = \Sigma_e \Sigma_e.
\end{aligned}
\tag{23}
$$

The system is just identified if the number of reduced-form parameters minus the structural-VAR parameters, $(N_e - 1)(N_f + 1)$, is equal to 0[15]. It follows that this is the case for $N_e = 1$. In the estimation procedure I assume the yields of maturities 1-month, 10-year and 30-year to be measured without error. The 1-month rate serves as the risk-free rate in the financial market model. The 10-year and 30-year yield both assure a good fit on the long end of the curve, where most of the risk of a pension fund generally lies. To assure a better shape of the curve for lower maturities I assume the 5-year yield to be measured with error.

The parameters of the affine term structure model are estimated by the Minimum Chi Square Estimation (MCSE) procedure of Fisher (1924) and Neyman and Pearson (1928), which was introduced in the affine term structure literature by Hamilton and Wu (2012). This method allows the model parameters to be inferred directly from the estimated OLS parameters of the reduced-form regressions via the parameter mapping. This requires a combination of analytical and numerical calculations, where the numerical part is much less complex than direct numerical optimisation of the likelihood function. MCSE is based on the Chi Square Difference test. Let π be a vector consisting of the reduced-form VAR parameters, $\mathcal{L}(\pi; Y)$ the full sample log likelihood function and $\hat{\pi}$ be the full-information maximum likelihood estimate of π. Furthermore, let $\hat{R}$ be a consistent estimate of the Fisher information matrix

$$
R = -T^{-1} E\left[\frac{\partial^2 \mathcal{L}(\pi; Y)}{\partial \pi \partial \pi'}\right], \tag{24}
$$

then the Wald statistic to test the hypothesis that $\pi = g(\theta)$ is calculated as

$$
T\left[\hat{\pi} - g(\theta)\right]' \hat{R}\left[\hat{\pi} - g(\theta)\right] \xrightarrow{d} \chi^2(q), \tag{25}
$$

[15]Even if this statement holds, still under some circumstances the parameters can be unidentified. Hamilton and Wu (2012) give a detailed explanation of these situations.

where q is the number of parameters in π. The MCSE estimate of the parameters $\hat{\theta}$ is the value that minimises this chi-square statistic. Hamilton and Wu (2012) provide a more detailed explanation of the MCSE procedure and a step by step pseudo-algorithm for the specific model that I use. They also show that this estimation method is asymptotically equivalent to standard Maximum Likelihood Estimation (MLE). In the case of exact identification, the minimum value of the Wald statistic is zero. In this case the MCSE estimate is identical to the MLE estimate and the Global Optimum is reached with certainty. This is a big advantage compared to direct numerical optimisation of the likelihood function, where there is no certainty that a Global Optimum is reached, irrespectively of the number of unique starting values used.

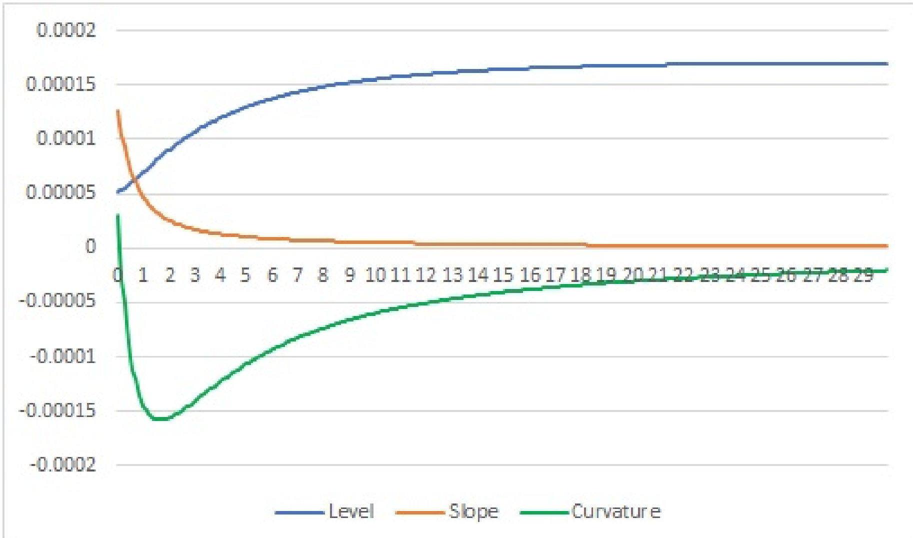
In the estimation procedure I set the unconditional expectation in the OLS step equal to the sample average yields. This assures realistic long term averages of the yield curve. The resulting parameter estimates of the MCSE procedure are reported in Table 4. Figure 3 shows the resulting factor loadings for different maturities of the curve. From this figure it becomes clear that the factors can be interpreted as a level, slope and curvature factor. An increase in the first factor (Level) increases the overall level of interest rates, while maintaining the general shape. An increase in the second factor (Slope) results in higher short term rates, while the long term rates remain practically unchanged. This factor thus influences the difference between long term rates and short term rates. At last an increase in the third factor (Curvature) causes medium term rates to decreases, whereas short and long term rates remain relatively unchanged. This influences the shape of the yield curve.

Table 4: Interest Rate Factor Model Parameters

The parameter estimates of the interest rate factor model of Equation 2 under the P measure, the estimated prices of interest rate factor risk and the short rate parameters.

	$\boldsymbol{f_{1,t}}$	$\boldsymbol{f_{2,t}}$	$\boldsymbol{f_{3,t}}$
$\boldsymbol{\nu_f'}$	0	0	0
$\boldsymbol{\Phi_f'}$			
$f_{1,t-1}$	0.9722	-0.0374	0.0100
$f_{2,t-1}$	-0.0485	0.9556	-0.3345
$f_{3,t-1}$	0.0018	0.0126	0.9199
$\boldsymbol{\Sigma_f'}$			
$f_{1,t}$	1		
$f_{2,t}$	0	1	
$f_{3,t}$	0	0	1
$\boldsymbol{\delta_0}$	0.0019		
$\boldsymbol{\delta_1'}$	2.88E-05	3.19E-05	1.04E-04
$\boldsymbol{\lambda_0'}$	-0.0431	0.2501	0.5206
$\boldsymbol{\Lambda_1'}$	-0.0270	-0.0029	0.0389
	-0.0485	0.0397	0.1021
	0.0018	0.0126	0.0040

Figure 3: Factor Loadings
The factor loadings for different maturities of the yield curve. This shows that the factors can be interpreted as level, slope and curvature.



3.3 Interest Rate Curve Extrapolation

The interest rate model described in the previous sections provides a good fit of the yields for maturities up to 30 years, which is the highest maturity used for fitting the curve. The model does not provide realistic values of yields for maturities beyond this point. To value liabilities of the pension fund for maturities greater than 30 years, I extrapolate the curve with two different methods. The first method constructs a nominal yield curve by following the method used by the DNB to extrapolate the yield curve under the FTK[16]. To extrapolate the yield curve, the 1-year forward rate is assumed to remain constant after the last observed maturity. If the last observed maturity is reasonably large, this is a realistic assumption. The 1-year forward rate for maturity n is the market expectation of the 1-year rate n years from now. For large maturities it is unlikely for the market to have substantially different expectations of the future 1-year rate. The yield curve resulting from this method will further be mentioned as the nominal yield curve.

Under the rules of the FTK pension funds are allowed to value the liabilities by using a risk-free curve with an Ultimate Forward Rate (UFR) incorporated. The UFR can be seen as the long term expectation of the forward rate. The DNB determines the value of the UFR as 120-months moving average of the 20-year maturity 1-year forward rate rounded to one decimal place. In my analysis I use annual scenarios, where I determine the UFR as the 10-year moving average of the 20-year forward rate. The UFR yield curve is constructed by extrapolating from the 20-year maturity point onward. For this purpose I use the Smith Wilson method adopted by EIOPA[17]. The level of convergence is set to

[16] Financieel Toetsings Kader or financial review framework. The prescribed method is described in this document: `http://www.toezicht.dnb.nl/binaries/50-212329.pdf`.

[17] European Insurance and Occupational Pension Authority

assure the extrapolated forward curve has approached the UFR at a maturity of 60 years up to one basis-point. This yield curve is further mentioned in this thesis as the DNB yield curve.

3.4 Asset Returns

The financial market has a total of four assets available. The first is a global diversified equity portfolio in the form of the MSCI world index. Secondly two bond funds are available, which are assumed to be constant maturity bond funds of the 1-month and 10-year maturity, where the 1-month bond is the risk free return. Lastly, the fund can enter into interest rate swap contracts. The returns of all asset classes are simulated under the real world measure P and the risk neutral measure Q. The risk neutral simulations are used to asses the market value of the pension entitlements and to detect value-transfers.

The equity returns in excess of the 1-month rate are generated directly by the MSVAR model of Section 3.1. Equity returns under the Q measure can be generated by simulating with the Q parameters given in Equation 12. Bond returns follow from the affine term structure model. For a n period bond with price P_t^n the one month log return r_t^n under the P measure is given by

$$r_t^n = \log\left(\frac{P_{t+1}^{n-1}}{P_t^n}\right) = (A_{n-1} + B_{n-1}' X_{t+1}) - (A_n + B_n' X_t). \tag{26}$$

In terms of the affine term structure parameters this can be deduced to

$$\begin{aligned} r_t^n = \; & \underbrace{\delta_0 + \delta_{1,f}' f_t}_{\text{Risk Free Rate}} + \underbrace{B_{n-1,f}'\left(\Sigma_f^{\frac{1}{2}}\lambda_{0,f} + \Sigma_f^{\frac{1}{2}}\Lambda_{1,f} f_t\right)}_{\text{Risk Premium}} \\ & \underbrace{-\frac{1}{2}B_{n-1,f}'\Sigma_f^{\frac{1}{2}}\Sigma_f^{\frac{1}{2}\prime}B_{n-1,f}}_{\text{Convexity Adjustment}} + \underbrace{B_{n-1,f}'\Sigma_f^{\frac{1}{2}}u_{f,t}}_{\text{Stochastic Shock}}. \end{aligned} \tag{27}$$

The proof of this equation can be found in Appendix E.3. Under the risk neutral measure the expected return for every asset is equal to the risk free rate. Now given this observation, it becomes clear from the above equation that bond returns under the Q measure, $r_t^{Q,n}$, are given by

$$r_t^{Qn} = \delta_0 + \delta_{1,f}' f_t - \frac{1}{2}B_{n-1,f}'\Sigma_f^{\frac{1}{2}}\Sigma_f^{\frac{1}{2}\prime}B_{n-1,f} + B_{n-1,f}'\Sigma_f^{\frac{1}{2}}u_{f,t}. \tag{28}$$

The last asset available is an interest rate swap, which the fund can use to hedge the interest rate risk of the liabilities. The amount of swap contracts is expressed as a percentage of the total interest rate risk that the swap hedges. The swap is based on the nominal yield curve and is constructed such that the value at time 0 is equal to 0. Let $\hat{L}_t^{fw_t}$ be the discounted value of the projected cash flows at time t based on the 1-period forward curve fw_t. $\hat{L}_t^{fw_t}$ is thus the forward price of these liabilities. Subsequently let $L_{t+1}^{sp_{t+1}}$ be the discounted value of the liabilities at time $t+1$ based on the spot rate or nominal yield curve at time $t+1$. The total value change of the liabilities due to interest rate risk is then given by

$$\Delta L_{t+1} = L_{t+1}^{sp_{t+1}} - \hat{L}_t^{fw_t}. \tag{29}$$

If the applied hedging percentage is given by κ_t, then the pay-off of the swap contract Y_t^{Swap} is

$$Y_t^{Swap} = \kappa_t \times \Delta L_{t+1}. \tag{30}$$

4 The Pension Fund

The aim of the research is to determine whether a carve-out can be an interesting de-risking solution for pension funds, where both the pensioners as the remaining participants benefit from the carve-out. To answer this question the expected pension benefits for both groups of participants should be compared between a situation where a carve-out is conducted and the situation where the fund continues as it is. Modelling both the development of the pension funds' assets and the liabilities over time provides insight into the expected consequences of such a buy-out. This section describes the policy of the fund on both the asset and liability side of the balance sheet. A carve-out is only applicable to Defined Benefit pension schemes. Therefore the pension fund is assumed to have only one pension scheme, which is an average wage defined benefit scheme with conditional indexation. This is the most common defined benefit scheme in the Netherlands[18].

4.1 Participants

For the composition of the pension fund I use self generated data. This allows me to analyse the effect of different durations of the liabilities on the attractiveness of a carve-out. For this purpose I use three stylised funds; Young, Average and Old. To be able to compare the utility of the participants of different ages, the entitlements need to be traceable per age. The participants are assumed to be either active and accruing pension entitlements or retired. A larger distinction can generally be made for the participants, this is however unlikely to have large implications for the results.

The initial distribution of the participants over age is generated by means of a truncated normal distribution. Participants are assumed to only enter the fund after being 20 years old. The maximum age a participant is able to reach is equal to 120. Based on data of pension fund demographics[19] the average age and standard deviation for the pension funds is chosen. The sample average ages used to generate the funds are for young, average and old respectively 45, 55, and 65. All funds are generated with an age sample standard deviation of 15. The funds are simulated with 5000 participants and with 10,000 simulations. The final participants per age is the average over these simulations.

The development of the fund is modelled similar to the Push-Pull Markov model introduced by Boender (1997), but non-stochasticly. The participants retire at the age 67. The mortality of the participants is calculated using the mortality rates from the 'Actuarieel Genootschap' (AG), where 56%[20] of the participants is assumed to be male. If $q^M_{x,t}$ and $q^F_{x,t}$ are the male and female mortality rates at time t for age x respectively, then the fund mortality rates $q^{fund}_{x,t}$ is

$$q^{fund}_{x,t} = 56\% \times q^M_{x,t} + 44\% \times q^F_{x,t}. \tag{31}$$

The number of active participants is assumed to be constant over time, which ensures a stable workforce. New entrants enter the fund according to specific entrance probabilities,

[18] Figure 4 in Appendix A.2 provides an overview of the pension schemes in the Netherlands.
[19] Available at the website of the DNB: `www.dnb.nl`
[20] Based on demographic data of Dutch pension funds of the DNB.

which are fund specific[21]. These probabilities and distribution plots of the participants can be found in Appendix A.4 and A.5. Table 5 provides an overview of the durations of the liabilities of the pension funds as a whole and split in actives and pensioners.

Table 5: Fund Durations

	Young	**Average**	**Old**
All	22.1	18.3	14.6
Actives	26.8	23.3	20.7
Pensioners	9.2	9.1	8.6

4.2 Liability Dynamics

The liabilities of the pension fund change due to various factors. This section describes how the entitlements of the participants are incorporated in the model. Thereafter the pension fund policies in the case of underfunding are discussed.

4.2.1 Entitlements

To determine the amount of liabilities involved in the buy-out, it is necessary to model the pension entitlements of the different age groups separately. To this end the expected cash flows are modelled per age. The expected cash flows are a function of the accumulated pension entitlements per age group and the survival probabilities of that particular age group. Let $C_{x,t}$ be the entitlement of a x years old participants of the fund at time t. The fund is then obliged to pay this person an amount of $C_{x,t}$ for as long as this person lives. Let $p_t^{x,x+n}$ be the probability of a participant of age x to be alive n years from now, then the expected cash flow $CF_t^{x,x+n}$ for this participant becomes

$$CF_t^{x,x+n} = p_t^{x,x+n} \times C_{x,t}. \tag{32}$$

The probability for this participant to reach an age of $x+n$, $p_t^{x,x+n}$, is in terms of the mortality rates of Equation 31 given by

$$p_t^{x,x+n} = \prod_{i=1}^{n} \left(1 - q_{x+i,t+i}^{fund}\right). \tag{33}$$

The value of the liabilities is then determined by the present value of these cash flows.

The accrual of new pension entitlements of the active participants increase the liabilities over time. In an average wage pension fund the participants accrue entitlements throughout their working life with as goal to assure an income including old age allowance (AOW) after retirement equal to the average of wages throughout their career. Pension entitlements are accrued as a fixed percentage of the pensionable income. This fixed percentage known as accrual rate is set to the maximum statutory value for this type of fund of 1.875% in 2016. The pensionable income is calculated as the participants wage minus

[21] These probabilities are provided by Willis Towers Watson.

an offset. This offset functions as a correction for the AOW and will be corrected for inflation over the simulated years in the model. The offset for 2016 equals €12,953 for married persons. The accrued benefits further depend on the wages of the participant. The wages per age follow the curve shown in Figure 1 and are indexed with the simulated inflation plus an additional 0.5% of wage inflation.

4.2.2 Recovery Policies

Pension funds in the Netherlands are regulatory required to report the financial position of the fund by means of the ratio of assets and the present value of liabilities known as the funding ratio. This regulatory funding ratio, further denoted by DNB funding ratio, is calculated as the 12-month average of the ratio of assets divided by the present value of the cash flows discounted with the DNB yield curve:

$$FR_t^{DNB} = \frac{A_t}{L_t^{DNB}}, \tag{34}$$

with FR_t^{DNB} the DNB funding ratio, A_t the assets and L_t^{DNB} the liabilities valued with the DNB yield curve. The pension fund is required to hold a capital buffer for the amount of risk it has on the balance sheet. The DNB prescribes a complete model to determine the size of this buffer, named the required own capital (VEV[22]). In the ALM study the level of this buffer will be kept constant at 20% additional to the value of the liabilities.

When the funding ratio of the pension fund falls below this level of 120%, the fund is said to be in shortfall. The regulator requires funds in shortfall to compose a recovery plan that assures the fund to recover the funding ratio minimally to the level of the VEV over a horizon of 10 years. To assure recovery, the fund in this model has two instruments to intervene. Either the amount of indexation can be reduced or, when that is not enough, reductions can be applied to the entitlements. These reductions are regulatory allowed to be spread over the course of 10 years. The recovery plan consists of a projection of the current liabilities and assets of the fund. Based on this projection, the required intervention can be determined to assure recovery over a horizon of 10 years. I incorporate a simplified version of this projection. In the projection the fund is assumed to be closed, meaning no new participants enter and no new entitlements are accrued. Asset returns and inflation expectations are set to the regulative maximal parameter values[23]. Furthermore, interest rates are assumed to be constant in the 10 year projection. The required intervention is determined by numerical search methods.

If the funding ratio of the fund falls even further, below the value of the minimum required capital (MVEV)[24] of 104%, more severe actions are required by the regulator. If the funding ratio is below the value of the MVEV for 5 consecutive years, the DNB requires the fund to apply a reduction to assure that the fund immediately satisfies the MVEV requirements. This reduction can be spread over a horizon of 10 years, but is unconditional. Thus even if the fund recovers to the MVEV requirements in less than 10

[22]In Dutch: Vereist Eigen Vermogen.

[23]Determined by the Commissie Parameters.

[24]In Dutch: Minimum Vereist Eigen Vermogen.

years, the reduction is still required. Additional deposits by the sponsoring company in case of underfunding are disregarded.

4.2.3 Indexation Policy

The real value of accrued entitlements decreases over time due to inflation. The pension fund strives to compensate the participants for this fact by indexing the entitlements. In a defined benefit scheme with conditional indexation the fund assures the nominal entitlements and grants indexation conditional on the financial position of the fund. Pension funds generally follow a realised inflation measure to determine the amount of compensation, where the granted indexation is expressed as a percentage of this measure. In this thesis the granted indexation at time t, I_t, is a percentage of the simulated European HICP inflation. The Dutch pension fund regulations, the FTK[25], state that the indexation policy should be future proof. This means that current indexation can not be at the expense of potential future indexations.

More specifically these regulations prescribe a framework in which the amount of indexation a fund is allowed to grant depends on the DNB funding ratio. A pension fund is only allowed to index entitlements if this funding ratio is at least equal to 110%. Above this level the rule applies that the current indexation must also be expected to be realisable in the future with currently available capital. The maximum indexation is equal to 100% of the applicable measure excluding possible compensation for previously missed indexation or reductions.

The pension fund can apply $I_t = 100\%$ indexation if the present value of all future indexations is smaller than the current assets available for indexation. If $PV_{i,t}$ is the present value of all future indexations then the required funding ratio for full indexation $FR_{100\%,t}$ is

$$FR_{100\%,t} = 110\% + \frac{PV_{i,t}}{L_t^{DNB}}, \tag{35}$$

where L_t^{DNB} is the value of the liabilities before indexation discounted with the DNB yield curve. The present value of inflation is calculated by indexing all future cash flows with the expected indexation, where a cash flows with maturity n is indexed n times. These cash flows are then discounted with the expected return on equity determined by the 'Commissie Parameters'. The expected inflation used in practice for this purpose is set to a value of 2%, also determined by the 'Commissie Parameters'. The fraction of indexation granted, I_t, is approximated linearly between the zero indexation and 100% indexation funding ratios as

$$I_t = \frac{FR_t - 110\%}{FR_{100\%,t} - 110\%}, \tag{36}$$

where FR_t is the current funding ratio.

The fund is allowed to compensate participants for missing indexations or reductions in previous years if the financial position allows to do so. This is the case if the funding ratio is above the level of $FR_{100\%}$ and the required own capital (VEV). The available capital

[25] Financieel Toetsingskader

for recovery indexation is the amount of assets above the maximum of either $FR_{100\%,t}$ or the VEV. The fund is allowed to use up to 20% of this available capital to recover previous reductions and missed indexations.

4.3 Asset Dynamics

The assets of the pension fund develop over time by return on investment, the gains of pension premiums and the payment of pension benefits. The pension premiums are a percentage of the accrued entitlements. I follow Hoevenaars (2008) by assuming the premium rate to be constant over time. The premium rate is fixed at 30% of the pension basis, defined as the wage minus the AOW offset, which is on average slightly more then the cost covering premium[26]. This premium is the regulatory minimum amount of premium income a fund must generate and consists of three elements. The first element is the actuarial required premium to finance the newly accrued pension entitlements. Secondly this amount is raised by an amount to cover the execution costs faced by the fund. The third element raises the premium by an amount equivalent to the required capital of the fund. The premiums are assumed to be payed by the employer. The operating costs are assumed to be incorporated in the premiums and are not further included in the model.

Return on investment is determined by the asset portfolio and the market developments. The fund can invest in a 1-month and 10-year constant maturity bond fund and a global equity portfolio. The yearly bond returns are calculated by rolling over the bonds each month. The yearly equity returns are the cumulative monthly returns. The asset scenarios are generated on a monthly interval, but transformed to annual scenarios to decrease the computational burden. Furthermore, the fund can enter into interest rate swap contracts to hedge the interest rate risks on the liability side of the balance sheet. The swap is discussed in more detail in Section 3.4. The asset returns follow from the simulated scenarios generated by the financial market model described in Section 3. The model will be used to simulate scenario's for a horizon of 15 years. This horizon is also prescribed by the DNB for pension fund stress testing.

4.4 Evaluation Criterion and Portfolio Optimisation

To determine the benefit of the participants and to be able to derive an optimal investment policy, the utility function of the participants must be specified. The participants benefit from a high real pension income and low contributions from their side. The participants thus benefit from high indexations and suffer from entitlement reductions. Furthermore, they benefit from being in a healthy pension fund, as this assures future indexations. This fact should be taken in consideration after the 15 year horizon of the simulation. The premiums are assumed to be payed by the employer. The utility of the participants is thus fully determined by the pension benefits. The participants are assumed to receive utility over their pension benefits and to have CRRA preferences. The CRRA utility

[26] I also analysed the carve-out with a variable premium equal to the cost covering premium. This did not influence the results significantly.

function is given by

$$U(b_t) = \delta^t \frac{b_t^{1-\gamma} - 1}{1-\gamma}, \tag{37}$$

where b_t denotes the benefit received t years from now divided by the fully indexed benefit that the participants could have potentially had at time t. The value of b_t can be interpreted as the pension result of this particular participant at time t. The parameter γ is the risk aversion and δ denotes the time preference of the participants. The value of δ is set to 1 in this thesis, unless it is stated otherwise[27]. The pensioners already get paid benefits during the 15 years of simulation. For the active participants this is generally not the case. To calculate the utility derived after the 15 simulation years, I assume that the participants no longer accrue entitlements after this time. For each year after $t = 15$ the utility of a participant is equal to the utility derived from the payment that this participant would receive, given the entitlements at time $t = 15$, times the probability of that participant still being alive. Given the survival probabilities in Equation 33, the utility after $t = 15$ becomes

$$U(b_t) = p_t^{x,x+n} \times \delta^t \frac{b_t^{1-\gamma} - 1}{1-\gamma}. \tag{38}$$

To incorporate the financial situation of the fund after these 15 years, the fund applies an one-off indexation or reduction. The fund determines this one-off mutation by ensuring the DNB funding ratio after 15 years is equal to the required own capital (VEV)[28]. The inflation after 15 years is assumed to be equal to the 2% determined by the 'Commissie Parameters'. For the carve-out population the indexation after 15 years is also determined by this 2% inflation and the percentage of inflation compensation purchased for the pensioners. The resulting utility is used as evaluation criterion to asses whether the actives and pensioned participants are better off with or without a carve-out.

To make a fair comparison of the carve-out scenario's and the scenario without carve-out, it is important that the pension fund allocates its capital optimally in terms of the evaluation criterion to the different available assets. The pension fund carries out a constant proportion strategy, where the strategic asset allocation is determined in terms of the available assets and the fund rebalances to this allocation after each period. This comes close to what pension funds implement in practice. Generally pension funds determine a long horizon strategic asset allocation, from which they can only deviate slightly. The pension fund can not short any of the assets and thus can also not leverage positions. The interest rate hedge percentage κ of Equation 30 is restricted between 0% and 100%.

[27] The results did not change drastically when a value of $\delta = 0.98$ was used.

[28] As alternative for the one-off indexation, I also considered the fund to pay the future proof indexation level determined at time $t = 15$ for all years following after $t = 15$, in accordance with how the fund determines the indexation each year in the simulations. This did not lead to significantly different results.

5 Carve-Out

This section discusses the important features of a carve-out in more detail. In the buy-out market annuity contracts with and without inflation compensation are available. For both parts the pricing is discussed in Section 5.1. Thereafter an important carve-out specific aspect is the distribution of the assets, which is discussed in Section 5.2.

5.1 Buy-Out Pricing

The price of a buy-out can be split in two parts; the nominal price of the pension entitlements and the price of additional indexation. Indexation can either be purchased as a fixed percentage or as a percentage of an inflation benchmark. Dutch insurers use European HICP inflation for this purpose.

To obtain useful results on the attractiveness of a carve-out, the pricing of a buy-out must be close to prices observed in the current market. To achieve this I apply the nominal pricing method used by Willis Towers Watson in their Buy-Out Monitor[29]. Figure 5 in Appendix A.3 provides insight in the historical development of the buy-out prices in the Netherlands. The nominal buy-out price is determined by discounting the projected cash flows with the nominal yield curve plus a buy-out spread. This spread can be seen as a discount on the nominal price and effectively determines the price of the buy-out. In this analysis the buy-out spread is set at a constant level of 35 basis points.

In my model I assume the pension fund chooses to buy indexation as a percentage of HICP. The price of this indexation is calculated by means of risk neutral simulations of inflation. Given an amount of assets available for the buy-out, the percentage of HICP indexation purchased is determined by numerical search, such that the present value equals the current assets. The amount of HICP indexation bought for the pensioners is maxed at 100%. If a asset distribution rule attributes more assets to the pensioners than required for 100% HICP indexation, then the surplus of the assets will be attributed to the remaining participants.

5.2 Capital Distribution

The most important aspect of a carve-out is the decision on how to divide the assets between the carve-out population and the remaining participants. The fund management should strive to divide the assets as fair as possible. In the actuarial literature fair is often taken to be value neutral in a no-arbitrage context, examples are Cui et al. (2005) and Hoevenaars and Ponds (2008a). As with all pension policy decisions, a carve-out inevitably leads to inter-generational value transfers (Hoevenaars and Molenaar (2010)). A fair distribution of the assets can be characterised as a solution where inter-generational value transfers are kept as small as possible. However, the solution must also be explainable to the funds participants. Therefore distribution based on a highly technical model would not be preferred. In order to find the most fair and optimal method to distribute the

[29] A monthly report for clients of Willis Towers Watson providing current observed buy-out prices in the market for some stylised pension funds.

assets I compare a total of 6 methods. Table 6 gives the abbreviations used further in this paper for each of the methods.

Table 6: Asset Distribution Methods

The notation used in this paper for the various methods to distribute the assets between active participants and pensioners.

DNB FR	*Denotes the method where assets are split based on the DNB funding ratio.*
Nominal FR	*Denotes the method where assets are split based on the nominal funding ratio.*
Real FR	*Denotes the method where assets are split based on the real funding ratio.*
Exp. Ind.	*Denotes the method where assets are split based on the expected indexation of the fund.*
Indiff.	*Denotes the method where assets are split such that pensioners are indifferent of the carve-out.*
Fair Value	*Denotes the method where assets are split such that the no-arbitrage value of the pension entitlements is unchanged.*

The most intuitive way for a pension fund to distribute the assets would be to divide them in the same proportion as the liability value of the accrued pension entitlements for each group. In practice this would mean that the assets are split based on the current funding ratio. This way the funding ratio of the fund after the carve-out remains unchanged. This idea can be applied to all definitions of the funding ratio of which I distinguish between three cases; the DNB funding ratio, the nominal funding ratio and the real funding ratio.

The DNB funding ratio is the ratio that results by valuing the liabilities based on the regulatory UFR yield curve. Splitting the assets based on this funding ratio does not affect the funding status of the fund in the eyes of the DNB. This way of splitting is thus expected to a have small impact on the expected indexation of fund. To calculate the nominal funding ratio, the liabilities are discounted with the nominal yield curve. This curve is generally lower for longer maturities, resulting in higher present values of the entitlements of the active participants. Under this measure the fund will likely have a funding benefit in terms of the DNB funding ratio, which will likely increase indexation potential. The real funding ratio is calculated by valuing the liabilities with the real term structure. This funding ratio gives insight in the indexation potential of the fund. By splitting according to this funding ratio the indexation potential of the fund is expected to be distributed in a fair manner. This method is likely to give even larger funding benefits in terms of DNB funding ratio.

Next to funding ratio based measures the assets can also be distributed in other ways. By means of an ALM study the expected indexation of the pension fund can be determined. This expected indexation can be used to buy the same expected amount of HICP inflation. In this setting the pensioners are guaranteed to be better of by eliminating the risk of reductions and maintaining the expected level of indexation. This method is very intuitive and has as big advantage that it is easily seen that the pensioners will benefit. Additionally I split the assets based on the no-arbitrage value of the pension contracts. This method is of course quite technical and difficult to explain to the participants. It will however give an indication of a fair distribution of the assets in a theoretical context. The insights can be used to select a more intuitive method. Lastly, I determine the amount of assets needed for the pensioners to be indifferent between a carve-out and remaining with the fund. This option is included to provide a decisive answer to the question whether a win-win situation is possible for both the active as the pensioned participants.

6 Results

The results are split in two main sections. First the results with homogeneous risk aversion are discussed. The carve-out results are discussed for the Average, Young and Old pension funds to assess the effect of duration on the carve-out proposition. Furthermore, I look at the impact of the initial funding ratio on the carve-out. Also several other parameter sensitivities are checked to gain insight in the robustness of the results. Thereafter, I differentiate the risk aversion of the pensioners versus the risk aversion of the active participants. Older people are generally more risk averse than young people (Campbell and Viceira (2002)). The heterogeneous risk aversion in the fund can strongly influence the success of a carve-out. The ALM model has a large variety of parameters, of which most are fixed at a constant level. Appendix A.6 provides an overview of the parameter settings used.

6.1 Homogeneous Risk Aversion

6.1.1 Basis Scenario

The first carve-out scenario regards an average pension fund with a funding ratio of 115%. This is below the level of the required capital and thus the fund is in shortfall. While the fund is in shortfall, the funding ratio is still higher than the MVEV. This means that reductions might not yet be necessary, but full indexation is also not very likely in the coming years. Table 7 shows the carve-out optimisation results for the pension fund. It contains the utility of the active and pensioned participants for each of the carve-out scenarios in the first two columns. The base utility without carve-out is coloured yellow. Utilities that are higher than the 'No Carve-Out' scenario are coloured green and lower utilities are coloured red. The third and fourth column give the amount of assets that is allocated to active participants and the pensioners under a given carve-out scenario. The distribution of the assets is given in terms of the regulatory funding ratio. The last columns contain the assets weights for respectively the 1-month bond fund, the 10-year bond fund and the global equity portfolio and the percentage of interest rate risk that is hedged by swap contracts.

Table 7: Carve-Out Optimisation Results

Carve-out optimisation results for an average pension fund with an initial regulatory funding ratio of 115%. The first two columns contain the utility derived by the participants, split in active and pensioned participants. The columns under 'Distribution in FR' show the amount of assets that is allocated to each group in terms of regulatory funding ratio. The last four columns show the optimised portfolio for the pension fund. Here B_{1M} and B_{10Y} denote the 1-month and 10-year constant maturity bond funds and xs denotes the global diversified equity fund. Lastly *Swap* is the amount of liability interest rate risk that is hedged with swap contracts.

	Utility		**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
No Carve-Out	-30580	-4172	115.0%	115.0%	0.0%	61.5%	38.5%	80.6%
Nominal FR	-33290	-3592	117.2%	109.9%	0.0%	61.6%	38.4%	79.8%
Real FR	-28805	-7543	125.1%	91.7%	0.0%	61.8%	38.2%	80.6%
DNB FR	-34673	-3080	115.0%	115.0%	0.0%	61.6%	38.4%	82.1%
Exp. Index.	-33207	-3628	117.3%	109.6%	0.0%	61.8%	38.2%	80.6%
Fair Value	-31578	-4564	120.1%	103.3%	0.0%	61.8%	38.2%	80.9%
Indiff.	-32179	-4172	119.1%	105.6%	0.0%	61.9%	38.1%	80.8%

Table 7 shows that for this pension fund none of the carve-out scenarios lead to a win-win situation, where both active participants and pensioners gain utility. Even when the pensioners are indifferent between a carve-out or no carve-out by allocating a funding ratio 105.6% to the pensioners, still the active participants are worse off. An interesting result is that the portfolio changes only marginally after a carve-out. Campbell and Viceira (2002) show that optimal portfolios for longer horizons should be allocated more to equity. This also holds true in this ALM model. Appendix D.1 shows how the optimal portfolio changes per age, duration of the fund and initial funding ratio. Portfolios are increasingly allocated to equity for lower ages and longer durations. However, in the homogeneous risk aversion setting the differences are small. The relative small portfolio changes after a carve-out are the result of the larger marginal utilities of the active participants compared to the pensioners. Due to the increasing life expectancy and longer horizon, an increase in the benefits of younger participants will have a larger impact on the total utility than an equal increase for a pensioner. This, combined with the fact that the largest part of the fund is not retired yet and the fact that the difference in preferences are already small, results in portfolios that are shifted more towards the preferences of the active participants than to the pensioners.

All the intuitive asset distribution policies lead to value transfers from the active participants to the pensioners. In the value neutral asset distribution case, the pensioners get 103.3% in terms of funding ratio. All other measures result in more assets being allocated to the pensioners. Distribution based on the nominal funding ratio and expected indexation lead to the smallest value transfers. The value-based asset distribution leads to the most well-balanced distribution in terms of utility. Of the more intuitive distribution rules, the expected indexation is closest to a well-balanced utility distribution.

Despite the value transfers, almost all of carve-out scenarios lead to an increased regulatory funding ratio for the remaining participants. As the pension fund policy decisions are all based on this funding ratio, one would expect the participants of the fund to benefit from the higher regulatory funding ratio through higher indexations. Table 8 shows various risk measures for the pension fund with and without carve-out. The risk measures give insight in the distributions of the funding ratio, indexation and reductions. The distribution of the funding ratio is summarised by the median, to indicate the level split in short and long term, and various downside risk measures. $P(FR_t < 100\%)$ and $Pw(FR < 100\%)$ give the probabilities that the funding ratio falls below 100% in a specific year and at least once within the 15-year horizon respectively.

The median funding ratios and probabilities of underfunding show that the pension fund benefits in the short term from a carve-out. The median funding ratio increases and the probability of underfunding decreases for the 1-year horizon for all except the 'DNB FR' carve-out scenario. For the 15-year horizon the opposite is true, the median funding ratio is lower and probability of underfunding higher for all carve-out scenarios. Only for the 'Real FR' carve-out the median funding ratio after 15 years is higher.

$FaR^{2.5\%}_{t\to t+15}$ gives the 2.5% probability funding ratio at risk and conditional funding for a horizon of 15 years and $CFaR^{2.5\%}_{t\to t+15}$ is the corresponding conditional funding ratio at risk. $CFaR^{2.5\%}_{t\to t+15}$ is the expected percentage loss in funding ratio given that a 2.5% tail probability loss occurs. Both the $FaR^{2.5\%}_{t\to t+15}$ and the $CFaR^{2.5\%}_{t\to t+15}$ increase quite

Table 8: ALM Risk Measures

Simulated risk measures based on the regulatory funding ratio, the indexation and the reduction results. FR_{t+i}, I_{t+i} and R_{t+i} denote the funding ratio, indexation fraction and reduction in the i-th simulation year. $P(FR_{t+i} < 100\%)$, $P(I_{t+i} < 80\%)$ and $P(R_{t+i} > 0)$ are the simulated probabilities that in year i the funding ratio is lower than 100%, the indexation fraction is lower than 80% and the reduction is greater than 0. $Pw(FR_{t\to t+15} < 100\%)$ and $Pw(R_{t\to t+15} > 0)$ are the probabilities that the funding ratio is lower than 100% and the reduction is greater than 0 at least once within the next 15 years. $FaR^{2.5\%}_{t\to t+15}$ denotes the 2.5% probability funding ratio at risk for the horizon of 15 years. $CFaR^{2.5\%}_{t\to t+15}$ is the corresponding conditional funding ratio at risk also known as expected shortfall. It is the expected loss given the fact that a 2.5% probability loss occurs. $P(I = 100\%)$ is the probability of having a 100% indexation result at any moment in the coming 15 years. $P(I_{t\to t+15} < 80\%)$ is the probability that the cumulative indexations and reductions over 15 years is smaller than 80% HICP inflation over the same period. Finally $E[R \mid R > 0]$ gives the expected reduction given that a reduction is required at any moment in the coming 15 years.

	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Fair Value	Indiff.
Regulatory Funding Ratio							
Median FR_{t+1}	116.8%	118.1%	125.2%	116.1%	118.2%	120.8%	119.9%
Median FR_{t+15}	139.8%	138.0%	141.6%	137.1%	138.0%	138.8%	138.5%
$P(FR_{t+1} < 100\%)$	4.4%	3.0%	0.4%	5.2%	2.9%	1.4%	1.6%
$P(FR_{t+15} < 100\%)$	3.2%	3.9%	3.3%	4.2%	3.9%	3.5%	3.5%
$Pw(FR_{t\to t+15} < 100\%)$	31.6%	32.6%	21.6%	36.4%	32.3%	27.8%	29.0%
$FaR^{2.5\%}_{t\to t+15}$	17.8%	20.5%	25.7%	19.2%	20.6%	22.3%	21.6%
$CFaR^{2.5\%}_{t\to t+15}$	28.0%	31.2%	36.6%	29.9%	31.3%	33.2%	32.5%
Indexation							
Median I_{t+1}	31.1%	31.3%	59.5%	23.3%	31.8%	41.7%	38.0%
Median I_{t+15}	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
$P(I_{t+1} < 80\%)$	93.5%	96.2%	75.2%	98.3%	96.0%	91.7%	93.9%
$P(I_{t+15} < 80\%)$	33.1%	37.8%	32.6%	39.5%	37.8%	35.5%	36.4%
$P(I = 100\%)$	30.5%	24.8%	33.8%	23.0%	24.9%	27.4%	26.3%
$P(I_{t\to t+15} < 80\%)$	47.1%	53.4%	41.8%	57.1%	53.3%	48.7%	50.2%
Reductions							
$P(R_{t+1} > 0)$	0.10%	0.12%	0.03%	0.18%	0.11%	0.08%	0.09%
$P(R_{t+15} > 0)$	0.19%	0.21%	0.18%	0.21%	0.21%	0.21%	0.21%
$Pw(R_{t\to t+15} > 0)$	26.0%	27.8%	18.5%	31.3%	27.4%	23.9%	25.3%
$E[R \mid R > 0]$	1.09%	1.05%	1.10%	1.03%	1.06%	1.05%	1.05%

significantly in each of the carve-out scenarios, indicating an increase in the downside risk. This increased downside risk can not be the result of increased portfolio risk as the portfolios changed only marginally after a carve-out. The main driver of the increased volatility of the funding ratio is the increased duration of the liabilities combined with the hedging mismatch of the swap. The increased duration make the liabilities more vulnerable for interest rate shocks. This increased risk can normally be hedged by means of swap overlays, but the swap can only be used to hedge this risk in terms of the nominal curve. The UFR results in a dampening of the shocks on the liability side, but no assets are available to mimic this on the asset side of the balance sheet. This mismatch adds to the volatility of the funding ratio (Duyvesteyn et al. (2013)). This increased funding ratio volatility will impact the indexations and reductions of the entitlements.

The indexation results are summarised by the median indexation and the probability of indexation below 80% HICP inflation in the short and long term. The indexation results in the short term do indicate an improvement in all except the 'DNB FR' carve-out. The median indexation results are higher and the downward risk following from $P(I_{t+1} < 80\%)$ is generally lower. The probability of indexation lower than 80% HICP in year 15, $P(I_{t+15} < 80\%)$, and the overall proportion of full indexations, $P(I = 100\%)$, show that the indexation quality in the long term is worse after a carve-out. Especially the long term is important for active participants. The long term indexation result and the real value of the entitlements are best summarised by the cumulative indexations and reductions combined over 15 years. $P(I_{t \to t+15} < 80\%)$ gives the probability of all indexations, including recovery indexations, and reductions to be lower than 80% of the HICP inflation over this same period. For all carve-out scenarios, except the 'Real FR' carve-out, this probability increases, indicating that the active participants are worse off in terms of real value of the entitlements after 15 years. These worse indexation results are mainly caused by the increased downside risk of the funding ratio. However, indexation also becomes more expensive as the duration of the fund increases. For the pension fund the price of the indexations is incorporated through the future proof indexation level, discussed in Section 4.2.3. With higher durations the funding ratio required for 100% indexation also becomes higher. With the same funding level this means that the participants in a fund with lower duration receive higher indexation than a fund with a higher duration. This effect becomes evident by comparing the 'DNB FR' carve-out scenario results to the 'No Carve-Out' results. With a median funding ratio of 116.8% the median indexation result is 31.1%, whereas with a DNB Funding Ratio carve-out this is only 23.3.% with a similar funding ratio of 116.1%. Additionally, the more expensive indexations cause the funding ratio to decline more at the moment the indexation is incorporated in the entitlements. These higher costs of indexation thereby lead to a declined growth potential of the funding ratio. For the carve-out to be beneficial for the remaining participants, a significant funding benefit is required to compensate the more expensive indexations.

The reduction results are summarised by the probability of a reduction in the first and last year, $P(R_{t+1} > 0)$ and $P(R_{t+15} > 0)$, the probability of a reduction within the next 15 years, $Pw(R_{t \to t+15} > 0)$, and the expected value of the reduction at any moment in time given a reduction is necessary, $E[R \mid R > 0]$. The probability of reductions in the first year after a carve-out is strongly dependent on the funding benefit of the fund. The Indifference carve-out scenario has the lowest funding ratio where the probability of a reduction in the first year is lower compared to no carve-out. The long term probability of reductions increases for all except the Real carve-out. The probability of a reduction within the next 15 years gives mixed results. For the Nominal, DNB and Expected Indexation carve-out the probability increases. For the other scenarios the funding benefit is high enough to result in a lower probability of a reduction in the next 15 years.

The overall conclusion of the results remains that a carve-out where the pensioners benefit, may also benefit the active participants in the short term. In the long term however, the active participants will be worse off. The increased duration after a carve-out makes indexations more expensive for the fund. This makes a significant funding benefit required for the carve-out to be attractive for the remaining participants. The results also indicate an increased funding ratio volatility, which can not be successfully hedged by a nominal swap due to the UFR. This increases the probability of reductions in the long term. With a carve-out the fund can no longer reduce the payments made to the pensioners in negative scenarios to ensure recovery. In these negative scenarios the amount of assets distributed to the pensioners is 'too large', compared to what they would receive in the fund. This causes the negative scenarios to hurt more for the remaining participants, as the risks can no longer be shared with the pensioners. In the setting discussed in this section a scenario where the remaining participants benefit from a carve-out, the pensioners are worse off and vice versa, making a win-win situation impossible in this setting and model assumptions. To determine whether a longer simulation horizon would lead to different results of I performed the same analysis with a 60 year horizon and 2000 simulations. This did not lead to different conclusions.

6.1.2 Sensitivity to the Initial Funding Ratio

The initial funding ratio of the pension fund has a large influence on the indexation and reduction probabilities of the pension fund. This section explores how this initial funding ratio also influences the pension results of the participants with and without a carve-out. For this purpose the Average pension fund is used, which has an average age of 55 years. The carve-out results of a pension fund with an initial funding ratio of 100% are compared with the results with a 130% initial funding ratio. The participants have homogeneous risk aversion, which is set to a value of $\gamma = 5$.

Table 9 shows the carve-out optimisation results for both the pension funds with an initial funding ratio of 100% and 130%. The total utilities in this table show that in both cases no win-win situation arises. With a funding ratio of 130% the pensioners get a maximum of 115.1% funding ratio for their entitlements. At this level 100% HICP indexation can be bought. This causes the funding ratios after a 'DNB FR' carve-out to be different for both groups. Also for the 'Nominal' carve-out this maximum 100% HICP indexation for the pensioners is reached.

The portfolios after carve-out change only marginally in both situations. With a higher funding ratio the optimal portfolio has more swap contracts. The fund thereby protects the strong financial position, whereas with a lower funding ratio less interest rate risk is covered to pursue recovery. With a funding ratio of 130% the funding benefits for the remaining participants are much larger. This would make a carve-out more likely to be beneficial with a higher funding ratio. This also follows from the resulting utilities. With a funding ratio of 100% both the active as the pensioned participants are worse off in most of the carve-out scenarios. With a funding ratio of 130% the pensioners still can benefit in some scenarios and loss in utility for the active participants seems less severe.

Table 10 contains the ALM risk measures for the carve-out scenarios with a funding ratio of 100% and 130%. For both levels of funding ratio a carve-out is generally not beneficial for the remaining participants. The median funding ratio after 15 years is lower in all cases and the conditional funding ratio at risk, $CFaR^{2.5\%}_{t+1\rightarrow t+15}$, shows an increased downward risk after a carve-out. For the case of a pension fund with a funding ratio of 130%, the probability of the first year indexation result to be lower than 80% indicates a benefit for the remaining participants in terms of indexation. In the long term this indexation benefit vanishes for both the low and high funding ratio. With a funding ratio of 100% the probability of reductions in the first year increases, despite the funding benefits of the fund. This is caused by the increased volatility of the funding ratio, making reductions in the short term more likely. In the 130% funding ratio case the likelihood of reductions in the long and short term does not change significantly. The probability of a reduction in the next 15 years does decrease for all carve-out scenarios, whereas these increase in the case of a funding ratio of 100%.

A carve-out seems to have a higher probability of success with a higher funding ratio. The higher funding ratio and funding benefits cause the fund to be more resistant to the increased downward risk that come with a carve-out. Also, with a higher funding ratio the fund is able to cope with the more expensive indexations for the remaining participants. As mentioned previously, the increased downside risk is mainly the result of the hedging mismatch caused by the UFR and the decreased ability to share the funding risk over multiple generations. In a negative scenario the pensioners would normally share the pain of reductions, whereas with a carve-out this is no longer the case.

Table 9: Carve-Out Optimisation Results With a Funding Ratio of 100% and 130%

The carve-out optimisation results for an average pension fund with an initial regulatory funding ratio of 115% and 130%. The first two columns contain the utility derived by the participants, split in active and pensioned participants, in the various carve-out scenarios. The columns under 'Distribution in FR' show the amount of assets that is allocated to each group in terms of regulatory funding ratio. The last four columns show the optimised portfolio for the pension fund. Here B_{1M} and B_{10Y} denote the 1-month and 10-year constant maturity bond funds and xs denotes the global diversified equity fund. Lastly $Swap$ is the amount of liability interest rate risk that is hedged with swap contracts.

	Utility		**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
No Carve-out	-44193	-5241	100.0%	100.0%	0.0%	60.7%	39.3%	77.4%
Nominal	-44913	-6396	101.9%	95.6%	0.0%	60.9%	39.1%	78.6%
Real	-39066	-13193	108.8%	79.8%	0.0%	61.4%	38.6%	78.9%
DNB	-46771	-5249	100.0%	100.0%	0.0%	60.8%	39.2%	78.4%
Exp. Index.	-48751	-4352	98.0%	104.5%	0.0%	60.7%	39.3%	78.2%
Value Based	-45986	-5705	100.8%	98.2%	0.0%	60.8%	39.2%	78.4%
Indiff.	-46786	-5241	100.0%	100.0%	0.0%	60.8%	39.2%	78.4%

(a) Funding Ratio of 100%

	Utility		**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
No Carve-Out	-21764	-3531	130.0%	130.0%	0.00%	60.87%	39.13%	87.61%
Nominal	-23551	-3073	136.5%	115.1%	0.00%	61.12%	38.88%	86.34%
Real	-21619	-4493	141.4%	103.7%	0.00%	61.07%	38.93%	88.02%
DNB	-23551	-3073	136.5%	115.1%	0.00%	61.12%	38.88%	86.34%
Exp. Index.	-23131	-3292	137.5%	112.7%	0.00%	61.10%	38.90%	86.81%
Value Based	-22383	-3787	139.4%	108.4%	0.00%	61.07%	38.93%	87.24%
Indiff.	-22740	-3531	138.5%	110.5%	0.00%	61.06%	38.94%	87.00%

(b) Funding Ratio of 130%

Table 10: ALM Risk Measures with a Funding Ratio of 100% and 130%

Simulated risk measures based on the regulatory funding ratio, the indexation and the reduction results. FR_{t+i}, I_{t+i} and R_{t+i} denote the funding ratio, indexation fraction and reduction in the i-th simulation year. $P(I_{t+i} < 80\%)$ and $P(R_{t+i} > 0)$ are the simulated probabilities that in year i the indexation fraction is lower than 80% and the reduction is greater than 0. $Pw(FR_{t \to t+15} < 100\%)$ and $Pw(R_{t \to t+15} > 0)$ are the probabilities that the funding ratio is lower than 100% and the reduction is greater than 0 at least once within the next 15 years. $CFaR^{2.5\%}_{t \to t+15}$ is the conditional funding ratio at risk also known as expected shortfall. $P(I_{t \to t+15} < 80\%)$ is the probability that the cumulative indexations and reductions over 15 years are smaller than 80% HICP inflation over the same period.

	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
median FR_{t+15}	133.3%	133.2%	135.2%	132.6%	131.9%	132.9%	132.6%
$Pw(FR_{t+1 \to t+15} < 100)$	75.4%	72.6%	50.7%	78.7%	84.1%	76.6%	78.8%
$CFaR^{2.5\%}_{t+1 \to t+15}$	17.4%	20.2%	25.3%	18.8%	17.2%	19.3%	18.8%
Indexation							
$P(I_{t+1} < 80\%)$	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
$P(I_{t+15} < 80\%)$	45.0%	46.7%	42.9%	47.8%	49.1%	47.2%	47.8%
$P(I_{t+1 \to t+15} < 80\%)$	74.9%	77.9%	67.4%	80.7%	83.1%	79.8%	80.8%
Reductions							
$P(R_{t+1} > 0)$	0.87%	0.99%	0.37%	1.28%	1.58%	1.16%	1.28%
$P(R_{t+15} > 0)$	0.35%	0.37%	0.28%	0.39%	0.40%	0.39%	0.39%
$Pw(R_{t+1 \to t+15} > 0)$	58.3%	60.1%	41.3%	65.8%	70.1%	63.8%	65.8%

(a) Funding Ratio of 100%

	No Carve-out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
median FR_{t+15}	149.9%	147.1%	151.0%	147.1%	147.9%	149.5%	148.7%
$Pw(FR_{t+1 \to t+15} < 100)$	13.1%	12.0%	9.6%	12.0%	11.5%	10.5%	10.8%
$CFaR^{2.5\%}_{t+1 \to t+15}$	36.1%	43.0%	45.2%	43.0%	43.4%	44.3%	43.9%
Indexation							
$P(I_{t+1} < 80\%)$	35.6%	29.7%	16.7%	29.7%	27.0%	22.3%	24.1%
$P(I_{t+15} < 80\%)$	21.5%	25.6%	22.4%	25.6%	24.9%	23.7%	24.7%
$P(I_{t+1 \to t+15} < 80\%)$	20.0%	25.2%	18.8%	25.2%	24.1%	21.5%	22.5%
Reductions							
$P(R_{t+1} > 0)$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
$P(R_{t+15} > 0)$	0.14%	0.14%	0.14%	0.14%	0.14%	0.14%	0.14%
$Pw(R_{t+1 \to t+15} > 0)$	12.7%	10.8%	9.3%	10.8%	10.5%	9.8%	10.0%

(b) Funding Ratio of 130%

6.1.3 Sensitivity to Duration

The duration of the liabilities of a pension fund heavily depend on the age distribution of the participants. This section explores how the duration affects the pension fund dynamics after a carve-out. Table 11 shows the Carve-Out optimisation results for both the Young and the Old fund. The Young fund is generated with an average age of 45 and the old fund with an average age of 65, which result in initial durations of 22.1 and 14.6 years. The pension funds start with a regulatory funding ratio of 115% and the participants have a risk aversion of $\gamma = 5$.

The utility values in Table 11 show that no win-win situation is achieved for both funds. For the Old fund the funding benefit in terms of the regulatory funding ratio is generally larger compared to the Young fund. This should have a positive impact on the indexation possibilities of the Old fund after a carve-out compared to the Young fund.

Table 11: Carve-Out Optimisation Results For a Young and an Old Fund

The carve-out optimisation results for a young and old aged pension fund with an initial regulatory funding ratio of 115%. The first two columns contain the utility derived by the participants, split in active and pensioned participants, in the various carve-out scenarios. The columns under 'Distribution in FR' show the amount of assets that is allocated to each group in terms of regulatory funding ratio. The last four columns show the optimised portfolio for the pension fund. Here B_{1M} and B_{10Y} denote the 1-month and 10-year constant maturity bond funds and xs denotes the global diversified equity fund. Lastly $Swap$ is the amount of liability interest rate risk that is hedged with swap contracts.

	Utility		**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
No Carve-Out	-31504	-1819	115.0%	115.0%	0.0%	63.4%	36.6%	80.5%
Nominal	-32801	-1714	117.1%	106.3%	0.0%	63.4%	36.6%	80.6%
Real	-30165	-4892	123.0%	82.3%	0.0%	63.5%	36.5%	81.0%
DNB	-33813	-1295	115.0%	115.0%	0.0%	63.2%	36.8%	80.3%
Exp. Index.	-33096	-1572	116.5%	108.7%	0.0%	63.4%	36.6%	80.6%
Value Based	-32449	-1921	117.9%	103.3%	0.0%	63.4%	36.6%	80.6%
Indiff.	-32613	-1819	117.5%	104.7%	0.0%	63.4%	36.6%	80.6%

(a) Young Fund

	Utility		**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
No Carve-Out	-21866	-7260	115.0%	115.0%	0.0%	61.1%	38.9%	79.0%
Nominal	-26177	-5967	117.3%	112.4%	0.0%	61.3%	38.7%	79.7%
Real	-20867	-9500	128.3%	99.8%	0.0%	61.0%	39.0%	83.4%
DNB	-27076	-5650	115.7%	114.2%	0.0%	61.3%	38.7%	78.7%
Exp. Index.	-25234	-6345	119.0%	110.4%	0.0%	61.2%	38.8%	80.2%
Value Based	-22726	-7768	124.1%	104.6%	0.0%	61.1%	38.9%	81.4%
Indiff.	-23480	-7260	122.5%	106.5%	0.0%	61.1%	38.9%	81.1%

(b) Old Fund

The pension risk measures for the Young and Old fund are shown in Table 12. The conditional funding ratio at risk over the course of 15 years, $CFaR^{2.5\%}_{t+1\rightarrow t+15}$, increases for both funds and every carve-out scenario. However, for the Young fund this increase is much more severe. The Young fund has a longer duration, which causes the UFR hedging mismatch effect for this fund to be larger. The larger probability of underfunding within 15 years, $Pw(FR_{t+1\rightarrow t+15} < 100)$, for the Young fund endorses this statement. The probability of the indexation in the first year to be below 80% increases for the Young fund for all except the Real carve-out. This indicates that the remaining participants do not benefit in the short term from a carve-out in a Young fund. The probability of a reduction in the first year after a carve-out shows the same pattern.

For the Old fund the short term benefits for the remaining participants in terms of indexation and reduction probabilities show mixed results for the various carve-out scenarios. For example in the Indifference scenario the probability of receiving indexation below 80% and the probability of a reduction both decrease. These short term benefits vanish in the long term, where the risk of receiving lower indexation and even reductions increase. The probability of the total indexations and reduction combined over 15 years to be lower than 80% of the HICP inflation, $P(I_{t+1\rightarrow t+15} < 80\%)$, increases for all scenarios. This shows that the particpants are more likely to be worse off in terms of real pension entitlements. The probability of receiving any reductions in the next 15 years increases for all except the Real carve-out for the Young fund. For the Old fund this probability also decreases for the Value Based and the Indifference carve-out. Overal the results show that a carve-out with homogeneous risk aversion is not beneficial for both the remaining participants as the pensioners. The results for the Old fund do look a bit more promising, due to the higher funding benefits and lower indexation price. The risk measures also show that for the Old fund the remaining participants can benefit in the short term, whereas this is generally not the case for the Young fund.

Table 12: ALM Risk Measures for a Young and Old Fund

Simulated risk measures based on the regulatory funding ratio, the indexation and the reduction results. FR_{t+i}, I_{t+i} and R_{t+i} denote the funding ratio, indexation fraction and reduction in the i-th simulation year. $P(I_{t+i} < 80\%)$ and $P(R_{t+i} > 0)$ are the simulated probabilities that in year i the indexation fraction is lower than 80% and the reduction is greater than 0. $Pw(FR_{t\rightarrow t+15} < 100\%)$ and $Pw(R_{t\rightarrow t+15} > 0)$ are the probabilities that the funding ratio is lower than 100% and the reduction is greater than 0 at least once within the next 15 years. $CFaR^{2.5\%}_{t\rightarrow t+15}$ is the conditional funding ratio at risk also known as expected shortfall. $P(I_{t\rightarrow t+15} < 80\%)$ is the probability that the cumulative indexations and reductions over 15 years are smaller than 80% HICP inflation over the same period.

	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
$medianFR_{t+15}$	141.7%	141.3%	143.0%	140.7%	141.2%	141.5%	141.4%
$Pw(FR_{t+1\rightarrow t+15} < 100)$	34.5%	33.8%	23.9%	37.3%	34.7%	32.4%	33.2%
$CFaR^{2.5\%}_{t+1\rightarrow t+15}$	32.1%	34.7%	38.9%	32.9%	34.3%	35.3%	35.1%
Indexation							
$P(I_{t+1} < 80\%)$	97.9%	98.2%	91.6%	99.1%	98.6%	97.9%	98.1%
$P(I_{t+15} < 80\%)$	36.4%	37.6%	35.3%	38.5%	37.9%	37.4%	37.6%
$P(I_{t+1\rightarrow t+15} < 80\%)$	55.3%	57.6%	49.4%	61.5%	58.7%	56.9%	57.3%
Reductions							
$P(R_{t+1} > 0)$	0.20%	0.22%	0.09%	0.28%	0.23%	0.20%	0.20%
$P(R_{t+15} > 0)$	0.21%	0.25%	0.20%	0.26%	0.25%	0.24%	0.24%
$Pw(R_{t+1\rightarrow t+15} > 0)$	29.1%	29.8%	21.6%	33.3%	30.7%	29.1%	29.4%

(a) Young Fund

	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
$medianFR_{t+15}$	142.3%	137.6%	143.7%	136.8%	138.2%	141.0%	140.1%
$Pw(FR_{t+1\rightarrow t+15} < 100)$	27.6%	30.2%	16.7%	33.5%	28.2%	21.3%	23.5%
$CFaR^{2.5\%}_{t+1\rightarrow t+15}$	24.4%	29.7%	36.0%	28.5%	30.8%	33.7%	32.9%
Indexation							
$P(I_{t+1} < 80\%)$	82.2%	93.1%	50.5%	95.9%	88.9%	70.0%	75.8%
$P(I_{t+15} < 80\%)$	26.6%	35.5%	27.4%	36.6%	33.3%	29.9%	31.4%
$P(I_{t+1\rightarrow t+15} < 80\%)$	34.4%	48.9%	30.6%	51.3%	46.0%	37.7%	40.0%
Reductions							
$P(R_{t+1} > 0)$	0.06%	0.07%	0.00%	0.09%	0.06%	0.02%	0.03%
$P(R_{t+15} > 0)$	0.17%	0.23%	0.17%	0.23%	0.21%	0.17%	0.19%
$Pw(R_{t+1\rightarrow t+15} > 0)$	22.7%	26.2%	15.4%	27.6%	23.7%	18.6%	20.1%

(b) Old Fund

6.2 Heterogeneous Risk Aversion

Various studies have shown that the level of risk aversion of an individual generally increases with age, e.g. Albert and Duffy (2012). This difference in risk aversion of younger and older participants of a pension fund influences the optimal asset allocation of the fund. While a portfolio might be optimal for the complete pension fund, on the individual level this portfolio is likely to be suboptimal. In the case of homogeneous risk aversions these optimal portfolio preferences already are variant per age, but with heterogeneous risk aversion this effect will be even larger.

To incorporate heterogeneous risk aversion per age, the active participants are assumed to have different risk preferences than the pensioners. Within these groups the risk preferences remain homogeneous. Riley and Chow (1992) study the risk aversion for several demographic and socioeconomic categories in an asset allocation setting. They reveal that risk aversion over age can be divided in a group younger than 65 and a group older than 65, which resembles the age of retirement. Both the active participants and the pensioners have CRRA utility preferences in this setting, but with heterogeneous risk aversion.

Due to the heterogeneity in the risk aversion the total utility of both groups can no longer be optimised in one step, because one unit utility does not resemble equal amounts for both groups anymore. Therefore, the fund portfolio without a carve-out is optimised for both groups separately. The final pension fund asset allocation is the weighted average of the individual group portfolios. The weights are chosen such that the portfolios for both groups, optimised with the same risk aversion level($\gamma = 5$), are as close as possible to the actual homogeneous optimal portfolio. This is best achieved by determining the weights by the amount of pension payments that each age is expected to receive. This expected amount is equal to the sum of the probabilities of reaching age x for x greater or equal to 67. This way the weights resemble the marginal utility per cash flow the participants receive in an accurate way[30]. The final weights differ per average age of the fund and can be found in Appendix D.2.

The carve-out results with heterogeneous risk aversion strongly depend on the levels of risk aversion assumed and especially the dispersion of the risk aversion. Therefore, I analyse the results for multiple combinations of the risk aversion parameters. For the actives the risk aversion parameter, γ_{67^-}, is set to a level 5. For the pensioners I vary the risk aversion parameter, γ_{67^+}, 9 and 11. These values are chosen based on the optimal asset allocations for active and pensioned participants shown in Appendix D.2. These risk aversion levels result in realistic portfolios for the individual groups. In this section I show the results with $\gamma_{67^-} = 5$ and $\gamma_{67^+} = 11$. The results with $\gamma_{67^+} = 9$ did not lead to any win-win situation. The utility tables for this setting can be found in Appendix D.3.

Table 13 shows the expected utilities before and after carve-out for the participants of a Young, Average and Old pension fund with initial funding ratios of 100%, 115% and 130%. The table shows that with an initial funding ratio of 115% in none of the carve-out scenarios a win-win situation arises. However, with the initial funding ratios of 100% and 130% these situations do occur. With a funding ratio of 100% a win-win

[30] Additionally I considered weights based on the number of participants and the present value of the liabilities, however these resulted in less accurate portfolios when compared to the homogeneous optimal portfolios.

Table 13: Carve-Out Utilities with $\gamma_{67^-} = 5$ and $\gamma_{67^+} = 11$

The utilities of the active and pensioned participants before and after a carve-out. The results are ordered by the initial funding ratio, which are from top to bottom 100%, 115% and 130%. Each table section shows the results for a Young, Average and Old fund. Utility gains are marked in green and utility losses are marked in red.

$\boldsymbol{FR_0 = 100\%}$	**Young**		**Average**		**Old**	
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	**Actives**	**Pensioners**
No Carve-Out	-41101	-7441	-44235	-11845	-35469	-16234
Nominal	-41337	-5140	-44913	-8648	-36853	-11920
Real	-38220	-66172	-39066	-52838	-29610	-40001
DNB	-42597	-2065	-46771	-4872	-38710	-8542
Exp. Index.	-43308	-1252	-48538	-2981	-42289	-5154
Value Based	-42413	-2396	-45986	-6285	-37569	-10390
Indiff.	-40842	-7441	-43709	-11845	-34839	-16234
$\boldsymbol{FR_0 = 115\%}$	**Young**		**Average**		**Old**	
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	**Actives**	**Pensioners**
No Carve-Out	-31506	-3552	-30593	-4948	-21892	-6119
Nominal	-32801	-1006	-33290	-1699	-26177	-2450
Real	-30165	-16357	-28805	-13061	-20867	-8751
DNB	-33813	-497	-34673	-1171	-27076	-2153
Exp. Index.	-33070	-815	-33132	-1784	-25111	-2910
Value Based	-32444	-1385	-31551	-3255	-22696	-4900
Indiff.	-31662	-3552	-30727	-4948	-21917	-6119
$\boldsymbol{FR_0 = 130\%}$	**Young**		**Average**		**Old**	
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	**Actives**	**Pensioners**
No Carve-Out	-24615	-1904	-21773	-2589	-14294	-3340
Nominal	-26038	-484	-23551	-1164	-15506	-2153
Real	-24160	-4800	-21619	-3096	-15177	-2366
DNB	-26038	-484	-23551	-1164	-15506	-2153
Exp. Index.	-25751	-624	-23119	-1376	-15192	-2355
Value Based	-25380	-892	-22378	-1950	-14604	-2859
Indiff.	-24790	-1904	-21889	-2589	-14199	-3340

situation is possible for all three types of pension funds, whereas with a funding ratio of 130% this only occurs for the Old pension fund. This result stands in contrast to the results obtained with homogeneous risk aversion, where a carve-out seemed less attractive for lower funding ratios. Unfortunately, the win-win situation is not achieved by any of the carve-out scenarios, but becomes evident from the Indifference carve-out.

Due to the higher risk aversion of the pensioners, the amount of assets needed for the pensioners to be indifferent is lower compared to the homogeneous case. The gain in utility from the optimal portfolio together with the larger amount of assets that can be allocated to the fund make a carve-out more attractive. To gain insight into why the carve-out seems to be more successful for an older fund, I first compare the results of the Old and the Young fund with an initial funding ratio of 130%. Thereafter I will compare the results of the Young fund with an initial funding ratio of 130% to the same fund with an initial funding ratio of 100%. This gives more insight to why a carve-out can be beneficial with the lower funding ratio, whereas this is not the case for the higher funding ratio. Table 14 contains the optimisation results for the aforementioned funds. Table 15 shows the pension fund risk measures.

6.2.1 Young Versus Old Fund

Table 14 shows that the funding benefits for the Old pension fund are much larger in terms of the regulatory funding ratio compared to the Young fund. This larger funding benefit causes full indexations to be more likely and lowers the probability of reductions in the short term. The lower probabilities of achieving indexations below 80% in Table 15 endorse this. For the old fund the probability of receiving less then 80% indexation, $P(I_{t+1} < 80\%)$, declines from 22.6% to a value in the range of 9.4%-5.5%. For the Young fund this probability changes from 56.6% to a range of 56.7%-42.8%. The long term probability of indexation below 80% suggests that this indexation benefit disappears in the long run.

The conditional value at risk in Table 15 shows that the Young fund has much more downside risk compared to the Old pension fund. As already mentioned for the homogeneous case, this is caused by a combination of the increased duration, which increase the volatility of the liabilities, and the UFR hedging mismatch, which restrains the ability of the fund to hedge this increased risk.

In the heterogeneous risk aversion setting the optimal asset allocation in an Old fund is more shifted towards the preferences of the pensioners when compared to the Young fund. In the Old pension fund the loss in utility due to the sub-optimality of the fund portfolio for their preferences is thus larger. In reality this gain in optimal portfolio compared to Young funds might be smaller. It would be for example more realistic if risk aversion increases gradually over time instead of assuming only two values. The further implications of this fact are beyond the scope of this research.

6.2.2 High Versus Low Funding Ratio

For the Young pension fund with an initial funding ratio of 130% there is no win-win situation possible from a carve-out. For the same fund with a funding ratio of 100% this situation is possible.

For the fund with a 130% initial funding ratio the probability of underfunding decreases slightly after a carve-out. However, the conditional funding ratio at risk increases quite significantly, indicating that the lower probability of underfunding is mainly the result of the funding benefit. For the fund with a funding ratio of 130% the active participants gain from an Indifference carve-out in terms of short term indexation results. In the long term the participants are expected to receive approximately the same indexations. In terms of the total indexation after 15 years this thus results in a benefit. The probability of receiving reductions in the short term was already near zero, but the probability of reductions in the long run remains about the same. The unconditional expected level of the reduction required does increase however. As the losses in terms of reductions hurt more than gains from increased indexations, the participants can not benefit from a carve-out in the Young fund with an intitial funding ratio of 130%.

For the fund with an initial funding ratio of 100% the probability of underfunding in the next 15 years decreases more significantly for the Indifference carve-out compared to the 130% funding ratio case. The probability of receiving indexations below 80% in the first year remain 100%, but after 15 years this probability is slightly lower. Together with the smaller probability of the total indexations and reduction after 15 year to be lower than 80% the indexation results indicate a benefit for the remaining participants. The largest benefit results from the decreased probability of reductions in the short run. This also results in a lower probability of receiving any reduction at all in the next 15 year. The average value of reduction, given that a reduction occurs is also lower. A win-win situation is thus possible with a funding ratio of 100% due to the decrease in loss in terms of reductions, which lead to larger benefits than only higher indexation. This effect is caused by the concavity of the CRRA utility function.

Table 14: Carve-Out Optimisation Results with Heterogeneous Risk Aversion

The carve-out optimisation results for a young and old aged pension fund with an initial regulatory funding ratio of 130% and for a young fund with an initial funding of 100%. The columns under 'Distribution in FR' show the amount of assets that is allocated to each group in terms of regulatory funding ratio. The last four columns show the optimised portfolio for the pension fund. Here B_{1M} and B_{10Y} denote the 1-month and 10-year constant maturity bond funds and xs denotes the global diversified equity fund. Lastly $Swap$ is the amount of liability interest rate risk that is hedged with swap contracts. The first two rows labelled 'Actives' and 'Pensioners' give the group specific optimal portfolio in the no carve-out case. The final optimal portfolio without a carve-out is a weighted average of the above portfolios.

Old Fund $FR_0 = 130\%$	**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
Actives	-	-	0.00%	61.10%	38.90%	90.93%
Pensioners	-	-	0.00%	76.46%	23.53%	69.44%
No Carve-Out	130.0%	-	**0.00%**	**63.88%**	**36.12%**	**87.04%**
Nominal	143.8%	114.2%	0.00%	60.73%	39.27%	90.73%
Real	145.0%	112.8%	0.00%	60.76%	39.24%	91.22%
DNB	143.8%	114.2%	0.00%	60.73%	39.27%	90.73%
Exp. Index.	145.0%	112.9%	0.00%	60.77%	39.23%	91.21%
Value Based	147.2%	110.3%	0.00%	60.87%	39.13%	92.23%
Indiff.	148.9%	108.5%	0.00%	60.90%	39.10%	93.18%

(a) Old Pension Fund with an Initial Funding Ratio of 130%

Young Fund $FR_0 = 130\%$	**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
Actives	-	-	0.00%	63.46%	36.54%	83.25%
Pensioners	-	-	0.00%	79.11%	20.89%	80.11%
No Carve-Out	130.0%	-	**0.00%**	**64.04%**	**35.96%**	**83.13%**
Nominal	133.6%	115.4%	0.00%	63.68%	36.32%	80.81%
Real	139.0%	93.0%	0.00%	63.36%	36.64%	83.23%
DNB	133.6%	115.4%	0.00%	63.68%	36.32%	80.81%
Exp. Index.	134.5%	111.8%	0.00%	63.60%	36.40%	82.03%
Value Based	135.5%	107.5%	0.00%	63.58%	36.42%	82.24%
Indiff.	137.2%	100.6%	0.00%	63.44%	36.56%	82.84%

(b) Young Pension Fund with an Initial Funding Ratio of 130%

Young Fund $FR_0 = 100\%$	**Distribution in FR**		**Asset Weights**			
Carve-Out Type	**Actives**	**Pensioners**	B_{1M}	B_{10Y}	xs	$Swap$
Actives	-	-	0.00%	62.86%	37.14%	79.39%
Pensioners	-	-	0.00%	79.23%	20.77%	75.20%
No Carve-Out	100.0%	-	**0.00%**	**63.47%**	**36.53%**	**79.23%**
Nominal	101.9%	92.4%	0.00%	62.51%	37.49%	79.16%
Real	106.9%	71.6%	0.00%	62.91%	37.09%	79.49%
DNB	100.0%	100.0%	0.00%	62.43%	37.57%	79.41%
Exp. Index.	99.0%	104.2%	0.00%	62.38%	37.62%	79.29%
Value Based	100.3%	98.9%	0.00%	62.46%	37.54%	79.41%
Indiff.	102.7%	89.1%	0.00%	62.64%	37.36%	79.37%

(c) Young Pension Fund with an Initial Funding Ratio of 100%

Table 15: ALM Risk Measures with Heterogeneous Risk Aversion

Simulated risk measures based on the regulatory funding ratio, the indexation and the reduction results. The results shown are for an Old and a Young fund with $FR_0 = 130\%$ and a Young fund with $FR_0 = 100\%$. FR_{t+i}, I_{t+i} and R_{t+i} denote the funding ratio, indexation fraction and reduction in the i-th simulation year. $P(I_{t+i} < 80\%)$ and $P(R_{t+i} > 0)$ are the simulated probabilities that in year i the indexation fraction is lower than 80% and the reduction is greater than 0. $Pw(FR_{t\rightarrow t+15} < 100\%)$ and $Pw(R_{t\rightarrow t+15} > 0)$ are the probabilities that the funding ratio is lower than 100% and the reduction is greater than 0 at least once within the next 15 years. $CFaR^{2.5\%}_{t\rightarrow t+15}$ is the conditional funding ratio at risk also known as expected shortfall. $P(I_{t\rightarrow t+15} < 80\%)$ is the probability that the cumulative indexations and reductions over 15 years are smaller than 80% HICP inflation over the same period.

Old Fund $FR_0 = 130\%$	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
$medianFR_{t+1}$	131.6%	143.0%	144.1%	143.0%	144.1%	146.3%	147.8%
$Pw(FR_{t+1\rightarrow t+15} < 100)$	7.4%	6.4%	5.7%	6.4%	5.7%	5.1%	4.8%
$CFaR^{2.5\%}_{t+1\rightarrow t+15}$	28.4%	43.0%	43.5%	43.0%	43.4%	44.4%	44.9%
Indexation							
$P(I_{t+1} < 80\%)$	22.6%	9.4%	8.0%	9.4%	8.0%	6.0%	5.5%
$P(I_{t+15} < 80\%)$	13.8%	17.2%	16.8%	17.2%	16.8%	15.5%	14.8%
$P(I_{t+1\rightarrow t+15} < 80\%)$	11.3%	11.8%	11.4%	11.8%	11.4%	10.1%	9.3%
Reductions							
$P(R_{t+1} > 0)$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
$P(R_{t+15} > 0)$	0.08%	0.13%	0.12%	0.13%	0.12%	0.10%	0.09%
$Pw(R_{t+1\rightarrow t+15} > 0)$	9.5%	8.3%	7.5%	8.3%	7.5%	7.1%	6.6%
$E[R\|R > 0]$	1.15%	1.16%	1.17%	1.16%	1.17%	1.12%	1.14%

(a) Old Pension Fund with an Initial Funding Ratio of 130%

Young Fund $FR_0 = 130\%$	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
$medianFR_{t+1}$	130.1%	132.6%	137.6%	132.6%	133.5%	134.4%	135.9%
$Pw(FR_{t+1\rightarrow t+15} < 100)$	14.6%	14.5%	12.2%	14.5%	14.2%	13.6%	13.2%
$CFaR^{2.5\%}_{t+1\rightarrow t+15}$	41.6%	45.5%	48.9%	45.5%	46.3%	47.0%	47.9%
Indexation							
$P(I_{t+1} < 80n\%)$	56.6%	56.7%	37.6%	56.7%	53.1%	49.5%	42.8%
$P(I_{t+15} < 80\%)$	29.3%	31.5%	28.5%	31.5%	31.1%	30.0%	29.2%
$P(I_{t+1\rightarrow t+15} < 80\%)$	32.7%	36.5%	29.5%	36.5%	34.6%	33.3%	31.2%
Reductions							
$P(R_{t+1} > 0)$	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
$P(R_{t+15} > 0)$	0.15%	0.15%	0.15%	0.15%	0.15%	0.15%	0.15%
$Pw(R_{t+1\rightarrow t+15} > 0)$	12.0%	11.4%	9.7%	11.4%	11.1%	10.6%	10.3%
$E[R\|R > 0]$	1.14%	1.17%	1.18%	1.17%	1.16%	1.19%	1.20%

(b) Young Pension Fund with an Initial Funding Ratio of 130%

Young Fund $FR_0 = 100\%$	No Carve-Out	Nominal	Real	DNB	Exp. Index.	Value Based	Indiff.
Regulatory Funding Ratio							
$medianFR_{t+1}$	101.9%	103.5%	108.5%	101.7%	100.7%	101.9%	104.3%
$Pw(FR_{t+1\rightarrow t+15} < 100)$	78.2%	74.4%	57.3%	80.0%	83.1%	79.4%	72.0%
$CFaR^{2.5\%}_{t+1\rightarrow t+15}$	20.2%	22.9%	27.0%	21.4%	20.6%	21.6%	23.6%
Indexation							
$P(I_{t+1} < 80n\%)$	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
$P(I_{t+15} < 80\%)$	44.9%	44.2%	42.0%	45.3%	45.6%	45.2%	44.2%
$P(I_{t+1\rightarrow t+15} < 80\%)$	80.0%	77.5%	71.2%	79.8%	81.0%	79.6%	76.2%
Reductions							
$P(R_{t+1} > 0)$	1.62%	1.57%	0.83%	1.93%	2.12%	1.91%	1.47%
$P(R_{t+15} > 0)$	0.31%	0.33%	0.31%	0.34%	0.35%	0.34%	0.33%
$Pw(R_{t+1\rightarrow t+15} > 0)$	66.5%	65.7%	52.4%	71.6%	74.0%	70.8%	63.8%
$E[R\|R > 0]$	1.06%	1.01%	0.98%	1.04%	1.05%	1.04%	1.01%

(c) Young Pension Fund with an Initial Funding Ratio of 100%

7 Conclusions

The increase in regulatory requirements and low interest rates have caused the financial statuses of pension funds to be under pressure. To help pension funds overcome this, Willis Towers Watson proposes a new de-risking solution: the carve-out. The carve-out is a partial buy-out, where instead of all entitlements only the entitlements of the pensioners are transferred to an insurer. The pension fund can then adjust its policy to be more in line with the preferences of the younger participants. The more risk averse pensioners benefit from insured benefits, which eliminates the possibility of reductions. For the sponsoring company a carve-out can decrease the pension obligations and risk buffers on the balance sheet. This releases capital that can be used for investments, which contributes to the profitability of the company.

I asses whether the carve-out can be beneficial for both the active as the retired participants by means of a value-based ALM study. To simulate equity returns and inflation I use a two-state Markov-Switching VAR model with switching intercepts and (co-)variances. The interest rates are modelled by a latent three factor affine term structure model. All pay-offs in the model can be priced by means of the pricing kernel. For the carve-out various asset distribution rules are compared in the search for the most even-minded distribution rule. For this purpose I define seven carve-out scenarios, where the assets are split based on the nominal, real and regulatory funding ratio, on the expected indexation in the fund, on the arbitrage free value of the entitlements per age and based on indifference for the pensioners. The success of the carve-out is determined by the utility of the participants, who are assumed to have CRRA preferences. First I perform my analysis in a homogeneous and thereafter in a heterogeneous risk aversion setting, where the active participants differ in preferences from the pensioners.

With homogeneous risk aversion the carve-out does not lead to a win-win situation, where both the remaining participants as the pensioners gain in utility. This holds for a pension fund with a low (100%), average (115%) and high (130%) funding ratio and for a Young, Average and Old pension fund. For these pension funds I show that a win-win situation is not possible, independent of the asset distribution. The pensioners can benefit from the insured entitlements after a carve-out in multiple carve-out scenarios. However, the amount of assets remaining in the fund in these cases is too low to compensate for the increased risk on the balance sheet of the pension fund.

In the short term the remaining participants do benefit from increased indexation probabilities caused by the increase in regulatory funding ratio. In the long term the increased funding ratio risk causes the remaining participants to be worse off with a carve-out. This increased risk is the result of the increased duration of the liabilities, which leads to higher sensitivity to changes in the interest rates and the absence of inter-generational risk sharing with the pensioners after a carve-out. The increased balance sheet risks can not be hedged away with the swap contracts available, because the long term liabilities are valued with regulatory interest rate which converges to the UFR.

The carve-out is more promising for pension funds with a higher funding ratio. In this case funding benefits for the remaining participants are higher in terms of the regulatory funding ratio, thereby increasing the likelihood of full indexations and lowering the probability of a reduction in the short term. The carve-out is also more promising for pension funds with a relatively low duration. With a relatively high percentage of the entitlements being transferred to an insurer, the funding benefits become larger for the remaining participants. This is of course only the case if the pensioners get less assets than the regulatory funding ratio attributes to them.

Distributing the assets based on the no arbitrage value of the entitlements can be considered as a fair way of distributing the assets, but also leads to a well-balanced redistribution in utility terms. For higher funding ratios distributing the assets based on the expected indexation comes closest to a value neutral approach and also results in relatively balanced distribution in terms of utility. With a lower funding ratio the expected indexation grants a too large proportion of the assets to the pensioners. The increased risk of reductions is not incorporated enough in the asset distribution in this case. After the expected indexation carve-out, the nominal carve-out is closest to being value neutral. This neutrality is also less sensitive to the initial funding for the nominal carve-out.

When the pensioners are assumed to be more risk averse than the active participants, a mutually beneficial carve-out is possible. The dispersion in the risk aversion strongly influences whether a win-win situation is possible. A higher dispersion in risk aversion causes a larger potential utility benefit from adjusting the policy to the preferences of the active participants. Furthermore, lower risk aversion of the pensioners lowers the required assets to attain the same level of utility after a carve-out. I show that with a risk aversion of 5 for the active participants and 11 for the pensioners, win-win scenarios are possible.

Again a relatively high funding ratio leads to higher funding benefits, making a carve-out more attractive at a funding ratio of 130% compared to 115%. With heterogeneous risk aversion a win-win situation also occurs with a funding ratio of 100%. A carve-out can decrease the likelihood of reductions in this case, leading to a utility benefit for all three stylised pension funds. When the pensioners are assumed to be more risk averse, splitting the assets based on the expected indexation generally grants the pensioners a too large proportion of the assets. Due to the increased risk aversion, the pensioners are satisfied with less already. With heterogeneous risk aversion the value-based asset distribution leads to a well-balanced distribution in terms of utility. With a funding ratio of 100% the nominal split leads to the best balance in utility.

Overall, the results indicate that a carve-out might not be an interesting de-risking solution. The carve-out is more attractive for older funds and with higher funding ratios. However, for older pension funds a carve-out might cause the pension fund to become too small, increasing the execution costs for the remaining participants. Whereas with higher funding ratios a de-risking solution might not be necessary. The results with heterogeneous risk aversion do look more promising for pension funds with lower funding ratios. To be able to determine whether a carve-out is an attractive de-risking method, further research is required.

In this model the financial market is quite limited. More asset classes can lead to gains from diversification of the portfolio. This can also influence the benefits of the remaining participants by having more benefit of the pension fund policy being in line with their preferences. Additionally, I assume the required capital to be constant. The required capital is greatly depended on the portfolio and the liability duration. After a carve-out, the required capital is likely to increase, which can have a negative result on the indexations after a carve-out. The main bottleneck for the carve-out seems to be the increased risk in terms of the regulatory funding ratio. This is caused by the increased duration in combination with a hedging mismatch. Introducing a complex swap contract that incorporates the UFR in a value neutral way, may overcome these difficulties encountered with a carve-out.

In the heterogeneous analysis I assume weights for the utility of the active and pensioned participants in the portfolio optimisation. These weights however do not guarantee optimality of the portfolio in terms of total utility derived by the participants. A more sophisticated approach can be found in the theory of asset pricing with heterogeneous beliefs. For example Basak (2005) make use of a central planner whose utility is a weighted average of the individual utility functions, where the weights are stochastic. Another approach is that of deriving a representative agent from the heterogeneous preferences as in Xiouros and Zapatero (2010). With these theories the heterogeneity in risk aversion can also be more conveniently modelled to gradually increase by age.

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Appendices

A Data Overview

A.1 Wage Data

The following table provides the raw data used to construct a wage curve. The curve is constructed by assuming that the wages stays constant for 50+ years. This smooths the curve and is a reasonable assumption as productivity is likely to stagnate after reaching an age of 50. The wages in the table are then assumed to hold exactly for the average wage of that particular group, e.g. wage at 22.5 years of age is assumed to be €20,900. A curve with wages for each age is then obtained by linear interpolation.

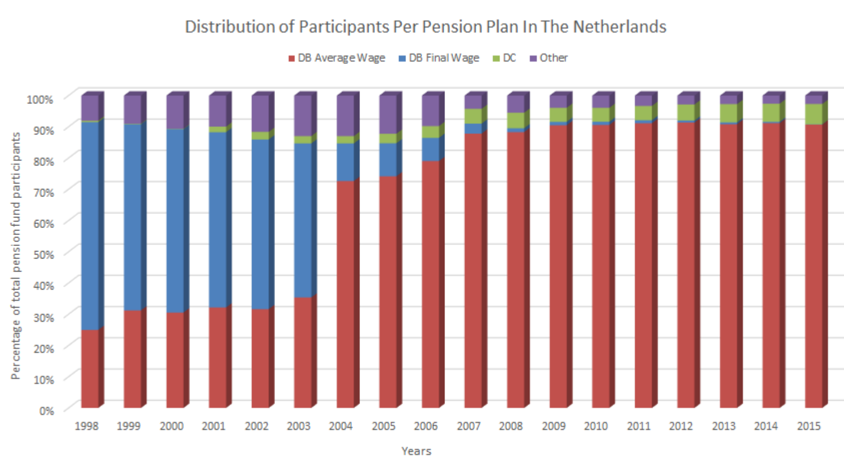
Table 16: Raw Wage Data

Age	Income
15 - 20	€4,500
20 - 25	€20,900
25 - 30	€29,700
30 - 35	€36,000
35 - 40	€39,800
40 - 45	€43,100
45 - 50	€44,400
50 - 55	€43,600
55 - 60	€44,000
60 - 65	€43,300
65 - 70	€51,000
70 - 75	€44,800

A.2 Dutch Pension System

Figure 4: Pension Scheme in the Netherlands

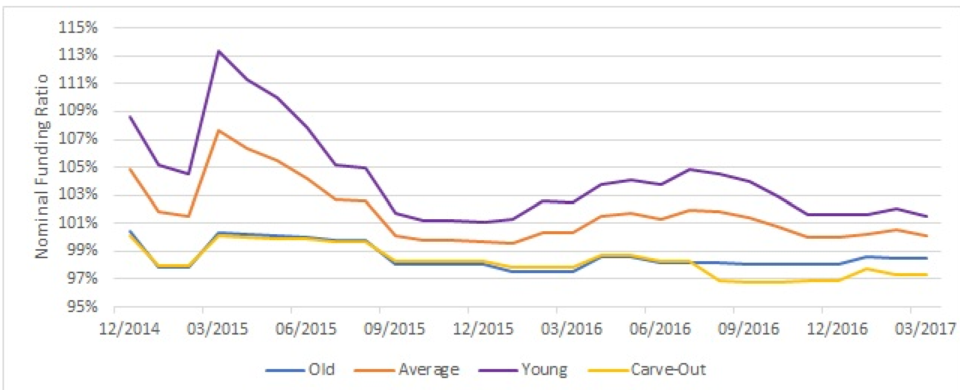
This figure shows the percentage of participants for different pension schemes in the Netherlands. The three categories are a defined benefit final wage schemes, a defined benefit average wage schemes and other. Other consists of all remaining schemes.



A.3 Nominal Buy-Out Prices

The figure below shows the historical nominal buy-out prices in the Netherlands. The prices are taken from the Buy-Out Monitor provided by Willis Towers Watson. The prices are based on three stylised pension funds with different durations and a stylised population of pensioners for the carve-out price. In the figure these are denoted by Old, Average, Young and Carve-Out. The prices are given as percentage of the liability values, where these values are determined based on the nominal curve plus UFR provided by the DNB. The figure shows that Buy-Outs are generally cheaper for funds with lower durations.

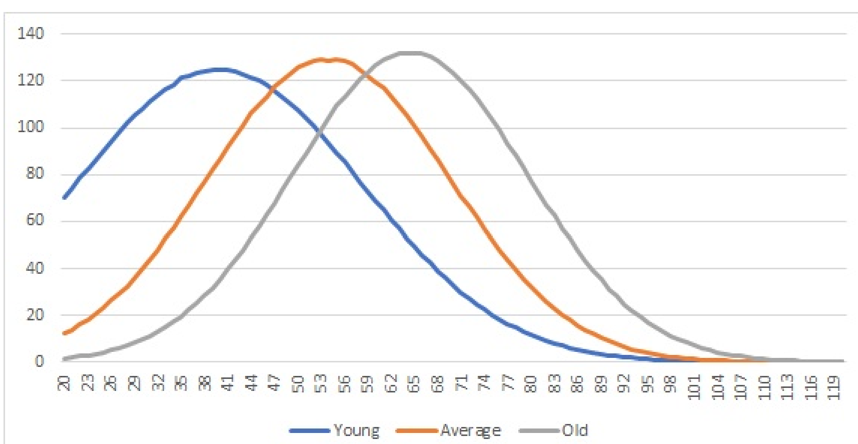
Figure 5: Nominal Buy-Out Prices



A.4 Participants Distribution

Figure 6: Distribution of Particpants

The age distributions of the fund participants for the three different funds.



A.5 Accession Probabilities

This section provides the accession probabilities for the three types of funds; Young, Average and Old.

Table 17: Accession Probabilities

Age	Young	Average	Old	Age	Young	Average	Old
20	0.15	0.01	0.01	44	0.015	0.03	0.04
21	0.05	0.01	0.01	45	0.01	0.03	0.03
22	0.05	0.01	0.01	46	0.01	0.03	0.03
23	0.05	0.01	0.01	47	0.01	0.03	0.03
24	0.05	0.01	0.01	48	0.01	0.03	0.03
25	0.05	0.02	0.01	49	0.01	0.03	0.03
26	0.05	0.02	0.01	50	0.005	0.02	0.03
27	0.04	0.02	0.01	51	0.005	0.02	0.03
28	0.04	0.02	0.01	52	0.005	0.02	0.03
29	0.04	0.02	0.01	53	0.005	0.02	0.03
30	0.04	0.03	0.02	54	0.005	0.02	0.03
31	0.04	0.03	0.02	55	0	0.01	0.02
32	0.03	0.03	0.02	56	0	0.01	0.02
33	0.03	0.03	0.02	57	0	0.01	0.02
34	0.03	0.03	0.02	58	0	0.01	0.02
35	0.03	0.04	0.03	59	0	0.01	0.02
36	0.02	0.04	0.03	60	0	0.01	0.01
37	0.02	0.04	0.03	61	0	0.01	0.01
38	0.02	0.04	0.03	62	0	0.01	0.01
39	0.02	0.04	0.03	63	0	0.01	0.01
40	0.015	0.03	0.04	64	0	0.01	0.01
41	0.015	0.03	0.04	65	0	0	0
42	0.015	0.03	0.04	66	0	0	0
43	0.015	0.03	0.04	67	0	0	0

A.6 ALM Model Parameters

The fixed parameters used in the ALM study. Most of the parameter names are self explanatory, the others will be explained here. $FR_{I=0}$ is the lower bound funding ratio from which level indexation can be granted by the fund. Recovery % is the percentage of the available capital that is regulatory allowed to be allocated to recovery indexations. The unconditional MSVAR model expectations for inflation and excess stock returns are denoted by $E[\pi]$ and $E[xs]$ respectively. $E[r_{B_{1M}}]$ and $E[r_{B_{10Y}}]$ are the unconditional expected annual returns of the 1-month and 10-year bond. Lastly δ denotes the utility discount rate, which is set to 1 unless stated otherwise.

Table 18: Fixed ALM Parameters

Regulatory parameters		**Model**	
MVEV	104%	$E[\pi]$	1.64%
VEV	120%	$E[xs]$	3.17%
$FR_{I=0}$	110%	$E[r_{B_{1M}}]$	2.24%
Accrual Rate	1.875%	$E[r_{B_{10Y}}]$	4.01%
AOW Offset	€12,953	# Simulations	10,000
Pension Age	67	Horizon	15 years
Max Age	120	Buy-Out Spread	35 bps
Min Age	20	δ	1
Male %	55%	**Pension fund**	
Exp. Inflation	2.00%	# Participants	5000
Wage Inflation	0.50%	Avr. Age Young	45
Bond Return	2.50%	Avr. Age Average	55
Equity Return	7.00%	Avr. Age Old	65
Recovery %	20.00%	Std. Dev. Age	15
Recovery Horizon	10 years	Premium %	30.00%

A.7 Financial Scenarios

The figures in this appendix section show the percentiles of the simulated financial scenarios. The figures give insight in the distribution of the simulated returns, rates and inflation.

Figure 7: Simulated Annual 1-Month Bond Return Percentiles

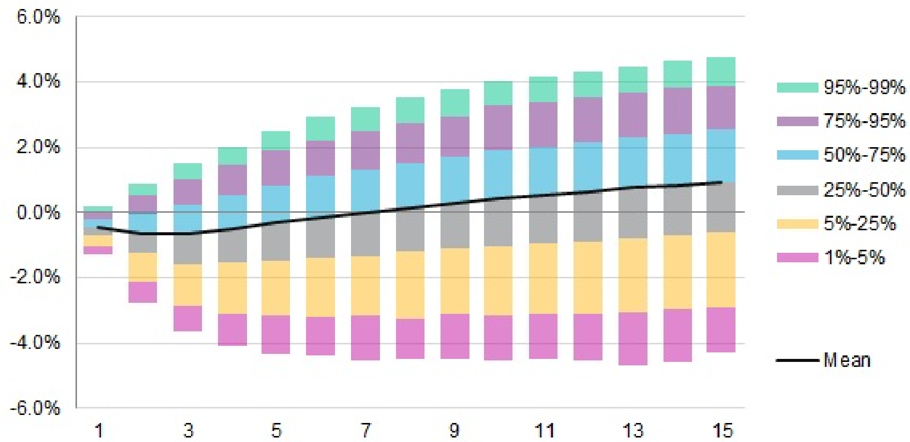


Figure 8: Simulated Annual 10-year Bond Return Percentiles

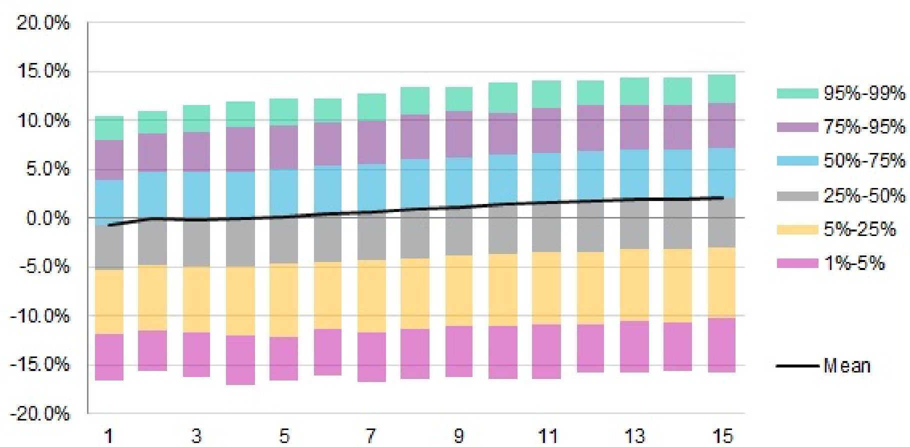


Figure 9: Simulated 10 Year Yield Percentiles

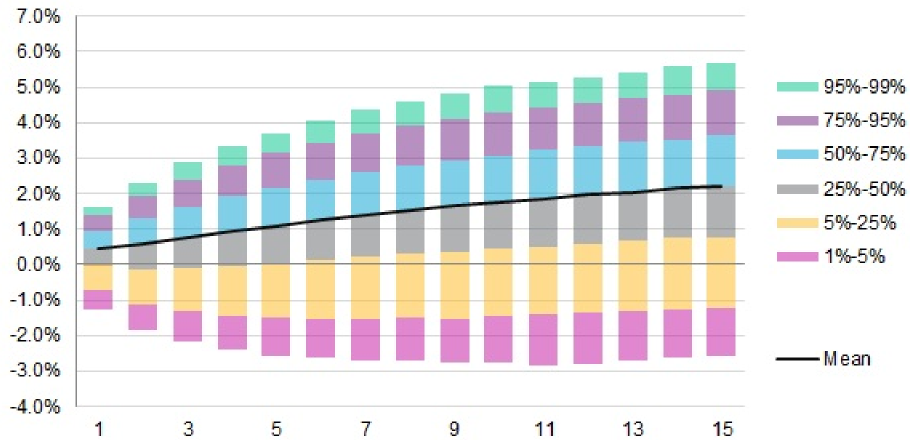


Figure 10: Simulated Total Equity Return Percentiles

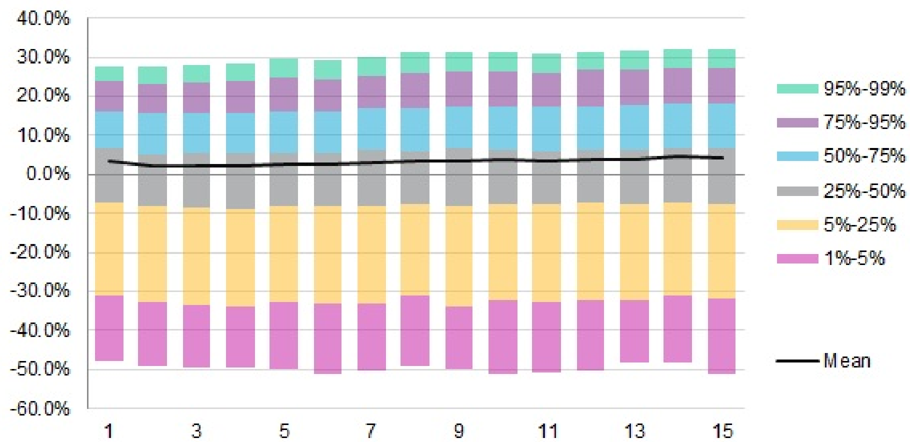


Figure 11: Simulated Inflation Percentiles

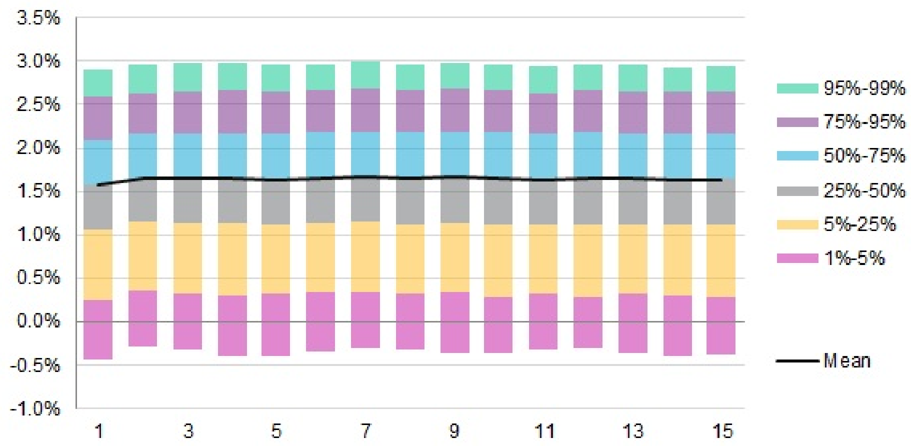
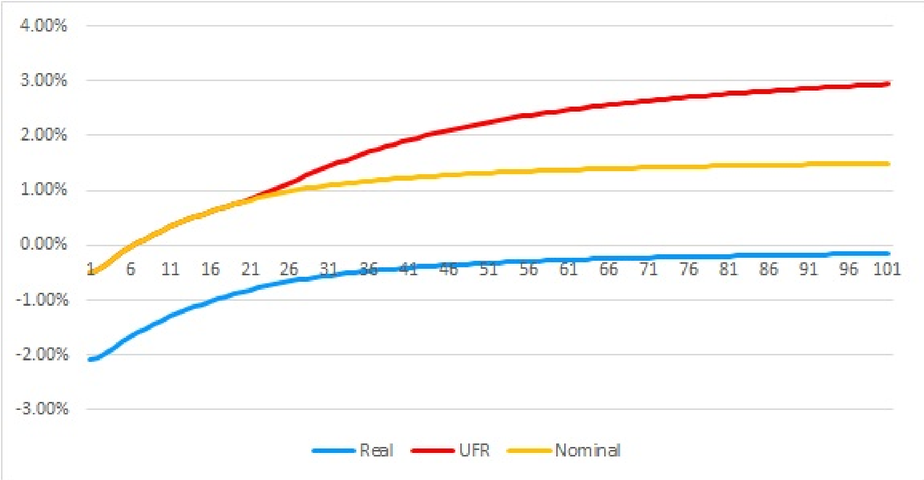


Figure 12: Initial Yield Curves



B MSVAR Model Estimation Results

B.1 Linear VAR Model

Figure 13: Linear VAR Estimates

Model parameters and statistics for the linear VAR model of inflation, equity and dividend.

Vector Autoregression Estimates
Date: 04/04/17 Time: 17:37
Sample (adjusted): 1997M03 2016M04
Included observations: 230 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	INFL	EQ	DVD
INFL(-1)	0.333023	-1.749096	0.012465
	(0.06205)	(1.94012)	(0.04431)
	[5.36673]	[-0.90154]	[0.28128]
EQ(-1)	0.004335	0.284942	-0.004962
	(0.00198)	(0.06194)	(0.00141)
	[2.18796]	[4.60003]	[-3.50731]
DVD(-1)	-0.042973	1.822772	0.984870
	(0.01797)	(0.56187)	(0.01283)
	[-2.39124]	[3.24413]	[76.7415]
C	0.001950	-0.050831	0.000266
	(0.00044)	(0.01375)	(0.00031)
	[4.43285]	[-3.69572]	[0.84678]
R-squared	0.162743	0.139532	0.964299
Adj. R-squared	0.151629	0.128110	0.963825
Sum sq. resids	0.000535	0.522766	0.000273
S.E. equation	0.001538	0.048095	0.001099
F-statistic	14.64302	12.21592	2034.797
Log likelihood	1165.392	373.6146	1242.830
Akaike AIC	-10.09906	-3.214040	-10.77244
Schwarz SC	-10.03927	-3.154247	-10.71265
Mean dependent	0.001370	-0.017693	0.022330
S.D. dependent	0.001670	0.051507	0.005776

Determinant resid covariance (dof adj.)	1.66E-15
Determinant resid covariance	1.58E-15
Log likelihood	2940.551
Akaike information criterion	-25.46566
Schwarz criterion	-25.28628

B.2 Model Selection Results

Table 19: MSVAR Model Selection Results

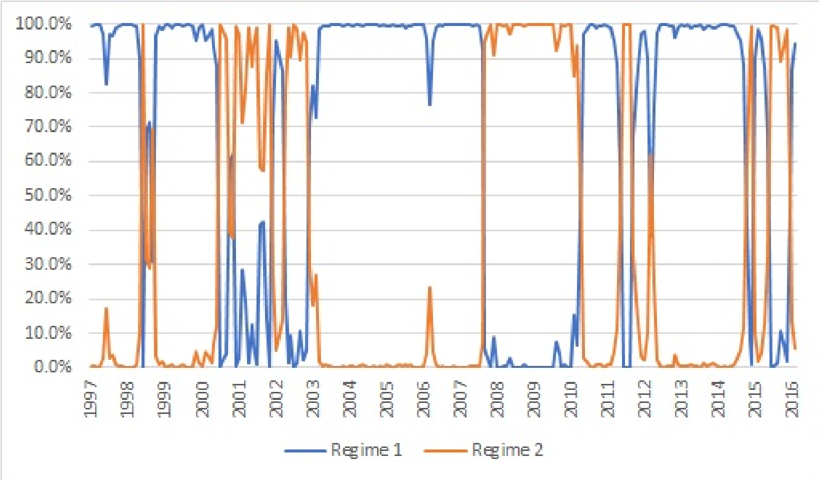
Model estimation results for MSVAR model with the number of lags p ranging from 0 to 4 and the number of regimes M ranging from 1 to 4. In the table AIC denotes the Akaike Information Criterion, BIC the Bayesian Information Criterion and HQC the Hannan-Quin Information Criterion. The models in the table are ranked based on these individual criteria, whereby the last column ranks the models based on the sum of the individual ranks.

Model Type	**Number of parameters**	**AIC**	**BIC**	**HQC**	**AIC Rank**	**BIC Rank**	**HQC Rank**	**Total Score**
VAR(0)	9	-20.695	-20.559	-20.640	50	50	50	50
VAR(1)	18	-25.771	-25.499	-25.661	40	19	34	33
VAR(2)	27	-25.815	-25.408	-25.651	38	25	36	34
VAR(3)	36	-25.779	-25.235	-25.559	39	34	40	40
VAR(4)	45	-25.856	-25.177	-25.582	37	37	38	39
MSI(2,0)	14	-21.573	-21.362	-21.488	47	46	46	46.5
MSI(2,1)	23	-25.926	-25.579	-25.786	34	15	31	28.5
MSI(2,2)	32	-25.914	-25.431	-25.719	35	24	32	32
MSI(2,3)	41	-25.901	-25.282	-25.651	36	33	35	36
MSI(2,4)	50	-25.973	-25.218	-25.668	32	36	33	35
MSH(2,0)	17	-21.261	-21.004	-21.157	49	49	49	49
MSH(2,1)	26	-26.261	-25.869	-26.103	21	2	8	8
MSH(2,2)	35	-26.254	-25.726	-26.041	22	7	12	12
MSH(2,3)	44	-26.238	-25.574	-25.970	23	16	21	20
MSH(2,4)	53	-26.205	-25.405	-25.882	28	26	30	31
MSIH(2,0)	20	-21.679	-21.377	-21.557	46	45	45	45
MSIH(2,1)	**29**	**-26.391**	**-25.954**	**-26.215**	**8**	**1**	**1**	**1**
MSIH(2,2)	38	-26.376	-25.803	-26.145	11	4	3	3
MSIH(2,3)	47	-26.345	-25.636	-26.059	16	12	10	11
MSIH(2,4)	56	-26.314	-25.469	-25.973	20	23	20	21.5
MSI(3,0)	21	-21.989	-21.672	-21.861	44	44	44	44
MSI(3,1)	30	-26.216	-25.764	-26.034	26	6	13	13
MSI(3,2)	39	-26.211	-25.623	-25.973	27	13	19	19
MSI(3,3)	48	-26.196	-25.472	-25.904	29	21	27	27
MSI(3,4)	57	-25.966	-25.106	-25.619	33	39	37	37
MSH(3,0)	27	-21.432	-21.025	-21.268	48	48	48	48
MSH(3,1)	36	-26.351	-25.808	-26.132	15	3	4	6
MSH(3,2)	45	-26.343	-25.664	-26.069	17	9	9	10
MSH(3,3)	54	-26.340	-25.525	-26.011	18	17	14	15
MSH(3,4)	63	-26.315	-25.365	-25.932	19	29	24	25
MSIH(3,0)	33	-22.358	-21.860	-22.157	43	43	43	43
MSIH(3,1)	42	-26.430	-25.796	-26.174	4	5	2	2
MSIH(3,2)	51	-26.418	-25.648	-26.107	6	10	6	6
MSIH(3,3)	60	-26.422	-25.517	-26.057	5	18	11	9
MSIH(3,4)	69	-26.388	-25.347	-25.968	10	31	22	21.5
MSI(4,0)	30	-22.807	-22.354	-22.624	42	41	42	42
MSI(4,1)	39	-26.227	-25.638	-25.989	25	11	17	17
MSI(4,2)	48	-26.196	-25.472	-25.904	30	22	28	28.5
MSI(4,3)	57	-26.232	-25.372	-25.885	24	28	29	30
MSI(4,4)	66	-25.977	-24.981	-25.575	31	40	39	38
MSH(4,0)	39	-21.690	-21.102	-21.453	45	47	47	46.5
MSH(4,1)	48	-26.397	-25.673	-26.105	7	8	7	6
MSH(4,2)	57	-26.357	-25.497	-26.010	14	20	16	16
MSH(4,3)	66	-26.358	-25.363	-25.957	13	30	23	23.5
MSH(4,4)	75	-26.366	-25.234	-25.909	12	35	26	26
MSIH(4,0)	48	-23.055	-22.331	-22.763	41	42	41	41
MSIH(4,1)	57	-26.457	-25.596	-26.109	2	14	5	4
MSIH(4,2)	66	-26.388	-25.393	-25.987	9	27	18	18
MSIH(4,3)	75	-26.467	-25.336	-26.011	1	32	15	14
MSIH(4,4)	84	-26.433	-25.165	-25.921	3	38	25	23.5

C Model Regimes

C.1 Multivariate Model Regimes

Figure 14: Full Model Smoothed Regime Probabilities
Estimated smoothed regime probabilities of the multivariate MSIH(2,1) model for inflation, equity returns and dividend yields.



C.2 Univariate Model Regimes

Figure 15: Inflation Smoothed Regime Probabilities
Estimated smoothed regime probabilities of the univariate MSIH(2,1) model for inflation.

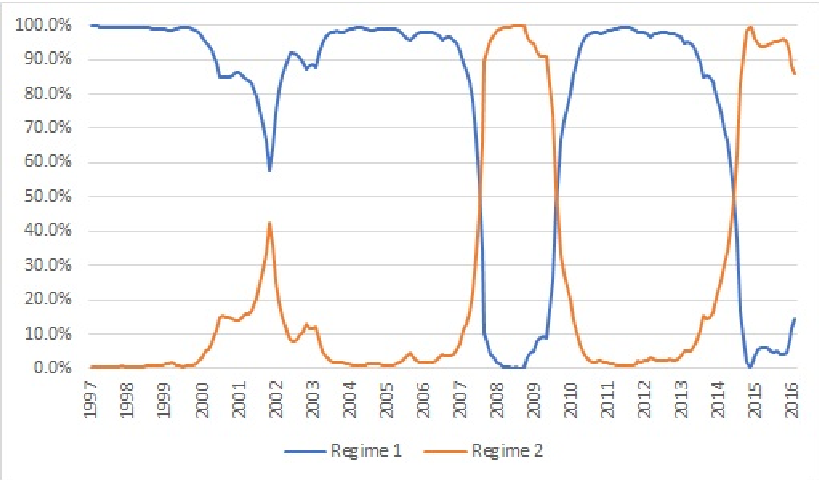


Figure 16: Equity Smoothed Regime Probabilities

Estimated smoothed regime probabilities of the univariate MSIH(2,1) model for equity returns.

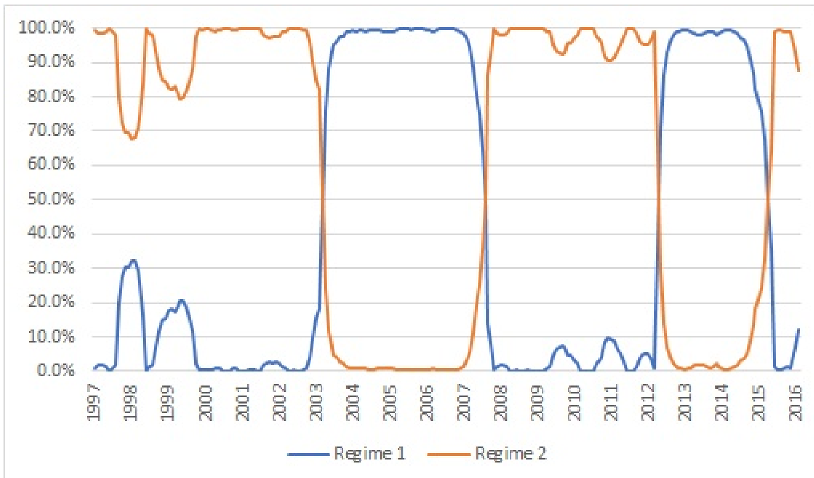
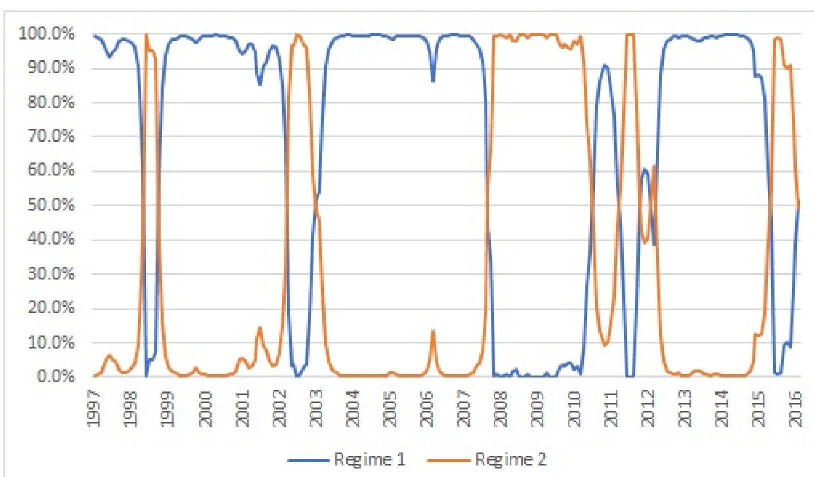


Figure 17: Dividend Yield Smoothed Regime Probabilities

Estimated smoothed regime probabilities of the univariate MSIH(2,1) model for dividend yields.



C.3 Regime Correlations

Table 21 shows the correlations between the estimated smoothed regime probabilities of the multivariate and univariate MSIH(2,1) models for inflation, equity returns and dividend yields. The estimated regimes for equity and dividend yields are strongly correlated. From Figures 16 and 17 this strong correlation is clearly seen. This strong regime correlation also holds between dividend yields and inflation. The correlation is less strong for inflation and equity returns. However, Figures 15 and 16 do show very similar patterns in the smoothed regimes probabilities. The strong correlations and similar patterns indicate that a multivariate regime switching model with the same underlying regimes for all variables, might be appropriate.

In the case that the underlying regimes do coincide for all variables, combining the information in all variables in the regime estimation makes the estimated regime probabilities more accurate. The full model smoothed regime probabilities indicate that the regimes of the multivariate model represent a combination of the univariate regimes. The full model regimes are strongly correlated with the estimated univariate regimes, with a correlation of at least 0.57. The full model regimes thus capture most of the information of the individual regimes. It follows that modelling the inflation, equity returns and dividend yields with the same underlying regimes does not lead to large information losses. It does reduce the dimensionality of the model immensely, as the number of regimes by modelling independent regimes would be 2^3.

Table 21: Regime Correlations

Correlations between the multivariate and univariate MSIH(2,1) model smoothed regime probabilities. Inflation is denoted by π, equity returns by xs and dividend yields by dy.

	π	xs	dy
Multivariate Model	0.575	0.578	0.766
π	1		
xs	0.221	1	
dy	0.524	0.593	1

D ALM Results

D.1 Optimal Portfolios

This appendix shows the results of various portfolio optimisations to provide insight in the sensitivity of the optimal portfolio to various parameters. Table 22 shows the optimal portfolios per age for participants with a risk aversion of 5 in a pension fund with an initial funding ratio of 130%. Older participants gradually invest less in equity and more in bonds, which is in line with the findings of Campbell and Viceira (2002).

Table 23 shows optimal portfolios for various settings of the average age in the participant simulation of the fund. For longer durations the optimal portfolio is generally more allocated to equity. For durations longer than 22.4 years there is a slight shift, where portfolio become less allocated to equity. Also the percentage of interest rate risk hedged with swaps becomes lower. This might be due to the fact that the pension fund policy is based on an interest rate curve with UFR, while swaps are priced with the nominal curve. For durations longer 20 years there is a hedging mismatch, which makes swaps less effective in lowering balance sheet risks. This is compensated by allocating more to bonds, which are less risky, and by lowering the amount of swap contracts.

Table 24 shows optimal portfolios for various initial funding ratios. The fund has an average age of 55 and the participants have a risk aversion of 5. With lower funding ratios the portfolio is more allocated to equity and the amount of swap contracts is lower relative to higher funding ratios. For lower funding ratios the participants pursue recovery of the funding ratio, whereas for higher funding ratios the solvency position is protected more.

Table 22: Optimal Portfolios Per Age

The optimal portfolios per age for a pension fund with an initial funding ratio of 130% and participants with a risk aversion of 5. The fund is simulated with an average age of 55.

Age	B_{1M}	B_{10Y}	xs	$Swap$
20	0.0%	59.1%	40.9%	85.7%
30	0.0%	60.1%	39.9%	86.6%
40	0.0%	60.8%	39.2%	87.0%
50	0.0%	61.1%	38.9%	87.1%
60	0.0%	62.4%	37.6%	86.0%
70	0.0%	64.5%	35.5%	81.2%
80	0.0%	76.4%	23.6%	65.3%
90	0.0%	90.2%	9.7%	47.4%
100	0.0%	94.9%	5.1%	40.5%
110	2.5%	93.5%	4.0%	39.6%
119	42.9%	46.5%	10.6%	56.6%

Table 23: Optimal Portfolios Per Duration

The optimal portfolios for various average ages of the participants. The funds are generated with a truncated normal distribution with the average age as specified in the first column. The initial funding ratio is equal to 130% and the risk aversion parameter is 5.

Avr. Age	**Duration**	B_{1M}	B_{10Y}	xs	$Swap$
30	29.3	0.0%	63.1%	36.9%	77.9%
35	27.0	0.0%	62.5%	37.5%	79.7%
40	24.7	0.0%	62.0%	38.0%	80.8%
45	22.4	0.0%	61.1%	38.9%	83.1%
50	20.2	0.0%	61.1%	38.9%	83.0%
55	18.2	0.0%	61.5%	38.4%	82.0%
60	16.3	0.0%	62.4%	37.6%	80.8%
65	14.6	0.0%	63.6%	36.4%	78.3%
70	13.1	0.0%	65.3%	34.7%	74.5%
75	11.7	0.0%	67.5%	32.5%	68.3%
80	10.4	0.0%	70.3%	29.7%	59.4%

Table 24: Optimal Portfolios Per Initial Funding Ratio

The optimal portfolios for different initial funding ratios ranging from 60% to 160%. The fund is generated with an average age of the participants of 55. The risk aversion is set to 5.

FR0	B_{1M}	B_{10Y}	xs	$Swap$
60%	0.0%	57.9%	42.1%	66.4%
70%	0.0%	58.1%	41.9%	69.6%
80%	0.0%	59.3%	40.7%	69.4%
90%	0.0%	60.1%	39.9%	73.5%
100%	0.0%	60.9%	39.1%	75.1%
110%	0.0%	61.2%	38.8%	77.4%
120%	0.0%	61.2%	38.8%	80.5%
130%	0.0%	60.9%	39.1%	86.4%
140%	0.0%	61.5%	38.5%	92.7%
150%	0.0%	61.8%	38.2%	99.1%
160%	0.0%	62.1%	37.9%	100.0%

D.2 Heterogeneous Risk Aversion Portfolio Choice

Heterogeneous Risk Aversion Group Weights

The table below contains the weights used to calculate the optimal portfolio in the heterogeneous risk aversion setting. The weights are specified per stylised fund; the Young, Average and Old fund.

Table 25: Heterogeneous Risk Aversion Group Weights

	Actives	**Pensioners**
Young	96.3%	3.7%
Average	91.4%	8.6%
Old	81.9%	18.1%

Optimal Portfolios Per Group

The following table contains optimised portfolios for both the active participants and the pensioners in an Average fund with a funding ratio of 130%. These optimal portfolios give an indication of realistic risk aversion parameter values for both groups.

Table 26: Optimal Portfolios Per Group

	$\boldsymbol{B_{1M}}$	$\boldsymbol{B_{10Y}}$	$\boldsymbol{xs}$	*Swap*
$\boldsymbol{\gamma = 2}$	0.0%	19.7%	80.3%	85.8%
3	0.0%	39.8%	60.2%	88.4%
4	0.0%	52.2%	47.8%	88.0%
5	0.0%	60.8%	39.2%	86.6%
6	0.0%	67.1%	32.9%	84.2%
7	0.0%	71.8%	28.2%	82.5%

(a) Active Participants

	$\boldsymbol{B_{1M}}$	$\boldsymbol{B_{10Y}}$	$\boldsymbol{xs}$	*Swap*
$\boldsymbol{\gamma = 5}$	0.0%	65.4%	34.6%	79.8%
6	0.0%	70.2%	29.8%	78.3%
7	0.0%	73.8%	26.2%	77.3%
8	0.0%	76.6%	23.4%	76.7%
9	0.0%	78.8%	21.2%	76.3%
10	0.0%	80.7%	19.3%	75.8%
11	0.0%	82.5%	17.5%	74.7%
12	0.0%	83.9%	16.1%	74.3%

(b) Pensioners

D.3 Heterogeneous Risk Aversion Additional Results

This section shows the utility results for the active participants and the pensioners with and without carve-out for the three stylised pension funds, where the risk aversion is: $\gamma_{67^-=5}$ and $\gamma_{67^+=9}$.

Table 27: Carve-Out Utilities with $\gamma_{67^-} = 5$ and $\gamma_{67^+} = 9$

Utilities of the active and pensioned participants before and after a carve-out. The results are ordered by the initial funding ratio, which are from top to bottom 100%, 115% and 130%. Each table section shows the results for a Young, Average and Old fund. Utility gains are marked in green and utility losses are marked in red.

$\boldsymbol{FR_0 = 100\%}$	**Young**		**Average**		**Old**	
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	**Actives**	**Pensioners**
No Carve-Out	-41099	-3230	-44195	-6556	-35408	-10589
Nominal	-41337	-3892	-44913	-7023	-36853	-10271
Real	-38220	-30058	-39066	-29879	-29610	-26938
DNB	-42597	-1913	-46771	-4527	-38710	-7997
Exp. Index.	-43319	-1298	-48610	-3070	-42560	-5312
Value Based	-42414	-2140	-45988	-5484	-37574	-9260
Indiff.	-41694	-3230	-45256	-6556	-36650	-10589
$\boldsymbol{FR_0 = 115\%}$	**Young**		**Average**		**Old**	
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	**Actives**	**Pensioners**
No Carve-Out	-31503	-1946	-30579	-3625	-21870	-5418
Nominal	-32801	-1101	-33290	-2003	-26177	-3029
Real	-30165	-9826	-28805	-9768	-20867	-8143
DNB	-33813	-628	-34673	-1484	-27076	-2728
Exp. Index.	-33078	-929	-33158	-2069	-25154	-3456
Value Based	-32446	-1410	-31569	-3323	-22707	-5221
Indiff.	-32050	-1946	-31322	-3625	-22528	-5418
$\boldsymbol{FR_0 = 130\%}$	**Young**		**Average**		**Old**	
Carve-Out Type	**Actives**	**Pensioners**	**Actives**	**Pensioners**	**Actives**	**Pensioners**
No Carve-Out	-24613	-1282	-21768	-2362	-14281	-3644
Nominal	-26038	-614	-23551	-1477	-15506	-2728
Real	-24160	-3685	-21619	-3202	-15177	-2945
DNB	-26038	-614	-23551	-1477	-15506	-2728
Exp. Index.	-25754	-754	-23123	-1690	-15193	-2934
Value Based	-25381	-1002	-22380	-2233	-14604	-3430
Indiff.	-25109	-1282	-22248	-2362	-14399	-3644

E Affine Term Structure

This section provides the derivations for the affine term structure model. First, the equations for the nominal bond prices are derived, followed by derivations for the real bond prices. The notation in this section will follow the same notation as in the main text.

E.1 Nominal Term Structure

Proposition E.1. *Let X_t be the state variables of the economy as defined in Equation 3 and let $\nu(j)$, Φ and $\Sigma^{\frac{1}{2}}(j)$ be the model parameters defined in Equations 4 and 5 for regime j. The regimes follow a Markov chain with transition probability matrix Π with on row i and column j the probability of switching from regime i to regime j noted by p_{ij}. With the pricing kernel defined as in Equation 7 and the prices of risk as in Equation 11, the zero coupon bond price for maturity n and given regime $s_t = i$, $P_t^n(s_t = i)$ equals*

$$P_t^n = exp\left(A_n + B_n' X_t\right). \tag{39}$$

A_n is a scalar that is independent of current regime and B_n is $N \times 1$ vector with N the total number of state variables. In this model the coefficients A_n and B_n are recursively given by

$$\begin{aligned} A_n &= -\delta_0 + A_{n-1} + B_{n-1,f}'(\nu_f - \Sigma_f^{\frac{1}{2}} \lambda_{0,f}) + \frac{1}{2} B_{n-1,f}' \Sigma_f B_{n-1,f} \\ B_n &= \begin{bmatrix} B_{n,f} \\ B_{n,x} \end{bmatrix} = \begin{bmatrix} -\delta_{1,f} + (\Phi_f - \Sigma_f^{\frac{1}{2}} \Lambda_{1,f})' B_{n-1,f} \\ 0 \end{bmatrix} \end{aligned} \tag{40}$$

The starting vales for A_n and B_n are

$$\begin{aligned} A_1 &= -\delta_0, \\ B_1 &= -\delta_1. \end{aligned} \tag{41}$$

Proof.
In order to derive closed form solutions I need to assume that the auto-regressive coefficients and B_n are constant across regimes. To show this, I start my proof with switching notation for these coefficients. It then follows later in the proof that these coefficients need to be restricted to derive a closed form solution. From this derivation it will also follow that Φ_x must be restricted to non-switching.
The proof starts by deriving the initial values of $A_n(i)$ and $B_n(i)$. The price of a zero coupon bond with maturity n can according to asset pricing theory be written as

$$P_t^n = E_t\left[M_{t+1} P_{t+1}^{n-1}\right]. \tag{42}$$

When generalised to the Markov switching case, the price of a one period bond with pay off one is equal to

$$
\begin{aligned}
P_t^1(i) &= \sum_j p_{ij}\ E_t\left[M_{t+1}|S_{t+1}=j\right], \\
&= \sum_j p_{ij}\ E_t\left[\exp\left(-\delta_0-\delta_1'X_t-\frac{1}{2}\lambda_t'(j)\lambda_t(j)-\lambda_t'(j)u_{t+1}\right)\right], \\
&= \sum_j p_{ij}\ \exp\left(-\delta_0-\delta_1'X_t-\frac{1}{2}\lambda_t'(j)\lambda_t(j)+\frac{1}{2}\lambda_t'(j)\lambda_t(j)\right), \\
&= \sum_j p_{ij}\ \exp\left(-\delta_0-\delta_1'X_t\right), \\
&= \exp\left(-\delta_0-\delta_1'X_t\right).
\end{aligned}
$$

From Equations E.1 it follows that the bond price in Equation 39 holds for the one-period bond, where the starting values are

$$
\begin{aligned}
A_1(i) &= -\delta_0, \\
B_1(i) &= -\delta_1.
\end{aligned}
\tag{43}
$$

The proof of the recursion of Equation 40 follows by induction. If I assume that Equation 39 holds for maturity n, then the price of a $n+1$ period bond, $P_t^{n+1}(i)$, is given by

$$
\begin{aligned}
P_t^{n+1}(i) &= \sum_j p_{ij}\ E_t\left[M_{t+1}P_{t+1}^n(j)|S_{t+1}=j\right], \\
&= \sum_j p_{ij}\ E_t\left[\exp\left(-\delta_0-\delta_1'X_t-\frac{1}{2}\lambda_t'(j)\lambda_t(j)-\lambda_t'(j)u_{t+1}+A_n(j)+B_n'(j)X_{t+1}\right)\right], \\
&= \sum_j p_{ij}\ E_t\Bigg[\exp\Bigg(-\delta_0-\delta_1'X_t-\frac{1}{2}\lambda_t'(j)\lambda_t(j)-\lambda_t'(j)u_{t+1}+A_n(j)+ \\
&\qquad B_n'(j)\left(\nu(j)+\Phi(j)X_t+\Sigma^{\frac{1}{2}}(j)u_{t+1}\right)\Bigg)\Bigg], \\
&= \sum_j p_{ij}\ E_t\Bigg[\exp\Bigg(-\delta_0-\delta_1'X_t-\frac{1}{2}\lambda_t'(j)\lambda_t(j)+A_n(j)+B_n'(j)\nu(j)+ \\
&\qquad B_n'(j)\Phi(j)X_t+\left(B_n'(j)\Sigma^{\frac{1}{2}}(j)-\lambda_t'(j)\right)u_{t+1}\Bigg)\Bigg], \\
&= \sum_j p_{ij}\ \exp\Bigg(-\delta_0-\delta_1'X_t-\frac{1}{2}\lambda_t'(j)\lambda_t(j)+A_n(j)+B_n'(j)\nu(j)+ \\
&\qquad B_n'(j)\Phi(j)X_t+\frac{1}{2}\left(B_n'(j)\Sigma(j)-\lambda_t'(j)\right)\left(B_n'(j)\Sigma^{\frac{1}{2}}(j)-\lambda_t'(j)\right)'\Bigg), \\
&= \sum_j p_{ij}\ \exp\Bigg(-\delta_0-\delta_1'X_t+A_n(j)+B_n'(j)\nu(j)+B_n'(j)\Phi(j)X_t+ \\
&\qquad \frac{1}{2}B_n'(j)\Sigma(j)B_n(j)-B_n'(j)\Sigma^{\frac{1}{2}}(j)\lambda_t(j)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_j p_{ij} \exp\bigg(- \delta_0 - \delta_1' X_t + A_n(j) + B_n'(j)\nu(j) + B_n'(j)\Phi(j)X_t + \\
&\quad \frac{1}{2} B_n'(j)\Sigma(j)B_n(j) - B_n'(j)\Sigma^{\frac{1}{2}}(j)\left(\lambda_0(j) + \Lambda_1(j)X_t\right)\bigg), \\
&= \sum_j p_{ij} \exp\bigg(- \delta_0 - \delta_1' X_t + A_n(j) + B_n'(j)\nu(j) + B_n'(j)\Phi(j)X_t + \\
&\quad \frac{1}{2} B_n'(j)\Sigma(j)B_n(j) - B_n'(j)\Sigma^{\frac{1}{2}}(j)\lambda_0(j) - B_n'(j)\Sigma^{\frac{1}{2}}(j)\Lambda_1(j)X_t\bigg),
\end{aligned}
$$

The nominal interest rates are independent of the state variables x_t, which also means that $B_{n,x} = 0 \quad \forall \quad n$. For Equation 39 to hold, the expression derived above should be written in the form $\exp\Big(A_{n+1}(j) + B_{n+1}(j)X_t\Big)$. In the above expression therefore all terms without X_t will determine A_{n+1} and all terms with X_t determine B_{n+1}. To show that $B_{n,x} = 0 \quad \forall \quad n$, I will split all terms concerning B_{n+1} into a part corresponding to f_t and a part corresponding to x_t. It will then become evident that if $B_{1,x} = 0$, this will be the case for all n.

From the model specification it is clear that

$$
-\delta_1' X_t = -\begin{bmatrix} \delta_{1,f} \\ 0 \end{bmatrix} X_t.
$$

Thus from this term $B_{n+1,x}$ can never achieve any other value than 0. Moreover, the term $B_n'(j)\Phi(j)X_t$ can be written as

$$
\begin{aligned}
B_n'(j)\Phi(j)X_t &= \begin{bmatrix} B_{n,f} \\ B_{n,x} \end{bmatrix}' \begin{bmatrix} \Phi_f & 0 \\ 0 & \Phi_x \end{bmatrix} \begin{bmatrix} f_t \\ x_t \end{bmatrix}, \\
&= \begin{bmatrix} B_{n,f}'\Phi_f \\ B_{n,x}'\Phi_x \end{bmatrix} \begin{bmatrix} f_t \\ x_t \end{bmatrix}.
\end{aligned}
$$

If $B_{1,x} = 0$, then of course $B_{n,x}'\Phi_x = 0$, which makes it clear that from this term $B_{n+1,x}$ again can never achieve any other value than 0.

For the final term containing X_t

$$
\begin{aligned}
B_n'(j)\Sigma^{\frac{1}{2}}(j)\Lambda_1(j)X_t &= \begin{bmatrix} B_{n,f} \\ B_{n,x} \end{bmatrix}' \begin{bmatrix} \Sigma_f^{\frac{1}{2}} & 0 \\ 0 & \Sigma_x^{\frac{1}{2}}(j) \end{bmatrix} \begin{bmatrix} \Lambda_{1,f} & 0 \\ 0 & \Lambda_{1,x}(j) \end{bmatrix} \begin{bmatrix} f_t \\ x_t \end{bmatrix}, \\
&= \begin{bmatrix} B_{n,f}'\Sigma_f^{\frac{1}{2}}\Lambda_{1,f} \\ B_{n,x}'\Sigma_x^{\frac{1}{2}}(j)\Lambda_{1,x}(j) \end{bmatrix} \begin{bmatrix} f_t \\ x_t \end{bmatrix},
\end{aligned}
$$

it again holds that $B_{n+1,x} = 0$ for all n. This greatly simplifies the last expression in the previous derivation to

$$
\begin{aligned}
P_t^{n+1}(i) = \sum_j p_{ij} \exp\bigg(&- \delta_0 - \delta_{1,f}' f_t + A_n(j) + B_{n,f}'\nu_f + B_{n,f}'\Phi_f f_t + \\
&\frac{1}{2} B_{n,f}'\Sigma_f B_{n,f} - B_{n,f}'\Sigma_f^{\frac{1}{2}}\lambda_{0,f} - B_{n,f}'\Sigma_f^{\frac{1}{2}}\Lambda_{1,f} f_t\bigg).
\end{aligned}
$$

To attain expressions for A_{n+1} and B_{n+1}, the above expression must be split in an intercept part without f_t terms and a part with f_t.

$$P_t^{n+1}(i) = \exp\Bigg(- \delta_0 - \delta_{1,f}' f_t + B_{n,f}' \nu_f + B_{n,f}' \Phi_f f_t + \frac{1}{2} B_{n,f}' \Sigma_f B_{n,f} - B_{n,f}' \Sigma_f^{\frac{1}{2}} \lambda_{0,f} - B_{n,f}' \Sigma_f^{\frac{1}{2}} \Lambda_{1,f} f_t \Bigg) \sum_j p_{ij} \exp\left(A_n(j)\right).$$

With the above expression and knowing that $A_1 = -\delta_0$ is not regime dependent, we now know that A_{n+1} is also not dependent on the regime. This leaves us with the recursive expressions

$$A_n = -\delta_0 + A_{n-1} + B_{n-1,f}'(\nu_f - \Sigma_f^{\frac{1}{2}} \lambda_{0,f}) + \frac{1}{2} B_{n-1,f}' \Sigma_f B_{n-1,f},$$
$$B_n = \begin{bmatrix} B_{n,f} \\ B_{n,x} \end{bmatrix} = \begin{bmatrix} -\delta_{1,f} + (\Phi_f - \Sigma_f^{\frac{1}{2}} \Lambda_{1,f})' B_{n-1,f} \\ 0 \end{bmatrix}.$$

□

E.2 Real Term Structure

Proposition E.2. *Let X_t be the state variables of the economy as defined in Equation 3 and let $\nu(j)$, Φ and $\Sigma^{\frac{1}{2}}(j)$ be the model parameters defined in Equations 4 and 5 for regime j. The regimes follow a Markov chain with transition probability matrix Π with on row i and column j the probability of switching from regime i to regime j, noted by p_{ij}. With the real pricing kernel defined as in Equation 17 and the prices of risk as in Equation 11, the price of a zero coupon inflation linked bond for maturity n and given regime $s_t = i$, $P_t^n(s_t = i)$ equals*

$$\widehat{P}_t^n(i) = exp\left(\widehat{A}_n(i) + \widehat{B}_n' X_t\right). \tag{44}$$

$\widehat{A}_n$ is a scalar that depends on the current regime and $\widehat{B}_n$ is $N \times 1$ vector with N the total number of state variables. In this model the coefficients $\widehat{A}_n$ and $\widehat{B}_n$ are recursively given by

$$\widehat{A}_{n+1}(i) = -\delta_0 - \widehat{B}_{n,f}' \Sigma_f^{\frac{1}{2}} \lambda_{0,f} + log\Bigg(\sum_j p_{ij}\ exp\Bigg(\widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right) \nu(j) + \frac{1}{2}\left(\widehat{B}_n' + e_\pi'\right) \Sigma(j) \left(\widehat{B}_n' + e_\pi'\right)'\Bigg)\Bigg), \tag{45}$$
$$\widehat{B}_{n+1} = -\delta_1 + \Phi'\left(\widehat{B}_n + e_\pi\right) - e_f' \Lambda_{1,f}' \Sigma_f^{\frac{1}{2}} \widehat{B}_{n,f}.$$

In this formulation the matrix e_f transforms the 3×1 vector $\Lambda_{1,f}' \Sigma_f^{\frac{1}{2}} \widehat{B}_{n,f}$ into a 6×1 vector equal to $[\Lambda_{1,f}' \Sigma_f^{\frac{1}{2}} \widehat{B}_{n,f} \quad 0]'$. The vectors e_π and $e_{\pi,x}$ are unit vectors with a one on *the index of inflation in respectively the full state variable vector X_t and the MSVAR state variables x_t. The starting values for this recursion are equal to*

$$\begin{aligned} \widehat{A}_1(i) &= -\delta_0 + log \sum_j p_{ij}\ exp\left(e_\pi' \nu(j) + \frac{1}{2} e_\pi' \Sigma(j) e_\pi \right), \\ \widehat{B}_1 &= -\delta_1 + \Phi' e_\pi. \end{aligned} \tag{46}$$

Proof.
The proof starts by deriving the initial values of $\widehat{A}_n(i)$ and $\widehat{B}_n(i)$. The price of a zero coupon inflation linked bond with maturity n, $\widehat{P}_t^n$, can according to asset pricing theory be written as

$$\widehat{P}_t^n = E_t\left[M_{t+1} \widehat{P}_{t+1}^{n-1} \right].$$

When generalised to the Markov switching case, the price of a one period inflation linked bond with pay off $\exp(\pi_{t+1})$ is equal to

$$\begin{aligned} \widehat{P}_t^1(i) &= \sum_j p_{ij}\ E_t\left[M_{t+1}\ \exp(\pi_{t+1}) | S_{t+1} = j \right], \\ &= \sum_j p_{ij}\ E_t\left[\exp\left(m_{t+1} + \pi_{t+1} \right) | S_{t+1} = j \right], \\ &= \sum_j p_{ij}\ E_t\left[\exp\left(-\delta_0 - \delta_1' X_t - \frac{1}{2} \lambda_t'(j) \lambda_t(j) - \lambda_t'(j) u_{t+1} + e_\pi' X_{t+1} \right) \right], \\ &= \sum_j p_{ij}\ E_t\left[\exp\left(-\delta_0 - \delta_1' X_t - \frac{1}{2} \lambda_t'(j) \lambda_t(j) - \lambda_t'(j) u_{t+1} + e_\pi' \left(\nu(j) + \Phi X_t + \Sigma^{\frac{1}{2}}(j) u_{t+1} \right) \right) \right], \\ &= \sum_j p_{ij}\ E_t\Bigg[\exp\Bigg(-\delta_0 - \delta_1' X_t - \frac{1}{2} \lambda_t'(j) \lambda_t(j) + e_\pi' \nu(j) + e_\pi' \Phi X_t + \left(e_\pi' \Sigma^{\frac{1}{2}}(j) - \lambda_t'(j) \right) u_{t+1} \Bigg) \Bigg], \\ &= \sum_j p_{ij}\ E_t\Bigg[\exp\Bigg(-\delta_0 - \delta_1' X_t - \frac{1}{2} \lambda_t'(j) \lambda_t(j) + e_\pi' \nu(j) + e_\pi' \Phi X_t + \\ &\quad \frac{1}{2} \left(e_\pi' \Sigma^{\frac{1}{2}}(j) - \lambda_t'(j) \right) \left(e_\pi' \Sigma^{\frac{1}{2}}(j) - \lambda_t'(j) \right)' \Bigg) \Bigg], \\ &= \exp\left(-\delta_0 - \delta_1' X_t + e_\pi' \Phi X_t \right) \sum_j p_{ij}\ \exp\left(e_\pi' \nu(j) + \frac{1}{2} e_\pi' \Sigma(j) e_\pi - e_\pi' \Sigma^{\frac{1}{2}}(j) \lambda_t(j) \right). \end{aligned}$$

The price of risk for inflation is assumed to be equal to zero in the model:

$$e_\pi' \Sigma^{\frac{1}{2}}(j) \lambda_t(j) = 0.$$

Leaving us with the expression

$$\widehat{P}_t^1(i) = \exp\left(-\delta_0 - \delta_1' X_t + e_\pi' \Phi X_t \right) \sum_j p_{ij}\ \exp\left(e_\pi' \nu(j) + \frac{1}{2} e_\pi' \Sigma(j) e_\pi \right).$$

It then follows that the bond price in Equation 44 holds for the one period bond, where the starting values are

$$\begin{aligned}\widehat{A}_1(i) &= -\delta_0 + \log \sum_j p_{ij} \exp\left(e'_\pi \nu(j) + \frac{1}{2} e'_\pi \Sigma(j) e_\pi\right),\\ \widehat{B}_1 &= -\delta_1 + \Phi' e_\pi.\end{aligned}$$

The proof of the recursion of Equation 19 follows by induction. If we assume that Equation 44 holds for maturity n than the price of a $n+1$ period bond, $\widehat{P}_t^{n+1}(i)$, is given by

$$\widehat{P}_t^{n+1}(i) = \sum_j p_{ij}\ E_t\left[M_{t+1} \widehat{P}_{t+1}^n(j) | S_{t+1} = j\right]. \tag{47}$$

In an arbitrage free economy the price of an inflation linked bond issued at time t with maturity $n-1$ is equal to the price of an inflation linked bond issued next period with the same maturity plus the inflation of the bond price over that period:[31]

$$\begin{aligned}\widehat{P}_{t+1|t}^n &= \exp(\pi_{t+1}) \widehat{P}_{t+1|t+1}^n,\\ &= \exp(\pi_{t+1}) \exp\left(\widehat{A}_n(j) + \widehat{B}'_n X_{t+1}\right),\\ &= \exp\left(e'_\pi X_{t+1} + \widehat{A}_n(j) + \widehat{B}'_n X_{t+1}\right).\end{aligned} \tag{48}$$

Combining Equations 47 and 48 it follows that

$$\begin{aligned}\widehat{P}_t^{n+1}(i) &= \sum_j p_{ij}\ E_t\left[\exp\left(-\delta_0 - \delta'_1 X_t - \frac{1}{2}\lambda'_t(j)\lambda_t(j) - \lambda'_t(j) u_{t+1} + e'_\pi X_{t+1} + \widehat{A}_n(j) + \widehat{B}'_n X_{t+1}\right)\right],\\
&= \sum_j p_{ij}\ E_t\Bigg[\exp\Bigg(- \delta_0 - \delta'_1 X_t - \frac{1}{2}\lambda'_t(j)\lambda_t(j) - \lambda'_t(j) u_{t+1} + \widehat{A}_n(j) +\\
&\qquad \left(\widehat{B}'_n + e'_\pi\right)\left(\nu(j) + \Phi X_t + \Sigma^{\frac{1}{2}}(j) u_{t+1}\right)\Bigg)\Bigg],\\
&= \sum_j p_{ij}\ E_t\Bigg[\exp\Bigg(- \delta_0 - \delta'_1 X_t - \frac{1}{2}\lambda'_t(j)\lambda_t(j) + \widehat{A}_n(j) + \left(\widehat{B}'_n + e'_\pi\right)\nu(j) +\\
&\qquad \left(\widehat{B}'_n + e'_\pi\right)\Phi X_t + \left(\left(\widehat{B}'_n + e'_\pi\right)\Sigma^{\frac{1}{2}}(j) - \lambda'_t(j)\right) u_{t+1}\Bigg)\Bigg],\\
&= \sum_j p_{ij}\ \exp\Bigg(- \delta_0 - \delta'_1 X_t - \frac{1}{2}\lambda'_t(j)\lambda_t(j) + \widehat{A}_n(j) + \left(\widehat{B}'_n + e'_\pi\right)\nu(j) + \left(\widehat{B}'_n + e'_\pi\right)\Phi X_t +\\
&\qquad \frac{1}{2}\left(\left(\widehat{B}'_n + e'_\pi\right)\Sigma^{\frac{1}{2}}(j) - \lambda'_t(j)\right)\left(\left(\widehat{B}'_n + e'_\pi\right)\Sigma^{\frac{1}{2}}(j) - \lambda'_t(j)\right)'\Bigg),\\
&= \sum_j p_{ij}\ \exp\Bigg(- \delta_0 - \delta'_1 X_t + \widehat{A}_n(j) + \left(\widehat{B}'_n + e'_\pi\right)\nu(j) + \left(\widehat{B}'_n + e'_\pi\right)\Phi X_t +\\
&\qquad \frac{1}{2}\left(\widehat{B}'_n + e'_\pi\right)\Sigma(j)\left(\widehat{B}_n + e_\pi\right) - \left(\widehat{B}'_n + e'_\pi\right)\Sigma^{\frac{1}{2}}(j)\lambda_t(j)\Bigg),\end{aligned}$$

[31] See Hoevenaars (2008).

$$= \sum_j p_{ij} \exp\Big(- \delta_0 - \delta_1' X_t + \widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right) \nu(j) + \left(\widehat{B}_n' + e_\pi'\right) \Phi X_t +$$

$$\frac{1}{2} \left(\widehat{B}_n' + e_\pi'\right) \Sigma(j) \left(\widehat{B}_n + e_\pi\right) - \left(\widehat{B}_n' + e_\pi'\right) \Sigma^{\frac{1}{2}}(j)\lambda_0(j) - \left(\widehat{B}_n' + e_\pi'\right) \Sigma^{\frac{1}{2}}(j)\Lambda_1(j) X_t \Big).$$

As the price of inflation risk is zero, the terms $e_\pi' \Sigma^{\frac{1}{2}}(j)\lambda_0(j)$ and $e_\pi' \Sigma^{\frac{1}{2}}(j)\Lambda_1(j)$ disappear.

$$\widehat{P}_t^{n+1}(i) = \sum_j p_{ij} \exp\Big(- \delta_0 - \delta_1' X_t + \widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right) \nu(j) + \left(\widehat{B}_n' + e_\pi'\right) \Phi X_t +$$

$$\frac{1}{2} \left(\widehat{B}_n' + e_\pi'\right) \Sigma(j) \left(\widehat{B}_n + e_\pi\right) - \widehat{B}_n' \Sigma^{\frac{1}{2}}(j)\lambda_0(j) - \widehat{B}_n' \Sigma^{\frac{1}{2}}(j)\Lambda_1(j) X_t \Big).$$

By restricting the parameter in Φ corresponding to the lagged equity return and dividend yield to zero, $\widehat{B_n}$ will also be zero on the indices of equity and dividend. This restriction allow the equity price of risk term in Λ_1 to be different across regimes, while maintaining closed form solutions for the real term structure. This restriction allows the pricing Equation 6 to hold for all X_t for equity returns. This restriction already holds for $\widehat{B_1} = -\delta_1 + \Phi' e_\pi$. Whether it also holds for $\widehat{B_{n+1}}$ is determined by the terms containing a X_t term in the above expression. The first is $-\delta_1$, which does not change this fact. For the second term $\left(\widehat{B}_n' + e_\pi'\right) \Phi X_t$ it is less obvious:

$$\left(\widehat{B}_n' + e_\pi'\right) \Phi X_t = \begin{bmatrix} \widehat{B}_{n,f} \\ \widehat{B}_{n,\pi} + 1 \\ 0 \\ 0 \end{bmatrix}' \begin{bmatrix} \Phi_f & 0 & 0 & 0 \\ 0 & \phi_{\pi,\pi} & 0 & 0 \\ 0 & \phi_{xs,\pi} & \phi_{xs,xs} & \phi_{xs,dy} \\ 0 & \phi_{dy,\pi} & \phi_{dy,xs} & \phi_{dy,dy} \end{bmatrix} X_t$$

$$= \begin{bmatrix} \Phi_f \widehat{B}_{n,f} \\ (\widehat{B}_{n,\pi} + 1)\phi_{\pi,\pi} \\ 0 \\ 0 \end{bmatrix}' X_t.$$

The last term containing X_t is $\widehat{B}_n' \Sigma^{\frac{1}{2}}(j)\Lambda_1(j) X_t$, which can be written as:

$$\widehat{B}_n' \Sigma^{\frac{1}{2}}(j)\Lambda_1(j) X_t = \begin{bmatrix} \widehat{B}_{n,f} \\ \widehat{B}_{n,x} \end{bmatrix}' \begin{bmatrix} \Sigma_f^{\frac{1}{2}} & 0 \\ 0 & \Sigma_x^{\frac{1}{2}}(j) \end{bmatrix} \begin{bmatrix} \Lambda_{1,f} & 0 \\ 0 & \Lambda_{1,x}(j) \end{bmatrix} X_t.$$

If $B_{n,x}$ is only non-zero on the index of inflation, then it follows by the fact that the inflation risk premium is zero that $\widehat{B}_{n,x}' \Sigma_x^{\frac{1}{2}}(j)\Lambda_{1,x}(j) = 0 \quad \forall \quad j$.
The restriction of the auto-regressive parameters of the inflation equation, $\phi_{\pi,xs} = 0$ and $\phi_{\pi,dy} = 0$, ensure that the price of inflation linked bonds is independent of the price of equity risk. This allows equity risk to be regime switching without loss of closed form solutions, making the price of equity risk much more accurate.

Using this information the inflation linked bond price can be expressed as:

$$\begin{aligned}
\widehat{P}_t^{n+1}(i) = \sum_j p_{ij} \exp\Bigg(& -\delta_0 - \delta_1' X_t + \widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right) \nu(j) + \left(\widehat{B}_n' + e_\pi'\right) \Phi X_t + \\
& \frac{1}{2}\left(\widehat{B}_n' + e_\pi'\right) \Sigma(j) \left(\widehat{B}_n + e_\pi\right) - \widehat{B}_{n,f}' \Sigma_f^{\frac{1}{2}} \lambda_{0,f} - \widehat{B}_{n,f}' \Sigma_f^{\frac{1}{2}} \Lambda_{1,f}\ e_f X_t \Bigg) \\
= \exp\Bigg(& -\delta_0 - \delta_1' X_t + \left(\widehat{B}_n' + e_\pi'\right) \Phi X_t - \widehat{B}_{n,f}' \Sigma_f^{\frac{1}{2}} \lambda_{0,f} - \widehat{B}_{n,f}' \Sigma_f^{\frac{1}{2}} \Lambda_{1,f}\ e_f X_t \Bigg) \\
& \sum_j p_{ij} \exp\left(\widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right) \nu(j) + \frac{1}{2}\left(\widehat{B}_n' + e_\pi'\right) \Sigma(j) \left(\widehat{B}_n + e_\pi\right) \right).
\end{aligned}$$

In these equations e_f is a matrix which transforms the vector X_t into f_t. From this last expression the recursive expression for the model parameters can be formulated:

$$\begin{aligned}
\widehat{A}_{n+1}(i) &= -\delta_0 - \widehat{B}_{n,f}' \Sigma_f^{\frac{1}{2}} \lambda_{0,f} + \log\Bigg(\sum_j p_{ij} \exp\bigg(\widehat{A}_n(j) + \left(\widehat{B}_n' + e_\pi'\right) \nu(j) + \\
& \qquad \frac{1}{2}\left(\widehat{B}_n' + e_\pi'\right) \Sigma(j) \left(\widehat{B}_n' + e_\pi'\right)' \bigg)\Bigg), \\
\widehat{B}_{n+1} &= -\delta_1 + \Phi' \left(\widehat{B}_n + e_\pi\right) - e_f' \Lambda_{1,f}' \Sigma_f^{\frac{1}{2}} \widehat{B}_{n,f}.
\end{aligned}$$

□

E.3 Bond Returns

Proposition E.3. *Given the model parameters defined in Section 3 and the insights from the proofs of the term structure equations, then the one period log return r_t^n on a n period bond with price P_t^n under the P measure is given by*

$$r_t^n = \underbrace{\delta_0 + \delta_{1,f}' f_t}_{\text{Risk Free Rate}} + \underbrace{B_{n-1,f}' \left(\Sigma_f^{\frac{1}{2}} \lambda_{0,f} + \Sigma_f^{\frac{1}{2}} \Lambda_{1,f} f_t \right)}_{\text{Risk Premium}} \underbrace{- \frac{1}{2} B_{n-1,f}' \Sigma_f^{\frac{1}{2}} \Sigma_f^{\frac{1}{2}\prime} B_{n-1,f}}_{\text{Convexity Adjustment}} + \underbrace{B_{n-1,f}' \Sigma_f^{\frac{1}{2}} u_{f,t}}_{\text{Stochastic Shock}}. \tag{49}$$

Moreover, the one period return under the Q measure of this same bond, $r_t^{Q,n}$, is given by

$$r_t^{Qn} = \delta_0 + \delta_{1,f}' f_t - \frac{1}{2} B_{n-1,f}' \Sigma_f^{\frac{1}{2}} \Sigma_f^{\frac{1}{2}\prime} B_{n-1,f} + B_{n-1,f}' \Sigma_f^{\frac{1}{2}} u_{f,t}. \tag{50}$$

Proof.
For a n-period bond with price P_t^n the one month log return r_t^n under the P measure is given by

$$r_t^n = \log \left(\frac{P_{t+1}^{n-1}}{P_t^n} \right) = (A_{n-1} + B_{n-1}' X_{t+1}) - (A_n + B_n' X_t)$$

From the proofs of the affine term structure recursions in the previous appendix section, we know that bond price only depends on the parameters of the interest rate factors f_t and not on the other state variables in x_t.

$$\begin{aligned}
r_t^n &= (A_{n-1} + B_{n-1,f}' f_{t+1}) - (A_n + B_{n,f}' f_t) \\
&= \left(A_{n-1} + B_{n-1,f}' \left(\nu_f + \Phi_f f_t + \Sigma_f^{\frac{1}{2}} u_{f,t+1} \right) \right) - (A_n + B_{n,f}' f_t) \\
&= \left(A_{n-1} + B_{n-1,f}' \left(\nu_f + \Phi_f f_t + \Sigma_f^{\frac{1}{2}} u_{f,t+1} \right) \right) - \\
&\left(A_{n-1} + B_{n-1,f}' \left(\nu_f - \Sigma_f^{\frac{1}{2}} \lambda_{1,f} \right) + \frac{1}{2} B_{n-1,f}' \Sigma_f \Sigma_f' B_{n-1,f} - \delta_0 \right) \\
&- \left(B_{n-1,f}' (\Phi_f - \Sigma_f^{\frac{1}{2}} \Lambda_{1,f}) - \delta_{1,f}' \right) f_t \\
&= \delta_0 + \delta_{1,f}' f_t + B_{n-1,f}' \nu_f + B_{n-1,f}' \Phi_f f_t + B_{n-1,f}' \Sigma_f^{\frac{1}{2}} u_{f,t+1} - B_{n-1,f}' \nu_f \\
&+ B_{n-1,f}' \Sigma_f^{\frac{1}{2}} \lambda_{0,f} - \frac{1}{2} B_{n-1,f}' \Sigma_f^{\frac{1}{2}} \Sigma_f^{\frac{1}{2}\prime} B_{n-1,f} - B_{n-1,f}' \Phi_f f_t + B_{n-1,f}' \Sigma_f^{\frac{1}{2}} \Lambda_{1,f} f_t \\
&= \delta_0 + \delta_{1,f}' f_t + B_{n-1,f}' \left(\Sigma_f^{\frac{1}{2}} \lambda_{0,f} + \Sigma_f^{\frac{1}{2}} \Lambda_{1,f} f_t \right) \\
&- \frac{1}{2} B_{n-1,f}' \Sigma_f^{\frac{1}{2}} \Sigma_f^{\frac{1}{2}\prime} B_{n-1,f} + B_{n-1,f}' \Sigma_f^{\frac{1}{2}} u_{f,t}.
\end{aligned}$$

By using the expectation of the lognormal distribution the expected bond return $E[\exp(r_t^n)]$ is equal to

$$
\begin{aligned}
E[\exp(r_t^n)] = E\Bigg[\exp\Bigg(&\delta_0 + \delta_{1,f}' f_t + B_{n-1,f}'\left(\Sigma_f^{\frac{1}{2}}\lambda_{0,f} + \Sigma_f^{\frac{1}{2}}\Lambda_{1,f} f_t\right) \\
&- \frac{1}{2}B_{n-1,f}'\Sigma_f^{\frac{1}{2}}\Sigma_f^{\frac{1}{2}\prime}B_{n-1,f} + B_{n-1,f}'\Sigma_f^{\frac{1}{2}}u_{f,t}\Bigg) \\
&= \exp\Bigg(\delta_0 + \delta_{1,f}' f_t + B_{n-1,f}'\left(\Sigma_f^{\frac{1}{2}}\lambda_{0,f} + \Sigma_f^{\frac{1}{2}}\Lambda_{1,f} f_t\right) \\
&- \frac{1}{2}B_{n-1,f}'\Sigma_f^{\frac{1}{2}}\Sigma_f^{\frac{1}{2}\prime}B_{n-1,f} + \frac{1}{2}B_{n-1,f}'\Sigma_f^{\frac{1}{2}}\Sigma_f^{\frac{1}{2}\prime}B_{n-1,f}\Bigg) \\
&= \exp\Bigg(\delta_0 + \delta_{1,f}' f_t + B_{n-1,f}'\left(\Sigma_f^{\frac{1}{2}}\lambda_{0,f} + \Sigma_f^{\frac{1}{2}}\Lambda_{1,f} f_t\right)\Bigg)
\end{aligned}
$$

The expected return under the risk neutral measure Q is equal to

$$
E[\exp(r_t^{Q,n})] = \exp\left(\delta_0 + \delta_{1,f}' f_t\right)
$$

The risk neutral bond return can thus be calculated as

$$
\begin{aligned}
r_t^{Q,n} &= r_t^n - B_{n-1,f}'\left(\Sigma_f^{\frac{1}{2}}\lambda_{0,f} + \Sigma_f^{\frac{1}{2}}\Lambda_{1,f} f_t\right) \\
&= \delta_0 + \delta_{1,f}' f_t - \frac{1}{2}B_{n-1,f}'\Sigma_f^{\frac{1}{2}}\Sigma_f^{\frac{1}{2}\prime}B_{n-1,f} + B_{n-1,f}'\Sigma_f^{\frac{1}{2}}u_{f,t}
\end{aligned}
$$

□