Bachelor Thesis – *Econometrics and Operations Research*

Robust Facility Location Problem Under Uncertainty

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Abstract

In this thesis we consider a multi-period fixed-charge facility location problem under uncertainty in demand to which we apply a robust optimization approach. We use a two-stage approach where in the first stage the locations and the capacities of the facilities to be opened are determined. These decisions are then tested in the second stage where the production and allocation of products are determined. We make use of sample paths, where the customers' demand is sampled over an uncertainty set. We consider this uncertain demand to be bounded within a multidimensional box. We show that the robust approach can provide improvements in the objective when partially immunizing against the uncertainty.

Contents

1 Introduction

Facilities, along with factories and distribution centers, have to function for many years. During this time, many circumstances like customer demand, travel times and distinct costs may be highly volatile. Optimal decisions regarding the locations of facilities, while being very costly and thus of big importance, are therefore difficult to optimally. This caused an interest in the development of models for facility location problems under uncertainty in the logistics fields. Many different techniques that have been developed to approach optimization under uncertainty have been applied to facility location problems.

Facility location problems deal with the decision making of placing potential facilities to fulfill customer demand. A standard formulation of the facility location problem consists of a set of customers to be serviced and a set of potential facility locations that can be opened. The aim is to minimize the sum of both the opening costs of the facilities and the cost of delivering units from facilities to customers, while satisfying the customers' demand. Since the determination of the locations and the sizes of facilities to be opened are long-term decisions, these decisions usually can not be reconsidered later on. This causes facility location problems to often be solved using a two-stage approach. In the first stage a decision maker, in this case a firm, has to make all strategic decisions regarding location and capacities of facilities. After this, in the second stage, operational decisions like production and transportation will be made. When all parameters are deterministic, this method can be used to obtain optimal results. However, when uncertainty is being accounted for, this may yield less efficient results. For example in case of uncertainty in demand, if the real demand in a period is larger than expected, not all demand may be satisfied. This is due to not anticipating this deviation when deciding on the maximum capacity in the first stage, resulting in a loss of potential revenue.

Studies that incorporated uncertainty in demand in facility location problems have traditionally done so by assuming stochastic distributions about the uncertain parameters, e.g., Sheppard (1974) and Snyder et al. (2007). The aim of stochastic optimization is to optimize the objective using the expected value of the uncertain parameters. This requires certain assumptions on the distribution of the uncertain parameters. An alternative approach is to apply robust optimization to the facility location problem. Here the objective is to obtain solutions that are immunized against all possible uncertainty. Applications of this method have been previously used in finance, computer science, engineering and different logistical problems. More recently, robust optimization has also been applied to facility problems by Baron et al. (2011) and Gülpınar et al. (2013). For an extensive literature review dealing with facility location problems under uncertainty, we refer to Snyder (2006).

This thesis focuses on the robust optimization approach applied to an adapted version of the facility location problem, in which we also allow for uncertainty in demand. We use a two-stage approach, where the locations and the sizes of the facilities to be opened have to be decided on at the beginning of the time horizon. Since the operational decisions are strongly limited by previously made strategic decisions, uncertainty in demand has a large effect on the objective. We will present two different models with the objective to maximize the total profit over the time horizon. The nominal model is formulated under the assumption that future demand is known and constant over time. For the robust optimization approach we formulate a model that immunizes against all possible uncertainty in demand, assuming that all possible values for demand are located within a multidimensional box uncertainty set. These models are then tested and compared using sample paths, created by sampling over the uncertainty set.

This thesis is organized as follows: In Section 2 we introduce the facility location problem and robust optimization in more detail. We also review previous research that has been done and we describe the techniques that will be used in this thesis. The models that will be used throughout the research are presented in Section 3. In Section 4, we introduce the test environment and compare the results provided by the different models. Furthermore, in Section 5 we will test the results under different conditions by performing a sensitivity analysis. Finally, we conclude our results and discuss possible future research directions in Section 6.

2 Problem Description

This section discusses the facility location problem as considered throughout this thesis and its deviations from the basic facility location problem. This is followed by an introduction to robust optimization, including a derivation similar to the case of facility location problem facing uncertain future demand.

2.1 Facility Location Problems

As stated in Section 1, the basic formulation of the facility location problem consists of customer locations and sites for potential facilities to be opened. The facility location problem can be seen as a generalized version of the transportation problem, which considers the optimal transportation and allocation of resources. In addition, the facility location problem incurs costs for opening facilities. Many different variations of the facility location problem exist. In the most basic form, p facilities can be selected with the goal to minimize the total distance between customers and their nearest open facility. This problem is called the p-median problem and has been widely studied (e.g. Daskin (2011), Drezner and Hamacher (2001)). The p-median problem ignores the fixed cost for opening facilities. The standard uncapacitated facility location problem does take these cost into consideration. In both the p-median problem and the uncapacitated facility location problem, each customer is assigned to its nearest facility. The capacitated facility location problem assigns a maximum capacity to each facility. This puts a restriction on the demand that can be supplied from each potential site. To approach a more realistic situation in which parameters can change over time, multi-period location problems have been proposed. For an extensive review on different types of facility location problems, we refer to Melo et al. (2009).

In this thesis we consider the capacitated multi-period facility location problem. This implies that a time horizon is included and that the facilities have a maximum capacity. The size of these capacities have to be decided on before observing future demand. The restriction that all customer demand has to be fulfilled is relaxed. Instead, the objective of this facility location problem is to maximize total profit instead of minimizing costs, where revenue is received for satisfied demand. And as previously stated we consider a case where demand is no longer deterministic, but uncertain, and apply a robust optimization approach. We will now provide an introduction to robust optimization.

2.2 Robust Optimization

In a typical linear program, parameters are deterministic and the problem can be solved to optimality. Robust optimization (RO) approaches consider some of the data to be uncertain, without taking a specific probability distribution into account. In this section, we give a brief introduction to robust optimization; for further details, we refer to Ben-Tal and Nemirovski (1999, 2000).

The goal of the RO approach is to immunize against uncertainty for which it defines a set that expresses limits on the amount of uncertainty. Using this uncertainty set, the RO approach is able to find a solution that guarantees feasibility even if the uncertain parameters take their worst-case values within the defined set. To adapt this approach into a mixed integer program (MIP) formulation, every constraint involving uncertain parameters is replaced by a constraint that incorporates the uncertainty set.

Consider the following linear program (LP):

$$\max_{x\in\mathbb{R}^n} \left\{ c^\top x : Ax \leq b \right\}, \tag{1}$$

where $c \in \mathbb{R}^n$ are the objective coefficients, $x \in \mathbb{R}^n$ are the decision variables, $A \in \mathbb{R}^{m \times n}$ is a constraint coefficient matrix with elements a_{ij} and $b \in \mathbb{R}^m$ are the right-hand side parameters. In a standard case, c, A and b are deterministic and (LP) can be solved to optimality. RO is applied when some of the data parameters are uncertain. The objective of RO is to find a solution that satisfies the constraints for all realizations of the uncertain parameters, which are limited by an uncertainty set. As an example we suppose A denotes an uncertain matrix, residing in a known uncertain set U. The robust version of a mathematical optimization problem is generally referred to as the robust counterpart (RC) problem. Below we present the RC of (1).

$$\max_{x \in \mathbb{R}^n} \left\{ c^\top x : Ax \leq b \quad \forall A \in U \right\},$$

The constraints $Ax \leq b$ must now be satisfied by all values for $A \in U$, including the worst-case values related to x.

Several unique classes of uncertainty sets have been introduced in Ben-Tal and Nemirovski (2002) and Ben-Tal et al. (2015). In this thesis we focus on the box uncertainty set, where an entry $\tilde{a}_{ij}$ in matrix A is bounded by a symmetric interval. Therefore, $\tilde{a}$, the set of all elements $\tilde{a}_{ij}$, is bounded by a multidimensional box $U^B = \{\tilde{a}_{ij} \in \mathbb{R}^{mxn} : |\tilde{a}_{ij} - \bar{a}_{ij}| \leq \varepsilon |\bar{a}_{ij}| \ \forall i = 1, \ldots, m, j = 1, \ldots, n\}$, where $\bar{a}_{ij}$ is the nominal value of the uncertain value $\tilde{a}_{ij}$ and $\varepsilon \in \mathbb{R}_{\geq 0}$ is the maximum deviation from the nominal demand. For a constraint i, $\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i$, the robust counterpart with uncertainty set U^B is given by

$$\max_{\tilde{a} \in U^B} \left\{ \sum_{j=1}^{n} \tilde{a}_{ij} x_j \right\} \leq b_i. \tag{2}$$

Since the minimum value of $\tilde{a}$ in U^B is found at one of the extreme values, (2) can be reformulated as

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \varepsilon \sum_{j=1}^{n} |\bar{a}_{ij}||x_j| \leq b_i. \tag{3}$$

Assume that $a_{ij} = \bar{a}_{ij}$ in (1) for all i, j. Since the second term of the left-hand side in (3) is positive, the RC provides tighter constraints than the original LP.

3 Model Formulations

In this section we first introduce the notation that will be used in the considered formulations of the facility location problem. Next, we will give a more detailed explanation on the objective of the two-stage approach. Then we present the formulation for the first stage of the nominal problem, followed by the first stage of the robust problem. Afterwards, we discuss the formulation of the second stage. Finally, we extend the existing models with the need to deploy trucks at facilities to deliver the products to customers.

3.1 Notation

We consider the capacitated multi-period fixed-charge facility location problem. The set $\mathcal{T} = \{1, \ldots, T\}$ contains the time periods t up to the horizon length T. Let $\mathcal{N} = \{1, \ldots, N\}$ be the set of nodes, with indices i and j denoting the locations of the facilities and the customers, respectively.

The essence of the problem is to open facilities and decide on the maximum capacity for each open facility. These decisions are made once at the beginning of the time horizon and are called strategic decisions. The decision variable I_i is equal to 1 if facility i is opened and 0 otherwise. The established capacity at facility i is denoted by Z_{i0}. Afterwards, the operational decisions, regarding the production and the allocation of products, have to be made at each time period. Let Z_{it} be the production of facility i in period t and let X_{ijt} be the proportion of demand of customer j provided by facility i in period t. These operational decisions are restricted by the previously made strategic choices, since products can only be delivered from open facilities and each facility has a maximum production per period depending on its capacity.

The decisions of opening facilities, establishing maximum capacities, production and product deliveries result in costs. Let K_i be the cost of opening facility i and let C_{i0} be the cost per unit of capacity established at facility i. The cost per unit of production at facility i in period t is denoted by c_{it} and the delivery cost of a product from facility i to customer j by d_{ij}. The revenue η is obtained for every delivered product to fulfill customer demand. The parameter D_{jt} denotes the demand of customer j at period t. The demand of every customer, however, can vary over time and future demand is not known a priori. The aim of the problem is to maximize the total profit τ over the time horizon, after being corrected for inflation using discount factor $0 < \delta \leq 1$. A complete overview of the notation used in the considered formulations is given in Appendix A.

3.2 Two Stage Approach

The two stage approach is a method to split the previously described problem into two parts. The goal of the first part is to decide on the strategic decisions at the start of the time horizon. The results regarding the locations of the open facilities and their maximum capacities are then used in the second stage to find the optimal operational decisions, after the demand per customer is observed. The application of the two stage approach on different variations of location problems under uncertainty in demand has been studied by Louveaux (1986).

The nominal problem differs from its robust counterpart in the first stage. The formulation of the nominal problem describes the future demand as if it is constant over time and therefore assumed to be known in advance. For the robust method the demand in each period is assumed to be uncertain, however, the nominal demand is known along with the box uncertainty set in which the real demand must be located. From the first stage, we only use the results of the variables I_i and Z_{i0}, which determine the strategic decisions. In the second stage we change the variables to fixed parameters equal to the outcomes of I_i and Z_{i0} provided by the first stage. The strategic decisions are then tested against different values for demand. Let $\bar{D}_{jt}$ be the nominal demand of customer j in period t. The demand $\tilde{D}_{jt}$ which is uncertain before stage 2 is located within the interval

$$[\bar{D}_{jt}(1-\varepsilon_t), \bar{D}_{jt}(1+\varepsilon_t)], \tag{4}$$

where $\varepsilon_t \in [0,1]$. We use sample paths to define these new demand values. These sample paths are created for each customer by generating values for demand by sampling over the interval specified in (4) for all $t \in \mathcal{T}$.

3.3 Strategic Decisions

We present the formulations of the first stage of both the nominal model and the box uncertainty model to decide on the strategic decisions. These formulations are based on those from Baron et al. (2011).

3.3.1 Nominal Problem Formulation

The nominal problem is formulated below. Recall that τ is the total profit over all nodes and all time periods, which is to be maximized. The objective function is represented in constraint (5) for convenience when deriving its robust counterpart. The three terms in (5) express the revenue minus the delivery cost for every delivered product, the total production cost and the cost for opening facilities and establishing capacities, respectively. Since we investigate the present value of the total profits, the revenue and costs in future period t are corrected for inflation by the term δ^{t-1}.

$$(P^{Nom}) \qquad \max_{X,Z,I,Z_0,\tau} \quad \tau$$

s.t.

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\delta^{t-1}(\eta-d_{ij})\bar{D}_{jt}X_{ijt} - \sum_{i=1}^{N}\sum_{t=1}^{T}\delta^{t-1}c_{it}Z_{it} - \sum_{i=1}^{N}(C_{i0}Z_{i0}+K_iI_i) \geq \tau \tag{5}$$

$$\sum_{j=1}^{N}\bar{D}_{jt}X_{ijt} \leq Z_{it} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T} \tag{6}$$

$$\sum_{i=1}^{N}X_{ijt} \leq 1 \qquad \forall\, j \in \mathcal{N},\ t \in \mathcal{T} \tag{7}$$

$$Z_{i0} \leq MI_i \qquad \forall\, i \in \mathcal{N} \tag{8}$$

$$Z_{it} \leq Z_{i0} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T} \tag{9}$$

$$X_{ijt} \geq 0 \qquad \forall\, i,j \in \mathcal{N},\ t \in \mathcal{T} \tag{10}$$

$$I_i \in \mathbb{B} \qquad \forall\, i \in \mathcal{N} \tag{11}$$

The term $\sum_{j=1}^{N}\bar{D}_{jt}X_{ijt}$, used in constraint (6) and (7) can be interpreted as the demand of customer i fulfilled by all facilities in period t. Constraint (6) assures that the supplied demand from a facility is limited by its production, while constraint (7) guarantees that no more than 100% of demand can be satisfied. The necessity that capacity can only be established at open facilities is ensured by constraint (8). The value of M has to be sufficiently large to allow a single facility to have enough capacity to fulfill all demand, which is in this case equal to $\sum_{j=1}^{N}\bar{D}_{jt}$.

Constraint (9) makes sure that the production can only be carried out at facilities with an established capacity and that it does not exceed this capacity.

3.3.2 Robust Problem Formulation Using Box Uncertainty

We now reformulate (P^{Nom}) using the robust optimization approach to include the uncertainty in demand. The uncertain demand $\tilde{D}_{jt}$ is assumed to be located in a symmetrically bounded interval around nominal demand $\bar{D}_{jt}$, as defined in (4). This uncertainty in demand changes the definitions of constraints (5) and (6) for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$. First we let $U_{jt}^B = [\bar{D}_{jt}(1-\varepsilon_t), \bar{D}_{jt}(1+\varepsilon_t)]$, $U_t^B = U_{1t}^B \times U_{2t}^B \times \cdots \times U_{Nt}^B$ and $U^B = U_1^B \times U_2^B \times \cdots \times U_T^B$. When substituting D_{jt} by $\tilde{D}_{jt}$ to express the uncertainty, we get the following augmented constraints for $\tilde{D}_{jt} \in U^B$:

$$\min_{\tilde{D} \in U^B} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \delta^{t-1}(\eta - d_{ij}) \tilde{D}_{jt} X_{ijt} \right\} - \sum_{i=1}^{N} \sum_{t=1}^{T} \delta^{t-1} c_{it} Z_{it} - \sum_{i=1}^{N} (C_{i0} Z_{i0} + K_i I_i) \geq \tau, \quad (12)$$

$$\max_{\tilde{D}_t \in U_t^B} \left\{ \sum_{j=1}^{N} \tilde{D}_{jt} X_{ijt} \right\} \leq Z_{it} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T}. \quad (13)$$

The extreme values of $\tilde{D}_{jt}$ are $\bar{D}_{jt}(1-\varepsilon_t)$ and $\bar{D}_{jt}(1+\varepsilon_t)$. Since δ, $X_{ijt} \geq 0$, the minimum value of constraint (12) depends on whether the term $(\eta - d_{ij})$ is either positive or negative. In case $(\eta - d_{ij}) \leq 0$, however, no delivery will be made, thus $X_{ijt} = 0$. We can therefore ignore this case and obtain the minimum value at $\bar{D}_{jt}(1-\varepsilon_t)$. Since $\tilde{D}_{jt}, X_{ijt} \geq 0$, the maximum value of the left-hand side of constraint (13) is attained at $\bar{D}_{jt}(1+\varepsilon_t)$. We can now rewrite the robust counterparts of constraints (5) and (6) as follows:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \delta^{t-1}(\eta - d_{ij}) \bar{D}_{jt}(1-\varepsilon_t) X_{ijt} - \sum_{i=1}^{N} \sum_{t=1}^{T} \delta^{t-1} c_{it} Z_{it} - \sum_{i=1}^{N} (C_{i0} Z_{i0} + K_i I_i) \geq \tau, \quad (14)$$

$$\sum_{j=1}^{N} \bar{D}_{jt}(1+\varepsilon_t) X_{ijt} \leq Z_{it} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T}. \quad (15)$$

Using these constraints, the robust counterpart of P^{Nom} is fully immunized against the uncertainty in the set U^B. We can also choose to partially immunize against the uncertainty in demand. To do so, we include the parameter $\rho \in [0,1]$ as being the fraction of the uncertainty set to immunize against. The final formulation of the box uncertainty method is as follows:

$$(P^{Box}) \qquad \max_{X,Z,I,Z_0,\tau} \quad \tau$$

s.t.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \delta^{t-1}(\eta - d_{ij}) \bar{D}_{jt}(1-\rho\varepsilon_t) X_{ijt} - \sum_{i=1}^{N} \sum_{t=1}^{T} \delta^{t-1} c_{it} Z_{it} - \sum_{i=1}^{N} (C_{i0} Z_{i0} + K_i I_i) \geq \tau \quad (16)$$

$$\sum_{j=1}^{N} \bar{D}_{jt}(1+\rho\varepsilon_t) X_{ijt} \leq Z_{it} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T} \quad (17)$$

(7) – (11)

The formulations of constraints (7)-(11) remain unchanged. Since the capacities of open facilities now have to allow for uncertainty, the value of M in constraint (8) is changed to $\max_{t \in \mathcal{T}} \{\sum_{j=1}^{N} D_{jt}(1+\varepsilon_t)\}$.

In the base case we set ρ equal to 1, hence immunizing against all uncertainty in demand. Note that in case $\rho = 0$, the complete uncertainty set will be ignored and P^{Box} becomes equal to P^{Nom}.

3.4 Operational Decisions

We introduce the formulation of the second stage of the facility location problem, which is to be performed after the first stage has been completed. The aim of the second stage is to maximize the total profit over the entire time horizon, restricted by the strategic decisions from the first stage. These decisions cannot be changed over time and therefore we introduce new parameters I_i^* and Z_{i0}^*, which are initialized as the results of I_i and Z_{i0} from the first stage. Furthermore, instead of nominal demand $\bar{D}_{jt}$, we use the realised demand $\hat{D}_{jt}$ provided by the sample paths. The remainder of the formulation of the second stage remains similar to formulation (P^{Nom}). For completeness we present the full model of the second stage below:

$$(P^{Opr}) \quad \max_{X,Z} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T} \delta^{t-1}(\eta - d_{ij})\hat{D}_{jt}X_{ijt} - \sum_{i=1}^{N}\sum_{t=1}^{T} \delta^{t-1}c_{it}Z_{it} - \sum_{i=1}^{N}(C_{i0}Z_{i0}^* + K_i I_i^*) \quad (18)$$

s.t.

$$\sum_{j=1}^{N} \hat{D}_{jt}X_{ijt} \leq Z_{it} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T} \quad (19)$$

$$\sum_{i=1}^{N} X_{ijt} \leq 1 \qquad \forall\, j \in \mathcal{N},\ t \in \mathcal{T} \quad (20)$$

$$Z_{it} \leq Z_{i0}^* \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T} \quad (21)$$

$$X_{ijt} \geq 0 \qquad \forall\, i, j \in \mathcal{N},\ t \in \mathcal{T} \quad (22)$$

We solve the second stage for every unique sample path. Note that the strategic decisions are already fixed and this stage can therefore equivalently be solved by maximizing the profit in each individual time period.

3.5 Transport Problem Formulations

So far we defined the delivery cost to be linear: every single delivered product incurred a cost based on the distance from the facility in question to the customer. A more realistic approach would be to require a certain manner of transport to deliver the products. In this section we introduce the addition of trucks to our models as a means of transport.

A number of trucks can drive from an open facility i to a customer j at period t and transport a maximum number of products. We introduce two new parameters: the capacity of a truck q and the fixed cost k_i for each established truck at facility i. We also add two new decision variables. The number of trucks that travel from facility i to customer j at period t is denoted by Y_{ijt}. This number, however, cannot be larger than the number of trucks available at this facility, represented by Y_{i0}. There are no trucks needed for transport from facility i to customer j in case $i = j$ since the distance is equal to zero. Note that the choices of Y_{i0} are new strategic decisions, while Y_{ijt} denote operational decisions. We assume that there is no maximum number of trucks that can be hired. We do not consider a routing problem: every delivery will have to be made directly from an open facility. For simplicity we assume that all trucks are homogeneous although different types can readily be included. The formulation of the nominal transport problem is as follows:

$$(P^{TrNom}) \qquad \max_{X,Y,Z,I,Z_0,Y_0,\tau} \quad \tau$$

s.t.

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\delta^{t-1}(\eta\bar{D}_{jt}X_{ijt} - d_{ij}qY_{ijt}) - \sum_{i=1}^{N}\sum_{t=1}^{T}\delta^{t-1}c_{it}Z_{it} - \sum_{i=1}^{N}(C_{i0}Z_{i0} + K_iI_i + k_iY_{i0}) \geq \tau \quad (23)$$

$$\bar{D}_{jt}X_{ijt} \leq qY_{ijt} \qquad \forall\, i,j \in \mathcal{N},\ i \neq j,\ t \in \mathcal{T} \quad (24)$$

$$Y_{i0} \leq MI_i \qquad \forall\, i \in \mathcal{N} \quad (25)$$

$$\sum_{j=1}^{N} Y_{ijt} \leq Y_{i0} \qquad \forall\, i \in \mathcal{N},\ t \in \mathcal{T} \quad (26)$$

$$Y_{ijt} \in \mathbb{N} \qquad \forall\, i,j \in \mathcal{N},\ t \in \mathcal{T} \quad (27)$$

(6) – (11)

The objective constraint (23) is adapted from (5) by changing the delivery cost from a linear function to a step-wise function. Furthermore, there are also incurred costs for the number of deployed trucks. Constraint (24) guarantees that there is sufficient truck capacity for the transport of all fulfilled demand, while no trucks are needed for transport over the same coordinates. Constraint (25) ensures that trucks are only available at open facilities. The minimum value needed for M to support this restriction is $\lceil \frac{1}{q}\sum_{j=1}^{N}\bar{D}_{jt}\rceil$. Finally, constraint (26) implies that no more trucks can be used than there are available per facility.

We can derive a robust approach for the transport problem in a similar way as before using box uncertainty. Whereas previously the total possible production had to be sufficient to immunize against the maximum possible demand, the same principle applies to the decisions regarding available trucks. Again, we substitute $\bar{D}_{jt}$ by $\tilde{D}_{jt}$ to express the uncertainty and obtain the following augmented constraints for $\tilde{D}_{jt} \in U^B$:

$$\min_{\tilde{D}\in U^B}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\delta^{t-1}(\eta\tilde{D}_{jt}X_{ijt} - d_{ij}qY_{ijt})\right\} - \sum_{i=1}^{N}\sum_{t=1}^{T}\delta^{t-1}c_{it}Z_{it} - \sum_{i=1}^{N}(C_{i0}Z_{i0} + K_iI_i + k_iY_{i0}) \geq \tau, \quad (28)$$

$$\max_{\tilde{D}_t\in U_t^B}\left\{\tilde{D}_{jt}X_{ijt}\right\} \leq qY_{ijt} \qquad \forall\, i,j \in \mathcal{N},\ i \neq j,\ t \in \mathcal{T}. \quad (29)$$

We know that the extreme values of $\tilde{D}_{jt}$ are $\bar{D}_{jt}(1-\varepsilon_t)$ and $\bar{D}_{jt}(1+\varepsilon_t)$. Since η, δ, $X_{ijt} \geq 0$, the minimum value of constraint (28) is attained at $\bar{D}_{jt}(1-\varepsilon_t)$. Since $\tilde{D}_{jt}$, $X_{ijt} \geq 0$, the maximum value of the left-hand side of constraint (29) is located at $\bar{D}_{jt}(1+\varepsilon_t)$. We can now rewrite the robust counterparts of constraints (23) and (24) as follows:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\delta^{t-1}(\eta\bar{D}_{jt}(1-\varepsilon_t)X_{ijt} - d_{ij}qY_{ijt}) - \sum_{i=1}^{N}\sum_{t=1}^{T}\delta^{t-1}c_{it}Z_{it} - \sum_{i=1}^{N}(C_{i0}Z_{i0} + K_iI_i + k_iY_{i0}) \geq \tau, \quad (30)$$

$$\bar{D}_{jt}(1+\varepsilon_t)X_{ijt} \leq qY_{ijt} \qquad \forall\, i,j \in \mathcal{N},\ i \neq j,\ t \in \mathcal{T}. \quad (31)$$

After the addition of ρ the box uncertainty transport problem is given by:

$$(P^{TrBox}) \qquad \max_{X,Z,I,Z_0,Y,Y_0,\tau} \quad \tau$$

s.t.

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\delta^{t-1}(\eta\bar{D}_{jt}(1-\rho\varepsilon_t)X_{ijt}-d_{ij}qY_{ijt})-\sum_{i=1}^{N}\sum_{t=1}^{T}\delta^{t-1}c_{it}Z_{it}-\sum_{i=1}^{N}(C_{i0}Z_{i0}+K_iI_i+k_iY_{i0})\geq\tau \tag{32}$$

$$\bar{D}_{jt}(1+\rho\varepsilon_t)X_{ijt}\leq qY_{ijt} \qquad \forall\ i,j\in\mathcal{N},\ i\neq j,\ t\in\mathcal{T} \tag{33}$$

$$(7)-(11),(17),(25)-(27)$$

The minimum value of M to ensure the functioning of constraint (25) is set to $\lceil\frac{1}{q}\max_{t\in\mathcal{T}}\{\sum_{j=1}^{N}\bar{D}_{jt}(1+\varepsilon_t)\}\rceil$ to allow for uncertainty in demand. Again we let ρ be equal to 1 and therefore immunize against all uncertainty in demand.

Finally, we present the formulation of the second stage of the transport models. In addition to the location of open facilities and their maximum capacity, the number of available trucks per facility is also a decision that has been made at the beginning of the time horizon. Therefore we use parameter Y^*_{i0}, equal to the outcomes of Y_{i0} from the first stage. Again, we use $\hat{D}_{jt}$ to denote the demand provided by the sample paths. The aim of the second stage is again to make the operational decisions that maximize the profit per time period, while being restricted by the strategic decisions. The second stage of the transport model is formulated as follows:

$$(P^{TrOpr}) \ \max_{X,Z,Y} \quad \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\delta^{t-1}(\eta\hat{D}_{jt}X_{ijt}-d_{ij}qY_{ijt})-\sum_{i=1}^{N}\sum_{t=1}^{T}\delta^{t-1}c_{it}Z_{it}-\sum_{i=1}^{N}(C_{i0}Z^*_{i0}+K_iI^*_i+k_iY^*_{i0}) \tag{34}$$

s.t.

$$\hat{D}_{jt}X_{ijt}\leq qY_{ijt} \qquad \forall\ i,j\in\mathcal{N},\ i\neq j,\ t\in\mathcal{T} \tag{35}$$

$$\sum_{j=1}^{N}Y_{ijt}\leq Y^*_{i0} \qquad \forall\ i\in\mathcal{N},\ t\in\mathcal{T} \tag{36}$$

$$(20)-(22),(27)$$

4 Computational Results

In this section we analyse how the solutions provided by the nominal models and the box uncertainty models differ. First, we give a description of the test environment. Then we review the results provided by the first stage of the non-transport models. Here we investigate the topologies, which are the number of open facilities along with their location and size. We then analyse the operational decisions and the total profit provided by the second stage. Finally, we investigate the solutions provided by the transport models.

4.1 Test Environment

The test environment consists of several coordinates of customers and potential facilities, fixed cost to incur depending on the different decisions and demand over time. We generate coordinates of customers on a unit square for $N = 15$ customers. The delivery cost d_{ij} is equal to the Euclidean distance between nodes i and j, for all $i, j \in \mathcal{N}$. Nominal demand $\bar{D}_{jt}$ is uniformly distributed over the interval [17500, 22500] and assumed constant over a time period of $T = 20$ periods. The uncertainty set is defined using the following recursive function: $\varepsilon_t = \gamma + (1-\gamma)\varepsilon_{t-1}$, with $\varepsilon_0 = 0$ and $\gamma = 0.15$. Uncertainty parameter ε_t is a concave and monotonically increasing function, allowing for larger possible deviations from the nominal demand in further periods of time. We assume that $U^B_{jt} = [\bar{D}_{jt}(1-\varepsilon_t), \bar{D}_{jt}(1+\varepsilon_t)]$, whose values are strictly positive since $\lim_{t\to\infty} \varepsilon_t = 1$. All parameter values in the base case are summarized in Table 1.

Table 1: Parameter values in base case

η	1
c_{it}	0.1
C_{i0}	0.1
k_i	10
K_i	50000
δ	1
ρ	1
q	3000
$\bar{D}_j$	$\sim U[17500, 22500]$

4.2 Topology Comparison

We start off with reviewing the results from the first stage of the non-transport models regarding the decisions to be made at the beginning of the time horizon. We solve for 250 instances for both the nominal and the box formulation problem, each instance having different node locations over the unit square and different deterministic nominal demands.

In Table 2 we show the mean number of open facilities and the mean capacities of these open facilities over all instances. We also present the mean number of connections from open facilities, which is defined as:

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{1}\left\{\sum_{t=1}^{T} X_{ijt} > 0\right\}}{\sum_{j=1}^{N} I_j}. \tag{37}$$

The indicator function $\mathbb{1}\{\cdot\}$ defines a connection from node i to j if any products over the entire time horizon are delivered from facility i to customer j. Note that connections between facilities do not need to be symmetric. Finally, we also report the average strategic cost. These are the costs incurred by the strategic decisions, which will not change after the first stage. For completeness, we also present the objective values of the first stage. Note that for the box uncertainty model these values do not equal the total profit due to the immunization against lower potential revenues.

Table 2: Comparison of the topology solutions of both non-transport models

	Non-transport Model	Mean number of open facilities	Mean capacity	Mean number of connections	Strategic cost	Objective value
$\eta = 1$	Nominal	10.97	27,303	1.37	578,346	4,686,304
	Box	3.48	135,718	4.46	221,230	477,022
$\eta = 3$	Nominal	10.97	27,303	1.37	578,346	16,664,740
	Box	4.04	139,430	3.77	258,074	3,445,601
$\eta = 6$	Nominal	10.97	27,303	1.37	578,346	34,632,393
	Box	4.16	139,841	3.62	266,430	8,276,814

The most striking difference between the topology results of both models is the decrease in the number of open facilities provided by the box uncertainty model. The mean capacities of these facilities are on the other hand larger, as well as the average total capacity. This is due to the immunization against demand uncertainty, forcing a larger maximum production by all facilities per period. The mean number of connections of open facilities in the nominal model is equal to N divided by the average number of open facilities. This means that all demand of every customer is satisfied by a single facility. This is due to the assumption of the nominal formulation that all future demand is constant and therefore all planned deliveries are the same in every period of time. We also see that the strategic cost for the box uncertainty model is on average only 40% of the cost in the nominal model. Finally, we observe that the initial characteristics of the facilities in the nominal model do not differ for the presented different revenues. The box uncertainty method opens more facilities when the revenue increases. This is caused by the diminishing effect of the response against lower potential revenue in the objective function. For larger revenues, the effect of lower potential revenue decreases.

4.3 Product Allocation and Profit Comparison

In this section we compare the performances of the box uncertainty model to that of the nominal model in terms of profit and the realization of the deliveries. For both models, we solve the second stage of the problem formulations for the first 100 of the previously generated 250 topologies. For each topology we simulate the *real* demand over time using 30 sample paths, sampled over the uncertainty set (4). We sample from the Uniform, the Bell-shaped Beta(2,2), and the U-shaped Beta(0.5,0.5) distribution, with ten sample paths per distribution. The nominal demand of all customers remains the same as in the first stage. The other parameters also remain unchanged, see Table 1. We first consider the performance of the topologies determined by the first stage and presented in Table 2 for both models, after which we present the different costs and the final profit.

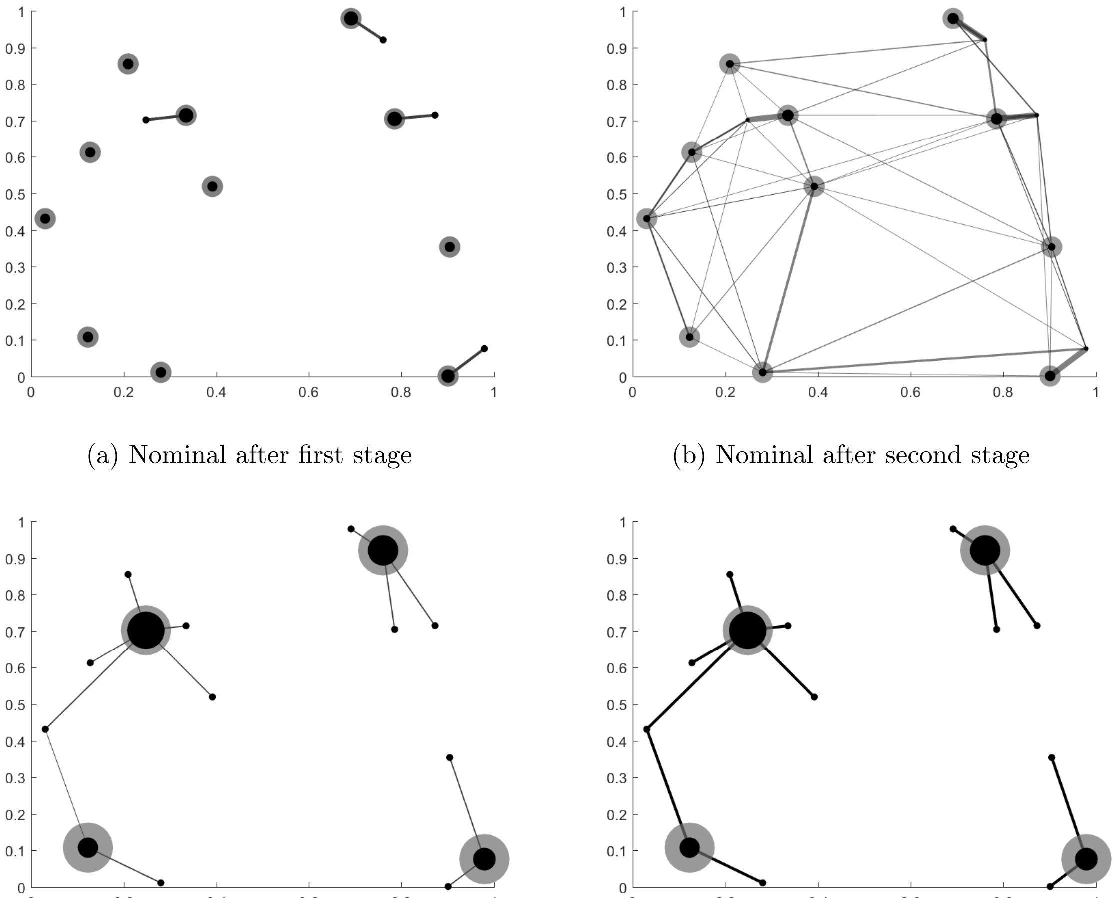
The solutions regarding the performance of both models after solving the second stage with the sample paths are presented in Table 3. We present the percentage of total demand which is covered, the fraction of capacity of all facilities which is used by production, and the average number of connections as defined in (37). Finally, we present the changes in the number of connections in relation to the first stage. Note that these changes do not exactly correspond with the values reported in Table 2, due to the decrease in the number of topologies. The esults are given for the three different distributions used to generate sample paths for both models.

Table 3: Comparison of the production and delivery performances of both non-transport models

Non-transport Model		% Demand covered	% Usage of capacity	Mean number of connections	% Increase in connectivity
Nominal	Bell-shaped Beta	96.62	96.53	6.12	341
	Uniform	95.53	95.62	6.15	343
	U-shaped Beta	94.66	94.50	6.28	352
Box	Bell-shaped Beta	100	63.55	4.36	−11.76
	Uniform	100	63.67	4.48	−9.35
	U-shaped Beta	100	63.51	4.71	−4.77

The first observation we can make is that for all sample methods in the box uncertainty model all of the demand is covered, while the nominal model covers around 95.6% of the demand. This lost demand is due to a lack of total capacity in periods with high total demand, while the box uncertainty model is immunized against this demand and therefore performs as expected. This is also supported by the average percentage of the capacity used by production. For the nominal model this is around 95%, implying that there is not much capacity available for production in periods with high total demand. Finally, we compare the real number of connections with its old values. The deviation in the number of connections of the box uncertainty model is relatively small. The large increase in the number of connections in the nominal model, however, is yet another indication that the model does not handle the deviation in demand well. Instead of all customers getting provided exclusively by a single facility, over time this is now an average of 4.5 facilities.

In Figure 1 we present the solutions of one topology obtained by both the nominal model and the box uncertainty model after both stages. The nodes in the figures represent the locations of customers and open facilities. The locations at which a facility is opened are represented by a grey border around the node. The size of the capacity of each open facility is represented by the size of the node. The edges show the deliveries from open facilities to customers. Finally, the thickness of the edges denote the number of periods in which a connection is used for transport. Figure 1a shows that the initial plan of the nominal problem is to have all demand over the time horizon of every customer satisfied by only one facility. However, we can see in Figure 1b that in reality more and longer connections are used to fulfill demand. The robustness of the box uncertainty method is shown in Figures 1c and 1d. The only difference between the planned and the executed transportation is the number of periods in which connections are used. This is a consequence of the lower potential revenue used in the objective function of the box uncertainty problem (16). This causes the potential revenue for fulfilled demand, starting from a certain period, to become smaller than the production cost, hence not resulting in any profit.



(a) Nominal after first stage (b) Nominal after second stage

(c) Box after first stage (d) Box after second stage

Figure 1: Topologies as obtained by the solutions of the nominal model and the box uncertainty model after both stages

In Table 4 we present the average values of the strategic cost, the operational cost, the revenue and the profit obtained by the different sample distributions for both non-transport models. The total profit can be calculated as the revenue minus both the strategic cost and the operational cost. We also show the percentage increase in profit of the box uncertainty model in relation to the nominal model. As shown before, the cost incurred by opening and establishing capacities of the facilities of the box uncertainty model is only about 40% of the strategic cost provided by the nominal model. Note that strategic cost between Table 2 and Table 4 slightly deviate since the number of sample topologies has decreased from 250 to 100. The operational cost of the nominal model however, is about 60% of the operational cost of the box uncertainty model, which is mainly due to the additional open facilities resulting in lower delivery cost. Furthermore, we find a slight increase in revenue for the box uncertainty method related to the higher demand coverage. Finally, we see that these costs and revenues do not result in significant improvements in terms of profit. The box uncertainty method only yields on average higher profits when the sample paths are generated by the U-shaped Beta distribution. On average, the box uncertainty method yields a decrease in profit of 0.69%. The standard deviations are displayed between brackets and show large deviations in profit based on different topologies.

Table 4: Comparison of the profit of both non-transport models

Non-transport Model		Strategic cost	Operational cost	Revenue	Profit	% Increase in profit
Nominal	Bell-shaped Beta	577,942	836,469	5,780,619	4,366,208	
	Uniform	577,942	872,792	5,725,794	4,275,060	
	U-shaped Beta	577,942	912,461	5,659,088	4,168,685	
Box	Bell-shaped Beta	220,599	1,523,047	5,982,691	4,239,045	−2.91 *(2.68)*
	Uniform	220,599	1,525,852	5,993,979	4,247,528	−0.64 *(2.77)*
	U-shaped Beta	220,599	1,523,583	5,978,803	4,234,621	1.58 *(2.87)*

4.4 Results for the Transport Models

We observed that the box uncertainty model does not result in significantly larger profits than the nominal model when immunizing against full uncertainty in demand. The addition of trucks as a manner of transport requires an extra strategic decision that may influence the impact of robust optimization. In this section we compare the results from the box uncertainty transport model to the nominal transport model. We reduced the number of sample topologies from 100 to the first 20 due to the increase in solving time. The number of sample paths per topology remains unchanged.

In Table 5 we present the main results of the nominal transport model and the box uncertainty transport model after performing both stages. We report the average number of open facilities and the mean number of trucks over all facilities. We also show the relative increase in profit of the box uncertainty transport model compared to the nominal transport model. Finally, we report on the average covered demand and the average use of the available trucks. We present the results for capacity sizes q equal to 1000, 3000 and 7000, to allow for variation in the number of trucks needed to transport products over the interval of the nominal demand [17500, 22500].

Table 5: Comparison of the results of both transport models

	Transport Model		Mean # of open facilities	Mean # of trucks	% Increase in profit	% Demand covered	% Usage of trucks
$q = 1000$	Nominal	Bell-shaped Beta	12.1	56.3		89.27	86.34
		Uniform	12.1	56.3		85.38	81.68
		U-shaped Beta	12.1	56.3		81.45	77.06
	Box	Bell-shaped Beta	5.2	260.2	8.52 *(0.99)*	99.91	74.76
		Uniform	5.2	260.2	14.16 *(1.36)*	99.82	74.66
		U-shaped Beta	5.2	260.2	20.07 *(2.02)*	99.49	74.54
$q = 3000$	Nominal	Bell-shaped Beta	12.2	17.2		88.00	87.69
		Uniform	12.2	17.2		84.29	83.86
		U-shaped Beta	12.2	17.2		80.30	80.26
	Box	Bell-shaped Beta	5.3	86.4	9.76 *(2.56)*	99.73	76.60
		Uniform	5.3	86.4	15.30 *(2.14)*	99.60	76.35
		U-shaped Beta	5.3	86.4	21.40 *(1.73)*	99.24	76.21
$q = 7000$	Nominal	Bell-shaped Beta	12.5	6.9		87.95	89.79
		Uniform	12.5	6.9		84.16	85.26
		U-shaped Beta	12.5	6.9		80.12	81.34
	Box	Bell-shaped Beta	5.5	36.5	9.25 *(3.65)*	99.36	80.35
		Uniform	5.5	36.5	14.84 *(3.60)*	99.18	80.07
		U-shaped Beta	5.5	36.5	20.89 *(3.48)*	98.71	79.70

First, we notice that the ratio of the average number of open facilities between the nominal model and the box uncertainty model is roughly the same as for both models without transport. The actual numbers, however, have slightly increased. This is due to the addition of needed transport in the form of trucks, which causes the average delivery cost to increase and incurs a fixed cost for the availability of trucks. The facilities as obtained by the box uncertainty transport model need more trucks to deliver products to all customers, to compensate for the smaller number of open facilities. As expected, the demand covered is higher for the robust approach, although it is not 100% anymore. Even though there are a sufficient number of trucks available to fulfill all demand at every period, the step-wise delivery cost function does not result in profits if a truck is not filled with a certain number of products. The demand covered by the nominal transport model on the other hand is much lower. The strategic decisions, which now also include the number of available trucks per open facility, limit the amount of demand that can be fulfilled in an extra way: a lack of available trucks. Finally, we can see that the box uncertainty transport model yields significantly higher profits compared to the nominal transport model for all sample path distributions. This is mainly caused by the high difference in the percentage of demand that can be covered, causing a lack of revenue for the nominal method. Our final observation is that the truck capacity has no big impact on the difference in profit, although the standard deviations increase for higher capacities. This is due to the effect of truck capacity to the step sizes in the delivery cost function.

While prior to the necessity of transport, a customer could be provided by all open facilities, they can now only be supplied by a facility at its own location or by other facilities that have trucks to their availability. We can therefore distinguish open facilities to be either a local facility or a global facility. Global facilities have at least one truck available and can therefore transport deliveries to all customers. Local facilities, however, can only fulfill demand of a customer at its own location. The average number of trucks per global facility is defined as:

$$\frac{\sum_{i=1}^{N} Y_{i0}}{\sum_{i=1}^{N} \mathbb{1}\left\{Y_{i0} > 0\right\}}.$$

We find the average number of trucks per global facility of the nominal transport model to be 20.5, 7.5 and 3.2 for capacity sizes $q = 1000$, 3000 and 7000, respectively. For the box uncertainty transport models these averages are 56.3, 18.4, 7.4. In case of $q = 3000$ we find that 18.8% of the open facilities provided by the nominal transport model are global facilities, against 88.6% using the box uncertainty transport model. Therefore, the reason that the average deployment of available trucks is only slightly higher for the nominal model is due to larger average distance from a customer to its closest global facility, resulting in a larger average delivery cost.

5 Sensitivity Analysis

In this section we will test the performance of the models with parameter values different than used in the base case. We first test the performances of the models when only immunizing against a subset of the uncertainty set. Then we compare the differences in the increase in profit relative to the nominal models when varying the discount factors and the length of the time horizon. We do not perform a sensitivity analysis for other parameters as they seem to have a linear effect on the objective. For example, higher values of K have a linear negative effect on the number of open facilities and the increase in profit of the box uncertainty models over the nominal models, as shown in Baron et al. (2011).

5.1 Performance of Different Degrees of Robustness

So far, we studied the case where the box uncertainty models were fully immunized against the uncertainty in demand, including the most extreme but unlikely deviations. In this section we compare the results when immunizing against a smaller subset of the uncertainty set, while still sampling from the initial box uncertainty set (4). We use ρ, as defined earlier, to denote the fraction of the symmetric interval to immunize against. This way the box uncertainty model immunizes against demand in the interval $[(1 - \rho\varepsilon_t)\bar{D}_{jt}, (1 + \rho\varepsilon_t)\bar{D}_{jt}]$. We first test the effect of partial immunization in the non-transport box uncertainty model, followed by the box uncertainty model including transport.

We observed that the box uncertainty model, without transport, does not obtain significant improvements in profit. While the robust approach performed as expected and obtained a 100% demand coverage, the increase in operational cost was higher than the reduced strategic cost and the improved revenue. We also observed that on average only about 64% of the total capacity was being used for production and that facilities barely needed to cooperate to deliver to a customer. This implied that fully immunizing against the uncertainty set might not be optimal. In Figure 2, we plot the increase in profit provided by the box uncertainty model, compared to the nominal model.

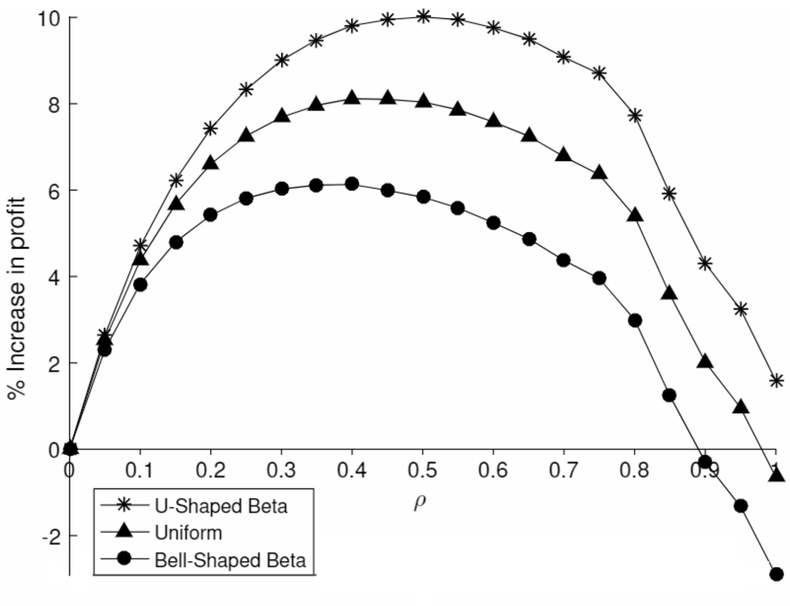


Figure 2: Percentage increase in profit for the box uncertainty model for different values of ρ compared to the nominal model ($\rho = 0$)

We observe that the relative profit increases when immunizing against more uncertainty up until a point. When ρ becomes too large, the model starts to immunize against uncertainties that are not likely to occur and the relative profits decrease. We also notice that all three sample distributions start yielding positive increase in profits when ρ is smaller than 0.9. Sample paths generated by the Uniform or Bell-shaped Beta distribution obtain their maximum increase in profit at $\rho = 0.4$, while the U-shaped Beta distribution yields highest improvements when $\rho = 0.5$, since the latter distribution allows for more uncertainty. Over all distributions, the maximum improvement is on average 8.02% and is obtained at $\rho = 0.4$. This is also the lowest value for ρ for which 99.99% of the demand still is covered, supporting the assumption that it may not be desired to immunize against all uncertainty.

We also investigate the increase in profit for the transport model when decreasing the fraction of immunized uncertainty. In Figure 3, we show the plot of the increase in profit provided by the box uncertainty transport model relative to the profits of the nominal transport model.

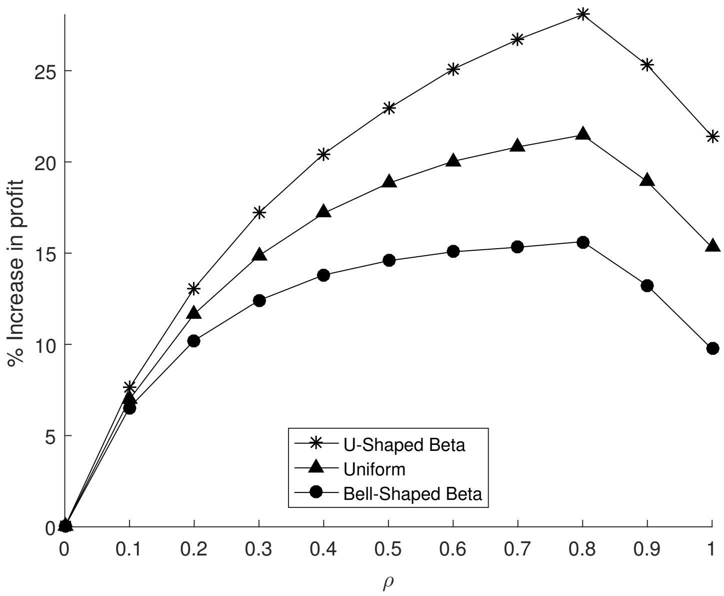


Figure 3: Percentage increase in profit for the box uncertainty transport model for different values of ρ compared to the nominal transport model ($\rho = 0$)

We see a similar concave shape of the performance under different fractions of immunization against uncertainty as with the basic models. The maximum increase in profit, however, is achieved at $\rho = 0.8$. This shows that with the introduction of transport, the robust model needs to immunize against more uncertainty to provide better solutions. The average increase in profit, however, is larger than before and achieves a maximum of 21.5% at $\rho = 0.8$.

5.2 Variation in Inflation and Time Horizon

In this section we investigate the differences in profit provided by the nominal solutions and the robust solutions with different values for the discount factor δ and the time horizon length T. So far we ignored inflation by keeping δ equal to 1. When studying revenues and costs over time it is interesting to take the deprecation of value over time, caused by inflation, into account, to mimic a more realistic setting. For example, with a constant inflation equal to $100(1 - \delta)$% per period, the present value of P in future period t is only $\delta^{(t-1)}P$. In Table 6 we present the increase in profit provided by the non-transport box uncertainty model compared to the nominal model for ρ equal to 1, and the earlier optimal case with $\rho = 0.4$ as obtained in Section 5.1.

Table 6: Percentage increase in profit for box uncertainty model when varying T and δ compared to the nominal model

$\boldsymbol{\rho = 0.4}$	δ					$\boldsymbol{\rho = 1}$	δ				
$\lvert T\rvert$	1	0.999	0.99	0.95	0.9	$\lvert T\rvert$	1	0.999	0.99	0.95	0.9
3	0.88	0.88	0.90	0.77	0.64	3	−1.08	−1.03	−1.38	−1.41	−2.31
5	2.11	2.11	2.00	1.94	1.72	5	−0.30	−0.29	−0.24	−1.74	−1.23
10	4.55	4.50	4.56	3.93	3.07	10	1.46	1.49	1.74	1.25	−0.04
20	8.02	7.99	6.81	5.09	4.76	20	−0.69	−0.10	0.30	0.75	−0.13
30	5.26	5.29	5.47	5.55	4.27	30	4.40	4.47	3.85	2.22	1.31

First off we observe that in most instances, the maximum increase in profit is attained at $\delta = 1$. However, a discount value of 0.99 seems to result in a higher increase in many instances. The discount factor $\delta = 0.999$ seems to be too small to change the topological results, hence only slightly reducing the present value of the higher revenues obtained by the box uncertainty model, compared to the case of $\delta = 1$. On average we see that higher discount values result in smaller increases in profits. This is due to a decrease in the number of open facilities provided by the nominal model, to reduce the strategic cost. When looking at the differences between ρ equal to 0.4 or 1, we observe that as the length of the time horizon increases, a larger fraction of immunization is required to achieve a better performance. For $\rho = 0.4$, the best improvements are obtained at $T = 20$, and in case of $\rho = 1$ at $T = 30$. However, the full immunization against uncertainty performs better at $T = 10$ than $T = 20$. This is caused by a bad performance of the nominal model at $T = 10$ due to high strategic cost. Finally, for $\rho = 1$, we observe a larger effect on the increase in profits from the discount factor for larger values of T. This is explained by the high operational cost when fully immunizing against all uncertainty.

We present the increase in profit provided by the transport box uncertainty model compared to the nominal model in Table 7. The results are provided for ρ equal to 1, and the earlier optimal case obtained in Section 5.1 with $\rho = 0.8$

Table 7: Percentage increase in profit for box uncertainty transport model when varying T and δ compared to the nominal transport model

$\boldsymbol{\rho = 0.8}$	δ					$\boldsymbol{\rho = 1}$	δ				
$\lvert T\rvert$	1	0.999	0.99	0.95	0.9	$\lvert T\rvert$	1	0.999	0.99	0.95	0.9
3	0.66	0.67	0.66	0.51	0.46	3	0.71	0.73	0.81	0.74	0.41
5	2.48	2.59	2.46	2.34	1.78	5	2.45	2.47	2.27	2.20	1.88
10	7.51	7.59	7.32	6.06	4.81	10	7.38	7.35	6.64	5.16	3.63
20	21.50	21.20	19.54	13.62	8.28	20	15.30	14.87	13.25	9.62	5.24
30	13.25	13.34	13.37	12.41	7.59	30	12.86	12.94	12.86	10.96	5.90

The highest increase in profit are obtained at $T = 20$ for both $\rho = 0.8$ and $\rho = 1$, while all instances yield a positive increase in profit compared to the nominal transport model. Similar as in Table 6, we observe a larger effect on the increase in profits caused by the discount factor δ for larger values of T. As a final remark we mention that for a long time horizon length T and a small discount factor δ, the topologies provided by the nominal model become more similar to those provided by the box uncertainty model, albeit with a smaller total capacity. Therefore, the nominal model also needs to establish more trucks and approaches the performance of the box uncertainty model. This results in relatively smaller differences between the solutions of these models. However, the box uncertainty model will still be able to handle the uncertain demand better, thus still providing higher profits.

6 Conclusion

In this thesis we considered a robust optimization (RO) approach to immunize against demand uncertainty in a multi-period fixed-charge facility location problem. We applied a two-stage approach where first the locations and the maximum capacities of the facilities to be opened are established. We show that in case the uncertain demand is bounded in a multidimensional box, the topology results differ from the nominal case by opening less, yet larger, facilities. After these strategic decisions have been made, we test the performance in the second stage using demand sampled from the uncertainty set in the form of sample paths. We show that when immunizing against all possible uncertainty in this box, the profit over the time horizon does not significantly improve.

We then contributed to this approach by extending the models to include trucks as a means of transport. This results in the additional strategic decision on the number of trucks to be established at open facilities. We can now distinguish open facilities between those that have trucks available and those that do not. This makes it more restrictive to deliver products in future time periods, since facilities without trucks cannot service customers at a different location. We show that the RO approach applied to this adapted problem results in large improvements in terms of profits compared to the nominal transport model. For both box uncertainty models we also show that larger increases in profit, relative to the nominal models, can be obtained when immunizing against a smaller subset of the uncertainty set.

Future research directions may include more variations in the immunization against uncertainty, including simultaneously using different subsets of the uncertainty set to immunize against. For example, we could investigate the effect on the objective when immunizing against different degrees of uncertainty in the objective function, the strategic decisions regarding production and the strategic decisions on the number of deployed trucks.

A List of Notation

Sets:

- $\mathcal{N} = \{1, \ldots, N\}$: the nodes in the graph. The indices $i, j \in \mathcal{N}$ are used for the facilities and the customers respectively.
- $\mathcal{T} = \{1, \ldots, T\}$: the time periods t up until time horizon T.

Parameters

- $\bar{D}_{jt} \in \mathbb{R}_{\geq 0}$: the nominal demand of customer j in period t.
- $\tilde{D}_{jt} \in \mathbb{R}_{\geq 0}$: the uncertain demand of customer j in period t.
- $\hat{D}_{jt} \in \mathbb{R}_{\geq 0}$: the realised demand of customer j in period t.
- $\eta \in \mathbb{R}_{>0}$: the revenue of every delivered product.
- $\delta \in (0, 1]$: the discount factor of value over a period.
- $d_{ij} \in \mathbb{R}_{\geq 0}$: the distance and delivery cost of a product from node i to j.
- $K_i \in \mathbb{R}_{\geq 0}$: the cost of opening facility i.
- $k_i \in \mathbb{R}_{\geq 0}$: the cost of deploying a truck at facility i.
- $C_{i0} \in \mathbb{R}_{\geq 0}$: the cost per unit of capacity established at facility i.
- $c_{it} \in \mathbb{R}_{\geq 0}$: the cost per unit of production at facility i in period t.
- $\varepsilon_t \in [0, 1]$: the uncertainty in demand in period t.
- $\gamma \in [0, 1]$: the uncertainty in demand in the first period.
- $q \in \mathbb{R}_{\geq 0}$: the capacity of a truck.
- $\rho \in [0, 1]$: the fraction of the uncertainty set to immunize against.

Variables

- $I_i \in \mathbb{B}$: equals 1 if facility i is opened, 0 otherwise.
- $Z_{i0} \in \mathbb{R}_{\geq 0}$: the maximum capacity of facility i.
- $Z_{it} \in \mathbb{R}_{\geq 0}$: the production of facility i in period t.
- $X_{ijt} \in [0, 1]$: the proportion of demand of customer j provided by facility i in period t.
- $Y_{i0} \in \mathbb{N}$: the number of trucks available at facility i.
- $Y_{ijt} \in \mathbb{N}$: the number of deployed trucks from facility i to customer j in period t.
- $\tau \in \mathbb{R}$: the total profit over the time horizon.

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