Forecasting the term structure interest rate of government bond yields

Bachelor Thesis Econometrics & Operational Research

Joost van Esch (419617)

Erasmus School of Economics, Erasmus University Rotterdam

Supervisor:
D. van Dijk

Second assessor:
X. Leng

July 2^{nd}, 2017

Abstract

In this paper, I introduce extensions on the dynamic Nelson-Siegel (DNS) framework for forecasting the yield-curve. These extensions regard time-varying volatility and including macroeconomic variables in the estimation of the beta's, which determine the forecasts of the yield-curve. I make forecast with all different models on different forecasting horizons, using both an extending out-of sample window and a moving out-of sample window. I assess the forecasts by their root mean squared error and I compare the forecasts of the different models against the DNS framework with the Diebold-Mariano test for equal prediction accuracy. The results show that the models which capture time-varying volatility produce better forecasts in almost all circumstances. Moreover, using a moving out-of sample window lead to better forecasts as well.

Keywords: Nelson-Siegel curve; Yield curve, Forecasting, ARCH

Contents

1 Introduction

The term structure interest rate of different financial products, including the yield curve of government bonds, is of great interest for portfolio management, derivative pricing, risk management and several other aspects in the world of finance. Therefore, forecasting the yield curve correctly is very important. Diebold and Li (2006) provide a powerful method to do such forecasts. This method is based on the dynamic Nelson-Siegel framework, as proposed by Nelson and Siegel (1987).

Although the method mentioned above seems to work very well, there are still some improvements which can be made. For instance, the variance of the yield is kept constant over time for all different government bonds. As it is questionably whether the volatility of the yield of the different government bonds is constant over time, it is useful to incorporate a volatility component in the DNS model. Koopman, Mallee, and van der Wel (2010) and Hautsch and Ou (2008) capture this time varying volatility by including GARCH components for the variance in their estimations. Their papers show that this gives significant improvements in forecasting the yield curve.

Moreover, including some macroeconomic variables can give an improvement as well, since it is very likely that the yield curve of government bonds is highly correlated with some macroeconomic variables. Diebold, Rudebusch and Aruoba (2006) and van Dijk, Koopman, van der Wel, and Wright(2014) show that including some relevant macroeconomic variables improves the yield forecasts. Exterkate, van Dijk, Heij and Groenen (2013) forecast the yield-curve using a large panel with macroeconomic variables.

Yet, there has not been done very much research on these two modifications of the DNS model. As these modifications of the general DNS framework show some significant improvements in forecasting the yield curve, further research on these modifications and other modifications are very useful. Therefore, in this paper I include different time varying volatility processes into the estimation of the factors of the original DNS framework. Moreover, I will investigate the effect of including several relevant macroeconomic variables on forecasting the yield curve and I will combine this extension with the first one, to check whether this gives some significant improvements as well. Whereas most papers only consider forecasts with an extending out-of sample window, I make forecasts with both an extending out-of sample window and a moving out-of sample window and asses the differences between the two methods. The main question which I try to answer in this paper, is whether these modifications of the dynamic Nelson-Siegel framework improve the forecasts of the yield curve of different government bonds.

The results in this paper show that several used models lead to improvements on forecasting the yield-curve. Despite there is not a model which is superior to the DNS model in all cases, the models which only capture time-varying volatility as an extension seem to outperform the DNS model, as they manage to produce better forecasts in almost all cases. On the other hand, the models in this paper which include some macroeconomic variables do not appear to be very useful.

The remainder of this paper is set up as follows. In section 2 I describe the dynamic Nelson-Siegel model and some natural competitors as described in Diebold and Li (2006). Hereafter, I introduce the modifications on the Nelson-Siegel model and the test to compare the different forecasts of the alternative models with the forecasts of the DNS model. Section 3 provides an overview of the data which I use throughout this paper. This includes both the yield data of government bonds and the macroeconomic variables. In Section 4 I present the forecasting results of the several models and examine their performance. Section 5 concludes.

2 Models & Methodology

In this section, I introduce the different models I use for my research. First I explain the dynamic Nelson-Siegel model (DNS) as well as two natural competitors, which Diebold and Li (2006) use in their paper. Hereafter, I describe extensions of the DNS model regarding time-varying volatility and including macroeconomic variables. Lastly, I introduce the test which I use to compare the forecast accuracy of the DNS model against all alternative models.

2.1 The Nelson-Siegel model

Diebold and Li (2010) use the Nelson and Siegel (1987) curve to fit and forecast the yield curve. The Nelson-Siegel curve includes three different factors, which can be interpret as latent level, slope and curvature factors.The Nelson-Siegel yield curve is defined as

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right) \tag{1}$$

where $y_t(\tau)$ denotes the yield at time t with time to maturity τ, where τ is in months. λ_t can be interpret as the exponential decay rate, where small values of λ_t imply a slow decay and therefore they fit the curve at long maturities better, while large values of λ_t imply a fast decay. Thus, they fit the curve at short maturities more accurately.

β_{1t}, β_{2t} and β_{3t} are defined as three latent dynamic factors. These three factors represent the level, slope and curvature of the yield curve, corresponding to the long-term, medium-term and short-term factors. β_{1t} has a constant loading of 1, and therefore doesn't converge to zero in the limit. Hence, this factor is the long-term factor and it governs the yield curve level. Since this factor is the long-term factor, it is approximately equal to $y_t(120)$. β_{2t} is the short-term factor as the loading on the factor begins at 1 and then decreases to 0 quickly. This factor is related to yield curve slope and $-\beta_{2t}$ is approximately equal to $y_t(120) - y_t(3)$. Lastly, β_{3t} is the medium-term factor and this factor is related to the curvature of the yield curve. This factor is approximately equal to $2y_t(24) - y_t(3) - y_t(120)$. This medium-term factor also determines the value of λ_t which is given above. λ_t takes the value, such that the loading of the medium-term factor is at its maximum. As usually bonds with a maturity of two or three years are referred to as medium-term bonds. Diebold and Li (2006) pick the average of these two bonds (30 months) and estimate the value of λ_t which maximizes the loading of β_{3t} at a maturity of 30 months. Throughout this paper, λ_t equals 0.0609, as proposed by Diebold and Li (2006).

2.2 Diebold and Li

In order to forecast the yield, first the Nelson-Siegel factors are forecasted as univariate AR(1) processes as described in Diebold and Li (2006). The estimated yield-curve is defined as

$$y_{t+h/t}(\tau) = \hat{\beta}_{1,t+h/t} + \hat{\beta}_{2,t+h/t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \hat{\beta}_{3,t+h/t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right) + \epsilon_{t+h}(\tau) \tag{2}$$

where

$$\beta_{i,t+h/t} = \hat{c}_i + \hat{\gamma}_i\beta_{it} + \eta_{i,t+h}, \qquad i = 1,2,3. \tag{3}$$

The residuals $\epsilon_{t+h}(\tau)$ and $\eta_{i,t+h}$ are supposed to be independent distributed with zero mean and constant variance. Moreover τ defines the maturity (months), t is the time and h is the forecasting horizon. In this paper, h is equal to 1, 6 and 12, similar to the forecasting horizons which Diebold and Li (2006) use. The coefficients for the yield factors can be obtained by using ordinary least-squares.

The two natural competitors I include for completeness are the random walk model and the AR(1) on yield levels. The random walk forecasts are described as

$$\hat{y}_{t+h/t}(\tau) = y_t(\tau), \tag{4}$$

whereas the forecasts of the AR(1) on yield levels are generated as

$$\hat{y}_{t+h/t}(\tau) = \hat{c}(\tau) + \hat{\gamma} y_t(\tau). \tag{5}$$

The random walk forecasts are thus always the same as the last observation. The AR(1) on yield levels model are again an autoregressive model. The parameters $\hat{c}(\tau)$ and $\hat{\gamma}$ can be obtained by regressing $\hat{y}_{t+h/t}(\tau)$ on an intercept and $y_t(\tau)$.

2.3 Time-varying volatility and macroeconomic variables

When analyzing the yield rates of government bond, one observes differences in the volatility over time, which may occur due to volatility on the financial markets. As the yield-forecasts only depend on the forecasts of the different beta's, this differing volatility can also be observed in the estimation of the beta's To check whether the residuals in the estimation of β_{it} are indeed different distributed over time, I use a Breusch-Pagan-Godfrey test for heteroskedasticity on all beta's and the different forecasting horizons.

In the DNS model, volatility is constant over time, so I modify the DNS model to capture time-varying volatility. I do this by adjusting the distribution of the error terms in the estimation of β_{it}. It is also possible to incorporate time-varying volatility in the estimation of the yield curve as has been done by Koopman et al. (2010), however less research has been done in modifying the estimation of β_{it}. Since the yield-forecasts directly depend on the forecasts of β_{it}, modifying this estimation should capture heteroskedasticity in the error terms of the yield forecasts as well. It appears that the errors in the estimation of β_{it} only are conditional heteroskedastic for the data set which I use, hence I will capture this feature. Therefore I include only ARCH lags in the models. I treat both symmetric and asymmetric effects of the lagged error terms, as periods of high volatility often begin with large positive or negative errors. The different models which I use are the ARCH, EGARCH and GJR-GARCH models, where the last two do not have GARCH components. Furthermore, instead of direct forecasting, I do iterative forecasts for the 6-month and 12-month forecasting horizons, because this may lead to a reduction in the variance according to Marcellino, Stock and Watson (2006). The estimation of $\beta_{i,t+h}$ becomes

$$\beta_{i,t+h} = \hat{c}_i + \hat{\gamma}_i \hat{\beta}_{i,t+h-1} + \eta_{i,t+h}, \qquad i = 1,2,3, \tag{6}$$

where η_{it} are distributed as $N(0, h_{it}^2)$ variables. h_{it}^2 is modeled as ARCH(1), EGARCH(0,1) and GJR-GARCH(0,1) processes. Due to iterative forecasting, for forecasting horizons 6 and 12. The estimation of a specific $\beta_{i,t+h}$ can be obtained by repeating formula 6 until the right forecasting horizon is achieved.

The different processes for h_{it}^2 are given by:

$$h_{it}^2 = \omega_i + \psi_i \epsilon_{i,t-1}^2 \tag{7}$$

for ARCH(1),

$$log(h_{it}^2) = \omega_i + \theta_i \left(\frac{|\epsilon_{i,t-1}|}{h_{i,t-1}} - E\left[\frac{|\epsilon_{i,t-1}|}{h_{i,t-1}}\right]\right) + \psi_i \left(\frac{|\epsilon_{i,t-1}|}{h_{i,t-1}}\right) \tag{8}$$

for EGARCH(0,1) and

$$h_{it}^2 = \omega_i + \theta_i \epsilon_{i,t-1}^2 I(\epsilon_{i,t-1} < 0) + \psi_i \epsilon_{i,t-1}^2 \tag{9}$$

for GJR-GARCH(0,1).

Moreover, the yield of government bonds is highly correlated with several other macroeconomic aspects. Including relevant macroeconomic variables may capture this correlation. Hence, including macroeconomic variables in the estimation of β_{it} improves the forecasts of β_{it} and therefore also the prediction of the yield curve. This leads to the following equation for $\beta_{i,t+h}$:

$$\beta_{i,t+h} = \hat{c}_i + \hat{\psi}_i X_t + \hat{\gamma}_i \hat{\beta}_{it} + \epsilon_{i,t+h}. \tag{10}$$

The matrix X consists all relevant macroeconomic variables, and $\hat{\psi}_i$ captures the loadings of the macroeconomic variables. All parameters can be estimated by regressing $\hat{\beta}_{i,t+h}$ on an intercept, X_t and $\hat{\beta}_{it}$.

Finally, I combine the two extensions regarding time-varying volatility and the inclusion of macroeconomic variables. In this case, $\beta_{i,t+h}$ is defined as

$$\beta_{i,t+h} = \hat{c}_i + \hat{\psi}_i X_t + \hat{\gamma}_i \hat{\beta}_{it} + \eta_{i,t+h} \tag{11}$$

where X is the matrix with the macroeconomic variables as described above and the innovations $\eta_{i,t+h}$ are modeled as $N(0, h_{it}^2)$ variables, where h_{it}^2 again follows ARCH(1), EGARCH(0,1) and GJR-GARCH(0,1) processes. Here, $\beta_{i,t+h}$ depends on both X_t and β_{it} and because of this, iterative forecasting is not possible as this also requires estimations for X. Thus, the yield-forecasts are are obtained by direct forecasting the latent factors.

In the remainder of this paper, I will refer to all models as described in table 1.

Table 1: Overview of the different models

DNS	Dynamic Nelson-Siegel framework as in Diebold and Li (2006).
RW	Random walk model.
AR-Y	AR(1) on yield levels.
ARCH	Dynamic Nelson-Siegel framework with ARCH(1) innovations in the estimation of β_{it}.
EGARCH	Dynamic Nelson-Siegel framework with EGARCH(0,1) innovations in the estimation of β_{it}.
GJR	Dynamic Nelson-Siegel framework with GJR-GARCH(0,1) innovations in the estimation of β_{it}.
DNS-X	Dynamic Nelson-Siegel framework with macroeconomic variables for β_{it}.
GARCHX	Dynamic Nelson-Siegel framework with ARCH(1) innovations and macroeconomic variables in the estimation of β_{it}.
EGARCHX	Dynamic Nelson-Siegel framework with EGARCH(0,1) innovations and macroeconomic variables in the estimation of β_{it}.
GJRX	Dynamic Nelson-Siegel framework with GJR-GARCH(0,1) innovations and macroeconomic variables in the estimation of β_{it}.

2.4 Forecasting windows and Diebold-Mariano test for equal predictive accuracy

Most papers on forecasting the yield curve only consider cases in which their out-of sample forecasting window is extending. Usually, very old observations are less relevant and therefore I use both an extending window and a moving window for the forecasts with all models and assess the differences.

The forecasts from the different models and with the different windows are tested on whether their forecast accuracy is equal to the DNS model. I use the Diebold-Mariano forecast comparison accuracy test to do this. The test statistic for this test is obtained as follows:

$$DM = \frac{\sqrt{\frac{1}{T}} * \sum_{t=1}^{T} \epsilon_{t,1}^2 - \epsilon_{t,2}^2}{\sigma_{\epsilon_1^2 - \epsilon_2^2}}, \tag{12}$$

and has the following distribution:

$$DM \backsim N(0,1). \tag{13}$$

In 12, T is the number of forecasts, $\epsilon_{t,1}$ are the residuals of the DNS forecasts and $\epsilon_{t,2}$ are the residuals of the alternative model. $\sigma_{\epsilon_1^2 - \epsilon_2^2}$ is the sample variance for the difference of the squared residuals, and this variance is corrected for autocorrelations at multi-period ahead forecasts. Negative values of DM indicate superiority of the DNS model, whereas positive values suggest outperformance by the alternative model. Significant superiority of the forecasts of one model against the other model is tested at a 5 percent level on each tail, similar to a test statistic of 1.65.

3 Data

This section provides an overview for both the yield data and the data of the macroeconomic variables which I include in the DNS model.

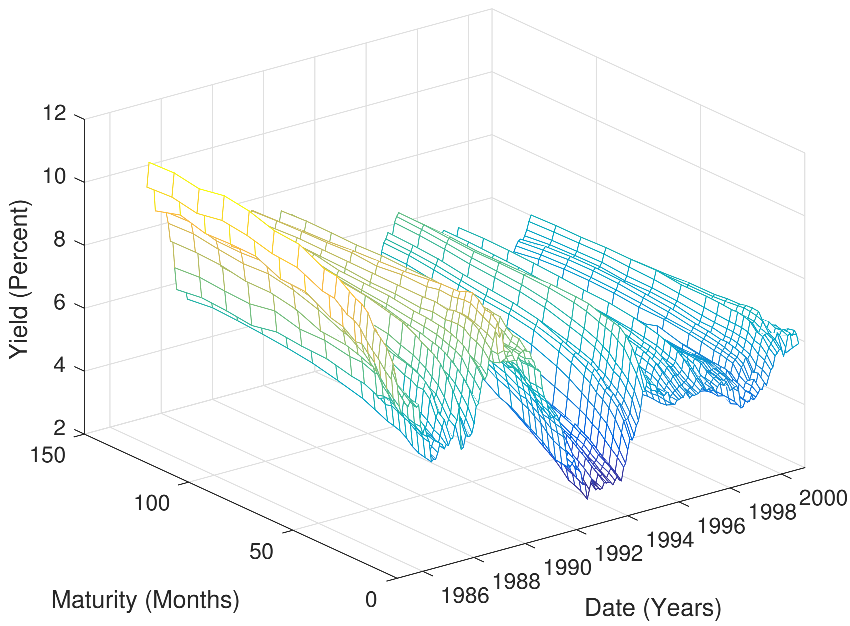

Figure 1: plot of the yield curves for the different maturities across time

3.1 Yield data

Throughout this paper, I make use of end-of-month price quotes for U.S. Treasuries from January 1985 until December 2000, equal to the horizon which Diebold and Li (2006) use. I obtain this data by making use of the data set which is used by van Dijk et al. (2014). This data set contains zero yield bonds from the CRSP unsmoothed Fama and Bliss (1987) forward rates. I use data of 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months government bond yields. Table 2 provides descriptive statistics for this data set. Besides the yields for the different maturities, the level, slope and curvature are also included. It is obvious that the yield curve increases as the time-to-maturity increases. Something which is reflected by the positive mean of the slope as well. Moreover, long-term yields are less volatile than short-term and they obviously are more stable over a long time. One can also see that there is quite some difference between the minimum yield and maximum yield throughout this period, indicating that the yield of the different maturities is volatile over time. This is also illustrated by figure 1. It also appears that the different yields are both less persistent and less variable relative relative to their mean when their maturity increases. Figures 2 - 4 display the relation between β_1, β_2 and β_3 and the level, slope and curvature.

Table 2: Descriptive statistics, yield US government bonds '85-'00

Maturity (months)	Mean	Std. dev.	Min.	Max.	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$
3	5.630	1.488	2.732	9.130	0.978	0.569	-0.079
6	5.786	1.483	2.891	9.324	0.975	0.555	-0.039
9	5.907	1.492	2.984	9.343	0.973	0.545	-0.005
12	6.052	1.494	3.062	9.559	0.969	0.535	0.019
15	6.214	1.501	3.278	9.954	0.968	0.528	0.057
18	6.296	1.493	3.467	10.145	0.966	0.514	0.085
21	6.361	1.480	3.617	10.230	0.964	0.505	0.111
24	6.395	1.465	3.753	10.371	0.961	0.486	0.135
30	6.525	1.458	4.003	10.661	0.958	0.481	0.179
36	6.614	1.435	4.168	10.715	0.957	0.474	0.213
48	6.795	1.428	4.314	11.126	0.952	0.460	0.277
60	6.882	1.415	4.343	11.196	0.952	0.462	0.322
72	7.005	1.432	4.372	11.450	0.953	0.462	0.356
84	7.064	1.413	4.348	11.571	0.952	0.459	0.373
96	7.131	1.405	4.415	11.425	0.954	0.468	0.393
108	7.173	1.424	4.414	11.562	0.954	0.466	0.401
120 (level)	7.150	1.411	4.419	11.531	0.953	0.460	0.404
Slope	1.520	1.114	-0.759	3.716	0.958	0.372	-0.044
Curvature	0.010	0.574	-1.546	1.590	0.876	0.264	0.000

This table shows the descriptive statistics for the yields at different maturities as well as the level, slope and curvature. I define the level as the 10-year government bond yield, the slope as the 10-year government bond yield minus the 3-month government bond yield and the curvature as two times the 2-year government bond yield minus the sum of the 10-year and the 3-month government bond yields. The last three columns show sample autocorrelations at different displacements.

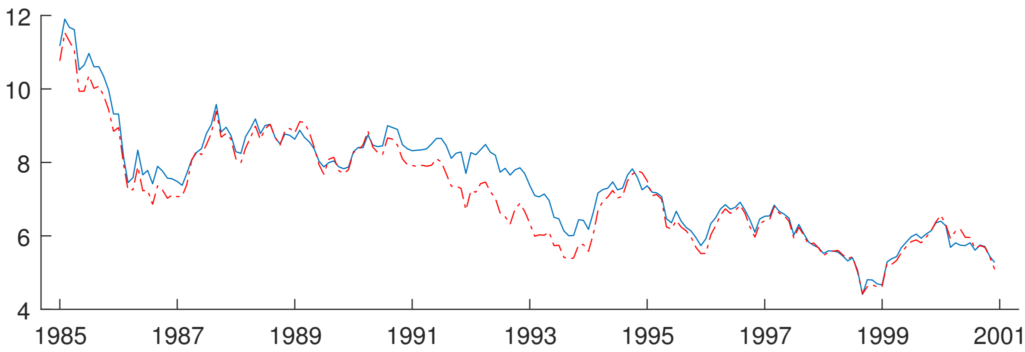

Figure 2: $\hat{\beta}_{1t}$ (solid line), Level (dotted line)

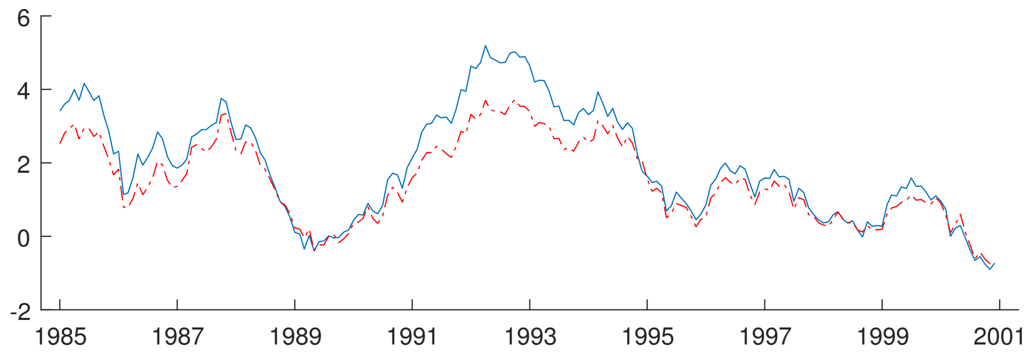

Figure 3: -$\hat{\beta}_{2t}$ (solid line), Slope (dotted line)

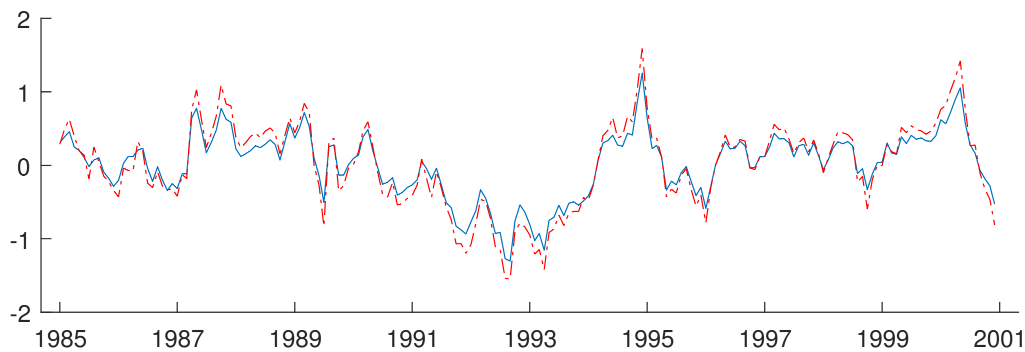

Figure 4: $0.3\hat{\beta}_{3t}$ (solid line), Curvature (dotted line)

3.2 Macroeconomic data

I use the FRED database to obtain monthly data of the US regarding several macroeconomic variables[1]. I make use of data from January 1985 until December 2000, similar to the period of the yield data. The different macroeconomic variables can be found in table 3. As most macroeconomic variables are non-stationary data sets, I transform the data with log transformations, such that I get year-on-year differences. Table 4 shows the correlations of the different macroeconomic variables with each other and β_1, β_2 and β_3.

[1]This data is available on research.stlouisfed.org/econ/mccracken/fred-databases/

Table 3: Macroeconomic variables

Abbreviation	Description	Data transformation
RPI	Real personal income	Log transformation
UNEMP	Unemployment rate	Real rate
IPI	Industrial production index	Log transformation
CONS	Real consumption	Log transformation
FED	Federal Funds rate	Real rate
PPI	Producer price index	Log transformation
CPI	Consumer price index	Log transformation

Table 4: Correlation factors β_1, β_2, β_3 and all macroeconomic variables

	β_1	β_2	β_3	RPI	UNEMP	IPI	CONS	FED	PPI	CPI
β_1	1.000	-0.483	0.023	-0.382	0.618	-0.410	-0.254	0.534	0.199	0.582
β_2		1.000	0.420	0.405	-0.829	0.110	0.157	0.465	0.299	0.008
β_3			1.000	0.329	-0.539	0.401	0.291	0.423	0.124	0.019
RPI				1.000	-0.509	0.591	0.698	0.021	-0.282	-0.500
UNEMP					1.000	-0.463	-0.320	-0.162	-0.258	0.112
IPI						1.000	0.446	-0.299	-0.223	-0.404
CONS							1.000	-0.088	-0.351	-0.536
FED								1.000	0.446	0.568
PPI									1.000	0.842
CPI										1.000

4 Results

In this section, first I show the results of the Breusch-Pagan-Godfrey test and hereafter I assess the different forecasting results, both on their RMSE and their Diebold-Mariano statistics, which I obtain from the different models and the different forecasting windows. More details on the forecasting results are in the Appendix.

4.1 Heteroskedasticity test on β_{it}

Table 5 shows the p-values for the Breusch-Pagan-Godfrey tests for heteroskedasticity for the error terms of the three beta's on the three different forecasting horizons. The tests on the residuals of β_1 rejects the assumption of homoskedasticity for the residuals at 6-month-ahead forecasts. The test on β_2 rejects the assumption of homoskedasticity for all different horizons. Lastly, the test rejects homoskedasticity for the error terms of β_3 at horizons h=1 and h=6.

Table 5: Breusch-Pagan-Godfrey test for heteroskedasticity for β

Horizon	β_1	β_2	β_3
1	0.001	0.433	0.946
6	0.203	0.141	0.793
12	0.022	0.094	0.001

p-values for the Breusch-Pagan-Godfrey test for heteroskedasticity, p-values larger than 0.05 indicate heteroskedasticity.

4.2 Forecasting results with an extending out-of sample window

Table 6: Ratio of root mean square prediction errors of government bond yield 1-month-ahead forecasts with an extending window with respect to the root mean square prediction errors of the DNS model

Maturity(τ)	3	12	36	60	120
DNS	1	1	1	1	1
RW	1.061	0.995	1.008	0.966	0.998
AR-Y	1.074	0.990	0.999	**0.961**	1.002
ARCH	0.967	1.016	0.986	0.972	0.990
EGARCH	0.954	1.003	0.980	**0.968**	0.987
GJR	0.967	1.015	0.986	0.972	0.990
DNS-X	*1.291*	1.077	*1.137*	*1.104*	*1.068*
ARCHX	*2.681*	*2.280*	*2.174*	*2.226*	*2.388*
EGARCHX	*2.374*	*2.117*	*2.017*	*2.057*	*2.229*
GJR-X	*2.611*	*2.253*	*2.188*	*2.248*	*2.409*

Significant outperformances of the DNS model are in bold whereas significant outperformances by the DNS model are in italic.

Table 7: Ratio of root mean square prediction errors of government bond yield 6-month-ahead forecasts with an extending window with respect to the root mean square prediction errors of the DNS model

Maturity(τ)	3	12	36	60	120
DNS	1	1	1	1	1
RW	1.241	1.248	1.232	1.157	1.117
AR-Y	1.179	1.141	1.084	1.027	1.040
ARCH	0.997	1.002	0.960	0.942	0.924
EGARCH	**0.946**	**0.956**	**0.948**	**0.948**	**0.947**
GJR	1.040	1.006	**0.970**	**0.962**	**0.950**
DNS-X	1.729	1.557	*1.454*	1.349	1.254
ARCHX	1.452	1.448	1.340	1.264	1.209
EGARCHX	1.326	1.395	1.302	1.224	1.168
GJR-X	1.350	1.351	1.259	1.196	1.148

Significant outperformances of the DNS model are in bold whereas significant outperformances by the DNS model are in italic.

Table 8: Ratio of root mean square prediction errors of government bond yield 12-month-ahead forecasts with an extending window with respect to the root mean square prediction errors of the DNS model

Maturity (τ)	3	12	36	60	120
DNS	1	1	1	1	1
RW	1.482	1.638	1.448	1.263	1.109
AR-Y	1.154	1.275	1.093	0.987	1.000
ARCH	0.973	1.016	0.904	0.855	0.807
EGARCH	0.905	0.953	0.893	0.866	**0.841**
GJR	0.991	1.011	0.917	0.880	0.844
DNS-X	1.797	1.823	1.564	1.363	1.160
ARCHX	1.330	1.390	1.162	1.021	0.903
EGARCHX	1.248	1.322	1.107	0.975	0.868
GJR-X	1.229	1.307	1.115	0.990	0.888

Significant outperformances of the DNS model are in bold whereas significant outperformances by the DNS model are in italic.

I start by assessing the root mean square prediction errors (RMSE) of the different government bond yields at 1-month-ahead forecasts with an extending out-of sample forecasting window. First of all, one can observe that the differences between the DNS model and the two natural competitors are very small. The models with time-varying volatility in the estimation of β_{it} seem to perform somewhat better than the DNS model. The performance is just slightly better, reflected by the fact that except for one, all of them do not have significant better forecasts according to the Diebold-Mariano test. Only the yield curve at a maturity of 12 months has a higher RMSE when using these models. Looking at these models, the EGARCH model has the lowest RMSE for all different maturities and therefore seems to be the most useful in this case. On the other hand, the models with macroeconomic variables perform pretty bad in comparison with the DNS model. The difference between the regular DNS model and the DNS model with macroeconomic variables is not very large with respect to the RMSE, however the Diebold-Mariano test shows that at four of the five different maturities the forecast are significant worse. The models with both heteroskedastic error terms and macroeconomic variables in the estimation of the latent dynamic factors have much worse forecasts than the DNS model. The RMSE of these models is more than twice as large as the DNS model and all of them are significantly outperformed by the DNS model.

Continuing with the 6-month-ahead forecasts, there appears to be both some differences and some similarities. The random walk model and the AR(1) on yield levels do no perform very good in comparison to the DNS model. Once again, the models which only capture heteroskedasticity seem to give the best forecasting results for the yield curve. The EGARCH model has a lower RMSE than the DNS model in all cases both in absolute as in relative terms. Moreover the ARCH and GJR model outperform the DNS model for medium term and long term yield, but only the GJR model has significant outperformances. The results for the models with macroeconomic variables are quite similar to the results from the 1-month-ahead forecasts. None of them outperforms the DNS model at any maturity. The DNS-X model even performs somewhat worse than in the previous case, whereas the models with macroeconomic variables and heteroskedastic error terms perform somewhat better with respect to the DNS model, however this is mainly due to the increase of the RMSE of the DNS model. Since the Diebold-Mariano test takes into account autocorrelations of the forecasting errors, most of these forecasts are not significantly worse than the DNS model.

The prediction results for the 12-month-ahead forecasts with an extending window show again both some differences and similarities in comparison with the results of the forecasts with other forecasting horizons. The random walk model has bad forecasting results, especially for the short- and medium term yield-curves. Since these yields are the most volatile, it is not surprising that the random walk model does not give good 12-month-ahead forecast. The models which capture heteroskedasticity in the estimation of β_{it} outperform the DNS model in terms of RMSE again in almost all cases. The EGARCH model is the only model which produces better forecasts than the DNS model for all different maturities and seems the best model, despite the ARCH model has a lower RMSE for the two longest maturities, which is in line with the 6-month-ahead forecasts. As the Diebold-Mariano test takes autocorrelations of the error terms into account, the result of the forecasts for the longest maturity with the EGARCH model is the only significant result. Lastly, the models with macroeconomic variables in the estimation of the latent factors show some surprising results. Whereas the DNS-X model gives bad forecasts for all different maturities, the models with both macroeconomic variables and heteroskedastic error terms outperform the DNS model for long term yields (except for the the ARCHX model at 60-month maturity yields).

4.3 Forecasting results with a moving out-of sample window

Table 9: Average root mean square error ratio of forecasts with a moving out-of sample window compared to an extending out-of sample window

Forecast horizon	1	6	12
DNS	0.980	0.950	0.889
RW	1.000	1.000	1.000
AR-Y	1.000	1.000	0.985
ARCH	1.002	0.966	0.925
EGARCH	1.003	0.964	0.926
GJR	1.002	0.942	0.900
DNS-X	0.943	0.794	0.754
ARCHX	0.918	0.955	0.961
EGARCHX	0.910	0.914	1.034
GJR-X	0.936	0.993	1.030

I first discuss the forecasting results of the moving out-of sample window in comparison to the forecasting results with an extending out-of sample window before I assess the results of the forecasts with a moving out-of sample window separately. Table 9 shows that the DNS model has better forecasts when using a moving out-of sample window instead of an extending out-of sample window, especially at forecasting horizons h=6 and h=12. The models which capture heteroskedasticity of the error terms in the estimation of β_{it} perform better when forecasting 6- or 12-month-ahead. The 1-month-ahead forecasts seem to be slightly worse for these models. The models with macroeconomic variables show quite some improvements as well, except for the EGARCHX and GJR-X model at 12-month-ahead forecasts.

Table 10: Ratio of root mean square prediction errors of government bond yield 1-month-ahead forecasts with a moving window with respect to the root mean square prediction errors of the DNS model

Maturity(τ)	3	12	36	60	120
DNS	1	1	1	1	1
RW	1.096	1.008	1.021	0.988	1.019
AR-Y	*1.126*	1.013	1.008	0.975	1.010
ARCH	1.000	1.028	1.005	0.997	1.013
EGARCH	0.989	1.014	1.001	0.994	1.010
GJR	0.999	1.026	1.005	0.997	1.013
DNS-X	1.124	1.026	*1.136*	*1.103*	1.056
ARCHX	*2.741*	*2.120*	*2.031*	*2.057*	*2.088*
EGARCHX	*5.235*	*4.073*	*3.491*	*3.303*	*3.311*
GJR-X	*5.153*	*4.029*	*3.516*	*3.354*	*3.385*

Significant outperformances of the DNS model are in bold whereas significant outperformances by the DNS model are in italic.

Table 11: Ratio of root mean square prediction errors of government bond yield 6-month-ahead forecasts with a moving window with respect to the root mean square prediction errors of the DNS model

Maturity(τ)	3	12	36	60	120
DNS	1	1	1	1	1
RW	1.318	1.276	1.274	1.224	1.216
AR-Y	1.299	1.212	1.124	1.056	1.073
ARCH	0.994	1.004	0.976	0.969	0.961
EGARCH	**0.934**	0.962	0.966	0.972	0.980
GJR	0.991	0.998	0.971	0.965	0.958
DNS-X	1.354	1.237	1.234	1.175	1.109
ARCHX	*1.595*	*1.454*	1.296	1.231	1.188
EGARCHX	1.362	1.314	1.214	1.165	1.118
GJR-X	*1.535*	1.414	1.269	1.212	1.174

Significant outperformances of the DNS model are in bold whereas significant outperformances by the DNS model are in italic.

Table 12: Ratio of root mean square prediction errors of government bond yield 12-month-ahead forecasts with a moving window with respect to the root mean square prediction errors of the DNS model

Maturity(τ)	3	12	36	60	120
DNS	1	1	1	1	1
RW	1.569	1.673	1.635	1.507	1.387
AR-Y	1.271	1.335	1.231	1.134	1.122
ARCH	0.963	0.984	0.949	0.935	0.904
EGARCH	0.906	0.929	0.936	0.943	0.936
GJR	0.961	0.974	0.939	0.926	0.898
DNS-X	1.401	1.445	1.395	1.237	1.026
GARCHX	1.401	1.421	1.261	1.145	1.028
EGARCHX	1.435	1.459	1.296	1.170	1.045
GJR-X	1.436	1.454	1.286	1.167	1.050

Significant outperformances of the DNS model are in bold whereas significant outperformances by the DNS model are in italic.

The results for the 1-month-ahead forecasts with a moving window show that the DNS model, the natural competitors and the models with heteroskedastic error terms in the estimation of the latent factors have a RMSE which do not differ much for all different maturities. This is different from the result of the forecasts with an extending window and this is mainly because the DNS has better forecasts, whereas the other models have forecasting results which are approximately similar to those with an extending window. The DNS model with macroeconomic variables has the biggest improvement by forecasts with a moving window, and the RMSE of the forecasts reflects this, however the model is still worse than the regular DNS model.

The forecasting results for the models which capture both macroeconomic variables and heteroskedasticity in the error terms are in line with the forecasting results with an extending window. The RMSE's for all models are far worse than the DNS model, and the forecasts of the DNS model are significantly superior to the forecasts of these models.

At the 6-month-ahead forecasts, we see some clear differences with respect to the 1-month-ahead forecasts. Both natural competitors produce forecasts which are worse than the forecasts of the DNS model. The models with heteroskedasticity have a lower RMSE in almost every case. The EGARCH model, which has a better RMSE in comparison to the DNS model for all maturities, seems to be the best model overall,despite both the ARCH and the GJR model outperform the EGARCH model slightly at long term yields. The Diebold-Mariano test statistics do only show significant outperformance of the EGARCH model at 3-month maturity. Although the models with macroeconomic variables produce still forecasts which are inferior to the DNS model, the results are far better than the 1-month-ahead forecasting results. Except for the medium- and long term yield forecasts with the ARCHX model, there is no significant RMSE superiority of the DNS model. Once again this might be due to the fact that the Diebold-Mariano test takes into account autocorrelations.

The results for the longest forecasting horizon, when using a moving out-of sample window show results which are in line with the results of 6-month-ahead forecasts with a moving window. The natural competitors do not appear to be good predictors for this case. Moreover, the models with heteroskedastic error terms outperform the DNS model, somewhat more than in the previous case. Finally, the models with macroeconomic variables still do not prove themselves to be very good predictors for the yield curve when using a moving window. This result contradicts the forecasting results with an extending window, as these models manage to outperform the DNS model with their 12-month-ahead forecasts for long-term yields.

5 Conclusion

The scope of this paper is to assess whether including time-varying volatility and relevant macroeconomic variables in the estimation of β_{it} for the DNS model improves the quality of the yield forecasts. First of all, I have shown that the assumption of heteroskedasticity in the regular estimation of β_{it} is appropriate. Thus, including time-varying volatility factors in this estimation is a good way to deal with this problem. Moreover, as the macroeconomic variables which I propose to include in the estimation of β_{it} are correlated with β_{it}, this could be an improvement for yield forecasts as well.

Several constructed models appear to be very good alternatives for forecasting the yield-curves. In the case of forecasting with an extending out-of sample window, the models which capture heteroskedasticity for the error terms in the estimation of β_{it} produce better forecasts than the model introduced by Diebold and Li (2006). Based on the results, one can conclude that the ARCH, EGARCH and GJR models are preferred over the DNS model. Especially the EGARCH model, which outperforms the DNS model in almost every case, is a better alternative to forecast the yield curve. So, asymmetric heteroskedastic error terms seem to increase the quality of the forecasts, which implies that large (negative) errors should be treated different. The Diebold-Mariano test statistics for equal forecast accuracy confirm this result. On the other hand, the macroeconomic variables which I include in the estimation of β_{it} do not appear to be very useful for improving the forecasts of the yield curve. These models are very bad predictors for the yield curve in almost all cases and in the cases where they outperform the DNS model, they are still outperformed by the models which only capture heteroskedasticity. Hence, the macroeconomic variables which I propose to include into the estimation of the latent factors do not improve the quality of forecasts of the yield curve despite they are correlated with these factors.

I have also shown that forecasting the yield curve with a moving out-of sample window instead of an extending out-of sample window lead to improvements of the yield curve forecasts. In many cases, the forecasts with a moving window are much better than those with an extending window. It appears that the oldest data points do lead to larger forecast errors and therefore using a moving out-of sample window should be preferred when forecasting the yield curve.

When we compare the results of the forecasts with a moving window, we can draw similar conclusions as in the case with an extending window. The models which capture heteroskedasticity in the error terms outperform all other models despite the difference with the DNS model is a little bit smaller than in the case with an extending window. Overall, the EGARCH model produces the best forecasts and therefore one can conclude that using this model is the best way to forecast the yield curve. Thus, this heteroskedasticity should be captured with asymmetric processes such as an EGARCH model.

Despite the DNS model seem to have very good yield curve forecasts, I have shown that including time-varying volatility in the estimation of β_{it} improves the forecasts of yield curves. On the other hand, including the macroeconomic variables, which I propose in this paper, in the estimation of β_{it} does not improve the prediction of the yield curve and using these variables in such a way should be rejected. Lastly, using a moving out-of sample does improve the quality of the yield forecasts as well.

Future work might consider the way in which the capture of heteroskedasticity in the error terms of the estimation β_{it} can be improved. Moreover, as using a moving window lead to better forecasts, one could investigate the optimal size of this out-of sample window.

Appendix

Table 13: 1-month-ahead forecasting results with an extending out-of sample window

Maturity (τ)	Mean	Std. dev.	RMSE	$\hat{\rho}(1)$	$\hat{\rho}(12)$
Nelson-Siegel with AR(1) factor dynamics					
3 months	-0.027	0.167	0.168	0.235	0.015
1 year	0.025	0.240	0.240	0.441	-0.217
3 years	-0.040	0.271	0.272	0.334	-0.122
5 years	-0.065	0.276	0.282	0.335	-0.137
10 years	-0.041	0.254	0.256	0.253	-0.126
Random walk					
3 months	0.033	0.177	0.179	0.220	0.053
1 year	0.022	0.239	0.239	0.357	-0.175
3 years	0.008	0.276	0.274	0.346	-0.135
5 years	-0.002	0.274	0.273	0.281	-0.136
10 years	-0.008	0.257	0.255	0.240	-0.151
Univariate AR(1)s on yield levels					
3 months	0.043	0.177	0.181	0.227	0.061
1 year	0.026	0.237	0.237	0.359	-0.168
3 years	-0.003	0.273	0.272	0.349	-0.128
5 years	-0.025	0.272	0.271	0.286	-0.132
10 years	-0.041	0.254	0.256	0.246	-0.148
Nelson-Siegel with AR(1)-ARCH(1) processes for β					
3 months	0.016	0.163	0.163	0.198	0.024
1 year	0.067	0.236	0.244	0.419	-0.217
3 years	0.001	0.270	0.268	0.324	-0.122
5 years	-0.024	0.275	0.274	0.320	-0.137
10 years	0.001	0.255	0.253	0.243	-0.118
Nelson-Siegel with AR(1)-EGARCH(0,1) processes for β					
3 months	0.030	0.159	0.161	0.157	0.055
1 year	0.075	0.230	0.241	0.385	-0.204
3 years	0.003	0.268	0.267	0.309	-0.111
5 years	-0.023	0.274	0.273	0.310	-0.129
10 years	0.002	0.254	0.252	0.236	-0.115
Nelson-Siegel with AR(1)-GJR-GARCH(0,1) processes for β					
3 months	0.016	0.163	0.163	0.198	0.025
1 year	0.067	0.236	0.244	0.417	-0.215
3 years	0.001	0.270	0.268	0.323	-0.120
5 years	-0.024	0.275	0.274	0.319	-0.136
10 years	0.001	0.255	0.253	0.243	-0.118
Nelson-Siegel with macroeconomic variables					
3 months	-0.149	0.159	0.217	0.182	0.208
1 year	-0.104	0.238	0.259	0.461	-0.079
3 years	-0.150	0.272	0.310	0.374	-0.042
5 years	-0.151	0.274	0.312	0.364	-0.101
10 years	-0.093	0.258	0.273	0.321	-0.082
GARCH(0,1) with macroeconomic variables					
3 months	0.008	0.454	0.451	0.728	-0.143
1 year	0.073	0.546	0.547	0.783	-0.200
3 years	0.041	0.594	0.592	0.783	-0.235
5 years	0.038	0.631	0.628	0.802	-0.240
10 years	0.088	0.608	0.611	0.794	-0.297
EGARCH(0,1) with macroeconomic variables					
3 months	0.042	0.400	0.400	0.811	-0.241
1 year	0.106	0.500	0.508	0.837	-0.279
3 years	0.074	0.547	0.549	0.808	-0.278
5 years	0.072	0.579	0.580	0.827	-0.290
10 years	0.123	0.560	0.570	0.816	-0.336
GJR-GARCH(0,1) with macroeconomic variables					
3 months	-0.053	0.439	0.440	0.860	-0.102
1 year	0.021	0.544	0.541	0.859	-0.185
3 years	0.002	0.599	0.596	0.841	-0.234
5 years	0.006	0.638	0.634	0.858	-0.237
10 years	0.062	0.617	0.616	0.854	-0.291

Table 14: 6-month-ahead forecasting results with an extending out-of sample window

Maturity (τ)	Mean	Std. dev.	RMSE	$\hat{\rho}(6)$	$\hat{\rho}(18)$
Nelson-Siegel with AR(1) factor dynamics					
3 months	-0.020	0.488	0.486	0.300	-0.196
1 year	0.018	0.624	0.621	0.177	-0.182
3 years	-0.125	0.699	0.706	0.038	-0.235
5 years	-0.215	0.707	0.735	0.057	-0.270
10 years	-0.273	0.632	0.685	0.035	-0.275
Random walk					
3 months	0.220	0.564	0.603	0.381	-0.214
1 year	0.181	0.758	0.775	0.157	-0.155
3 years	0.102	0.868	0.869	0.022	-0.209
5 years	0.054	0.853	0.850	0.011	-0.245
10 years	0.001	0.769	0.765	0.019	-0.252
Univariate AR(1)s on yield levels					
3 months	0.224	0.530	0.573	0.414	-0.217
1 year	0.158	0.694	0.708	0.196	-0.162
3 years	-0.020	0.769	0.765	0.062	-0.215
5 years	-0.141	0.746	0.755	0.049	-0.256
10 years	-0.266	0.665	0.712	0.059	-0.265
Nelson-Siegel with AR(1)-ARCH(1) processes for β					
3 months	0.018	0.486	0.484	0.327	-0.241
1 year	0.067	0.622	0.622	0.203	-0.193
3 years	-0.054	0.679	0.677	0.047	-0.233
5 years	-0.132	0.684	0.693	0.056	-0.264
10 years	-0.175	0.611	0.633	0.029	-0.264
Nelson-Siegel with AR(1)-EGARCH(0,1) processes for β					
3 months	0.002	0.462	0.459	0.292	-0.222
1 year	0.029	0.596	0.593	0.177	-0.184
3 years	-0.116	0.662	0.669	0.034	-0.234
5 years	-0.202	0.671	0.697	0.050	-0.268
10 years	-0.249	0.602	0.649	0.032	-0.264
Nelson-Siegel with AR(1)-GJR-GARCH(0,1) processes for β					
3 months	-0.052	0.505	0.505	0.333	-0.236
1 year	-0.004	0.628	0.625	0.222	-0.199
3 years	-0.124	0.677	0.684	0.068	-0.238
5 years	-0.202	0.681	0.707	0.080	-0.268
10 years	-0.246	0.606	0.650	0.051	-0.273
Nelson-Siegel with macroeconomic variables					
3 months	-0.570	0.620	0.839	0.469	-0.215
1 year	-0.525	0.816	0.967	0.367	-0.295
3 years	-0.531	0.883	1.026	0.263	-0.404
5 years	-0.490	0.867	0.992	0.254	-0.456
10 years	-0.379	0.775	0.859	0.238	-0.472
ARCH(1) with macroeconomic variables					
3 months	-0.003	0.709	0.705	0.450	-0.249
1 year	0.118	0.897	0.899	0.383	-0.244
3 years	0.083	0.948	0.946	0.289	-0.291
5 years	0.033	0.934	0.929	0.268	-0.315
10 years	0.007	0.833	0.828	0.225	-0.316
EGARCH(0,1) with macroeconomic variables					
3 months	0.124	0.635	0.644	0.469	-0.249
1 year	0.226	0.841	0.866	0.403	-0.245
3 years	0.163	0.909	0.918	0.325	-0.299
5 years	0.100	0.899	0.899	0.307	-0.335
10 years	0.063	0.802	0.799	0.279	-0.334
GJR-GARCH (0,1) with macroeconomic variables					
3 months	-0.003	0.659	0.655	0.422	-0.207
1 year	0.120	0.835	0.838	0.378	-0.223
3 years	0.086	0.889	0.888	0.300	-0.284
5 years	0.035	0.884	0.879	0.290	-0.315
10 years	0.009	0.790	0.786	0.258	-0.318

Table 15: 12-month-ahead forecasting results with an extending out-of sample window

Maturity (τ)	Mean	Std. dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
Nelson-Siegel with AR(1) factor dynamics					
3 months	-0.133	0.675	0.684	-0.238	0.025
1 year	-0.118	0.724	0.729	-0.286	-0.008
3 years	-0.347	0.778	0.848	-0.368	0.008
5 years	-0.505	0.790	0.934	-0.376	-0.004
10 years	-0.644	0.709	0.955	-0.415	-0.009
Random walk					
3 months	0.416	0.930	1.013	-0.118	-0.109
1 year	0.383	1.138	1.195	-0.274	-0.020
3 years	0.244	1.211	1.228	-0.413	0.056
5 years	0.141	1.178	1.179	-0.475	0.070
10 years	0.011	1.066	1.059	-0.498	0.072
Univariate AR(1)s on yield levels					
3 months	0.269	0.747	0.789	-0.111	-0.118
1 year	0.203	0.913	0.930	-0.224	-0.037
3 years	-0.092	0.928	0.927	-0.339	0.038
5 years	-0.311	0.873	0.922	-0.405	0.044
10 years	-0.605	0.744	0.955	-0.419	0.018
Nelson-Siegel with AR(1)-ARCH(1) processes for β					
3 months	0.035	0.669	0.666	-0.238	-0.087
1 year	0.073	0.742	0.741	-0.265	-0.047
3 years	-0.105	0.764	0.767	-0.381	0.025
5 years	-0.230	0.769	0.798	-0.413	0.021
10 years	-0.332	0.700	0.771	-0.462	0.034
Nelson-Siegel with AR(1)-EGARCH(0,1) processes for β					
3 months	0.009	0.622	0.619	-0.304	-0.052
1 year	0.019	0.699	0.696	-0.304	-0.025
3 years	-0.190	0.737	0.757	-0.392	0.036
5 years	-0.325	0.745	0.809	-0.416	0.029
10 years	-0.432	0.681	0.803	-0.452	0.041
Nelson-Siegel with AR(1)-GJR-GARCH(0,1) processes for β					
3 months	-0.058	0.679	0.678	-0.220	-0.091
1 year	-0.020	0.742	0.738	-0.236	-0.060
3 years	-0.197	0.757	0.777	-0.346	0.011
5 years	-0.322	0.761	0.822	-0.374	0.007
10 years	-0.424	0.689	0.806	-0.425	0.016
Nelson-Siegel with macroeconomic variables					
3 months	-0.631	1.061	1.229	-0.215	-0.162
1 year	-0.646	1.169	1.330	-0.304	-0.113
3 years	-0.738	1.109	1.326	-0.355	-0.078
5 years	-0.735	1.045	1.273	-0.345	-0.074
10 years	-0.659	0.896	1.108	-0.321	-0.065
ARCH(1) with macroeconomic variables					
3 months	0.093	0.911	0.910	0.020	-0.155
1 year	0.146	1.010	1.014	-0.040	-0.108
3 years	-0.008	0.992	0.986	-0.155	-0.060
5 years	-0.123	0.951	0.953	-0.185	-0.073
10 years	-0.214	0.840	0.862	-0.180	-0.099
EGARCH(0,1) with macroeconomic variables					
3 months	0.317	0.798	0.854	-0.074	-0.135
1 year	0.315	0.917	0.964	-0.099	-0.095
3 years	0.085	0.940	0.938	-0.190	-0.041
5 years	-0.061	0.914	0.911	-0.217	-0.054
10 years	-0.181	0.814	0.829	-0.217	-0.079
GJR-GARCH(0,1) with macroeconomic variables					
3 months	0.121	0.837	0.840	-0.052	-0.136
1 year	0.162	0.945	0.954	-0.085	-0.093
3 years	-0.009	0.951	0.945	-0.185	-0.044
5 years	-0.132	0.921	0.925	-0.208	-0.059
10 years	-0.230	0.821	0.848	-0.197	-0.085

Table 16: 1-month-ahead forecasting results with a moving out-of sample window

Maturity (τ)	Mean	Std. dev.	RMSE	$\hat{\rho}(1)$	$\hat{\rho}(12)$
Nelson-Siegel with AR(1) factor dynamics					
3 months	-0.007	0.164	0.163	0.194	0.054
1 year	0.045	0.234	0.237	0.403	-0.220
3 years	-0.020	0.270	0.269	0.320	-0.119
5 years	-0.045	0.274	0.276	0.315	-0.135
10 years	-0.020	0.251	0.250	0.227	-0.134
Random walk					
3 months	0.033	0.177	0.179	0.220	0.053
1 year	0.022	0.239	0.239	0.357	-0.175
3 years	0.008	0.276	0.274	0.346	-0.135
5 years	-0.002	0.274	0.273	0.281	-0.136
10 years	-0.008	0.257	0.255	0.240	-0.151
Univariate AR(1)s on yield levels					
3 months	0.047	0.179	0.184	0.240	0.040
1 year	0.034	0.239	0.240	0.364	-0.180
3 years	0.010	0.272	0.271	0.347	-0.134
5 years	-0.010	0.270	0.269	0.279	-0.136
10 years	-0.026	0.253	0.253	0.236	-0.154
Nelson-Siegel with AR(1)-ARCH(1) processes for β					
3 months	0.014	0.163	0.163	0.196	0.021
1 year	0.064	0.236	0.243	0.414	-0.216
3 years	-0.001	0.272	0.270	0.326	-0.122
5 years	-0.025	0.275	0.275	0.318	-0.139
10 years	0.000	0.255	0.254	0.246	-0.119
Nelson-Siegel with AR(1)-EGARCH(0,1) processes for β					
3 months	0.019	0.161	0.161	0.175	0.036
1 year	0.067	0.232	0.240	0.393	-0.212
3 years	-0.001	0.271	0.269	0.318	-0.117
5 years	-0.026	0.275	0.274	0.313	-0.135
10 years	0.000	0.255	0.253	0.241	-0.119
Nelson-Siegel with AR(1)-GJR-GARCH(0,1) processes for β					
3 months	0.016	0.163	0.163	0.198	0.025
1 year	0.067	0.236	0.244	0.417	-0.215
3 years	0.001	0.270	0.268	0.323	-0.120
5 years	-0.024	0.275	0.274	0.319	-0.136
10 years	0.001	0.255	0.253	0.243	-0.118
Nelson-Siegel with macroeconomic variables					
3 months	-0.096	0.157	0.183	0.123	0.279
1 year	-0.075	0.232	0.243	0.390	0.042
3 years	-0.139	0.274	0.305	0.317	0.071
5 years	-0.138	0.273	0.304	0.294	0.009
10 years	-0.072	0.256	0.265	0.244	0.017
ARCH(1) with macroeconomic variables					
3 months	-0.212	0.396	0.447	0.803	0.054
1 year	-0.173	0.474	0.502	0.811	-0.091
3 years	-0.232	0.497	0.546	0.775	-0.147
5 years	-0.240	0.517	0.567	0.795	-0.153
10 years	-0.190	0.490	0.523	0.786	-0.212
EGARCH(0,1) with macroeconomic variables					
3 months	-0.095	0.376	0.386	0.726	0.034
1 year	-0.069	0.450	0.453	0.775	-0.140
3 years	-0.141	0.486	0.504	0.762	-0.204
5 years	-0.152	0.508	0.528	0.793	-0.226
10 years	-0.103	0.487	0.495	0.788	-0.291
GJR(0,1) with macroeconomic variables					
3 months	-0.200	0.409	0.454	0.810	0.046
1 year	-0.167	0.484	0.509	0.824	-0.104
3 years	-0.232	0.508	0.556	0.791	-0.152
5 years	-0.242	0.527	0.577	0.809	-0.159
10 years	-0.193	0.499	0.532	0.796	-0.214

Table 17: 6-month-ahead forecasting results with a moving out-of sample window

Maturity (τ)	Mean	Std. dev.	RMSE	$\hat{\rho}(6)$	$\hat{\rho}(18)$
Nelson-Siegel with AR(1) factor dynamics					
3 months	0.124	0.443	0.457	0.093	-0.009
1 year	0.169	0.587	0.607	-0.003	-0.065
3 years	0.032	0.685	0.682	-0.082	-0.178
5 years	-0.058	0.696	0.695	-0.040	-0.229
10 years	-0.116	0.622	0.629	-0.062	-0.258
Random walk					
3 months	0.220	0.564	0.603	0.381	-0.214
1 year	0.181	0.758	0.775	0.157	-0.155
3 years	0.102	0.868	0.869	0.022	-0.209
5 years	0.054	0.853	0.850	0.011	-0.245
10 years	0.001	0.769	0.765	0.019	-0.252
Univariate AR(1) on yield levels					
3 months	0.278	0.528	0.594	0.382	-0.245
1 year	0.229	0.704	0.736	0.176	-0.176
3 years	0.064	0.768	0.766	0.062	-0.214
5 years	-0.059	0.735	0.733	0.037	-0.251
10 years	-0.177	0.655	0.675	0.002	-0.266
Nelson-Siegel with AR(1)-ARCH(1) processes for β					
3 months	0.123	0.440	0.455	0.223	-0.150
1 year	0.174	0.588	0.610	0.110	-0.143
3 years	0.056	0.667	0.666	-0.018	-0.220
5 years	-0.022	0.676	0.673	0.008	-0.259
10 years	-0.065	0.604	0.604	-0.016	-0.277
Nelson-Siegel with AR(1)-EGARCH(0,1) processes for β					
3 months	0.108	0.416	0.427	0.190	-0.068
1 year	0.147	0.569	0.584	0.084	-0.101
3 years	0.017	0.663	0.659	-0.033	-0.205
5 years	-0.064	0.676	0.675	-0.002	-0.249
10 years	-0.109	0.610	0.616	-0.018	-0.263
Nelson-Siegel with AR(1)-GJR-GARCH(0,1) processes for β					
3 months	-0.052	0.505	0.505	0.333	-0.236
1 year	-0.004	0.628	0.625	0.222	-0.199
3 years	-0.124	0.677	0.684	0.068	-0.238
5 years	-0.202	0.681	0.707	0.080	-0.268
10 years	-0.246	0.606	0.650	0.051	-0.273
Nelson-Siegel with macroeconomic variables					
3 months	-0.396	0.479	0.619	0.229	-0.186
1 year	-0.377	0.653	0.751	0.200	-0.275
3 years	-0.420	0.734	0.842	0.108	-0.361
5 years	-0.396	0.718	0.816	0.114	-0.425
10 years	-0.299	0.634	0.698	0.086	-0.452
ARCH(1) with macroeconomic variables					
3 months	-0.011	0.734	0.729	0.396	-0.226
1 year	0.040	0.887	0.882	0.349	-0.243
3 years	-0.078	0.886	0.884	0.276	-0.290
5 years	-0.154	0.846	0.855	0.261	-0.318
10 years	-0.196	0.725	0.747	0.215	-0.320
EGARCH(0,1) with macroeconomic variables					
3 months	0.097	0.619	0.623	0.434	-0.300
1 year	0.127	0.793	0.798	0.358	-0.268
3 years	-0.005	0.833	0.828	0.272	-0.308
5 years	-0.080	0.810	0.809	0.261	-0.341
10 years	-0.113	0.698	0.703	0.232	-0.349
GJR-GARCH(0,1) with macroeconomic variables					
3 months	0.014	0.706	0.702	0.388	-0.240
1 year	0.061	0.861	0.858	0.348	-0.259
3 years	-0.065	0.868	0.866	0.277	-0.307
5 years	-0.146	0.834	0.842	0.263	-0.338
10 years	-0.193	0.717	0.738	0.217	-0.347

Table 18: 12-month-ahead forecasting results with a moving out-of sample window

Maturity (τ)	Mean	Std. dev.	RMSE	$\hat{\rho}(12)$	$\hat{\rho}(24)$
Nelson-Siegel with AR(1) factor dynamics					
3 months	0.232	0.607	0.646	-0.397	0.173
1 year	0.274	0.663	0.714	-0.452	0.147
3 years	0.060	0.753	0.751	-0.496	0.123
5 years	-0.105	0.780	0.783	-0.486	0.088
10 years	-0.259	0.723	0.764	-0.521	0.068
Random walk					
3 months	0.416	0.930	1.013	-0.118	-0.109
1 year	0.383	1.138	1.195	-0.274	-0.020
3 years	0.244	1.211	1.228	-0.413	0.056
5 years	0.141	1.178	1.179	-0.475	0.070
10 years	0.011	1.066	1.059	-0.498	0.072
Univariate AR(1)s on yield levels					
3 months	0.445	0.694	0.821	-0.230	-0.109
1 year	0.389	0.875	0.953	-0.309	-0.044
3 years	0.063	0.928	0.924	-0.379	0.046
5 years	-0.172	0.876	0.888	-0.430	0.075
10 years	-0.404	0.760	0.857	-0.512	0.093
Nelson-Siegel with AR(1)-ARCH(1) processes for β					
3 months	0.205	0.591	0.622	-0.360	-0.015
1 year	0.252	0.660	0.703	-0.361	0.018
3 years	0.079	0.713	0.713	-0.449	0.089
5 years	-0.050	0.734	0.732	-0.457	0.072
10 years	-0.158	0.676	0.691	-0.509	0.074
Nelson-Siegel with AR(1)-EGARCH(0,1) processes for β					
3 months	0.163	0.565	0.585	-0.341	-0.004
1 year	0.196	0.637	0.663	-0.355	0.041
3 years	0.008	0.707	0.703	-0.449	0.100
5 years	-0.125	0.732	0.738	-0.458	0.076
10 years	-0.235	0.679	0.715	-0.502	0.073
Nelson-Siegel with AR(1)-GJR-GARCH(0,1) processes for β					
3 months	-0.058	0.679	0.678	-0.220	-0.091
1 year	-0.020	0.742	0.738	-0.236	-0.060
3 years	-0.197	0.757	0.777	-0.346	0.011
5 years	-0.322	0.761	0.822	-0.374	0.007
10 years	-0.424	0.689	0.806	-0.425	0.016
Nelson-Siegel with macroeconomic variables					
3 months	-0.531	0.737	0.905	-0.380	-0.142
1 year	-0.567	0.867	1.031	-0.431	-0.061
3 years	-0.647	0.829	1.048	-0.436	-0.004
5 years	-0.616	0.752	0.969	-0.446	0.004
10 years	-0.497	0.610	0.784	-0.435	0.019
ARCH(1) with macroeconomic variables					
3 months	0.355	0.837	0.905	-0.176	-0.128
1 year	0.336	0.963	1.014	-0.196	-0.078
3 years	0.088	0.949	0.947	-0.253	-0.004
5 years	-0.064	0.900	0.897	-0.273	0.006
10 years	-0.187	0.767	0.785	-0.291	0.019
EGARCH(0,1) with macroeconomic variables					
3 months	0.424	0.830	0.927	-0.116	-0.135
1 year	0.405	0.965	1.042	-0.156	-0.086
3 years	0.157	0.966	0.973	-0.236	-0.013
5 years	0.006	0.922	0.916	-0.273	-0.004
10 years	-0.115	0.794	0.798	-0.302	0.008
GJR-GARCH(0,1) with macroeconomic variables					
3 months	0.378	0.853	0.928	-0.158	-0.133
1 year	0.355	0.981	1.038	-0.188	-0.079
3 years	0.097	0.967	0.966	-0.250	-0.005
5 years	-0.059	0.917	0.914	-0.274	0.007
10 years	-0.186	0.785	0.802	-0.296	0.021

Table 19: Diebold-Mariano forecast comparison statistics for 1-month-ahead forecasts with an extending out-of sample window

Maturity(τ)	3	12	36	60	120
RW	-1.05	0.16	-0.34	1.03	0.09
AR-Y	-1.12	0.33	0.07	1.73	-0.12
ARCH	1.12	-0.85	0.76	1.57	0.49
EGARCH	1.22	-0.16	1.07	1.78	0.65
GJR	1.13	-0.81	0.77	1.58	0.52
DNS-X	-3.97	-1.36	-3.07	-2.98	-2.67
ARCHX	-5.68	-6.49	-7.02	-6.66	-6.63
EGARCHX	-6.20	-6.54	-7.03	-7.01	-7.13
GJR-X	-6.56	-6.80	-7.29	-6.89	-6.75

Negative values indicate a better performance of the original DNS model.

Table 20: Diebold-Mariano forecast comparison statistics for 6-month-ahead forecasts with an extending out-of sample window

Maturity(τ)	3	12	36	60	120
RW	-0.90	-1.27	-1.38	-1.09	-0.85
AR-Y	-0.76	-0.92	-0.86	-0.43	-1.59
ARCH	0.06	-0.05	1.10	1.41	1.51
EGARCH	1.78	2.05	1.99	1.95	1.85
GJR	-0.91	-0.37	1.85	1.94	1.92
DNS-X	-1.52	-1.55	-1.66	-1.61	-1.60
ARCHX	-1.44	-1.42	-1.51	-1.49	-1.28
EGARCHX	-0.97	-1.21	-1.28	-1.22	-1.04
GJR-X	-1.50	-1.48	-1.44	-1.25	-0.92

Negative values indicate a better performance of the original DNS model.

Table 21: Diebold-Mariano forecast comparison statistics for 12-month-ahead forecasts with an extending out-of sample window

Maturity(τ)	3	12	36	60	120
RW	-0.99	-1.14	-1.14	-0.99	-0.59
AR-Y	-0.59	-0.85	-0.60	0.20	-0.01
ARCH	0.22	-0.16	1.05	1.39	1.57
EGARCH	0.92	0.62	1.39	1.56	1.67
GJR	0.07	-0.16	1.31	1.52	1.64
DNS-X	-1.22	-1.41	-1.46	-1.42	-1.18
ARCHX	-0.93	-1.01	-0.91	-0.21	1.02
EGARCHX	-0.65	-0.81	-0.55	0.20	1.15
GJR-X	-0.71	-0.87	-0.70	0.10	1.16

Negative values indicate a better performance of the original DNS model.

Table 22: Diebold-Mariano forecast comparison statistics for 1-month-ahead forecasts with a moving out-of sample window

Maturity(τ)	3	12	36	60	120
RW	-1.53	-0.23	-0.85	0.41	-0.87
AR-Y	-1.85	-0.39	-0.34	1.20	-0.56
ARCH	0.01	-1.54	-0.30	0.20	-0.76
EGARCH	0.49	-0.88	-0.06	0.39	-0.61
GJR	0.03	-1.47	-0.30	0.22	-0.73
DNS-X	-1.62	-0.41	-2.08	-1.82	-1.33
ARCHX	-5.97	-5.79	-5.73	-6.07	-6.23
EGARCHX	-6.38	-6.21	-6.31	-6.80	-7.19
GJR-X	-5.97	-5.82	-5.79	-6.05	-6.05

Negative values indicate a better performance of the original DNS model.

Table 23: Diebold-Mariano forecast comparison statistics for 6-month-ahead forecasts with a moving out-of sample window

Maturity(τ)	3	12	36	60	120
RW	-1.17	-1.30	-1.48	-1.44	-1.50
AR-Y	-1.26	-1.22	-0.93	-0.58	-1.08
ARCH	0.15	-0.09	0.52	0.69	0.80
EGARCH	2.07	0.89	0.76	0.62	0.40
GJR	0.29	0.05	0.62	0.77	0.83
DNS-X	-1.03	-0.86	-1.04	-0.91	-0.77
ARCHX	-2.06	-1.65	-1.43	-1.39	-1.26
EGARCHX	-1.63	-1.35	-1.21	-1.20	-1.04
GJR-X	-1.94	-1.59	-1.37	-1.33	-1.21

Negative values indicate a better performance of the original DNS model.

Table 24: Diebold-Mariano forecast comparison statistics for 12-month-ahead forecasts with a moving out-of sample window

Maturity(τ)	3	12	36	60	120
RW	-1.17	-1.29	-1.44	-1.58	-1.69
AR-Y	-1.13	-1.12	-0.95	-0.81	-1.01
ARCH	0.25	0.28	1.72	1.66	1.72
EGARCH	0.70	1.18	1.40	1.20	1.33
GJR	0.27	0.44	1.79	1.75	1.78
DNS-X	-0.94	-1.20	-1.36	-1.15	-0.29
ARCHX	-1.02	-1.07	-1.17	-1.28	-0.99
EGARCHX	-1.03	-1.07	-1.12	-1.10	-0.53
GJR-X	-1.07	-1.10	-1.21	-1.34	-1.17

Negative values indicate a better performance of the original DNS model.

Table 25: RMSE ratio's of the forecasts with a different out-of sample window. (RMSE Moving window divided by RMSE Extending window)

Maturity(τ)	3	12	36	60	120
Forecasting horizon	DNS				
1	0,969	0,986	0,988	0,977	0,979
6	0,942	0,978	0,967	0,945	0,918
12	0,945	0,979	0,886	0,838	0,800
Forecasting horizon	Random Walk				
1	1,000	1,000	1,000	1,000	1,000
6	1,000	1,000	1,000	1,000	1,000
12	1,000	1,000	1,000	1,000	1,000
Forecasting horizon	AR(1) on yield levels				
1	1,016	1,009	0,997	0,992	0,988
6	1,037	1,040	1,002	0,972	0,947
12	1,041	1,025	0,998	0,963	0,897
Forecasting horizon	AR(1)-ARCH(1)				
1	1,001	0,998	1,006	1,002	1,002
6	0,940	0,980	0,983	0,971	0,955
12	0,935	0,948	0,930	0,917	0,896
Forecasting horizon	AR(1)-EGARCH(0,1)				
1	1,005	0,997	1,009	1,004	1,003
6	0,930	0,984	0,985	0,969	0,950
12	0,946	0,954	0,928	0,913	0,890
Forecasting horizon	AR(1)-GJR-GARCH(0,1)				
1	1,001	0,997	1,006	1,002	1,002
6	0,898	0,970	0,968	0,948	0,927
12	0,916	0,942	0,907	0,882	0,851
Forecasting horizon	AR(1) model with macroeconomic variables				
1	0,843	0,939	0,986	0,976	0,968
6	0,738	0,777	0,821	0,823	0,812
12	0,736	0,776	0,790	0,761	0,708
Forecasting horizon	AR(1)-ARCH(1) model with macroeconomic variables				
1	0,990	0,917	0,923	0,903	0,856
6	1,035	0,982	0,935	0,921	0,902
12	0,995	1,000	0,961	0,940	0,911
Forecasting horizon	AR(1)-EGARCH(0,1) model with macroeconomic variables				
1	0,966	0,891	0,917	0,909	0,868
6	0,968	0,921	0,902	0,900	0,880
12	1,086	1,080	1,037	1,006	0,962
Forecasting horizon	AR(1)-GJR-GARCH(0,1) model with macroeconomic variables				
1	1,032	0,942	0,934	0,911	0,863
6	1,071	1,024	0,975	0,958	0,939
12	1,104	1,088	1,022	0,988	0,946

References

[1] Diebold, F. and C. Li (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics 130*, 337-364.

[2] Diebold, F., G. Rudebusch and S. Aruoba (2006) The macroeconomy and the yield curve. *Journal of Econometrics 131*, 309–338.

[3] Exterkate, P., D. van Dijk, C. Heij, P. Groenen (2013) Forecasting the Yield Curve in a Data-Rich Environment Using the Factor-Augmented Nelson–Siegel Model. *Journal of Forecasting 32*, 193-214.

[4] Fama E, and R. Bliss (1987). The information in long-maturity forward rates. *American Economic Review 77*, 680–692.

[5] Hautsch, N. and Y. Ou (2008). Yield Curve Factors, Yield Volatility, and the Predictability of Bond Excess Returns. *Discussion Paper 2008-053*, CRC 649.

[6] Koopman, S., M. Mallee, and M. van der Wel (2010). Analyzing the term structure of interest rates using Dynamic Nelson-Siegel model with time-varying parameters. *Journal of Business and Economic Statistics 28*, 329-343.

[7] Marcellino, M., J. Stock, and M. Watson (2006). A comparison of direct and iterated multistep AR methods for forecasting macroeconomic time series. *Journal of Econometrics 135*, 499-526

[8] Nelson, C. and A. Siegel (1987). Parsimonious modelling of yield curves. *Journal of Business 60*, 473-489.

[9] Van Dijk, D., S. Koopman, M. van der Wel and J. Wright (2014). Forecasting interest rates with shifting endpoints. *Journal of Applied Econometrics 29*, 693-712.