ERASMUS UNIVERSITY ROTTERDAM
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Bachelor Thesis Quantitative Finance

Modeling dependence in high dimensions using Factor Copulas

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Abstract

This paper builds a model for the dependence between the top 10 stocks of S&P 100. We follow the method introduced by Oh and Patton (2017) in building a Latent Factor model for the Copulas of a set of random variables. To estimate the Copula models parameters we use a Simulation Method of Moments (SMM). A simulation study shows that the SMM provides accurate estimates of the Factor Copula models. We consider restrictive models that force equidependence and do not allow for tail dependence between variables and than relax those restrictions to compare the results. In the works of Oh and Patton (2017) they used the Spearman correlation and quantile tail dependence. We introduce in this paper another moment condition calculated on the base of the Spectral measure. This new moment condition seems to perform adequately.

1 Introduction

Building a reliable characterization of dependence between financial variables is of crucial importance in risk management and financial analysis. Models like the Pearson correlation proved to be somewhat unreliable at giving an accurate picture of the dependence, as they assume linear dependence and do not allow for (asymmetric) tail dependence characterization. A clear dangerous repercussion of this is underestimating a joint crash. This issue became all the more critical for the industry after the 2007 financial crisis. Many assets that seemed to exhibit very little dependence in regular days proved to be highly dependent in crashes. When it comes to alternative measures to the Pearson correlation several suggestions exist in previous literature. For example the tail dependence coefficient χ is a pairwise tail dependence measure, it measures also the probability of one variable being large while the other variable is large. It gives an important intuition for dependence in the tails without making a strong assumption .

While these measures have a closed form and provide very powerful results they only describe the dependence in the tails while we would want a more global model for dependence that captures dependence

around the mean and dependence in both tails. Added to that these measures run into a serious problem of performance when the set of variables we are modeling the dependence between gets large, the curse of dimensionality. When we analyze highly dimensional data the required amount of data to come up with reliable information increases exponentially as the dimensions of the data increase. Furthermore the estimator's variance often also increases therefore making estimation methods not very reliable.
A solution to this crucial challenge is proposed by Oh and Patton (2017) to model dependence in high dimension using Factor Copulas. A copula is a joint probability distribution for which each marginal distribution is uniform. The main idea is that to assume that the copula of our financial variables could be modeled with varables (in our case the financial variables) that are generated by a common latent factors. The remaining less challenging work is to estimate the parameters of their data generating process. To be able to compare variables they need to be on the same scale, that is why we convert them to the uniform scale and use Copulas to model their joint distribution. Using a factor model for the joint distribution is to tackle the curse of dimensionality. If we find a limited set K of factors that influence all variables we can build a model that requires substantially less data. If the factors K influence all variables the same way the estimations of the model becomes even more convenient, we are in the case of equidependence. In equidependence the number of parameters to estimate is substantially decreased as we impose and equal factor leading across all variables. This is a rather restrictive assumption but it allows to require less data to estimate the parameters of the model with good accuracy. After building a model with equidependence we will also look into models with multi-factor copula specifications
Section 2 of the paper does a preliminary study of the data. Section 3 and 4 describes the theoretical background used in the study and the estimation methods used. Section 5 presents our findings while Section 6 concludes.

2 Data

The data we use is the daily stock returns of the top 10 stocks from the S&P 100 as of September 1 ,2017, from 6 July 2015 to 9 May 2018. Table 1 below shows the list of stocks included in the study .

Table 1: Stocks used in empirical analysis

Ticker	Name	SIC
AAPL	Apple	357
AMZN	Amazon.com	737
FB	Facebook.com	737
GOOGL	Google Alphabet	737
JPM	JP Morgan	672
MSFT	Microsoft	737
WFC	Wells Fargo	602
XOM	Exxon	291
JNJ	Johnson&J	283
WFC	Wells Fargo	602

Financial variables are known to exhibit serial correlation and high volatility. So in order to be able to analyze the data we first need to filter out the serial correlation and to standardize the data. In order to do so we use a an autoregressive AR(1) model and GJR-GARCH(1,1) volatility model :

$$
\begin{aligned}
r_{it} &= \phi_{0i} + \phi_{1i} r_t \\
\sigma_{it}^2 &= \omega_i + \beta_i \sigma_{i,t-1}^2 + \alpha_i \varepsilon_{i,t-1}^2 I(\varepsilon_{i,t-1} \leq 0),\ where\ I(A) = 1\ if\ A\ occurs.
\end{aligned} \tag{1}
$$

To estimate this model we use the packages with solvers in Matlab. The dependence measure we will use in this paper often compare a pair of variables against the same threshold. Such a comparison will make sense only if the variables are on the same scale. Therefore we will put all variables on the uniform scale. For a set of variables $[Y_i, \ldots, Y_n]$ the set $[\eta_1, \ldots, \eta_n]$ are the set of filtered and scaled residuals.r_{it} and σ^{it} represent the returns and volatility of variable Y_i at time t. Below we can see some scatter plots of few pairs of our financial variables after filtering and scaling.

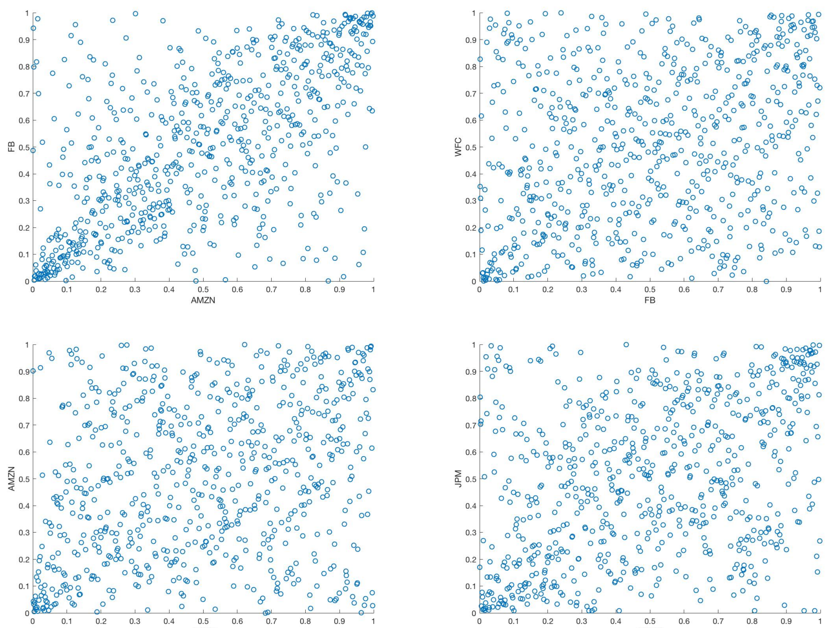

Figure 1: Scatter plots of daily log returns scaled to the uniform scale, of pairs from Amazon, JP Morgan, Google , Facebook and Wells Fargo

The scatter plots above suggest that the dependence between the various pairs of our 7 stocks is very different. We can see that some stocks that are from very different industries such as Wells Fargo and Facebook exhibit very little dependence while others like Amazon and Facebook that are in the same industry seem to be rather correlated. An interesting note nonetheless is that for stocks that seem to be independent such as Microsoft and Amazon we see evidence of dependence in the tails, with the evidence being more clear in the lower tails. This is in line with previous literature that suggest that dependence in the tails can be very different from dependence around the mean. Furthermore, we see that the dependence in crashes (lower tail) is higher than the dependence in booms (upper tail). We need to keep into consideration that the picture above is of 4 pairs of variables amongst 21 pairs that this study includes. So the intuitions that are given by the scatter plots need to be taken with much care as they might be misrepresentative and further analysis is needed.

3 Methodology

3.1 Skewed t distribution

A probability distribution that is of particular interest in this study is the *Skew t* distribution. The *Skew t* has different specifications in literature, in this paper we use the model distribution introduced by Hansen (1994). It is known for its flexibility and can characterize skewness in the data. Such a distribution is often used in heterogeneous and asymmetric data. This will allow us to capture the skewness that we suspect in the data generating process of the copula model for our financial stocks we are using. The probability density function of the *Skew t* is :

$$g(z|\upsilon,\lambda) = \begin{cases} bc\left(1+\frac{1}{\upsilon-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(\upsilon+1)/2}, & z < -a/b, \\ \left(1+\frac{1}{\upsilon-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(\upsilon+1)/2}, & z \geq -a/b, \end{cases} \tag{2}$$

where the constants a, b and c are given by

$$a = 4\lambda c \frac{\upsilon-2}{\upsilon-1} \tag{3}$$

$$b = 1+3\lambda^2 - a^2 \tag{4}$$

$$c = \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\pi(\upsilon-2)}\Gamma\left(\frac{n}{2}\right)} \tag{5}$$

3.2 Dependence measures

As mentioned in Section 2 to characterize the dependence between pairs of variables we put them on the same uniform scale. This means that we need to use dependence measures that are robust to this data transformation, so called "pure" dependence measures, this for example rules out the common Pearson correlation measure. We therefore will use rank correlations and tail dependence measures. Rank correlations examine the relationship between the ranking of the variables. Putting the variables on the uniform scale does not change the order of the variables, therefore, the rank correlation of the original variables will be the same as the one of the transformed variables.

Another type of "pure" dependence measures are the tail dependence measures. These are measures that characterize the dependence between variables in their tails. Several tail dependence measures exist but they all examine the comovements of variables in their tails.

3.2.1 Rank correlations

We will use in our analysis Spearman's ρ as a rank correlation measure. Spearman's ρ between two variables describes the Pearson Correlation between the rank of those two variables. For two variables $X = [X_1, ..., X_n]$

and $Y = [Y_1, ..., Y_n]$ we have rgX_i and rgY_i that are the rank values of these observations then ρ is defined as ,

$$\rho = corr(rgX_i, rgY_i) = \frac{cov(rgX_i, rgY_i)}{\sigma_{rgX_i}\sigma_{rgY_i}}, \tag{6}$$

where $corr(rgX, rgY)$ is the Pearson Correlation between the rank variables rgX and rgY, $cov(rgX, rgY)$ is the covariance of the rank variables, σ_{rgX} and σ_{rgY} are the standard deviation of rgX and rgY. It is clear from Formula 6 that $-1 < \rho < 1$. The intuitive interpretation of ρ is that if two variables have a Spearman correlation close to 1, their observations will have a similar ranking.

3.2.2 Tail dependence

Financial variables usually exhibit heavy tails. It means that a lot of the critical dependence information is in the tails. Thereof, it is important to characterize this dependence by using some tail dependence measures. We will use quantile dependence measures and the spectral measure..

For two random variables X_i, X_j with marginal distribution G_i, G_j :

$$\begin{aligned} \tau_{ij}^L &= \lim_{q \to 0} \frac{\Pr\left[X_i \leq G_i^{-1}(q)\ ,\ X_j \leq G_j^{-1}(q)\right]}{q} \\ \tau_{ij}^U &= \lim_{q \to 0} \frac{\Pr\left[X_i > G_i^{-1}(q)\ ,\ X_j > G_j^{-1}(q)\right]}{1-q}, \end{aligned} \tag{7}$$

where q is a quantile, τ_{ij}^L and τ_{ij}^U being the lower and upper tail dependence for variable X_i and X_j. It is clear from the formula above that $0 < \tau_{ij}^L < 1$ and $0 < \tau_{ij}^U < 1$. They can therefore be interpreted in terms of probabilities with a rather intuitive interpretation. τ_{ij}^L is the probability of X_i and X_j both lie below a low quantile, the probability of a joint crash.

The Spectral measure measures tail dependence but with a different interpretation. It looks at the contribution of one variable to the sum of variables given that the sum is large (or low). For this analysis to be meaningful, the variables need of course to be on the same scale. The idea is that for example, in two dimensions, if two variables are very dependent in the tails then when their sum is large we expect the ratio of each component to the sum to the sum to be around 0.5. If that is the case, it means they jointly exceeded a threshold, if it is not the case it means only one of them exceeded a threshold, therefore casting doubt on their extremal dependence. This distribution of the ratios of the components to their sum is the *spectral measure,* and is denoted by H (de Haan and Resnick, 1977). H is defined as follows :

$$P[\mathbf{W} \in \cdot | S > u] \xrightarrow{d} H(\cdot), \qquad \text{as} \quad u \to \infty, \tag{8}$$

where S is the sum of all transformed variables and W is the vector of ratios. They are defined as follow :

$$S = \eta_1 + ... + \eta_d \quad \text{and} \quad W = (\eta_1/S, ..., \eta_d/S) \tag{9}$$

The pair (S, W) are the radius and angle for every observation. The angle W is a value between 0 and 1. The variables should be on the same scale for this measure to provide any make sense.It is clear from the definition that if the variables are perfectly asymptotically dependent their spectral measure should converge to $1/d$ as the sum gets large. They will equally contribute to the sum. In our case since we will be looking at pairs of variables, so $d = 2$.

3.3 Copulas

A copula $\mathbf{C}$ is a multivariate probability distribution for which each marginal distribution is on the uniform scale. Let $\mathbf{Y}$ be a N dimensional vector of N variables $[Y_1, ..., Y_N]$ and let $\mathbf{F_Y}$ be the joint distribution of $\mathbf{Y}$ and F_{y_i} be the marginal distribution of Y_i than

$$\mathbf{Y} \sim \mathbf{F_Y} = \mathbf{C}(F_{y_1}, ..., F_{y_N}). \tag{10}$$

The copula thoroughly describes the dependence structure between $Y_1, ..., Y_N$ and its distribution is what interests us in this paper. While we have a lot of well-established techniques to estimate marginal distributions, the tasks get more complicated for joint distributions. As we go up in dimension the amount of data needed to make reliable estimation goes up exponentially. A small inaccuracy in estimating the parameters of a marginal distribution can proliferate and cause substantial inaccuracies when moving up to the joint distribution level. This phenomenon is widely known in statistics and often referred to as the "curse of dimensionality." A way of tackling this challenge would be to find common factors that generate the copula mentioned in Equation 10. This is the motivation behind the use of Latent Factor described in Section 3.3.1.

3.3.1 Latent Factor Copulas

The main idea of a latent factor model is that we assume that the copula is generated by some latent factors and an idiosyncratic random variable.We estimate the factor loading of the corresponding Factor Copula model. The model we use in this is study is generated by the following process,

$$\text{Let} \quad X_i = \sum_{k=1}^{K} \beta_{ik} Z_k + \varepsilon_i, \quad i = 1, 2, ..., N \tag{11}$$

$$\begin{aligned} \text{Where} \quad & \varepsilon_i \sim \text{iid } F_\varepsilon(\gamma_\varepsilon) \\ & Z_k \sim \text{inid } F_z(\gamma_k) \end{aligned}$$

$$\text{Then } \mathbf{X} \sim \mathbf{F}_x = \mathbf{C}(G_1(\theta), ..., G_N(\theta)),$$

X_i is a latent variable. Z_k is the latent factors of our model with probability distribution $F_z(\gamma_k)$ where γ_k are the parameters of that probability distribution. β_{ik} is the factor loading of parameter k for random variable X_i, ε_i is the idiosyncratic variable, its distribution is $F_\varepsilon(\gamma_\varepsilon)$ with parameters γ_ε. $\mathbf{C}(\theta)$ is the copula of the latent variables $\mathbf{X}$ and $G_1(\theta)$ is the distribution of X_i $\theta \equiv [vec(\mathbf{B})', \gamma_\varepsilon', \gamma_1', ..., \gamma_K']$ is the vector of parameters of the model that we need to estimate. It can be seen that model is rather flexible, it can still be a very complicated model with a high number of factors, but with an adequate number of factors, the number of parameters in θ that needs to be estimated becomes manageable. We use the copula of $\mathbf{X}$ as an estimation for the copula of $\mathbf{Y}$. It is important to note that the $G_i(\theta)$ can be different from F_{Y_i}, that is we do not estimate the marginal distribution of Y_i in this model. To estimate the marginal distribution of Y_i we use standard time series model. The Factor model is a model for the Copula of $\mathbf{Y}$.

The assumption about the probability distribution of factors Z_k and the idiosyncratic variable ε_i is crucial and challenging. Often a trade-off needs to be made between the simplicity of the model and how appropriate the model describes the reality. One of the most commonly used Copulas models is the Gaussian Copula. The Gaussian Copula is the model we get when both Z_k and ε_i follow a standard normal distribution. This model is prevalent in the financial sector. The Gaussian Copula became very popular due to its closed form formula presented by Li (1999). Generally, the copula of $\mathbf{X}$ is not known in closed form, thereof calculating its implied dependence characteristics of the data becomes an easy task. This very appealing advantage of the Gaussian copula comes at a very expensive cost sometimes. The Gaussian copula assumes that the distribution of the variables is symmetric and that the dependence in the tails between variables is the same as dependence around the mean. The few examples in Figure 1 show that stock markets do not behave like that at all.

The limitations of the Gaussian Copula model motivates the use of common factors that allow for asymmetric and tail dependence. We use a model presented by Oh and Patton (2017) where Z_k follows *Skew t* and ε_i follows a Normal distribution. We will first evaluate the Gaussian Copula and than evaluate a more the *Skew t* $-$ $N(0,1)$ model.

3.3.2 Dependence measures implied by the Copula

Even though most copulas do not have a closed form we can still simulate from their model and estimate the dependence measures of the data generated from the simulation, measuring therefore the dependence properties implied by the Factor Copula. The dependence measures we are using to characterize the dependence structure of the data, Spearman ρ, quantile dependence are functions of the copula (Oh and Patton, 2013). They are defined for the pair of scaled variables $(\eta_i, \eta_j$ as

$$\rho^{ij} = 12E\left[F_i(\eta_i)F_j(n_j)\right] - 3 = 12\iint uvC_{ij}(u,v) - 3, \tag{12}$$

$$\lambda_q^{ij} = \begin{cases} P[F_i(\eta_i) \leq q | F_j(\eta_j) \leq q] = \frac{C_{ij}(q,q)}{q}, & q \in (0, 0.5], \\ \\ P[F_i(\eta_i) > q | F_j(\eta_j) > q] = \frac{1-2q+C_{ij}(q,q)}{1-q}, & q \in [0.5, 1), \end{cases} \tag{13}$$

When it comes to tail dependence we can under certain regularities still have have an analytical formula for the tail dependence measures of a copula for which we have no closed formula. This very interesting result is presented by Oh and Patton (2017). Let the number of common factor be $K = 1$, F_Z and F_ε have a common tail index $\alpha > 0$. This means that

$$\begin{aligned} Pr\left[Z > s\right] \sim A_z^U s^{-\alpha} \; and \; Pr\left[\varepsilon_i > s\right] \sim A_\varepsilon^U s^{-\alpha}, \; as \; s \to \infty \\ Pr\left[Z < -s\right] \sim A_z^L s^{-\alpha} \; and \; Pr\left[\varepsilon_i < -s\right] \sim A_\varepsilon^L s^{-\alpha}, \; as \; s \to \infty \end{aligned} \tag{14}$$

where $A_z^L, A_z^U, A_\varepsilon^L,$ and A_ε^U are positive constants, $x_s \sim y_s$ if $\frac{x_s}{y_s} \to 1$ as $s \to \infty$, the tail index characterizes how fast the pdf of a random variables converges to 0 when $x \to \infty$. Then (a) if $\beta_i, \beta_j > 0$ the lower and upper tail coefficient can analytically be derived by

$$\tau_{ij}^{L} = \frac{\text{minn}(\beta_i, \beta_j)^{\alpha} A_z^L}{\text{minn}(\beta_i, \beta_j)^{\alpha} A_z^L + A_\varepsilon^L},$$
$$\tau_{ij}^{U} = \frac{\text{minn}(\beta_i, \beta_j)^{\alpha} A_z^U}{\text{minn}(\beta_i, \beta_j)^{\alpha} A_z^U + A_\varepsilon^U}, \tag{15}$$

(b) if $\beta_i, \beta_j < 0$ the lower and upper tail dependence coefficients are :

$$\tau_{ij}^{L} = \frac{\text{minn}(|\beta_i|, |\beta_j|)^{\alpha} A_z^U}{\text{minn}(|\beta_i|, |\beta_j|)^{\alpha} A_z^U + A_\varepsilon^L},$$
$$\tau_{ij}^{U} = \frac{\text{minn}(|\beta_i|, |\beta_j|)^{\alpha} A_z^L}{\text{minn}(|\beta_i|, |\beta_j|)^{\alpha} A_z^L + A_\varepsilon^U}, \tag{16}$$

(c) if $\beta_i\beta_j = 0$ or if $\beta_i\beta_j < 0$ than the lower and the upper and lower coefficients are equal to 0. We can see that if the latent factor loadings have the same sign , the factors probability distributions have the same tail index than the idiosyncratic variables than the model for the factor copulas implies a model for the tail dependence measures. This finding is not necessarily true for all Copula models, certain copula models might not have an implied model for tail dependence. It therefore adds to the appeal of a Latent Factor Model for the Copula.

4 Estimation

4.1 Simulation Method of Moments

To estimate the parameters of the model in Equation 11 we use the Simulated Method of Moments presented by Oh and Patton (2013). Since financial variables often have a time varying conditional mean and volatility the first step will be to filter the data following the process described in Section 2. $\eta = [\eta_1, ..., \eta_n]$ is the set of standardized residuals we obtain after filtering $[Y_1, ..., Y_n]$, the dependence between the standardized residuals is the same as the one between the original data . The initial approach would be to compute pairwise the dependence measures of (η_i, η_j). The dependence measures are our moment conditions. The choice of dependence measure is flexible, the unique requirement is that they need to be robust to data transformations. Added to that we should choose dependence measures that give an exhaustive characterization of the dependence of the data. That is they should describe dependence around the mean and also dependence in the tails.

Let $\hat{\mathbf{m}}_T$ be the vector of dependence measures of the data. We simulate data from $\mathbf{F}_x(\theta)$ and we measure the same dependence measures for the simulated data, $\hat{\mathbf{m}}_S(\theta)$ is the vector of dependence measures. The SMM estimator is defined as

$$\hat{\theta}_{T,S} \equiv \arg\min_{\theta \in \Theta} Q_{T,S}(\theta), \tag{17}$$

where

$$
\begin{aligned}
Q_{S,T}(\theta) &\equiv \mathbf{g}'_{T,S}(\theta)\hat{\mathbf{W}}_T\mathbf{g}_{T,S}(\theta), \\
\mathbf{g}_{T,S}(\theta) &= \hat{\mathbf{m}}_T - \hat{\mathbf{m}}_S(\theta),
\end{aligned}
$$

where $\hat{\mathbf{W}}_T$ is a positive definite weight matrix that can depend on the data. It can give a different weight to each dependence measure and can be chosen to improve the standard error standard error of the estimates. A simulation study conducted by Oh and Patton (2013) suggest that choosing an optimized weight matrix is not worth the corresponding trade off in computational time. We therefore in this paper will choose the identity matrix as a weight matrix. Oh and Patton (2013) show that if $S/T \to \infty$ as $T \to \infty$, the SMM estimator is consistent and asymptotically normal:

$$
\sqrt{T}(\hat{\theta}_{T,S} - \theta_0) \to N(0, \Omega_0) \; as \; T \; , \; S \to \infty, \tag{18}
$$

where

$$
\Omega_0 = (\mathbf{G}'_0\mathbf{W}_0\mathbf{G}_0)^{-1}\mathbf{G}'_0\mathbf{W}_0\Sigma_0\mathbf{W}_0\mathbf{G}_0(\mathbf{G}'_0\mathbf{W}_0\mathbf{G}_0)^{-1}. \tag{19}
$$

We will go through the estimation of each element of the matrix Ω. $\mathbf{W}_0$ is the weight matrix, in our case we will use the identity matrix. To estimate Σ_0 we (i) sample with replacement from η creating B iid bootstrap samples $\eta^{(b)}$, (ii) we calculate the bootstrap samples moment conditions (our list dependence measures) $\hat{\mathbf{m}}_T^{(b)}$. (iii) we calculate he sample covariance matrix with

$$
\hat{\Sigma}_{T,S} = \frac{T}{B}\sum_{b=1}^{B}(\hat{\mathbf{m}}_T^{(b)} - \hat{\mathbf{m}}_T)(\hat{\mathbf{m}}_T^{(b)} - \hat{\mathbf{m}}_T)'. \tag{20}
$$

The final element that we need to estimate from the matrix Ω_0 is $\mathbf{G}_0$. $\mathbf{G}_0$ is the derivative of $\mathbf{g}_{T,S}$ at $\hat{\theta}_{T,S}$, we can use standard numerical derivation techniques. The matrix $\mathbf{G}$ has the same number of columns as the number of parameters in θ with the k_{th} column

$$
\hat{\mathbf{G}}_{T,S,k} = \frac{\mathbf{g}_{T,S}(\hat{\theta} + \mathbf{e}_k\varepsilon_{T,S}) + \mathbf{g}_{T,S}(\hat{\theta} - \mathbf{e}_k\varepsilon_{T,S})}{2\varepsilon_{T,S}} \qquad k = 1, ..., p \tag{21}
$$

where p is the number of parameters in the copula, $\mathbf{e}_k$ is the k_th unit vector and ε is the step-size. Combining these results we can now have an estimate for Ω_0,

$$
\hat{\Omega}_{T,S,B} = (\hat{\mathbf{G}}'_{T,S}\hat{\mathbf{W}}_T\hat{\mathbf{G}}_{T,S})^{-1}\hat{\mathbf{G}}'_{T,S}\hat{\mathbf{W}}_T\Sigma_{T,B}\hat{\mathbf{W}}_T\mathbf{G}_{T,S}(\hat{\mathbf{G}}'_{T,S}\hat{\mathbf{W}}_T\hat{\mathbf{G}}_{T,S})^{-1}. \tag{22}
$$

4.2 Dependence measures

4.2.1 Rank Correlation

To measure the Spearman ρ we use the following non-parametric estimate :

$$
\rho^{ij} = \frac{12}{T}\sum_{t=1}^{T}\hat{F}_i(\hat{\eta}_{it})\hat{F}_j(\hat{\eta}_{jt}) - 3, \tag{23}
$$

$$\tau^{ij} = \frac{4}{T}\sum_{t=1}^{T}\hat{C}_{ij}(\hat{F}_i(\hat{\eta}_{it}), \hat{F}_j(\hat{\eta}_{ij})) - 1, \tag{24}$$

where

$$\hat{F}_i(y) = \frac{\sum_{t=1}^{T} 1\{\hat{\eta}_{it} \leq y\}}{T+1}.$$

4.2.2 Tail dependence measures

For the quantile dependence we use the following estimators ,

$$\hat{\lambda}_q^{ij} = \begin{cases} \frac{1}{Tq}\sum_{t=1}^{T} 1\{\hat{F}_i(\hat{\eta}_{it}) \leq q, \hat{F}_j(\hat{\eta}_{ij}) \leq q\}, & q \in (0, 0.5], \\ \\ \frac{1}{T(1-q)}\sum_{t=1}^{T} 1\{\hat{F}_i(\hat{\eta}_{it}) > q, \hat{F}_j(\hat{\eta}_{ij}) > q\}, & q \in [0.5, 1), \end{cases} \tag{25}$$

To estimate the spectral measure between $(\hat{\eta}_i, \hat{\eta}_j)$ we start by constructing the pairs

$$\hat{S} = \hat{\eta}_i + \hat{\eta}_j, \qquad \hat{W} = \hat{\eta}_i/(\hat{\eta}_i + \hat{\eta}_j). \tag{26}$$

We than use the measure proposed by De Carvalho et al. (2013) ,

$$\widehat{H}^{(i,j)}(x) = \sum_{t \in I_k} \widehat{p}_t I\{\widehat{W}_t^{()} \leq x\}, \quad x \in [0, 1], \tag{27}$$

where $\widehat{p}_i$ the weights given to $\widehat{W}_i$. Several weights exist in literature but De Carvalho et al. (2013) propose $\widehat{p}_i$ is defined as

$$p_i = \frac{1 - (\bar{W} - 1/2)S_W^{-2}(W_i - \bar{W})}{T}, \quad i \in I_T, \tag{28}$$

where W and S_W^2 denote the sample mean and the sample variance of W_i respectively.

5 Results

5.1 Simulation study

Before estimating the parameters of our Copula models an important step is to start by conducting a simulation study. This simulation study is crucial in seeing if our estimation method provides adequate results. The idea is that we simulate data from a known data generating process and then apply our estimation method to see if we manage to estimate the parameters correctly. This step is crucial to assess the efficiency and accuracy of our estimation model. We should therefore conduct a simulation study on all the models that we will use on our data later. We start with the simplest model ,

$$\begin{aligned} X_i &= \beta Z + \varepsilon_i, \quad i = 1, 2, ..., N \\ \text{Where} \quad & \varepsilon_i \sim \text{iid } F_\varepsilon(\gamma_\varepsilon) \\ & Z \sim \text{inid } F_z(\gamma), \end{aligned} \tag{29}$$

equidependence model. The equidependence model is when the factor loading β is the same for all variables, this is a very restrictive assumption that we will relax later in our study. Our first equidependence model has two variants. The first one with $Z \sim N(0,1)$ and $\varepsilon_i \sim N(0,1)$. The second model is with $Z \sim skewt(\upsilon, \lambda)$. For the skew t Factor model, we estimate υ^{-1}. The true parameters $\Theta = [\beta, \upsilon^{-1}, \lambda] = [1, 0.2, -2.5]$. We start by generating a sample of $T = 1000$ and estimate the parameters with a simulation sample size of$S = 25 * T$ .The moment conditions that we use are Spearman ρ, upper and lower quantile tail dependence. For the tail dependence measures the quantile we use is $q = 0.1$,$q = 0.05$, $q = 0.9$ and $q = 0.95$ for the lower and upper tail respectively. Throughout the simulation study we use $N = 10$ variables since that is the relevant size for our data later on. We run the simulation 100 times and obtain the following results. Results are shown in Table 2.

Table 2: Estimation results for multi-factor

	Normal		Skew t	
	Bias	Std Dev	Bias	Std Dev
υ^{-1}	-	-	0.0248	0.0891
λ	-	-	-0.0043	0.0521
β	-0.0054	0.0397	0.0026	0.0152

Note : The moment conditions are Spearman ρ, the quantile dependence for q = 0.05, 0.1, 0.9, 0.95.

Table 2 shows that the estimates converge to the true parameter values.Few things nonetheless need to be handled with caution with this simulation technique. The objective function $Q_{T,S}(\theta)$ is not smooth, and we need therefore to use a numerical optimization technique. We used the fminsearch function of Matlab. Since the simulation includes draw from a random variable, it is essential to keep the seed of the random number generator constant across the simulation. If we do not keep the seed constant, the simulation will not converge. The objective function in Equation 17 will be very unstable and produce a very unreliable minimum. With a large number of parameters, fminsearch will not even be able to minimize Equation 17. This is because the fminsearch has a termination tolerance for the function value of 0.0001, that is this is the lower bound on the change in the value of the objective function during a step. If we do not keep the same seed, the function will be very jittery, and even around the true minimum it will have a function value change greater than 0.0001, therefore never reaching a minimum even with a substantial number of iterations.

Another concern that is worth taking close care to is the fact that the objective function is not smooth. Therefore the algorithm runs the risk to get trapped into a local minimum. To remedy this, we need to run the minimization several times with several starting points and choose the solution that leads to the smallest local minimum giving us the global minimum. For the case of a model with one parameter is not an issue at all since the objective function is rather smooth (we have many dependence measures) and the minimization is very fast (we have only one parameter). On the other hand as the number of parameters increase, this becomes a considerable drawback to the estimation method. Because the number of starting points to reasonably cover the solution region will grow exponentially and added to that the minimization algorithm also becomes slower. To handle this issue of numerical optimization we start with preliminary work, we started with a large number of starting points and recorded the function value of the closest minimum to these points. This process was repeated several times until we had an idea where is the region around which the objective function value is the closest to zero, we know that the optimal solution should lead a function value close to

zero. Once that groundwork is done we start the simulation choose the starting points of fminsearch to be around the region that we determined in the preliminary work. This gives us a reasonable trade-off between the speed and the reliability of the optimization method.
It is also important to see how our simulation performs for more complicated problems than the one with only one parameter. We will look at three more settings that relax the equidependence restriction. The multi-factor model becomes as as follow

$$\begin{aligned} & X_i = \beta_{I(i)} Z_0 + \gamma_{I(i)} Z_{I(i)} + \varepsilon_i, \quad i = 1..10 \\ \text{Where} \quad & \varepsilon_i \sim \text{iid } F_\varepsilon(\gamma_\varepsilon) \\ & Z \sim \text{inid } F_z(\gamma). \end{aligned} \tag{30}$$

Here divide our 10 variables into group 1 and group 2. We still have a common factor Z_o but we allow variables to have a different factor loading $\beta_{I(i)}$ depending on their group. $\beta_{I(i)}$ indicates if a stock belongs to group 1 or group 2. Added to that we force a group equidependence with a group common factor $Z_{I(i)}$. The intuition behind this setting is that we believe that the variables dependence can be split into a dependence stemming from a common market factor and a dependence stemming from the fact that they belong to the same group. An obvious example of a group is if they belong to the same industry. Here again we consider two cases for the distribution of the common factors and the idiosyncratic variable. The first case we take $Z_o \sim N(0,1)$, $Z_{I(i)} \sim N(0,1)$ and the idiosyncratic variable $\varepsilon_i \sim N(0,1)$, this model does not allow for asymetric dependence. A second case is with $Z_o \sim \mathit{Skewt}(\upsilon^{-1},\lambda)$, $Z_{I(i)} \sim N(0,1)$ and the idiosyncratic variable $\varepsilon_i \sim N(0,1)$, this model allows for asynmetric dependence only in the market common factor. For the first two models we use spearman ρ , upper and lower tail dependence as our moment conditon. The quantiles used for the tail dependence are $q = 0.05, 0.1, 0.90, 0.95$. Finally we do a last simulation similar to the $\mathit{Skewt}(\upsilon^{-1},\lambda)$ mentioned earlier except that this time we try different moment conditions. We use the spearman ρ and a spectral measure moment. The spectral measure is defined for a sum of variables being large, so we will use only the data points for which the sum is considered large. We consider the sum of data to be large if it is past k = 1.2. The choice of this threshold k is a compromise done between how far we push off in the tails and how much data will remain for us to estimate our measures on. After trying several values of k on the actual data of S & P stocks that we will use later, we select 1.2, it is a sum that is large enough, and for all pairs of data we will have between 277 and 238 data points remaining after clearing up the pairs below k, this would be enough to construct a reliable Spectral measure. Another aspect that needs to be handled is the interpretation of the spectral density function in Equation 27. It can be interpreted as the probability distribution of the angle of the spectral measure, so the interpretation of quantile x is different from the interpretation of the quantile in the quantile tail dependence. We are interested in the area where w_i is around 0.5. Therefore we construct the following measure ,

$$\hat{\zeta}_{ij} = \hat{H}^{(i,j)}(0.5-q) - \hat{H}^{(i,j)}(0.5+q), \tag{31}$$

where ζ_{ij} is the spectral moment condition between η_i^* and η_j^*. $\hat{\zeta}_{ij}$ is the probability that $0.5-q < \frac{\eta_i^*}{\eta_i^*+\eta_j^*} < 0.5+q$. Since the interval is symmetric around 0.5 it is clear that $\hat{\zeta}_{ij} = \hat{\zeta}_{ji}$. If a variable η_i^* contribution to the sum of $\eta_i^*+\eta_j^*$ is between 0.5-q and 0.5 + q it directly implies that the contribution of η_j^* to the sum is also between 0.5-q and 0.5 + q. We use in this study q = 0.1 and q = 0.2. The main intuition of this measure is to try an recreate a moment condition that has a similar interpretation of the quantile tail dependence and see if it can be used as an alternative to the quantile tail dependence. The spectral moment condition procedure just described above will describe only upper tail dependence. To describe lower tail dependence we use the data negative log returns, therefore the lower tail becomes the upper tail and we do the same procedure again on the transformed data. The results of the simulation are shown in Table 3.

Table 3: Simulaion results for factor copula

	Normal		Skew t		Skew t	
	Bias	Stdv	Bias	Stdv	Bias	Stdv
υ^{-1}	-	-	0.0191	0.0219	0.0121	0.0265
λ	-	-	-0.0140	0.1291	-0.0045	0.0185
β_1	-0.0072	0.1402	0.0042	0.0977	0.0073	0.0641
β_2	0.0119	0.0438	0.0061	0.0184	0.0052	0.0107
γ_1	-0.0006	0.0788	0.0135	0.0249	-0.0272	0.1824
γ_2	0.0184	0.1126	0.0081	0.0101	0.0178	0.0272

Note : The model on the left has moment conditions as Spearman ρ and spectral measures with q = 0.5 and q = 0.1. The other models have moment conditions as Spearman ρ and quantile dependence with q = 0.05, 0.1, 0.90, 0.95.

We see in Table 3 that the estimation model still produces satisfactory results. The parameter estimates converge towards the true value. If we compare the *Skewt* model with the spectral measure and the *Skewt* model with the tail dependence we see that they give similar results. It is to note though that in general the standard deviation of the model with the Spectral measure is higher than the one of the model. It is to note that λ a strange exception to this. The model with the spectral measure moment condition seems to estimate λ much more precisely. There is no clear explanation as to why that might be the case.

5.2 Dependence measures

5.2.1 Rank Correlation

Table 4 show the results for the Spearman correlations of our data.

Table 4: Spearman Correlation

	APPL	AMZN	FB	GOOG	GOOGL	JNJ	JPM	MSFT	WFC	XOM
AAPL	1.0000	0.4258	0.4475	0.4745	0.4653	0.2669	0.3939	0.4952	0.3214	0.3140
AMZN	0.4258	1.0000	0.5891	0.6414	0.6467	0.3041	0.3031	0.5543	0.2667	0.2257
FB	0.4475	0.5891	1.0000	0.6461	0.6460	0.3058	0.2986	0.4894	0.2527	0.2255
GOOG	0.4745	0.6414	0.6461	1.0000	0.9509	0.3872	0.3688	0.5863	0.3485	0.2632
GOOGL	0.4653	0.6467	0.6460	0.9509	1.0000	0.3953	0.3546	0.5822	0.3424	0.2706
JNJ	0.2669	0.3041	0.3058	0.3872	0.3953	1.0000	0.3418	0.3629	0.3118	0.3241
JPM	0.3939	0.3031	0.2986	0.3688	0.3546	0.3418	1.0000	0.4533	0.7616	0.4744
MSFT	0.4952	0.5543	0.4894	0.5863	0.5822	0.3629	0.4533	1.0000	0.4023	0.3436
WFC	0.3214	0.2667	0.2527	0.3485	0.3424	0.3118	0.7616	0.4023	1.0000	0.4140
XOM	0.3140	0.2257	0.2255	0.2632	0.2706	0.3241	0.4744	0.3436	0.4140	1.0000

The rank correlation between financial variables depicts an expected picture. Companies from the same industries have a high rank-correlation and companies from different sectors have a lower rank correlation. An extreme case is the rank-correlation between Google and Google Alphabet that is at 0.9596. The companies belong to the same structure. On the other hand, we see that for Wells Fargo and Amazon the correlation is much lower at 0.2667.

5.2.2 Tail dependence

5.2.3 Quantile dependence

Table 5 and 6 show the results of the quantile dependence of pairs of the data. The quantile we use for the upper tail is 0.9 and for the lower tail is 0.1 in Table 5 and $q = 0.05, 0.95$ in Table 6. The motivation behind this choice is that we need to find a compromise between how far off in the tails we set the threshold and the amount of data we have.

Table 5: Lower tail / Upper tail

	APPL	AMZN	FB	GOOG	GOOGL	JNJ	JPM	MSFT	WFC	XOM
AAPL	1.0000	0.3217	0.3497	0.3497	0.3077	0.2378	0.2657	0.3217	0.2378	0.2797
AMZN	0.4755	1.0000	0.4895	0.4615	0.4615	0.2098	0.2797	0.4476	0.1958	0.3357
FB	0.4755	0.5874	1.0000	0.5175	0.5175	0.2517	0.3077	0.3916	0.2657	0.3357
GOOG	0.4895	0.6014	0.6434	1.0000	0.8671	0.3217	0.3776	0.5175	0.3077	0.2937
GOOGL	0.4336	0.5874	0.5734	0.8252	1.0000	0.2937	0.3776	0.5035	0.2797	0.3217
JNJ	0.3077	0.3077	0.3357	0.3916	0.4196	1.0000	0.3636	0.2797	0.3357	0.2937
JPM	0.4056	0.4056	0.3636	0.3916	0.3636	0.4056	1.0000	0.3916	0.6154	0.3776
MSFT	0.4336	0.5035	0.5315	0.5035	0.5035	0.3636	0.3916	1.0000	0.3217	0.3077
WFC	0.3916	0.3916	0.3357	0.4056	0.3636	0.3497	0.6573	0.3916	1.0000	0.3077
XOM	0.3217	0.3217	0.2797	0.3357	0.3357	0.3357	0.4056	0.3357	0.3916	1.0000

Table 6: Lower tail / Upper tail

	APPL	AMZN	FB	GOOG	GOOGL	JNJ	JPM	MSFT	WFC	XOM
AAPL	1.0000	0.3357	0.2517	0.3357	0.2797	0.1958	0.2517	0.3636	0.2238	0.1119
AMZN	0.4476	1.0000	0.4196	0.4476	0.4196	0.1399	0.2517	0.3077	0.1958	0.1678
FB	0.4196	0.5035	1.0000	0.4755	0.3636	0.1399	0.1958	0.3357	0.1399	0.3357
GOOG	0.3636	0.4476	0.4476	1.0000	0.8392	0.1678	0.2517	0.4476	0.1678	0.2238
GOOGL	0.3357	0.4196	0.3916	0.7552	1.0000	0.1399	0.2797	0.3916	0.1678	0.1678
JNJ	0.2238	0.3077	0.3077	0.3357	0.3357	1.0000	0.2797	0.2797	0.2517	0.2238
JPM	0.2238	0.3916	0.3077	0.2797	0.2238	0.3077	1.0000	0.3357	0.5594	0.2238
MSFT	0.3357	0.4196	0.3357	0.4196	0.3916	0.3636	0.3916	1.0000	0.2797	0.2238
WFC	0.3077	0.4476	0.3636	0.3916	0.3357	0.3357	0.5035	0.4196	1.0000	0.1678
XOM	0.2517	0.3357	0.1958	0.2238	0.2238	0.2517	0.2797	0.2517	0.3077	1.0000

It is clear from the results that the upper tail is always lower that the lower tail. This highlights a commonly known fact, stocks are more dependent in crashes that in tails. If we take for example the pair that exhibited the lowest rank correlation , Facebook and Wells Fargo. We notice that they still exhibit a low Upper Tail dependence of 0.266 but have a much higher Lower Tail dependence of 0.336 in Table 5. While they seem to be unrelated in normal times and during booms we see that they have a stronger dependence during crashes. Findings like this are at the core of the importance of tail dependence analysis.

5.2.4 Spectral measure

Before making the moment conditions for the Spectral measure, we start by constructing the pairs $\hat{S}$ and $\hat{W}$ and plotting them. This gives an initial intuition about the extremal dependence captured by the spectral measures. We put our variables on the Unit-Frechet scale to have a better visibility. The graphs below show the scatter plots of the pairs $\hat{S}$ and $\hat{W}$. For the visibility, we plot log(S) instead of S.

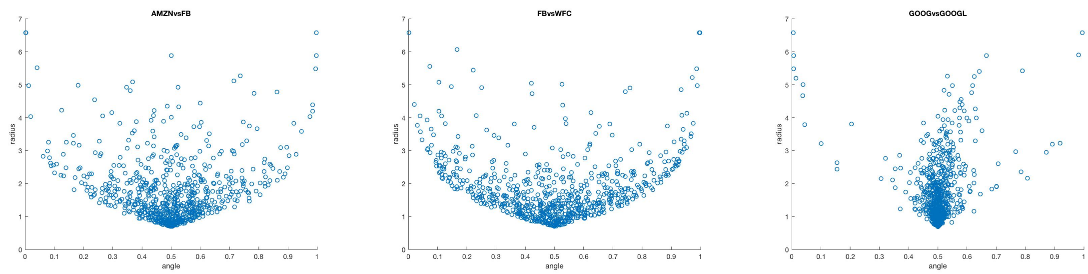

Figure 2: Scatter plot of radius and polar Spectral coordinates in Lower Tail. The sum of returns is plotted on the y-axes on the exponential scale and the percentage contributed by one factor on the x-axis

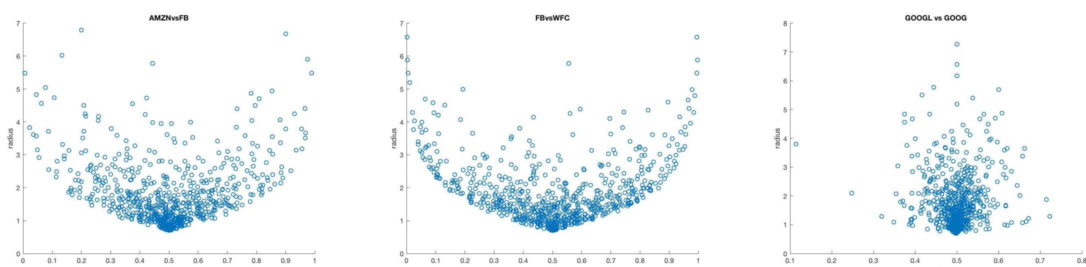

Figure 3: Scatter plot of radius and polar Spectral coordinates in Upper Tail. The sum of returns is plotted on the y-axes on the exponential scale and the percentage contributed by one factor on the x-axis

First of all the plot of the pair Google Alphabet and Google is to illustrate a case of very high dependence. We see indeed that observations are heavily concentrated around 0.5. For the other two pairs, the evidence is not overwhelming, but we can still see that Facebook seem to be more dependent in the tail with Amazon than with Facebook. Furthermore, we still see that dependence in the lower tail is higher than dependence in the upper tail. This result was first elucidated by the quantile tail dependence measures.

The above analysis gives us a visual intuition but we need to generate spectral measures that will be our moment conditions in Equation 17. We construct the moment condition following the procedure described in Section 5.1. Table 7 and 8 show the results for q = 10 and q = 20.

Table 7: Lower tail / Upper Tail

	APPL	AMZN	FB	GOOG	GOOGL	JNJ	JPM	MSFT	WFC	XOM
AAPL	1.0000	0.5121	0.5341	0.5847	0.5691	0.4777	0.5082	0.5512	0.5021	0.4435
AMZN	0.6091	1.0000	0.5946	0.5911	0.6023	0.4531	0.4733	0.5259	0.4333	0.4670
FB	0.5847	0.6900	1.0000	0.6602	0.6453	0.4855	0.4812	0.5598	0.4890	0.5022
GOOG	0.5911	0.6914	0.7016	1.0000	0.9683	0.5591	0.5152	0.5670	0.5081	0.4408
GOOGL	0.5823	0.6751	0.6795	0.9181	1.0000	0.5476	0.4854	0.5769	0.4960	0.4321
JNJ	0.4257	0.5228	0.5021	0.5267	0.5369	1.0000	0.5120	0.5020	0.4609	0.4322
JPM	0.5142	0.4750	0.4754	0.5375	0.5630	0.4776	1.0000	0.5451	0.7409	0.4830
MSFT	0.5547	0.5878	0.5953	0.6053	0.6374	0.5079	0.5569	1.0000	0.5000	0.4486
WFC	0.5060	0.5129	0.4936	0.5353	0.5294	0.4792	0.6882	0.5100	1.0000	0.4487
XOM	0.5145	0.5351	0.4851	0.5263	0.5152	0.4032	0.5361	0.5223	0.5287	1.0000

Table 8: Lower tail / Upper tail

	APPL	AMZN	FB	GOOG	GOOGL	JNJ	JPM	MSFT	WFC	XOM
AAPL	1.0000	0.7661	0.7631	0.7903	0.8008	0.6923	0.7500	0.8110	0.7280	0.7056
AMZN	0.7942	1.0000	0.8649	0.8439	0.8409	0.7306	0.7284	0.7889	0.7125	0.7181
FB	0.8185	0.8339	1.0000	0.8687	0.8830	0.7095	0.6987	0.7915	0.7048	0.7249
GOOG	0.8097	0.8476	0.8992	1.0000	0.9930	0.7559	0.7749	0.8697	0.7561	0.6939
GOOGL	0.8032	0.8592	0.8842	0.9488	1.0000	0.7738	0.7531	0.8115	0.7339	0.6790
JNJ	0.6506	0.7386	0.7597	0.7531	0.7787	1.0000	0.7160	0.7331	0.7243	0.7542
JPM	0.7571	0.7500	0.7090	0.7792	0.7605	0.7143	1.0000	0.7910	0.9197	0.7547
MSFT	0.7962	0.8282	0.7860	0.7970	0.8168	0.7421	0.7725	1.0000	0.7621	0.7325
WFC	0.7012	0.7241	0.7362	0.7593	0.7689	0.7458	0.8602	0.7331	1.0000	0.6958
XOM	0.7552	0.7368	0.7021	0.7588	0.7403	0.7075	0.7567	0.7449	0.7280	1.0000

The interpretation of the above results is that for example if the sum of the scaled returns of Amazon an Apple is above 1.2 than Apple's share of that sum is between 0.45 and 0.55 with probability 0.512.

5.3 Equidependence models

We start our analysis with the simplest model, the equidependence model with one common factor Z. We impose that all the stocks have the same factor loading. We assume that the idiosyncratic variable $\varepsilon_i \sim N(0,1)$ and consider two cases for the common factor. The first case where $Z \sim N(0,1)$, this copula model does not capture any tail dependence. The second case is with $Z \sim Skewt(\upsilon,\lambda)$, this model captures asymmetric tail dependence. The results are reported in Table 9.

Table 9: Estimation results for multi-factor

	Normal		Skew t	
	Est	Std Err	Est	Std Err
υ^{-1}	-	-	0.0274	0.0145
λ	-	-	-0.3145	0.0980
β	1.0947	0.0975	1.0591	0.0672

Note : The moment conditions are Spearman ρ, the quantile dependence with q = 0.05, 0.1, 0.9, 0.95.

We can see that both models infer a coefficient of β around 1. The standard errors are computed using the result a step-size $\varepsilon_{T,S} = 0.1$ and a 100 bootstrap samples for $\Sigma_{T,S}$. The standard error estimation could be improved in two ways. Firstly an obvious possibility is to choose a larger value of S, the number of bootstrap iterations. The result in Equation 18 holds for large values of S. Another area where they could be improved is with the choice of the weight matrix W. We choose W as the identity matrix but if we instead choose $W = \Sigma^{-1}$ and plug it in Equation 19 we will have

$$\Omega_0 = (\mathbf{G}_0'\Sigma_0^{-1}\mathbf{G}_0)^{-1}\mathbf{G}_0'\Sigma_0^{-1}\mathbf{G}_0(\mathbf{G}_0'\Sigma_0^{-1}\mathbf{G}_0)^{-1}.$$

The matrix Ω_0 is shorter now. We can therefore build an estimator

$$\hat{\Omega}_{T,S,B} = (\hat{\mathbf{G}}_{T,S}'\Sigma_{T,B}^{-1}\hat{\mathbf{G}}_{T,S})^{-1}\hat{\mathbf{G}}_{T,S}'\Sigma_{T,B}^{-1}\mathbf{G}_{T,S}(\hat{\mathbf{G}}_{T,S}'\Sigma_{T,B}^{-1}\hat{\mathbf{G}}_{T,S})^{-1}.$$

These possible improvements were tried by Oh and Patton (2013) and did not considerably improve the results. We will therefore stick in this paper to the identity weight matrix.

5.4 Multi-Factor Copula model

In order to relax the very restrictive assumption of equidependence we propose two multi-factor copula models generated by the following model

$$X_i = \beta_{S(i)}Z_0 + \gamma_{S(i)}Z_{S(i)} + \varepsilon_i, \quad i = 1..10 \tag{32}$$

where $S(i) = 1$ if the stock of a technology company and $S(i) = 2$ otherwise. Z_0 is the market factor towards which each group of stocks can have a different loading. We also impose an equidependence in withing each group of stock through the group common factor Z_S, γ_S is the group common factor. For both models the idiosyncratic variable $\varepsilon_i \sim$ *iid* $N(0,1)$ and $Z_S \sim$ *iid* $N(0,1)$.In the first model $Z_0 \sim$ *iid* $N(0,1)$ while in the second model $Z_0 \sim$ *iid* $Skew(\upsilon, \lambda)$. The results are presented in Table 10.

Table 10: Estimation results for multi-factor

	Normal		Skew t	
	Est	Std Err	Est	Std Err
υ^{-1}	-		0.0561	0.00725
λ	-	-	-10.0639	2.1715
β_1	1.1401	0.0402	1.0705	0.0452
β_2	1.0678	0.0202	0.9751	0.0124
γ_1	0.5913	0.0377	0.5673	0.0094
γ_2	0.4732	0.0179	0.4639	0.0187

Note : The moment conditions are Spearman ρ, quantile dependence with q = 0.05, 0.1, 0.90, 0.95

For the non-tech stocks we see that both models give a market factor around 0.95 and an additional intra-group dependence of 0.51 and 0.4574 for the Normal and Skew t models respectively. For the stocks from the tech industry both models give quite different coefficients, but they both show an intra-group dependence coefficient much higher than for the group of non-tech companies. The results are in line with expectations, for group one the stocks are from the same industry, an important part of their dependence structure comes from that. For the second group the stocks are from heterogeneous industries. It is to note that the standard error of skewness parameter λ remains quite high.
Another model that we propose is a model where we change our moment conditions. Namely we take out the quantile dependence moment conditions and replace them with the Spectral moment conditions presented in Section 5.2.4. Results from this model are presented in Table 11.

Table 11: Estimation results for multi-factor

	Est	Std Err
υ^{-1}	0.0643	0.0129
λ	-8.4783	2.4673
β_1	1.1355	0.0906
β_2	0.9831	0.0137
γ_1	0.6137	0.0176
γ_2	0.3989	0.0165

Note : The moment conditions are Spearman ρ, and the Spectral measures with q = 0.1, 0.2.

We see that the results above are similar to ones in Table 10. We can nonetheless see that the result of table 11 seem to have substantially higher standard errors than the result of Table 11. This result was unexpected since during the simulation study we did have a significant difference in standard error between the two models. This difference nonetheless suggests that the objective function with the spectral moment condition might be less smooth. Nonetheless, drawing such a conclusion from the table above might be far fetched.

6 Discussion

This research tackles a very crucial topic, modeling dependence between financial variables in high dimensions. It is an essential area for our understanding and assessment of the risk associated with a portfolio, yet not a lot of progress has been made, the task is challenging. We follow the works of Oh and Patton (2017). This work is novel and was presented only in 2017. It forms a substantial and very promising advance in the area of copula modeling. Oh and Patton (2017) offer an elegant and flexible way of model Factor Copulas using a Simulation Method of Moments. While the estimation method proves to be powerful, specific challenges remain and are worth a further clarification.

The biggest problem in this work was the computational cost that comes with the estimation method. The estimation method is very time-consuming. To be able to carry out the computation in a reasonable timeframe, a special effort was made in code optimization and added to that we used a Distributed Computing Engine called Techila. It integrates with Matlab and allows us to run our simulations on a Google Cloud server. For smaller models, we used servers with 16 cores and for larger ones with used servers that are up to 96 cores.

Another issue also that is of crucial importance is the numerical optimization of the objective function. The objective function is not smooth. Therefore, the fminsearch will return only a local minimum. The choice of starting point becomes than crucial. We need to choose a lot of starting points and take the smallest local minima. It is quite tricky to develop a reliable heuristic for the choice of starting points, too many starting points are computationally prohibitive, too little starting points make the results unreliable. For the Normal Copula in the equidependence case, the Objective function seemed to be somewhat smooth, and therefore the minimization result is quite reliable. But for the other models that have more parameters, unfortunately, we lose this smoothness property, especially for the Factor Skew t model. Hence the results produced by this study should be taken with a lot of care. Further analysis on a heuristic for choosing the starting points of the objective function is necessary to make the results more reliable.

7 Conclusion

Financial products in today's economy are becoming complex at an alarming rate. Part of the complexity of financial products is that we have very few copula models for high dimensions. It becomes challenging therefore to track the risk associated with a tremendous combination of stocks. This paper uses models proposed by Oh and Patton (2017) to characterize dependence in high-dimensions. We replicate parts of the methods of Oh and Patton (2017) for the top 10 stocks of the S&P 100. After this replication, we introduce a new moment condition to Oh and Patton (2013) based on the Spectral measure. We took out the quantile dependence and replaced it with the Spectral measure moment condition. The results stayed similar which suggests that the Spectral measures capture tail dependence just as well as quantile dependence.

For most factor copulas we do not have a closed-form representation. Therefore, the simulation method is very convenient to bypass this restriction. On the other hand, simulation-based estimation pauses its own set of challenges. For the parameters to converge, we need a large number of iterations, and that can be very expensive computationally. An essential aspect in that computational task is the calculation of moment conditions. On the other hand, the more we moment conditions, the more dependence information we have and therein the better the estimation. It will then be an interesting area of study to find the optimal compromise to do between moment conditions that cover all aspects of the data dependence and that jointly are the least computationally expensive.

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