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Title Thesis: Ambiguity and deviations from standard game theory

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Abstract

Incorporating ambiguity into decision-making analysis creates a more refined and more realistic picture of actual human behavior. In this paper ambiguity is incorporated in to strategic interactions, namely two games from Goeree and Holt (2001). Unfortunately, these games are somewhat abstract and that is why in this paper the futures market is also studied game theoretically with ambiguity. The analysis will make use of the notion of capacities and the Choquet integral. One thing that becomes very clear from the analysis is the fact that ambiguity definitely influences and shapes the behavior of decision makers in strategic interactions. It is also shown that with incorporating ambiguity, observed deviations from Nash equilibria predictions can be explained and new equilibria can be found. As an extra the effect of risk attitudes on ambiguity attitudes is examined as well.

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1 Introduction

An essential part of economic decision-making is making decisions under risk. This means that individuals make decisions in which different scenarios can occur with certain objective probabilities. Within the paradigm of decision-making under risk it is assumed that decision makers know these objective probabilities. The normative model of decision-making that goes along with this view is Expected Utility Theory. However, Expected Utility Theory received a lot of criticism for not being realistic. It is more a model that shows how people should behave when rational, but it does a poor job at predicting how individuals actually behave.

The notion of objective probabilities is of importance here. These probabilities cannot be different for different decision makers. An example of objective probabilities is the probability of getting heads when flipping a fair coin. Many economic situations, in reality, do not have objective probabilities, even if we like them to have. Another form of probabilities is subjective probabilities. Subjective probabilities, as the word subjective says, can vary across different decision makers. Here, a decision maker behaves as if her subjective assessment of different events can be described by a unique probability distribution. An example of a subjective probability assessment is an investor's belief about whether the stock market is going up or down, in the coming week.

Next to objective and subjective probabilities, there is also uncertainty, or alternatively ambiguity, this is when probabilities are not known or agents do no not have full confidence in probabilities. What does ambiguity mean when a decision maker is faced with it? Suppose a decision maker's subjective knowledge about the likelihoods of probable events can be represented by more than one probability distribution (which represents the decision maker's belief), and further, that the decision maker's knowledge does not provide him a precise second-order probability distribution over the set of possible probabilities, then the decision maker's belief about probable events is characterized by ambiguity. In fact for many economic or political situations, it is not obvious why decision makers should know probabilities (Eichberger, Kelsey, & Schipper, 2009). So it may be beneficial and definitely more realistic to try and model individuals' behavior with ambiguity.

Savage's theory of Subjective Expected Utility maximization is one of the most well received paradigms for modeling decision-making under subjective uncertainty in economics (Mukerji, 2000). Savage's theory of subjective expected utility transforms a situation under ambiguity/uncertainty into a situation under risk. Unfortunately, Savage's theory received criticism as well. Most notable of the critics is Ellsberg (1961) with his famous Ellsberg paradox. In Savage's theory a decision maker is ambiguity-neutral, but as the Ellsberg paradox shows individuals in reality are certainly not ambiguity-neutral.

In the famous Ellsberg experiment subjects were presented two urns both containing 100 balls. The balls can either be black or red. In urn I, the composition of the balls is known, 50 black and 50 red balls. The subject has no information about the composition in urn II. So the composition in urn II is ambiguous. Placed in a choice situation, it is plausible that a subject strictly prefers receiving a prize upon drawing a red ball from urn I than receiving the same prize upon drawing a red ball from urn II, the same if red is changed with black. This behavior seems highly reasonable, but unfortunately it is not compatible with the idea that the subject has probabilistic beliefs on the composition of urn II, and this is not compatible with Savage's theory. The choice pattern creates a difficulty to any decision criterion based on probability. Following probabilistic sophistication the subject should be indifferent between betting on urn I or urn II. Since the ball can only be either black or red, a ball has a 50% chance of being black and a 50% chance of being red. There are 100 balls in urn II, thus an individual should expect $0.50 \times 100 = 50$ black balls and thus 50 red balls. Exactly the same as in urn I. However, preferring to bet on urn I instead of urn II implies the following: the individual thinks that the probability of a ball being red drawn from urn II is smaller then $\frac{1}{2}$, however, then the subject should choose betting on urn II drawing black. So we have a contradiction.

The above observation shows that in order to model ambiguity, new methods are needed. The Ellsberg paradox shows that individuals are, most of the time, ambiguity averse. It has been claimed that, while most are indifferent with respect to the color they bet on, they are not indifferent with respect to the urn they choose (Gilboa, 1987). Thus ambiguity plays a significant role in reality, hence the importance of a model, which describes it correctly. Schmeidler (1989) provided an axiomatic approach to model ambiguity. He makes use of capacities, to represent an individual's belief instead of an additive probability distribution, and the Choquet integral.

In standard game theory when players face decisions under risk (probabilities are known), the model that is used in order to predict the behavior of the players is the expected utility model. Interest in expected utility theory was revised thanks to John von Neumann and Oskar Morgenstern (1947), as they used it in their theory of games and economic behavior.

However, as I already mentioned, expected utility theory is not very realistic. This prompted theorist to search for new models to use when individuals face risk. Several new models were created to better describe how individuals behave when faced with risk. Most of them incorporate decision weights instead of just probabilities. There is evidence for the view that individuals have subjective attitudes to probabilities, which are distinct from attitudes to consequences (Starmer, 2000), hence, the use of decision weights.

One well-known model that made use of decision weights is Rank-Dependent Expected Utility Theory by John Quiggin (1982). Another model is Prospect Theory by Daniel Kanheman and Amos Tversky (1979), which also makes use of decision weights, however another important feature of Prospect Theory is its reference point dependence. Gains and losses are weighted differently. Despite these new models the model, which is mainly used in game theoretical modeling is still Expected Utility Theory.

The objective of this paper is to analyze different game theoretical situations where ambiguity is incorporated, in order to describe behavior in these games better. Simultaneously, more realistic equilibria will be derived. First, I will investigate two games from the paper *Ten little treasures of game theory and ten intuitive contradictions* by Goeree and Holt (2001). These games are somewhat abstract, and thus no real life comparison can be made, easily. Therefore, I will also look at a real life situation, namely the futures market. When ambiguity is incorporated it is more easily explained why deviations from Nash equilibrium predictions occur in reality. This is where this paper contributes to existing literature. It incorporates ambiguity into the analysis. This may lead to the fact that Expected Utility Theory may still be usable, despite its criticism. The main question of concern in this paper is: can incorporating ambiguity into the analysis explain deviations from Nash equilibrium predictions?

Another objective of this paper is to combine ambiguity attitudes with risk attitudes. In previous literature, the two have been studied separately, but not together as far as I know. I am interested how risk attitudes affect ambiguity attitudes.

The remainder of this paper goes as follows. In the next section I will describe capacities and the Choquet integral, in more detail. Section 3, contains an exploration of the type of capacity used in the analysis of this paper. Followed by a section devoted to the analysis of the experimental games. In section 5 ambiguity attitudes and risk attitudes are combined. Section 6, provides the analysis of the futures market game. Section 7 provides a discussion. Last the conclusion. In the main text only final values are given, for all derivations and intermediate algebraic steps I would like to point the reader to the appendix.

2 Preliminaries

In this section, I will define the ingredients needed in order to model situations under ambiguity, game theoretically. Let Ω denote the state space, and the elements of this set are called states of nature. Further there is an outcome space denoted by X. This outcome space has elements representing all the possible results of all the conceivable situations. Both Ω and X can be finite or infinite. Here, I will assume that both are finite, so we have the following

$$\Omega = \{\omega_1, \dots, \omega_n\}$$
$$X = \{x_1, \dots, x_n\}$$

An element $\omega \in \Omega$ is called a state of nature. Sets of states of nature, $E \subseteq \Omega$ are called events. Denote the set of events by $\mathcal{E}$. Further, players have preferences over acts, acts are mappings from Ω to X, $f: \Omega \to X$, and denote the set of acts by F. So players have a preference ordering $\succcurlyeq$ over F. Last, players have a utility function $u: X \to \mathbb{R}$.

A game consists of $\langle S_i, u_i \rangle_{i=1,2}$, when there are two players. The utility function of player i represents the payoff to player i depending on his chosen strategy and the strategies chosen by i's opponents, thus $u_i(s_i, s_{-i})$. Here s_i is a strategy of player i and s_{-i} denotes the strategy combination chosen by i's opponents. The set of strategies, of player i is S_i, where $s_i \in S_i$, and the set of strategies of player i's opponents is S_{-i}, where $s_{-i} \in S_{-i}$. The total set of all strategies is denoted by S and is defined as $S = S_i \times S_{-i}$.

In standard game theory, the belief of a player i is represented by an additive probability distribution over the pure strategies of i's opponents. But in the introduction I already mentioned that in many economic situations an agent may not know probabilities and therefore does not have an additive probability distribution to represent his beliefs. In the literature it has been proposed to use capacities to represent a player's belief when faced with ambiguity/uncertainty. A characteristic of capacities in modeling ambiguity is the fact that capacities are not necessarily additive. So a capacity is a non-additive measure of beliefs. A not so formal definition of capacity is that it is a measure of the size of a set. Here is a more formal definition of a capacity.

DEFINITION 2.1: A capacity is a function $v: \mathcal{E} \rightarrow \mathbb{R}$ which assigns real numbers to events, such that

(i.) $E, F \in \mathcal{E}, E \subseteq F$ implies $v(E) \leq v(F)$,

(ii.) $v(\emptyset) = 0$ and $v(\Omega) = 1$.

In order to model preferences under ambiguity, previous literature made use of the Choquet expected utility (CEU) model. In order to use the CEU model we first need to rank the outcomes from best to worst. Hence, for instance:

$$x_1 > \cdots > x_i > \cdots > x_n$$

Then the Choquet expected value of an act becomes:

$$\int f dv = \sum_{x \in f(\omega)}^{n} x \cdot [v(\{\omega | f(\omega) \geq x\}) - v(\{\omega | f(\omega) > x\})]$$

Here, $v(\{\omega | f(\omega) \geq x\})$ means the capacity/belief of the player of obtaining an outcome higher or equal than x in state of nature ω and $v(\{\omega | f(\omega) > x\})$ means the capacity/belief of the player of obtaining an outcome higher than x in state of nature ω. We get the Choquet expected value by multiplying the difference of the two capacities with the outcome x and take the sum of this multiplication over all n possible outcomes. So the Choquet expected utility is similar to Rank-dependent utility, however, the Choquet expected utility is taken with respect to capacities and not a standard probability weighting function.

3 Neo-additive capacities, Choquet expected utility and equilibrium

In the previous section, I discussed general capacities. A special case of capacities is neo-additive capacities, which I will use in this paper. This is because with general capacities a too broad range of behavior can be explained, and thus no accurate predictions can be made.

Neo-additive capacities are additive on non-extreme outcomes, and non-additive on extreme outcomes. Neo-additive capacities are a convex combination of an additive capacity and two capacities, (i.) one reflects full ambiguity, and (ii.) one that reflects full confidence (Chateauneuf, Eichberger & Grant, 2007). So neo-additive capacities can be used to represent an individual whose beliefs are described by an additive probability distribution π, but the decision maker lacks confidence in this belief. I will use the definition of neo-additive capacities from the paper *Ambiguity and social interaction* by Eichberger, Kelsey and Schipper (2009).

DEFINITION 3.1: Let α, δ be real numbers such that $0 \leq \alpha \leq 1$, $0 \leq \delta \leq 1$, define a neo-additive capacity v by $v(\emptyset) = 0$, $v(S_{-i}) = 1$, $v(A) = \delta\alpha + (1-\delta)\pi(A)$, $\emptyset \subset A \subset S_{-i}$, where π is an additive probability distribution on S_{-i}.

Neo-additive capacities can then be represented in the following manner:

$$v(A|\alpha,\delta,\pi) = \begin{cases} 1 & for\ A = S_{-i} \\ \delta\alpha + (1-\delta)\pi(A) & for\ \emptyset \subset A \subset S_{-i} \\ 0 & for\ A = \emptyset \end{cases}$$

Here $\alpha, \delta \in [0,1]$, π_i is an additive probability distribution on S_{-i} of player i which represents the belief of player i about his opponents' strategies.

With neo-additive capacities the following pattern arises. The decision maker has a belief represented by π, but lacks confidence in this belief. Then the decision maker can react in an optimistic or a pessimistic way, when optimistic the decision maker partially overweighs the best outcome, $max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$, and when pessimistic the decision maker partially overweighs the worst outcome, $min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$. The overweighting of $max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ and $min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ are measured by $\delta\alpha$ and $\delta(1-\alpha)$, respectively.

A convenient characteristic of the neo-additive capacity representation is that ambiguity and attitudes towards ambiguity are measured separately. Ambiguity is measured by δ and, conversely, confidence is measured by $(1-\delta)$. As for attitudes, α captures optimism and $(1-\alpha)$ captures pessimism.

Now that we have a clearer picture of what neo-additive capacities are we can continue and combine it with the Choquet integral to get the formula for the Choquet expected utility with neo-additive capacities.

DEFINITION 3.2: The Choquet expected utility with respect to the neo-additive capacity $v = \delta\alpha + (1-\delta)\pi$ from playing $s_i \in S_i$ is given by

$$V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i\left[\alpha_i max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha_i) min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right]$$
$$+ (1-\delta_i)\int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

(see appendix point 2 for derivation).

We can see that the Choquet expected utility, with neo-additive capacities, is a weighted average of utilities from the maximum, minimum, and average payoff. However, the weights are not probabilities but decision weights. The maximum payoff, $max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$, gets a weight $\delta\alpha + (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$, the minimum payoff, $min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$, gets a weight $\delta(1-\alpha) + (1-\delta)\pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$ (see appendix point 1), this is only true for acts with two outcomes, and any other outcome x_i, gets a decision weight of $(1-\delta)\pi(x_i)$.

Last, we need to define the notion of the support of beliefs, before we can move on to the equilibrium. The support of beliefs is the set of strategies that are assigned a positive probability by the belief of a player. So strategies that receive a belief that assigns a probability of zero to that particular strategy are not included in the support. The notion of support is necessary to ensure that only best responses are considered and played, as is required by equilibrium. Here, I will use the same definition of a support of a capacity as Eichberger, Kelsey and Schipper (2009).

DEFINITION 3.3: The support of the neo-additive capacity $v(A) = \delta\alpha + (1-\alpha)\pi(A)$, is defined by $supp(v) = supp(\pi)$.

Define the best-response function of player i given his/her beliefs, that are represented by a neo-additive capacity v, by $R_i(v_i) = R_i(\alpha_i, \delta_i, \pi_i) :=$ $\arg\max\{V_i(s_i, v_i) | s_i \in S_i\}$. Now we can define a condition for equilibria where ambiguity plays a role.

DEFINITION 3.4: A pair of neo-additive capacities (v_1^*, v_2^*) is an Equilibrium Under Ambiguity (EUA) if for $i = 1,2$, $supp(v_i^*) \subseteq R_{-i}(v_{-i}^*)$.

Now, to summarize, what does actually occur in an Equilibrium under Ambiguity (EUA)? Each player assigns only positive likelihoods to his/her opponent's best responses given the opponent's beliefs. But, players lack confidence in their likelihood assessment, so they respond in an optimistic or pessimistic way by overweighting the best outcome, $max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$, or overweighting the worst outcome, $min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$, respectively.

The Equilibrium under Ambiguity (EUA) concept is also related to the original Nash equilibrium. In fact if players exhibit/feel no ambiguity, so $\delta_i = 0$, the first half of the Choquet expected value in the equation from definition 3.2 becomes zero, and we are left with the expected utility. According the equilibrium definition players maximize their expected utility by choosing a strategy given his beliefs about the strategies and beliefs of his opponent. This solution concept exactly coincides with the Nash equilibrium. Thus, without ambiguity the Choquet expected utility model gives the same prediction as the Nash equilibrium.

4 Experimental games

In this section I will study two games from the paper *Ten little treasures of game theory and ten intuitive contradictions* by Goeree and Holt (2001). I will apply the Choquet expected utility model with neo-additive capacities, from the previous section, in order to predict the players' behavior. Both games are dynamic. Goeree and Holt (2001) show that behavior by their subjects moves away from the Nash equilibrium prediction when a seemingly irrelevant parameter is changed. The objective of this section is to clarify if models that incorporate ambiguity can explain this observed pattern.

4.1 Should you believe others to be rational?

The first game I will study is a dynamic game and is concerned with the fact if a player should trust his opponent to be rational. Henceforth, this game will be called 'not rational game'. In this game there are two players, player 1 and player 2, player 1 has the following strategy set $S_1 = \{S, R\}$ and player 2 has the strategy set $S_2 = \{PP, PN, NP, NN\}$. Here the strategy $s_2 = PP$ means that if player 1 plays $s_1 = S$, player 2 will play P and if player 1 plays $s_1 = R$, player 2 will play P as well. The game tree with payoffs is given in the figure below. The percentage between the parentheses refers to the actual percentage of subjects that ended up at that particular outcome.

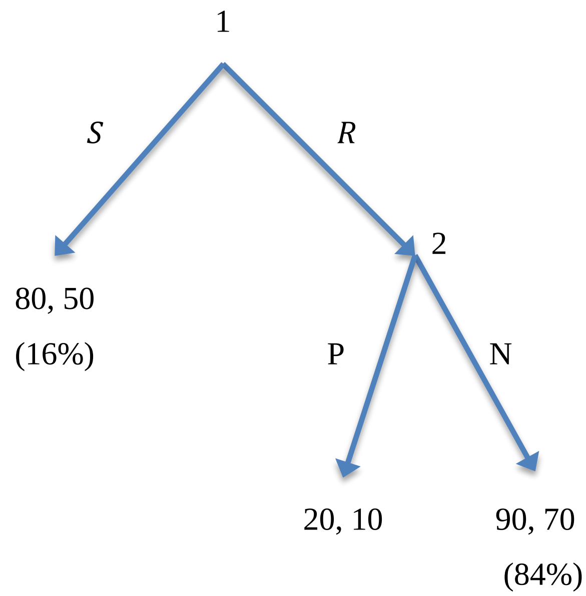

Figure 1. Version (a.) of 'not rational game'

Now, the analysis without ambiguity is pretty straightforward and it goes as follows. In order to determine the Nash equilibria it is convenient to represent the game in its normal form.

Table 1: Normal form representation of version (a.) of 'not rational game'

		2			
		PP	PN	NP	NN
1	S	80, 50	80, 50	80, 50	80, 50
	R	20, 10	90, 70	20, 10	90, 70

To find the Nash equilibria we look at the best responses of both players given the strategy of his opponent. If player 1 plays $s_1 = S$, player 2 will be indifferent between all four of his strategies. If player 2 plays $s_2 = PP$, player 1 will prefer $s_1 = S$ over

$s_1 = R$, because $80 > 20$. So when player 1 plays $s_1 = S$ and player 2 plays $s_2 = PP$ both players have no incentive to deviate, thus we have our first Nash equilibrium, namely (S, PP). The same reasoning applies to the situation where player 1 plays $s_1 = S$ and player 2 plays $s_2 = NP$, thus our second Nash equilibria is (S, NP). Now consider player 1 playing $s_1 = R$, player 2 is indifferent between $s_2 = PN$ and $s_2 = NN$, but prefers them above $s_2 = PP$ and $s_2 = PN$, because $70 > 10$. Then if player 2 plays $s_2 = PN$ or $s_2 = NN$, in both instances, player 1 would prefer $s_1 = R$ over $s_1 = S$, because $90 > 80$. Again the two players have no incentive to deviate unilaterally, and thus we have our other two Nash equilibria, namely (R, PN) and (R, NN). This gives us four Nash equilibria in total: (S, PP), (S, NP), (R, PN), and (R, NN).

Which of these Nash equilibria are perfect Nash equilibria? In order to find the ones that are, it will be convenient to look at figure 1. For a perfect Nash equilibrium, the strategies chosen must be an optimal response in every subgame. I will use backward induction to check which of the four Nash equilibria are perfect Nash equilibria. In the subgame where 2 may choose an action, $s_2 = N$ dominates $s_2 = P$, because $70 > 10$. Thus playing $s_2 = P$ is never an optimal strategy for player 2. Player 1 can deduce this. So if he expects player 2 to play $s_2 = NN$, player 1 will play $s_1 = R$. This is because if player 1 plays $s_1 = S$ he receives a payoff of 80, and if player 1 plays $s_1 = R$ when he believes player 2 will play $s_2 = N$, he receives a payoff of 90. Since $90 > 80$ player 1 will play $s_1 = R$. So the only Nash equilibrium that survives backward induction is (R, NN). This is also the outcome that most of the subjects, in the paper by Goeree and Holt (2001), arrive at (84%).

After this Goeree and Holt (2001) let their subjects play the game again but now one payoff to player 2 is changed, namely $x_2 = (s_1 = R, s_2 = P) = 10$ is increased to $x_2 = (s_1 = R, s_2 = P) = 68$. So now it may be more likely that player 2 plays $s_2 = P$ after player 1 played $s_1 = R$. I will call this game with the increased outcome version (b.). The previous game will be called version (a.). For version (b.), the analysis without ambiguity is exactly the same as the analysis of version (a.). However, in version (b.) a significant proportion of the subjects in the player 1 role, play $s_1 = S$, namely 52%, and thus behavior moves away from the Nash equilibrium solution. Now I will examine if this observed behavior could be explained by ambiguity. First the game tree of version (b.) of this game is given.

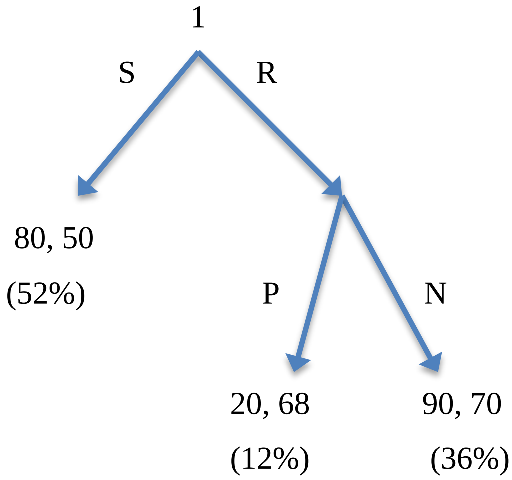

Figure 2. Version (b.) of 'not rational game'

In order for subjects, in the role of player 1, to play $s_1 = S$ instead of $s_1 = R$ the Choquet expected utility for player 1 from $s_1 = S$ must be larger than the Choquet expected utility for player 1 from $s_1 = R$. Thus:

CONDITION 4.1:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

To calculate $V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ and $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I use the formula of the Choquet expected utility with neo-additive capacities. For $V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I get the following expression:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 80$$

(See appendix point 3 for derivation).

And for $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$:

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 90 - \delta_1(1 - \alpha_1)70$$

(See appendix point 3 for derivation).

If we then plug in the values into condition 4.1, I get the following:

$$80 > 90 - \delta_1(1 - \alpha_1)70$$

Which in turn leads to

$$\delta_1(1 - \alpha_1) > 0.143$$

(See appendix point 4 for derivation).

What exactly does this finding entail? If we look at the term $\delta_1(1 - \alpha_1)$ it is δ_1 (ambiguity perceived by player 1) times $(1 - \alpha_1)$, which is the term for a pessimistic view on ambiguity, so the two combine represent ambiguity aversion. So if the ambiguity aversion of player 1 is larger than 0.143, he/she prefers playing $s_1 =$

S over $s_1 = R$. Which would lead to a deviation from the Nash equilibrium prediction.

4.2 Should you believe a threat that is not credible?

The first game considered is somewhat unusual, in the sense that, a punishment by player 2 cannot be justified, because there is an absence of a relative payoff effect. To understand what I mean, look again at the game trees of the first game (both from version (a.) and (b.)). Player 1 can play $s_1 = S$ and secure a payoff of $x_1(s_1 = S) = 80$ to himself and $x_2(s_1 = S) = 50$ to player 2. However, if player 1 plays $s_1 = R$ in expectation that player 2 is rational and chooses to play $s_2 = N$, both players receive a higher payoff, $x_1(s_1 = R, s_2 = N) = 90 > x_1(s_1 = S) = 80$ and $x_2(s_1 = R, s_2 = N) = 70 > x_2(s_1 = S) = 50 > x_2(s_1 = R, s_2 = P) = 10$ and in version (b.) $x_2(s_1 = R, s_2 = N) = 70 > x_2(s_1 = R, s_2 = P) = 68 > x_2(s_1 = S) = 50$, so player 2 is also better off when playing N. Thus no justifiable reason for player 2 to punish player 1, after playing $s_1 = R$, by playing $s_2 = P$.

In the second game (game trees are given below) there is a relative payoff effect. Now if player 1 plays $s_1 = R$ player 2's payoff is decreased for sure (the relative payoff effect). Thus fear of punishment might play a role in this game. The strategy set for player 1 and player 2 are, again, $S_1 = \{S, R\}$ and $S_2 = \{PP, PN, NP, NN\}$, respectively. Again the observed percentage of subjects, that Goeree and Holt (2001) find displaying certain behavior is given in parentheses.

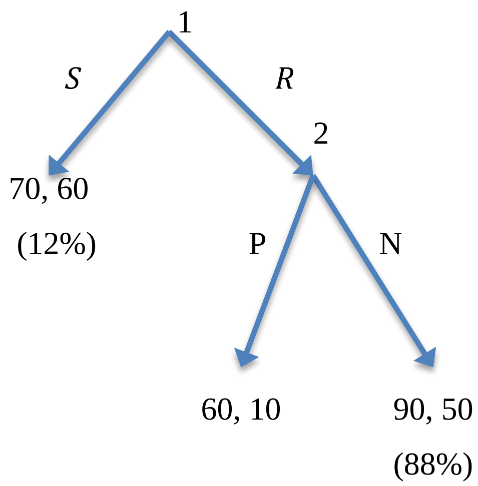

Figure 3. Version (a.) of 'non-credible threat game'

In version (a.) of this game, however, punishment by player 2 is rather costly for player 2. So it highly unlikely that player 2 will punish here. For the normal analysis of this game (without ambiguity), we use the normal form representation again.

Table 2: Normal form representation of Version (a.) of 'non-credible threat game'

		2			
		PP	PN	NP	NN
1	S	70, 60	70, 60	70, 60	70, 60
	R	60, 10	90, 50	60, 10	90, 50

The way of reasoning to find the Nash equilibria and perfect Nash equilibrium in version (a.) of the game in section 4.1 applies here as well. Again we get the following four Nash equilibria: (S, PP), (S, NP), (R, PN), and (R, NN). And again we get the same perfect Nash equilibrium, namely (R, NN). We see that in version (a.) of this game the outcome predicted by the perfect Nash equilibrium is played most of the time by the subjects in Goeree and Holt (2001), namely 88%.

Again, the payoff to player 2 is changed in version (b.) of this game in the same manner as in version (b.) of the game in the previous section only the value is different. In this game $x_2 = (s_1 = R, s_2 = P) = 10$ is increased to $x_2 = (s_1 = R, s_2 = P) = 48$. The game tree is given below.

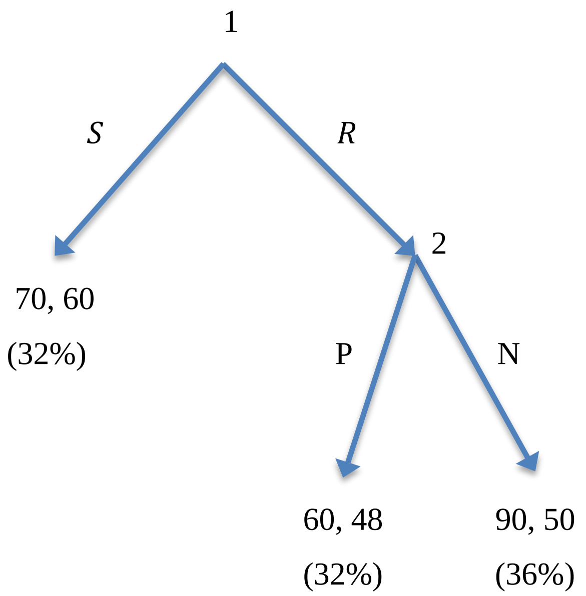

Figure 4. Version (b.) of 'non-credible threat game'

Here, again, player 1 is the decision maker who faces ambiguity. Goeree and Holt (2001) find that more of their subjects, in the player 1 role, play $s_1 = S$. This might be explainable with ambiguity. So, again I use the Choquet expected utility model with neo-additive capacities to check if this behavior by player 1 can be justified.

If player 1 plays $s_1 = S$ instead of $s_1 = R$ the following condition must hold:

CONDITION 4.2:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

For $V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I get the following expression:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 70$$

(See appendix point 5 for derivation).

And for $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I get the expression:

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 90 - \delta_1(1 - \alpha_1)30$$

(See appendix 5 for derivation).

Plugging in these values into condition 4.2, I get the following condition:

$$70 > 90 - \delta_1(1 - \alpha_1)30$$

This in turn, after some algebraic steps, leads to:

$$\delta_1(1 - \alpha_1) > 0.667$$

(See appendix point 6 for derivation).

So in order for player 1 to deviate from the Nash prediction $s_1 = R$, his ambiguity aversion must be larger than 0.667. This is a pretty high level and that is because in this game if player 2 decides to punish player 1 by playing $s_2 = P$ player 1 still receives a payoff of 60. If this value and therefore the difference between $max_{s_2 \in S_2} u_1(s_1 = R; s_2)$ and $min_{s_2 \in S_2} u_1(s_1 = R; s_2)$ had been lower and larger, respectively, then $\delta_1(1 - \alpha_1)$ would have been lower as well.

4.3 Varying the additive probability distribution

The analysis of the two games with ambiguity is similar to the analysis in Eichberger and Kelsey (2008). However, in their analysis they assume that $\int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$ is equal to $max\{u_i(s_i = 1, s_{-i}), u_i(s_i = 2, s_{-i})\}$, where $s_i = 1$ stands for the first strategy of the strategy set of player i and $s_i = 2$ stands for the second strategy of the strategy set of player i. This implies $\pi_1(s_2) = \{p_1(s_2 = P) = 0; p_1(s_2 = N) = 1\}$ as long as $u_2(s_1 = R; s_2 = P) < u_2(s_1 = R; s_2 = N)$, since with the additive probability distribution full rationality is assumed. Then player 2 will never play $s_2 = P$ as long as $s_2 = N$ grants player 2 a higher payoff. So if full rationality is assumed, player 1 will deduce this and therefore believe that player 2 will not play $s_2 = P$ as long as playing $s_2 = N$ after $s_1 = R$ gives a higher payoff to player 2. This logic predicts the same ambiguity aversion cut-off value in version (a.) and (b.) of both games. In version (b.) of both games punishment is relatively cheap to player 2. Therefore, if player 1 faces ambiguity over the strategies of player 2, rationality may not be likely thanks to the ambiguity. Thus in version (b.) player 1 must deem it more likely that player 2 may play the punishment strategy, and this should be reflected in the additive probability distribution, $\pi_1(s_2)$, of player 1.

In the 'non-credible threat game' there is a relative payoff effect for player 2 when player 1 plays $s_1 = R$. By playing $s_1 = R$ the payoff to player 2 is decreased from 60 to 50 with certainty. Player 2 may deem this as unfair and will therefore punish player 1, since the payoff difference for player 2 by playing $s_2 = P$ or $s_2 = N$ is very small, in version (b.), player 1 should deem it even more likely that player 2 will deviate from the standard Nash equilibrium prediction ($s_2 = N$), in the 'non-credible threat game'.

As a result $\int u_1(s_1, s_2)d\pi_1(s_2)$ should be lower in version (b.) than in version (a.), because now more weight should be put on the lower payoff outcome, and thus player 1's choice will not only depend on his ambiguity attitudes with his belief set at $\pi_1(s_2) = \{p_1(s_2 = P) = 0; p_1(s_2 = N) = 1\}$, now his belief has more impact, since $\pi_1(s_2)$ is allowed to vary.

In this section I will investigate what happens with the ambiguity parameters when the additive probability distribution can take different values and not just $\pi_1 = \{p_1(s_2 = P) = 0;\ p_1(s_2 = N) = 1\}$ as before. So we treat $p_1(s_2 = P)$ and $p_1(s_2 = N)$ as variables. Here $p_1(s_2 = P)$ and $p_1(s_2 = N)$ stand for the belief player 1 has about player 2 playing his strategy $s_2 = P$ and $s_2 = N$, respectively. I will use an additive probability distribution of the following form:

$$\pi_1(s_2) = \{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$$

with $0 \leq p \leq 1$ and $p_1(s_2 = P) + p_1(s_2 = N) = 1$.

The equation from definition 3.2 can be rewritten as:

EQUATION 1:

$$V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R;\ s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R;\ s_2)\right]$$
$$+ (1 - \delta_1)[p_1(s_2 = P) u_1(s_1 = R;\ s_2 = P)$$
$$+ p_1(s_2 = N) u_1(s_1 = R;\ s_2 = N)]$$

(See appendix point 7 for derivation).

4.3.1 Should you trust others to be rational?

Here I will use equation 1 and the additive probability distribution $\pi_1(s_2) = \{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$, which represents the belief of player 1 about S_2 that lacks full confidence. Then if player 1 plays $s_1 = S$ instead of $s_1 = R$ the following condition must hold:

CONDITION 4.3.1:

$$V_1(s_1 = S | s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

$V_1(s_1 = S | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ becomes:

$$V_1(s_1 = S | s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 80$$

(See appendix point 8 for derivation).

And $V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I get the following equation:

$$V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1[70\alpha_1 + 20 - p_1(s_2 = P)20 - p_1(s_2 = N)90]$$
$$+ [p_1(s_2 = P)20 + p_1(s_2 = N)90]$$

Since $\pi_1(s_2)$, is $\{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$ instead of $\{p_1(s_2 = P) = 0;\ p_1(s_2 = N) = 1\}$, so $V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ in its final form can be written as:

$$V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = (90 - 70p) - \delta_1(1 - \alpha_1 - p)70$$

(See appendix point 8 for derivation).

Now we can return to the condition that must be satisfied in order for player 1 to play $s_1 = S$ instead of $s_1 = R$. I get:

$$80 > (90 - 70p) - \delta_1(1 - \alpha_1 - p)70$$

This results in the following expression:

$$\delta_1(1 - \alpha_1 - p) > \frac{80 - (90 - 70p)}{-70}$$

(See appendix point 9 for derivation).

If we fill in $p = 0$, then we get the same outcome as in section 4.1, namely $\delta_1(1 - \alpha_1) > 0.143$. At the other end, we can fill in $p = 1$, then we get the following: $\delta_1\alpha_1 < \frac{6}{7}$.

If $\delta_1(1 - \alpha_1 - p) > \frac{80-(90-70p)}{-70}$ holds then player 1 plays $s_1 = S$ instead of $s_1 = R$. To interpret this finding, I will try to define a relationship between δ_1 and p, and between α_1 and p.The relation between $1 - \alpha_1$ and p directly follows from the relation between α_1 and p. For the relationship between α_1 and p and between $1 - \alpha_1$ and p, I will fix δ_1 at a reasonable level found in previous literature. As for the relationship between δ_1 and p, I will fix α_1 at a reasonable level found in previous literature. In Eichberger & Kelsey (2011), they state that Kilka & Weber (2001) found reasonable levels of optimism and ambiguity, experimentally. These values will be used here to fix δ_1 and α_1 at reasonable levels. Below a table of their findings is given.

Table 3: Optimism and ambiguity values from Kilka & Weber (2001)

	α	δ	$\alpha\delta$
Average	0.5	0.52	0.26
Max.	0.62	0.61	0.34
Min.	0.4	0.41	0.18

In order to find a relationship between δ_1 and p and between α_1 and p, first I fix δ_1 at a value from the KW-range (Table 3) and then show how α_1 depends on p and second I fix α_1 at a value from the KW-range (Table 3) and then show how δ_1 depends on p.

First, the relationship between α_1 and p when the value for δ_1 is fixed at the average from the KW-range, thus $\delta_1 = 0.52$, is examined. We can rewrite $\delta_1(1 - \alpha_1 - p) > \frac{80-(90-70p)}{-70}$ to find an equation for α_1 which depends on δ_1 and p. This gives:

$$\alpha_1 < 1 - \frac{80 - (90 - 70p)}{-36.4} - p$$

(See appendix point 10 for derivation).

Where $0 \leq p \leq 1$, so let's take steps of 0.10 for p and look what happens to α_1, where $\alpha_1 \in [0,1]$. So we get a relationship between α_1 and p when δ_1 is fixed.

What can we learn, from table A.11 in the appendix, about the relationship between α_1 and p when δ_1 is fixed at 0.52? If p increases (the belief of player 1 about player 2 playing $s_2 = P$), the upper bound of α_1 for player 1 to prefer $s_1 = S$ to $s_1 = R$ increases as well. This means that the more likely player 1 beliefs player 2 will punish (play $s_2 = P$), more values of optimism allow player 1 to play $s_1 = S$ for his given fixed ambiguity value. Figure 5 presents the relationship between p (x-axis) and the upper bound of α_1 (y-axis).

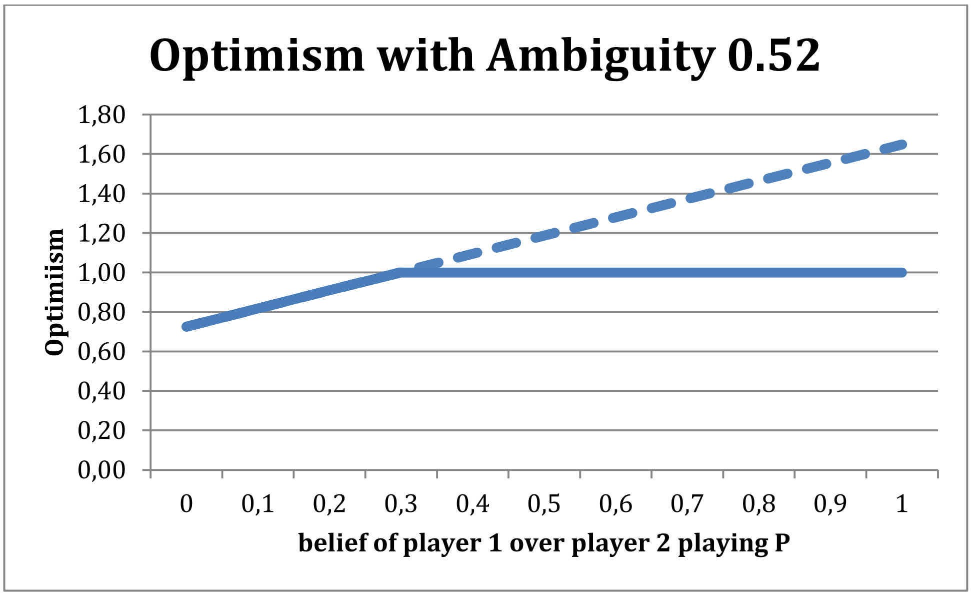

Figure 5. Relationship between p and upper bound of α_1

The relationship between $1 - \alpha_1$ and p is opposite to the relationship between α_1 and p, because optimism and pessimism are each other's opposites (see appendix). From table A.13 in the appendix we see that if p (the belief of player 1 about player 2 playing $s_2 = P$) increases, the pessimism $(1 - \alpha_1)$ needed to induce $s_1 = S \succ s_1 = R$ decreases. The lower bound of $(1 - \alpha_1)$ decreases. So a less pessimistic attitude towards ambiguity is needed for player 1 to play his safer strategy when p increases. Figure 6 presents the relationship between p (x-axis) and the lower bound of $1 - \alpha_1$ (y-axis).

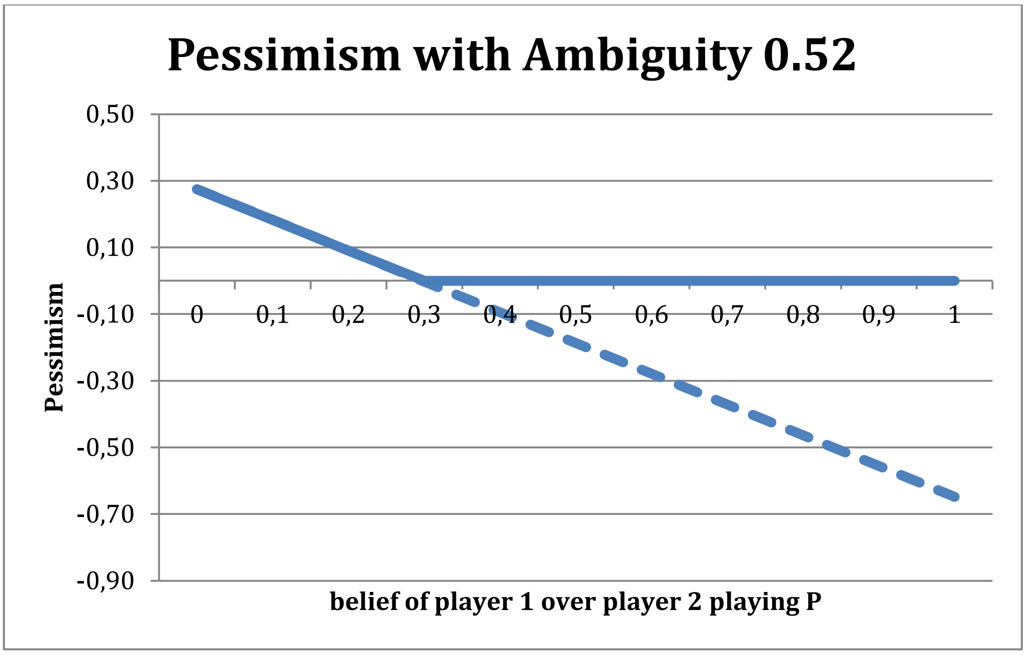

Figure 6. Relationship between p and lower bound of $1 - \alpha_1$

Now that the relationship between the belief of player 1 about player 2 playing $s_2 = P$ and optimism (α_1) and pessimism ($1 - \alpha_1$) is clear, I will move on to the relation between p (the belief of player 1 about player 2 playing $s_2 = P$) and ambiguity (δ_1), where $\delta_1 \in [0,1]$, when α_1 is fixed at a value from the KW-range (Table 3). The condition $\delta_1(1 - \alpha_1 - p) > \frac{80-(90-70p)}{-70}$ needs to be rewritten into a condition where δ_1 depends on α_1 and p. I arrive at the expression when $0 \leq p < 0.5$:

$$\delta_1 > \frac{80 - (90 - 70p)}{-70(0.5 - p)}$$

And the expression becomes the following when $0.5 < p \leq 1$:

$$\delta_1 < \frac{80 - (90 - 70p)}{-70(0.5 - p)}$$

(See appendix point 14 for derivation).

Where, yet again, $0 \leq p \leq 1$, so again let's take steps of 0.10 for p and look what happens to δ_1, where $\delta_1 \in [0,1]$. So we get a relationship between δ_1 and p when α_1 is fixed.

A few remarkable things happen in the relationship between δ_1 and p, which become clear in table A.15 in the appendix. First, in the range $p \in [0,0.1]$, the lower bound of δ_1 decreases, from 0.286 to 0.107, if p increases from 0 to 0.1, this means that less ambiguity is needed to induce $s_1 = S \succ s_1 = R$, meaning player 1 must be more confident in his belief p.

Second, in the range $p \in [0.2,0.4]$ it can be seen that the ambiguity needed to make sure that $s_1 = S \succ s_1 = R$ must be larger than a negative number. Since $\delta_1 \in [0,1]$, it is always satisfied, because δ_1 is always larger than zero. So if we take $\delta_1 = 0$ as the cut off point, then $\delta_1 > 0$ means all values of ambiguity induce $s_1 = S \succ s_1 = R$. It can also mean that even more confidence (than in the first range) is needed when $0.107 > \delta_1 > 0$.

Third, for $p = 0.5$ we cannot compute an ambiguity cut off value, since we divide by 0 which is mathematically not possible. So at $p = 0.5$, δ_1 is not well defined.

Last, in the range $p \in [0.6,1.0]$, as p increases further we have that δ_1 must be smaller than a number greater than 1, since $\delta_1 \in [0,1]$ this is always satisfied. Because if we have $\delta_1 < 4.571$ for instance then it must also be the case that $\delta_1 < 1$, so for $p \in [0.6,1.0]$ we have $\delta_1 < 1$ this means that all levels of ambiguity will induce $s_1 = S \succ s_1 = R$, but now it is the other way around, when compared to $p \in [0.2,0.4]$. Now less confidence will also suffice. So if p increases and it is above $p = 0.5$ confidence in p is not necessarily needed in order for player 1 to have the preference ordering $s_1 = S \succ s_1 = R$.

Figure 7 shows the relationship between the lower bound of δ_1 and p, with ambiguity (δ_1) on the y-axis and p on the x-axis. And figure 8 shows the relationship between the upper bound of δ_1 and p, with ambiguity (δ_1) on the y-axis and p on the x-axis.

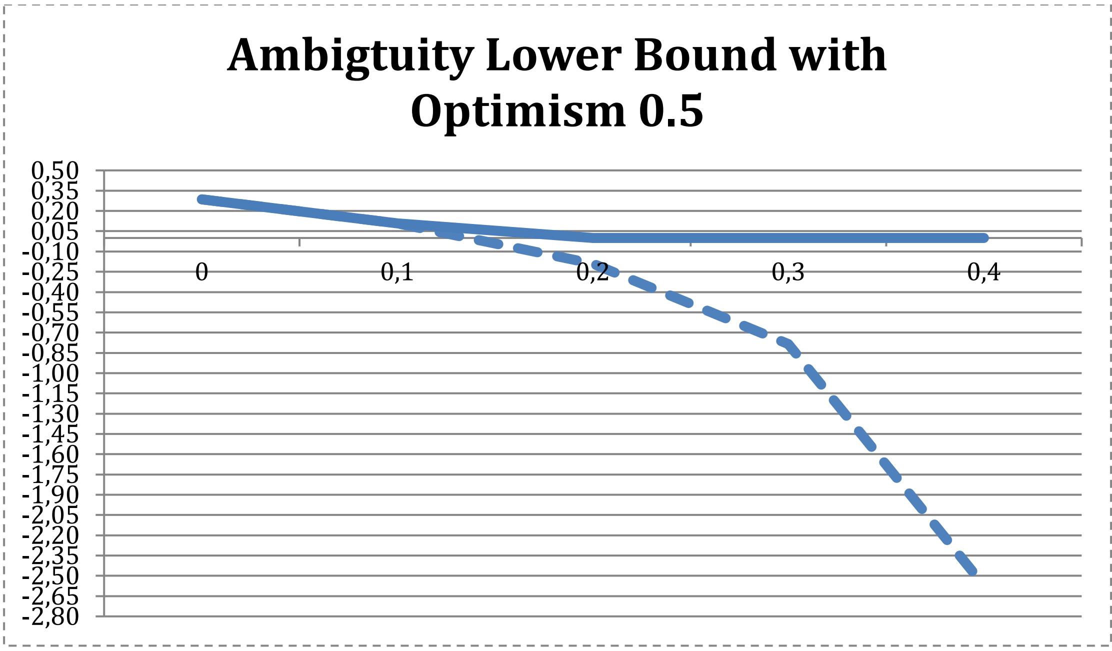

Figure 7. Relationship between the lower bound of δ_1 and p

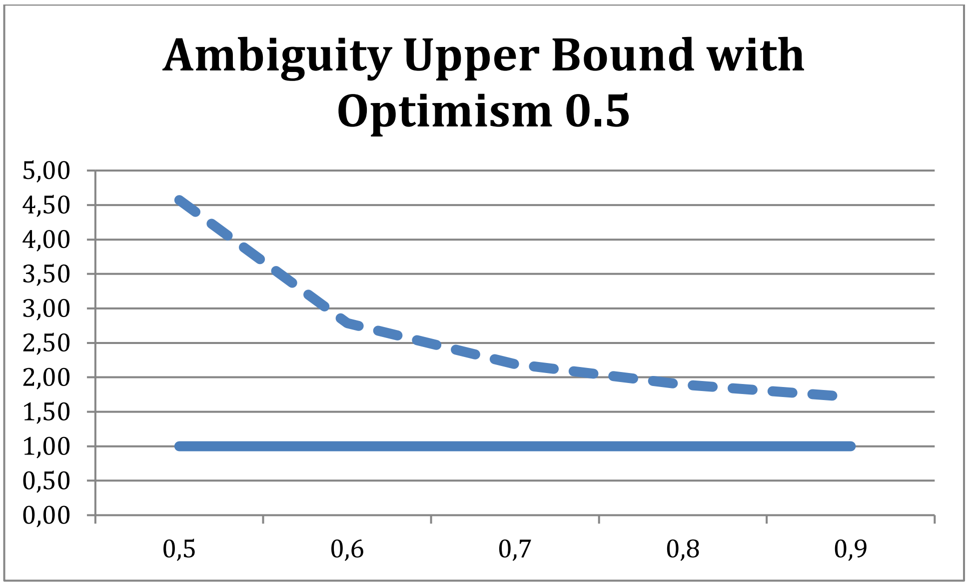

Figure 8. Relationship between the upper bound of δ_1 and p

4.3.2 Should you believe a threat that is not credible?

In this part of section 4, I will apply the same analysis of section 4.3.1 on the 'Should you believe a threat that is not credible?' game, henceforth "non-credible threat game'. So just to recap we have the additive probability distribution $\pi_1(s_2) = \{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$. Instead of $\pi_1(s_2) = \{p_1(s_2 = P) = 0;\ p_1(s_2 = N) = 1\}$.

For all derivations I would like to point the reader to the appendix point 16 until point 23. The patterns that are found for the relationships between δ_1 and p, when α_1 is fixed, between α_1 and p, and between $1 - \alpha_1$ and p when δ_1 is fixed are the same as in section 4.3.1. Since the 'non-credible threat game' has different payoffs than the 'not rational game' it will give different values for δ_1, α_1 and $1 - \alpha_1$. This is the only difference between section 4.3.1 and 4.3.2.

5 Ambiguity and Risk attitudes

In all the literature that I have read on the analysis of ambiguity in games, the authors assume a linear utility function of the form: $u_i(x) = x$, however with such a utility function a decision maker is risk-neutral. This may not be so realistic. Therefore, in this section, I will analyze the two games from section 4 again but now with different utility functions, one that resembles risk-aversion and one risk-loving. The utility functions for a risk-averse and risk-loving decision maker, that I will use, are $u_i = \sqrt{x}$ and $u_i = x^2$, respectively.

5.1 Should you believe others to be rational?

5.1.1 Should you believe others to be rational? with a risk-averse decision maker

Here, I will look at the first game of section 4 again, but now the players have a utility function of the following form $u_i = \sqrt{x}$, where $i = 1,2$, in order to incorporate risk-aversion. I expect that the critical ambiguity aversion value for player 1 will now be lower, as player 1 is now also risk-averse less ambiguity aversion is needed for him to play his safer option $s_1 = S$. At least this is what I expect.

For player 1 to prefer playing $s_1 = S$ over $s_1 = R$, his Choquet expected utility from playing $s_1 = S$ must be larger than the Choquet expected utility from playing $s_1 = R$. Here I will use the Choquet expected utility with neo-additive capacities formula as derived in section 3. So if $s_1 = S \succ s_1 = R$ must be true for player 1, the following condition must hold:

CONDITION 5.1.1:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

For $V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I, now, get the following expression:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = \sqrt{80} = 8.94$$

(See appendix point 24 for derivation).

As for $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ I get:

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 9.487 - \delta_1(1 - \alpha_1)5.015$$

(See appendix point 24 for derivation).

Putting these Choquet expected values into condition 5.1.1, I get the following:

$$8.94 > 9.487 - \delta_1(1 - \alpha_1)5.015$$

This leads to the critical ambiguity aversion value of:

$$\delta_1(1 - \alpha_1) > 0.109$$

(See appendix point 25 for derivation).

So if we look at the two critical values from this game, where we assume risk-neutrality and where we assume risk-aversion, we have $\delta_1(1 - \alpha_1) > 0.143$ and $\delta_1(1 - \alpha_1) > 0.109$, respectively. It is easy to see that $0.143 > 0.109$. So my intuition seemed to be right, as a decision maker is risk-averse his ambiguity aversion critical value is somewhat lower compared to a risk-neutral decision maker. This means that if a decision maker is risk-averse, a lower level of ambiguity aversion that he/she perceives is needed in order for him/her to deviate from the Nash equilibrium strategy, when compared to a risk-neutral decision maker.

5.1.2 Should you believe others to be rational? with a risk-loving decision maker

Now, I will look at the first game of section 4 again, but with a player that is risk-loving. Therefore I will use the utility function $u_i = x^2$, where $i = 1,2$, to represent risk-loving behavior. Here, I expect to observe the opposite of what was concluded in section 5.1.1. This means that a decision maker that is risk-loving needs to perceive a higher level of ambiguity aversion in order to deviate from the Nash equilibrium strategy and move to the Equilibrium under Ambiguity, in comparison to a risk-neutral decision maker.

In the equilibrium under ambiguity it is more likely that player 1 plays $s_1 = S$ instead of $s_1 = R$. This implies the following preference ordering for player 1 $s_1 = S \succ s_1 = R$. For this preference ordering to hold, the following condition must hold, again:

CONDITION 5.1.2:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

The equation for $V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$, becomes:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 80^2 = 6{,}400$$

(See appendix point 26 for derivation).

As for $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$, I get the following equation:

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = 8{,}100 + \delta_1(1 - \alpha_1)7{,}700$$

(See appendix point 26 for derivation).

So the above inequality becomes:

$$6{,}400 > 8{,}100 + \delta_1(1 - \alpha_1)7{,}700$$

Then thanks to some algebraic steps, I get a critical ambiguity aversion value of

$$\delta_1(1 - \alpha_1) > 0.221$$

(See appendix point 27 for derivation).

If we compare the critical ambiguity aversion values from this game with a player that is risk-loving and risk-neutral, $\delta_1(1 - \alpha_1) > 0.221$ and $\delta_1(1 - \alpha_1) > 0.143$, respectively, we see that $0.221 > 0.143$. This means that a decision maker who is risk-loving needs to perceive more ambiguity aversion in order for him to deviate from the standard Nash equilibrium solution, when compared to a risk-neutral decision maker. The finding coincides with my intuitive thought from the beginning of this section.

5.2 Should you believe a threat that is not credible?

5.2.1 Should you believe a threat that is not credible? with a risk-averse decision maker

In this section, I will study the game from section 4.2 again, but now the players have different attitudes towards risk, instead of risk-neutrality. I will first look at risk-aversion, and I incorporate this by using a different utility function, namely $u_i = \sqrt{x}$. Intuitively, I argue that a risk-averse player needs to perceive less ambiguity aversion in order for him to play his somewhat safer option (it has a certain outcome) and move away from the Nash equilibrium prediction. All the mathematical derivations, relevant for this section, can be found in the appendix point 28 and 29.

Again, if we look at the two critical ambiguity aversion values, the one from section 4.2 and the one derived in this section, we have $\delta_1(1 - \alpha_1) > 0.667$ and $\delta_1(1 - \alpha_1) > 0.643$. Again, we observe a lower ambiguity aversion level for a risk-averse player than for a risk-neutral player. Thus again a risk-averse player needs to perceive a lower level of ambiguity aversion in order to deviate from the Nash equilibrium prediction.

5.2.2 Should you believe a threat that is not credible? with a risk-loving decision maker

Last, I will look at how a risk-loving attitude interacts with ambiguity in the game from section 4.2. What I suspect is that the risk-loving player needs to perceive a higher level of ambiguity aversion in order to deviate from the Nash equilibrium solution, compared to a risk-neutral player. That is $\delta_1(1-\alpha_1)$ will be higher for a risk-loving person than for a risk-neutral person. All the mathematical derivations, relevant for this section, can be found in the appendix point 30 and 31.

Let's compare the critical ambiguity aversion values one last time. The critical value for a risk-neutral player comes from section 4.2 and is $\delta_1(1-\alpha_1) > 0.667$ and the critical value for a risk-loving player is $\delta_1(1-\alpha_1) > 0.711$. So again we see that $0.711 > 0.667$, the critical value for a risk-loving player is higher than the critical value for a risk-neutral player. Thus, a risk-loving player needs to have a higher aversion to ambiguity in order to deviate from the Nash equilibrium solution in comparison to a risk-neutral player.

6 Futures Market

Since the games in the previous sections are somewhat abstract, I will analyze a real life situation in this section. This is to make it more practical. In this section, the futures market will be analyzed, game theoretically.

A futures contract is an agreement between two investors, where both now agree to buy or sell, a certain commodity/asset in the future at a price determined now. The agreed upon price is called the strike price and denoted by F_0. When an investor buys or sells, we say the investor goes long or short, respectively. Last, the actual price of the commodity, P_t, called the spot price, need not be equal to F_0.

For every investor who takes a long position in a futures contract there must be another investor willing to take the opposite position, in this case a short position. Otherwise an agreement won't be reached. In order to take a long position the investor's beliefs must justify it, this is also true for the investor taking the short position. As the positions of the two investors in the agreement are opposite, the beliefs of the investors must be opposite as well.

Profits to a short position are as follows:

$$F_0 - P_t$$

With a short position you get the strike price, because this is the agreed upon price for which you would be willing to sell and you must deduct the value of the commodity at maturity. If $F_0 > P_t$ we get $F_0 - P_t > 0$ and then the short position earns a positive profit. Profits to a long position are opposite compared to a short position, namely

$$P_t - F_0$$

You pay the agreed upon strike price, F_0, while the underlying commodity, that you then own, is worth P_t. So when $P_t > F_0$ we have $P_t - F_0 > 0$, then a long position earns a positive profit. You can see that for a long position to be profitable we need $P_t > F_0$ and for a short position $F_0 > P_t$. As the spot price can only be higher or lower than the strike price at maturity, it can't be both, one position will be profitable and the other will not. From the profit formulas of the two positions it can be seen that the profits of one position are equal to the loss of the other position. Thus, a futures contract can be seen as a zero-sum game.

In reality, the range in which the price of the underlying commodity of the futures contract can fluctuate is $[0, \infty^+)$. This range, I will call the state space, is denoted by Ω. In this game there is no uncertainty about the player's opponents' strategies, but there is uncertainty over the state space. Players don't know for sure which state of nature will materialize. We have $\Omega = \{0, \ldots, F_0, \ldots, \infty^+\}$, here the numbers stand for the spot price, of the underlying commodity, at maturity of the futures contract. In the range $[0, F_0)$ a short position makes a positive profit and a long position makes a loss, because in this range $F_0 > P_t$ holds. In the range (F_0, ∞^+) a long position makes a positive profit and a short position makes a loss, since $P_t > F_0$ holds for that range.

To make it somewhat more visible I will assume that P_t can take on only two values, H and L, where H stands for a high value and L for a low value. For F_0, H and L the following is true $H > F_0 > L$.

So if H materializes, the state of nature that arises will be denoted by ω_H, and if L materializes, the state of nature that arises will be denoted by ω_L. The state space here can be seen as $\Omega = \{\omega_L, \omega_H\}$, investors have beliefs over Ω which can be represented by an additive probability distribution over $\Omega = \{\omega_L, \omega_H\}$. This will have the following representation:

$$\pi_i(\omega) = \{p_i(\omega_L);\ p_i(\omega_H)\}$$

Here $p_i(\omega_L)$ and $p_i(\omega_H)$ are the probabilities that the belief of investor i assign to the states of nature, ω_L and ω_H, respectively. For both investors it must be the case that $\sum_{\omega\in\Omega} \pi_i(\omega) = 1$.

Lastly, I assume that for every investor who takes one position there is a investor out there who is willing to take the opposite position. Then there will be no ambiguity over the strategy chosen by one's opponent, but only over the state space $\Omega = \{\omega_L, \omega_H\}$. So investors lack full confidence in their belief about the state of nature, $\pi_i(\omega) = \{p_i(\omega_L);\ p_i(\omega_H)\}$. Only two investors can enter in an agreement a third party cannot join one of the two earlier investors in their futures contract. Choices will be made simultaneously; therefore the analysis will be static and not dynamic.

Now that the game is sufficiently explained, for the remainder of this section I will analyze the game without ambiguity and with ambiguity, to see if any differences in behavior can be found.

6.1 Without ambiguity

First, I will look at this situation without ambiguity. This means that the investors have a belief about the likelihoods of the states of nature in which they have full confidence. However, only one of the two investors is right eventually in the agreement. This does not matter, because given the beliefs of both investors, both believe that their position is the right one and therefore both believe they play an optimal strategy given the opponent's strategy and their belief.

Both investors can either take a short position or a long position, in the agreement, so $S_i = \{short, long\}$ where $i = 1, 2$. For instance if investor 1 plays $s_1 = short$ and investor 2 plays $s_2 = long$, what kind of beliefs would induce this behavior? First, if both expected payoffs to investor 1 are the same, investor 1 will be indifferent between his two strategies. So in order to find the cut off point in investor 1's beliefs, the following condition must hold:

CONDITION 6.1.1:

$$E[u_1(s_1 = short, s_2 = long)] = E[u_1(s_1 = long, s_2 = short)]$$

This equation solved leads to the following belief of investor 1, $\pi_1(\omega)$:

$$\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H - F_0}{H - L}; p_1(\omega_H) = \frac{F_0 - L}{H - L}\right\}$$

(See appendix point 32 for derivation).

The same question can be asked for investor 2. For what $\pi_2(\omega)$ will investor 2 be indifferent between $s_2 = long$ and $s_2 = short$? Then

CONDITION 6.1.2:

$$E[u_2(s_2 = long, s_1 = short)] = E[u_2(s_2 = short, s_1 = long)]$$

Again, this leads to the following belief, $\pi_2(\omega)$:

$$\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H - F_0}{H - L}; p_2(\omega_H) = \frac{F_0 - L}{H - L}\right\}$$

(See appendix point 33 for derivation).

Like I mentioned before, for both investors to take a different position their beliefs need to be opposite in order to justify it. If I take the same example where investor 1 plays $s_1 = short$ and investor 2 plays $s_2 = long$, intuitively investor 1 must belief that the probability of ω_L is higher than the cut off point of $\frac{H-F_0}{H-L}$ and that the probability of ω_H is lower than the cut off point of $\frac{F_0-L}{H-L}$. For investor 2, then it must be the other way around. Let's find out if this is the case, formally. In order for investor 1 to prefer playing $s_1 = short$ over $s_1 = long$ the following condition must hold:

CONDITION 6.1.3:

$$E[u_1(s_1 = short, s_2 = long)] > E[u_1(s_1 = long, s_2 = short)]$$

This leads to the belief of investor 1 which can be represented as

$$\pi_1(\omega) = \left\{p_1(\omega_L) > \frac{H - F_0}{H - L}; p_1(\omega_H) < \frac{F_0 - L}{H - L}\right\}$$

(See appendix point 34 for derivation).

The same question can be asked for investor 2. What condition needs to hold in order for investor 2 to prefer playing $s_2 = long$ over $s_2 = short$?

CONDITION 6.1.4:

$$E[u_2(s_2 = long, s_1 = short)] > E[u_2(s_2 = short, s_1 = long)]$$

This leads to the following belief for investor 2 that induces $s_2 = long \succ s_2 = short$:

$$\pi_2(\omega) = \left\{p_2(\omega_L) < \frac{H - F_0}{H - L}; p_2(\omega_H) > \frac{F_0 - L}{H - L}\right\}$$

(See appendix point 35 for derivation).

So if the beliefs of investor 1 and 2 satisfy $\pi_1(\omega) = \left\{p_1(\omega_L) > \frac{H-F_0}{H-L}; p_1(\omega_H) < \frac{F_0-L}{H-L}\right\}$ and $\pi_2(\omega) = \left\{p_2(\omega_L) < \frac{H-F_0}{H-L}; p_2(\omega_H) > \frac{F_0-L}{H-L}\right\}$, respectively, then investor 1 believes playing $s_1 = short$ is best and investor 2 believes playing $s_2 = long$ is optimal. Therefore, an agreement can be reached between the two players and both believe they are playing an optimal strategy, which will give them a positive payoff in their eyes. However, in reality, only one of the two will be right and thus will earn a positive payoff, while the other will earn a negative payoff. This depends on the real state of nature that materializes, ω_L or ω_H.

Let's assume $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$ then investor 1 plays $s_1 = short$, and if $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$ then investor 2 plays $s_2 = long$, where $a \leq \frac{F_0-L}{H-L}$ for investor 1 and $a \leq \frac{H-F_0}{H-L}$ for investor 2 (see appendix point 38). Then an agreement is reached, but what payoff do the investors expect to get?

$$E\left[u_1\left(s_1 = short, s_2 = long \middle| \pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}\right)\right] = a(H-L)$$

(See appendix point 36 for derivation).

The same can be done for investor 2:

$$E\left[u_2\left(s_2 = long, s_1 = short \middle| \pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}\right)\right] = a(H-L)$$

(See appendix point 37 for derivation).

Thus, both investors believe they get a positive payoff of $a(H-L)$ and therefore they believe to play an optimal strategy given his opponent's strategy and their beliefs. Thus an futures contract where investor 1 takes a short position and investor 2 takes a long position, with $\pi_1(\omega) = \left\{p_1(\omega_L) > \frac{H-F_0}{H-L}; p_1(\omega_H) < \frac{F_0-L}{H-L}\right\}$ and $\pi_2(\omega) = \left\{p_2(\omega_L) < \frac{H-F_0}{H-L}; p_2(\omega_H) > \frac{F_0-L}{H-L}\right\}$, as the belief of investor 1 and investor 2, respectively, neither of the two investors have an incentive to unilaterally deviate. So we have an equilibrium.

6.2 With ambiguity

Now, I will perform the same analysis as in section 6.1 only the crucial difference is that ambiguity/uncertainty about the state space $\Omega = \{\omega_L, \omega_H\}$ plays a role. I will examine how this ambiguity plays a role in the behavior of the two investor, who want to enter into a futures contract.

In the previous section, both investors had a certain belief about $\Omega = \{\omega_L, \omega_H\}$ in which they had 100% confidence, that this belief does not correspond with the true probability distribution over $\Omega = \{\omega_L, \omega_H\}$, does not matter. The equilibrium in the previous section relies on the concept of rationalizability, which is less stringent than the Nash equilibrium.

Here, beliefs will be represented by neo-additive capacities as discussed in section 3. $v_1 = \delta_1\alpha_1 + (1 - \delta_1)\pi_1(\omega)$ and $v_2 = \delta_2\alpha_2 + (1 - \delta_2)\pi_2(\omega)$ will be the capacity for investor 1 and 2, respectively, which represents their beliefs. The formula for the Choquet expected utility can now be used, like the one in section 3, but with a small adjustment. Here the ambiguity is not about a investor's opponent's strategy, but over the real state of nature that will materialize. So, the Choquet expected utility could be represented in the following way:

EQUATION 6.2.1:

$$\begin{aligned} V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i) \\ &= \delta_i[\alpha_i max_{\omega\in\Omega} u_i(s_i, s_{-i}) + (1 - \alpha_i) min_{\omega\in\Omega} u_i(s_i, s_{-i})] \\ &+ (1 - \delta_i) \int u_i(s_i, s_{-i}) d\pi_i(\omega) \end{aligned}$$

where $i = 1,2$.

First, I start with the behavior of investor 1 when faced with ambiguity over the state space Ω. In order for investor 1 to still prefer playing $s_1 = short$ over playing $s_1 = long$, the Choquet expected utility from $s_1 = short$ must be larger than the Choquet expected utility from $s_1 = long$. Thus

CONDITION 6.2.1:

$$V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = long|s_2 = short, \alpha_1, \delta_1, \pi_1)$$

With $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$ and for a, $a \leq \frac{F_0-L}{H-L}$ (see appendix point 38) must hold. First of all we need to define the term $V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1)$, $V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1)$ can be represented in the following way:

$$V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1[\alpha_1 max_{\omega\in\Omega} u_1(s_1 = short, s_2 = long)$$
$$+ (1-\alpha_1) min_{\omega\in\Omega} u_1(s_1 = short, s_2 = long)]$$
$$+ (1-\delta_1)\int u_1(s_1 = short, s_2 = long) d\pi_1(\omega)$$

Which can be written as:

$$V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1)$$
$$= a(H-L) - \delta_1[(a-\alpha_1)(H-L) + (H-F_0)]$$

(See appendix point 39 for derivation).

Further we need to define $V_1(s_1 = long|s_2 = short, \alpha_1, \delta_1, \pi_1)$, which can be written as:

$$V_1(s_1 = long|s_2 = short, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1[\alpha_1 max_{\omega\in\Omega} u_1(s_1 = long, s_2 = short)$$
$$+ (1-\alpha_1) min_{\omega\in\Omega} u_1(s_1 = long, s_2 = short)]$$
$$+ (1-\delta_1)\int u_1(s_1 = long, s_2 = short) d\pi_1(\omega)$$

This can be shortened to:

$$V_1(s_1 = long|s_2 = short, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1[(a+\alpha_1)(H-L) - (F_0-L)] - a(H-L)$$

(See appendix point 40 for derivation).

Now, condition 6.2.1 can be represented as follows:

$$a(H-L) - \delta_1[(a-\alpha_1)(H-L) + (H-F_0)]$$
$$> \delta_1[(a+\alpha_1)(H-L) - (F_0-L)] - a(H-L)$$

After some algebraic derivation, I get to the following condition which must be satisfied in order for investor 1 to prefer playing $s_1 = short$ over playing $s_1 = long$.

$$\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

This is only true if $(2a+1)H-(2a-1)L-2F_0 > 0$, when $(2a+1)H-(2a-1)L-2F_0 < 0$ the following condition is true:

$$\delta_1 > \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

(See appendix point 41 for derivation).

To summarize, investor 1 will prefer playing $s_1 = short$ over playing $s_1 = long$ if the perceived ambiguity satisfies $\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ if $(2a+1)H -$

$(2a-1)L-2F_0>0$ and $\delta_1>\frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ if $(2a+1)H-(2a-1)L-2F_0<0$ with $\pi_1(\omega)=\left\{p_1(\omega_L)=\frac{H-F_0}{H-L}+a; p_1(\omega_H)=\frac{F_0-L}{H-L}-a\right\}$ and for a, $a\leq\frac{F_o-L}{H-L}$ must hold.

To continue with the behavior of investor 2, he will prefer $s_2=long$ over $s_2=short$ as long as the Choquet expected utility from playing $s_2=long$ exceeds the Choquet expected utility from playing $s_2=short$. This can formally represented by the condition:

CONDITION 6.2.2:

$$V_2(s_2=long|s_1=short,\alpha_2,\delta_2,\pi_2)>V_2(s_2=short|s_1=long,\alpha_2,\delta_2,\pi_2)$$

With $\pi_2(\omega)=\left\{p_2(\omega_L)=\frac{H-F_0}{H-L}-a; p_2(\omega_H)=\frac{F_0-L}{H-L}+a\right\}$ and for a, $a\leq\frac{H-F_0}{H-L}$ (see appendix point 38) must hold. So we need to define the Choquet expected utility from playing $s_2=long$ to investor 2 and the Choquet expected utility from playing $s_2=short$ to investor 2. First, the Choquet expected utility from playing $s_2=long$ to investor 2 can be written as:

$$\begin{aligned}V_2(s_2&=long|s_1=short,\alpha_2,\delta_2,\pi_2)\\&=\delta_2[\alpha_2 max_{\omega\in\Omega}u_2(s_2=long,s_1=short)\\&+(1-\alpha_2)min_{\omega\in\Omega}u_2(s_2=long,s_1=short)]\\&+(1-\delta_2)\int u_2(s_2=long,s_1=short)d\pi_2(\omega)\end{aligned}$$

By filling in all the right values, I arrived at

$$\begin{aligned}V_2(s_2&=long|s_1=short,\alpha_2,\delta_2,\pi_2)\\&=a(H-L)-\delta_2[(a-\alpha_2)(H-L)+(F_0-L)]\end{aligned}$$

(See appendix point 42 for derivation).

Second, for the Choquet expected utility from playing $s_2=short$ I get the following formula:

$$\begin{aligned}V_2(s_2&=short|s_1=long,\alpha_2,\delta_2,\pi_2)\\&=\delta_2[\alpha_2 max_{\omega\in\Omega}u_2(s_2=short,s_1=long)\\&+(1-\alpha_2)min_{\omega\in\Omega}u_2(s_2=short,s_1=long)]\\&+(1-\delta_2)\int u_2(s_2=short,s_1=long)d\pi_2(\omega)\end{aligned}$$

This can be rewritten as:

$$\begin{aligned}V_2(s_2&=short|s_1=long,\alpha_2,\delta_2,\pi_2)\\&=\delta_2[(a+\alpha_2)(H-L)-(H-F_0)]-a(H-L)\end{aligned}$$

(See appendix point 43 for derivation).

Filling in the values for the Choquet expected utility when investor 2 plays $s_2 = long$ and when investor 2 plays $s_2 = short$ into condition 6.2.2 leads to the following condition for the perceived ambiguity, in order for investor 2 to prefer $s_2 = long$ over $s_2 = short$:

$$\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$$

This is only true if $(2a-1)H-(2a+1)L+2F_0 > 0$ is satisfied if not and we have $(2a-1)H-(2a+1)L+2F_0 < 0$ then the following condition is true:

$$\delta_2 > \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$$

(See appendix point 44 for derivation).

Again to summarize, investor 2 prefers $s_2 = long$ over $s_2 = short$ when the perceived ambiguity satisfies $\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$ if $(2a-1)H-(2a+1)L+2F_0 > 0$ and $\delta_2 > \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$ if $(2a-1)H-(2a+1)L+2F_0 < 0$ with $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$ and for a, $a \leq \frac{H-F_0}{H-L}$ must hold.

To summarize, the beliefs of investor i can be represented in the following manner:

$$\pi_i(\omega) = \begin{cases} p_i(\omega_L) > \dfrac{H-F_0}{H-L};\ p_i(\omega_H) < \dfrac{F_0-L}{H-L} \ then\ s_i = short \succ s_i = long \\ p_i(\omega_L) < \dfrac{H-F_0}{H-L};\ p_i(\omega_H) > \dfrac{F_0-L}{H-L} \ then\ s_i = long \succ s_i = short \end{cases}$$

For the ambiguity value to induce $s_i = short \succ s_i = long$ the following condition must hold:

$$\delta_i < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

if $(2a+1)H-(2a-1)L-2F_0 > 0$ and if $(2a+1)H-(2a-1)L-2F_0 < 0$ the following condition for the ambiguity must hold:

$$\delta_i > \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

with

$$\pi_i(\omega) = \left\{p_i(\omega_L) = \frac{H-F_0}{H-L} + a;\ p_i(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$$

To induce $s_i = long \succ s_i = short$ we have that the belief of player i must be $\pi_i(\omega) = \left\{p_i(\omega_L) < \frac{H-F_0}{H-L};\ p_i(\omega_H) > \frac{F_0-L}{H-L}\right\}$. The ambiguity value must then satisfy the following condition:

$$\delta_i < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$$

if $(2a-1)H-(2a+1)L+2F_0 > 0$ and if $(2a-1)H-(2a+1)L+2F_0 < 0$ the following condition for the ambiguity must hold:

$$\delta_i > \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$$

with

$$\pi_i(\omega) = \left\{p_i(\omega_L) = \frac{H-F_0}{H-L} - a;\ p_i(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$$

So investor 1 only has the preference ordering $s_1 = short \succ s_2 = long$ if the following conditions hold:

1. $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$
2. $a \leq \frac{F_0-L}{H-L}$
3. $\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ if $(2a+1)H-(2a-1)L-2F_0 > 0$ and $\delta_1 > \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ if $(2a+1)H-(2a-1)L-2F_0 < 0$.

The same can be done for investor 2, for investor 2 to have the preference ordering $s_2 = long \succ s_2 = short$ the following three conditions must hold:

4. $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$
5. $a \leq \frac{H-F_0}{H-L}$
6. $\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$ if $(2a-1)H-(2a+1)L+2F_0 > 0$ and $\delta_2 > \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$ if $(2a-1)H-(2a+1)L+2F_0 < 0$.

Then if conditions 1 through 6 are satisfied then we get an equilibrium under ambiguity where investor 1 plays $s_1 = short$, in other words takes a short position, and where investor 2 plays $s_2 = long$, in other words takes a long position. Then the two players can agree upon the same futures contract.

To check how these conditions interact, I will use some numerical examples. These numerical examples will also make it less abstract. First, I will look at the conditions for investor 1. Let's take the following values, $H = 15, L = 5, F_0 = 10$. We know that $a \leq \frac{F_O - L}{H - L}$ must be satisfied as well. Filling in the values into $a \leq \frac{F_O - L}{H - L}$ gives $a \leq \frac{1}{2}$ (see appendix point 45), so if we would take $a = 0.2$ it satisfies $a \leq \frac{1}{2}$. Now what value must the ambiguity level be in order for investor 1 to play $s_1 = short$?

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

becomes,

$$\delta_1 < 1$$

(See appendix point 46 for derivation).

So for all values of ambiguity will investor 1 play $s_1 = short$ when $H = 15, L = 5, F_0 = 10$.

Now let's take a look at what happens if I take a different value for a. So we had $H = 15, L = 5, F_0 = 10$ and $a \leq \frac{1}{2}$. Let's take $a = 0.1$.

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

becomes,

$$\delta_1 < 1$$

(See appendix point 47 for derivation).

So still $\delta_1 < 1$. What if we take $a = 0.5$?

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

becomes,

$$\delta_1 < 1$$

(See appendix point 48 for derivation).

The pattern that becomes clear is that taking different degrees for a does not matter, when H, F_0, and L remain unchanged.

We can also change the payoff, we can change the downside payoff, H, or the upside payoff, L. Increasing the downside for investor 1 would mean increasing H, because then he loses more if he is wrong or increasing L, because then he wins less if he is right. So let's increase H only, $H = 20, L = 5, F_0 = 10$ and $a \leq \frac{F_O - L}{H - L}$. Now the condition for a changes, namely $a \leq \frac{F_O - L}{H - L}$ now becomes $a \leq \frac{1}{3}$ (see appendix point 49). So $a = 0.2$ will do. We have $H = 20, L = 5, F_0 = 10$ and $a = 0.2$:

$$\delta_1 < \frac{2a(H-L)}{(2a+1)H - (2a-1)L - 2F_0}$$

becomes,

$$\delta_1 < \frac{6}{11}$$

(See appendix point 50 for derivation).

We see that if the downside increases, ambiguity needs to be smaller and thus more confidence is needed in the belief of investor 1 by investor 1 in order to take his position. I will look at what happens to δ_1 one more time, but now by increasing L (also increasing downside for investor 1 if he takes a short position). We have $H = 15, L = 7, F_0 = 10$ and $a \leq \frac{F_O - L}{H - L}$. The new condition for $a \leq \frac{F_O - L}{H - L}$ will be $a \leq 0.375$ (see appendix point 51), $a = 0.2$ will do. We have $H = 15, L = 7, F_0 = 10$ and $a = 0.2$:

$$\delta_1 < \frac{2a(H-L)}{(2a+1)H - (2a-1)L - 2F_0}$$

becomes,

$$\delta_1 < 0.615$$

(See appendix point 52 for derivation).

Again, we see that if the downside increases, making the short position less profitable, ambiguity needs to be smaller and thus more confidence is needed in the belief of investor 1 by investor 1 in order to take his position. Therefore I conclude that the downside risk of the position matters for the ambiguity when taking a position.

The last thing we can do, in order to check if the downside here really only matters, is to increase the upside and see what happens to the ambiguity value, δ_1. This can be done in two ways: decreasing H or decreasing L. First, let's decrease L. The new values that I take are $H = 15, L = 3, F_0 = 10$ and $a \leq \frac{F_O - L}{H - L}$. Thus we get $a \leq 0.583$ (see appendix point 53). So again $a = 0.2$ will do, thus we have $H = 15, L = 3, F_0 = 10$ and $a = 0.2$:

$$\delta_1 < \frac{2a(H-L)}{(2a+1)H - (2a-1)L - 2F_0}$$

becomes,

$$\delta_1 < 1.714$$

(See appendix point 54 for derivation).

This is the same as $\delta_1 < 1$, since we have $\delta_1 \in [0,1]$.

Second, let's decrease H, the new values that I take are $H = 13, L = 5, F_0 = 10$ and $a \leq \frac{F_0 - L}{H - L}$. Thus a must satisfy $a \leq 0.625$ (see appendix point 55). So again $a = 0.2$ will do, thus we have $H = 13, L = 5, F_0 = 10$ and $a = 0.2$:

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

becomes,

$$\delta_1 < 2.667$$

(See appendix point 56 for derivation).

This is the same as $\delta_1 < 1$, since we have $\delta_1 \in [0,1]$.

So the pattern that arises is clear. If the downside is increased (increasing H or increasing L) the ambiguity that justifies the position must decrease, meaning that investor 1 must be more confident in his belief, $\pi_1(\omega)$ (the additive probability distribution). If the upside is increased (decreasing H or decreasing L) this results in the fact that the ambiguity can be higher and thus less confidence in his belief, $\pi_1(\omega)$ (the additive probability distribution) is needed.

For player 2 the same pattern arises. The analysis can be found in the appendix point 57 until 68. If the downside is increased (decreasing H or decreasing L) the ambiguity that justifies the position must decrease, meaning that investor 2 must be more confident in his belief, $\pi_2(\omega)$ (the additive probability distribution). If the upside is increased (increasing H or increasing L) this results in the fact that the ambiguity can be higher and thus less confidence in his belief, $\pi_2(\omega)$ (the additive probability distribution) is needed.

The last aspect of the ambiguity conditions that we can check is when and why the inequality sign flips from the upper bound to the lower bound. For $\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ to be true we need $(2a + 1)H - (2a - 1)L - 2F_0 > 0$, this in turn leads to the following condition for a:

$$a > \frac{2F_0 - H - L}{2(H - L)}$$

(See appendix point 69 for derivation).

For player 1 we also have another condition for a, namely $a \leq \frac{F_o-L}{H-L}$. Combining the two conditions for a, we get the following, $\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ is true if:

$$\frac{2F_0 - H - L}{2(H-L)} < a \leq \frac{F_o - L}{H - L}$$

If this is not fulfilled then the inequality sign flips from upper bound to lower bound. The condition for the lower bound is always satisfied, because then $(2a+1)H - (2a-1)L - 2F_0 < 0$ and $2a(H-L) > 0$ is always true, combining the two leads to $\frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0} < 0$. The condition now states that ambiguity must be larger than a negative number, since I have stated that $\delta_1 \in [0,1]$ the inequality is always satisfied.

In order to make it less abstract, in point 70 in the appendix, I examined some numerical examples for $a > \frac{2F_0-H-L}{2(H-L)}$. What becomes clear is only when H and F_0 are high and close to each other and L is low and far away from H and F_0, simultaneously, then it can be possible that the inequality sign, from $\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$, flips from upper bound to lower bound. This only happens when the increment in investor 1's belief in ω_L is smaller than $\frac{2F_0-H-L}{2(H-L)}$ thus needing ambiguity. However, if H and F_0 are high and close to each other and L is low and far away from H and F_0 then the upside is high and downside is low, simultaneously. Which may cause that every value of ambiguity suffices.

The way of analysis and pattern that is found will be the same for investor 2. So when the upside is high and the downside is small, then the inequality sign may flip in $\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$. Since investor 2 takes the opposite position of investor 1 the upside and downside for investor 2 are opposite to the upside a downside for investor 1. That's the difference, however the result of the pattern will be the same.

7 Discussion

First of all the analysis in section 4 has shown that ambiguity definitely influences agent's behavior in strategic interaction. For the 'not rational game' a very low level of ambiguity aversion already can lead to deviations from the Nash equilibrium prediction. As for the ‘non-credible threat game the ambiguity aversion value is higher although still very possible that people may exhibit such a value in reality. The difference in the ambiguity aversion levels is the result of the payoff difference in the two games. Something that becomes clear if the two games are compared is that the larger the difference between the payoffs for player 1, when player 2 does not play the punishment strategy and when player 2 does play the punishment strategy, the lower the ambiguity aversion value needs to be for player 1 to deviate from the Nash equilibrium prediction.

In the first part of the analysis, I assumed that the belief of player 1 in player 2 playing $s_2 = P$ is zero. Thus the additive probability distribution of player 1, representing his beliefs about player 2, has the form: $\pi_1(s_2) = \{p_1(s_2 = P) = 0, p_1(s_2 = N) = 1\}$. This is thanks to the rationality assumption, but as player 1 faces ambiguity the rationality assumption may not hold and thus the belief $\pi_1(s_2) = \{p_1(s_2 = P) = 0, p_1(s_2 = N) = 1\}$ may not be optimal/realistic for player 1. This is why I also looked at what happened if we allowed the belief of player 1 to vary. Meaning, the additive probability distribution of player 1, representing his beliefs, now has the form: $\pi_1(s_2) = \{p_1(s_2 = P) = p, p_1(s_2 = N) = 1 - p\}$ where $0 \leq p \leq 1$.

With $\pi_1(s_2) = \{p_1(s_2 = P) = p, p_1(s_2 = N) = 1 - p\}$, I looked at the interactions between p, δ_1 and α_1 and a clear pattern came to the surface. If p increases then α_1, the optimism that allows $s_1 = S \succ s_1 = R$, increases as well. As for $1 - \alpha_1$, if p increases then $1 - \alpha_1$, the pessimism needed to induce $s_1 = S \succ s_1 = R$, decreases. This is true for both games. A little explanation to make it clearer may be beneficial here. In both games optimism by player 1 means that player 1 assumes that player 2 is rational and pessimism by player 1 means that player 1 assumes player 2 is not rational. So if p, the belief of player 1 in player 2 playing $s_2 = P$ (the punishment strategy, which is not rational), increases, meaning player 1 deems it more likely that player 2 will play this strategy, then a more optimistic player 1, so a player 1 who initially more strongly assumes that player 2 is rational, will prefer playing $s_1 = S$ (his safe option) instead of playing $s_1 = R$. Conversely if p, the belief of player 1 in player 2 playing $s_2 = P$ (the punishment strategy, which is not rational), increases, meaning player 1 deems it more likely that player 2 will play this strategy, then a pessimistic player 1, so a player 1 who initially more strongly assumes that player 2 is not rational, now needs lower levels of pessimism to prefer playing $s_1 = S$ (his safe option) instead of playing $s_1 = R$.

Last, I investigated the relation between ambiguity, δ_1, and the belief of player 1 in player 2 playing $s_2 = P$, p, when α_1 is fixed. This relation is not as clear cut as the ones between optimism and the belief of player 1 in player 2 playing $s_2 = P$ and between pessimism and the belief of player 1 in player 2 playing $s_2 = P$. For the 'not rational game' I found the following. For smaller values of p, δ_1 needs to be lower, meaning if the belief of player 1 in player 2 playing $s_2 = P$ is low then the ambiguity in this belief must be low as well to induce $s_1 = S \succ s_1 = R$. Thus for low levels of p player 1 must be confident in his belief. As p increases and becomes larger then 0.5 and keeps increasing then more ambiguity is allowed to induce $s_1 = S \succ s_1 = R$. Thus for high levels of p player 1 can be less confident in this belief in order to prefer playing $s_1 = S$ over playing $s_1 = R$. For the 'non-credible threat game' the results are somewhat the same, but not entirely. For lower levels of p I get $\delta_1 > 1$ which is not possible. Therefore, here for low levels of p $s_1 = S \succ s_1 = R$ cannot be induced. Then as p increases, δ_1 increases as well, so again if p increases less confidence in this belief is needed for player 1 to have the preference ordering $s_1 = S \succ s_1 = R$. Overall, for low levels of p $s_1 = S \succ s_1 = R$ cannot happen or player 1 must be very confident in this belief ($1 - \delta_1$ must be high), then as p increases to higher levels more ambiguity is allowed and thus less confidence is needed.

When the 'not rational game' and 'non-credible threat game' are compared another interesting finding comes up. Ambiguity values must be lower and thus players must be more confident in their beliefs, when the downside is higher. This can be seen from the 'not rational game', where the payoff difference for player 1 when player 2 does not play his punishment strategy and when he does is larger than in the 'non-credible threat game'. In the 'non-credible threat game' this difference is not so large and the analysis shows that the ambiguity aversion level must be higher. If we would fix $(1 - \alpha_1)$ it would lead to a higher ambiguity value, δ_1 that is allowed, in the 'non-credible threat game' than in the 'not rational game'. All this, thanks to a larger downside in the 'not rational game'.

Further, I investigated the relationship between ambiguity aversion and risk attitudes. For the two games in section 4 in the original analysis the player's attitude towards risk was neutral. A clear pattern arises when risk aversion and risk loving is incorporated. When the player is risk averse his ambiguity aversion level needed to deviate from the Nash equilibrium prediction is lowered. Thus a risk-averse player would deviate sooner than a risk-neutral player. As for risk loving and ambiguity aversion the opposite effect takes form. When a player is risk loving his ambiguity aversion level needed to deviate from the Nash equilibrium prediction is higher. Meaning that a risk-loving player would deviate later than a risk neutral player.

As for the futures market, the analysis without ambiguity shows what conditions the beliefs of the investors must satisfy to come to an agreement. But when ambiguity over the state space is incorporated other conditions are added that also must be satisfied. The conditions from section 6.1 only are not enough anymore. If $\pi_i(\omega)$ and $\pi_{-i}(\omega)$ satisfy the necessary conditions, then for ambiguity the following pattern arises. When the downside to a position increases, ambiguity in the belief of an investor must decrease, in other words with the risk of a higher loss the investor must be more confident. As for an increase in the upside it is the other way around. When the upside is increased, so the potential gains are increased, then ambiguity can increase as well. In other words when the potential gains of a position are increased less confidence in an investor's belief is needed in order to justify that position.

Overall, an important feature that is found thanks to this analysis is the fact that when downsides of a position or a strategy are increased the ambiguity in an individual's belief must be lower otherwise the individual will not take that position or strategy.

8 Conclusion

In this thesis decision making under uncertainty is examined in order to investigate why deviations from Nash equilibrium predictions occur in reality. Since Expected Utility Theory and Subjective Expected Utility Theory do a poor job at predicting actual behavior, combined with the fact that in many economic situations it is not clear why agents should know probabilities, the need for new models is high.

In previous literature, it has been suggested to use capacities instead of an additive probability distribution to represent beliefs. This is because capacities are not necessarily additive, which reflects the uncertainty/ambiguity. In order to compute the expected value with capacities, the Choquet integral should be applied. Using capacities is not enough, since with capacities a too broad range of behavior can be explained/justified. Therefore a special type of capacities was used in this paper, namely neo-additive capacities.

The analysis of the games in this paper shows that attitudes towards ambiguity and ambiguity itself indeed influence behavior and can explain deviations from Nash equilibrium. I investigated the relationship between the belief of a player, ambiguity and its attitude towards ambiguity. The pattern that becomes clear between ambiguity and the belief of a player is when the belief of a player in something decreases, ambiguity in this belief needs to decrease as well, in other words he needs to be more confident, in order to justify the corresponding action.

For the games in section 4, the following is the result from the analysis: when the belief of player 1 in player 2 playing $s_2 = P$ increases, optimism of player 1 that allows $s_1 = S \succ s_1 = R$ increases as well and conversely pessimism of player 1 that is needed to induce $s_1 = S \succ s_1 = R$ decreases. This is when the ambiguity is fixed at a certain level. If α is fixed the relationship between p and δ becomes clear. If p (the belief of player 1 in player 2 playing $s_2 = P$) is low, ambiguity in this belief (δ) must be low as well, meaning more confidence is required, and if p is high, ambiguity can be higher as well.

As for the relationship between ambiguity and risk attitudes it becomes visible that when a player is risk averse the ambiguity aversion level needed to deviate from the Nash equilibrium prediction is lowered. When a player is risk-loving the ambiguity aversion level needed to deviate from the Nash equilibrium prediction is increased.

In the futures market game, the pattern is the same as the pattern from the ‘not rational game’ and ‘non-credible threat game’. When the downside is increased, the confidence in his belief must increase and thus ambiguity must decrease. In the games of section 4 the downside is the payoff (to player 1) difference when player 2 does not play the punishment strategy and when player 2 does play his punishment strategy. For the futures market game the downside is decreasing the payoff if one is right and increasing the loss when one is wrong. The bottom line here is if the downside is increased, ambiguity needs to be lower.

So from this paper it becomes clear that ambiguity is indeed important in decision-making and can rationally lead to different equilibria. Also the relationship between its different components has been made clear.

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Appendix

1. Derivation of decision weight for the minimal outcome:

The decision weight for the maximal outcome, $max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$, is given by $v\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) = \delta\alpha + (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$. Then the decision weight for the minimal outcome, $min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$, is given by

$$v\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$
$$= 1 - v\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$
$$= 1 - \left(\delta\alpha + (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)\right)$$
$$= 1 - \ \delta\alpha - (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$

We know that $\pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) + \pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) = 1$, thus $\pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) = 1 - \pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$. We can fill this in, in the above equation. Then we get

$$= 1 - \ \delta\alpha - (1-\delta)\left(1 - \pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)\right)$$
$$= 1 - \delta\alpha - 1 + \ \pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) + \delta - \delta\,\pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$
$$= \delta - \delta\alpha + \ \pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) - \delta\,\pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$
$$= \delta(1-\alpha) + (1-\delta)\pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$

2. Derivation of Choquet expected utility with neo-additive capacity $v = \delta\alpha + (1-\alpha)\pi$:

$$CEU = V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i) = \ \int u_i(s_i, s_{-i})dv$$
$$= v\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right) \cdot max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$$
$$+ \int u_i(s_i, s_{-i}|x_{min} < x < x_{max})dv + v\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)$$
$$\cdot\, min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$$
$$= \left(\delta\alpha + (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)\right) \cdot max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$$
$$+ \left(\delta(1-\alpha) + (1-\delta)\pi\left(min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right)\right)$$
$$\cdot\, min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i}) + \sum v(x) \cdot u_i(s_i, s_{-i}|x_{min} < x < x_{max})$$

$$= \left(\delta\alpha + (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)\right)\cdot max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ \left(\delta(1-\alpha) + (1-\delta)\pi\left(min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)\right)$$

$$\cdot\, min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ \sum(1-\delta)\pi(x)\cdot u_i(s_i,s_{-i}|x_{min} < x < x_{max})$$

$$= \delta\alpha\cdot max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ \delta(1-\alpha)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ (1-\delta)\pi\left(min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ (1-\delta)\sum\pi(x)\cdot u_i(s_i,s_{-i}|x_{min} < x < x_{max})$$

$$= \delta\alpha\cdot max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i}) + \delta(1-\alpha)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ (1-\delta)\sum\pi(x)\cdot u_i(s_i,s_{-i}|x_{min} < x < x_{max})$$

$$+ (1-\delta)\pi\left(max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$+ (1-\delta)\pi\left(min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$= \delta\left[\alpha\cdot max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i}) + (1-\alpha)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right]$$

$$+ (1-\delta)\left(\sum\pi(x)\cdot u_i(s_i,s_{-i}|x_{min} < x < x_{max})\right.$$

$$+ \pi\left(max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})$$

$$\left. + \pi\left(min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right)$$

$$= \delta\left[\alpha\cdot max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i}) + (1-\alpha)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right]$$

$$+ (1-\delta)\sum\pi(x)\cdot u_i(s_i,s_{-i})$$

$$= \delta\left[\alpha\cdot max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i}) + (1-\alpha)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right]$$

$$+ (1-\delta)\int u(s_i,s_{-i})d\pi(s_{-i})$$

Which is equal to

$$\delta_i\left[\alpha_i max_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i}) + (1-\alpha_i)min_{s_{-i}\in S_{-i}}u_i(s_i,s_{-i})\right]$$

$$+ (1-\delta_i)\int u_i(s_i,s_{-i})d\pi_i(s_{-i})$$

And this is equal to the equation from definition 3.2 in the main text.

3. Derivation of Choquet expected utilities for player 1 in the game of section 4.1. I make use of the following function:

$$V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i\left[\alpha_i max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha_i) min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right]$$
$$+ (1-\delta_i)\int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2\in S_2} u_1(s_1 = S, s_2) + (1-\alpha_1) min_{s_2\in S_2} u_1(s_1 = S, s_2)\right]$$
$$+ (1-\delta_1)\int u_1(s_1 = S, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 \cdot 80 + (1-\alpha_1)\cdot 80] + (1-\delta_1)\cdot 80$$
$$= \delta_1[80\alpha_1 - 80\alpha_1 + 80] + 80 - 80\delta_1$$
$$= 80\delta_1 - 80\delta_1 + 80$$
$$= 80$$

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2\in S_2} u_1(s_1 = R, s_2) + (1-\alpha_1) min_{s_2\in S_2} u_1(s_1 = R, s_2)\right]$$
$$+ (1-\delta_1)\int u_1(s_1 = R, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 \cdot 90 + (1-\alpha_1)\cdot 20] + (1-\delta_1)\cdot 90$$
$$= \delta_1[90\alpha_1 - 20\alpha_1 + 20] + 90 - 90\delta_1$$
$$= \delta_1[70\alpha_1 + 20] + 90 - 90\alpha_1$$
$$= 90 + 20\delta_1 - 90\delta_1 + 70\delta_1\alpha_1$$
$$= 90 - 70\delta_1 + 70\delta_1\alpha_1$$
$$= 90 - \delta_1(1-\alpha_1)70$$

In order to get the term $(1-\delta_1)\cdot 90$ full rationality is assumed. To see how this works look at the game in section 4.1 again. If it's player 2's turn, he can choose between $s_2 = P$ and $s_2 = N$. $s_2 = P$ yields player 2 a payoff of 68 and $s_2 = N$ a payoff of 70. Since $68 < 70$, if rational, player 2 plays $s_2 = N$. Player 1 can deduce this. He has a choice between $s_1 = S$ and $s_1 = R$. $s_1 = S$ yields a payoff of 80, whereas $s_1 = R$ can lead to a payoff of 20 or 90. But, since full rationality is assumed player 1 expects player 2 to play $s_2 = N$ (because $s_2 = N$ leads to a higher payoff for player 2). This yields a payoff of 90 for player 1 if he plays $s_1 = R$. So if player 1 plays $s_1 = R$, he expects a payoff of 90, but he does not have full confidence in this belief, hence $(1-\delta_1)\cdot 90$. This logic is also true for points 5, 24, 26, 28, 30 in this appendix.

4. Derivation of ambiguity aversion critical value for player 1 in the game of section 4.1:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

$$80 > 90 - \delta_1(1-\alpha_1)70$$

$$-10 > -\delta_1(1-\alpha_1)70$$

$$-10/_{-70} < \delta_1(1-\alpha_1)$$

$$\delta_1(1-\alpha_1) > \frac{1}{7}$$

$$\delta_1(1-\alpha_1) > 0.143$$

5. Derivation of Choquet expected utilities for player 1 in the game of section 4.2. I make use of the following function:

$$V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i\left[\alpha_i max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha_i) min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right]$$
$$+ (1-\delta_i)\int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2\in S_2} u_1(s_1 = S, s_2) + (1-\alpha_1) min_{s_2\in S_2} u_1(s_1 = S, s_2)\right]$$
$$+ (1-\delta_1)\int u_1(s_1 = S, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 \cdot 70 + (1-\alpha_1)\cdot 70] + (1-\delta_1)\cdot 70$$
$$= \delta_1[70\alpha_1 - 70\alpha_1 + 70] + 70 - 70\delta_1$$
$$= 70\delta_1 - 70\delta_1 + 70$$
$$= 70$$

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2\in S_2} u_1(s_1 = R, s_2) + (1-\alpha_1) min_{s_2\in S_2} u_1(s_1 = R, s_2)\right]$$
$$+ (1-\delta_1)\int u_1(s_1 = R, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 \cdot 90 + (1-\alpha_1)\cdot 60] + (1-\delta_1)\cdot 90$$
$$= \delta_1[90\alpha_1 - 60\alpha_1 + 60] + 90 - 90\delta_1$$
$$= \delta_1[30\alpha_1 + 60] + 90 - 90\delta_1$$
$$= 90 + 60\delta_1 - 90\delta_1 + 30\delta_1\alpha_1$$
$$= 90 - 30\delta_1 + 30\delta_1\alpha_1$$
$$= 90 - \delta_1(1-\alpha_1)30$$

6. Derivation of ambiguity aversion critical value for player 1 in the game of section 4.2:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

$$70 > 90 - \delta_1(1 - \alpha_1)30$$

$$-20 > -\delta_1(1 - \alpha_1)30$$

$$\frac{-20}{-30} < \delta_1(1 - \alpha_1)$$

$$\delta_1(1 - \alpha_1) > \frac{2}{3}$$

$$\delta_1(1 - \alpha_1) > 0.667$$

7. Derivation of $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ with $\pi_1(s_2) = \{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$ for section 4.3

$$V_1(s_1 = R|\alpha_1, \delta_1, \pi_1) = \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R, s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R, s_2)\right] + (1 - \delta_1)\int u_1(s_1, s_2) d\pi_1(s_2)$$

We know that $\int u_1(s_1, s_2) d\pi_1(s_2)$ with $\pi_1(s_2) = \{p_1(s_2 = P);\ p_1(s_2 = N)\}$ is:

$$p_1(s_2 = P)u_1(s_1 = R;\ s_2 = P) + p_1(s_2 = N)u_1(s_1 = R;\ s_2 = N)$$

So, for $V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$ we get:

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R;\ s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R;\ s_2)\right] + (1 - \delta_1)[p_1(s_2 = P)u_1(s_1 = R;\ s_2 = P) + p_1(s_2 = N)u_1(s_1 = R;\ s_2 = N)]$$

8. Derivation of Choquet expected utilities for player 1 in the game of section 4.3.1. I make use of the following formula:

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) = \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R;\ s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R;\ s_2)\right] + (1 - \delta_1)[p_1(s_2 = P)u_1(s_1 = R;\ s_2 = P) + p_1(s_2 = N)u_1(s_1 = R;\ s_2 = N)]$$

and $\pi_1(s_2) = \{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$.

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\big[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R;\ s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = R;\ s_2)\big]$$
$$+ (1-\delta_1)[p \cdot u_1(s_1 = R;\ s_2 = P) + (1-p) \cdot u_1(s_1 = R;\ s_2 = N)]$$
$$= \delta_1[\alpha_1 \cdot 90 + (1-\alpha_1) \cdot 20] + (1-\delta_1)[p \cdot 20 + (1-p) \cdot 90]$$
$$= \delta_1(90\alpha_1 - 20\alpha_1 + 20) + (1-\delta_1)(20p - 90p + 90)$$
$$= \delta_1(70\alpha_1 + 20) + (1-\delta_1)(-70p + 90)$$
$$= \delta_1(70\alpha_1 + 20) + (1-\delta_1)(90 - 70p)$$
$$= \delta_1(70\alpha_1 + 20) + (90 - 70p) - \delta_1(90 - 70p)$$
$$= \delta_1(70\alpha_1 + 20 - 90 + 70p) + (90 - 70p)$$
$$= (90 - 70p) + \delta_1(-70 + 70\alpha_1 + 70p)$$
$$= (90 - 70p) + \delta_1(-1 + \alpha_1 + p)70$$
$$= (90 - 70p) - \delta_1(1 - \alpha_1 - p)70$$
$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\big[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = S;\ s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = S;\ s_2)\big]$$
$$+ (1-\delta_1)[p \cdot u_1(s_1 = S;\ s_2 = P) + (1-p) \cdot u_1(s_1 = S;\ s_2 = N)]$$
$$= \delta_1[\alpha_1 \cdot 80 + (1-\alpha_1) \cdot 80] + (1-\delta_1)(p \cdot 80 + (1-p) \cdot 80)$$
$$= \delta_1[\alpha_1 \cdot 80 - \alpha_1 \cdot 80 + 80] + (1-\delta_1)(p \cdot 80 - p \cdot 80 + 80)$$
$$= \delta_1 \cdot 80 - \delta_1 \cdot 80 + 80$$
$$= 80$$

9. Derivation of ambiguity aversion critical value for player 1 in the game of section 4.3.1:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$80 > (90 - 70p) - \delta_1(1 - \alpha_1 - p)70$$
$$80 - (90 - 70p) > -\delta_1(1 - \alpha_1 - p)70$$
$$\frac{80 - (90 - 70p)}{-70} < \delta_1(1 - \alpha_1 - p)$$
$$\delta_1(1 - \alpha_1 - p) > \frac{80 - (90 - 70p)}{-70}$$

Now, let's fill in $p = 0$:

$$\delta_1(1 - \alpha_1 - 0) > \frac{80 - (90 - 70 \cdot 0)}{-70}$$
$$\delta_1(1 - \alpha_1 - 0) > \frac{80 - 90}{-70}$$

$$\delta_1(1-\alpha_1-0) > \frac{-10}{-70}$$

$$\delta_1(1-\alpha_1-0) > 0.143$$

Now, let's fill in $p = 1$:

$$\delta_1(1-\alpha_1-1) > \frac{80-(90-70\cdot 1)}{-70}$$

$$\delta_1(-\alpha_1) > \frac{80-20}{-70}$$

$$-\delta_1\alpha_1 > -\frac{60}{70}$$

$$\delta_1\alpha_1 < \frac{6}{7}$$

10. Derivation of the condition for α_1 for which $s_1 = S \succ s_1 = R$ applies with p as a variable and δ_1 is fixed at 0.52, for the game in section 4.3.1:

$$\delta_1(1-\alpha_1-p) > \frac{80-(90-70p)}{-70}$$

$$1-\alpha_1-p > \frac{80-(90-70p)}{-70\delta_1}$$

$$-\alpha_1 > -1 + \frac{80-(90-70p)}{-70\delta_1} + p$$

$$\alpha_1 < 1 - \frac{80-(90-70p)}{-70\delta_1} - p$$

So for δ_1 I take the average, which is 0.52, filling this in leads to:

$$\alpha_1 < 1 - \frac{80-(90-70p)}{-70\cdot 0.52} - p$$

$$\alpha_1 < 1 - \frac{80-(90-70p)}{-36.4} - p$$

11. Table A.11: Relationship between α_1 and p, when $\delta_1 = 0.52$, for the game in section 4.3.1:

$p = 0$	$p = 0.1$	$p = 0.2$
$\alpha_1 < 1 - \frac{80-(90-70\cdot 0)}{-36.4} + 0$ $\alpha_1 < 0.726$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.1)}{-36.4} + 0.1$ $\alpha_1 < 0.818$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.2)}{-36.4} + 0.1$ $\alpha_1 < 0.910$
$p = 0.3$	$p = 0.4$	$p = 0.5$

$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.3)}{-36.4} + 0.3$ $\alpha_1 < 1.002$ $\alpha_1 < 1.000$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.4)}{-36.4} + 0.4$ $\alpha_1 < 1.095$ $\alpha_1 < 1.000$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.5)}{-36.4} + 0.5$ $\alpha_1 < 1.185$ $\alpha_1 < 1.000$
$p = 0.6$	$p = 0.7$	$p = 0.8$
$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.6)}{-36.4} + 0.6$ $\alpha_1 < 1.279$ $\alpha_1 < 1.000$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.7)}{-36.4} + 0.7$ $\alpha_1 < 1.371$ $\alpha_1 < 1.000$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.8)}{-36.4} + 0.8$ $\alpha_1 < 1.464$ $\alpha_1 < 1.000$
$p = 0.9$	$p = 1.0$	
$\alpha_1 < 1 - \frac{80-(90-70\cdot 0.9)}{-36.4} + 0.9$ $\alpha_1 < 1.556$ $\alpha_1 < 1.000$	$\alpha_1 < 1 - \frac{80-(90-70\cdot 1.0)}{-36.4} + 1.0$ $\alpha_1 < 1.648$ $\alpha_1 < 1.000$	

12. Derivation of the condition for $1 - \alpha_1$ for which $s_1 = S \succ s_1 = R$ applies with p as a variable and δ_1 is fixed at 0.52, for the game in section 4.3.1:

$$\delta_1(1 - \alpha_1 - p) > \frac{80 - (90 - 70p)}{-70}$$

$$1 - \alpha_1 - p > \frac{80 - (90 - 70p)}{-70\delta_1}$$

$$1 - \alpha_1 > \frac{80 - (90 - 70p)}{-70\delta_1} + p$$

So for δ_1 I take the average, which is 0.52, filling this in leads to:

$$1 - \alpha_1 > \frac{80 - (90 - 70p)}{-70 \cdot 0.52} + p$$

$$1 - \alpha_1 > \frac{80 - (90 - 70p)}{-36.4} + p$$

13. Table A.13: Relationship between $1 - \alpha_1$ and p, when $\delta_1 = 0.52$, for the game in section 4.3.1:

$p = 0$	$p = 0.1$	$p = 0.2$
$1 - \alpha_1 > \frac{80-(90-70\cdot 0)}{-36.4} + 0$ $1 - \alpha_1 > 0.275$	$1 - \alpha_1 > \frac{80-(90-70\cdot 0.1)}{-36.4} + 0.1$ $1 - \alpha_1 > 0.182$	$1 - \alpha_1 > \frac{80-(90-70\cdot 0.2)}{-36.4} + 0.2$ $1 - \alpha_1 > 0.090$
$p = 0.3$	$p = 0.4$	$p = 0.5$
$1 - \alpha_1 > \frac{80-(90-70\cdot 0.3)}{-36.4} + 0.3$ $1 - \alpha_1 > -0.002$	$1 - \alpha_1 > \frac{80-(90-70\cdot 0.4)}{-36.4} + 0.4$ $1 - \alpha_1 > -0.095$	$1 - \alpha_1 > \frac{80-(90-70\cdot 0.5)}{-36.4} + 0.5$ $1 - \alpha_1 > -0.187$

$1-\alpha_1 > 0$	$1-\alpha_1 > 0$	$1-\alpha_1 > 0$
$p = 0.6$	$p = 0.7$	$p = 0.8$
$1-\alpha_1 > \frac{80-(90-70\cdot 0.6)}{-36.4}+0.6$ $1-\alpha_1 > -0.279$ $1-\alpha_1 > 0$	$1-\alpha_1 > \frac{80-(90-70\cdot 0.7)}{-36.4}+0.7$ $1-\alpha_1 > -0.371$ $1-\alpha_1 > 0$	$1-\alpha_1 > \frac{80-(90-70\cdot 0.8)}{-36.4}+0.8$ $1-\alpha_1 > -0.464$ $1-\alpha_1 > 0$
$p = 0.9$	$p = 1.0$	
$1-\alpha_1 > \frac{80-(90-70\cdot 0.9)}{-36.4}+0.9$ $1-\alpha_1 > -0.556$ $1-\alpha_1 > 0$	$1-\alpha_1 > \frac{80-(90-70\cdot 1.0)}{-36.4}+1.0$ $1-\alpha_1 > -0.648$ $1-\alpha_1 > 0$	

14. Derivation of the condition for δ_1 for which $s_1 = S \succ s_1 = R$ applies with p as a variable and α_1 is fixed at 0.5, for the game in section 4.3.1:

$$\delta_1(1-\alpha_1-p) > \frac{80-(90-70p)}{-70}$$

$$\delta_1 > \frac{80-(90-70p)}{-70(1-\alpha_1-p)}$$

The value for α_1 that I will use will be the average from the KW-range so, $\alpha_1 = 0.5$.

$$\delta_1 > \frac{80-(90-70p)}{-70(1-0.5-p)}$$

$$\delta_1 > \frac{80-(90-70p)}{-70(0.5-p)}$$

This formula only holds if $0 \leq p < 0.5$ because then $-70(0.5-p) < 0$ and the sign does not flip. But when $0.5 < p \leq 1$ then $-70(0.5-p) > 0$ and the sign flips from $>$ to $<$. Thus when $0 \leq p < 0.5$ the following condition is correct:

$$\delta_1 > \frac{80-(90-70p)}{-70(0.5-p)}$$

And when $0.5 < p \leq 1$ the following condition is correct:

$$\delta_1 < \frac{80-(90-70p)}{-70(0.5-p)}$$

15. Table A.15: Relationship between δ_1 and p, when $\alpha_1 = 0.5$, for the game in section 4.3.1:

$p = 0$	$p = 0.1$	$p = 0.2$
$\delta_1 > \frac{80-(90-70\cdot 0)}{-70(0.5-0)}$ $\delta_1 > 0.286$	$\delta_1 > \frac{80-(90-70\cdot 0.1)}{-70(0.5-0.1)}$ $\delta_1 > 0.107$	$\delta_1 > \frac{80-(90-70\cdot 0.2)}{-70(0.5-0.2)}$ $\delta_1 > -0.190$

		$\delta_1 > 0$
$p = 0.3$	$p = 0.4$	$p = 0.5$
$\delta_1 > \frac{80 - (90 - 70 \cdot 0.3)}{-70(0.5 - 0.3)}$ $\delta_1 > -0.786$ $\delta_1 > 0$	$\delta_1 > \frac{80 - (90 - 70 \cdot 0.4)}{-70(0.5 - 0.4)}$ $\delta_1 > -2.571$ $\delta_1 > 0$	$\delta_1 > \frac{80 - (90 - 70 \cdot 0.5)}{-70(0.5 - 0.5)}$ $\delta_1 > \frac{25}{0}$ Not possible
$p = 0.6$	$p = 0.7$	$p = 0.8$
$\delta_1 > \frac{80 - (90 - 70 \cdot 0.6)}{-70(0.5 - 0.6)}$ $\delta_1 < 4.571$ $\delta_1 < 1$	$\delta_1 > \frac{80 - (90 - 70 \cdot 0.7)}{-70(0.5 - 0.7)}$ $\delta_1 < 2.786$ $\delta_1 < 1$	$\delta_1 > \frac{80 - (90 - 70 \cdot 0.8)}{-70(0.5 - 0.8)}$ $\delta_1 < 2.190$ $\delta_1 < 1$
$p = 0.9$	$p = 1.0$	
$\delta_1 > \frac{80 - (90 - 70 \cdot 0.9)}{-70(0.5 - 0.9)}$ $\delta_1 < 1,893$ $\delta_1 < 1$	$\delta_1 > \frac{80 - (90 - 70 \cdot 1.0)}{-70(0.5 - 1.0)}$ $\delta_1 < 1.714$ $\delta_1 < 1$	

16. Derivation of Choquet expected utilities for player 1 in the game of section 4.3.2.

I make use of the following formula:

$$V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R;\ s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R;\ s_2)\right]$$
$$+ (1 - \delta_1)[p_1(s_2 = P) u_1(s_1 = R;\ s_2 = P)$$
$$+ p_1(s_2 = N) u_1(s_1 = R;\ s_2 = N)]$$

and $\pi_1(s_2) = \{p_1(s_2 = P) = p;\ p_1(s_2 = N) = 1 - p\}$.

$$V_1(s_1 = R | s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R;\ s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R;\ s_2)\right]$$
$$+ (1 - \delta_1)[p \cdot u_1(s_1 = R;\ s_2 = P) + (1 - p) \cdot u_1(s_1 = R;\ s_2 = N)]$$
$$= \delta_1[\alpha_1 \cdot 90 + (1 - \alpha_1) \cdot 60] + (1 - \delta_1)[p \cdot 60 + (1 - p) \cdot 90]$$
$$= \delta_1(90\alpha_1 - 60\alpha_1 + 60) + (1 - \delta_1)(60p - 90p + 90)$$
$$= \delta_1(30\alpha_1 + 60) + (1 - \delta_1)(-30p + 90)$$
$$= \delta_1(30\alpha_1 + 60) + (1 - \delta_1)(90 - 30p)$$
$$= \delta_1(30\alpha_1 + 60) + (90 - 30p) - \delta_1(90 - 30p)$$
$$= \delta_1(30\alpha_1 + 60 - 90 + 30p) + (90 - 30p)$$
$$= (90 - 30p) + \delta_1(-30 + 30\alpha_1 + 30p)$$
$$= (90 - 30p) + \delta_1(-1 + \alpha_1 + p)30$$
$$= (90 - 30p) - \delta_1(1 - \alpha_1 - p)30$$

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\big[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = S;\ s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = S;\ s_2)\big]$$
$$+ (1-\delta_1)[p \cdot u_1(s_1 = S;\ s_2 = P) + (1-p) \cdot u_1(s_1 = S;\ s_2 = N)]$$
$$= \delta_1[\alpha_1 \cdot 70 + (1-\alpha_1) \cdot 70] + (1-\delta_1)(p \cdot 70 + (1-p) \cdot 70)$$
$$= \delta_1[\alpha_1 \cdot 70 - \alpha_1 \cdot 70 + 70] + (1-\delta_1)(p \cdot 70 - p \cdot 70 + 70)$$
$$= \delta_1 \cdot 70 - \delta_1 \cdot 70 + 70$$
$$= 70$$

17. Derivation of ambiguity aversion critical value for player 1 in the game of section 4.3.2:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$70 > (90 - 30p) - \delta_1(1 - \alpha_1 - p)30$$
$$70 - (90 - 30p) > -\delta_1(1 - \alpha_1 - p)30$$
$$\frac{70 - (90 - 30p)}{-30} < \delta_1(1 - \alpha_1 - p)$$
$$\delta_1(1 - \alpha_1 - p) > \frac{70 - (90 - 30p)}{-30}$$

Now, let's fill in $p = 0$:

$$\delta_1(1 - \alpha_1 - 0) > \frac{70 - (90 - 30 \cdot 0)}{-30}$$
$$\delta_1(1 - \alpha_1) > \frac{70 - 90}{-30}$$
$$\delta_1(1 - \alpha_1) > \frac{-20}{-30}$$
$$\delta_1(1 - \alpha_1) > 0.667$$

Now, let's fill in $p = 1$:

$$\delta_1(1 - \alpha_1 - 1) > \frac{70 - (90 - 30 \cdot 1)}{-30}$$
$$\delta_1(-\alpha_1) > \frac{70 - 60}{-30}$$
$$-\delta_1\alpha_1 > \frac{10}{-30}$$
$$\delta_1\alpha_1 < \frac{1}{3}$$

18. Derivation of the condition for α_1 for which $s_1 = S \succ s_1 = R$ applies with p as a variable and δ_1 is fixed at 0.52, for the game in section 4.3.2:

$$\delta_1(1 - \alpha_1 - p) > \frac{70 - (90 - 30p)}{-30}$$

$$1 - \alpha_1 - p > \frac{70 - (90 - 30p)}{-30\delta_1}$$

$$-\alpha_1 > -1 + \frac{70 - (90 - 30p)}{-30\delta_1} + p$$

$$\alpha_1 < 1 - \frac{70 - (90 - 30p)}{-30\delta_1} - p$$

I said before, δ_1 will be fixed at 0.52, filling 0.52 in leads to:

$$\alpha_1 < 1 - \frac{70 - (90 - 30p)}{-30 \cdot 0.52} - p$$

$$\alpha_1 < 1 - \frac{70 - (90 - 30p)}{-15.6} - p$$

19. Table B.19: Relationship between α_1 and p, when $\delta_1 = 0.52$, for the game in section 4.3.2:

$p = 0$	$p = 0.1$	$p = 0.2$
$\alpha_1 < 1 - \frac{70-(90-30\cdot 0)}{-15.6} - 0$ $\alpha_1 < -0.282$ $\alpha_1 < 0$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.1)}{-15.6} - 0.1$ $\alpha_1 < -0.190$ $\alpha_1 < 0$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.2)}{-15.6} - 0.2$ $\alpha_1 < -0.097$ $\alpha_1 < 0$
$p = 0.3$	$p = 0.4$	$p = 0.5$
$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.3)}{-15.6} - 0.3$ $\alpha_1 < -0.005$ $\alpha_1 < 0$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.4)}{-15.6} - 0.4$ $\alpha_1 < 0.087$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.5)}{-15.6} - 0.5$ $\alpha_1 < 0.179$
$p = 0.6$	$p = 0.7$	$p = 0.8$
$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.6)}{-15.6} - 0.6$ $\alpha_1 < 0.272$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.7)}{-15.6} - 0.7$ $\alpha_1 < 0.364$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.8)}{-15.6} - 0.8$ $\alpha_1 < 0.456$
$p = 0.9$	$p = 1.0$	
$\alpha_1 < 1 - \frac{70-(90-30\cdot 0.9)}{-15.6} - 0.9$ $\alpha_1 < 0.549$	$\alpha_1 < 1 - \frac{70-(90-30\cdot 1.0)}{-15.6} - 1.0$ $\alpha_1 < 0.641$	

The pattern that arises is the same as the pattern for the game from section 4.3.1. The only difference is that the value for which the optimism, α_1, needs to be lower, to induce $s_1 = S \succ s_1 = R$, is lower in the 'non-credible threat game' compared to the 'not rational game'. This is because of the payoff difference between the two games. However, the same pattern arises.

So if p increases, the upper bound of the optimism values that will result in $s_1 = S \succ s_1 = R$ increases as well. So if the belief of player 1 about player 2 playing $s_2 = P$ increases, more values of optimism will lead to player 1 preferring $s_1 = S$ over $s_1 = R$. One important thing must be mentioned here, for $p \in [0,0.3]$ we have $\alpha_1 < 0$ which is not possible, since $\alpha_1 \in [0,1]$. So unfortunately for $p \in [0,0.3]$ no values of optimism can induce the preference ordering $s_1 = S \succ s_1 = R$.

To make it clearer a graph can be made, where the dependent variable is optimism and the independent variable the belief of player 1 about player 2 playing his punishment strategy, this is done in figure 9.

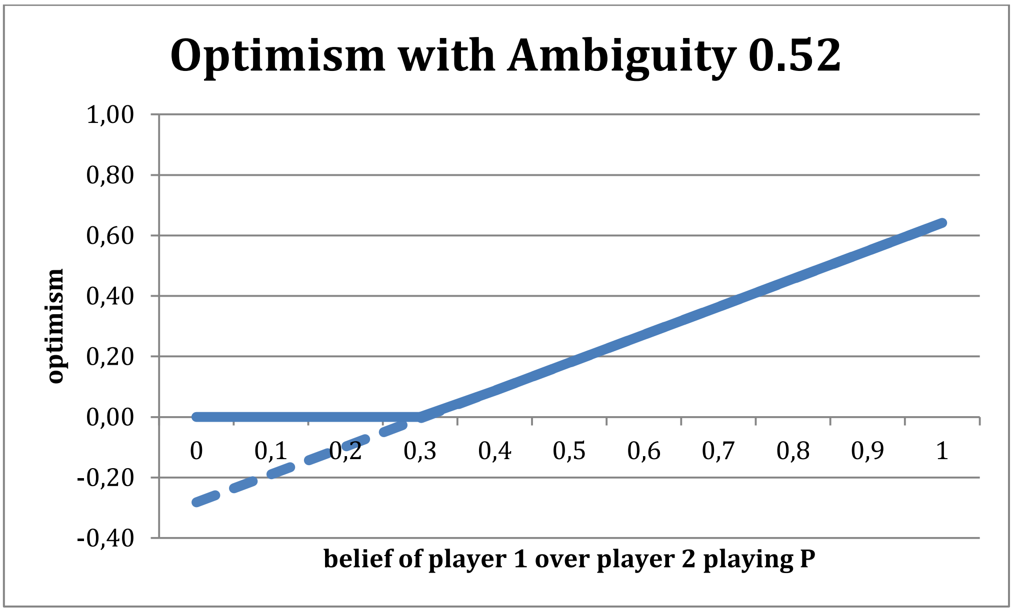

Figure 9. Relationship between p and the upper bound of α_1

20. Derivation of the condition for $1 - \alpha_1$ for which $s_1 = S \succ s_1 = R$ applies with p as a variable and δ_1 is fixed at 0.52, for the game in section 4.3.2:

$$\delta_1(1 - \alpha_1 - p) > \frac{70 - (90 - 30p)}{-30}$$

$$1 - \alpha_1 - p > \frac{70 - (90 - 30p)}{-30\delta_1}$$

$$1 - \alpha_1 > \frac{70 - (90 - 30p)}{-30\delta_1} + p$$

Again, δ_1 will be fixed at 0.52.

$$1 - \alpha_1 > \frac{70 - (90 - 30p)}{-30 \cdot 0.52} + p$$

$$1 - \alpha_1 > \frac{70 - (90 - 30p)}{-15.6} + p$$

21. Table B.21: Relationship between $1 - \alpha_1$ and p, when $\delta_1 = 0.52$, for the game in section 4.3.2:

$p = 0$	$p = 0.1$	$p = 0.2$
$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0)}{-15.6} + 0$ $1 - \alpha_1 > 1.282$ $1 - \alpha_1 > 1$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.1)}{-15.6} + 0.1$ $1 - \alpha_1 > 1.190$ $1 - \alpha_1 > 1$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.2)}{-15.6} + 0.2$ $1 - \alpha_1 > 1.097$ $1 - \alpha_1 > 1$
$p = 0.3$	$p = 0.4$	$p = 0.5$
$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.3)}{-15.6} + 0.3$ $1 - \alpha_1 > 1.005$ $1 - \alpha_1 > 1$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.4)}{-15.6} + 0.4$ $1 - \alpha_1 > 0.913$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.5)}{-15.6} + 0.5$ $1 - \alpha_1 > 0.821$
$p = 0.6$	$p = 0.7$	$p = 0.8$
$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.6)}{-15.6} + 0.6$ $1 - \alpha_1 > 0.728$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.7)}{-15.6} + 0.7$ $1 - \alpha_1 > 0.636$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.8)}{-15.6} + 0.8$ $1 - \alpha_1 > 0.544$
$p = 0.9$	$p = 1.0$	
$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 0.9)}{-15.6} + 0.9$ $1 - \alpha_1 > 0.451$	$1 - \alpha_1 > \frac{70 - (90 - 30 \cdot 1.0)}{-15.6} + 1.0$ $1 - \alpha_1 > 0.359$	

If p increases the optimism value that leads to $s_1 = S \succ s_1 = R$ increases as well. Again, we see that if p increases the lower bound of the pessimism value that leads to $s_1 = S \succ s_1 = R$ decreases. So if the belief of player 1 about payer 2 playing $s_2 = P$ increases, less pessimism is needed for player 1 to prefer playing $s_1 = S$ instead of playing $s_1 = R$. Which is opposite to the relationship between α_1 and p. Figure 10, shows the relationship between p (x-axis) and the lower bound of $1 - \alpha_1$ (y-axis).

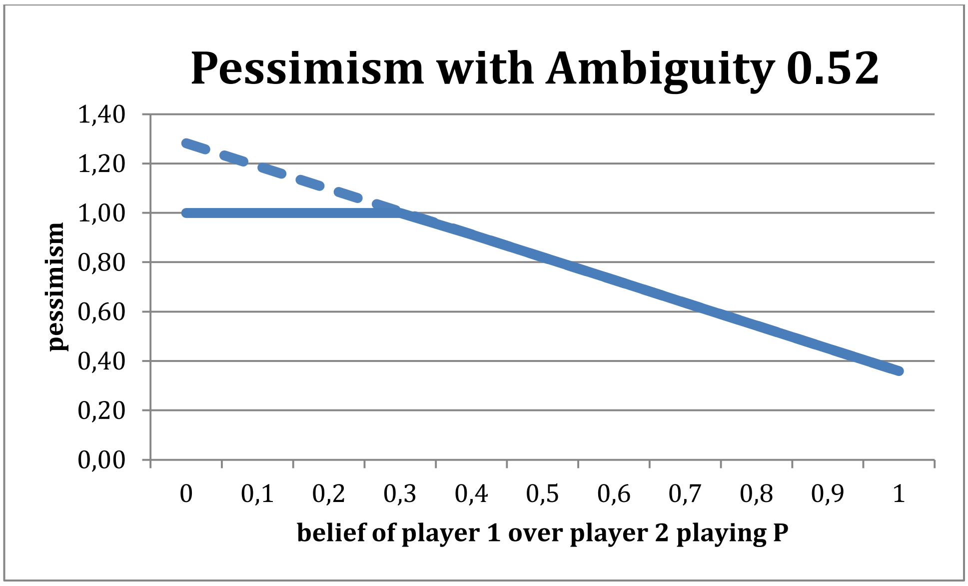

Figure 10. Relationship between p and the lower bound of $1 - \alpha_1$

22. Derivation of the condition for δ_1 for which $s_1 = S \succ s_1 = R$ applies with p as a variable and α_1 is fixed at 0.5, for the game in section 4.3.2:

$$\delta_1(1 - \alpha_1 - p) > \frac{70 - (90 - 30p)}{-30}$$

$$\delta_1 > \frac{70 - (90 - 30p)}{-30(1 - \alpha_1 - p)}$$

As stated above, the value for α_1 will be fixed at the KW-range average, so 0.5.

$$\delta_1 > \frac{70 - (90 - 30p)}{-30(1 - 0.5 - p)}$$

$$\delta_1 > \frac{70 - (90 - 30p)}{-30(0.5 - p)}$$

This formula only holds if $0 \leq p < 0.5$ because then $-30(0.5 - p) < 0$ and the sign does not flip. But when $0.5 < p \leq 1$ then $-30(0.5 - p) > 0$ and the sign flips from $>$ to $<$. Thus when $0 \leq p < 0.5$ the following condition is correct:

$$\delta_1 > \frac{70 - (90 - 30p)}{-30(0.5 - p)}$$

And when $0.5 < p \leq 1$ the following condition is correct:

$$\delta_1 < \frac{70 - (90 - 30p)}{-30(0.5 - p)}$$

23. Table B.23: Relationship between δ_1 and p, when $\alpha_1 = 0.5$, for the game in section 4.3.2:

$p = 0$	$p = 0.1$	$p = 0.2$

$\delta_1 > \frac{70-(90-30\cdot 0)}{-30(0.5-0)}$ $\delta_1 > 1.333$ $\delta_1 > 1$	$\delta_1 > \frac{70-(90-30\cdot 0.1)}{-30(0.5-0.1)}$ $\delta_1 > 1.417$ $\delta_1 > 1$	$\delta_1 > \frac{70-(90-30\cdot 0.2)}{-30(0.5-0.2)}$ $\delta_1 > 1.556$ $\delta_1 > 1$
$p = 0.3$	$p = 0.4$	$p = 0.5$
$\delta_1 > \frac{70-(90-30\cdot 0.3)}{-30(0.5-0.3)}$ $\delta_1 > 1.833$ $\delta_1 > 1$	$\delta_1 > \frac{70-(90-30\cdot 0.4)}{-30(0.5-0.4)}$ $\delta_1 > 2.667$ $\delta_1 > 1$	$\delta_1 > \frac{70-(90-30\cdot 0.5)}{-30(0.5-0.5)}$ $\delta_1 > \frac{-5}{0}$ Not possible
$p = 0.6$	$p = 0.7$	$p = 0.8$
$\delta_1 > \frac{70-(90-30\cdot 0.6)}{-30(0.5-0.6)}$ $\delta_1 < -0.667$ $\delta_1 < 0$	$\delta_1 > \frac{70-(90-30\cdot 0.7)}{-30(0.5-0.7)}$ $\delta_1 < 0.167$	$\delta_1 > \frac{70-(90-30\cdot 0.8)}{-30(0.5-0.8)}$ $\delta_1 < 0.444$
$p = 0.9$	$p = 1.0$	
$\delta_1 > \frac{70-(90-30\cdot 0.9)}{-30(0.5-0.9)}$ $\delta_1 < 0.583$	$\delta_1 > \frac{70-(90-30\cdot 1.0)}{-30(0.5-1.0)}$ $\delta_1 < 0.667$	

Looking at the above table, a few interesting things come up in the relation between ambiguity (δ_1) and the belief of player 1 about player 2 playing $s_2 = P$ (p) when optimism (α_1) is fixed in the 'non-credible threat game'. First, if we have $p \in [0,0.4]$ I find that $\delta_1 > 1$ which is not possible since $\delta_1 \in [0,1]$. The fact that ambiguity must be larger than 1 for $p \in [0,0.4]$ to induce $s_1 = S \succ s_1 = R$ implies that the confidence of player 1 must be negative ($1 - \delta_1 < 0$) which is also not possible. So for $p \in [0,0.4]$ no value for ambiguity can result in player 1 preferring $s_1 = S$ over $s_1 = R$.

Second, for $p = 0.5$ we cannot compute an ambiguity cut off value, since we divide by 0 which is mathematically not possible. So at $p = 0.5$, δ_1 is not well defined.

Third, for $p = 0.6$, I found that the ambiguity value must be smaller than 0, $\delta_1 < 0$, which is also not possible thanks to the fact $\delta_1 \in [0,1]$. Thus, again, no value for ambiguity results in the preference ordering $s_1 = S \succ s_1 = R$ for player 1, when $p = 0.6$. Just like $p \in [0,0.4]$.

Last, for $p \in [0.7,1.0]$, the following pattern arises: if p increases, the upper bound of δ_1 increases as well. Meaning if the belief of player 1 about player 2 playing $s_2 = P$ increases, the ambiguity level that allows $s_1 = S \succ s_1 = R$ increases as well, when $p \in [0.7,1.0]$. The logic behind this is as follows as player 1 deems it more likely that player 2 will play $s_2 = P$ (his punishment strategy), the increased fear of being punished, leads to the fact that player 1 will need to have less confidence in this belief. Figure 11 shows the relationship between the lower bound of δ_1 and p, with δ_1 on the y-axis and p on the x-axis. While figure 12 shows the relationship between the upper bound of δ_1 and p, with δ_1 on the y-axis and p on the x-axis.

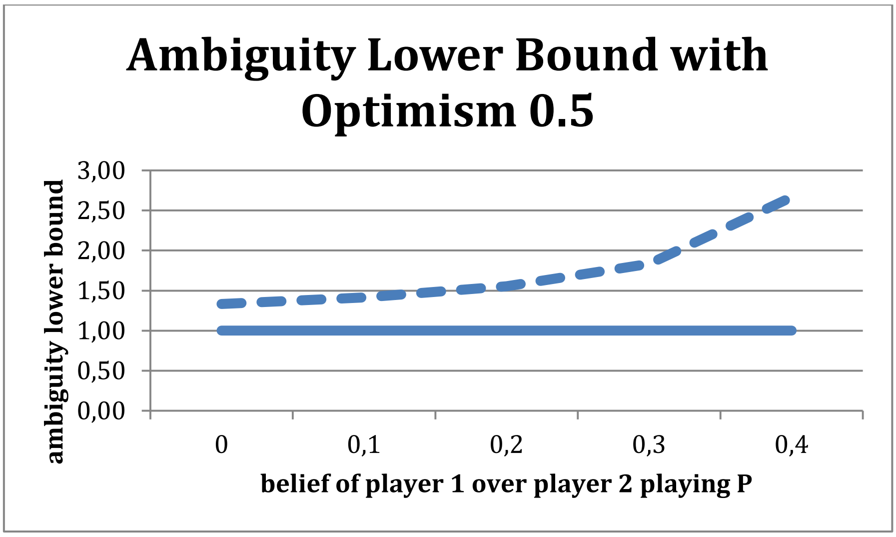

Figure 11. Relationship between the lower bound of δ_1 and p

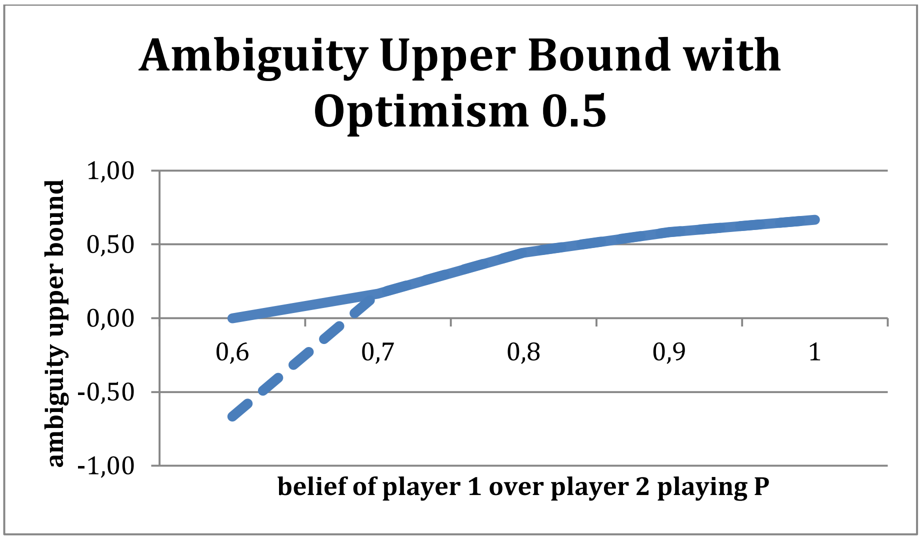

Figure 12. Relationship between the upper bound of δ_1 and p

24. Derivation of Choquet expected utilities for player 1 in the game of section 5.1.1.

I make use of the following function:

$$V_i(s_i | s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i \left[\alpha_i max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1 - \alpha_i) min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \right]$$
$$+ (1 - \delta_i) \int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

Now, we have $u_i(x) = \sqrt{x}$ instead of $u_i(x) = x$.

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = S, s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = S, s_2)\right]$$

$$+ (1 - \delta_1) \int u_1(s_1 = S, s_2) d\pi_1(s_2)$$

$$= \delta_1\left[\alpha_1 \cdot \sqrt{80} + (1 - \alpha_1) \cdot \sqrt{80}\right] + (1 - \delta_1) \cdot \sqrt{80}$$

$$= \delta_1\left[\alpha_1 \cdot \sqrt{80} - \alpha_1 \cdot \sqrt{80} + \sqrt{80}\right] + \sqrt{80} - \delta_1 \cdot \sqrt{80}$$

$$= \delta_1 \cdot \sqrt{80} - \delta_1 \cdot \sqrt{80} + \sqrt{80}$$

$$= \sqrt{80}$$

$$= 8.94$$

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R, s_2) + (1 - \alpha_1) min_{s_2 \in S_2} u_1(s_1 = R, s_2)\right]$$

$$+ (1 - \delta_1) \int u_1(s_1 = R, s_2) d\pi_1(s_2)$$

$$= \delta_1\left[\alpha_1 \sqrt{90} + (1 - \alpha_1)\sqrt{20}\right] + (1 - \delta_1)\sqrt{90}$$

$$= \delta_1\left[\alpha_1 \sqrt{90} - \alpha_1 \sqrt{20} + \sqrt{20}\right] + \sqrt{90} - \delta_1 \sqrt{90}$$

$$= \delta_1[\alpha_1 \cdot 9.487 - \alpha_1 \cdot 4.472 + 4.472] + 9.487 - \delta_1 \cdot 9.487$$

$$= \delta_1[\alpha_1 \cdot 5.015 + 4.472] + 9.487 - \delta_1 \cdot 9.487$$

$$= 9.487 + \delta_1 \cdot 4.472 - \delta_1 \cdot 9.487 + \delta_1 \alpha_1 \cdot 5.015$$

$$= 9.487 - \delta_1 \cdot 5.015 + \delta_1 \alpha_1 \cdot 5.015$$

$$= 9.487 - \delta_1 (1 - \alpha_1) 5.015$$

25. Derivation of ambiguity aversion critical value for player 1 in the game of section 5.1.1:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$

$$8.94 > 9.487 - \delta_1 (1 - \alpha_1) 5.015$$

$$-0.547 > -\delta_1 (1 - \alpha_1) 5.015$$

$$\frac{-0.547}{-5.015} < \delta_1 (1 - \alpha_1)$$

$$\delta_1 (1 - \alpha_1) > 0.109$$

26. Derivation of Choquet expected utilities for player 1 in the game of section 5.1.2. I make use of the following function:

$$V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i\left[\alpha_i max_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha_i) min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\right]$$
$$+ (1-\delta_i)\int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

Now, we have $u_i(x) = x^2$ instead of $u_i(x) = x$.

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2\in S_2} u_1(s_1 = S, s_2) + (1-\alpha_1) min_{s_2\in S_2} u_1(s_1 = S, s_2)\right]$$
$$+ (1-\delta_1)\int u_1(s_1 = S, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 80^2 + (1-\alpha_1)80^2] + (1-\delta_1)80^2$$
$$= \delta_1[\alpha_1 80^2 - \alpha_1 80^2 + 80^2] + 80^2 - \delta_1 80^2$$
$$= \delta_1 80^2 - \delta_1 80^2 + 80^2$$
$$= 80^2 = 6{,}400$$

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2\in S_2} u_1(s_1 = R, s_2) + (1-\alpha_1) min_{s_2\in S_2} u_1(s_1 = R, s_2)\right]$$
$$+ (1-\delta_1)\int u_1(s_1 = R, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 \cdot 90^2 + (1-\alpha_1)\cdot 20^2] + (1-\delta_1)\cdot 90^2$$
$$= \delta_1[\alpha_1 \cdot 8{,}100 + (1-\alpha_1)\cdot 400] + (1-\delta_1)\cdot 8{,}100$$
$$= \delta_1[\alpha_1 8{,}100 - \alpha_1 400 + 400] + 8{,}100 - \delta_1 8{,}100$$
$$= 8{,}100 + \delta_1 400 - \delta_1 8{,}100 + \delta_1\alpha_1 7{,}700$$
$$= 8{,}100 - \delta_1 7{,}700 + \delta_1\alpha_1 7{,}700$$
$$= 8{,}100 - \delta_1(1-\alpha_1)7{,}700$$

27. Derivation of ambiguity aversion critical value for player 1 in the game of section 5.1.2:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$6{,}400 > 8{,}100 - \delta_1(1-\alpha_1)7{,}700$$
$$-1{,}700 > -\delta_1(1-\alpha_1)7{,}700$$
$$\frac{-1{,}700}{-7{,}700} < \delta_1(1-\alpha_1)$$
$$\delta_1(1-\alpha_1) > 0.221$$

28. Derivation of Choquet expected utilities for player 1 in the game of section 5.2.1. I make use of the following function:

$$V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i\left[\alpha_i max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha_i) min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\right]$$
$$+ (1-\delta_i) \int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

Now, we have $u_i(x) = \sqrt{x}$ instead of $u_i(x) = x$.

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = S, s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = S, s_2)\right]$$
$$+ (1-\delta_1) \int u_1(s_1 = S, s_2) d\pi_1(s_2)$$
$$= \delta_1\left[\alpha_1 \cdot \sqrt{70} + (1-\alpha_1) \cdot \sqrt{70}\right] + (1-\delta_1) \cdot \sqrt{70}$$
$$= \delta_1\left[\alpha_1 \cdot \sqrt{70} - \alpha_1 \cdot \sqrt{70} + \sqrt{70}\right] + \sqrt{70} - \delta_1 \cdot \sqrt{70}$$
$$= \delta_1 \cdot \sqrt{70} - \delta_1 \cdot \sqrt{70} + \sqrt{70}$$
$$= \sqrt{70}$$
$$= 8.367$$

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R, s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = R, s_2)\right]$$
$$+ (1-\delta_1) \int u_1(s_1 = R, s_2) d\pi_1(s_2)$$
$$= \delta_1\left[\alpha_1\sqrt{90} + (1-\alpha_1)\sqrt{60}\right] + (1-\delta_1)\sqrt{90}$$
$$= \delta_1\left[\alpha_1\sqrt{90} - \alpha_1\sqrt{60} + \sqrt{60}\right] + \sqrt{90} - \delta_1\sqrt{90}$$
$$= \delta_1[\alpha_1 9.487 - \alpha_1 7.746 + 7.746] + 9.487 - \delta_1 9.487$$
$$= \delta_1[\alpha_1 1.741 + 7.746] + 9.487 - \delta_1 9.487$$
$$= 9.487 + \delta_1 7.746 - \delta_1 9.487 + \delta_1\alpha_1 1.741$$
$$= 9.487 - \delta_1 1.741 + \delta_1\alpha_1 1.741$$
$$= 9.487 - \delta_1(1-\alpha_1)1.741$$

29. Derivation of ambiguity aversion critical value for player 1 in the game of section 5.2.1:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$8.367 > 9.487 - \delta_1(1-\alpha_1)1.741$$
$$-1.12 > -\delta_1(1-\alpha_1)1.741$$
$$\frac{-1.12}{-1.741} < \delta_1(1-\alpha_1)$$
$$\delta_1(1-\alpha_1) > 0.643$$

30. Derivation of Choquet expected utilities for player 1 in the game of section 5.2.2. I make use of the following function:

$$V_i(s_i|s_{-i} \in S_{-i}, \alpha_i, \delta_i, \pi_i)$$
$$= \delta_i\left[\alpha_i max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1-\alpha_i) min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\right]$$
$$+ (1-\delta_i) \int u_i(s_i, s_{-i}) d\pi_i(s_{-i})$$

Now, we have $u_i(x) = x^2$ instead of $u_i(x) = x$.

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = S, s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = S, s_2)\right]$$
$$+ (1-\delta_1) \int u_1(s_1 = S, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 70^2 + (1-\alpha_1)70^2] + (1-\delta_1)70^2$$
$$= \delta_1[\alpha_1 70^2 - \alpha_1 70^2 + 70^2] + 70^2 - \delta_1 70^2$$
$$= \delta_1 70^2 - \delta_1 70^2 + 70^2$$
$$= 70^2 = 4{,}900$$

$$V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$= \delta_1\left[\alpha_1 max_{s_2 \in S_2} u_1(s_1 = R, s_2) + (1-\alpha_1) min_{s_2 \in S_2} u_1(s_1 = R, s_2)\right]$$
$$+ (1-\delta_1) \int u_1(s_1 = R, s_2) d\pi_1(s_2)$$
$$= \delta_1[\alpha_1 \cdot 90^2 + (1-\alpha_1) \cdot 60^2] + (1-\delta_1) \cdot 90^2$$
$$= \delta_1[\alpha_1 \cdot 8{,}100 + (1-\alpha_1) \cdot 3{,}600] + (1-\delta_1) \cdot 8{,}100$$
$$= \delta_1[\alpha_1 8{,}100 - \alpha_1 3{,}600 + 3{,}600] + 8{,}100 - \delta_1 8{,}100$$
$$= 8{,}100 + \delta_1 3{,}600 - \delta_1 8{,}100 + \delta_1 \alpha_1 4{,}500$$
$$= 8{,}100 - \delta_1 4{,}500 + \delta_1 \alpha_1 4{,}500$$
$$= 8{,}100 - \delta_1(1-\alpha_1)4{,}500$$

31. Derivation of ambiguity aversion critical value for player 1 in the game of section 5.2.2:

$$V_1(s_1 = S|s_2 \in S_2, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = R|s_2 \in S_2, \alpha_1, \delta_1, \pi_1)$$
$$4{,}900 > 8{,}100 - \delta_1(1-\alpha_1)4{,}500$$
$$-3{,}200 > -\delta_1(1-\alpha_1)4{,}500$$
$$\frac{-3{,}200}{-4{,}500} < \delta_1(1-\alpha_1)$$
$$\delta_1(1-\alpha_1) > 0.711$$

32. Derivation of $\pi_1(\omega) = \{p_1(\omega_L); p_1(\omega_H)\}$ for which $E[u_1(s_1 = short, s_2 = long)] = E[u_1(s_1 = long, s_2 = short)]$ holds:

$$E[u_1(s_1 = short, s_2 = long)] = E[u_1(s_1 = long, s_2 = short)]$$

$$p_1(\omega_L) \cdot (F_0 - L) + p_1(\omega_H) \cdot (F_0 - H) = p_1(\omega_L) \cdot (L - F_0) + p_1(\omega_H) \cdot (H - F_0)$$

We know that $p_1(\omega_L) + p_1(\omega_H) = 1$ so $p_1(\omega_H) = 1 - p_1(\omega_L)$, filling this in I get:

$$p_1(\omega_L) \cdot (F_0 - L) + \left(1 - p_1(\omega_L)\right) \cdot (F_0 - H)$$
$$= p_1(\omega_L) \cdot (L - F_0) + \left(1 - p_1(\omega_L)\right) \cdot (H - F_0)$$

$$p_1(\omega_L) \cdot (F_0 - L) + (F_0 - H) - p_1(\omega_L) \cdot (F_0 - H)$$
$$= -p_1(\omega_L) \cdot (F_0 - L) + (H - F_0) - p_1(\omega_L) \cdot (H - F_0)$$

$$p_1(\omega_L) \cdot (F_0 - L) + p_1(\omega_L) \cdot (H - F_0) - (H - F_0)$$
$$= -p_1(\omega_L) \cdot (F_0 - L) - p_1(\omega_L) \cdot (H - F_0) + (H - F_0)$$

$$p_1(\omega_L) \cdot (H - F_0 + F_0 - L) - (H - F_0) = -p_1(\omega_L) \cdot (H - F_0 + F_0 - L) + (H - F_0)$$

$$p_1(\omega_L) \cdot (H - L) - (H - F_0) = -p_1(\omega_L) \cdot (H - L) + (H - F_0)$$

$$2p_1(\omega_L) \cdot (H - L) = 2(H - F_0)$$

$$p_1(\omega_L) \cdot (H - L) = (H - F_0)$$

$$p_1(\omega_L) = \frac{H - F_0}{H - L}$$

$$\left.\begin{array}{r} p_1(\omega_H) = 1 - p_1(\omega_L) \\ p_1(\omega_L) = \dfrac{H - F_0}{H - L} \end{array}\right\} p_1(\omega_H) = 1 - \frac{H - F_0}{H - L} = \frac{H - L}{H - L} - \frac{H - F_0}{H - L} = \frac{F_0 - L}{H - L}$$

So for $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L}; p_1(\omega_H) = \frac{F_0-L}{H-L}\right\}$ we have $s_1 = short \sim s_1 = long$, because $E[u_1(s_1 = short, s_2 = long)] = E[u_1(s_1 = long, s_2 = short)]$.

33. Derivation of $\pi_2(\omega) = \{p_2(\omega_L); p_2(\omega_H)\}$ for which $E[u_2(s_2 = long, s_1 = short)] = E[u_2(s_2 = short, s_1 = long)]$ holds:

$$E[u_2(s_2 = long, s_1 = short)] = E[u_2(s_2 = short, s_1 = long)]$$

$$p_2(\omega_L) \cdot (L - F_0) + p_2(\omega_H) \cdot (H - F_0) = p_2(\omega_L) \cdot (F_0 - L) + p_2(\omega_H) \cdot (F_0 - H)$$

We know that $p_2(\omega_L) + p_2(\omega_H) = 1$ so $p_2(\omega_H) = 1 - p_2(\omega_L)$ filling this in I get:

$$-p_2(\omega_L) \cdot (F_0 - L) + \left(1 - p_2(\omega_L)\right) \cdot (H - F_0)$$
$$= p_2(\omega_L) \cdot (F_0 - L) + \left(1 - p_2(\omega_L)\right) \cdot [-(H - F_0)]$$

$$-p_2(\omega_L) \cdot (F_0 - L) - p_2(\omega_L) \cdot (H - F_0) + (H - F_0)$$
$$= p_2(\omega_L) \cdot (F_0 - L) + p_2(\omega_L) \cdot (H - F_0) - (H - F_0)$$

$$-p_2(\omega_L) \cdot (H - F_0 + F_0 - L) + (H - F_0) = p_2(\omega_L) \cdot (H - F_0 + F_0 - L) - (H - F_0)$$

$$-p_2(\omega_L)\cdot(H-L)+(H-F_0)=p_2(\omega_L)\cdot(H-L)-(H-F_0)$$
$$-2p_2(\omega_L)\cdot(H-L)=-2(H-F_0)$$
$$p_2(\omega_L)\cdot(H-L)=(H-F_0)$$
$$p_2(\omega_L)=\frac{H-F_0}{H-L}$$
$$\left.\begin{aligned}p_2(\omega_H)&=1-p_2(\omega_L)\\ p_2(\omega_L)&=\frac{H-F_0}{H-L}\end{aligned}\right\}p_2(\omega_H)=1-\frac{H-F_0}{H-L}=\frac{H-L}{H-L}-\frac{H-F_0}{H-L}=\frac{F_0-L}{H-L}$$

So for $\pi_2(\omega)=\left\{p_2(\omega_L)=\frac{H-F_0}{H-L};p_2(\omega_H)=\frac{F_0-L}{H-L}\right\}$ we have $s_2=long\sim s_2=short$, because $E[u_2(s_2=long,s_1=short)]=E[u_2(s_2=short,s_1=long)]$.

34. Derivation of $\pi_1(\omega)=\{p_1(\omega_L);p_1(\omega_H)\}$ for which $E[u_1(s_1=short,s_2=long)]>E[u_1(s_1=long,s_2=short)]$ holds:

$$E[u_1(s_1=short,s_2=long)]>E[u_1(s_1=long,s_2=short)]$$
$$p_1(\omega_L)\cdot(F_0-L)+p_1(\omega_H)\cdot(F_0-H)>p_1(\omega_L)\cdot(L-F_0)+p_1(\omega_H)\cdot(H-F_0)$$

We know that $p_1(\omega_L)+p_1(\omega_H)=1$ so $p_1(\omega_H)=1-p_1(\omega_L)$, filling this in I get:

$$p_1(\omega_L)\cdot(F_0-L)+\left(1-p_1(\omega_L)\right)\cdot(F_0-H)$$
$$>p_1(\omega_L)\cdot(L-F_0)+\left(1-p_1(\omega_L)\right)\cdot(H-F_0)$$
$$p_1(\omega_L)\cdot(F_0-L)+(F_0-H)-p_1(\omega_L)\cdot(F_0-H)$$
$$>-p_1(\omega_L)\cdot(F_0-L)+(H-F_0)-p_1(\omega_L)\cdot(H-F_0)$$
$$p_1(\omega_L)\cdot(F_0-L)+p_1(\omega_L)\cdot(H-F_0)-(H-F_0)$$
$$>-p_1(\omega_L)\cdot(F_0-L)-p_1(\omega_L)\cdot(H-F_0)+(H-F_0)$$
$$p_1(\omega_L)\cdot(H-F_0+F_0-L)-(H-F_0)>-p_1(\omega_L)\cdot(H-F_0+F_0-L)+(H-F_0)$$
$$p_1(\omega_L)\cdot(H-L)-(H-F_0)>-p_1(\omega_L)\cdot(H-L)+(H-F_0)$$
$$2p_1(\omega_L)\cdot(H-L)>2(H-F_0)$$
$$p_1(\omega_L)\cdot(H-L)>(H-F_0)$$
$$p_1(\omega_L)>\frac{H-F_0}{H-L}$$

We know $p_1(\omega_L)=1-p_1(\omega_H)$ and $p_1(\omega_L)>\frac{H-F_0}{H-L}$ combining the two gives:

$$1-p_1(\omega_H)>\frac{H-F_0}{H-L}$$
$$-p_1(\omega_H)>-1+\frac{H-F_0}{H-L}$$
$$p_1(\omega_H)<1-\frac{H-F_0}{H-L}$$

$$p_1(\omega_H) < \frac{H-L}{H-L} - \frac{H-F_0}{H-L}$$

$$p_1(\omega_H) < \frac{F_0-L}{H-L}$$

So for $\pi_1(\omega) = \left\{p_1(\omega_L) > \frac{H-F_0}{H-L}; p_1(\omega_H) < \frac{F_0-L}{H-L}\right\}$ we have $s_1 = short \succ s_1 = long$, because $E[u_1(s_1 = short, s_2 = long)] > E[u_1(s_1 = long, s_2 = short)]$.

35. Derivation of $\pi_2(\omega) = \{p_2(\omega_L); p_2(\omega_H)\}$ for which $E[u_2(s_2 = long, s_1 = short)] > E[u_2(s_2 = short, s_1 = long)]$ holds:

$$E[u_2(s_2 = long, s_1 = short)] > E[u_2(s_2 = short, s_1 = long)]$$

$$p_2(\omega_L) \cdot (L - F_0) + p_2(\omega_H) \cdot (H - F_0) > p_2(\omega_L) \cdot (F_0 - L) + p_2(\omega_H) \cdot (F_0 - H)$$

We know that $p_2(\omega_L) + p_2(\omega_H) = 1$ so $p_2(\omega_H) = 1 - p_2(\omega_L)$ filling this in I get:

$$-p_2(\omega_L) \cdot (F_0 - L) + \left(1 - p_2(\omega_L)\right) \cdot (H - F_0)$$
$$> p_2(\omega_L) \cdot (F_0 - L) + \left(1 - p_2(\omega_L)\right) \cdot [-(H - F_0)]$$

$$-p_2(\omega_L) \cdot (F_0 - L) - p_2(\omega_L) \cdot (H - F_0) + (H - F_0)$$
$$> p_2(\omega_L) \cdot (F_0 - L) + p_2(\omega_L) \cdot (H - F_0) - (H - F_0)$$

$$-p_2(\omega_L) \cdot (H - F_0 + F_0 - L) + (H - F_0) > p_2(\omega_L) \cdot (H - F_0 + F_0 - L) - (H - F_0)$$

$$-p_2(\omega_L) \cdot (H - L) + (H - F_0) > p_2(\omega_L) \cdot (H - L) - (H - F_0)$$

$$-2p_2(\omega_L) \cdot (H - L) > -2(H - F_0)$$

$$p_2(\omega_L) \cdot (H - L) < (H - F_0)$$

$$p_2(\omega_L) < \frac{H-F_0}{H-L}$$

We know that $p_2(\omega_L) = 1 - p_2(\omega_H)$ and $p_2(\omega_L) < \frac{H-F_0}{H-L}$ combining the two gives

$$1 - p_2(\omega_H) < \frac{H-F_0}{H-L}$$

$$-p_2(\omega_H) < -1 + \frac{H-F_0}{H-L}$$

$$p_2(\omega_H) > 1 - \frac{H-F_0}{H-L}$$

$$p_2(\omega_H) > \frac{H-L}{H-L} - \frac{H-F_0}{H-L}$$

$$p_2(\omega_H) > \frac{F_0-L}{H-L}$$

So for $\pi_2(\omega) = \left\{p_2(\omega_L) < \frac{H-F_0}{H-L}; p_2(\omega_H) > \frac{F_0-L}{H-L}\right\}$ we have $s_2 = long \succ s_2 = short$, because $E[u_2(s_2 = long, s_1 = short)] > E[u_2(s_2 = short, s_1 = long)]$.

36. Derivation of expected payoff to player 1 from playing $s_1 = short$ with an additive probability distribution as belief of $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$ which satisfies $\pi_1(\omega) = \left\{p_1(\omega_L) > \frac{H-F_0}{H-L}; p_1(\omega_H) < \frac{F_0-L}{H-L}\right\}$.

$$E\big[u_1\big(s_1 = short, s_2 = long|\pi_1(\omega)\big)\big] = p_1(\omega_L)\cdot(F_0 - L) + p_1(\omega_H)\cdot(F_0 - H)$$

$$= \left(\frac{H-F_0}{H-L} + a\right)\cdot(F_0 - L) + \left(\frac{F_0 - L}{H-L} - a\right)\cdot(F_0 - H)$$

$$= \frac{(H-F_0)(F_0-L)}{H-L} + \frac{(F_0-L)(F_0-H)}{H-L} + a(F_0 - L) - a(F_0 - H)$$

$$= \frac{(H-F_0)(F_0-L)}{H-L} - \frac{(H-F_0)(F_0-L)}{H-L} + a(F_0 - L) + a(H - F_0)$$

$$= a(H - F_0 + F_0 - L)$$

$$= a(H - L)$$

37. Derivation of expected payoff to player 2 from playing $s_2 = long$ with an additive probability distribution as belief of $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$ which satisfies $\pi_2(\omega) = \left\{p_2(\omega_L) < \frac{H-F_0}{H-L}; p_2(\omega_H) > \frac{F_0-L}{H-L}\right\}$.

$$E\big[u_2\big(s_2 = long, s_1 = short|\pi_2(\omega)\big)\big] = p_2(\omega_L)\cdot(L - F_0) + p_2(\omega_H)\cdot(H - F_0)$$

$$= \left(\frac{H-F_0}{H-L} - a\right)\cdot(L - F_0) + \left(\frac{F_0 - L}{H-L} + a\right)\cdot(H - F_0)$$

$$= \frac{(H-F_0)(L-F_0)}{L-F_0} + \frac{(H-F_0)(F_0-L)}{H-L} - a(L - F_0) + a(H - F_0)$$

$$= -\frac{(H-F_0)(F_0-L)}{H-L} + \frac{(H-F_0)(F_0-L)}{H-L} + a(F_0 - L) + a(H - F_0)$$

$$= a(H - F_0 + F_0 - L)$$

$$= a(H - L)$$

38. Conditions which a must satisfy:

(1.) For $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$:

$$\frac{H-F_0}{H-L} + a \leq 1$$

$$a \leq 1 - \frac{H-F_0}{H-L}$$

$$a \leq \frac{H-L}{H-L} - \frac{H-F_0}{H-L}$$

$$a \leq \frac{F_0 - L}{H - L}$$

and

$$\frac{F_0 - L}{H - L} - a \geq 0$$

$$-a \geq -\frac{F_0 - L}{H - L}$$

$$a \leq \frac{F_0 - L}{H - L}$$

(2.) For $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$:

$$\frac{H - F_0}{H - L} - a \geq 0$$

$$-a \geq -\frac{H - F_0}{H - L}$$

$$a \leq \frac{H - F_0}{H - L}$$

and

$$\frac{F_0 - L}{H - L} + a \leq 1$$

$$a \leq 1 - \frac{F_0 - L}{H - L}$$

$$a \leq \frac{H - L}{H - L} - \frac{F_0 - L}{H - L}$$

$$a \leq \frac{H - F_0}{H - L}$$

39. Derivation of the Choquet expected utility for player 1 from playing $s_1 = short$, where $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$ and for a, $a \leq \frac{F_0-L}{H-L}$ must hold:

$$\begin{aligned} V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1) \\ &= \delta_1[\alpha_1 max_{\omega\in\Omega} u_1(s_1 = short, s_2 = long) \\ &+ (1-\alpha_1) min_{\omega\in\Omega} u_1(s_1 = short, s_2 = long)] \\ &+ (1-\delta_1)\int u_1(s_1 = short, s_2 = long) d\pi_1(\omega) \end{aligned}$$

$$= \delta_1[\alpha_1(F_0 - L) + (1-\alpha_1)(F_0 - H)]$$

$$+ (1-\delta_1)\left[\left(\frac{H - F_0}{H - L} + a\right)\cdot(F_0 - L) + \left(\frac{F_0 - L}{H - L} - a\right)\cdot(F_0 - H)\right]$$

$$= \delta_1[\alpha_1(F_0 - L) - \alpha_1(F_0 - H) + (F_0 - H)] + (1-\delta_1)\left[\frac{(H - F_0)(F_0 - L)}{H - L} + \frac{(F_0 - L)(F_0 - H)}{H - L} + a(F_0 - L) - a(F_0 - H)\right]$$

$$= \delta_1[\alpha_1(F_0 - L) + \alpha_1(H - F_0) - (H - F_0)] + (1-\delta_1)\left[\frac{(H - F_0)(F_0 - L)}{H - L} - \frac{(H - F_0)(F_0 - L)}{H - L} + a(F_0 - L) + a(H - F_0)\right]$$

$$= \delta_1[\alpha_1(H - F_0 + F_0 - L) - (H - F_0)] + (1-\delta_1)[a(F_0 - L) + a(H - F_0)]$$

$$= \delta_1[\alpha_1(H - L) - (H - F_0)] + (1-\delta_1)[a(H - F_0 + F_0 - L)]$$

$$= \delta_1[\alpha_1(H - L) - (H - F_0)] + (1-\delta_1)a(H - L)$$

$$= \delta_1[\alpha_1(H - L) - (H - F_0)] - \delta_1 a(H - L) + \ a(H - L)$$

$$= \ a(H - L) + \delta_1[-a(H - L) + \alpha_1(H - L) - (H - F_0)]$$

$$= \ a(H - L) - \delta_1[a(H - L) - \alpha_1(H - L) + (H - F_0)]$$

$$= \ a(H - L) - \delta_1[(a - \alpha_1)(H - L) + (H - F_0)]$$

40. Derivation of the Choquet expected utility for player 1 from playing $s_1 = long$, where $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H - F_0}{H - L} + a; p_1(\omega_H) = \frac{F_0 - L}{H - L} - a\right\}$ and for a, $a \le \frac{F_o - L}{H - L}$ must hold:

$$V_1(s_1 = long | s_2 = short, \alpha_1, \delta_1, \pi_1) = \delta_1[\alpha_1 max_{\omega\in\Omega} u_1(s_1 = long, s_2 = short) + (1-\alpha_1) min_{\omega\in\Omega} u_1(s_1 = long, s_2 = short)] + (1-\delta_1)\int u_1(s_1 = long, s_2 = short) d\pi_1(\omega)$$

$$= \delta_1[\alpha_1(H - F_0) + (1-\alpha_1)(L - F_0)] + (1-\delta_1)\big(p_1(\omega_L)\cdot(L - F_0) + p_1(\omega_H)\cdot(H - F_0)\big)$$

$$= \delta_1[\alpha_1(H - F_0) - \alpha_1(L - F_0) + (L - F_0)] + (1-\delta_1)\left[\left(\frac{H - F_0}{H - L} + a\right)(L - F_0) + \left(\frac{F_0 - L}{H - L} - a\right)(H - F_0)\right]$$

$$= \delta_1[\alpha_1(H - F_0) + \alpha_1(F_0 - L) - (F_0 - L)]$$
$$+ (1 - \delta_1)\left[\frac{(H - F_0)(L - F_0)}{H - L} + a(L - F_0) + \frac{(H - F_0)(F_0 - L)}{H - L} - a(H - F_0)\right]$$
$$= \delta_1[\alpha_1(H - F_0 + F_0 - L) - (F_0 - L)]$$
$$+ (1 - \delta_1)\left[-\frac{(H - F_0)(F_0 - L)}{H - L} - a(F_0 - L) + \frac{(H - F_0)(F_0 - L)}{H - L} - a(H - F_0)\right]$$
$$= \delta_1[\alpha_1(H - L) - (F_0 - L)] + (1 - \delta_1)[-a(F_0 - L) - a(H - F_0)]$$
$$= \delta_1[\alpha_1(H - L) - (F_0 - L)] + (1 - \delta_1)[-a(H - F_0 + F_0 - L)]$$
$$= \delta_1[\alpha_1(H - L) - (F_0 - L)] + (1 - \delta_1)[-a(H - L)]$$
$$= \delta_1[\alpha_1(H - L) - (F_0 - L)] + \delta_1 a(H - L) - a(H - L)$$
$$= \delta_1[a(H - L) + \alpha_1(H - L) - (F_0 - L)] - a(H - L)$$
$$= \delta_1[(a + \alpha_1)(H - L) - (F_0 - L)] - a(H - L)$$

41. Derivation of ambiguity value for which player 1 has the following preference order $s_1 = short \succ s_1 = long$ with $\pi_1(\omega) = \left\{p_1(\omega_L) = \frac{H-F_0}{H-L} + a; p_1(\omega_H) = \frac{F_0-L}{H-L} - a\right\}$ and $a \leq \frac{F_0-L}{H-L}$:

$$V_1(s_1 = short|s_2 = long, \alpha_1, \delta_1, \pi_1) > V_1(s_1 = long|s_2 = short, \alpha_1, \delta_1, \pi_1)$$
$$a(H - L) - \delta_1[(a - \alpha_1)(H - L) + (H - F_0)]$$
$$> \delta_1[(a + \alpha_1)(H - L) - (F_0 - L)] - a(H - L)$$
$$-\delta_1[(a - \alpha_1)(H - L) + (H - F_0)] - \delta_1[(a + \alpha_1)(H - L) - (F_0 - L)]$$
$$> -a(H - L) - a(H - L)$$
$$-\delta_1[(a - \alpha_1)(H - L) + (H - F_0) + (a + \alpha_1)(H - L) - (F_0 - L)] > -2a(H - L)$$
$$-\delta_1[(a - \alpha_1 + a + \alpha_1)(H - L) + (H - F_0) - (F_0 - L)] > -2a(H - L)$$
$$-\delta_1[2a(H - L) + H - F_0 - F_0 + L] > -2a(H - L)$$
$$-\delta_1[2aH + H - 2aL + L - 2F_0] > -2a(H - L)$$
$$-\delta_1[(2a + 1)H - (2a - 1)L - 2F_0] > -2a(H - L)$$
$$\delta_1[(2a + 1)H - (2a - 1)L - 2F_0] < 2a(H - L)$$
$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

This condition is true as long as $(2a+1)H-(2a-1)L-2F_0>0$, but when $(2a+1)H-(2a-1)L-2F_0<0$ then $2a(H-L)$ will be divided by a negative number and thus the inequality sign flips from $<$ to $>$ and thus we get the following expression when $(2a+1)H-(2a-1)L-2F_0<0$:

$$\delta_1 > \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

If $(2a+1)H-(2a-1)L-2F_0<0$ then the condition states that ambiguity must be larger than a negative number, since $2a(H-L)>0$ is always true. But we have $\delta_1 \in [0,1]$, thus when the condition says that ambiguity must be larger than a negative number this is always satisfied.

42. Derivation of the Choquet expected utility for player 2 from playing $s_2 = long$ where $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$ and for a, $a \leq \frac{H-F_0}{H-L}$ must hold:

$$\begin{aligned}
&V_2(s_2 = long|s_1 = short, \alpha_2, \delta_2, \pi_2)\\
&= \delta_2[\alpha_2 max_{\omega\in\Omega} u_2(s_2 = long, s_1 = short)\\
&+ (1-\alpha_2) min_{\omega\in\Omega} u_2(s_2 = long, s_1 = short)]\\
&+ (1-\delta_2)\int u_2(s_2 = long, s_1 = short) d\pi_2(\omega)\\
&= \delta_2[\alpha_2(H-F_0) + (1-\alpha_2)(L-F_0)]\\
&+ (1-\delta_2)[p_2(\omega_L)\cdot(L-F_0) + p_2(\omega_H)\cdot(H-F_0)]\\
&= \delta_2[\alpha_2(H-F_0) - \alpha_2(L-F_0) + (L-F_0)]\\
&+ (1-\delta_2)\left[\left(\frac{H-F_0}{H-L} - a\right)(L-F_0) + \left(\frac{F_0-L}{H-L} + a\right)(H-F_0)\right]\\
&= \delta_2[\alpha_2(H-F_0) + \alpha_2(F_0-L) - (F_0-L)]\\
&+ (1-\delta_2)\left[\frac{(H-F_0)(L-F_0)}{H-L} - a(L-F_0) + \frac{(H-F_0)(F_0-L)}{H-L}\right.\\
&\left. + a(H-F_0)\right]\\
&= \delta_2[\alpha_2(H-F_0+F_0-L) - (F_0-L)]\\
&+ (1-\delta_2)\left[-\frac{(H-F_0)(F_0-L)}{H-L} + a(F_0-L) + \frac{(H-F_0)(F_0-L)}{H-L}\right.\\
&\left. + a(H-F_0)\right]\\
&= \delta_2[\alpha_2(H-L) - (F_0-L)] + (1-\delta_2)[a(F_0-L) + a(H-F_0)]
\end{aligned}$$

$$= \delta_2[\alpha_2(H-L)-(F_0-L)]+(1-\delta_2)[a(H-F_0+F_0-L)]$$
$$= \delta_2[\alpha_2(H-L)-(F_0-L)]+(1-\delta_2)a(H-L)$$
$$= \delta_2[\alpha_2(H-L)-(F_0-L)]-\delta_2 a(H-L)+a(H-L)$$
$$= a(H-L)+\delta_2[-a(H-L)+\alpha_2(H-L)-(F_0-L)]$$
$$= a(H-L)-\delta_2[a(H-L)-\alpha_2(H-L)+(F_0-L)]$$
$$= a(H-L)-\delta_2[(a-\alpha_2)(H-L)+(F_0-L)]$$

43. Derivation of the Choquet expected utility for player 2 from playing $s_2 = short$ where $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$ and for a, $a \leq \frac{H-F_0}{H-L}$ must hold:

$$V_2(s_2 = short|s_1 = lo\square g, \alpha_2, \delta_2, \pi_2)$$
$$= \delta_2[\alpha_2 max_{\omega\in\Omega} u_2(s_2 = short, s_1 = long)$$
$$+ (1-\alpha_2) min_{\omega\in\Omega} u_2(s_2 = short, s_1 = long)]$$
$$+ (1-\delta_2)\int u_2(s_2 = short, s_1 = long) d\pi_2(\omega)$$
$$= \delta_2[\alpha_2(F_0-L)+(1-\alpha_2)(F_0-H)]$$
$$+ (1-\delta_2)[p_2(\omega_L)\cdot(F_0-L)+p_2(\omega_H)\cdot(F_0-H)]$$
$$= \delta_2[\alpha_2(F_0-L)-\alpha_2(F_0-H)+(F_0-H)]$$
$$+ (1-\delta_2)\left[\left(\frac{H-F_0}{H-L}-a\right)(F_0-L)+\left(\frac{F_0-L}{H-L}+a\right)(F_0-H)\right]$$
$$= \delta_2[\alpha_2(F_0-L)+\alpha_2(H-F_0)-(H-F_0)]$$
$$+ (1-\delta_2)\left[\frac{(H-F_0)(F_0-L)}{H-L} - a(F_0-L) + \frac{(F_0-H)(F_0-L)}{H-L}\right.$$
$$\left. + a(F_0-H)\right]$$
$$= \delta_2[\alpha_2(H-F_0+F_0-L)-(H-F_0)]$$
$$+ (1-\delta_2)\left[\frac{(H-F_0)(F_0-L)}{H-L} - a(F_0-L) - \frac{(H-F_0)(F_0-L)}{H-L}\right.$$
$$\left. - a(H-F_0)\right]$$
$$= \delta_2[\alpha_2(H-L)-(H-F_0)]+(1-\delta_2)[-a(F_0-L)-a(H-F_0)]$$
$$= \delta_2[\alpha_2(H-L)-(H-F_0)]+(1-\delta_2)[-a(H-F_0+F_0-L)]$$
$$= \delta_2[\alpha_2(H-L)-(H-F_0)]+(1-\delta_2)[-a(H-L)]$$
$$= \delta_2[\alpha_2(H-L)-(H-F_0)]+\delta_2 a(H-L)-a(H-L)$$

$$= \delta_2[a(H-L) + \alpha_2(H-L) - (H-F_0)] - a(H-L)$$
$$= \delta_2[(a+\alpha_2)(H-L) - (H-F_0)] - a(H-L)$$

44. Derivation of ambiguity value for which player 2 has the following preference order $s_2 = long \succ s_2 = short$ with $\pi_2(\omega) = \left\{p_2(\omega_L) = \frac{H-F_0}{H-L} - a; p_2(\omega_H) = \frac{F_0-L}{H-L} + a\right\}$ and $a \leq \frac{H-F_0}{H-L}$:

$$V_2(s_2 = long|s_1 = short, \alpha_2, \delta_2, \pi_2) > V_2(s_2 = short|s_1 = long, \alpha_2, \delta_2, \pi_2)$$
$$a(H-L) - \delta_2[(a-\alpha_2)(H-L) + (F_0-L)]$$
$$> \delta_2[(a+\alpha_2)(H-L) - (H-F_0)] - a(H-L)$$
$$-\delta_2[(a-\alpha_2)(H-L) + (F_0-L)] - \delta_2[(a+\alpha_2)(H-L) - (H-F_0)]$$
$$> -a(H-L) - a(H-L)$$
$$-\delta_2[(a-\alpha_2)(H-L) + (F_0-L) + (a+\alpha_2)(H-L) - (H-F_0)] > -2a(H-L)$$
$$-\delta_2[(a-\alpha_2+a+\alpha_2)(H-L) + (F_0-L) - (H-F_0)] > -2a(H-L)$$
$$-\delta_2[2a(H-L) - H - L + F_0 + F_0] > -2a(H-L)$$
$$-\delta_2(2aH - H - 2aL - L + 2F_0) > -2a(H-L)$$
$$-\delta_2[(2a-1)H - (2a+1)L + 2F_0] > -2a(H-L)$$
$$\delta_2[(2a-1)H - (2a+1)L + 2F_0] < 2a(H-L)$$
$$\delta_2 < \frac{2a(H-L)}{(2a-1)H - (2a+1)L + 2F_0}$$

This condition is true as long as $(2a-1)H - (2a+1)L + 2F_0 > 0$, but when $(2a-1)H - (2a+1)L + 2F_0 < 0$ then $2a(H-L)$ will be divided by a negative number and thus the inequality sign flips from $<$ to $>$ and thus we get the following expression when $(2a-1)H - (2a+1)L + 2F_0 < 0$:

$$\delta_2 > \frac{2a(H-L)}{(2a-1)H - (2a+1)L + 2F_0}$$

If $(2a-1)H - (2a+1)L + 2F_0 < 0$ then the condition states that ambiguity must be larger than a negative number, since $2a(H-L) > 0$ is always true. But we have $\delta_1 \in [0,1]$, thus when the condition says that ambiguity must be larger than a negative number this is always satisfied.

45. Derivation, value for $a \leq \frac{F_o-L}{H-L}$ when $H = 15, L = 5, F_0 = 10$:

$$a \leq \frac{F_o - L}{H-L}$$

$$a \leq \frac{10 - 5}{15 - 5}$$

$$a \leq \frac{5}{10}$$

$$a \leq 0.5$$

46. Derivation ambiguity value for player 1 with $H = 15, L = 5, F_0 = 10$ and $a = 0.2$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

$$\delta_1 < \frac{2 \cdot 0.2 \cdot (15 - 5)}{(2 \cdot 0.2 + 1) \cdot 15 - (2 \cdot 0.2 - 1) \cdot 5 - 2 \cdot 10}$$

$$\delta_1 < \frac{0.4 \cdot 10}{(0.4 + 1) \cdot 15 - (0.4 - 1) \cdot 5 - 20}$$

$$\delta_1 < \frac{4}{1.4 \cdot 15 - (-0.6) \cdot 5 - 20}$$

$$\delta_1 < \frac{4}{21 + 3 - 20}$$

$$\delta_1 < \frac{4}{24 - 20}$$

$$\delta_1 < \frac{4}{4}$$

$$\delta_1 < 1$$

47. Derivation ambiguity value for player 1 with $H = 15, L = 5, F_0 = 10$ and $a = 0.1$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

$$\delta_1 < \frac{2 \cdot 0.1 \cdot (15 - 5)}{(2 \cdot 0.1 + 1) \cdot 15 - (2 \cdot 0.1 - 1) \cdot 5 - 2 \cdot 10}$$

$$\delta_1 < \frac{0.2 \cdot 10}{(0.2 + 1) \cdot 15 - (0.2 - 1) \cdot 5 - 20}$$

$$\delta_1 < \frac{2}{1.2 \cdot 15 - (-0.8) \cdot 5 - 20}$$

$$\delta_1 < \frac{2}{18 + 4 - 20}$$

$$\delta_1 < \frac{2}{22 - 20}$$
$$\delta_1 < \frac{2}{2}$$
$$\delta_1 < 1$$

48. Derivation ambiguity value for player 1 with $H = 15, L = 5, F_0 = 10$ and $a = 0.5$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$
$$\delta_1 < \frac{2 \cdot 0.5 \cdot (15 - 5)}{(2 \cdot 0.5 + 1) \cdot 15 - (2 \cdot 0.5 - 1) \cdot 5 - 2 \cdot 10}$$
$$\delta_1 < \frac{1 \cdot 10}{(1 + 1) \cdot 15 - (1 - 1) \cdot 5 - 20}$$
$$\delta_1 < \frac{10}{2 \cdot 15 - (0) \cdot 5 - 20}$$
$$\delta_1 < \frac{10}{30 + 0 - 20}$$
$$\delta_1 < \frac{10}{30 - 20}$$
$$\delta_1 < \frac{10}{10}$$
$$\delta_1 < 1$$

49. Derivation, value for $a \leq \frac{F_o - L}{H - L}$ when $H = 20, L = 5, F_0 = 10$:

$$a \leq \frac{F_o - L}{H - L}$$
$$a \leq \frac{10 - 5}{20 - 5}$$
$$a \leq \frac{5}{15}$$
$$a \leq \frac{1}{3}$$

50. Derivation ambiguity value for player 1 with $H = 20, L = 5, F_0 = 10$ and $a = 0.2$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

$$\delta_1 < \frac{2\cdot 0.2\cdot(20-5)}{(2\cdot 0.2+1)\cdot 20-(2\cdot 0.2-1)\cdot 5-2\cdot 10}$$

$$\delta_1 < \frac{0.4\cdot 15}{(0.4+1)\cdot 20-(0.4-1)\cdot 5-20}$$

$$\delta_1 < \frac{6}{1.4\cdot 20-(-0.6)\cdot 5-20}$$

$$\delta_1 < \frac{6}{28+3-20}$$

$$\delta_1 < \frac{6}{31-20}$$

$$\delta_1 < \frac{6}{11}$$

51. Derivation, value for $a \leq \frac{F_o-L}{H-L}$ when $H = 15, L = 7, F_0 = 10$:

$$a \leq \frac{F_o-L}{H-L}$$

$$a \leq \frac{10-7}{15-7}$$

$$a \leq \frac{3}{8}$$

$$a \leq 0.375$$

52. Derivation ambiguity value for player 1 with $H = 15, L = 7, F_0 = 10$ and $a = 0.2$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$$

$$\delta_1 < \frac{2\cdot 0.2\cdot(15-7)}{(2\cdot 0.2+1)\cdot 15-(2\cdot 0.2-1)\cdot 7-2\cdot 10}$$

$$\delta_1 < \frac{0.4\cdot 8}{(0.4+1)\cdot 15-(0.4-1)\cdot 7-20}$$

$$\delta_1 < \frac{3.2}{(1.4)\cdot 15-(-0.6)\cdot 7-20}$$

$$\delta_1 < \frac{3.2}{(1.4)\cdot 15-(-0.6)\cdot 7-20}$$

$$\delta_1 < \frac{3.2}{21+4.2-20}$$

$$\delta_1 < \frac{3.2}{25.2 - 20}$$

$$\delta_1 < \frac{3.2}{5.2}$$

$$\delta_1 < 0.615$$

53. Derivation, value for $a \leq \frac{F_o - L}{H - L}$ when $H = 15, L = 3, F_0 = 10$:

$$a \leq \frac{F_o - L}{H - L}$$

$$a \leq \frac{10 - 3}{15 - 3}$$

$$a \leq \frac{7}{12}$$

$$a \leq 0.583$$

54. Derivation ambiguity value for player 1 with $H = 15, L = 3, F_0 = 10$ and $a = 0.2$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

$$\delta_1 < \frac{2 \cdot 0.2 \cdot (15 - 3)}{(2 \cdot 0.2 + 1) \cdot 15 - (2 \cdot 0.2 - 1) \cdot 3 - 2 \cdot 10}$$

$$\delta_1 < \frac{0.4 \cdot 12}{(0.4 + 1) \cdot 15 - (0.4 - 1) \cdot 3 - 20}$$

$$\delta_1 < \frac{4.8}{1.4 \cdot 15 - (-0.6) \cdot 3 - 20}$$

$$\delta_1 < \frac{4.8}{21 + 1.8 - 20}$$

$$\delta_1 < \frac{4.8}{22.8 - 20}$$

$$\delta_1 < \frac{4.8}{2.8}$$

$$\delta_1 < 1.714$$

55. Derivation, value for $a \leq \frac{F_o - L}{H - L}$ when $H = 13, L = 5, F_0 = 10$:

$$a \leq \frac{F_o - L}{H - L}$$

$$a \leq \frac{10 - 5}{13 - 5}$$

$$a \leq \frac{5}{8}$$

$$a \leq 0.625$$

56. Derivation ambiguity value for player 1 with $H = 13, L = 5, F_0 = 10$ and $a = 0.2$ to induce $s_1 = short \succ s_1 = long$:

$$\delta_1 < \frac{2a(H - L)}{(2a + 1)H - (2a - 1)L - 2F_0}$$

$$\delta_1 < \frac{2 \cdot 0.2 \cdot (13 - 5)}{(2 \cdot 0.2 + 1) \cdot 13 - (2 \cdot 0.2 - 1) \cdot 5 - 2 \cdot 10}$$

$$\delta_1 < \frac{0.4 \cdot 8}{(0.4 + 1) \cdot 13 - (0.4 - 1) \cdot 5 - 20}$$

$$\delta_1 < \frac{0.4 \cdot 8}{1.4 \cdot 13 - (-0.6) \cdot 5 - 20}$$

$$\delta_1 < \frac{3.2}{18.2 + 3 - 20}$$

$$\delta_1 < \frac{3.2}{21.2 - 20}$$

$$\delta_1 < \frac{3.2}{1.2}$$

$$\delta_1 < 2.667$$

57. Derivation, value for $a \leq \frac{H - F_0}{H - L}$ with $H = 15, L = 5, F_0 = 10$:

$$a \leq \frac{H - F_0}{H - L}$$

$$a \leq \frac{15 - 10}{15 - 5}$$

$$a \leq \frac{5}{10}$$

$$a \leq 0.50$$

58. Derivation ambiguity value for player 2 with $H = 15, L = 5, F_0 = 10$ and $a = 0.2$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H - L)}{(2a - 1)H - (2a + 1)L + 2F_0}$$

$$\delta_2 < \frac{2 \cdot 0.2 \cdot (15 - 5)}{(2 \cdot 0.2 - 1) \cdot 15 - (2 \cdot 0.2 + 1) \cdot 5 + 2 \cdot 10}$$

$$\delta_2 < \frac{0.4 \cdot 10}{(0.4 - 1) \cdot 15 - (0.4 + 1) \cdot 5 + 20}$$

$$\delta_2 < \frac{4}{(-0.6) \cdot 15 - (1.4) \cdot 5 + 20}$$

$$\delta_2 < \frac{4}{(-0.6) \cdot 15 - (1.4) \cdot 5 + 20}$$

$$\delta_2 < \frac{4}{-9 - 7 + 20}$$

$$\delta_2 < \frac{4}{-16 + 20}$$

$$\delta_2 < \frac{4}{4}$$

$$\delta_2 < 1$$

59. Derivation ambiguity value for player 2 with $H = 15, L = 5, F_0 = 10$ and $a = 0.1$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H - L)}{(2a - 1)H - (2a + 1)L + 2F_0}$$

$$\delta_2 < \frac{2 \cdot 0.1 \cdot (15 - 5)}{(2 \cdot 0.1 - 1) \cdot 15 - (2 \cdot 0.1 + 1) \cdot 5 + 2 \cdot 10}$$

$$\delta_2 < \frac{0.2 \cdot 10}{(0.2 - 1) \cdot 15 - (0.2 + 1) \cdot 5 + 20}$$

$$\delta_2 < \frac{2}{(-0.8) \cdot 15 - (1.2) \cdot 5 + 20}$$

$$\delta_2 < \frac{2}{-12 - 6 + 20}$$

$$\delta_2 < \frac{2}{-18 + 20}$$

$$\delta_2 < \frac{2}{2}$$

$$\delta_2 < 1$$

60. Derivation ambiguity value for player 2 with $H = 15, L = 5, F_0 = 10$ and $a = 0.5$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$$

$$\delta_2 < \frac{2\cdot 0.5\cdot(15-5)}{(2\cdot 0.5-1)\cdot 15-(2\cdot 0.5+1)\cdot 5+2\cdot 10}$$

$$\delta_2 < \frac{1\cdot 10}{(1-1)\cdot 15-(1+1)\cdot 5+20}$$

$$\delta_2 < \frac{10}{0\cdot 15-2\cdot 5+20}$$

$$\delta_2 < \frac{10}{-10+20}$$

$$\delta_2 < \frac{10}{10}$$

$$\delta_2 < 1$$

61. Derivation, value for $a \leq \frac{H-F_0}{H-L}$ with $H = 12, L = 5, F_0 = 10$:

$$a \leq \frac{H-F_0}{H-L}$$

$$a \leq \frac{12-10}{12-5}$$

$$a \leq \frac{2}{7}$$

$$a \leq 0.286$$

62. Derivation ambiguity value for player 2 with $H = 12, L = 5, F_0 = 10$ and $a = 0.2$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$$

$$\delta_2 < \frac{2\cdot 0.2\cdot(12-5)}{(2\cdot 0.2-1)\cdot 12-(2\cdot 0.2+1)\cdot 5+2\cdot 10}$$

$$\delta_2 < \frac{0.4\cdot 7}{(0.4-1)\cdot 12-(0.4+1)\cdot 5+20}$$

$$\delta_2 < \frac{2.8}{(-0.6)\cdot 12-(1.4)\cdot 5+20}$$

$$\delta_2 < \frac{2.8}{-7.2-7+20}$$

$$\delta_2 < \frac{2.8}{-14.2+20}$$

$$\delta_2 < \frac{2.8}{5.8}$$

$$\delta_2 < 0.483$$

63. Derivation, value for $a \leq \frac{H-F_0}{H-L}$ with $H = 15, L = 1, F_0 = 10$:

$$a \leq \frac{H - F_0}{H - L}$$

$$a \leq \frac{15 - 10}{15 - 1}$$

$$a \leq \frac{5}{14}$$

$$a \leq 0.357$$

64. Derivation ambiguity value for player 2 with $H = 15, L = 1, F_0 = 10$ and $a = 0.2$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H - L)}{(2a - 1)H - (2a + 1)L + 2F_0}$$

$$\delta_2 < \frac{2 \cdot 0.2 \cdot (15 - 1)}{(2 \cdot 0.2 - 1) \cdot 15 - (2 \cdot 0.2 + 1) \cdot 1 + 2 \cdot 10}$$

$$\delta_2 < \frac{0.4 \cdot 14}{(0.4 - 1) \cdot 15 - (0.4 + 1) \cdot 1 + 20}$$

$$\delta_2 < \frac{5.6}{-0.6 \cdot 15 - 1.4 \cdot 1 + 20}$$

$$\delta_2 < \frac{5.6}{-9 - 1.4 + 20}$$

$$\delta_2 < \frac{5.6}{-10.4 + 20}$$

$$\delta_2 < \frac{5.6}{9.6}$$

$$\delta_2 < 0.583$$

65. Derivation, value for $a \leq \frac{H-F_0}{H-L}$ with $H = 20, L = 5, F_0 = 10$:

$$a \leq \frac{H - F_0}{H - L}$$

$$a \leq \frac{20 - 10}{20 - 5}$$

$$a \leq \frac{10}{15}$$

$$a \leq \frac{2}{3}$$

$$a \leq 0.667$$

66. Derivation ambiguity value for player 2 with $H = 20, L = 5, F_0 = 10$ and $a = 0.2$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H - L)}{(2a - 1)H - (2a + 1)L + 2F_0}$$

$$\delta_2 < \frac{2 \cdot 0.2 \cdot (20 - 5)}{(2 \cdot 0.2 - 1) \cdot 20 - (2 \cdot 0.2 + 1) \cdot 5 + 2 \cdot 10}$$

$$\delta_2 < \frac{0.4 \cdot 15}{(0.4 - 1) \cdot 20 - (0.4 + 1) \cdot 5 + 20}$$

$$\delta_2 < \frac{6}{(-0.6) \cdot 20 - (1.4) \cdot 5 + 20}$$

$$\delta_2 < \frac{6}{-12 - 7 + 20}$$

$$\delta_2 < \frac{6}{-19 + 20}$$

$$\delta_2 < \frac{6}{1}$$

$$\delta_2 < 6$$

67. Derivation, value for $a \leq \frac{H-F_0}{H-L}$ with $H = 15, L = 7, F_0 = 10$:

$$a \leq \frac{H - F_0}{H - L}$$

$$a \leq \frac{15 - 10}{15 - 7}$$

$$a \leq \frac{5}{8}$$

$$a \leq 0.625$$

68. Derivation ambiguity value for player 2 with $H = 15, L = 7, F_0 = 10$ and $a = 0.2$ to induce $s_2 = long \succ s_2 = short$:

$$\delta_2 < \frac{2a(H - L)}{(2a - 1)H - (2a + 1)L + 2F_0}$$

$$\delta_2 < \frac{2 \cdot 0.2 \cdot (15 - 7)}{(2 \cdot 0.2 - 1) \cdot 15 - (2 \cdot 0.2 + 1) \cdot 7 + 2 \cdot 10}$$

$$\delta_2 < \frac{0.4 \cdot 8}{(0.4 - 1) \cdot 15 - (0.4 + 1) \cdot 7 + 20}$$

$$\delta_2 < \frac{3.2}{-0.6 \cdot 15 - 1.4 \cdot 7 + 20}$$

$$\delta_2 < \frac{3.2}{-9 - 9.8 + 20}$$

$$\delta_2 < \frac{3.2}{-18.8 + 20}$$

$$\delta_2 < \frac{3.2}{1.2}$$

$$\delta_2 < 2.667$$

69. Condition for a to induce $(2a+1)H - (2a-1)L - 2F_0 > 0$:

$$(2a+1)H - (2a-1)L - 2F_0 > 0$$

$$2aH + H - 2aL + L > 2F_0$$

$$2a(H - L) + H + L > 2F_0$$

$$2a(H - L) > 2F_0 - H - L$$

$$2a > \frac{2F_0 - H - L}{(H - L)}$$

$$a > \frac{2F_0 - H - L}{2(H - L)}$$

70. Numerical examples for $a > \frac{2F_0 - H - L}{2(H-L)}$ in order to examine its properties:

First of all we must have $H > F_0 > L > 0$.

(a.) $H = 15 > F_0 = 10 > L = 5 > 0$ then $a > \frac{2F_0 - H - L}{2(H-L)}$ becomes:

$$a > \frac{2 \cdot 10 - 15 - 5}{2(15 - 5)}$$

$$a > \frac{20 - 20}{2 \cdot 10}$$

$$a > \frac{0}{20}$$

$$a > 0$$

This is always satisfied.

(b.) $H = 3 > F_0 = 2 > L = 1 > 0$ then $a > \frac{2F_0 - H - L}{2(H-L)}$ becomes:

$$a > \frac{2 \cdot 2 - 3 - 1}{2(3-1)}$$

$$a > \frac{4-4}{2 \cdot 2}$$

$$a > \frac{0}{4}$$

$$a > 0$$

This is always satisfied.

(c.) $H = 10 > F_0 = 9 > L = 8 > 0$ then $a > \frac{2F_0 - H - L}{2(H-L)}$ becomes:

$$a > \frac{2 \cdot 9 - 10 - 8}{2(10-8)}$$

$$a > \frac{18-18}{2 \cdot 2}$$

$$a > \frac{0}{4}$$

$$a > 0$$

This is always satisfied.

(d.) $H = 10 > F_0 = 2 > L = 1 > 0$ then $a > \frac{2F_0 - H - L}{2(H-L)}$ becomes:

$$a > \frac{2 \cdot 2 - 10 - 1}{2(10-1)}$$

$$a > \frac{4-11}{2 \cdot 9}$$

$$a > \frac{-7}{18}$$

$$a > -\frac{7}{18}$$

Since $a \in [0,1]$, $a > -\frac{7}{18}$ is always satisfied.

(e.) $H = 15 > F_0 = 14 > L = 1 > 0$ then $a > \frac{2F_0 - H - L}{2(H-L)}$ becomes:

$$a > \frac{2 \cdot 14 - 15 - 1}{2(15-1)}$$

$$a > \frac{28-16}{2 \cdot 14}$$

$$a > \frac{12}{28}$$

$$a > 0.43$$

So when H and F_0 are high and close to each other and L is low and far away from H and F_0, simultaneously, then the upside (payoff if investor 1 is right), $F_0 - L = 14 - 1 = 13$ is high and the downside (payoff if investor 1 is wrong), $F_0 - H = 14 - 15 = -1$ is negative but small. Thus the upside is high and downside is low simultaneously. This may lead to the flipping of the inequality sign in $\delta_1 < \frac{2a(H-L)}{(2a+1)H-(2a-1)L-2F_0}$ from the upper bound ($<$) to the lower bound ($>$) only when $a < 0.43$. So if the increment in investor 1's belief in ω_L materializing is smaller than 0.43, the increment is too small and thus ambiguity is needed. However, since the upside is positive and high, and the downside is negative but low, simultaneously, every value of ambiguity suffices.

The way of analysis and pattern that is found will be the same for investor 2. So when the upside is high and the downside is small, simultaneously, then the inequality sign may flip in $\delta_2 < \frac{2a(H-L)}{(2a-1)H-(2a+1)L+2F_0}$. Since investor 2 takes the opposite position of investor 1, the upside and downside for investor 2 are opposite to the upside and downside for investor 1. That's the difference, however the pattern that arises will be the same.