Applying an Active Trading Strategy to Financial Bubbles.

Forward Looking Risk Management

Master Thesis Quantitative Finance

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January 24, 2018

Abstract

This paper examines the performance of the The Log-Periodic Power-Law (LPPL) model and the Generalized Hurst Exponent (GHE) in forecasting a bubble's crash date. Furthermore, it is investigated whether applying a market-timing strategy to financial bubbles results in obtaining higher returns compared to a passive trading strategy. The LPPL model and the GHE approach are fitted to historical price time series of several indexes. Based on the information gathered from this calibration, an active trading strategy is applied. The aim of the active trading strategy is to obtain significant higher returns, whilst reducing the risk, compared to a passive trading strategy. Furthermore, the performance of the LPPL and the GHE model on the Bitcoin is examined. Both the LPPL model and the GHE approach are accurate when fitting historical bubbles. When applying the LPPL model and GHE approach to a trading strategy, the models overall show to be beneficial. The LPPL model leaving the market at the OLS estimated critical time provides the best trading strategy. For the Bitcoin, the models do not provide a higher return.

Keywords : Active Trading Strategy, Financial Bubbles, Log-Periodic Power-Law Model, Generalized Hurst Exponent, Critical Time, Forward Looking Risk Management, Cryptocurrency, Bitcoin

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S/o to Robbert for putting up with me the from day one of the bachelor. To Kimberlee for being my MB and joining me on the OM journey. To Leander for all the coffee and all life lessons learned. To Bella for accepting all forced affection and returning the favour. And most important to my lovely mother for being the best role model a child can have.

1 Introduction

In this research, the performance of bubble crash estimation models are verified and incorporated in different trading strategies. This paper contributes by comparing the performance of the different models. Setting appropriate parameter conditions and boundaries, the models are used in a trading strategy. The performance of these strategies is determined using the Sharpe ratio of the trading strategies applied to historical bubbles.

A bubble can be defined as "An economic cycle characterized by rapid escalation of asset prices, followed by a contraction" (Investopia, 2017). The bubble is created by hearding market behaviour, leading to overpriced assets. The collapse of the bubble occurs when the investors massively sell the asset as consequence of unreasonable high prices. In the past, many financial bubbles have occurred. The first large bubble was the Dutch Tulip Bubble in the 1630s. Other large well known bubbles are the South Sea bubble in 1720s, Japan's Real Estate and Stock Market Bubble in 1989, the Dot-Com Bubble in 2000. Especially after the financial crisis of 2007-2008, the importance and dangers of bubbles have acquired more attention.

When companies hold assets that are affected by a bubble, passive trading strategies can lead to excessive risk. These risks can be reduced by using bubble crash detection models and applying an active trading strategy when the assumed bubble's crash date is detected. The Log-Periodic Power-Law (LPPL) model and the Generalized Hurst Exponent (GHE) are two means to predict the moment of the bubble collapse. One of the parameters of the LPPL model is the crash date of the bubble. The GHE approach provides a signal in the form of a maximum multifractality, indicating that traders change their strategies. Using the critical time, the moment the bubble collapses, as estimated by the models, investors are able to act and take their money out before bubbles collapse while maximizing profits riding the bubble. In theory this approach will reduce the risk and lead to excessive returns, compared to the passive trading strategies.

The contribution of this research to existing literature is in threefold. The first contribution is the comparison between the LPPL model based on Maximum Likelihood Estimation and the GHE approach. Both the LPPL model and the GHE approach are tested for accuracy using 12 different time series during historical bubbles. The performance is discussed for both model separately to get a indication of the use and flaws. Based on these observations, the models can be compared to each other.

The second contribution is by incorporating the LPPL model and the GHE approach in active trading strategies. Nowadays, passive trading strategies are preferred by investors. However, an inefficient market allows for arbitrage, and therefore larger excess returns. The estimated crash date of a financial bubble can be used in an active trading strategy, in this case market timing. The LPPL model and GHE are applied to real time trading data. For three different risky assets, the application of bubble detecting strategies are tested. The selected assets are: the Shanghai asset Exchange Composite (SSEC), the EUROSTOXX 50 (ES50) and the S&P 500 indices. All indices are affected by a bubble during certain time periods. As risk-free asset benchmark, the United States government bond with a one month maturity is used. If a model shows presence of a bubble, changes are made in the tactical asset allocation. The goal of the bubble detection strategies, are to beat the risk-free asset while limiting the risks. In other words, ride the bubble to obtain excess returns, but leave the asset before the crash occurs.

The third contribution is applying the models and a new strategy to current data on the Bitcoin currency. The chosen indices all contain several financial bubbles in their time series. Therefore, using this information it can be tested if the market timing trading strategy empirically offers higher returns. Currently, many have speculated that the Bitcoin prices are following a bubble (i.e.Helms (2017)). Both models are fitted to time series of the Bitcoin during historical bubbles. Similar as to the indices, a market timing investment strategy is provided for the Bitcoin using the models.

The notion of not able to beat the market by nothing other than luck is the conclusion of the efficient-market hypothesis (EMH) of Fama (1970). The hypothesis states that the asset prices are based on certain risk factors and all information is processed in the prices. According to the theory, an efficient market qualifies itself in one of three levels: weak form, semi-strong form or a strong form. In the weak form extraordinary profits can be obtained by using fundamental analyses. For markets in semi-strong form, higher returns compared to the market can only be obtained via inside information, as the price does not only contain information on the historical price evaluation but also on all other public information. For strong markets the use of inside information will not yield excess returns over the market in the long run.

The hypothesis of a full efficient market is often rejected in the literature. In the foreword of Smithers (2009), Jeremy Grantham claims the global financial crisis of 2008 is caused by the belief in the efficient-market hypothesis, as the hypothesis underestimates asset bubbles and the risks arising with them. A market is not efficient when it is in a bubble state. A rejection of the EMH indicates that it is possible to obtain excess returns on the long term by deviating from the market, which is not linked to risk taking. Therefore, profitable arbitrage opportunities exist and market timing can be used to exploit them. The focus of this paper is on the latter form of active trading as the goal is to forecast a market bubble and use this information to beat the market.

This research aims to detect a bubble and predict the time it collapses. When it comes to the theoretical side of (predicting) financial bubbles, there are three well-known frameworks: the Efficient Market Hypothesis, Rational bubble view and the LPPL model. Another more recent method is using the GHE.

As one of the goals of this research is to apply a market timing strategy, the critical time of the bubble is crucial. To detect bubbles and their critical time, the Johansen-Ledoit-Sornette (JLS) framework has been proved to be accurate (Johansen & Sornette (1999), Johansen et al. (1999), Johansen et al. (2000) and Sornette (2003)). The JLS framework follows a LPPL model to mathematically describe the evolution of the price prior to a crash or regime switch. The LPPL model is based on a hypothesis of collective heading behavior within rational expectations. When selecting the appropriate event window, the model LPPL can capture a time jump in log-periodic oscillations of the prices (Johansen et al. (1999) and Lillo & Mantegna (2004)). Furthermore, the model is able to detect excessive volatility before a crash occurs (Choudhry (1996) and Levy (2008)) Using the price evolution, a method is provided which can predict the time the bubble bursts. Knowing this critical time makes it interesting for traders to apply an active trading strategy and outperform the market.

The LPPL model received some criticism. For example, Laloux et al. (1999) state that fitting a seven-parameter model could result in overfitting, as the data contains a high level of noise. Also, Feigenbaum (2001) shows in a sensitivity analysis that the time of the crash depends on the size of the selected event window. However, according to Kaizoji (2006) the model shows to fit the data, as the changes in asset price are captured in the tail of the power-law distribution. The research of Kaizoji (2010) shows that financial crashes are indeed a consequence of the herding behavior of noise traders and that the momentum in noise trading during the life cycle of active noise trading determines the increase or decrease of the asset price. The LPPL model has accurately predicted the critical time for multiple historical bubbles (i.e. Zhang et al. (2016)). As such, this model will be used in order to determine the critical time of a bubble.

Recently, an alternative method to detect the crash time has come more to the attention. This so called Hurst Exponent (HE) measures the long-range dependence in a fractional Brownian motion process (Hurst, 1951). There are various applications to the HE. One of them is the GHE approach, in which the HE is generalized, providing a test method for long-range correlation in time series. The GHE approach finds the time of the bubble burst based on the multifractality. In their paper, Morales et al. (2012) adjust the GHE method so that larger weights are given to more recent events. Placing restrictions on the weights results in the model being able to capture the multifractality in the time series better. Despite the limited amount of research regarding the weighted GHE approach, promising empirical results have been provided (Watorek & Stawiarski, 2016). Grech & Mazur (2005) compares an approach which measures the fluctuations of a time series around the local trend, against a method which uses moving average instead of linear or polynomial detrendisation. These two methods tend to either be noisy or overestimate the Hurst exponent.

According to the EMH, financial bubbles do not exist. As it is assumed by the hypothesis that the prices contain all information available, it is not possible for the prices to deviate from the fundamental value and bubbles are not able to emerge. However, further research has shown that under the EMH, it is possible to obtain higher returns than the market. Novak & Beirlant (2006) show that within the EMH framework, it is possible to predict the magnitude of a crash using extreme value theory (EVT). Research on a conditional EVT show to outperform bubble detection methods that rely on asset price movements with the normal distribution (Bali, 2007). However, the distribution of the EVT does not provide a semi-martingale process. In other words, the integral over the distribution function does not always exist, making the resulting asset predictions unreliable.

In the framework of the rational bubble view, Shiller (1981) detects rational bubbles using the variance of the asset's fundamental value as an boundary. The variance of the observed asset price will be larger than this boundary in the presence of a rational bubble. According to Friedman & Abraham (2007), the rational bubble view assumes an unobservable financial asset value and a observable transaction price. The deviation of the transaction price from the asset value then represents the bubble. The downfall of this approach is that there is no clear approach on how to determine the financial asset value. The results from empirical test show that the rational bubble model does not identify the price bubbles accurately before a large price drop (i.e. Blancard & Watson (1982), West (1987)).

Two frameworks have been chosen for this paper to estimate the critical time of a bubble, the LPPL model and the GHE approach. Watorek & Stawiarski (2016) compared the LPPL model based on OLS with the GHE approach and concluded that they show to perform equally good. It is stated by Watorek & Stawiarski (2016) that the GHE approach has the advantage of predicting a burst based on the selected weights and multifractality poses, making it more appealing to use. However, Filmonov et al. (2016) provide a new and improved estimation method for the LPPL model, based on a modified profile likelihood. This method returns a more accurate estimation of the crash time compared to estimation by OLS regressions. Because of the estimation improvement on the LPPL model, this research will compare the GHEapproach with the improved LPPL model.

One of the objectives of this research is to apply an active trading strategy when a bubble is detected. The trading strategies are based on the critical time estimations of the LPPL model and the GHE approach, the trading strategy can be determined. As the market is inefficient in the presence of a bubble and one knows the market is in a bubble, then it could be possible to use this knowledge as an advantage. The key element of the strategy is to change the asset allocation before the bubble bursts. As the LPPL model and GHE approach detect the crash date, where other methods are only informative on the crash size, this research applies two approaches to financial assets. The resulting returns from the trading strategy using the LPPL and GHE model will be compared in order to choose the best model in an active trading strategy framework.

The results show that the LPPL model and the GHE approach are both able to accurately estimate the critical time when choosing an appropriate time window. The LPPL model is however sensitive to the number of observations in the time window and to different growth trends during the time series. Also, the model often finds a local minimum rather than a global minimum. The GHE approach is also sensitive tho the number of observations in the time window. The crash date of the Bitcoin time series are estimated accurately using the GHE approach. The LPPL model fails to estimate the crash date for the Bitcoin mainly due to the large (daily) volatility.

The trading strategies using the LPPL model and GHE approach show that it is overall profitable to adjust the trading strategy to a possible crash when the asset is assumed to be in a bubble. Using the OLS estimate of the LPPL model overall provides good combinations for the cumulative excess returns, Sharpe ratio and Value at Risk. Adding a minimal investment period of six months to the strategies in some cases improves the GHE strategies. Using the

LPPL model or GHE approach in a trading strategy for the Bitcoin time series does not result in profitable returns. The models detect a crash in the time series, but the Bitcoin price increases again after the deterioration. Therefore, selling the Bitcoin as implied by the models leads to lower cumulative excess returns compared to not selling the cryptocurrency.

In this paper, section 2 will explain the methodology of the LPPL model, the calibration procedure of the model and the GHE approach. Thereafter, section 3 provides case studies for the two frameworks. In these studies, 9 historical bubbles from financial markets and three historical Bitcoin bubbles are tested. Section 4 discusses the performance of the LPPL model or GHE approach in combination with a trading strategy. In total, six different trading strategies are applied to three different indices and to the Bitcoin. The performance of the strategies are examined without and with a holding period, before applying the strategies. Section 5 gives the conclusion of the research.

2 Methodology

This section discusses the methodology used to predict the date of a bubble crash. Section 2.1 discusses the Log-Periodic Power-Law (LPPL) model, followed by the calibration technique in section 2.2. As alternative to the LPPL model, section 2.3 explains the Generalized Hurst Exponent (GHE). A comparison between the LPPL model and GHE approach is provided in section 2.4.

2.1 Log-Periodic Power-Law Model

The Johansen Ledoit Sornette (JLS) model describes the dynamics of financial markets during bubbles and crashes (Johansen et al., 2000). The model states that bubbles are not characterized by an exponential increase of price, but rather by a faster-than-exponential growth of price. This phenomenon is generated by behaviors of investors and traders that create positive feedback in the valuation of assets leading to unsustainable growth ending with a finite-time singularity at some future time t_c.

According to the JLS model, the asset price $p(t)$ can be denoted as an jump-diffusion process with drift $\mu(t)$ and volatility $\sigma(t)$ as

$$\frac{dp}{p} = \mu(t)dt + \sigma(t)dW - \kappa dj, \tag{1}$$

where dW a standard Wiener process and dj denotes the discontinuous jump as $j = \Xi(t - t_c)$. The function $\Xi(\cdot)$ is a heavy-side function and t_c represents the time for which a market crash or regime switch is most probable, referred to as the "critical time". The hazard rate is the probability per unit of time that the crash will happen in the next instant if it has not happened yet. Here, the crash hazard rate $h(t)$ is defined by the expected value of the discontinuous jump diffusion, $E[dj] = h(t)dt$ and given as

$$h(t) = \alpha(t_c - t)^{m-1}\big(1 + \beta\cos(w\ln(t_c - t) - \phi')\big). \tag{2}$$

The dynamics of the hazard rate combine the complex actions of the noise traders. The term $(t_c - t)$ states the singular power law behavior, which causes the super-exponential price growth.

Under the assumption of no arbitrage, $E[dp] = 0$, and no crash has occurred yet, $dj = 0$, the expected value of a log price can be derived to the Log-Periodic Power Law equation given as

$$\mathrm{LPPL}(t) \equiv E[\ln p(t)] = A + B|t_c - t|^m + C|t_c - t|^m\cos(w\ln|t_c - t| - \phi), \tag{3}$$

where $B = -\kappa\alpha/m$ and $C = -\kappa\alpha\beta/\sqrt{m^2 + w^2}$. A represents the expected log price value at the time of the crash, B gives the magnitude of the power law acceleration and C gives the amplitude of the log-periodic oscillations. The absolute brackets are allowed when assuming time-inversion symmetry of the price trajectory around t_c is valid. Note that under the no arbitrage condition, the excess return is proportional to the crash hazard rate, with a factor κ.

Calibration of the LPPL model in (3) requires solving for four nonlinear parameters. To simplify the calibration, Filimonov & Sornette (2013)[1] propose a change in the variable C as

$$C_1 = C\cos\phi, \tag{4}$$

$$C_2 = C\sin\phi. \tag{5}$$

Changing the variable results in a reformulation of (3) to

$$\begin{aligned}\mathrm{LPPL}(t) \equiv E[\ln p(t)] = A + B|t_c - t|^m + C_1|t_c - t|^m\cos(w\ln|t_c - t|)\\ + C_2|t_c - t|^m\sin(w\ln|t_c - t|),\end{aligned} \tag{6}$$

consisting only three nonlinear parameters (t_c, w, m). Reducing the number of nonlinear parameters improves the stability of the model, as the smoothing properties are improved and in turn the amount of local minimums reduced.

The parameters of the LPPL model are bound by certain constraints. Taking the integral of the hazard rate over $[t_0, t_c]$ returns the probability that a crash occurs. As the probability is bounded by 1, the super-exponential growth m is smaller than 1. Additionally, $m > 0$ as the expected log price is finite for any $t \leq t_c$. The acceleration of the hazard rate is captured by the constraint $B < 0$. Furthermore, the damping parameter $D = m|B|/w|C| > 1$ to prevent the hazard rate becoming negative (van Bothmer & Meister, 2003). Resulting from, among others, Huang et al. (2000), Sornette & Johansen (2010), Zhou & Sornette (2002), Johansen & Sornette (2010), Lin & Sornette (2013), Sornette et al. (2013), Lin et al. (2014) and Sornette et al. (2014),

[1] The results of Filimonov & Sornette (2013) have been reproduced where possible.

the so called “stylized features of the LPPL” model are given by

$$0.1 \leq m \leq 0.9, \qquad 6 \leq w \leq 13, \qquad B < 0, \qquad |C| < 1, \qquad D = \frac{m|B|}{w\sqrt{C_1^2 + C_2^2}} \geq 0.8. \tag{7}$$

2.2 Calibration of the LPPL model

This section describes the calibration methods of the LPPL model. Section 2.2.1 explains the Ordinary Least Squares method to fit the model. Instead of Least Squares, the model can be estimated using a Maximum Likelihood Estimation method. Section 2.2.2 discusses the Profile Likelihood method and section 2.2.3 continues with the Modified Profile Likelihood method.

2.2.1 Ordinary Least Squares

The formula in (3) requires estimating seven parameters. Estimating four non-linear and three linear parameters is extremely difficult, as discussed in Filimonov & Sornette (2013). By reducing the number of non-linear parameters to three and linear parameters to four, Filimonov & Sornette (2013) propose a more sophisticated manner of calibration of the reformulated LPPL model in (6). Estimation of the parameters is done by the Ordinary Least Squares (OLS) method. This requires minimizing the sum of squared residuals as

$$\{\widehat{t}_c, \widehat{m}, \widehat{w}\} = \arg \min_{t_c, m, w} F_1(t_c, m, w), \tag{8}$$

$$F_1(t_c, m, w) = \min_{A,B,C_1,C_2} \text{SSE}(t_c, \psi), \tag{9}$$

$$\text{SSE}(t_c, \psi) = \sum_{i=1}^{n} (\varepsilon(\tau_i; t_c, \psi))^2 = \sum_{i=1}^{n} (\ln p(\tau_i) - \text{LPPL}(\tau_i; t_c, \psi))^2, \tag{10}$$

where $\psi = \{m, w, A, B, C_1, C_2\}$, τ_i is the time in a certain window of analysis $[t_1, t_2]$ and the SSE is the Sum of Squared Errors. It is possible that the resulting optimum is found in a local minimum. To prevent the use of a local minimum, the optimization of (9) is performed using different starting values for m and w. The parameters are estimated using each of the starting values. When different minima are found, this indicates at least one of them is a local minimum. The minimization problem first minimizes the SSE over the linear parameters. As an advantage of rewriting LPPL model with 4 nonlinear parameters to a model with 3 nonlinear parameters, an unique solution for (9) can be obtained from the first-order condition

$$\begin{pmatrix} n & \sum f_i & \sum g_i & \sum h_i \\ \sum f_i & \sum f_i^2 & \sum f_i g_i & \sum f_i h_i \\ \sum g_i & \sum f_i g_i & \sum g_i^2 & \sum g_i h_i \\ \sum h_i & \sum f_i h_i & \sum g_i h_i & \sum h_i^2 \end{pmatrix} \begin{pmatrix} \widehat{A} \\ \widehat{B} \\ \widehat{C_1} \\ \widehat{C_2} \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum y_i f_i \\ \sum y_i g_i \\ \sum y_i h_i \end{pmatrix}, \tag{11}$$

where

$$y_i = \ln p(\tau_i) \qquad g_i = |t_c - \tau_i|^m \cos(w \ln|t_c - \tau_i|)$$
$$f_i = |t_c - \tau_i|^m \qquad h_i = |t_c - \tau_i|^m \sin(w \ln|t_c - \tau_i|).$$

Filimonov & Sornette (2013) state that an estimate of t_c can be obtained by using a subordination idea, separating the critical time from the other parameters in (8) and (9). This results in the optimization problem

$$\widehat{t_c} = \arg\min_{t_c} F_2(t_c), \tag{12}$$
$$F_2(t_c) = \min_{w,m} F_1(t_c, m, w), \tag{13}$$
$$\{\widehat{m}(t_c), \widehat{w}(t_c)\} = \arg\min_{m,w} F_1(t_c, m, w). \tag{14}$$

The new function additionally provides an indication on the dependence of the estimates m and w on the critical time. Similar to $F_1(t_c, m, w)$, it is possible that the found minimum of cost function $F_2(t_c)$ is a local minimum. Therefore, different starting values of t_c are used to find the global minimum.

2.2.2 Profile Likelihood

Filmonov et al. (2016)[2] propose to use Maximum Likelihood Estimation (MLE) over OLS estimation, as the obtained point estimates of the parameters hold a estimation uncertainty. The biggest advantage of using MLE over OLS is obtaining a range estimate rather than a point estimate. This is due to the estimated uncertainty incorporated in the MLE parameters. Because the error term in (10) is normally distributed, the likelihood function is given by

$$L(t_c, \psi, s^2) = (2\pi s)^{-n/2} \exp\left(- \frac{SSE(t_c, \psi)}{2s} \right) \to \max_{t_c, \psi, s^2}, \tag{15}$$

where n the number of data points and s^2 is the variance of the residuals. The likelihood function provides the estimates $\widehat{t_c}$ and $\widehat{\psi}$, and from the log likelihood it can be derived that

$$\widehat{\sigma}^2 = \widehat{s} = \frac{1}{n} \mathrm{SSE}(\widehat{t_c}, \widehat{\psi}). \tag{16}$$

The estimate for $\widehat{\sigma}^2$ shows to be equivalent with the OLS method (8) under the assumption that the error terms $\varepsilon(\tau_i; t_c, \psi)$ are normally distributed.

As the parameter of interest is t_c, Filmonov et al. (2016) propose using the profile likelihood $L_p(t_c)$ as a method to eliminate the nuisance parameters $\eta = \{\psi, s\}$. Eliminating η requires concentrating the likelihood around one parameter, while taking into account parameter uncertainty. Keeping the parameter of interest fixed, the nuisance parameters are replaced by their

[2] The results of Filmonov et al. (2016) have been reproduced where possible.

MLE as

$$L_p(t_c) = \max_{\eta} L(t_c, \eta) \equiv L(t_c, \widehat{\eta}_{t_c}), \tag{17}$$

where $\widehat{\eta}_{t_c} = \arg\max_{\eta} L(t_c, \eta)$ is a MLE for η for a fixed t_c. The parameter MLE can now all be obtained as follows: $\widehat{m}_{t_c}$ and $\widehat{w}_{t_c}$ are obtained from (14), the linear parameters $\widehat{A}_{t_c}$, $\widehat{B}_{t_c}$, $\widehat{C}_{1,t_c}$ and , $\widehat{C}_{2,t_c}$ are derived from (11). As the estimated variance van be rewritten in terms of $F_2(t_c)$, it holds for the profile likelihood that

$$L_p(t_c) \propto (\widehat{s}_{t_c})^{-n/2} \propto \big(F_2(t_c)\big)^{-n/2} \tag{18}$$

The relative profile likelihood $R(t_c)$ can be obtained by dividing a likelihood for one constant $L(t_c)$ by the maximum likelihood $\max_{t_c} L(t_c)$.

The profile likelihood is an improvement compared to the OLS estimation, but still has some limitations. The nuisance parameters are assumed to be fixed. Treating the parameters as fixed implies adding extra information to the model. When the data in fact does not contains this kind of information, the estimated likelihood is biased. Also, Filmonov et al. (2016) remark that certain conditions may lead to unstable estimates when the data undergoes small changes.

2.2.3 Modified Profile Likelihood

The modified profile likelihood solves for the limitations of the profile likelihood. By adding another modulation factor $M(t_c)$ to the profile likelihood, the so called modified profile likelihood $L_m(t_c)$ (Eriksen & Barndorff-Nielsen, 1983) is given by

$$L_m(t_c) = M(t_c)L_p(t_c) = \left| -\frac{\partial^2 \ln L(t_c, \eta)}{\partial \eta \partial \eta^T} \,\Big|_{\eta=\widehat{\eta}_{t_c}} \right|^{-1/2} \left| \frac{\partial \widehat{\eta}}{\partial \widehat{\eta}_{t_c}} \right| L_p(t_c), \tag{19}$$

where $\widehat{\eta}_{t_c}$ is the MLE of the parameters η for a time t_c. The modified profile likelihood is invariant to reparametrisation, does not require specification of statistics and approximates integrated likelihood functions. However, Filmonov et al. (2016) recon that solving the Jacobian in (19) is often difficult.

To determine $L_m(t_c)$, Severini (1998) provides an approximation of the likelihood that only uses the covariance of the nuisance parameters their score functions as

$$C(t_c, \widehat{\eta}_{t_c}; \widehat{t}_c, \widehat{\eta}) \approx \Sigma(t_c, \widehat{\eta}_{t_c}; \widehat{t}_c, \widehat{\eta}), \tag{20}$$

$$\Sigma(t_c, \widehat{\eta}_{t_c}; \widehat{t}_c, \widehat{\eta}) = \mathrm{E}_{(2)} \left[\frac{\partial \ln L(t_c, \eta)}{\partial \eta} \,\Big|_{\substack{t_c = t_{c,1} \\ \eta = \eta_1}} \frac{\partial \ln L(t_c, \eta)}{\partial \eta^T} \,\Big|_{\substack{t_c = t_{c,2} \\ \eta = \eta_2}} \right]. \tag{21}$$

Using the expression for Σ, the approximation of the modified log likelihood can be derived as

$$L_m(t_c) \propto \frac{\left| X^T(t_c, \widehat{\psi}) X(t_c, \widehat{\psi}) - H(t_c, \widehat{\psi}) \right|^{1/2}}{\left| X^T(\widehat{t}_c, \widehat{\psi}) X(t_c, \widehat{\psi}) \right|} (\widehat{s}_{t_c})^{-(n-p-2)/2}, \tag{22}$$

where $p = \dim \psi = 6$ and X and H are computed by

$$X_{ij}(t_c, \psi) = \frac{\partial \text{LPPL}(\tau_i; t_c, \psi)}{\partial \psi_j}, \tag{23}$$

$$H_{ij}(t_c, \psi) = \sum_{k=1}^{n} \big(\ln p - \text{LPPL}(\tau_k; t_c, \psi) \big) \frac{\partial^2 \text{LPPL}(\tau_k; t_c, \psi)}{\psi_i \partial \psi_j}. \tag{24}$$

The first and second order partial derivatives of the $LPPL$ function to ψ are given in Filmonov et al. (2016).

The biggest advantage of using MLE over OLS is obtaining a range estimate rather than a point estimate. This is due to the estimated uncertainty incorporated in the MLE parameters. The likelihood interval is obtained using the relative modified profile likelihood R_m. If the likelihood ratio, $R(\theta)$, for a parameter is sufficient small, the null hypothesis $\theta = \theta_0$ can be rejected. This sufficiency is determined by a likelihood interval at 5% cutoff (Fisher, 1956). In other words, the critical time estimate is significant when the relative modified likelihood is larger than 5%.

$$\text{LI}(t_c) = \left\{ t_c : R_m(t_c) = \frac{L_m(t_c)}{L_m(\widehat{t_c})} > 0.05 \right\}. \tag{25}$$

2.3 General Hurst Exponent

The Hurst Exponent is named after Hurst (1951) and generalized by (Barabási & Vicsek (1991), Mandelbrot (1997)). The GHE is able to analyze the long-term correlation of time series. The scaling properties of the time series are analyzed using the q^{th} order moments of its distribution using

$$K_q(\tau) = \frac{1}{T - \tau + 1} \sum_{t=0}^{T-\tau} |X(t+\tau) - X(t)|^q, \tag{26}$$

$$K_q(\tau) \sim \tau^{qH(q)}, \tag{27}$$

where $X(t)$ is the stochastic process of the time series, and $\tau \in \{\tau_{min}, \tau_{max}\}$. Using the scaling behavior of (26) and the relation in (27), the GHE $H(q)$ can be derived as

$$K_q(\tau) = C\tau^{qH(q)}. \tag{28}$$

The GHE can be either constant or time-varying. In case $H(q) = H$, it is independent of the order moments and from the unique Hurst coefficient or so called self-affine index. The non-constant process depending on order moments q are called multi-scaling and scale every moment using a different exponent. Financial time series often have the property of being multi-scaled. Furthermore, there are some special features. For $q = 1$, the GHE explains the scaling behavior of the absolute values of the increments. The second order moment represents scaling of the auto-correlation function.

In (26), it is assumed that all observations in the time series are equally important to the scaling behaviour $K_q(\tau)$. The growth of the asset price during a bubble does not move similar to the price of a asset without a bubble. Furthermore, the information of the asset moving with the bubble is more important compared to the information before. In the recent paper of Morales et al. (2014), it is proposed to use a smoothing procedure using weights

$$w_t = w_0 \exp \frac{-t}{\theta}, \tag{29}$$

$$w_0 = \frac{1 - \exp \alpha}{1 - \exp -\alpha T}, \tag{30}$$

where $0 < t < T$, θ is the weights characteristic time and $\alpha = 1/\theta$ is a exponential shrinkage factor. The weighted GHE can be determined by computing the weighted averages instead of the normal averages in (26) as

$$K_q^w(\tau) = \frac{1}{T - \tau + 1} \sum_{t=0}^{T-\tau} |X(t+\tau) - X(t)|^q w_t, \tag{31}$$

$$\ln K_q^w(\tau) = \ln C + qH^w(q) \ln \tau. \tag{32}$$

where (32) gives the weighted scaling relation.

The critical time of the bubble is determined by analysing the degree in which the multifractality changes in a certain time period. The degree of multifractality is computed as

$$\Delta H(1, 2) = H(1) - H(2). \tag{33}$$

In their research, Morales et al. (2014) state that $\Delta H > 0.015$ is considered as a significant multifractal.

Following the research of Watorek & Stawiarski (2016), market multifractality grows once investors apply a specific trading strategy. Investors react to these growths, enlarging the volatility of the time series and causing more frequent log price oscillations. A bubble bursts once investors stop reacting to the growths. At this point, the multifractality decreases. finding the maximum multifractality gives a sign that traders change their investment strategy.

Another interpretation for the multifractality can be given using the HE based on the scaling behaviour of the first en second order moment. As stated before, $H(1)$ gives the HE based on the scaling behaviour of the absolute values of the growth. When the price increases or decreases, the value of $H(1)$ increases. A stable price will decrease the $H(1)$ value, as there is no more growth. The HE based on the auto-correlation $H(2)$ increases when the price increases, as traders are reacting to former growth. When traders change their strategy, the scaling behavior of the auto-correlation decreases. Therefore, when a price experiences large growth, the difference ΔH is small. However, when the traders stop reacting to the bubble, the price decreases and the multifractality increases. This can imply the bubble is about to burst.

The most important information on the crash moves with the bubble. A moving window can be applied to see the impact of the estimate when approaching the critical time (Δt_c) If the estimates move with Δt_c, this gives the impression that traders have not stopped changing their investment strategy and a crash is not expected in the near future.

2.4 LPPL Model versus GHE Approach

The LPPL model and GHE approach are two very different methods to detect the crash of a bubble. As discussed in section 2.1, the LPPL model fits a 7 parameter model to a time series, with one of the estimated parameters being the critical time. As the model assumes that $t_c > t_2$, the estimate of the critical time is a forecast. The GHE approach does not forecast the critical time, but determines the maximal multifractality. the multifractality is obtained from within the known time sample. When the multifractality is high, the investors change the trading strategy they use during the bubble. This gives a signal that the bubble is about to burst.

The LPPL model fits a model. Therefore, when the time series shows a expected bubble growth regime, the model will in theory provide accurate estimates. However, when there is a change in growth pattern, the fit of the model deteriorates. The estimation methods of the model are based on OLS and MLE. Within optimization of the model fit, decisions are made on the minimization algorithm (here 'levenberg-marquardt'). The fit of the model is dependent on all of the methodology and model assumptions.

The GHE approach determines the maximum multifractality in the time series of interest. As there is no use of forecasting, there is less uncertainty on the maximum ΔH. It is important to set a appropriate threshold for when the multifractality is large enough to point out a true bubble crash rather than a smaller deterioration. Furthermore, when the maximal multifractality is not stable when looking at moving windows, this may indicate that the traders are still investing in a bubble growth.

3 Case studies

The goal of this research is to estimate the critical time of bubbles and use this information in an active trading strategy. Most important is the accuracy of the LPPL model and the GHE approach when applied to real data time series. To examine the accuracy of the LPPL model and GHE approach, nine time series are selected from indices from different markets.

The Bitcoin cryptocurrency is often speculated to be in a bubble. Previous research has been conducted to test the performance of the LPPL model to financial market data, but not on cryptocurrencies. To apply the models to the Bitcoin, three different time series on the Bitcoin log price are selected. In each of these time series, it has been speculated that the series was in a bubble-regime.

For both the LPPL model and the GHE approach, all series are examined using either a moving window or expanding window. Section 3.1 discusses the data used, section 3.2 shows the results on the LPPL model, section 3.3 examines the accuracy of the GHE approach and section 3.4 compares the results of the LPPL model to the results of the GHE approach.

3.1 Data

This section discusses the characteristics of the log price time series during a bubble. As data for the case studies, seven indices and the Bitcoin price time series are used. For the selected time series, the LPPL model and GHE approach are fitted to the historical bubble. Section 3.1.1 discusses the data characteristics of the indices and section 3.1.2 examines the Bitcoin time series.

3.1.1 Indices

Table 1 shows nine different time series selected from seven indices, the Dow Jones (DJI), the Shanghai asset Exchange Composite (SSEC), the NASDAQ, the Warsaw asset Exchange (WIG 20), the S&P 500, the Deutscher Aktienindex (DAX) and the Nikkei 225. The table shows the start date of the bubble t_{start}, the date the bubble collapsed t_c, amount of days during the bubble (n) and the price time series' statistics. Furthermore, table 1 shows statistics on the log price time series. The given statistics are the levelled mean (μ), the levelled standard deviation (σ), the skewness (γ_1), the kurtosis (γ_2) and the p-value of the Jarque-Bera (JB) test. Additional to the characteristics, appendix B shows plots of the price time series of the indices.

Table 1: **Historical Bubbles** This table shows the Indices discussed in section 3. The table presents the start date of the time series, the crash date and the number of observations in the time series. Furthermore, the characteristics (levelled) mean (μ), (levelled) standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) are given per indices. To test for normally distributed time series, the p-value of a Jarque-Bera test is given (JB).

	t_{start}	t_c	n	μ (lev)	σ (lev)	γ_1	γ_2	JB
DJI	13/06/2006	29/09/2008	578	9.4271 (1.0130)	0.0698 (0.0074)	-0.2033	2.1238	0.0006
SSEC 1	03/06/2005	01/11/2007	592	7.5903 (1.0969)	0.5432 (0.0785)	0.5673	1.9327	0.0000
SSEC 2	19/05/2014	12/06/2015	263	7.9520 (1.0461)	0.2824 (0.0371)	0.4681	1.9512	0.0018
NASDAQ	19/10/1999	01/04/2000	115	8.2681 (1.0471)	0.1682 (0.0213)	-0.4109	2.1716	0.0360
WIG20	06/03/2003	21/01/2008	1225	7.7281 (1.1079)	0.3552 (0.0509)	-0.1704	1.8734	0.0000
S&P 500 1	24/07/1984	19/10/1987	818	5.3992 (1.0807)	0.2165 (0.0433)	0.1986	1.8329	0.0000
S&P 500 2	09/03/2009	04/08/2011	608	7.0170 (1.0767)	0.1366 (0.0210)	-0.7362	3.2421	0.0000
DAX	15/10/2014	11/08/2015	205	9.2858 (1.0253)	0.0929 (0.0103)	-0.6764	2.2392	0.0023
Nikkei 225	18/10/1988	01/04/1990	358	10.4165 (1.0203)	0.0831 (0.0081)	-0.4275	2.6947	0.0076

The time series of the indices are chosen from different years. For example, the crashes of the DJI and the WIG20 were during the sub-prime mortgage crisis, the SSEC 1 series represents the Chinese asset bubble of 2007 and the NASDAQ time series is chosen during the Dot-com bubble. Another well know bubble burst is Black Monday. The crash of the S&P 500 1 is on Black Monday. As can be observed in table 1, the chosen historical bubbles show diversity in duration. The NASDAQ bubble lasted only 115 trading days, while the WIG20 bubble lasted 1225 trading days. To compare the statistics of the different time series, all log price series are divided by their first observation. These characteristics are referred to as “levelled time series”.

The levelled time series of the indices show to have comparable means. All of the levelled time series show a mean larger than 1. This indicates that, on average, the log prices grow in the selected time windows. The volatility of the time series ranges between 6.98% and 54.32%. When the asset price is in a bubble, the prices show a exponential growth. Therefore, the standard deviation of the log-prized time series is rather high. The high volatility is a indicator that they are in a bubble regime. During a bubble, the log price is expected to show exponential growth, creating a high volatility. The standard deviation of the levelled time series ranges between 0.74% and 7.85%. These values show that there is a difference in growth during a bubble regime (DJI and WIG20 differ).

The skewness ranges between -0.8215 and 0.5673. A positive skewness implies that log price values below the mean occur more often than log price values above the mean and for a negative skewness vice versa. Except for the S&P 500 2 time series, the kurtosis of the indices are below 2.7. As the skewness and kurtosis are close to the values of the normal distribution (0 and 3 respectively), a Jarque-Bera test is conducted. From the p-values in table 1 it can be concluded that for all of the indices, the ‘log time series differ significantly from a normal distribution.

Except for the DJI, DAX and Nikkei 225, the time series show to grow during the full sample period. The asset price of the DJI starts decreasing at the end of 2007. This early decrease in price results in a time series that does not follow a exponetial increase in price as is expected during the bubble. As a consequence, the LPPL model will have difficulties to fit the time series. Similar, the DAX price becomes rather stable at the beginning of April 2015. The change in growth may affect the fit of the LPPL model and cause the GHE approach to estimate the crash as well early. The Nikkei 225 price drops severely a few months before the crash date, which may affect the estimates of the models.

3.1.2 Bitcoin

In lifetime of the Bitcoin, four bubbles have already past (Helms, 2017). The first large increase in value of the Bitcoin occurred in 2010. The value of one Bitcoin, then $0.008 ten folded over five days. The ”Great bubble of 2011” was the second bubble, which brought the Bitcoin to a value of $31.91. After the burst of the bubble, the value of the Bitcoin decreased by 93%. In 2013, two bubbles were present in the Bitcoin. The first bubble took the price to a value above $250, but did not last long. At the end of 2013 another bubble raised the value to $1,242 per Bitcoin. The bubble burst as users could not withdraw their money, bringing the value back below $200.

Table 2 shows three different time series selected from the Bitcoin. The table shows the start date of the bubble t_{start}, the date the bubble collapsed t_c, amount of days during the bubble (n) and the price time series’ statistics. Furthermore, table 1 shows statistics on the log price time series. The given statistics are the levelled mean (μ), the levelled standard deviation (σ), the skewness (γ_1), the kurtosis (γ_2) and the p-value of the Jarque-Bera (JB) test. Additional to the characteristics, appendix B shows plots of the price time series of the bitcoin.

Table 2: **Historical Bubbles Bitcoin** The table presents the start date of the time series, the crash date and the number of observations in the time series. Furthermore, the characteristics (levelled) mean (μ), (levelled) standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) are given per indices. To test for normally distributed time series, the p-value of a Jarque-Bera test is given (JB).

	t_{start}	t_c	n	μ (lev)	σ (lev)	γ_1	γ_2	JB
Bitcoin 1	06/10/2010	10/06/2011	177	-0.3878 (0.1307)	1.3541 (0.4813)	-0.8215	3.1559	0.0026
Bitcoin 2	18/11/2011	17/08/2012	195	1.6807 (2.3468)	0.3451 (0.4808)	0.0532	3.7878	0.0606*
Bitcoin 3	26/10/2012	10/04/2013	118	3.0945 (1.3418)	0.7730 (0.3333)	1.2164	3.5616	0.0009

For the levelled Bitcoin 1 time series, a mean of 0.1307 is observed. The first value of the time series is levelled to 1, thus an value below 1 indicates a average decrease. As can be seen in figure 1, the log price of the Bitcoin 1 time series is mostly negative and increases becoming positive during the bubble. Due to an increasing negative log price with a few positive outliers, the mean is close to zero. The Bitcoin 2 series shows a very large mean respectively to the other bubbles, also compared to the indices. This indicates that compared to the other time series, the log price has increased most on average.

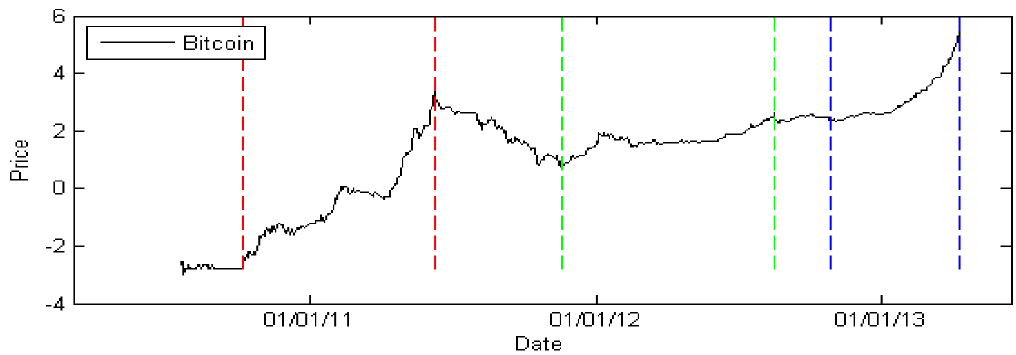


Figure 1: **Time-series of the Bitcoin prices** The time series of the Bitcoin cryptocurrency prices are given over the period July 19, 2010 to April 10, 2013. The red, green and blue dashed lines mark the start and end period for the Bitcoin 1, 2 and 3 series respectively. The second line of each color is also the crash date.

When comparing the volatility's of the indices and the Bitcoin series, the Bitcoin series show to be extremely volatile with levelled values 48.13%, 48.08% and 33.33%. It is well-known that the price of cryptocurrencies fluctuate by large amounts. Contrary to the indices, the Bitcoin price series are exposed to more risks as the value is not guaranteed and they are digitally purchased. Additional risks for the cryptocurrency are for example fraud risk and security risk. However, the coin is of interest to many investors as it serves as substitute for traditional commodities or national fiat money.

The skewness of the tree time series are -0.8215, 0.0532 and 1.2164 respectively. The third time series shows a very high positive skewness. A positive skewness implies that log price values below the mean occur more often than log price values above the mean and for a negative skewness vice versa. For the Bitcoin series, the kurtosis ranges between 3.1559 and 3.7878. The Bitcoin series are, in contradiction to most of the indices rather fat-tailed than thin-tailed. As the kurtosis of a normal distribution is approximately 3, the Bitcoin time series are either Normally distributed or fat-tailed. Fat-tailed indicates that outliers occur more often than under normality. The p-value obtained from performing a JB test provides that the Bitcoin 1 and Bitcoin 3 time series differ significantly from a normal distribution. For the Bitcoin 2, there is no significant evidence that the time series is not normally distributed.

The three time series all show the exponential growth in price. The Bitcoin 1 series experiences a price drop a month before the true crash. This could influence the estimation of the critical time for both the LPPL model as the GHE approach by estimating the crash to early. Also the Bitcoin 2 price is rather stable in the middle of the time series. The LPPL model may have difficulties to fit these bubbles. The GHE approach may be triggered by the change in price-growth after the stable period in the Bitcoin 2 time series.

3.2 LPPL Model

The goal of the case studies is to determine the accuracy of the LPPL model in estimating the crash date of historical bubbles. For the LPPL model, the number of trading days in the time window is set to range in $dt = [l, 100, 120, 180, 210, 250, 375, 500]$, where l is the amount of days during the bubble (full sample). The base of the LPPL model estimation relies on OLS estimation. Increasing the amount of observations in the estimation results in more accurate estimations. However, information moving with the bubble is most important on the date of the crash. Therefore, using smaller sample sizes may estimate the crash date more accurately as only the key information is taken into account. For each time window, a moving window is applied as $\Delta t_c = t_c - t_{end} = \{5, 10, 15, 20\}$. In words, 4 different time windows are observed in which t_2 is chosen as 5,10,15 or 20 days before the true crash date. This will provide insight how the results change when approaching the bubble crash.

In the paper of Filimonov & Sornette (2013), it is stated that no more than 20 starting values are needed when optimizing the seven parameters in the LPPL model. To be sure a global minimum is found, 50 start values for the parameters m and w are randomly drawn within the intervals $0.1 \leq m \leq 0.9$ and $6 \leq w \leq 13$, which are based on the stylized features in (7). Furthermore, the model considers the critical time as $t_{c,0} = [t_{end} + 1; t_{end} + 42]$. In words, the model estimates the critical time using a starting value for 1 to 42 trading days (or 2 months) in the future. The model is estimated for each combination of $[m_0, w_0]$ and $t_{c,0}$. In the optimization algorithm, boundary conditions have been set as $0.001 \leq m \leq 0.999$ as discussed in section 2.1. Furthermore, $0 \leq w \leq 25$ to prevent $\widehat{w}$ estimates that are to different from the stylized features.

The stylized features interval of w is based on the the research of Huang et al. (2000). They state that a value lower than 6 indicates that the log-periodic oscillations may contribute to the trend. On the other hand, if the log-periodic oscillation are as well fast, which is approximately above 13, it is possible they fit the random component of the time series. Values of w that differ from the stylized features can therefore be seen as unreliable.

In section 3.2.1, the results of the indices are discussed when fitting the LPPL model to the entire bubble time series. Following, section 3.2.2 examines the fit of the LPPL model when using the past 150 or 210 trading days. Finally, section 3.2.3 shows the fit of the LPPL model to the Bitcoin time series.

3.2.1 Full sample size

Table 3 shows the result for the LPPL model when using the whole time sample for the time series of each index. The table shows the results for the OLS estimation and the modified profile likelihood estimation. For each estimation method, the distance from the estimated critical time to the true critical time is given as $\Delta\widehat{t_c} = \widehat{t_c} - t_c$, followed by the SSE and the estimates of parameters m and w. Furthermore, LI gives the likelihood interval obtained from the modified profile likelihood estimation method. This interval can be translated to the period in which the bubble is likely to burst. If the LI contains only one value, the interval is given as $[-]$. This would imply except for the optimal value, the $R_m(t_c)$ is smaller than 5%.

For the indices, the estimated critical time based on OLS is in most cases close to the true critical time. The estimates mostly fluctuate four weeks before or after the true crash date. For some of the time series, the $\widehat{t_c}$ estimate lies a few days before the true crash date. For example, the S&P 500 2 estimations are the day after the t_2 value. In other words, the model estimates the crash directly after the end of the sample period. The prices of the S&P 500 2 time series become more stable towards the end of the sample period. This triggers the model as there is a change in growth. The SSEC 1 prices do not become stable, but the model does show a price decrease before the true crash. The LPPL model fits the crash to this deterioration rather than the true crash a few weeks later. For the DJI the estimation is extremely poor. Analyzing the time series in section 3.1.1, it appeared that the DJI does not show the exponential growth in price as is expected from a price time series during a bubble. The LPPL model is based on the assumption of a faster-than-exponential growth in price. This results in a poor fit of the model.

Overall, the indices with a large amount of observations show a larger SSE. The SSE results to be very large for the the WIG20, which is the larges time series consisting of 1225 trading days, indicating the LPPL model does not provide a good fit to the time series. Fitting a seven parameter model to larger time series gives room for more outliers and therefore a larger SSE.

What stands out is that most of the OLS estimates of m are at the end of the boundary conditions (0.001 and 0.9990). This could indicate that the estimates are based on a local optimum instead of a global optimum. The values of $\widehat{w}$ are lower than 6 for the S&P 500 2 series, the DAX and the Nikkei 225. What these time series have in common is the relative low (levelled) volatility during the bubble. As stated before, estimating a value of w below 6 may indicate that the oscillation are part of the trend.

Table 3: **Historical Bubbles full sample** This table presents the results obtained from fitting the LPPL model to the full time series available. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	-275	0.2729	0.8880	7.2914	[-]	-275	0.2729	0.8880	7.2914
	10	1419	0.4174	0.9433	12.9129	[-]	1419	0.4174	0.9433	12.9129
	15	16	0.4274	0.9990	1.3123	[-]	16	0.4274	0.9990	1.3123
	20	1605	0.4150	0.8495	14.4271	[-]	1605	0.4150	0.8495	14.4271
SSEC 1	5	-19	1.9960	0.6766	3.0586	[-]	-19	1.9960	0.6766	3.0586
	10	-19	1.9886	0.6824	3.0520	[-]	-19	1.9886	0.6824	3.0520
	15	-28	1.9631	0.7080	2.9164	[-]	-28	1.9631	0.7080	2.9164
	20	-32	1.8992	0.7030	2.7283	[-]	-32	1.8992	0.7030	2.7283
SECC 2	5	40	0.2380	0.4505	9.6360	[-]	40	0.2380	0.4505	9.6360
	10	19	0.2465	0.5711	8.0986	[-]	19	0.2465	0.5711	8.0986
	15	14	0.2456	0.5878	7.7114	[-]	14	0.2456	0.5878	7.7114
	20	10	0.2463	0.5737	7.3961	[-]	10	0.2463	0.5737	7.3961
NASDAQ	5	11	0.0736	0.9990	6.6857	[-4,24]	11	0.0736	0.9990	6.6857
	10	5	0.0698	0.9990	5.8494	[-4,24]	4	0.0698	0.9990	5.7507
	15	-2	0.0529	0.9990	4.0892	[-4,24]	0	0.0530	0.9990	4.3140
	20	10	0.0516	0.9990	4.6504	[-4,24]	10	0.0516	0.9990	4.6504
WIG20	5	-128	2.7798	0.9990	7.1292	[-]	-128	2.7798	0.9990	7.1292
	10	-128	2.7669	0.9990	7.1153	[-]	-128	2.7669	0.9990	7.1153
	15	-128	2.7662	0.9990	7.1128	[-]	-128	2.7662	0.9990	7.1128
	20	-128	2.7651	0.9990	7.1158	[-]	-128	2.7651	0.9990	7.1158
S&P 500 1	5	24	0.8052	0.8848	8.8581	[-]	24	0.8052	0.8848	8.8581
	10	19	0.7971	0.8703	8.8137	[-]	19	0.7971	0.8703	8.8137
	15	14	0.7986	0.8685	8.7469	[-]	14	0.7986	0.8685	8.7469
	20	10	0.7981	0.8633	8.6544	[-]	10	0.7981	0.8633	8.6544
S&P 500 2	5	-4	0.5629	0.9990	5.5640	[-]	-4	0.5629	0.9990	5.5640
	10	-9	0.5506	0.9990	5.4090	[-]	-9	0.5506	0.9990	5.4090
	15	-14	0.5417	0.9990	5.2665	[-]	-14	0.5417	0.9990	5.2665
	20	-19	0.5327	0.9990	5.1321	[-]	-19	0.5327	0.9990	5.1321
DAX	5	-5	0.1151	0.9938	3.0748	[-]	-5	0.1151	0.9938	3.0748
	10	-6	0.1145	0.9488	2.9705	[-5,24]	-7	0.1145	0.9332	2.9362
	15	14	0.1109	0.5789	3.7711	[0,24]	14	0.1109	0.5789	3.7711
	20	10	0.1104	0.7470	3.5854	[-1,24]	10	0.1104	0.7470	3.5854
Nikkei 225	5	-6	0.2083	0.0475	0.0047	[-]	-6	0.2083	0.0475	0.0047
	10	-10	0.2016	0.4839	0.0028	[-]	-10	0.2016	0.4839	0.0028
	15	-16	0.1947	0.5418	0.0011	[-]	-16	0.1947	0.5418	0.0011
	20	-35	0.1566	0.9990	3.6801	[-]	-35	0.1566	0.9990	3.6801

The profile likelihood estimation method corrects the OLS estimation for the number of observations. From the case studies, it appears that the profile likelihood estimation does not improve the performance of the OLS, as was expected from a theoretical perspective. Changing the number number of observations incorporated in the estimation does not change this result. Therefore, the results of the profile likelihood estimations are not provided separately to the result tables.

The modified profile likelihood estimation method improves the OLS estimation for the NASDAQ and the DAX. What these bubbles have in common is that they have a smaller duration compared to the other indices, 115 and 205 trading days respectively. The SSE is smallest for these series, indicating the LPPL model provides a better fit to these series compared to the other historical bubble time series. The indices do not have any similarities apart from a shorter time series.

An analyses on different time windows provide insight in the role of the number of observations for the modified profile likelihood estimation method. The resulting estimates for the full sample size show that for large time series, the SSE is relatively large compared to smaller time series. For smaller time series the modified profile likelihood is able to estimate a LI. Additionally, fixing the number of observations gives insight in the estimates between indices differing due to other factors.

3.2.2 Selected sample sizes

From the full sample size analyses in the previous section, it seems that the modified profile likelihood estimation method works better for smaller size time windows. To see the impact of changing the window, tables 4 and 5 show the resulting estimations of the LPPL model using a time window of 150 and 210 trading days respectively. Furthermore, Appendix C.1 also gives the results for time window sizes 100, 120, 180, 250, 375 and 500.

The SSE are overall much smaller for 150 trading days compared to large sample series, indicating that reducing the time series lengths does provide a better fit of the LPPL model. Fitting a seven parameter model to 150 observations rather than 500 observations reduces the number of possible deviations from true values and therefore also the SSE. Together with the reduction of the SSE, the critical time estimations are also improved. The observations closer to the bubble hold more information on the crash date compared to observations several years before. All information in these observations is removed and the model only focuses on the observations consisting the most relevant information.

The number of observation seems to play a large role in the goodness-of-fit of the LPPL model. Compared to the full sample results, using a time window of 150 days show large improvements in the estimations of the DJI. Using only the final 150 observations for these price time series results in a decreasing price. The linear parameter B gives the magnitude of the power law acceleration. During bubble regimes, this estimate is overall negative. However, for the DJI price time series the value is positive. The model does not fit the growth of the log price, but rather tries to fit a reverse bubble. Furthermore, reducing the number of observations also affects the S&P 500 2 series. Using 150 trading days, the model is fitted to the more stable period only. This results in a better estimation as the model only focuses on the accelerations and log price oscillations during this time series in stead of estimating the crash at the beginning of this stable period. Overall, the estimates improve by reducing sample size, as the model is more likely to fit the price time series. Incorporating more observations could lead to differences in the growth trend of the price, and deteriorate the goodness-of-fit.

Table 4: **Historical Bubbles 150 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 150 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.0371	0.9990	7.6889	[19,24]	24	0.0371	0.9990	7.6889
	10	19	0.0363	0.9990	7.3551	[13,19]	19	0.0363	0.9990	7.3551
	15	14	0.0354	0.9990	6.9372	[10,14]	14	0.0354	0.9990	6.9372
	20	8	0.0335	0.9874	6.5200	[3,8]	8	0.0335	0.9874	6.5200
SSEC 1	5	6	0.1186	0.9990	6.5183	[1,13]	6	0.1186	0.9990	6.5183
	10	-7	0.1161	0.8460	5.7696	[-9,19]	19	0.1165	0.7435	7.8302
	15	14	0.1153	0.6910	7.4828	[-10,14]	14	0.1153	0.6910	7.4828
	20	5	0.1405	0.4164	7.3966	[-1,5]	5	0.1405	0.4164	7.3966
SECC 2	5	24	0.0916	0.6353	8.0917	[9,24]	24	0.0916	0.6353	8.0917
	10	19	0.0949	0.5475	7.7576	[14,19]	19	0.0949	0.5475	7.7576
	15	14	0.1022	0.4438	7.4938	[12,14]	14	0.1022	0.4438	7.4938
	20	10	0.1141	0.3312	7.3940	[8,10]	10	0.1141	0.3312	7.3940
WIG20	5	-4	0.1038	0.0010	7.0958	[-4,-3]	-4	0.1038	0.0010	7.0958
	10	-8	0.0997	0.0010	6.4903	[-9,-1]	-8	0.0997	0.0010	6.4903
	15	1	0.1004	0.0010	8.2826	[-6,3]	1	0.1005	0.0010	8.3833
	20	5	0.0951	0.4889	8.7006	[-1,5]	5	0.0951	0.4889	8.7006
S&P 500 1	5	-4	0.0494	0.8769	3.0228	[-4,24]	-4	0.0494	0.8769	3.0228
	10	-8	0.0457	0.4373	2.7994	[7,19]	8	0.0464	0.0010	4.0934
	15	-6	0.0503	0.0010	2.9745	[-6,14]	-4	0.0504	0.0010	3.0946
	20	-12	0.0517	0.0010	2.4450	[-14,6]	-11	0.0518	0.0010	2.4952
S&P 500 2	5	1	0.0319	0.2311	3.4934	[-3,11]	1	0.0319	0.2160	3.5811
	10	-8	0.0301	0.1411	3.2289	[-]	-8	0.0301	0.1411	3.2289
	15	-14	0.0283	0.2188	2.4067	[-14,14]	-14	0.0283	0.2188	2.4067
	20	-19	0.0276	0.4422	1.4765	[-12,-8]	-8	0.0283	0.0010	1.5719
DAX	5	-5	0.0606	0.9990	2.9249	[-5,24]	-5	0.0606	0.9990	2.9249
	10	-6	0.0646	0.9990	2.8976	[-9,19]	-6	0.0646	0.9990	2.8976
	15	14	0.0630	0.9990	4.2218	[1,14]	14	0.0630	0.9990	4.2218
	20	10	0.0732	0.9990	4.2423	[1,10]	10	0.0732	0.9990	4.2423
Nikkei 225	5	24	0.0370	0.9990	2.5139	[-4,1]	-2	0.0377	0.9990	1.6695
	10	17	0.0325	0.9463	2.6033	[-11,-8]	-8	0.0333	0.9990	1.7082
	15	12	0.0323	0.9286	2.4033	[-16,-13]	-14	0.0330	0.9990	1.5576
	20	8	0.0329	0.9071	1.3758	[-]	8	0.0329	0.9071	1.3758

The estimations are overall improved compared to the full sample windows, but improvement is still possible. The parameter estimate for m lies at the boundary condition in almost half of the cases. The critical time estimations are therefore likely to be a local minimum. Finding a global minimum rather than a local minimum can increase the goodness-of-fit even more.

The S&P series, the DAX and the Nikkei 225 show small values for $\widehat{w}$. The small values may point out that the model is not fitted to the underlying jump-diffusion process, but to a trend in the log price time series In section 3.2.1, the time series with a small volatility showed to follow a trend. Fixing the time series at 150 trading days changes the (levelled) volatility. The time series with a small estimated value for w have a relatively low volatility. Therefore, it is possible that a low volatility may contribute to the LPPL model fitting a trend.

Increasing the amount of observations from 150 to 210, shows minimal improvement in the OLS estimates. It is well known that OLS overall performs better when increasing the amount of observations, as it makes the estimates less sensitive to large outliers. However, as discussed before, including as well much observations may lead to a bad fit of the LPPL model. Therefore, improving the OLS estimate through more data does not hold for the LPPL model in practice.

Looking at the parameter estimates for m increasing the number of observations to 210 trading days improves the estimates. The estimates are not meeting the boundary conditions as often as for smaller time series. Furthermore, the value of $\widehat{w}$ overall lie in the interval $6 \leq \widehat{w} \leq 13$ indicating they are a good fit according to the stylized features. The model seems to be fitted following the bubble regime instead of a trend or random component.

Based on 150 trading days, the modified profile likelihood provides a LI and improves several OLS estimations. This shows that the estimation method works better for smaller time windows. The approximation formula of the modified profile likelihood in (22) shows that the Likelihood is corrected for the number of observations. The MLE of the variance ($\widehat{s}_{t_c}$) results to be a value smaller than 1. By taking a large negative exponent, $\widehat{s}_{t_c}^{-n/2} \to \infty$, which will not improve the OLS estimation. Therefore, smaller sample size are considered when estimating the LPPL model with modified profile likelihood estimation.

As the modified profile likelihood estimation method shows to be sensitive to the number of observations, table 5 shows the results when using a window of 210 trading days. For windows larger than 210 trading days, the estimation method does not provide any LIs (see Appendix C.1). Additionally, estimating the critical time based on the modified profile likelihood does not show any improvements compared to the OLS estimations. As discussed before, a large amount of trading days results in overcorrecting the OLS estimations. This resolves the modified profile likelihood estimation method not being able to obtain a critical time estimate.

From the results over different time-widow sizes, it is obtained that increasing the number of observations could lead to better OLS estimations at first, but detoriorate the results of the modified profile likelihood estimations. All together, the complex model shows to have difficulties in robust fitting the time series. The model is sensitive to data changes and often finds a local minimum, even with 50 different starting values for the non-linear parameters. To use the LPPL model, a trade-off has to be made regarding the accuracy of the OLS estimations and being able to fit the LPPL model and a LI to the time series.

Table 5: **Historical Bubbles 210 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 210 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta \widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta \widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta \widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.0820	0.4996	7.1541	[-]	24	0.0820	0.4996	7.1541
	10	19	0.0883	0.3732	6.7810	[-]	19	0.0883	0.3732	6.7810
	15	14	0.0887	0.1696	6.5871	[-]	14	0.0887	0.1696	6.5871
	20	8	0.0799	0.0155	6.3490	[-]	8	0.0799	0.0155	6.3490
SSEC 1	5	8	0.4707	0.6664	6.8283	[1,19]	8	0.4707	0.6662	6.8109
	10	-2	0.5217	0.6601	5.6198	[-9,19]	-8	0.5226	0.6857	5.3574
	15	-12	0.5653	0.7074	4.6318	[-14,11]	-12	0.5653	0.7095	4.6077
	20	-23	0.5736	0.7494	3.7297	[-2,5]	-2	0.5821	0.9990	16.1347
SECC 2	5	24	0.1622	0.4897	8.5296	[-]	24	0.1622	0.4897	8.5296
	10	19	0.1902	0.4638	7.9748	[-]	19	0.1902	0.4638	7.9748
	15	14	0.2097	0.4649	7.5029	[11,14]	14	0.2097	0.4649	7.5029
	20	10	0.2256	0.4619	7.1389	[4,10]	10	0.2256	0.4619	7.1389
WIG20	5	-3	0.1981	0.0154	7.6179	[-]	-3	0.1981	0.0154	7.6179
	10	-7	0.1954	0.3415	7.0560	[-9,-1]	-7	0.1954	0.3415	7.0560
	15	-5	0.1776	0.3984	7.2750	[-]	-5	0.1776	0.3984	7.2750
	20	5	0.1616	0.4764	7.8473	[-]	5	0.1616	0.4764	7.8473
S&P 500 1	5	-4	0.0977	0.9990	8.9939	[-]	-4	0.0977	0.9990	8.9939
	10	-8	0.0848	0.9990	8.5946	[-]	-8	0.0848	0.9990	8.5946
	15	-7	0.0856	0.9990	8.8503	[-]	-7	0.0856	0.9990	8.8503
	20	-10	0.0845	0.9646	8.4138	[-]	-10	0.0845	0.9646	8.4138
S&P 500 2	5	-4	0.0442	0.8762	2.2080	[-]	-4	0.0442	0.8762	2.2080
	10	-9	0.0431	0.8261	1.9712	[-]	-9	0.0431	0.8261	1.9712
	15	-14	0.0414	0.7449	1.7514	[-]	-14	0.0414	0.7449	1.7514
	20	-12	0.0380	0.1466	0.0200	[-]	-12	0.0380	0.1466	0.0200
Nikkei 225	5	6	0.0557	0.0010	2.1049	[-]	6	0.0557	0.0010	2.1049
	10	0	0.0513	0.0010	2.1526	[-]	0	0.0513	0.0010	2.1526
	15	-16	0.0505	0.5511	1.4400	[-]	-16	0.0505	0.5511	1.4400
	20	8	0.0502	0.0212	1.6047	[-]	8	0.0502	0.0212	1.6047

3.2.3 Bitcoin

Overall, the LPPL model is not able to fit the Bitcoin time series. The time series show the exponential growth in price, which gives the impression that the LPPL model should be able to fit to the time series. However, the largest price growth is in the final two months. This results in a very large (levelled) volatility compared to the indices. Also, the daily price change is more extreme for the cryptocurrency. The high volatility and extreme daily price changes during the time series may cause the LPPL model to provide a bad fit to the time series.

The critical time estimates are very poor. For example, the $\widehat{t_c}$ estimates for the Bitcoin 2 time series are approximately 190 days before the crash date. However, the sample period only consists out of 195 observations. This indicates that the model estimates the crash of the first bubble rather than the second bubble, while taking into account the information after the crash. Contrary, the Bitcoin 1 time series estimates the second time series' crash for the $\Delta t_c = 15$ and 20 windows. The Bitcoin 3 estimates are several years in the future and very different for the moving windows. This results in unreliable estimates.

In table 6, the SSE shows to be very high for the Bitcoin 1 series. However, the resulting SSE for the Bitcoin 2 and 3 time series is similar to the fit of the indices. The LPPL model does provide a reasonable fit to the series. The critical time estimates however do not improve with the improvement of the SSE as discussed previously. This shows that even when the larger-than-exponential growth of the price during the bubble is captured by the LPPL model, the crash is still not accurately estimated for the Bitcoin.

Table 6: **Historical Bubbles Bitcoin full sample** This table presents the results obtained from fitting the LPPL model to the full time series available. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
Bitcoin 1	5	-168	8.7336	0.9990	10.8902	[-168,-164]	-168	8.7336	0.9990	10.8902
	10	-168	8.2533	0.9990	11.1142	[-168,4]	-168	8.2533	0.9990	11.1142
	15	171	7.5528	0.0010	25.0000	[3,172]	172	7.5549	0.0010	25.0000
	20	166	6.5482	0.0221	25.0000	[-168,5]	168	6.5537	0.0182	25.0000
Bitcoin 2	5	-187	1.3266	0.5497	2.6607	[-195,-187]	-187	1.3266	0.5497	2.6607
	10	-199	1.3739	0.6031	2.9873	[-3482,7992]	-199	1.3739	0.6031	2.9873
	15	-187	1.1684	0.3644	2.8607	[-980,-187]	-187	1.1684	0.3644	2.8607
	20	-187	1.1458	0.3290	2.8834	[-187,6734]	-187	1.1458	0.3290	2.8834
Bitcoin 3	5	2544	0.3329	0.9961	25.0000	[2401,2549]	2549	0.3330	0.9235	25.0000
	10	634	0.2579	0.9990	25.0000	[0,2549]	543	0.2643	0.9990	21.7475
	15	381	0.1896	0.0010	24.4931	[-1,370]	365	0.1974	0.0010	24.0172
	20	-21	0.1080	0.4537	3.6394	[-25,1]	-21	0.1080	0.4537	3.6394

The values of $\widehat{m}$ and $\widehat{w}$ are mostly meeting the boundary conditions. The model therefore finds the critical time estimate in a local minimum rather than a global minimum. As discussed before, the estimates of w may indicate that the LPPL model is fitted to the trend in the time series or to the random component of the data, resulting in unreliable critical time estimates.

Similar as for the indices, reducing the number of observations to 150 trading days improves the critical time estimations for the Bitcoin time series. As can be observed from table 7, the SSE is decreased for most estimations, indicating a better fit of the model. However, the SSE estimates of the Bitcoin 1 time series are still high and despite the estimates of the Bitcoin 2 time series being improved, they are still inaccurate.

The critical time estimations differ much between the full sample size and smaller sample sizes. Even using the same amount of observations shows large differences in critical time estimates, making the series dependent on the selected time window additional to the amount of observations. Adding the observations of the 14 to 10 before the crash for the Bitcoin 1 time series provides a large setback in the estimated crash date. Contrary, adding the observations 9 to 5 days before the crash improve the third week before the crash of the Bitcoin 2 series severely. As the Bitcoin series shows large daily volatility, including or excluding certain observations can have a large influence on the estimations. This contributes to the inaccurate estimations for the Bitcoin time series.

Table 7: **Historical Bubbles Bitcoin 150 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 150 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
Bitcoin 1	5	-132	3.5240	0.9990	6.8424	[-]	-132	3.5240	0.9990	6.8424
	10	-136	4.4888	0.9990	7.3861	[-]	-136	4.4888	0.9990	7.3861
	15	-18	4.8839	0.0893	3.8981	[-]	-18	4.8839	0.0893	3.8981
	20	-19	6.0987	0.1658	3.3471	[-20,-9]	-9	6.4690	0.0010	4.0871
Bitcoin 2	5	-5	0.5354	0.9990	0.0037	[-]	-5	0.5354	0.9990	0.0037
	10	-93	0.4620	0.9990	2.0651	[-]	4	0.5560	0.9990	0.0187
	15	-159	0.6585	0.7011	0.0021	[-]	-164	0.7520	0.4682	0.0137
	20	-164	0.7554	0.6419	1.1616	[-]	-164	0.7554	0.6419	1.1616

The values of $\widehat{m}$ differ over the moving windows. For both series, the parameter meets the upper boundary for two windows. The poor estimates may be due to finding a local minimum rather than a global minimum. Furthermore, the small estimates values of w may indicate LPPL model follows a trend in the time series.

For most of the Bitcoin bubbles, which also have a relative small time series, a likelihood interval is provided by the modified profile likelihood estimation. The resulting critical time estimates are however very poor. The modified profile likelihood is not able to improve the OLS estimations, independent of the amount of trading days. The LI are very extreme, sometimes ranging several years before or after the true crash date. The likelihood estimation corrects for the OLS estimations. As these estimations are overall extremely poor, there is few improvement.

From the results over different time-widow sizes, it is obtained that the model has difficulties to fit log price time series with a high (daily) volatility, such as the Bitcoin time series. Decreasing the number observations compared to the full sample size shows to improve the estimations with a small amount. Some of the estimates show to be promising, but overall the model is not able to cope with the large volatility in the series, leading to unstable and inaccurate results. Also, the modified profile likelihood does not improve the OLS estimations, but result in extreme large LI.

3.3 GHE Approach

The purpose of the case studies is to determine performance of the GHE approach for estimating the crash date of historical bubbles. As discussed in section 2.3, the HE is determined by the parameters τ, T, q and α. Following Di Matteo (2007) and Pozzi et al. (2012), $\tau \in \{5, 19\}$ and $\alpha = 83$ respectively. According to Pozzi et al. (2012), the value of T is optimal for 250 trading days. However, in this research samples smaller than and close to 250 trading days are used.

The GHE approach uses a moving window over the sample size, taking 250 observations per window and moving the window with one day. To make sure enough window comparison can be made, time series smaller than 300 observations use a T which is 50 days shorter than the time window. As the interest lies in finding the maximum multifractality, the GHE is estimated for $q = 1, 2$

In contrast to estimating the LPPL model, estimating the HE has a very low computation time. Therefore, more window sizes can be examined and used in order to determine the distance to the crash date. For each of the time series, an expanding window size is applied starting with 100 observations increasing to the full sample size. For each of the historical bubbles $\Delta t_c = t_c - t_{end} \in \{5, 10, 15, 20\}$. This creates a moving window towards the bubble.

Section 3.3.1, discusses the results of the indices when applying the GHE approach to the entire bubble time series and section 3.3.2 examines the effect on the multifractality estimate when changing the sample size to 120 or 250 trading days. Finally, section 3.3.3 shows the fit of the GHE approach to the Bitcoin time series. Additional, appendix D shows plots of the multifractality estimates during the time series using the full sample size.

3.3.1 Full sample size

Table 8 shows the results for the GHE approach when using the whole time sample. In the table, the distance from the estimated critical time to the true critical time is given as $\Delta\widehat{t_c} = \widehat{t_c} - t_c$ and $\Delta H = H(1) - H(2)$. As discussed in section 2.3, the multifractal is significant for values larger than 0.015. This indicates that all values are significant in table 8.

The multifractality is determined by reducing the HE based on the scaling behavior of the absolute values with the estimated HE based on the scaling of the auto-correlation. When the multifractality is high, the scaling behavior of the absolute value is larger than the scaling of the autocorrelation. At the beginning of the bubble, the scaling behavior of the absolute value increases as the asset price is growing. Investors react to these growths which increases the scaling of the autocorrelation as well. As both HE estimates increase, the multifractality is small. When the price becomes stable, this implies that the investors change their reaction to the growth. As a result, the $H(2)$ decreases. A stable price also decreases $H(1)$ providing a small ΔH. When price of the asset decreases, the scaling of the absolute value increases. However, the decrease in price is a result of investors changing their strategy, which deteriorates the scaling of the autocorrelation. Therefore, when prices decrease the multifractality grows.

Table 8: **Historical Bubbles full sample** This table presents the results obtained from fitting the GHE approach to full sample time series. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		NASDAQ		WIG20	
Δt_c	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0817	-166	0.0794	-343	0.0832	-32	0.0609	-33	0.0747	-543
10	0.0817	-166	0.0794	-343	0.0768	-38	0.0761	-26	0.0747	-543
15	0.0817	-166	0.0794	-343	0.0741	-43	0.0838	-31	0.0747	-543
20	0.0817	-166	0.0794	-343	0.0731	-48	0.0885	-36	0.0747	-543
	S&P 500 1		S&P 500 2		DAX		Nikkei 225			
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$		
5	0.0775	-73	0.0786	-117	0.0496	-25	0.0372	-103		
10	0.0775	-73	0.0786	-117	0.0495	-30	0.0372	-103		
15	0.0775	-73	0.0786	-117	0.0472	-35	0.0372	-103		
20	0.0775	-73	0.0786	-117	0.0502	-32	0.0372	-103		

It can be observed that the maximum multifractality estimates are quite poor as they are estimated several weeks (or even years) before the real crash. This is as the maximum multifractality could be during a small deterioration rather than an upcoming bubble crash. As discussed in section 3.1.1, the DJI becomes rather stable halfway the time-period and starts declining in the last few months before the crash. The GHE estimate captures this decrease in price and therefore estimates the multifractality prior to the crash. However, the maximum multifractality of the SSEC 1 price is estimated at the beginning of the time series. The price of the SSEC 1 grows severely after this estimated time, indicating the GHE approach finds a smaller price decrease.

Analyzing the estimates per index, they show to be very volatile for the different moving windows. Several indices show different estimates for the different time windows, indicating that there is a uncertainty about the true crash date. When a new maximum multifractality is found when moving towards the window, this indicates that the relation between the scaling behavior of the absolute value and the scaling of the autocorrelation is still changing. In other words, investors have start changing their investment strategy, but are still reacting to the growth of the bubble. Once the multifractality estimate is stable, the crash of a bubble is most likely to be in the near future.

Table 8 shows that time series with a large amount of observations are dependent on the window size. Time-series with a smaller sample size, those are the Nasdaq, the DAX and the Nikkei 225, estimate the multifractality closer to the critical time. Large time series overall show poor results, for example the SSEC 1 and the WIG20 time series.

3.3.2 Selected sample sizes

To see the impact of window-size on the multifractality, several time windows are discussed. Tables 9 and 10 show the resulting estimations of the GHE model using a time window of 120 and 250 trading days respectively. Furthermore, Appendix C.2 also gives the results for time window sizes 100, 150, 180, 210, 375 and 500.

When considering a smaller sample size of 120 trading days, the values for $\Delta\widehat{t_c}$ are closer to the critical time estimates. Overall, the estimates show that the maximum multifractality still occurs several weeks before the bubble crashes. The multifractality estimates result to be stable per index over the different time windows.

Table 9: **Historical Bubbles 120 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 120 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		WIG20	
Δt_c	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0641	-7	0.0812	-45	0.0826	-37	0.0667	-14
10	0.0640	-59	0.0812	-45	0.0826	-37	0.0667	-14
15	0.0680	-64	0.0812	-45	0.0826	-37	0.0595	-16
20	0.0680	-64	0.0812	-45	0.0826	-37	0.0654	-70
	S&P 500 1		S&P 500 2		DAX		Nikkei 225	
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0666	-38	0.0390	-34	0.0552	-54	0.0817	-12
10	0.0675	-60	0.0521	-56	0.0552	-54	0.0817	-12
15	0.0675	-60	0.0651	-62	0.0552	-54	0.0708	-22
20	0.0675	-60	0.0651	-62	0.0552	-54	0.0708	-22

Determining the HE occurs using a moving window. As discussed in section 2.3, the moving window sizes are 250 trading days for time series larger than 300 observations. When the sample size is smaller, the window sizes are taken as sample size - 50 observations. For a sample size of 120 trading days, this implies using 70 observations for 50 moving-windows. Reducing the number of observations results in affecting the scaling behavior of the autocorrelation. This affects the multifractality estimates. Overall, the estimates are approximately the same or slightly decreased.

As shown in table 10, increasing the sample size to 250 trading days improves the estimates for a part of the historical bubble time series, but show worst estimates for the other half of the time series. These results show that the performance of the GHE model does not always improve by increasing or decreasing the amount of observations. The results overall are again stable over the different time windows. However, the values of the multifractality mainly decreased compared to results based on 120 trading days and the full sample.

From analyzing the results over different time windows, it is clear that a trade-off has to be made between multifractality estimations closer to the critical time and more stable estimations. Comparing all of the tables in this section and section C.2 (tables 29 to 34), it is observed that a time window of 180 trading days provides a good trade-off results. The time series show to be stable in the weeks before the crash and the estimates are not extremely far ahead of the critical time. Also, the multifractality estimates are overall quite high respectively to the other time windows.

Table 10: **Historical Bubbles 250 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 250 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		WIG20	
Δt_c	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0529	-37	0.0700	-35	0.0749	-41	0.0259	-6
10	0.0529	-37	0.0700	-35	0.0749	-41	0.0369	-59
15	0.0529	-37	0.0700	-35	0.0741	-43	0.0369	-59
20	0.0578	-70	0.0700	-35	0.0731	-48	0.0369	-59
	S&P 500 1		S&P 500 2		Nikkei 225			
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$		
5	0.0534	-52	0.0752	-29	0.0312	-24		
10	0.0534	-52	0.0752	-29	0.0474	-58		
15	0.0534	-52	0.0752	-29	0.0474	-58		
20	0.0534	-52	0.0752	-29	0.0562	-68		

3.3.3 Bitcoin

The GHE approach shows similar estimations for the Bitcoin as for the indices. Considering the full-sample, table 11 shows that the maximum multifractality is estimated several weeks before the crash date. Comparing the estimations per time series over different time windows, the results show to be very volatile.

Decreasing the amount of observations to 120 trading days improves both the estimations as the stability of the estimates. The multifractality value increased as well. This indicates that either the HE for the scaling behavior of the absolute value increased and/or the HE for the scaling behavior of the autocorrelation decreased. As discussed in section 3.3.2, decreasing the sample size decreases the time series for which the HE is estimated. This affects the autocorrelation.

The main difference between the Bitcoin time series and the indices is the (daily) volatility. However, this aspect does not seem to have a large affect on the estimates of the GHE approach. Analyzing the resulting multifractality values of the indices and the Bitcoin time series, the values for the Bitcoin time series are larger. The large daily volatility may contribute to a larger estimate for $H(1)$, resulting in larger multifractality estimates.

Table 11: **Historical Bubbles Bitcoin** This table presents the results obtained from fitting the GHE approach to full sample time series. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

		Bitcoin 1		Bitcoin 2		Bitcoin 3	
		ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
Full sample	5	0.0562	-31	0.0873	-41	0.1045	-33
	10	0.0696	-30	0.0834	-58	0.1354	-57
	15	0.0747	-35	0.0843	-63	0.1353	-62
	20	0.0706	-40	0.0873	-36	0.1392	-67
		Bitcoin 1		Bitcoin 2			
		ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$		
120 trading days	5	0.0988	-11	0.0929	-26		
	10	0.0988	-11	0.0929	-26		
	15	0.0911	-47	0.1130	-64		
	20	0.0911	-47	0.1130	-64		

3.4 LPPL model versus GHE Approach

The LPPL model and GHE approach are two very different methods to determine the critical time of a bubble. The LPPL model fits a complex seven parameter model to the time series. One of these parameters in the model is the critical time. The LPPL model tries to forecast the exact date of the critical time and returns this as $\widehat{t_c}$. The GHE does not make any forecasts. The approach examines the time series and determines the maximum multifractality. At this date, there is a severe change in investment strategies of the investors. Once the date is known, this provides a signal that the bubble is about the burst.

Both methods show to be promising, but also show flaws. The LPPL model is sensitive to the number of observations in the time window and slightly to the volatility. When the time series shows a change in growth regime, the model will react to this change rather than the crash. Furthermore, the extension of the modified profile likelihood estimation method shows to be very dependent on the amount of observations in the time window. The model is theoretically well built and accurate, but is not that optimal empirically. When an appropriate time window is selected, the model is able to return estimates close to the true crash date. Also, a LI is provided to which an investor can base a trading strategy.

Similar to the LPPL model, the GHE approach is sensitive to the number of observations in the time window. Incorporating to few observations may make the optimal multifractality estimations volatile and important information on the series autocorrelation may be lost. On the other hand, using to many observations can give estimates of an increasing bubble instead of signalling a crash of the bubble. For the GHE approach, it is therefore also crucial that the right amount of trading days are selected in the time window.

From the case studies, it results that both the LPPL model and the GHE model are sensitive to the price becoming stable or a price decrease. Observing the full sample period, both models estimate the crash of the DJI and the WIG 20 early due the prices start decreasing prior to the crash. For the SSEC 1, the GHE approach is triggered by a small price decrease early in the time series. The LPPL model does not suffer from this decrease and estimates the critical time more accurate. The LPPL model overall performs a little bit better than the GHE approach. Unfortunately, the model often estimates the crash days after the true crash, which will lead to large losses. This is not the case for the GHE approach as there is no forecast.

In case of the Bitcoin time series, there is a clear distinction between the performance of the crash detect methods. The LPPL model is as well sensitive to the large daily volatility of the Bitcoin time series. Therefore, the model does not provide a good fit to the time series and is unable to accurately estimate the critical time. The GHE approach does not show these limitations. The accuracy of the indices and the Bitcoin time series show similarities indicating the volatility of the Bitcoin does not affect the results of the maximum multifractality.

4 Trading Strategy

As shown in section 3, the models are able to provide a good indication when a bubble crashes given the right settings. Knowing that market is inefficient during bubbles and bubbles can be detected with models as the LPPL model or the GHE, tactical asset allocation is useful to apply. Short term changes are made in the allocation, based on the knowledge of the investors.

In this research, three indices and a cryptocurrency are examined while they are in a bubble. For each of the four time series, a different historical bubble is chosen. By observing different bubbles per index, the effect of using different trading strategies can be directly observed. The trading strategies can be divided into three categories, which are no bubble crash model, the LPPL model or the GHE approach.

To get a good comparison, six investors with different trading strategies are considered. Each of the investors put all of their capital in the asset. When they leave the index due to a bubble crash, they switch to a government bond with a 1 month maturity. In this research it is examined whether using the information from the LPPL model of GHE approach to a trading strategy leads to higher returns during a financial bubble.

The first trading strategy acts as a benchmark and does not consider methods to detect the bubble crash. When the bubble has crashed, the investor does not react and keeps all capital in the asset. The second trading strategy neither uses a model approach. This investor reacts to a large price change. Examining all historical bubbles listed in Appendix E, during the crash the daily price change is (at least) -5% when investing in an index. For the Bitcoin time series the price change starts with -10%. In case the price change is smaller than these thresholds, the investor sells the asset and invests in the bond.

The third and fourth trading strategies revolve around the LPPL model. The third investor uses the LPPL model to estimate the bubble's critical time. In case the model detects a crash, the investor sells all of his stocks. This method only considers the critical time estimate. After leaving the asset, the capital is invested in the bond. The fourth investor follows a similar strategy. Instead of selling all stocks at once, this investor uses a smoothing approach when leaving the market. The investor starts smoothing at the beginning of the LI. After selling the asset, the investor invests in the risk-free asset. These two approaches determine the advantage of the LI provided by the modified profile likelihood estimation method.

The final two investors consider the GHE approach rather than the LPPL model. Comparable to the LPPL investors, the first trading strategy sells the asset when a crash is assumed by the GHE approach. The investor changes to the risk-free asset. The second GHE investor does not leave the asset at once, but uses a smoothing principle to switch to the risk-free asset. For the GHE approach, the investors reacts when a certain amount of weeks in a row the same multifractality date is returned. After this stability, the trader leaves the asset at once or based on a smoothed approach.

For each of the trading strategies, the end capitals are compared. Additional to the six investors, there is a risk-free investor which only invests in the government bond. The Cumulative Excess Returns (CER), average Excess returns (μ_{ER}), the Sharpe ratio (Sharpe), the Sortino ratio (Sortino) and the Value at Risk (VaR) are computed using the risk-free asset returns. The Sharpe ratio shows the risk that comes along with the returns. The Sortino ratio is similar to the Sharpe ratio, but considers the downside deviation of the asset rather than the standard deviation. This provides the return per unit of downside risk.

Section 4.1 discusses indices chosen for the trading strategy, the bubble of interest and the characteristics of these indices. Thereafter, section 4.2 explains the parameter settings of the LPPL model and the GHE approach. The section also goes into the details of the triggers the hypothetical investors need to change their asset allocation. Section 4.3 discusses the returns characteristics obtained from performing the trading strategy.

4.1 Data

The trading strategies as discussed in section 4 are applied to 4 different stocks during different bubbles. The first index is the SSEC. In November 2007 the Chinese asset market bubble crashed affecting the SSEC. Furthermore, from the European market the EUROSTOXX 50 (ES50) is selected. On the 21st of January 2008, the US Housing bubble burst, affecting even the European market. The third bubble takes place in 2010 and 2011 ending with the US asset market crash on August 4, 2011. This bubble affected the S&P 500 is used. The fourth bubble takes place in the cryptocurrency market. The Bitcoin price has shown large fluctuations from the moment it came to the market (see section 3.1). On the 7th of July 2017, the '2017 Summer Selloff' lead to a large crash in the price of the cryptocurrency.

If the investors sell the risky asset, they buy a government bond. For this research, the United States government bond 1-Month Yield (US-1M) is chosen. From the moment of investing in the bond until the end of the trading period, the position in the bond is renewed every month.

Table 12 shows the period considered per market and the data characteristics of the log price. Furthermore, the table shows the percentage price change on the crash date and the three days after. Figure 2, 3, 4 and 5 show the log price time series of the SSEC, ES50, S&P 500 and the Bitcoin respectively. Also, the series their returns are given during the time-period. With the returns, the returns of the US-1M are plotted.

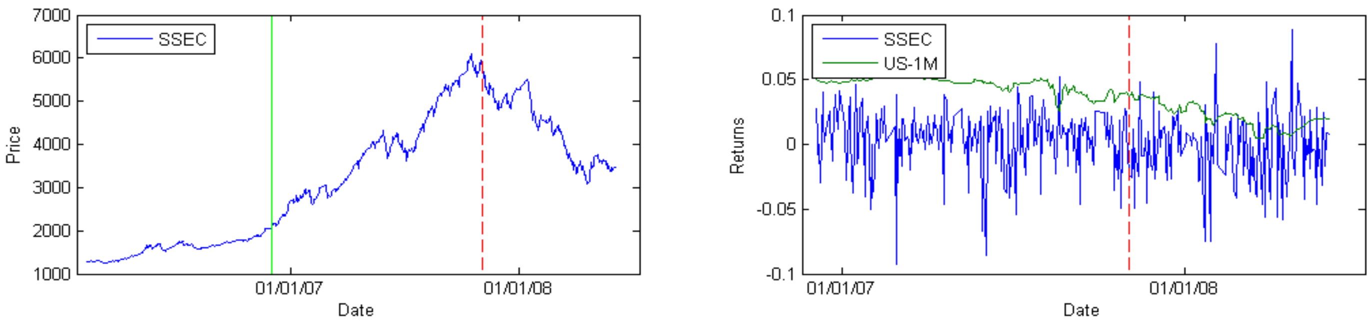
Table 12: **Selected Markets** This table shows the Indices and Bitcoin discussed in section 4. The table presents the start date of the time series, the crash date and investment period (l). Also, the standard deviation (stdev) of the log price and the change in price on the crash date and the 3 days after ($t_c + 0, 1, 2, 3$) are given.

	t_{start}	t_c	l	stdev	t_c	$t_c + 1$	$t_c + 2$	$t_c + 3$
SSEC	01/12/2006	02/11/2007	1.5	0.2564	-2.3076%	-2.4812%	-1.7372%	1.1779%
ES50	02/10/2006	21/01/2008	2	0.1020	-7.3118%	1.3673%	-4.6805%	6.4584%
S&P 500	01/10/2010	04/08/2011	1	0.0534	-4.7820%	-0.0575%	-6.6634%	4.7407%
Bitcoin	02/01/2017	07/07/2017	0.8986	0.6750	-3.1878%	-5.1821%	-2.3318%	2.2790%

The Chinese asset market crash is observed over the period December 1, 2006 to June 2, 2008 using the SSEC. Table 12 show that price change does not exceed 5%. The price does seem to decrease begin and mid 2007. If the price change is larger than 5%, this will trigger investor 2 as well early. About a month after the crash, the price change is certainly large enough to trigger investor 2.

From figure 2, it is observed that the returns are very volatile and consist of large negative outliers before the crash. When a bubble bursts, investing in the asset can lead to excessive losses. The figure shows that the returns are not extreme begin November. They are however large during the two price decreases in January and April 2008. Finally, the figure shows that the yield of the US-1M is decreasing. This is due to the US housing bubble in 2007-2008.

The time series show two small decreases begin and mid 2007. It is possible that both the LPPL model and the GHE approach are triggered by these decreases. As the trading strategy is based on forecasting, it is possible that LPPL traders and GHE traders sell the SSEC index as well soon, missing out on excess returns.



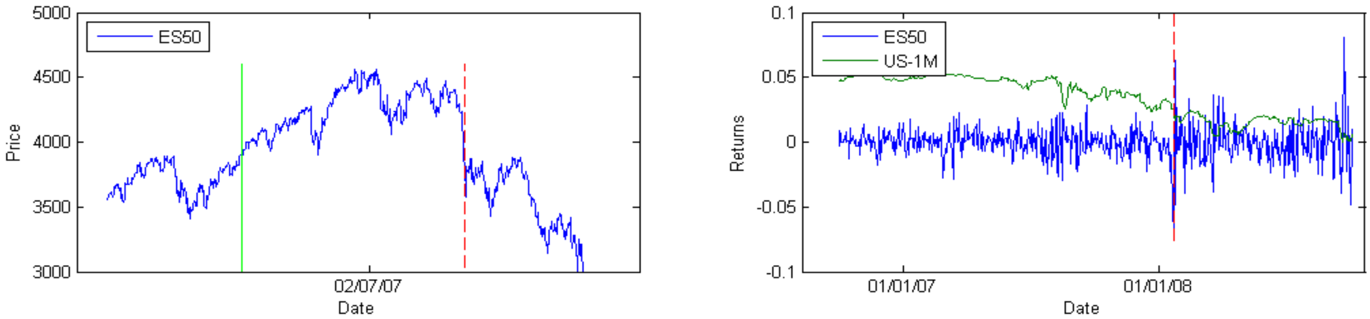
(a) Index price (b) Returns

Figure 2: **Prices and Returns SSEC** The price time series (2a) of the SSEC is given over the period February 9, 2006 to June 2, 2008. The green solid line marks the start date and the red dashed line marks the crash date. The returns (2b) are given for the SSEC index and the US-1M over the period December 1, 2006 to June 2, 2008. The red dashed line marks the crash date.

The ES50 time series is observed during the US Housing bubble. The series are obtained from October 2, 2006 to September 30, 2007. From table 12 it is observed that the change in price is -7.3118% on the crash date. The decrease in price is large enough to trigger investor 2. The time series experiences a large crash and keeps decreasing in the next months.

The returns in figure 3b are overall fluctuating around zero. When the bubble bursts, the returns become more volatile. This is due the increasing risk paired with investing in the ES50. During the US housing bubble, the yield of the US-1M decreases. Investing in the government bond after the crash may not be more profitable compared to the ES50, but it will be less risky.

The price time series in figure 3a shows that there is a smaller crash at the end of 2006. After this small crash the index recovers. It is possible that the estimates of the LPPL model and GHE approach trigger the investors.



(a) Index price

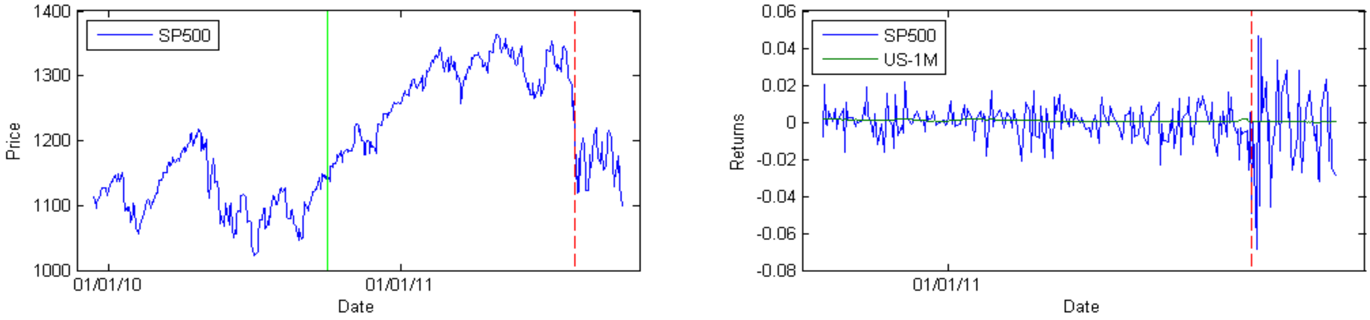
(b) Returns

Figure 3: **Prices and Returns ES50** The price time series (3a) of the ES50 is given over the period December 20, 2005 to September 30, 2008. The green solid line marks the start date and the red dashed line marks the crash date. The returns (3b) are given for the ES50 index and the US-1M over the period October 2, 2006 to September 30, 2008. The red dashed line marks the crash date.

The S&P 500 time series during the Stocks Market Fall ranges from October 1, 2010 to October 3, 2011. On the crash date, the change in price is -4.7820%, decreasing with another 6.6634% again two days later. If investor 2 is not triggered before the crash, he will be two days after the crash.

Figure 4 shows that the returns is volatile at first. Moving towards the crash date, the volatility of the return increases. After the bubble bursts, the returns are extremely volatile. The large fluctuations in the returns after the crash are due to the increase of risk in the asset. In the selected time window, the government bond is very small. Investing in the risk-free asset will not lead to large returns.

Table 12 gives a volatility of 0.0534, which is low relative to the other tree time series. From section 2.1, it showed there is a possibility that LPPL model may fit a trend of the time series, leading to unreliable estimates. Furthermore, the final months of the time series, the growth of the price decreases and the price becomes stable on average. The GHE approach may be triggered before the crash as the traders stop reacting to the large growths in the time series.



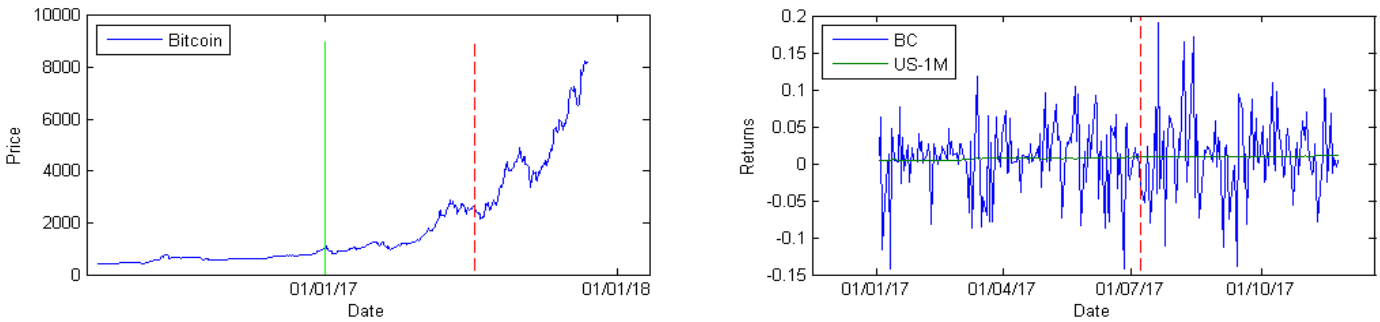
(a) Index price

(b) Returns

Figure 4: **Prices and Returns S&P 500** The log price time series (4a) of the S&P 500 is given over the period December 14, 2009 to October 3, 2011. The green solid line marks the start date and the red dashed line marks the crash date. The returns (4b) are given for the S&P 500 index and the US-1M over the period October 1, 2010 to October 3, 2011. The red dashed line marks the crash date.

The Bitcoin time series ranges from January 1, 2017 to November 24, 2017, which is little under a year. The change in price during and after the crash is low relative to other Bitcoin bubbles (see table 2. The price decreases, but the decrease is less than 10% at one day. Investor 2 will not be triggered by this crash. The Bitcoin price falls again about a month later. This crash however may be large enough to trigger the investor.

Looking at figure 5, it is seen that the price again grows after the crash mid July. It almost seems as if a new bubble emerges. The time series can also be interpreted as if it currently still in a bubble regime. Keeping all investments in the Bitcoin may lead to the largest returns. The returns of the cryptocurrency are very volatile, as shown in figure 5b. The large fluctuations indicate investing in the Bitcoin is risky. The returns on the US-1M are extremely low, fluctuating around 0. Due to these small returns, it is very easy to outperform the risk-free asset when investing in the Bitcoin.



(a) Bitcoin price

(b) Returns

Figure 5: **Prices and Returns Bitcoin** The log price time series (5a) of the Bitcoin is given over the period March 25, 2016 to November 24, 2017. The green solid line marks the start date and the red dashed line marks the crash date. The returns (5b) are given for the Bitcoin index and the US-1M over the period January 2, 2017 to November 24, 2017. The red dashed line marks the crash date.

The LPPL model has showed to be inaccurate when fitting the Bitcoin time series. Therefore, it is not expected that the LPPL traders find a critical time estimate that complies with the restrictions set by the trader. Based on the price time series, the GHE approach either estimates the critical time before the large price increase before the crash, or close to the crash date.

4.2 LPPL model and GHE Approach parameter settings

Section 4 discusses six types of trading strategies. For each strategy, the excess returns are discussed. As the risk-free asset the US-1M government bond is chosen. Additional to the six investors, there is a risk-free investor which only invests in the government bond. The start capital for all investors in €10,000.

Investor 1 never sells the chosen asset and therefore needs no parameter settings. Investor 2 does sell the asset when a possible crash occurs. Based on the price changes during the crash of historical bubbles, the investor sells the asset when price change is (at least) -5% (-10% for the Bitcoin). For the LPPL and GHE investors, each week the models are used to determine if they sell the asset or hold their investments in the asset.

The LPPL investors have to consider multiple aspects. The model has to provide a good fit. The investor wants the model to fit the bubble regime and not a trend or random component. To find a good cut-off value for the parameters m and w, 14 historical bubbles are examined additional to eight bubbles discussed in section 3. For each bubble the parameter estimates up to the crash date are examined. Section E gives the list of bubbles.

From the analyses on historical bubbles, it results that for m there is no difference between the fit before the crash and during the crash. To prevent the estimate of a local minimum, the trader only considers a estimated m with a value between 0.1 and 0.9. The estimated $\widehat{w}$ increases when approaching the critical time. The most accurate estimates of the crash date have a value of w that lies between 6.5 and 9. Therefore, the investor sets boundaries for w around these values. If an estimate coincides with these constraints and improves the SSE value so far obtained, the LPPL investor reacts to the estimated critical time. Investor 3 leaves the market on the crash date, if in the meantime no better estimation is found. Investor 4 determines the LI interval from the modified profile likelihood estimation method. During this interval, the investor proportionally leaves the market and re-allocates this capital to the US-1M.

The GHE investor also uses the historical bubbles given in E. From the analysis, it is obtained that most of the crashes happen after a multifractality around 0.07. Therefore, the GHE investors set a boundary condition of 0.07 or higher on the estimation of ΔH. Furthermore, to make sure the estimate is reliable. The investors require the multifractality to be estimated the same 4 weeks in a row. Investor 5 will sell the asset directly when the GHE approach estimates the same multifractality 6 weeks in a row. From the case studies in section 3.3 and the historical bubbles, the multifractality is on average estimated 30 trading days before the crash date, translating to approximately 6 weeks. After leaving the asset, the capital is reallocated to the US-1M bond. Investor 6 reacts to the same signal, but does not sell the asset all at once. This investor estimates the same multifractality 5 weeks in a row. The final week, the investor leaves the asset exponentially, re-allocating to the risk-free asset.

Additional to applying the trading strategies to the whole time series, the performance of the investors is examined with a minimal investment period in the asset. From the data analyses in section 4.1, small decreases are often observed before the true crash. The LPPL model and GHE approach may be sensitive to these deteriorations. Analyzing the historical bubbles in appendix E, it shows that none of them crash within the first six months. The investors will invest in the risky assets at least six months, and apply their strategy afterwards. This strategy will be referred to as using a 'holding period' of six months.

4.3 Returns

For each of the assets, the end capital is given. Also, the CER, μ_{ER}, the Sharpe, the Sortino and the VaR_{95} and VaR_{99} are discussed per asset, per investor. For the LPPL traders (investors 3 and 4) and the GHE traders (investors 5 and 6), the estimated critical time, SSE, the estimates of the parameters m and w and maximum multifractality are provided. The results are provided for the trading strategy without and with a holding period of 6 months (H0 and H6 respectively).

The returns are discussed in different sections for each of the assets. Section 4.3.1 gives the results of the SSEC index, section 4.3.2 follows with the results of the ES50. The results of the S&P 500 are discussed in section 4.3.3 and section 4.3.4 shows the results for applying the trading strategies to the Bitcoin. For all of the results, figures showing the capital changes are given in appendix F.

4.3.1 Returns SSEC

For investor 1, the CER and μ_{ER} are negative. The other five investors all have positive CER. This implies that when the market is assumed to be in a bubble, it is beneficial to adjust the trading strategy to a possible bubble burst. This holds for both traders without and with a holding period.

Table 13: **Model estimates SSEC** This table shows the estimated crash dates and the parameters of investor 2 (I2), the LPPL and GHE traders (Investors 3 to 6). The table presents the estimated crash date ($\widehat{t_c}$). For investor 2, the triggered price change (pc) is given. For the LPPL traders, the LI, the SSE, $\widehat{m}$ and $\widehat{w}$ are given and from the GHE estimation the ΔH value is provided. The results are given for with and without a holding period (H0 and H6 respectively).

	I2		LPPL					GHE	
	$\widehat{t_c}$	pc	$\widehat{t_c}$	LI	SSE	$\widehat{m}$	$\widehat{w}$	$\widehat{t_c}$	ΔH
H0	28/02/2007	-8.8406%	11/06/2007	[-4,0]	0.0898	0.7532	6.6718	22/01/2007	0.0822
H6	05/06/2007	-8.2570%	16/07/2007	[-21,0]	0.1388	0.7585	7.4686	22/10/2007	0.1259

Table 13 shows the estimated critical time of investor 2, the LPPL traders and the GHE traders. The true crash date of the Chinese stock bubble is November 1, 2007. The poorest estimation is for the GHE approach, estimating the crash approximately 9 months in advance.

As expected, the GHE approach is triggered by the small decreases at the beginning of 2007. Investor 2 is also triggered by these price changes. The LPPL model performs a little bit better than the GHE approach. However, the price drop mid 2007 provide estimates that comply with the boundary conditions set by the Investor. The trader leaves the market at the beginning of June 2007. The LI covers a small period of 5 days around the OLS estimate.

As can be observed in table 14, for the H0 traders, the highest CER are obtained when using the OLS estimate from the LPPL model. Using the strategy of Investor 5 results in the second largest CER. However, as the investor leaves the market as well soon, the CER are not from riding the bubble. The Sharpe ratio of investor 3 is largest, indicating that the trading strategy of investor 3 has the largest average ER per unit of extra volatility. The larger Sortino ratio from investor 5 is due to the small amount of negative returns at the beginning of the time series, resulting in a relative small downside deviation.

Dividing the VaR by the end capital gives the fraction capital at risk. Comparing the ratio of investor 5 to investor 3, who sells the risky asset after only 3 months, the difference is relatively small. With a 95% (99%) probability, the investor may maximum lose an additional 3% (4%) of the total capital in one day. Investing in the SSEC an additional 5 months does not lead to a large extra risk. As the US-1M is affected by the US housing bubble during the investment period, the VaR is larger than expected from a risk-free asset.

Comparing the CER, Sharpe and Sortino ratio and VaR over the investors without a holding period, the LPPL trader acting to the OLS estimate performs best. However, the model estimates the crash date to far in advance and therefore the strategy is not optimal.

Table 14: **Returns SSEC** This table shows the results from the trading strategies without and with a 6 month hold period (H0 and H6 respectively). For each investor, the end capital is given together with the Cumulative Excess Returns (CER), the average Excess Returns (μ_{ER}), the Sharpe Ratio (Sharpe), the Sortino ratio (Sortino) and the 95% and 99% Value at Risk (VaR_{95} and VaR_{99}).

		End capital	CER	μ_{ER}	Sharpe	Sortino	VaR_{95}	VaR_{99}
H0	Investor 1	14778.60	-4236.97	-0.0004	-0.2661	-0.3477	-7149.29	-10111.38
	Investor 2	21494.12	2478.55	0.0004	0.3813	0.2880	-5241.54	-7413.22
	Investor 3	25450.87	6435.30	0.0009	0.7881	0.6069	-7251.82	-10256.39
	Investor 4	21752.51	2736.94	0.0004	0.4210	0.3001	-5279.48	-7466.87
	Investor 5	23139.59	4124.02	0.0006	0.6388	0.6147	-5911.77	-8361.13
	Investor 6	19395.22	379.65	0.0001	0.0699	0.0546	-3790.7	-5361.26
H1	Investor 1	14778.60	-4236.97	-0.0004	-0.2661	-0.3477	-7149.29	-10111.38
	Investor 2	24196.14	5180.57	0.0007	0.6655	0.5059	-6433.91	-9099.61
	Investor 3	23528.96	4513.39	0.0007	0.5859	0.4946	-6059.49	-8570.05
	Investor 4	20492.35	1476.78	0.0003	0.2559	0.1985	-4755.65	-6726.01
	Investor 5	30553.91	11538.34	0.0014	1.1623	1.0906	-10733.73	-15180.93
	Investor 6	28430.91	9415.34	0.0012	1.0007	0.9500	-9602.10	-13580.43

Incorporating a holding period of 6 months to the trading strategies changes the critical time estimates of all investors. The estimate of the GHE traders show a large improvement, estimating the crash only 2 weeks in advance. Table 13 shows that the multifractality increased compared to the H0 strategy, indicating the estimate is better. The LPPL model still estimates the critical time during the decrease mid 2007. Investor 2 also leaves the risky asset during this decrease.

Similar to the H0 strategy, investor 1 performs worse compared to the risk-free asset investor. The other investors are able to obtain positive CER. At the end of the investment period, the GHE traders show to perform the other investors. Both traders ride along with the bubble and sell the asset during the peak period. This results in the largest CER. Both the Sharpe ratio and the Sortino ratio are approximately 1 for the traders, which is commonly known to be a good ratio.

The fraction VaR per end capital are most in favor of investor 4. With a 95% (99%) probability, investor 5 loses maximum 35% (49%) of the total capital in one day. This is only 11% (16%) more compared to investor 4, who obtains a severe less CER. Investor 5 obtains larger CER compared to investor 4 while having slightly more risk.

Comparing the CER, Sharpe and Sortino ratio and VaR over the investors with a holding period of 6 months, the GHE trader leaving the market at once performs best. The critical time estimation is close to the true crash date. The investor gains the CER during the bubble, but leaves in time to avoid the losses. Incorporating a holding period improves the estimate for investor 2, but does not optimize the strategy observing the Chinese Stock market crash.

4.3.2 Returns ES50

For all investors, the CER is negative. This indicates that during the investment period, the yield of the ES50 is not high compared to the US-1M bond. Comparing the results for the different investors, the investors gain more when adjusting the trading strategy to a possible bubble burst. This holds for both traders without and with a holding period.

Table 15: **Model estimates ES50** This table shows the estimated crash dates and the parameters of investor 2 (I2), the LPPL and GHE traders (Investors 3 to 6). The table presents the estimated crash date ($\widehat{t_c}$). For investor 2, the triggered price change (pc) is given. For the LPPL traders, the LI, the SSE, $\widehat{m}$ and $\widehat{w}$ are given and from the GHE estimation the ΔH value is provided. The results are given for with and without a holding period (H0 and H6 respectively).

	I2		LPPL					GHE	
	$\widehat{t_c}$	pc	$\widehat{t_c}$	LI	SSE	$\widehat{m}$	$\widehat{w}$	$\widehat{t_c}$	ΔH
H0	22/01/2008	-7.3118%	28/11/2006	[0,3]	0.0614	0.7320	7.8108	19/05/2008	0.0733
H6	22/01/2008	-7.3118%	06/08/2007	[-14,0]	0.0340	0.6967	7.1014	19/05/2008	0.0733

Investor 2 estimates the critical time best, when not incorporating a holding period. Table 15 shows that the investor leaves the risky asset only one day after the true crash on January 21, 2008. The investor loses money during the crash, but prevents the negative returns the months thereafter. The GHE traders estimate the critical time far after the true crash date. Contrary to investor 2, investor 5 and 6 experience more negative returns. The GHE traders perform better compared to investor 1, as the asset is sold just before the large decrease at the end of the trading period. The multifractality estimation is 0.0733, which is relatively small. The critical time estimation of LPPL traders is very poor. Table 15 shows that the LPPL model estimates the critical time on November 28, 2006, which is only two months after the start of the investment period. The model is triggered by a small decrease in price.

Table 16 shows that for the H0 traders, the LPPL investors show the most favorable CER. However, this is due to the traders leaving the market more than a year before the bubble burst. As investing in the risk-free asset results in a larger yield than investing in the ES50, the CER improves when leaving the market sooner rather than later. Comparing investor 2 with the GHE traders, investor 2 has a smaller CER. This translates in it being more profitable to leave the market during the crash. The Sharpe ratio and Sortino ratio are negative for all investors as the average ER are negative.

Investor 1 never sells the risky asset, while the LPPL traders leave the market almost immediately. Comparing the percentage VaR to end capital for investor 1 and investor 3, the additional loss in one day is -5% and -7% respectively. This indicates that only investing in the risky asset reduces the risk compared to investing in the government bond. This is due to the government bond also being hit by the US housing bubble Investor 2 shows only 9% and 13% of the end capital being at risk based on the VaR_{95} and VaR_{99} respectively.

Not taking into account the overall low yield from investing in the ES50, investor 2 performs best in estimating the bubble crash without taking into account a holding period. The investment strategy of investor 2 results in a combination of low risk with favorable CER. The LPPL model estimates the crash very poor, being triggered by a small decrease at the beginning of the time series and the GHE approach in not triggered until after the crash.

Table 16: **Returns ES50** This table shows the results from the trading strategies without and with a 6 month hold period (H0 and H6 respectively). For each investor, the end capital is given together with the Cumulative Excess Returns (CER), the average Excess Returns (μ_{ER}), the Sharpe Ratio (Sharpe), the Sortino ratio (Sortino) and the 95% and 99% Value at Risk (VaR_{95} and VaR_{99}).

		End capital	CER	μ_{ER}	Sharpe	Sortino	VaR_{95}	VaR_{99}
H0	Investor 1	7472.86	-14961.57	-0.0021	-2.1527	-2.5326	-1787.21	-2527.69
	Investor 2	10558.40	-11876.03	-0.0015	-1.9408	-1.6311	-998.01	-1411.51
	Investor 3	21230.60	-1203.83	-0.0001	-0.1528	-0.0870	-6081.04	-8600.53
	Investor 4	15014.48	-7419.95	-0.0008	-1.4579	-0.6421	-2613.38	-3696.16
	Investor 5	10291.66	-12142.77	-0.0015	-1.7395	-1.7332	-1142.04	-1615.21
	Investor 6	9910.97	-12523.46	-0.0016	-1.8300	-1.8278	-1209.45	-1710.54
H6	Investor 1	7472.86	-14961.57	-0.0021	-2.1527	-2.5326	-1787.21	-2527.69
	Investor 2	10558.40	-11876.03	-0.0015	-1.9408	-1.6311	-998.01	-1411.51
	Investor 3	15096.88	-7337.55	-0.0008	-1.1277	-0.8353	-2591.98	-3665.89
	Investor 4	13036.33	-9398.10	-0.0011	-1.7475	-1.1543	-1495.11	-2114.56
	Investor 5	10291.66	-12142.77	-0.0015	-1.7395	-1.7332	-1142.04	-1615.21
	Investor 6	9910.97	-12523.46	-0.0016	-1.8300	-1.8278	-1209.45	-1710.54

Adding a holding period of 6 months to the trading strategies only affects only the LPPL estimation. From table 15 it can be observed that the estimation show an improvement as the

SSE is reduced from 0.0614 to 0.0340. The start of the LI is estimated at the day the ES50 price peaked. The estimation is not during the true crash, but during the decrease in price 3 month before. The price showed a little increase after the decrease, but crashed again the end of January. The estimations of investor 2 and the GHE traders are not affected by incorporating a holding period.

Only the estimations of the LPPL traders changed, indicating the other investment results are similar to the H0 strategies. The CER of investor 3 and 4 have decreased compared to the H0 strategy, as they leave the ES50 asset almost a year later. Compared to investor 2, the LPPL traders obtain higher CER. The LPPL traders sell the asset a little bit early, but are not hit by the large price decrease during the crash, unlike investor 2. The Sharpe ratio for investor 3 is less negative compared to investor 2. This also holds for the Sortino ratio. In other words, the trading strategy of investor 3 has the largest average ER per unit of extra volatility as well as per unit of bad risk.

Similar to the H0 traders, the ratio is smallest for investor 2. The percentage of the end capital at risk for the LPPL investors decreased after using the holding period. With a 95% (99%) probability, investor 3 loses maximum 17% (24%) of the total capital in one day. This is only 7% (11%) more compared to investor 2, while investor 3 obtains a much larger CER.

Comparing the results of the investors with a holding period of 6 months, the LPPL trader leaving the market at once performs best. The critical time estimation improves by using a holding period, but could be slightly improved. The investor gains the CER during the bubble, but leaves in time to avoid the losses during and after the crash. Also, the trade off between risk and CER is favorable for investor 3.

4.3.3 Returns S&P 500

The CER and μ_{ER} are negative for investor 1 and 2. Incorporating the LPPL model or GHE approach to the Stocks Market Fall bubble results to be difficult, as the models do not always detect a crash. When they do return a critical time estimate, the average ER is positive. This indicates that applying a trading strategy to a possible bubble burst can be profitable for both traders without and with a holding period.

Table 17: **Model estimates S&P 500** This table shows the estimated crash dates and the parameters of investor 2 (I2), the LPPL and GHE traders (Investors 3 to 6). The table presents the estimated crash date ($\widehat{t}_c$). For investor 2, the triggered price change (pc) is given. For the LPPL traders, the LI, the SSE, $\widehat{m}$ and $\widehat{w}$ are given and from the GHE estimation the ΔH value is provided. The results are given for with and without a holding period (H0 and H6 respectively).

	I2		LPPL					GHE	
	$\widehat{t}_c$	pc	$\widehat{t}_c$	LI	SSE	$\widehat{m}$	$\widehat{w}$	$\widehat{t}_c$	ΔH
H0	09/08/2011	-6.6634%	07/02/2011	[-13,15]	0.0475	0.7514	7.8347	-	-
H6	09/08/2011	-6.6634%	-	[-]	-	-	-	01/08/2011	0.0776

The critical time estimates for investor 2, the LPPL traders and the GHE traders are given in table 17. The LPPL model estimates the critical time 4 months in advance, when not incorporating a holding period. Just after the estimated critical time, the S&P 500 price deteriorates. After this decrease, the price does not show the super-exponential price growth pattern a bubble is known for, but stays volatile around a certain mean. The LI is very broad, consisting 28 days in total. Investor 4 therefore leaves the market gradually over a longer period. After the price volatility, the price crashes with -6.6634%, triggering investor 2. The GHE approach is not able to estimate a proper critical time. The fluctuations after the bubble growth may contribute to the GHE approach not being able to provide stable results 6 weeks in a row. If the ΔH value before or during these fluctuations is not exceeded, the investor will not react to any signals resulting in not leaving the risky asset.

Table 18 shows that the LPPL traders are the only investors with a positive CER. There is only a small difference between the LPPL trader using the OLS estimate and the LPPL trader using the LI, where the first trader performs slightly better. The LPPL traders ride the main part of the bubble, but leave when the exponential growth stops. Investor 2 leaves the market just after the crash, which shows not to be beneficial. As the average ER are negative for investors 1, 2, 5 and 6, the Sharpe and Sortino ratios are as well. The Sharpe and Sortino ratios of the LPPL traders are approximately 1.9 and 1.7 respectively, which is commonly known to be acceptable to good ratios.

The ratio VaR to end capital, is low for the LPPL traders. Investor 3 loses maximum 6% and 9% of the total capital in one day with a 95% and 99% probability respectively. The percentage VaR of the other investors are very comparable, being approximately 11% and 15%. Investor 2 shows a slightly larger ratio, which is due to the large losses during the crash and the switch to the risk-free asset afterwards. Using the LPPL model provides larger CER against less risk. The yield of the risk-free asset is almost 0, and therefore staying in the S&P 500 may result is larger returns after the crash.

Overall, the LPPL traders perform best. Especially the Sharpe and Sortino ratio show that the investment strategy based on the LPPL model estimates provides a higher yield. Furthermore, using the LPPL model provides larger CER against less risk compared to the other investors.

Incorporating a holding time of 6 months to the trading strategies changes the critical time estimates of the LPPL and GHE traders. Where at first the GHE traders did not provide a critical time estimation, they now do. The crash is estimated only 3 days before the true crash date. The value of ΔH is rather low, being 0.0776. The LPPL model does not provide a critical time estimate. The holding period exceeds the critical time estimate of the H0 strategy. After this date, the model is not able to fit the time series while providing parameter estimates within the set boundaries. As stated before, the last few months of the crash, the price growth is not exponentially. The LPPL model is based on the assumption of the price growing super-exponentially. The lack of this growth results in no critical time estimate.

Table 18: **Returns S&P 500** This table shows the results from the trading strategies without and with a 6 month hold period (H0 and H6 respectively). For each investor, the end capital is given together with the Cumulative Excess Returns (CER), the average Excess Returns (μ_{ER}), the Sharpe Ratio (Sharpe), the Sortino ratio (Sortino) and the 95% and 99% Value at Risk (VaR_{95} and VaR_{99}).

		End capital	CER	μ_{ER}	Sharpe	Sortino	VaR_{95}	VaR_{99}
H0	Investor 1	9393.43	-695.51	-0.0002	-0.2509	-0.2815	-987.60	-1396.79
	Investor 2	9662.34	-426.60	-0.0001	-0.2211	-0.2084	-1135.99	-1606.65
	Investor 3	11442.12	1353.18	0.0005	1.8879	1.6134	-698.05	-987.26
	Investor 4	11416.03	1327.09	0.0005	1.9240	1.7527	-683.77	-967.07
	Investor 5	9393.43	-695.51	-0.0002	-0.2509	-0.2815	-987.60	-1396.79
	Investor 6	9393.43	-695.51	-0.0002	-0.2509	-0.2815	-987.60	-1396.79
H6	Investor 1	9393.43	-695.51	-0.0002	-0.2509	-0.2815	-987.60	-1396.79
	Investor 2	9662.34	-426.60	-0.0001	-0.2211	-0.2084	-1135.99	-1606.65
	Investor 3	9393.43	-695.51	-0.0002	-0.2509	-0.2815	-987.60	-1396.79
	Investor 4	9393.43	-695.51	-0.0002	-0.2509	-0.2815	-987.60	-1396.79
	Investor 5	11215.13	1126.19	0.0004	0.9719	1.2129	-718.04	-1015.54
	Investor 6	10583.15	494.21	0.0002	0.4515	0.5315	-790.15	-1117.52

The most interesting comparison is the LPPL trader based on the H0 strategy to the GHE trader with a H6 strategy. This provides the contribution in leaving the risky asset before the crash instead of after the exponential growth. The CER of the LPPL investors are larger compared to the GHE traders. During the last 4 months before the crash, the CER decrease with €226.99. The yield in this period is not in favor of the investors. This also shows in the Sharpe ratio, which is only 0.9719 for investor 5.

The difference in percentage of the end capital at risk is very small. Investor 5 loses maximum 6% (9%) of the total capital in one day with a 95% (99%) probability. These losses are about 0.3% larger compared to the LPPL traders with a H0 strategy. The LPPL traders show a less risky investment but obtain larger CER. However, differences are small enough to neglect.

Based on the CER, the Sharpe and Sortino ratios and the VaR, investors with a holding period of 6 months, the GHE trader leaving the market at once performs best. The critical time estimate is very accurate, being only 3 days before the true crash date. The investor gains the CER during the bubble, but leaves in time to avoid the losses. However, the investment of the GHE trader does not outperform the LPPL traders investment strategy without a holding period.

4.3.4 Returns Bitcoin

Investing in the Bitcoin during the full trading period provides the largest yield. This implies that for the Bitcoin, during this time period, it is not beneficial to adjust the trading strategy to a possible bubble burst. This holds for both traders without and with a holding period.

The critical time estimates of the GHE trader, LPPL trader and investor 2 are given in table 19. The crash date of the Bitcoin is July 7, 2017. None of the models using a H0 strategy are able to detect this crash. Investor 2 is triggered by a price decrease of 11.0291% only 4 days after starting the investment. The LPPL traders estimate the critical time at the beginning of February, which is also as well far ahead of the critical time. In section 3.2, it already showed that the LPPL model overall is not able to capture the true crash. The GHE approach is not able to estimate a maximum multifractality stable over 6 weeks. Therefore, they do not sell the risky asset.

Table 19: **Model estimates Bitcoin** This table shows the estimated crash dates and the parameters of investor 2 (I2), the LPPL and GHE traders (Investors 3 to 6). The table presents the estimated crash date ($\widehat{t}_c$). For investor 2, the triggered price change (pc) is given. For the LPPL traders, the LI, the SSE, $\widehat{m}$ and $\widehat{w}$ are given and from the GHE estimation the ΔH value is provided. The results are given for with and without a holding period (H0 and H6 respectively).

	I2		LPPL					GHE	
	$\widehat{t}_c$	pc	$\widehat{t}_c$	LI	SSE	$\widehat{m}$	$\widehat{w}$	$\widehat{t}_c$	ΔH
H0	06/01/2017	-11.0291%	07/02/2017	[0,3]	0.4779	0.4493	7.1279	-	-
H6	27/07/2017	-10.5607%	16/10/2017	[-7,0]	0.7441	0.4418	7.5540	25/09/2017	0.0909

For the H0 traders, table 20 shows that the highest CER are obtained from investing only in the Bitcoin. This is mainly due to the large price growth after the small crash in July. The Sharpe and Sortino ratios are very high for investor 1, being 2.7896 and 3.7736 respectively. These ratios are rated as very good to excellent. The LPPL trader based on the OLS estimate has a very small, but positive average ER. Therefore, the Sharpe ratio is positive as well, but very small.

The results show that investing in Bitcoin is quite risky. For investor 1, the percentage of end capital at risk is 37% and 53% respectively for the VaR_{95} and VaR_{99}. The LPPL traders and investor 2 show a very small ratio, all being less than 10%. As the traders leave the cryptocurrency market at the beginning of the investment period, they mostly invest in the risk-free asset. The additional CER that investor 1 gains compared to the low-risk traders outweighs the extra risk.

Investors get a higher yield only investing in the Bitcoin. The large price increases after the crash date are the main contribution to this result. Therefore, despite the Bitcoin being in a bubble regime, critical time estimations do not improve the trading strategies.

Incorporating a holding period of 6 months changes all estimations. However, the investors all leave the market rather soon after they are able to apply their trading strategies. Investor 2 leaves at the end of July, which is a little after the burst of the bubble. The LPPL traders estimate the crash date mid October, with a large SSE. Slightly after the critical time estimation, there is indeed a small decrease. However, the Bitcoin starts growing again. The LPPL model does not estimate the crash of interest, which in July 7, 2017. Using a holding period makes it possible for the GHE approach to estimate a critical time. The estimate lies close to the LPPL estimation. The value of ΔH is 0.0909, which is overall a good estimate.

Similar to the traders not using a holding period, investor 1 has the largest CER. The difference with the LPPL and GHE traders is small, as they leave the market around October, 2017 and the investment period ends November 24, 2017. In the last month, the CER of investor

Table 20: **Returns Bitcoin** This table shows the results from the trading strategies without and with a 6 month hold period (H0 and H6 respectively). For each investor, the end capital is given together with the Cumulative Excess Returns (CER), the average Excess Returns (μ_{ER}), the Sharpe Ratio (Sharpe), the Sortino ratio (Sortino) and the 95% and 99% Value at Risk (VaR_{95} and VaR_{99}).

		End capital	CER	μ_{ER}	Sharpe	Sortino	VaR_{95}	VaR_{99}
H0	Investor 1	59912.76	49121.65	0.0086	2.7896	3.7736	-22459.66	-31765.12
	Investor 2	10262.66	-528.45	-0.0002	-0.3118	-0.0855	-387.99	-548.74
	Investor 3	10548.97	-242.14	0.0000	0.0368	0.0139	-795.22	-1124.70
	Investor 4	10227.71	-563.40	-0.0001	-0.0953	-0.0373	-694.64	-982.44
	Investor 5	59912.76	49121.65	0.0086	2.7896	3.7736	-22459.66	-31765.12
	Investor 6	59912.76	49121.65	0.0086	2.7896	3.7736	-22459.66	-31765.12
H6	Investor 1	59912.76	49121.65	0.0086	2.7896	3.7736	-22459.66	-31765.12
	Investor 2	20970.29	10179.18	0.0036	1.4998	1.5320	-8622.86	-12195.47
	Investor 3	42452.86	31661.75	0.0070	2.3715	2.9698	-19035.29	-26921.97
	Investor 4	36295.91	25504.80	0.0063	2.1662	2.6636	-16857.82	-23842.33
	Investor 5	27860.90	17069.79	0.0052	1.7931	2.1540	-14106.22	-19950.69
	Investor 6	31571.38	20780.27	0.0057	1.9693	2.3881	-15316.96	-21663.07

1 increases with another €17459,90. The Sharpe ratio increases when the investor invests in the Bitcoin over a longer period. Investor 2, who sells the risky asset first, still has a Sharpe ratio of 1.4998, which is a good ratio. The average ER per unit per bad risk is also high for all of the investors. In particular for the LPPL traders and investor 1, the Sortino ratio is considered excellent.

As all investors mainly invest in the cryptocurrency, the ratios increased severely. With a probability of 95% (99%), investor 3 loses maximum 45% (63%) of their capital in 1 day. The other traders leaving the market have ratios even larger, ranging up to 72%. The ratios of investor 1 are smaller, as the investor obtains a very high yield in the final period of the investment. The investor therefore does not only obtain a larger CER, but also reduces the risk by keeping the investments in the Bitcoin time series.

Similar to the H0 strategy, investors with a holding period obtain the highest yield when keeping their investments in the Bitcoin. None of the models are able to provide a accurate estimate of the crash in July, but the LPPL model and GHE approach do capture a small crash at the beginning of October.

4.3.5 Overall performance

For the indices, it is overall profitable to adjust the trading strategy to a possible crash when the asset is assumed to be in a bubble. There are strategies that return a good CER and Sharpe ratio, but the strategies are not (always) optimal.

When not applying a holding period, there is a risk that the traders estimate a crash to a smaller price decrease earlier in the time series. Incorporating a holding period into the strategies overall provides better estimates for investor 2, the LPPL trader and the GHE traders. Comparing the performance of the investors over the indices, the LPPL trader estimating the critical time with OLS estimation overall performs best. When also using a holding period, the GHE approach can be profitable as well.

For the Bitcoin, none of the trading strategies are able to detect the crash of interest. The investors leave the market either as well soon or estimate the crash during the next price decrease. The poor CER results are mainly due to the large price increase after the crash. Looking to the time series, it can be questionable if the Bitcoin has experienced several bubbles in 2017. Overall the cryptocurrency seems to still be in a bubble showing an exponential growth pattern.

5 Conclusion

This research contributes to existing literature by comparing accuracy of the LPPL model and the GHE approach when estimating the crash date of a bubble. Furthermore, the models are used in different active trading strategy for financial bubbles. Finally, the models and trading strategies are applied to a the Bitcoin cryptocurrency.

The LPPL model is a seven parameter model, with one being the crash date of a financial bubble. The model is based on a hypothesis of collective heading behavior within rational expectations and assumes a faster-than-exponential growth in the asset price during a bubble. A case studies on historical bubbles using indices shows that the model is able to estimate the critical time close to the true crash date when choosing an appropriate time window. Limitations of the model are the sensitivity to the number of observations in the time window and to different growth trends during the time series. Also, when calibrating the model, a local minimum is often found rather than a global minimum. Considering the Bitcoin time series, the model fails to estimate an accurate crash date. The model is not able to fit the large (daily) volatility of the Bitcoin.

The GHE approach provides a test method for long-range correlation in time series. The multifractality, which is the difference between the scaling behavior of the absolute values and the scaling of the autocorrelation, signals a change in investment strategy. This is likely to result in a crash of the bubble. A case studies on historical bubbles using indices shows that, using an appropriate time window, the model is able to estimate the critical time close to the true crash date. The model is however sensitive to the number of observations in the time-sample. Contrary to the LPPL model, the GHE approach is able to estimate the crash date of the Bitcoin time series.

The market is inefficient when a bubble is detected. When the LPPL model and the GHE approach estimate the crash date, an investor can ride the bubble until the estimate, and leave the risky asset prior to the crash. This can lead to excessive returns. 6 different trading strategies are compared. Strategy 1 never leaves the market, strategy 2 leaves the market after a price crash of 5%. Strategies 3 and 4 are based on the OLS estimation and Likelihood interval respectively of the LPPL model. The LPPL strategies incorporate boundary conditions on the non-linear parameters to make sure the model estimates a crash after a bubble time series. The GHE strategies are triggered when the multifractality is larger than a certain treshold and this estimate is stable several weeks in a row. Strategy 5 leaves the market at based once, while strategy 6 uses a exponential-smoothing approach. For the indies, it is overall profitable to adjust the trading strategy to a possible crash when the asset is assumed to be in a bubble. The LPPL model using the OLS estimated critical time provides the best combination of Cumulatives Excess Returns, Sharpe ratio and Value at Risk. When incorporating a minimal investment period in the risky asset of 6 months to the strategies, the GHE approach can be profitable as well. In case of the Bitcoin, the LPPL model and GHE approach are not profitable compared to keep investing in the risky asset.

When the models are used in a trading strategy, they are dependent on the parameter boundaries chosen by the investor. More research can be done on the optimal performance of the models when used in a trading strategy. For example, changing the boundaries of the parameters or adding additional constraints to the strategies. Especially for the cryptocurrencies, the boundary conditions of the LPPL parameters can be looked into. The bubbles in the Bitcoin prices show to increase faster during a smaller time interval. Increasing the boundaries of the w parameter and reducing the number of observations may improve the crash estimates. Extensive research can be done in this area. Also interesting further research is combining models that estimate the size of the crash with the models that estimate the crash date. This may prevent the investors leaving the bubble before a small price deterioration rather than a large crash.

The use of the models in investment strategies has a downside. When a high percentage of traders in the market use these models, and act upon these estimations, the market will move along with the behavior. This will provide different crash dates compared with a market without investors using the models.

References

Bali, T. (2007). A generalized extreme value approach to financial risk measurement. *Journal of Money, Credit and Banking*, *39*, 1613-1649.

Barabási, A. L., & Vicsek, T. (1991). Multifractality of self-affine fractals. *Physical Review A*, *44*(4), 2730-2733.

Blancard, O., & Watson, M. (1982). Bubbles, rational expectations, and financial markets. In P. Wacher (Ed.), *Crises in the economic and financial structure* (p. 295-315). Lexington MA: Lexington Books.

Carhart, M. M. (1997). On persistence in mutual fund performance. *Journal of Finance*, *52*(1), 57-82.

Choudhry, C. (1996). Stock market volatility and the crash of 1897: evidence from six emerging markets. *International Money & Finance*, *15*, 969-981.

DeMiguel, V., Garlappi, L., & Uppal, R. (2009). Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *The Review of Financial Studies*, *22*(5), 1915-1953.

Di Matteo, T. (2007). Multi-scaling in finance. *Quantitative Finance*, *7*(1), 21-36.

Eriksen, O., & Barndorff-Nielsen. (1983). On a formula for the distribution of the maximum likelihood estimator. *Biometrika*, *70*(2), 343-365.

Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work. *Journal of Finance*, *25*(2), 383-417.

Fama, E. F., & French, K. R. (1992). The cross-section of expected stock returns. *Journal of Finance*, *47*(2), 427-465.

Feigenbaum, J. (2001). A statistical analysis of log-periodic precursor to financial crashes. *Quantitative Finance*, *1*, 333-383.

Filimonov, V., & Sornette, D. (2013). A stable and robust calibration scheme of the log-periodic power law model. *Physica A: Statistical Mechanics and its Applications*, *392*(17), 3698-3707.

Filmonov, V., Demos, G., & Sornette, D. (2016). Modified profile likelihood inference and interval forecast of the burst of financial bubbles. *Swiss Finance Institure research paper series*, *11*(16).

Fisher, R. A. (1956). Statistical methods and scientific inference. *Oliver & Boyd*.

Friedman, D., & Abraham, R. (2007). *Bubbles and crashes: Escape dynamics in financial markets* (Working Papers Santa Cruz Center for International Economics No. 07-03). Retrieved from `http://hdl.handle.net/10419/64112`

Garcia-Zarate, J. (2017, February). *Why passive funds are growing in popularity.* Retrieved from `http://www.morningstar.co.uk/uk/news/156449/why-passive-funds-are-growing-in-popularity.aspx`

Grech, D., & Mazur, Z. (2005). Statistical properties of old and new techniques in detrended analysis of time series. *Acta Physica Polonica B*, *33*(8), 2403-2413.

Helms, K. (2017, jan). A look at bitcoin bubbles, when will the next one be? Retrieved from `https://news.bitcoin.com/look-bitcoin-bubble-will-next-one/`

Huang, Y., Johansen, A., Lee, M., Saleur, H., & Sornette, D. (2000). Artifactual log-periodicity in finite size data: Relevance for earthquake aftershocks. *Journal of Geophysical Research-Solid Earth*, *13*, 4417-4428.

Hurst, H. E. (1951). Long-term stroage capacity in reservoirs. *Transactions of the American Society of Civil Engineers*, *55*, 400-410.

Investopia. (2017). Bubble. Retrieved from `http://www.investopedia.com/terms/b/bubble.asp`

Johansen, A., Ledoit, O., & Sornette, D. (2000). Crashes as critical points. *International Journal of Theoretical and Applied Finance*, *3*(1), 219-255.

Johansen, A., & Sornette, D. (1999). Critical crashes. *Risk*, *12*(1), 91-94.

Johansen, A., & Sornette, D. (2001). Large stock market price drawdowns are outliers. *Journal of Risk*, *4*(2), 69-110.

Johansen, A., & Sornette, D. (2010). Shocks, crashes and bubbles in financial markets. *Brussels Economic Review*, *53*(2), 201-253.

Johansen, A., Sornette, D., & Ledoit, O. (1999). Predicting financial crashes using discrete scale invariance. *Risk*, *1*(5), 5-32.

Kaizoji, T. (2006). A precursor of market crashes: Empirical laws of japans internet bubble. *European Physical B*, *50*, 123-127.

Kaizoji, T. (2010). *Stock volatility in the periods of booms and stagnations* (EERI Research Paper Series No. EERI_RP_2010_07). Economics and Econometrics Research Institute (EERI), Brussels. Retrieved from `https://ideas.repec.org/p/eei/rpaper/eeri_rp_2010_07.html`

Kaizoji, T., & Kaizoji, M. (2004). Power law for ensembles of stock prices. *Physica A: Statistical Mechanics and its Applications*, *344*, 240-243.

Knufken, D. (2008, oct). 10 countries least affected by the us financial crisis. Retrieved from http://www.businesspundit.com/10-countries-least-affected-by-the-us-financial-crisis/

Laloux, L., Potters, M., & Cont, R. (1999). Are financial crashes predicable? *Europhysics Letters*, *45*, 1-5.

Levy, M. (2008). Stock market crashes as social phase transitions. *Economic Dynamics & Control*, *32*, 137-155.

Lillo, F., & Mantegna, R. (2004). Dynamics of a financial market index after a crash. *Physica A: Statistical Mechanics and its Applications*, *338*, 125-134.

Lin, L., Ren, R. E., & Sornette, D. (2014). A consistent model of 'explosive' financial bubbles with mean-reversing residuals. *International Review of Financial Analysis*, *33*, 210-225.

Lin, L., & Sornette, D. (2013). Diagnostics of rational expectation financial bubbles with stochastic mean-reverting termination times. *The European Journal of Finance*, *19*(5), 344-365.

LLC, S. B. (2017). https://www.bitcoin.com/you-need-to-know. Retrieved from https://www.bitcoin.com/you-need-to-know

Mandelbrot, B. (1997). Fractals and scaling in finance: discontinuity, concentration, risk. *Springer-Verlag*.

Martinez, M. F., Granero, M. A. S., Torrecillas, M. J. M., & McKelvey, B. (2017). A comparison among some hurst exponent approaches to predict nascent bubbles in 500 company stocks. *Fractals*, *25*(1).

Morales, R., DiMatteo, T., , & Aste, T. (2014). Dynamical generalized hurst exponent as a tool to monitor unstable periods in financial time series. *Physica A*, *391*, 3180-3189.

Morales, R., DiMatteo, T., Grammatica, R., & Aste, T. (2012). Dynamial generalized hurst exponent as a tool to monitor unstable periods in financial time series. *Physica A: Statistical Mechanics and its Applications*, *391*, 3180-3189.

Novak, S. Y., & Beirlant, J. (2006). The magnitude of a market crash can be predicted. *Journal of Banking & Finance*, *30*(2), 453-462.

Phillips, P. C. B., & Yu, J. (2009). *Limit theory for dating the origination and collapse of mildy explosive periods in time series data.*

Pozzi, F., Di Matteo, T., & Aste, T. (2012). Exponential smoothing weighted correlations. *European Physical Journal B*, *85*, 175.

Severini, T. A. (1998). An approximation to the modified profile likelihood function. *Biometrika*, *85*(2), 403-411.

Shiller, R. (1981). Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review*, *71*(3), 421-436.

Smithers, A. (2009). *wall street revalued, imperfect markets and inept central bankers.* John Wiley & Sons, Ltd.

Sornette, D. (2003). Why stock markets crash (critical events in complex financial systems). *Princeton University Press*.

Sornette, D., Demos, G., Zhang, Q., Cauwels, P., Filimonov, V., & Zhang, Q. (2014). Real-time prediction and post-mortem analysis of the shanghai 2015 stock market bubble and crash. *Journal of Investment Strategies*, *4*(4), 77-95.

Sornette, D., & Johansen, A. (2010). Significance of log-periodic precursors to financial crashes. *Quantitative Finance*, *1*(4), 452-471.

Sornette, D., Woodard, R., Yan, W., & Zhou, W.-X. (2013). Clarifications to questions and criticisms on the johansen ledoit sornette financial bubble model. *Physica A: Statistical Mechanics and its Applications*, *329*(19), 4417-4428.

Terekhova, M. (2017, may). Bitcoin may be headed for a bubble. Retrieved from `http://www.businessinsider.com/bitcoin-may-be-headed-for-a-bubble-2017-5?international=true&r=US&IR=T`

van Bothmer, H. C. G., & Meister, C. (2003). Predicting critical crashes? a new restriction for the free variables. *Physica A: Statistical Mechanics and its Applications*, *320*, 539-547.

Watorek, M., & Stawiarski, B. (2016). Log-periodic power law and generalized hurst exponent analysis in estimating an asset bubble bursting time. *Financial Internet Quarterly*, *12*(3), 49-58.

West, K. (1987). A specification test for speculative bubbles. *Quarterly Journal of Economics*, *102*, 553-580.

Zhang, Q., Sornette, D., Balcilar, M., Gupta, R., Ozdemir, Z. A., & Yetkiner, H. (2016). Lppls bubble indicators over two centuries of the s&p 500 index. *Physica A: Statistical Mechanics and its Applications*, *458*, 126-139.

Zhou, W.-X., & Sornette, D. (2002). Statistical significance of periodicity and log-periodicity with heavy-tailed correlated noise. *International Journal of Modern Physics C*, *13*(2).

Appendix

In this section the appendix is given. In the report references will be made to the subsections below.

A Notation overview

In this section, table 21 provides an overview of all the symbols in the paper, and their description.

Table 21: **notation** this table provides an overview of all symbols used in this paper.

symbol	subscription
$y_t > 0$	log price at time t
$a > 0$	value of y_t if the bubble lasts to t_c
$b < 0$	decrease in y_t before the crash if c almost 0
c	proportional magnitude of fluctuations around exponential growth
$t_c > 0$	critical time
$t < t_c$	time during bubble
m	exponent of the power law growth
w	frequency of fluctuations during the bubble
A	expected log price value at time of crash
B	magnitude of the power law acceleration
C	amplitude of the log-periodic oscillations
ϕ	shift parameter. $0 \leq \phi \leq 2\pi$
η	nuisance parameters
F_1	minimal SSE for fixed t_c, m and w
F_2	minimal SSE for fixed t_c
n	number of observations
s^2	variance of the residuals
L_p	profile LogLikelihood
L_m	modified profile LogLikelihood
R	(modified) profile likelihood ratio q
order moment	
$H(q)$	Hurst exponent for q^{th} moment
ΔH	multifractality

B Data time series historical bubbles

Figures 6 to 14 show the log price time series of the indices discussed in section 3.1.1.

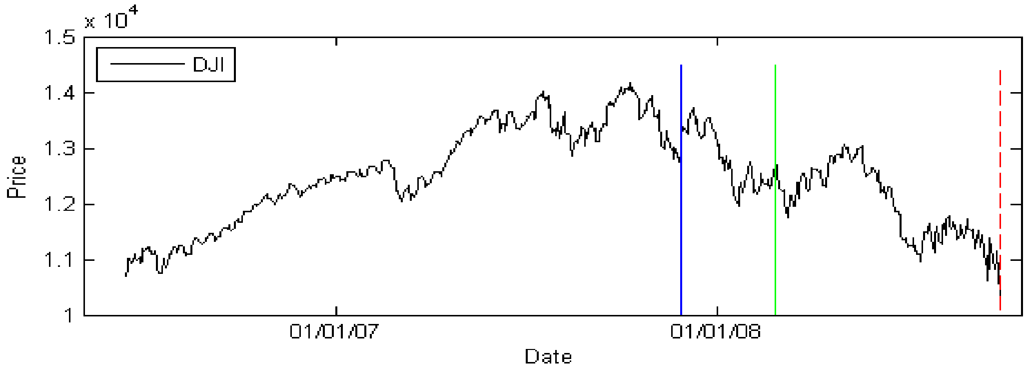


Figure 6: **Time-series of the DJI log prices** The time series of the DJI log prices are given over the period June 13, 2006 to September 29, 2008. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

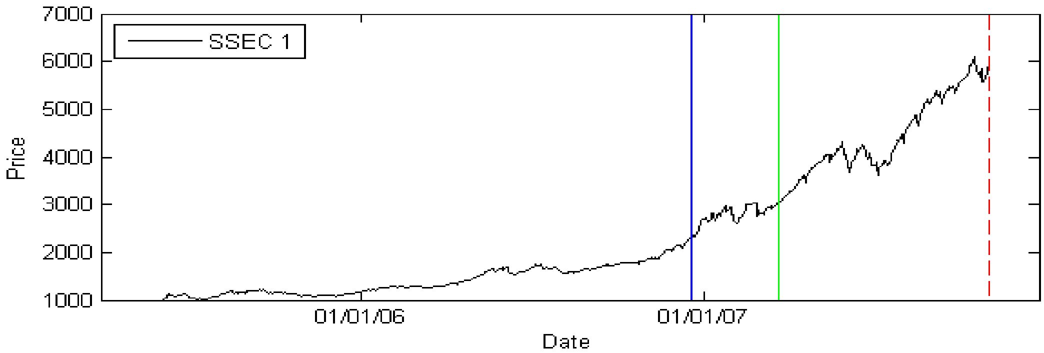


Figure 7: **Time-series of the SSEC 1 prices** The time series of the SSEC prices are given over the period June 3, 2005 to November 1, 2007. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

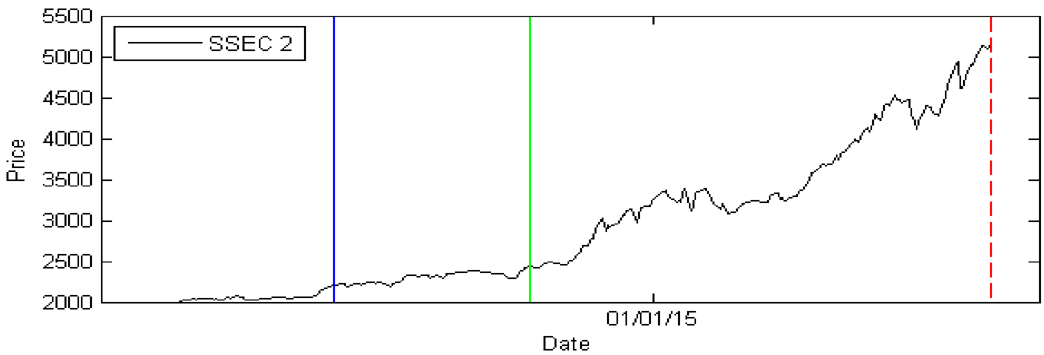


Figure 8: **Time-series of the SSEC 2 prices** The time series of the SSEC 2 prices are given over the period May 19, 2014 to June 12, 2015. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

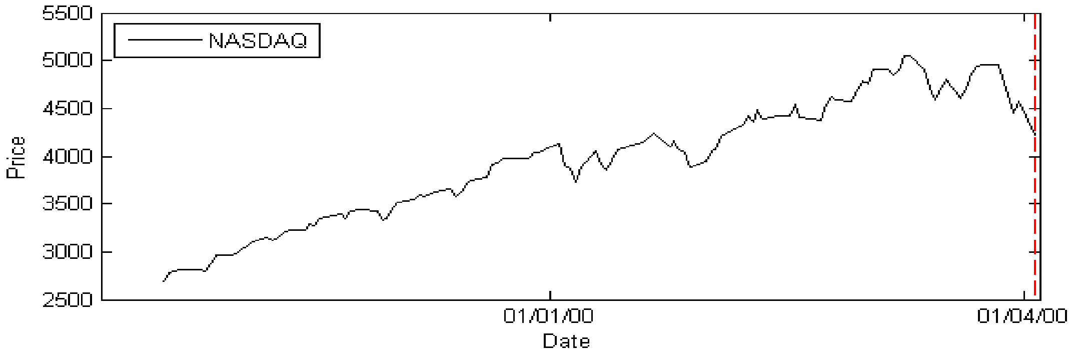


Figure 9: **Time-series of the NASDAQ prices** The time series of the NASDAQ prices are given over the period October 19, 1999 to April 1, 2000. The red dashed line marks the crash date.

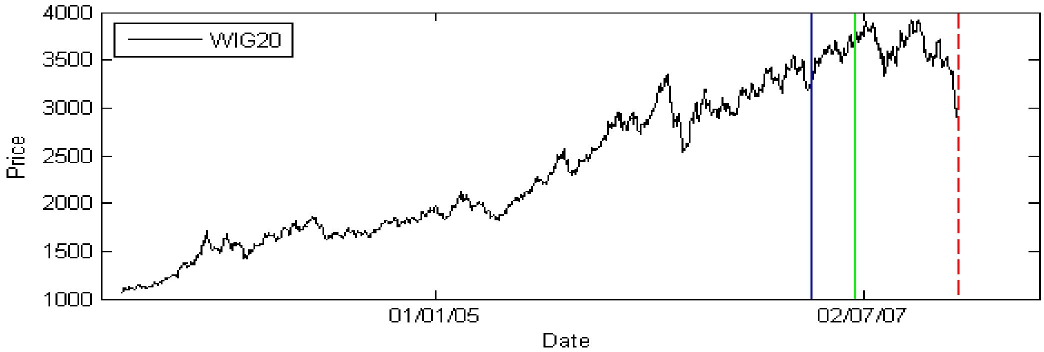


Figure 10: **Time-series of the WIG20 prices** The time series of the WIG20 prices are given over the period March 6, 2003 to January 1, 2008. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

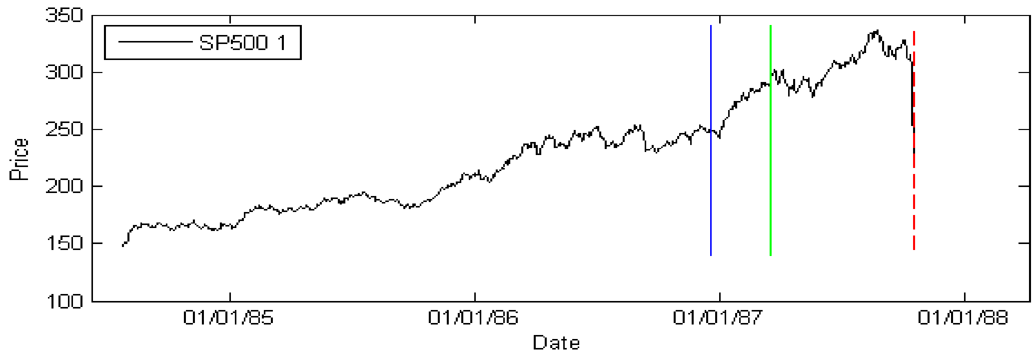


Figure 11: **Time-series of the S&P 500 1 prices** The time series of the S&P 500 1 prices are given over the period June 24, 1984 to October 19, 1987. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

C Case Studies Results

This section contains results from the LPPL model and GHE approach as discussed in section 3.2 and section 3.3.

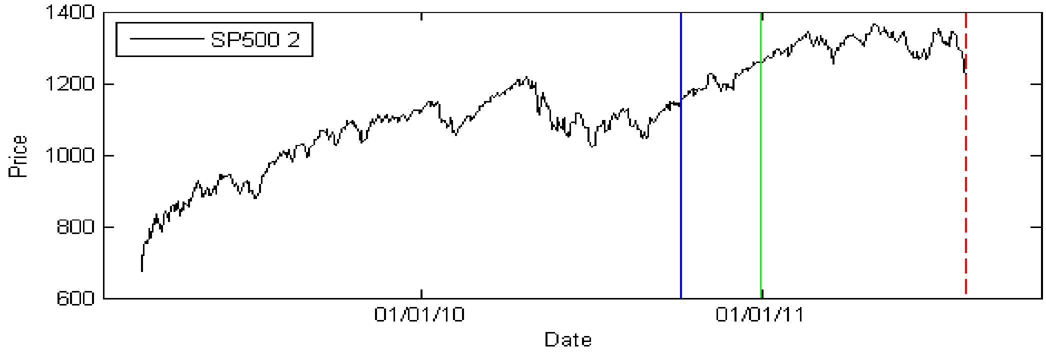


Figure 12: **Time-series of the S&P 500 2 prices** The time series of the S&P 500 2 prices are given over the period March 9, 2009 to August 4, 2011. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

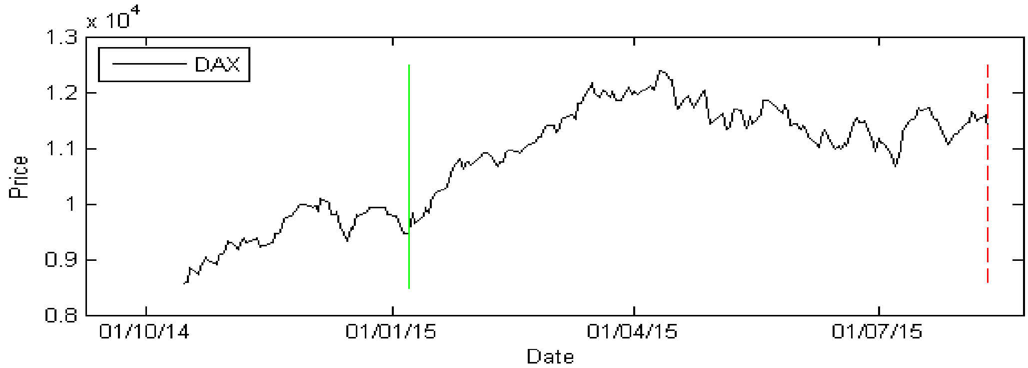


Figure 13: **Time-series of the DAX prices** The time series of the DAX prices are given over the period October 15, 2014 to August 11, 2015. The green line marks the start date using 150 trading days and the red dashed line is the crash date.

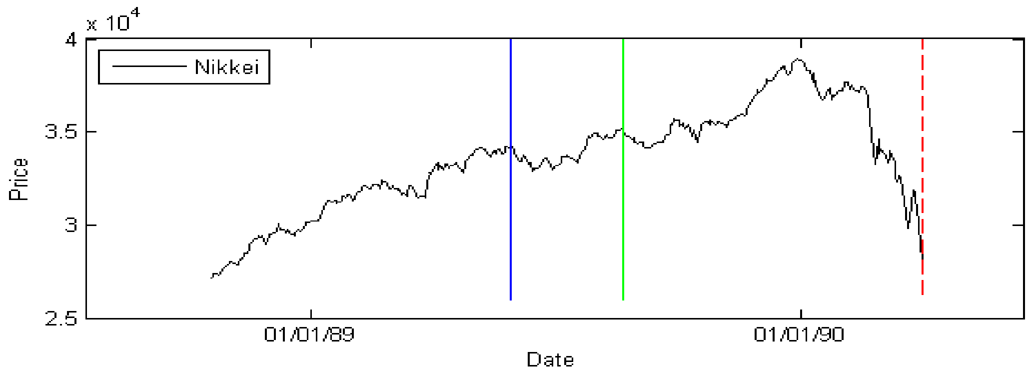


Figure 14: **Time-series of the Nikkei prices** The time series of the Nikkei prices are given over the period October 18, 1988 to April 1, 1990. The green and blue line mark the start date using 150 and 210 trading days respectively and the red dashed line is the crash date.

C.1 LPPL model

This section contains results from the LPPL model as discussed in section 3.2. The tables in this section show the results from fitting the LPPL model to time series containing 100, 120,

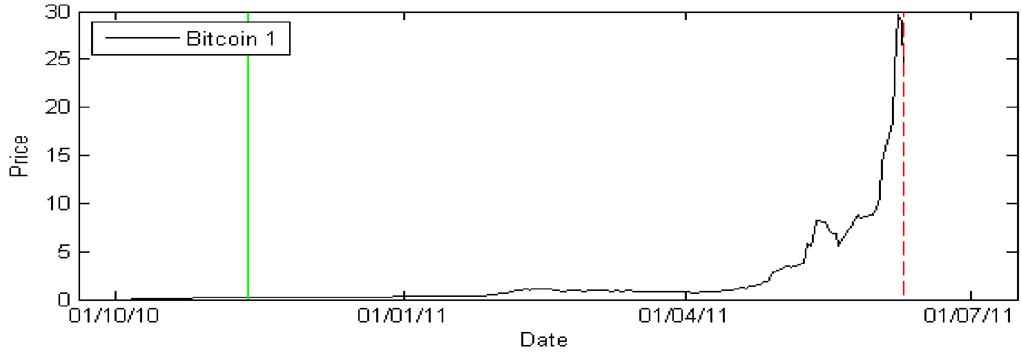


Figure 15: **Time-series of the Bitcoin 1 prices** The time series of the Bitcoin 1 prices are given over the period October 6, 2010 to June 10, 2011. The green line marks the start date using 150 trading days and the red dashed line is the crash date.

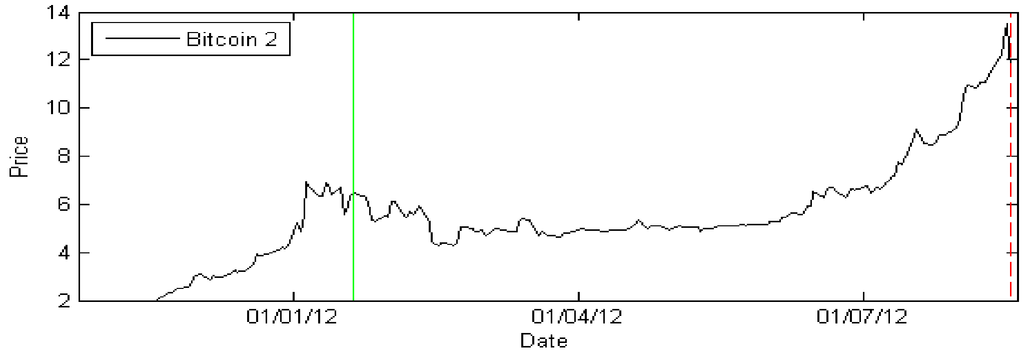


Figure 16: **Time-series of the Bitcoin 2 prices** The time series of the Bitcoin 2 prices are given over the period November 18, 2011 to August 17, 2012. The green line marks the start date using 150 trading days and the red dashed line is the crash date.

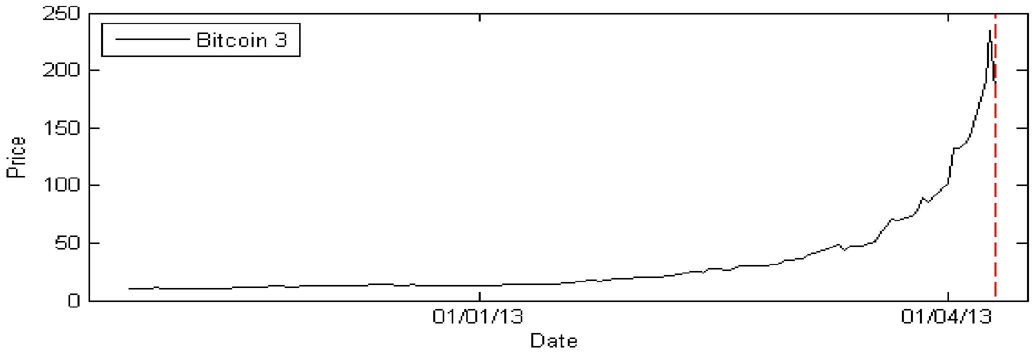


Figure 17: **Time-series of the Bitcoin 3 prices** The time series of the Nikkei prices are given over the period October 26, 2012 to April 10, 2013. The red dashed line marks the crash date.

180, 250, 375 and 500 trading days.

Table 22: **Historical Bubbles** This table presents the results obtained from fitting the LPPL model to time series obtaining 100 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.0210	0.9990	7.0691	[-4,24]	24	0.0210	0.9990	7.0691
	10	-6	0.0178	0.9990	4.7732	[-9,19]	-6	0.0178	0.9990	4.7732
	15	-14	0.0159	0.9990	4.2792	[-14,14]	-14	0.0159	0.9990	4.2792
	20	-19	0.0150	0.9990	3.7763	[-21,-15]	-19	0.0150	0.9990	3.7763
SSEC 1	5	10	0.0982	0.9990	7.2896	[-1,24]	10	0.0982	0.9990	7.2896
	10	-8	0.0994	0.9764	5.5869	[-9,19]	10	0.1012	0.9990	7.0217
	15	-11	0.0978	0.9990	5.1071	[-14,14]	-11	0.0978	0.9990	5.1071
	20	-21	0.0972	0.9990	4.4314	[-22,3]	3	0.1003	0.9990	6.3312
SECC 2	5	-4	0.0409	0.6979	4.2728	[-4,-1]	-4	0.0409	0.6979	4.2728
	10	-8	0.0453	0.5920	3.6118	[-8,11]	-8	0.0453	0.5920	3.6118
	15	10	0.0474	0.9990	6.3764	[-1,14]	11	0.0474	0.9990	6.4451
	20	10	0.0461	0.9990	6.2913	[-3,10]	10	0.0461	0.9990	6.2913
NASDAQ	5	24	0.0626	0.9990	9.1615	[3,24]	24	0.0626	0.9990	9.1615
	10	11	0.0672	0.9990	6.8834	[-8,19]	10	0.0672	0.9990	6.7834
	15	-2	0.0529	0.9990	4.0892	[-14,14]	0	0.0530	0.9990	4.3140
	20	-	-	-	-	-	-	-	-	-
WIG20	5	-4	0.0491	0.2418	6.9471	[6,24]	19	0.0703	0.9990	0.0264
	10	-9	0.0407	0.4796	5.7836	[-9,-8]	-8	0.0409	0.4507	5.9182
	15	-17	0.0322	0.3338	4.7465	[-17,-14]	-17	0.0322	0.3338	4.7465
	20	-16	0.0339	0.5025	4.9658	[-19,-10]	-16	0.0339	0.5025	4.9658
S&P 500 1	5	-4	0.0317	0.9990	2.8905	[-4,24]	24	0.0325	0.4257	3.0626
	10	-9	0.0265	0.4649	2.4028	[4,19]	4	0.0277	0.0010	3.3316
	15	-14	0.0272	0.4145	2.2588	[-6,14]	-4	0.0278	0.0010	3.0524
	20	-11	0.0273	0.0010	2.4231	[-12,10]	-8	0.0274	0.0010	2.6363
S&P 500 2	5	-4	0.0134	0.1792	4.5737	[-4,24]	24	0.0139	0.1005	7.3418
	10	-8	0.0128	0.0010	3.9439	[-8,19]	19	0.0134	0.0010	7.6472
	15	14	0.0110	0.0010	7.5772	[3,14]	14	0.0110	0.0010	7.6655
	20	10	0.0111	0.0010	7.1540	[-1,10]	10	0.0111	0.0010	7.1540
DAX	5	-4	0.0342	0.3904	2.8514	[14,24]	24	0.0349	0.9990	19.2634
	10	19	0.0356	0.9990	17.3969	[-8,-6]	-8	0.0372	0.2459	2.6420
	15	14	0.0341	0.9990	0.0364	[-7,12]	-1	0.0349	0.9990	0.0328
	20	10	0.0340	0.9990	0.0414	[-]	0	0.0347	0.9990	0.0554
Nikkei 225	5	-4	0.0265	0.5058	0.0172	[-2,-3]	-3	0.0273	0.4308	0.0077
	10	17	0.0259	0.9990	2.2881	[-]	17	0.0259	0.9990	2.2881
	15	-15	0.0254	0.2775	0.0449	[-10,-8]	-10	0.0260	0.0010	0.4878
	20	8	0.0271	0.9990	1.5892	[-]	8	0.0271	0.9990	1.5892

Table 23: **Historical Bubbles 100 trading days (continued)** This table presents the results obtained from fitting the LPPL model to time series obtaining 100 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
Bitcoin 1	5	-2	1.4473	0.9534	3.7604	[-2,-1]	-2	1.4473	0.9536	3.7596
	10	48	1.5643	0.0010	9.2778	[48,49]	49	1.5644	0.0010	9.3433
	15	19	1.6748	0.0010	6.7735	[-]	19	1.6755	0.0010	6.7629
	20	-21	0.1080	0.4537	3.6394	[-23,-21]	-21	0.1080	0.4537	3.6394
Bitcoin 2	5	2	0.1195	0.0010	8.9956	[1,3]	2	0.1195	0.0010	8.9956
	10	-9	0.1372	0.3236	2.2453	[-7,0]	-2	0.1431	0.0010	7.6062
	15	-14	0.1524	0.6994	1.1799	[0,14]	14	0.1555	0.0010	3.0843
	20	-17	0.0984	0.0010	4.4482	[-17,-15]	-17	0.0984	0.0010	4.4482
Bitcoin 3	5	-5	0.3011	0.9988	0.0013	[-1,-2]	-2	0.3395	0.0010	2.2065
	10	-902	0.2435	0.0010	25.0000	[-902,-8]	-902	0.2435	0.0010	25.0000
	15	395	0.1812	0.9990	25.0000	[5,16]	16	0.2140	0.9990	3.8277
	20	-	-	-	-	-	-	-	-	-

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Table 24: **Historical Bubbles** 120

	Δt_c	OLS $\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	Modified Profile Likelihood LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.0240	0.9990	7.2114	[-4,24]	24	0.0240	0.9990	7.2114
	10	-6	0.0216	0.9990	4.7697	[-9,19]	-6	0.0216	0.9990	4.7697
	15	-14	0.0217	0.9990	4.1254	[-14,14]	14	0.0219	0.9990	6.4426
	20	-19	0.0200	0.8825	3.6170	[-21,8]	-19	0.0200	0.8825	3.6170
SSEC 1	5	7	0.1112	0.9990	6.6604	[1,15]	7	0.1112	0.9990	6.6604
	10	-7	0.1045	0.9990	5.7512	[-9,19]	-8	0.1045	0.9990	5.6728
	15	-10	0.1035	0.9990	5.4463	[-14,14]	-10	0.1035	0.9990	5.3897
	20	-21	0.1016	0.9990	4.6800	[-23,5]	5	0.1044	0.9990	6.5103
SECC 2	5	10	0.0589	0.8867	6.4762	[1,21]	10	0.0589	0.8867	6.4762
	10	6	0.0608	0.9990	6.0065	[-2,19]	5	0.0608	0.9990	5.9400
	15	13	0.0567	0.9990	6.7433	[3,14]	14	0.0567	0.9990	6.8022
	20	10	0.0574	0.9990	6.4894	[0,10]	10	0.0574	0.9990	6.4894
WIG20	5	-4	0.0623	0.3413	6.6826	[4,16]	9	0.0801	0.9990	6.9533
	10	-9	0.0448	0.5127	5.9015	[-9,-8]	-8	0.0449	0.4952	6.0080
	15	-16	0.0403	0.4649	5.0405	[-17,-12]	-16	0.0403	0.4649	5.0405
	20	-16	0.0467	0.3065	5.0337	[-19,-10]	-16	0.0467	0.3065	5.0337
S&P 500 1	5	-4	0.0378	0.9464	2.9717	[-4,24]	-4	0.0378	0.9464	2.9717
	10	-8	0.0317	0.4344	2.5753	[3,19]	4	0.0328	0.0010	3.4668
	15	-14	0.0332	0.4651	2.3089	[-5,14]	-3	0.0340	0.0010	3.1334
	20	-19	0.0354	0.9990	1.6610	[-19,-18]	-18	0.0354	0.9990	1.6919
S&P 500 2	5	24	0.0183	0.0190	7.9322	[22,24]	24	0.0183	0.0190	7.9322
	10	19	0.0190	0.0010	7.4655	[10,19]	19	0.0190	0.0010	7.4655
	15	-1	0.0189	0.0010	5.1905	[-6,9]	-1	0.0189	0.0010	5.1905
	20	-10	0.0201	0.0010	3.8579	[-18,-17]	-17	0.0209	0.0010	2.3366
DAX	5	-4	0.0454	0.9990	3.1313	[-4,10]	-4	0.0454	0.9990	3.1313
	10	-8	0.0455	0.9990	2.6356	[-9,19]	-8	0.0455	0.9990	2.6356
	15	14	0.0407	0.8933	2.5996	[-11,14]	-2	0.0409	0.9990	1.4877
	20	10	0.0410	0.9990	2.9349	[-16,-12]	-15	0.0425	0.9990	0.0060
Nikkei 225	5	24	0.0339	0.9990	2.8861	[-4,3]	3	0.0349	0.9990	1.9610
	10	17	0.0298	0.9990	2.8900	[-10,-9]	-9	0.0311	0.9990	1.6645
	15	12	0.0307	0.9990	2.4293	[-]	12	0.0307	0.9990	2.4293
	20	8	0.0302	0.9990	0.5567	[-]	8	0.0302	0.9990	0.5567
Bitcoin 1	5	7	2.5255	0.0010	5.2121	[6,7]	7	2.5256	0.0010	5.2204
	10	-124	2.3650	0.9990	5.5380	[-]	-124	2.3650	0.9990	5.5380
	15	-125	2.2151	0.9990	5.8060	[-]	-125	2.2151	0.9990	5.8060
	20	-19	2.8436	0.1358	3.6676	[3,10]	1	2.9179	0.0010	5.3486
Bitcoin 2	5	25	0.1778	0.9990	2.4710	[-3,8]	5	0.1805	0.9990	1.6771
	10	-9	0.2065	0.3615	2.3769	[-6,-4]	-6	0.2102	0.0010	2.9759
	15	41	0.2861	0.0078	5.0849	[41,42]	41	0.2861	0.0078	5.0849
	20	-21	0.3747	0.9183	0.0010	[8,10]	8	0.3856	0.2996	0.0998

Table 25: **Historical Bubbles 180 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 180 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.0487	0.8882	7.1242	[-]	24	0.0487	0.8882	7.1242
	10	19	0.0480	0.9990	6.8784	[-]	19	0.0480	0.9990	6.8784
	15	14	0.0477	0.9579	6.6309	[-]	14	0.0477	0.9579	6.6309
	20	8	0.0456	0.7063	6.3109	[-]	8	0.0456	0.7063	6.3109
SSEC 1	5	19	0.1932	0.5764	8.8840	[17,24]	18	0.1933	0.5746	8.8174
	10	19	0.1899	0.5137	8.8658	[1,19]	19	0.1899	0.5137	8.8658
	15	14	0.1916	0.4727	8.3806	[-3,14]	3	0.1918	0.4775	7.6762
	20	5	0.1820	0.3578	7.5774	[1,5]	5	0.1820	0.3578	7.5774
SECC 2	5	24	0.1276	0.4551	8.8769	[21,24]	24	0.1276	0.4551	8.8769
	10	19	0.1317	0.4012	8.2449	[18,19]	19	0.1317	0.4012	8.2449
	15	14	0.1326	0.3497	7.7168	[13,14]	14	0.1326	0.3497	7.7168
	20	10	0.1320	0.3173	7.3845	[8,10]	10	0.1320	0.3173	7.3845
WIG20	5	-4	0.1404	0.0010	7.6723	[-3,10]	-3	0.1406	0.0010	7.8240
	10	-1	0.1226	0.0010	8.1744	[-3,2]	-1	0.1226	0.0010	8.1744
	15	3	0.1114	0.0010	8.6599	[-]	-17	0.1504	0.3178	7.1724
	20	5	0.1062	0.0076	8.5782	[-12,-9]	-9	0.1156	0.0010	7.0361
S&P 500 1	5	-4	0.0855	0.9990	9.1388	[-4,21]	-1	0.0855	0.9990	9.5093
	10	19	0.0694	0.9990	6.5170	[14,19]	19	0.0694	0.9990	6.5170
	15	14	0.0707	0.9990	6.3070	[10,14]	14	0.0707	0.9990	6.3070
	20	-16	0.0702	0.9990	7.4852	[-19,-5]	-16	0.0702	0.9990	7.4852
S&P 500 2	5	-4	0.0408	0.9841	2.0273	[-]	-4	0.0408	0.9841	2.0273
	10	-9	0.0417	0.7855	2.1153	[-]	-9	0.0417	0.7855	2.1153
	15	-14	0.0396	0.6598	1.9307	[-]	-14	0.0396	0.6598	1.9307
	20	-10	0.0357	0.0011	0.1147	[-]	-10	0.0357	0.0011	0.1147
DAX	5	0	0.0917	0.9990	3.8117	[-4,13]	0	0.0917	0.9990	3.8117
	10	0	0.0930	0.9990	3.7197	[-6,19]	1	0.0930	0.9990	3.7555
	15	15	0.0920	0.9990	4.3537	[-2,14]	15	0.0920	0.9990	4.3537
	20	11	0.0982	0.9990	3.8992	[-10,11]	11	0.0982	0.9990	3.8992
Nikkei 225	5	9	0.0498	0.0010	2.3270	[-]	9	0.0498	0.0010	2.3270
	10	-1	0.0473	0.0010	2.1134	[-]	-1	0.0473	0.0010	2.1134
	15	-13	0.0466	0.0354	1.5692	[-]	-13	0.0466	0.0354	1.5692
	20	-6	0.0467	0.0010	1.4101	[-]	-6	0.0467	0.0010	1.4101
Bitcoin 2	5	-185	1.1881	0.6295	2.5241	[-]	-185	1.1881	0.6295	2.5241
	10	-182	1.1203	0.5842	2.3412	[-]	-182	1.1203	0.5842	2.3412
	15	-187	1.1684	0.3644	2.8607	[-]	-187	1.1684	0.3644	2.8607
	20	-187	1.1458	0.3290	2.8834	[-]	-187	1.1458	0.3290	2.8834

Table 26: **Historical Bubbles 250 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 250 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.1005	0.2797	7.1963	[-]	24	0.1005	0.2797	7.1963
	10	19	0.0998	0.2276	6.8683	[-]	19	0.0998	0.2276	6.8683
	15	14	0.1025	0.0916	6.7742	[-]	14	0.1025	0.0916	6.7742
	20	8	0.1015	0.0010	6.5758	[-]	8	0.1015	0.0010	6.5758
SSEC 1	5	23	0.6189	0.9990	13.3430	[-]	23	0.6189	0.9990	13.3430
	10	19	0.6135	0.9990	12.9702	[-]	19	0.6135	0.9990	12.9702
	15	188	0.4654	0.9990	25.0000	[-]	188	0.4654	0.9990	25.0000
	20	5	0.6276	0.9990	11.9424	[-]	5	0.6276	0.9990	11.9424
SECC 2	5	24	0.2325	0.6027	8.7437	[-]	24	0.2325	0.6027	8.7437
	10	40	0.2324	0.4502	9.7856	[-]	40	0.2324	0.4502	9.7856
	15	14	0.2456	0.5878	7.7114	[-]	14	0.2456	0.5878	7.7114
	20	10	0.2463	0.5737	7.3961	[-]	10	0.2463	0.5737	7.3961
WIG20	5	-4	0.2789	0.2908	7.2034	[-]	-4	0.2789	0.2908	7.2034
	10	-8	0.2549	0.4503	6.6922	[-]	-8	0.2549	0.4503	6.6922
	15	-12	0.2315	0.4243	6.3047	[-]	-12	0.2315	0.4243	6.3047
	20	-12	0.2191	0.4928	6.1961	[-]	-12	0.2191	0.4928	6.1961
S&P 500 1	5	-4	0.1188	0.9990	8.4370	[-]	-4	0.1188	0.9990	8.4370
	10	-9	0.1059	0.9990	7.9313	[-]	-9	0.1059	0.9990	7.9313
	15	-14	0.1070	0.9990	7.3921	[-]	-14	0.1070	0.9990	7.3921
	20	-19	0.1070	0.9990	6.8901	[-]	-19	0.1070	0.9990	6.8901
S&P 500 2	5	-4	0.0745	0.9548	2.1551	[-]	-4	0.0745	0.9548	2.1551
	10	-9	0.0778	0.8910	2.1286	[-]	-9	0.0778	0.8910	2.1286
	15	-14	0.0774	0.7596	1.9098	[-]	-14	0.0774	0.7596	1.9098
	20	-19	0.0769	0.5509	1.5183	[-]	-19	0.0769	0.5509	1.5183
Nikkei 225	5	0	0.0738	0.0010	1.7098	[-]	0	0.0738	0.0010	1.7098
	10	-7	0.0748	0.0010	1.6058	[-]	-7	0.0748	0.0010	1.6058
	15	-14	0.0762	0.0010	1.3214	[-]	-14	0.0762	0.0010	1.3214
	20	-21	0.0757	0.0373	0.9893	[-]	-21	0.0757	0.0373	0.9893

Table 27: **Historical Bubbles 375 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 375 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	24	0.2086	0.1261	7.4622	[-]	24	0.2086	0.1261	7.4622
	10	1	0.2179	0.1880	6.3676	[-]	1	0.2179	0.1880	6.3676
	15	21	0.2309	0.0010	7.2259	[-]	21	0.2309	0.0010	7.2259
	20	8	0.2396	0.0010	6.6497	[-]	8	0.2396	0.0010	6.6497
SSEC 1	5	-12	0.8994	0.9337	9.7328	[-]	-12	0.8994	0.9337	9.7328
	10	-12	0.9033	0.9298	9.6946	[-]	-12	0.9033	0.9298	9.6946
	15	-17	0.8981	0.9383	9.4003	[-]	-17	0.8981	0.9383	9.4003
	20	-28	0.9051	0.9476	8.6971	[-]	-28	0.9051	0.9476	8.6971
WIG20	5	-5	0.3981	0.9990	0.3605	[-]	-5	0.3981	0.9990	0.3605
	10	21	0.4001	0.9990	1.2215	[-]	21	0.4001	0.9990	1.2215
	15	14	0.4007	0.9990	1.4845	[-]	14	0.4007	0.9990	1.4845
	20	264	0.4237	0.9990	2.2322	[-]	264	0.4237	0.9990	2.2322
S&P 500 1	5	24	0.2002	0.9990	7.6305	[-]	24	0.2002	0.9990	7.6305
	10	19	0.1882	0.9338	7.3118	[-]	19	0.1882	0.9338	7.3118
	15	-14	0.1899	0.9863	6.8147	[-]	-14	0.1899	0.9863	6.8147
	20	-19	0.1851	0.9399	6.5590	[-]	-19	0.1851	0.9399	6.5590
S&P 500 2	5	23	0.3347	0.9990	4.0080	[-]	23	0.3347	0.9990	4.0080
	10	6	0.3355	0.9990	7.1125	[-]	6	0.3355	0.9990	7.1125
	15	3	0.3323	0.9990	6.9359	[-]	3	0.3323	0.9990	6.9359
	20	5	0.3312	0.9990	6.7983	[-]	5	0.3312	0.9990	6.7983

Table 28: **Historical Bubbles 500 trading days** This table presents the results obtained from fitting the LPPL model to time series obtaining 500 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results are given based on OLS estimation, profile likelihood estimation and modified profile likelihood estimation. $\Delta\widehat{t_c}$ represents the distance from the estimated crash date to the true crash date and LI gives the Likelihood interval.

		OLS				Modified Profile Likelihood				
	Δt_c	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$	LI	$\Delta\widehat{t_c}$	SSE	$\widehat{m}$	$\widehat{w}$
DJI	5	-458	0.3902	0.9011	2.1061	[-]	-458	0.3902	0.9011	2.1061
	10	1138	0.3989	0.9990	13.2737	[-]	1.138	0.3989	0.9990	13.2737
	15	1899	0.3972	0.9990	21.5218	[-]	1.899	0.3972	0.9990	21.5218
	20	-95	0.3802	0.2998	2.1516	[-]	-95	0.3802	0.2998	2.1516
SSEC 1	5	-12	1.4516	0.8526	4.2538	[-]	-12	1.4516	0.8526	4.2538
	10	-12	1.5066	0.8386	4.1571	[-]	-12	1.5066	0.8386	4.1571
	15	-18	1.5791	0.8314	3.8555	[-]	-18	1.5791	0.8314	3.8555
	20	-29	1.6662	0.8271	3.3816	[-]	-29	1.6662	0.8271	3.3816
WIG20	5	459	0.8350	0.1270	0.8241	[-]	459	0.8350	0.1270	0.8241
	10	19	0.8349	0.9990	1.9382	[-]	19	0.8349	0.9990	1.9382
	15	13	0.8450	0.9990	2.1670	[-]	13	0.8450	0.9990	2.1670
	20	6	0.8587	0.9990	2.2935	[-]	6	0.8587	0.9990	2.2935
S&P 500 1	5	24	0.2638	0.8446	7.3841	[-]	24	0.2638	0.8446	7.3841
	10	19	0.2430	0.8070	7.2310	[-]	19	0.2430	0.8070	7.2310
	15	-39	0.2046	0.7750	5.8212	[-]	-39	0.2046	0.7750	5.8212
	20	-39	0.2016	0.7697	5.8135	[-]	-39	0.2016	0.7697	5.8135
S&P 500 2	5	-4	0.3968	0.9990	5.5718	[-]	-4	0.3968	0.9990	5.5718
	10	-9	0.3876	0.9990	5.4004	[-]	-9	0.3876	0.9990	5.4004
	15	-14	0.3815	0.9990	5.2577	[-]	-14	0.3815	0.9990	5.2577
	20	-23	0.3800	0.9990	5.0571	[-]	-23	0.3800	0.9990	5.0571

C.2 GHE Approach

This section contains results from the GHE approach as discussed in section 3.3. The tables in this section show the results from conducting the GHE approach to time series containing 100, 150, 180, 210, 375 and 500 trading days.

Table 29: **Historical Bubbles 100 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 100 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		NASDAQ	
Δt_c	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0777	-27	0.0741	-54	0.0763	-8	0.0838	-31
10	0.0777	-27	0.0741	-54	0.0671	-53	0.0838	-31
15	0.0777	-27	0.0741	-54	0.0676	-61	0.0838	-31
20	0.0777	-27	0.0741	-54	0.0676	-61	-	-
	WIG20		S&P 500 1		S&P 500 2		DAX	
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0709	-9	0.0605	-40	0.0379	-52	0.0658	-30
10	0.0695	-11	0.0605	-40	0.0379	-52	0.0658	-30
15	0.0581	-16	0.0792	-62	0.0379	-52	0.0658	-30
20	0.0498	-59	0.0792	-62	0.0390	-69	0.0658	-30
	Nikkei 225		Bitcoin 1		Bitcoin 2		Bitcoin 3	
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0843	-10	0.0880	-23	0.1040	-37	0.1083	-46
10	0.0757	-12	0.0880	-23	0.1040	-37	0.1083	-46
15	0.0968	-64	0.0880	-23	0.1040	-37	0.1483	-65
20	0.0968	-64	0.0880	-23	0.1041	-69	-	-

Table 30: **Historical Bubbles 150 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 150 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		NASDAQ	
Δt_c	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$
5	0.0758	-49	0.1105	-54	0.0844	-22	-	-
10	0.0758	-49	0.1105	-54	0.0844	-22	-	-
15	0.0758	-49	0.1105	-54	0.0844	-22	-	-
20	0.0758	-49	0.1201	-69	0.0844	-22	-	-
	WIG20		S&P 500 1		S&P 500 2		DAX	
	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$
5	0.0445	-44	0.0735	-33	0.0556	-53	0.0617	-24
10	0.0445	-44	0.0735	-33	0.0670	-60	0.0617	-24
15	0.0445	-44	0.0735	-33	0.0686	-64	0.0617	-24
20	0.0445	-44	0.0735	-33	0.0686	-64	0.0617	-24
	Nikkei 225		Bitcoin 1		Bitcoin 2		Bitcoin 3	
	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$
5	0.0933	-6	0.0713	-35	0.1449	-27	-	-
10	0.0702	-12	0.0713	-35	0.1449	-27	-	-
15	0.0658	-16	0.0713	-35	0.1449	-27	-	-
20	0.0577	-52	0.0713	-35	0.1449	-27	-	-

Table 31: **Historical Bubbles 180 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 180 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		NASDAQ	
Δt_c	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$
5	0.0723	-19	0.1252	-39	0.0664	-54	-	-
10	0.0723	-19	0.1252	-39	0.0664	-54	-	-
15	0.0723	-19	0.1252	-39	0.0664	-54	-	-
20	0.0714	-37	0.1252	-39	0.0664	-54	-	-
	WIG20		S&P 500 1		S&P 500 2		DAX	
	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$
5	0.0396	-16	0.0696	-31	0.0823	-34	-	-
10	0.0396	-16	0.0696	-31	0.0823	-34	-	-
15	0.0396	-16	0.0696	-31	0.0823	-34	-	-
20	0.0306	-46	0.0696	-31	0.0823	-34	-	-
	Nikkei 225		Bitcoin 1		Bitcoin 2		Bitcoin 3	
	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$	ΔH	$\Delta \widehat{t_c}$
5	0.0589	-6	-	-	0.0768	-51	-	-
10	0.0574	-22	-	-	0.0768	-51	-	-
15	0.0574	-22	-	-	0.0843	-63	-	-
20	0.0574	-22	-	-	-	-	-	-

Table 32: **Historical Bubbles 210 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 210 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		NASDAQ	
Δt_c	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$
5	0.0771	-6	0.1072	-10	0.0866	-25	-	-
10	0.0479	-14	0.1049	-25	0.0866	-25	-	-
15	0.0479	-51	0.1049	-25	0.0866	-25	-	-
20	0.0479	-51	0.1049	-25	0.0866	-25	-	-
	WIG20		S&P 500 1		S&P 500 2		DAX	
	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$
5	0.0348	-16	0.0541	-16	0.0764	-39	-	-
10	0.0348	-16	0.0541	-16	0.0764	-39	-	-
15	0.0445	-63	0.0541	-16	0.0764	-39	-	-
20	0.0445	-63	0.0500	-58	0.0764	-39	-	-
	Nikkei 225		Bitcoin 1		Bitcoin 2		Bitcoin 3	
	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$
5	0.0421	-6	-	-	-	-	-	-
10	0.0321	-27	-	-	-	-	-	-
15	0.0321	-27	-	-	-	-	-	-
20	0.0321	-27	-	-	-	-	-	-

Table 33: **Historical Bubbles 375 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 375 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

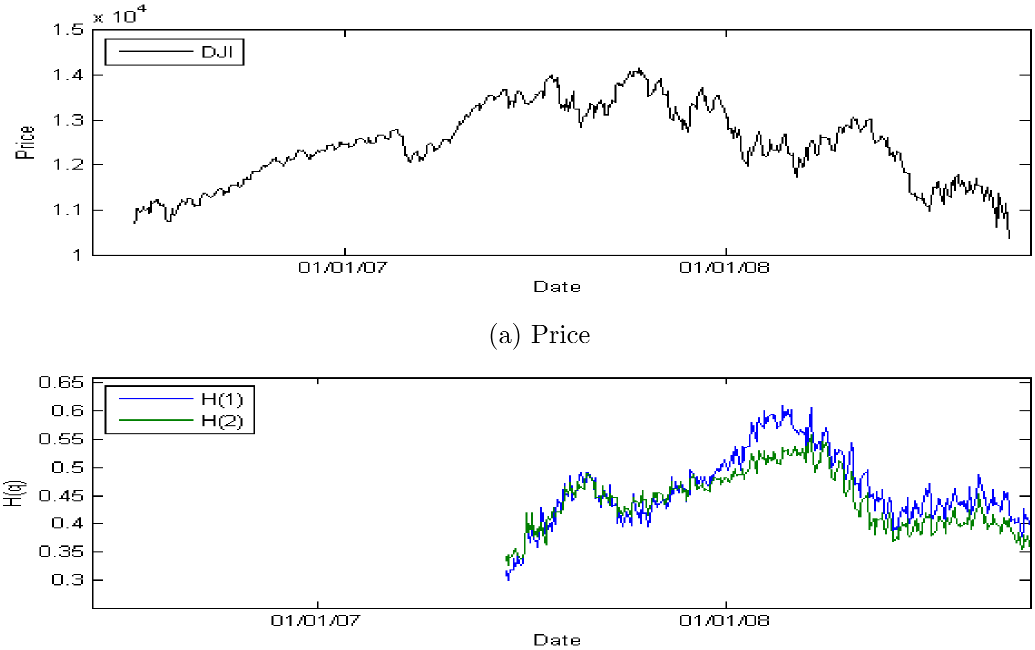
	DJI		SSEC 1		SSEC 2		NASDAQ	
Δt_c	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$
5	0.0553	-104	0.0713	-20	-	-	-	-
10	0.0553	-104	0.0713	-20	-	-	-	-
15	0.0553	-104	0.0713	-20	-	-	-	-
20	0.0553	-104	0.0711	-95	-	-	-	-
	WIG20		S&P 500 1		S&P 500 2		DAX	
	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$
5	0.0437	-49	0.0775	-73	0.0786	-117	-	-
10	0.0437	-49	0.0775	-73	0.0786	-117	-	-
15	0.0437	-49	0.0775	-73	0.0786	-117	-	-
20	0.0437	-49	0.0775	-73	0.0786	-117	-	-
	Nikkei 225		Bitcoin 1		Bitcoin 2		Bitcoin 3	
	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$	ΔH	$\widehat{\Delta t_c}$
5	-	-	-	-	-	-	-	-
10	-	-	-	-	-	-	-	-
15	-	-	-	-	-	-	-	-
20	-	-	-	-	-	-	-	-

Table 34: **Historical Bubbles 500 trading days** This table presents the results obtained from fitting the GHE approach to time series based on 500 trading days. Per time series the estimation is performed taking observations until Δt_c days before the bubble burst. The results show the maximum multifractality $\Delta H = H(1) - H(2)$ and the distance from the day with a maximum multifractality to the true crash date.

	DJI		SSEC 1		SSEC 2		NASDAQ	
Δt_c	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0817	-166	0.0713	-20	-	-	-	-
10	0.0817	-166	0.0713	-20	-	-	-	-
15	0.0817	-166	0.0713	-20	-	-	-	-
20	0.0817	-166	0.0711	-95	-	-	-	-
	WIG20		S&P 500 1		S&P 500 2		DAX	
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	0.0437	-49	0.0775	-73	0.0786	-117	-	-
10	0.0437	-49	0.0775	-73	0.0786	-117	-	-
15	0.0437	-49	0.0775	-73	0.0786	-117	-	-
20	0.0437	-49	0.0775	-73	0.0786	-117	-	-
	Nikkei 225		Bitcoin 1		Bitcoin 2		Bitcoin 3	
	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$	ΔH	$\Delta\widehat{t_c}$
5	-	-	-	-	-	-	-	-
10	-	-	-	-	-	-	-	-
15	-	-	-	-	-	-	-	-
20	-	-	-	-	-	-	-	-

D Hurst Exponent estimations

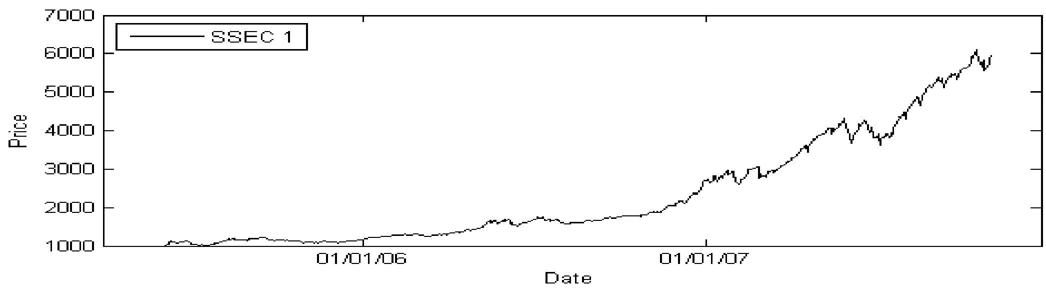
Figures 18 to 26 show the price time series of the indices discussed in section 3.1.1 together with the estimated Hurst Exponents for the order moments 1 and 2.



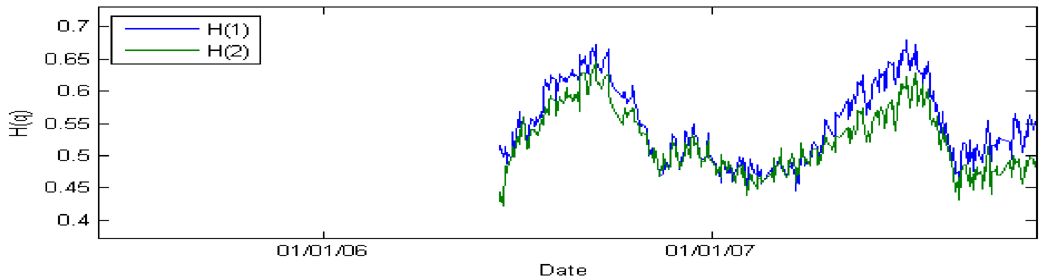
(a) Price

(b) HE

Figure 18: **Estimated HE DJI** The time series of the DJI is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

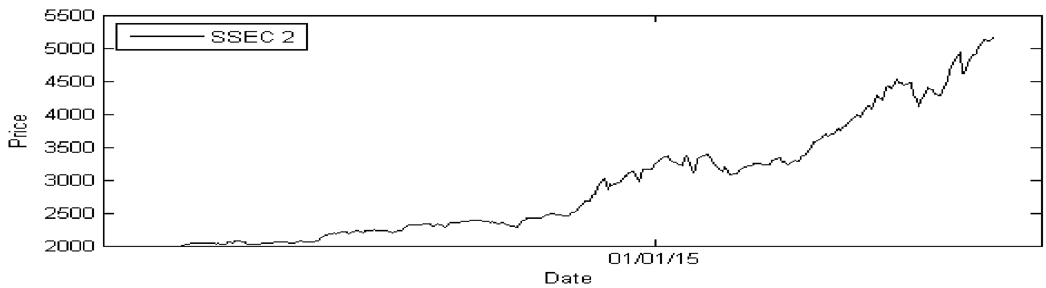


(a) Price

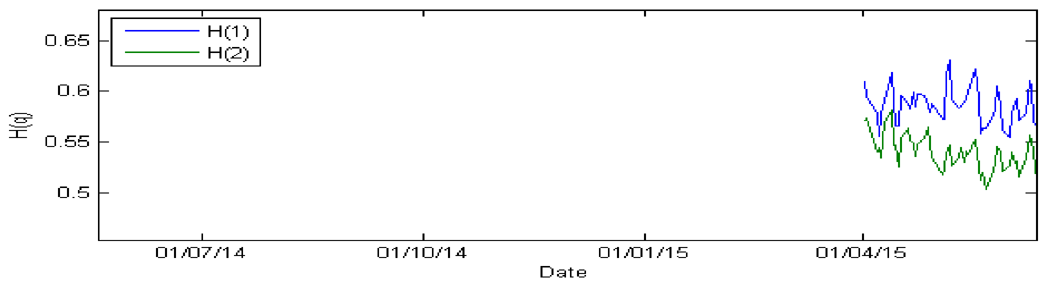


(b) HE

Figure 19: **Estimated Hurst Exponent SSEC 1** The time series of the SSEC 1 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

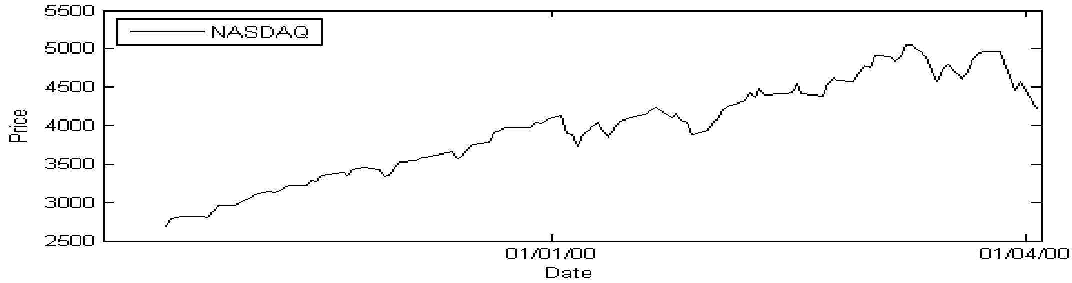


(a) Price

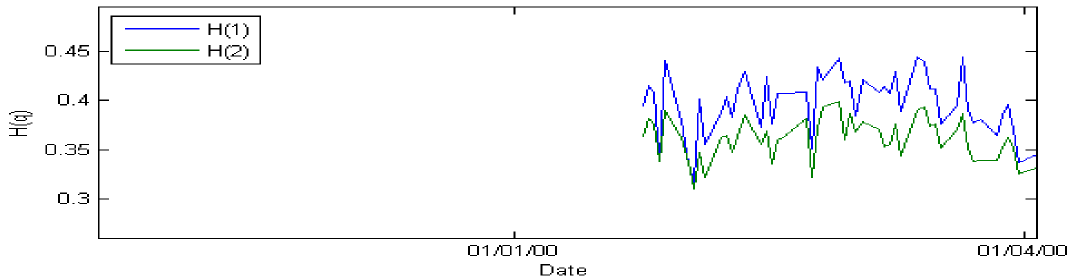


(b) HE

Figure 20: **Estimated Hurst Exponent SSEC 2** The time series of the SSEC 2 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

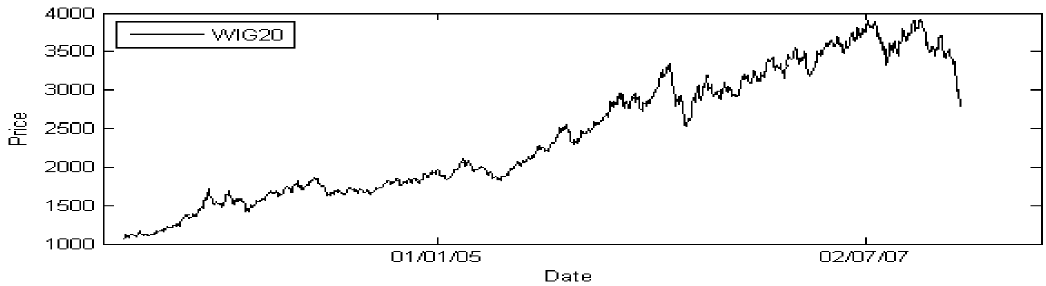


(a) Price

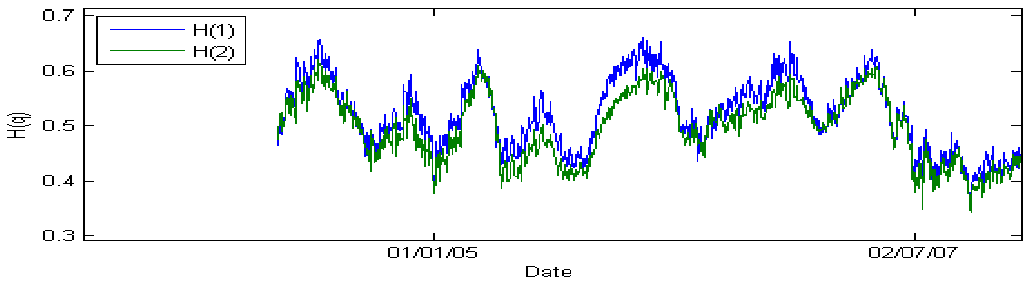


(b) HE

Figure 21: **Estimated Hurst Exponent NASDAQ** The time series of the NASDAQ is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

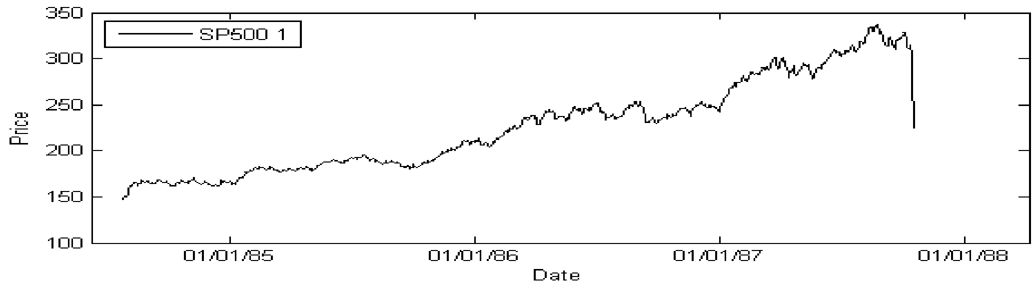


(a) Price

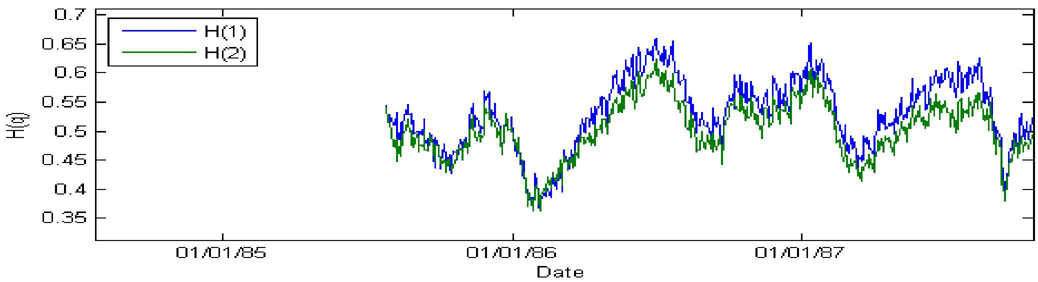


(b) HE

Figure 22: **Estimated Hurst Exponent WIG 20** The time series of the WIG 20 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

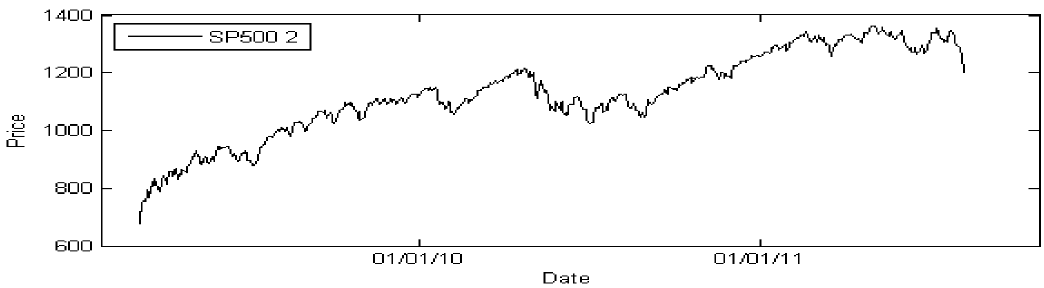


(a) Price

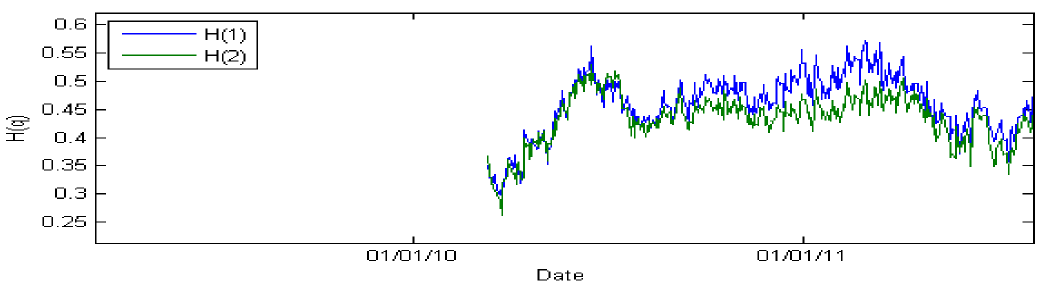


(b) HE

Figure 23: **Estimated Hurst Exponent S&P 500 1** The time series of the S&P 500 1 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

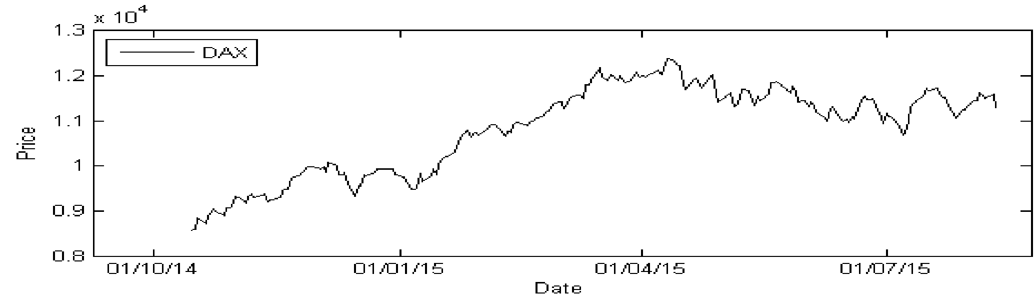


(a) Price

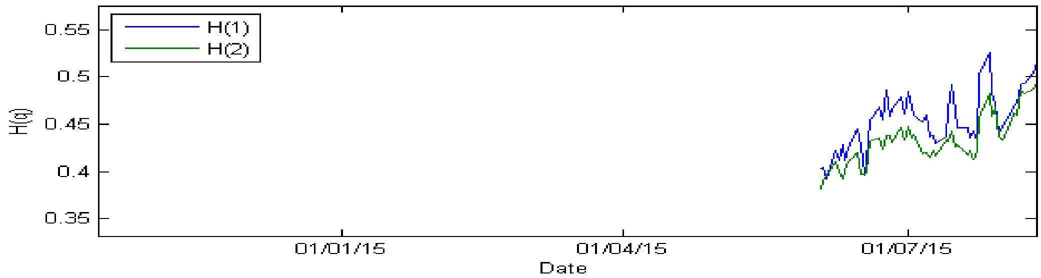


(b) HE

Figure 24: **Estimated Hurst Exponent S&P 500 2** The time series of the S&P 500 2 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

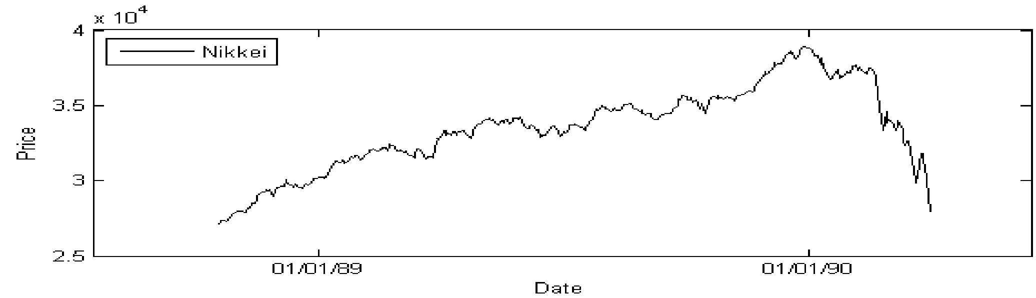


(a) Price

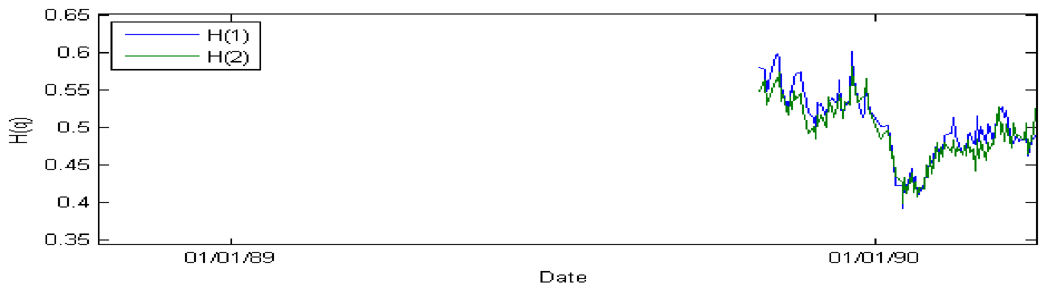


(b) HE

Figure 25: **Estimated Hurst Exponent DAX** The time series of the DAX is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

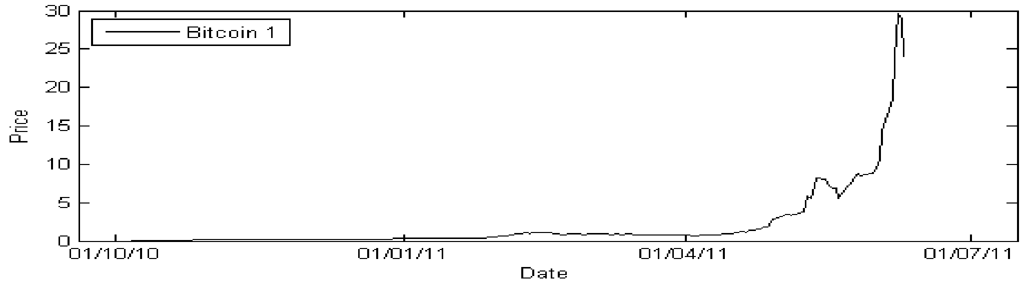


(a) Price

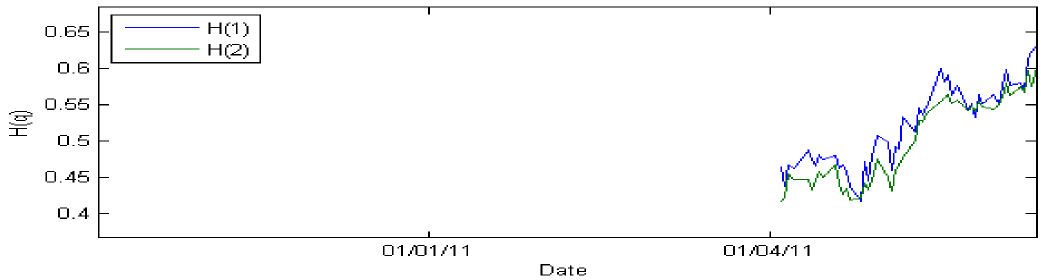


(b) HE

Figure 26: **Estimated Hurst Exponent Nikkei 225** The time series of the Nikkei 225 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

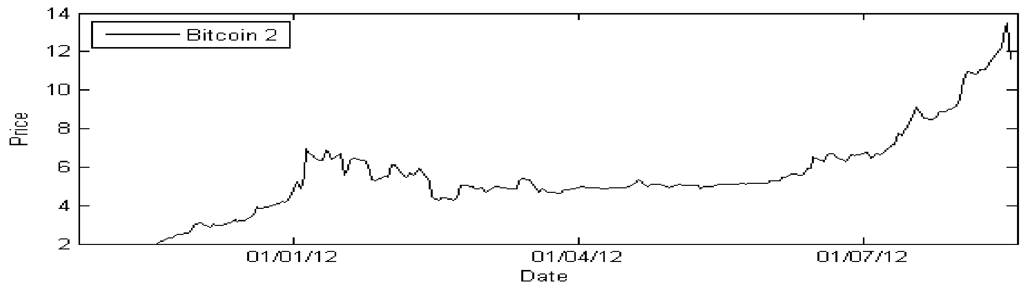


(a) Price

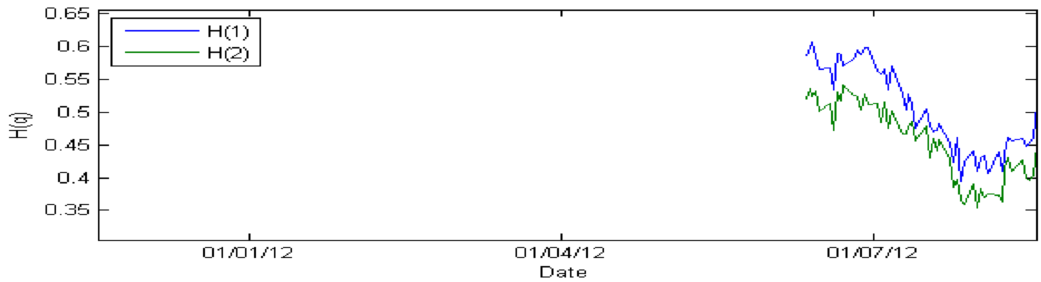


(b) HE

Figure 27: **Estimated Hurst Exponent Bitcoin 1** The time series of the Bitcoin 1 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

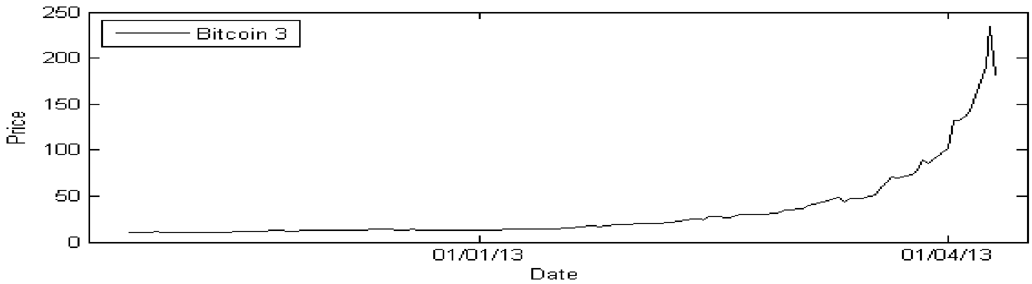


(a) Price

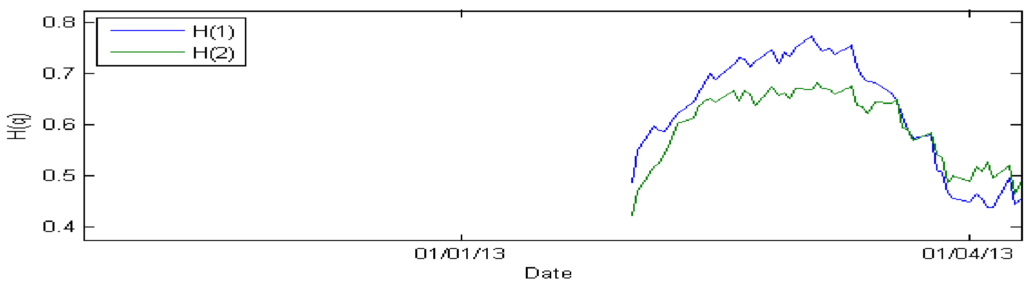


(b) HE

Figure 28: **Estimated Hurst Exponent Bitcoin 2** The time series of the Bitcoin 2 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.



(a) Price



(b) HE

Figure 29: **Estimated Hurst Exponent Bitcoin 3** The time series of the Bitcoin 3 is given over its full sample period, together with the estimated Hurst Exponents for the first en second moment order, H(1) and H(2) respectively.

E Trading strategy fitting bubbles

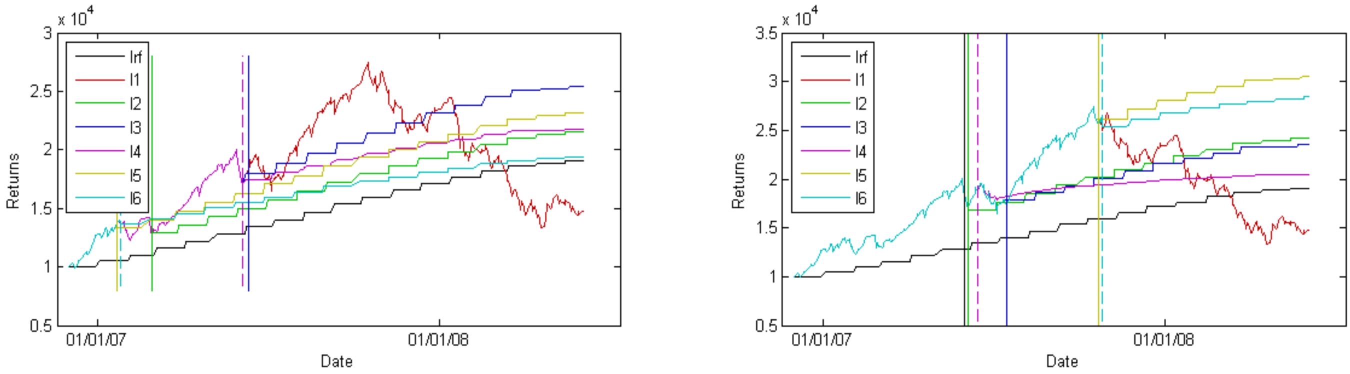
In this section, table 35 gives an overview of the bubbles considered in forecasting. The bubbles are tested on estimates resulting in boundary conditions for the GHE and LPPL model.

Table 35: **Historical Bubbles Trading strategy** This table presents the historical bubbles used to obtain boundary conditions for the LPPL trader and GHE trader.

Index	critical time	bubble
DJI	29-9-2008	US asset market crash
	19-10-1987	Black Monday
SSEC	12-6-2015	Chinese asset market crash
	12-5-1997	Asian Financial crisis
WIG20	21-1-2008	US housing bubble
	4-8-2011	Stocks Market Fall
SP500	19-10-1987	Black Monday
	29-9-2008	US asset market crash
	1-4-2000	Dotcom bubble
DAX	11-8-2015	Migrant crisis
	4-8-2011	Stocks Market Fall
	13-8-1998	Russian financial crisis
Nikkei	2-4-1990	Real estate bubble
	4-8-2011	Stocks Market Fall
	1-4-2000	Dotcom bubble
Nasdaq	29-9-2008	US asset market crash
	1-4-2000	Dotcom bubble
	4-8-2011	Stocks Market Fall
AEX	4-8-2011	Stocks Market Fall
CAC40	13-8-1998	Russian financial crisis
	4-8-2011	Stocks Market Fall
RTSI	13-8-1998	Russian financial crisis

F Capital Changes Trading Strategies

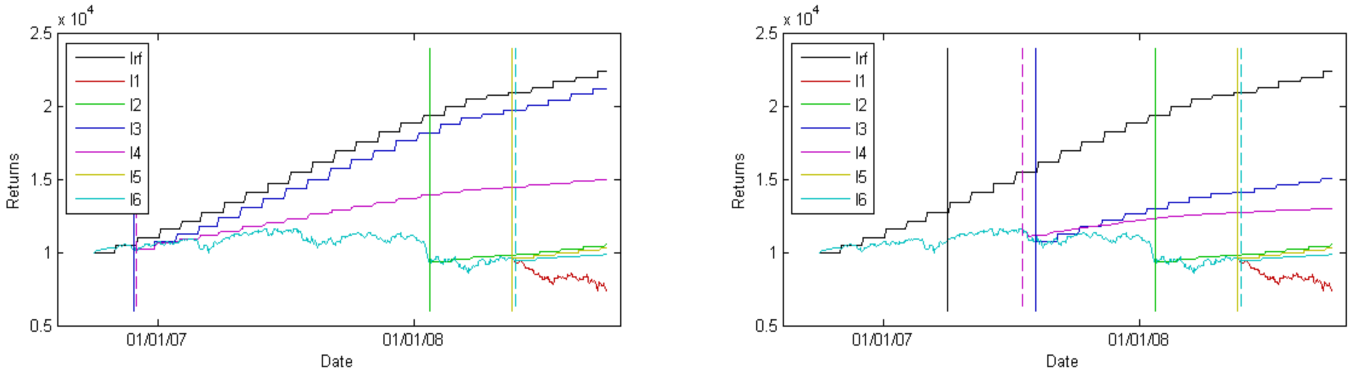
In this section, figures 30, 31, 32 and 33 give the capital time series during each of their investment periods.



(a) H0

(b) H6

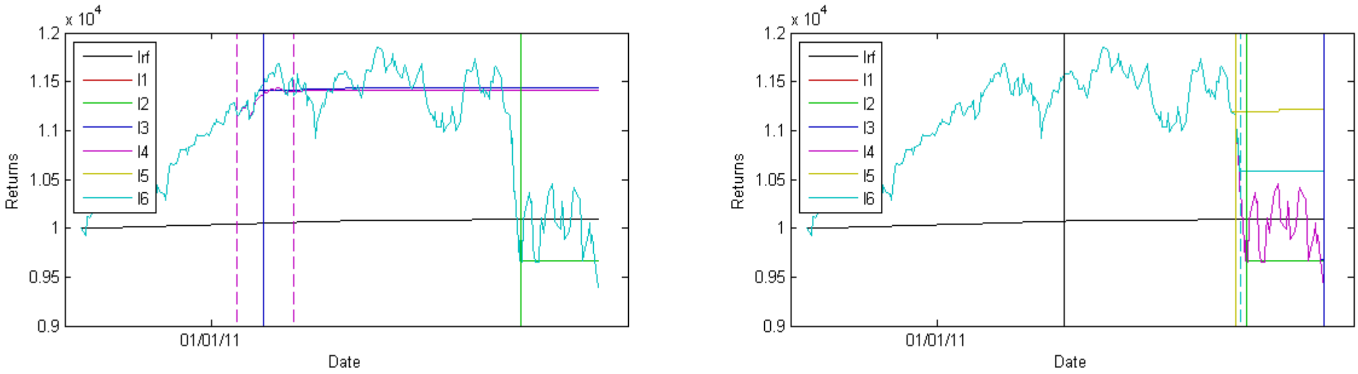
Figure 30: **Trading strategy results SSEC** The capital time series is given for the SSEC index per investor (I1 to I6) together with the US-1M investor (Irf). The colored lines mark the estimated crash dates, the dashed lines give the beginning or end of the smoothing intervals for investor 4 and 6. The capital time series are given without(30a) and with (30b) a holding period. The black line marks the end of the holding period.



(a) H0

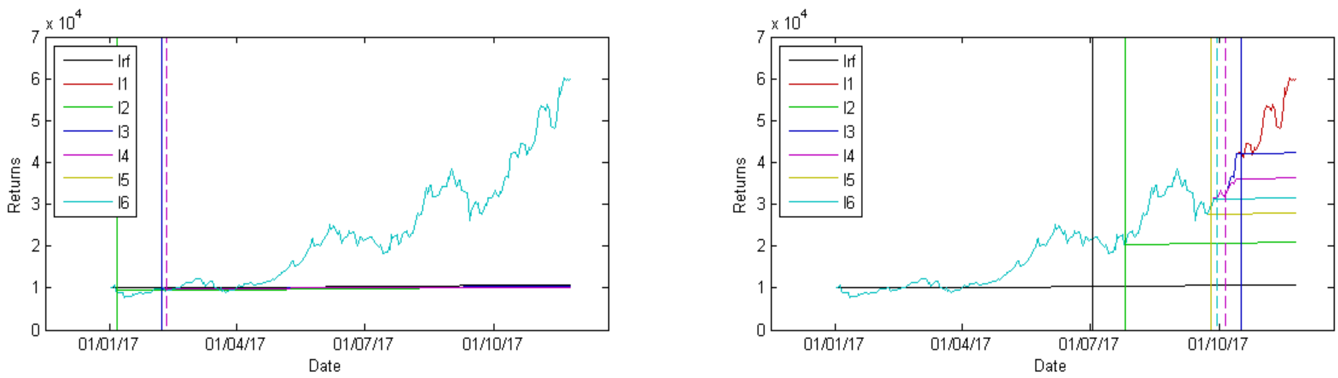
(b) H6

Figure 31: **Trading strategy results ES50** The capital time series is given for the ES50 index per investor (I1 to I6) together with the US-1M investor (Irf). The colored lines mark the estimated crash dates, the dashed lines give the beginning or end of the smoothing intervals for investor 4 and 6. The capital time series are given without(31a) and with (31b) a holding period. The black line marks the end of the holding period.



(a) H0 (b) H6

Figure 32: **Trading strategy results S&P 500** The capital time series is given for the S&P 500 index per investor (I1 to I6) together with the US-1M investor (Irf). The colored lines mark the estimated crash dates, the dashed lines give the beginning or end of the smoothing intervals for investor 4 and 6. The capital time series are given without(32a) and with (32b) a holding period. The black line marks the end of the holding period.



(a) H0 (b) H6

Figure 33: **Trading strategy results Bitcoin** The capital time series is given for the Bitcoin per investor (I1 to I6) together with the US-1M investor (Irf). The colored lines mark the estimated crash dates, the dashed lines give the beginning or end of the smoothing intervals for investor 4 and 6. The capital time series are given without(33a) and with (33b) a holding period. The black line marks the end of the holding period.