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Modeling Dependence in High Dimensions:
A Factor Copula Approach

Bachelor Thesis submitted by

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In this paper we study the dependence structure, or copula, of a large set of economic variables. We employ recently proposed factor copula models, that offer a flexible, parsimonious approach to high-dimensional applications. We augment the factor copula models with Markov-switching dynamics to allow for time-varying dependence, resulting in a more realistic model specification. We apply the models to daily returns on 101 constituents of the S&P 100 equity index and find that a model that allows for non-normal features of dependence is needed to adequately model the dependence structure of the returns. Specifically, we show that the returns display heterogeneous dependence, tail dependence and asymmetric dependence. Furthermore, we find strong evidence of regime shifts in the dependence structure, where periods of high dependence seem to alternate with periods of substantially less dependence between the asset returns.

Keywords: Factor model, Copulas, Dependence, Tail dependence, Asymmetric dependence, Markov-switching.

1 Introduction

The dependence between asset returns plays a central role in assessing portfolio risk, and consequently in optimal portfolio allocation. For financial institutions holding a specific portfolio, assessing the portfolio risk is essential in constructing value-at-risk estimates for the portfolio. Hence, accurate modeling of the dependence between returns is of vital importance for risk management. The financial crisis of 2007 to 2008 showed that many models that tried to capture the dependence across numerous financial assets were inadequate, and the use of these poor models caused great problems for specific financial markets. However, when analyzing risk across a large number of variables, such as asset returns, the number of suitable models available is relatively limited. Moreover, the problem with most available models is that they are correlation-based and often assume a joint normal distribution for the variables. Due to this assumption, possible tail dependence of variables - in case of returns, that is, the tendency of stocks to crash simultaneously - is neglected. As a result, when investment decisions are based on these models the diversification benefits are often overestimated, as the risk of large correlated crashes is ignored. Furthermore, when normality is assumed the variables are assumed to have a symmetric dependence. For returns this means: ignoring the fact that correlations across assets tend to increase substantially during crashes compared to booms.

The central question in this paper is how to model the dependence structure - also known as the copula function - of economic variables in high dimensions, while taking into account the possibility of, among others, tail dependence and asymmetric dependence. In recent years, copula models have gained great popularity in financial econometrics and time series analysis, especially in high-dimensional applications. This is due to their ability to describe complicated dependence structures. More specifically, copulas allow us to separately model the marginal properties and the dependence structure of the data, resulting in more flexible and realistic modeling. The research in this papers builds on the research done by Oh and Patton (2017), that examined the use of factor copula models in a high-dimensional setting. In their paper they presented new flexible models for the dependence structure, or copula, of economic variables based on a latent factor structure. The use of this factor structure makes the factor copulas particularly attractive for relatively high-dimensional applications. Besides, the copulas allow for both tail dependence - non-zero probability of joint extreme events - and asymmetric dependence of asset returns. The models also allow for heterogeneous dependence - dependence can differ for each pair of variables - and interpretable constraints on dependence. They performed estimation of copulas of dimension as high as 100, using the simulation-based estimation method proposed by Oh and Patton (2013). This estimation method is closely related to simulated method of moments (SMM) estimation, though not strictly SMM, as the "moments" that are used in estimation are functions of rank statistics.

Several studies have already considered certain types of factor copulas. Krupskii and Joe (2013) and Nikoloulopoulos and Joe (2015) both investigated factor copula models where the factor structure is implied by the choice of copula linking each variable to the latent factor(s). "Structured" factor copula models were proposed by Krupskii and Joe (2015), suitable for modeling high-dimensional data consisting of several groups of variables with homogeneous dependence in each group. The structured multivariate copula models are built from a sequence of bivariate copulas. This is similar to the way vine copulas are constructed, see Aas et al. (2009), for example, where bivariate copulas are used to build up a higher-dimensional copula. Brechmann and Czado (2013) proposed a factor-vine copula model, referred to as the "regular vine market sector" model, that combines regular vine copulas with GARCH models for the marginal distribution.

We employ the factor copula model of Oh and Patton (2017) and propose to augment it by Markov-switching dynamics, inspired by the Markov switching model of Hamilton (1989), to allow for time-varying dependence between the variables. While the original Markov switching model focuses on mean behaviour of variables, a number of studies have considered Markov-switching copula models, see Chollete et al. (2009), Garcia and Tsafack (2011), and Stöber and Czado (2014), for example. However, most studies impose unrealistic restrictions on the dependence structure and few studies, if any, consider dimensions greater than ten. We aim to fill the gap in the existing literature by considering Markov-switching in the context of factor copula models, adding to the literature on time-varying copula models for high-dimensional applications.

We apply the models to daily returns of the individual constituents of the S&P 500 equity index over the period July 2015 - May 2018. We consider a market-wide common factor and find that industry-specific loadings on the common factor are needed to adequately model the dependence structure of the returns, implying heterogeneous dependence. Furthermore, we find evidence of a fat-tailed and left-skewed common factor, implying tail dependence and asymmetric dependence. The Markov-switching factor copula model reveals the presence of two regimes, with a distinct difference in dependence structure of the returns across the regimes. We show that the Markov-switching model is preferred to the factor copula model proposed by Oh and Patton (2017) for modeling the dependence between the daily asset returns, providing a substantially better fit. Combining the above, we empirically show that the daily returns on the S&P 100 constituents exhibit a substantial degree of heterogeneous dependence, tail dependence and asymmetric dependence, and find strong support for the hypothesis of time-varying dependence structure of the returns.

The remainder of this paper is organized as follows. Section 2 presents the class of factor copulas and their tail dependence properties, provides an illustration of the flexibility of the considered copulas and describes the simulated methods of moments estimation method we use. Section 3 presents the concept of Markov-switching in the context of factor copulas. In Section 4 we present a simulation study on the finite-sample properties of the employed moment-based estimation method. Section 5 presents an empirical study of daily returns on the S&P 100 index constituents over the period 2015 - 2018. Finally, Sections 6 and 7 conclude the paper, discuss our findings and provide suggestions for further research.

2 Factor Copulas

This section consists of four parts. In the first part we introduce the notion of copula functions and specify the class of copulas we consider in our analysis. The second part provides analytic results on tail dependence properties of the considered copulas, while the third part gives an illustration of the flexibility of the considered copulas. Lastly, part four discusses the employed estimation method and the associated measures of dependence.

2.1 Copulas of latent factor structure

Consider $\mathbf{Y} = [Y_1, \ldots, Y_N]$, an N-variate stochastic process with some joint distribution function $\mathbf{F}_y$. Let F_i denote the marginal distribution function of Y_i, and let f_i be the corresponding density function. The copula $\mathbf{C}$: $[0,1]^N \to [0,1]$ contains all information about the dependence structure of $\mathbf{Y}$, such that

$$[Y_1, \ldots, Y_N]' \equiv \mathbf{Y} \sim \mathbf{F}_y = \mathbf{C}(F_1(Y_1), \ldots, F_N(Y_N)). \tag{1}$$

The existence of the copula $\mathbf{C}$ is ensured by Sklar's Theorem (Sklar (1959)), that states that for every multivariate distribution there exists a copula with uniform margins that completely describes the dependence between the variables. In essence, we decompose the joint distribution into marginal distributions and a copula, and the copula can thus be understood as a multivariate distribution that combines $F_1, \ldots, F_N$ to $\mathbf{F}_y$. This decomposition allows the researcher to use existing models to estimate the marginal distributions, on the one hand, and on the other hand focus on constructing useful new models for the dependence structure or copula of the variables, which is a simpler task. As a result, a great deal of flexibility in modeling joint distributions is gained when considering copula-based models, compared to existing multivariate distributions. Furthermore, the decomposition of the joint distribution facilitates the study of high-dimensional problems, reducing the computational load by allowing for multi-stage estimation.

Following Oh and Patton (2017), we consider the class of copulas that can be generated by a simple linear, additive factor structure, based on a set of latent variables. Suppose we have K factors $Z_1, \ldots, Z_K$, and consider the vector of latent variables $\mathbf{X} = [X_1, \ldots, X_N]$, whose components are defined by

$$X_i = \sum_{k=1}^{K} \beta_{ik} Z_k + \varepsilon_i, \quad i = 1, 2, \ldots, N, \tag{2}$$

where $Z_k \overset{inid}{\sim} F_{Z_k}(\,\cdot\,;\gamma_k)$, $\varepsilon_i \overset{iid}{\sim} F_\varepsilon(\,\cdot\,;\gamma_\varepsilon)$ and Z_k is independent of ε_i for each i, k. Then $\mathbf{X}$ follows a K-factor model, and if we denote the corresponding copula by $\mathbf{C}_x$ we have

$$[X_1, \ldots, X_N]' \equiv \mathbf{X} \sim \mathbf{F}_x = \mathbf{C}_x(G_1(X_1), \ldots, G_N(X_N); \boldsymbol{\theta}_c). \tag{3}$$

Here, the parameter vector $\boldsymbol{\theta}_c$ contains all model parameters, or factor copula parameters. That is, $\boldsymbol{\theta}_c \equiv (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K, \gamma_1, \ldots, \gamma_K, \gamma_\varepsilon)$, where $\boldsymbol{\beta}_i$ is the N-dimensional vector of $\beta_{1k}, \ldots, \beta_{Nk}$, for $k = 1, \ldots, K$. $G_1, \ldots, G_N$ are the marginals corresponding to $\mathbf{F}_x$. $\mathbf{C}_x$, the copula of the latent variables $\mathbf{X}$, is used as the model for the copula $\mathbf{C}$ in Equation (1). Note that in general $F_i \neq G_i$, and the structure for the vector $\mathbf{X}$ is used only for its copula. We only impose a factor structure on the copula, achieving dimension-reduction in the component of the joint distribution that is difficult to estimate in high dimensions. The resulting marginal distributions are completely disregarded.

The factor structure provides us great flexibility, in the sense that we can freely choose the distributions for the common factor(s) and the idiosyncratic variables. For example, tail dependence can be captured by considering fat-tailed distributions. Skewed distributions for the factors and the idiosyncratic variables allow us to capture possible asymmetric dependence - the difference in dependence structure during "crashes" and during "booms". Besides, by increasing the number of factors we can add more flexibility to the model, while this also increases the computational burden. We will discuss such choices in the upcoming sections.

2.2 Tail dependence properties

To examine how a specific copula models the dependence between variables we can investigate the tail dependence of the variables implied by the copula. For this we rely on extreme value theory (EVT), as most factor copulas do not have a closed-form expression[1]. As shown by Oh and Patton (2017), using EVT we can analytically obtain tail dependence properties of a given factor copula model. For two variables X_i, X_j with marginal distributions G_i, G_j the tail dependence is defined as

$$\tau_{ij}^L = \lim_{q\downarrow 0} \frac{\Pr\Big[X_i \le G_i^{-1}(q), X_j \le G_j^{-1}(q)\Big]}{q}$$
$$\tau_{ij}^U = \lim_{q\uparrow 1} \frac{\Pr\Big[X_i > G_i^{-1}(q), X_j > G_j^{-1}(q)\Big]}{1-q},$$

where τ_{ij}^L is the lower tail dependence and τ_{ij}^U the upper tail dependence. Note that if $X_i \sim G_i$, $U_i \equiv G_i(X_i) \sim Unif(0,1)$. Hence, we can write the lower and upper tail dependence as

$$\tau_{ij}^L = \lim_{q\downarrow 0} \frac{\Pr[U_i \le q, U_j \le q]}{q} = \lim_{q\downarrow 0} \frac{\mathbf{C}_x(q,q)}{q}$$
$$\tau_{ij}^U = \lim_{q\uparrow 1} \frac{\Pr[U_i > q, U_j > q]}{1-q} = \lim_{q\uparrow 1} \frac{1-2q+\mathbf{C}_x(q,q)}{1-q}.$$

In case $\tau_{ij}^L = \tau_{ij}^U$, the copula $\mathbf{C}_x$ is said to be symmetric and does not capture asymmetric dependence between returns. We can also immediately examine whether the copula allows for nonzero tail dependence. Oh and Patton (2017) report tail dependence results for a single factor copula model, presented in Proposition 1 in Appendix A.

The empirical analysis in Section 5 will focus on the Skew t distribution of Hansen (1994) as a model for the common factor. The density function of this distribution is given by

$$g(z|\nu,\lambda) = \begin{cases} bc\Big(1 + \frac{1}{\nu-2}\big(\frac{bz+a}{1-\lambda}\big)^2\Big)^{-(\nu+1)/2}, & z < -a/b \\ bc\Big(1 + \frac{1}{\nu-2}\big(\frac{bz+a}{1+\lambda}\big)^2\Big)^{-(\nu+1)/2}, & z \ge -a/b \end{cases}$$

where

$$a = 4\lambda c\Big(\frac{\nu-2}{\nu-1}\Big), \; b = \sqrt{1+3\lambda^2-a^2},$$
$$c = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)}}, \tag{4}$$

and Γ denotes the gamma function. The Skew t distribution has zero mean and a variance of one. The density function is parameterized by the degree of freedom parameter $\nu \in (2,\infty)$, that determines the fatness of the tails, and the skewness parameter $\lambda \in (-1,1)$. For $\lambda = 0$ the density is symmetric, while for $\lambda < 0$ ($\lambda > 0$) the distribution displays negative (positive) skewness. We combine the Skew t distribution for the common factor with the standardized t distribution for the idiosyncratic variables[2], resulting in the Skew t-t factor copula. This factor copula is found by Oh and Patton (2017) to be a good choice for financial data. Proposition 2 in Appendix A presents expressions for the lower and upper tail coefficients for the Skew t-t factor copula, as verified by Oh and Patton (2017), that can be combined with Proposition 1 to determine the lower and upper tail dependence implied by the Skew t-t factor copula model. We will use the analytic results presented in the two propositions in our empirical analysis in Section 5.

2.3 Illustration of some factor copulas

As mentioned in Section 2.1, the factor structure that generates the class of copulas we consider provides us a lot of flexibility. To illustrate this flexibility, we examine random draws from bivariate distributions constructed using four different factor copulas, generated by Equation (3) with $K = 1$ common factor Z. For each copula Figure 1 presents 1000

[1] When $F_{Z_1}, \dots, F_{Z_K}, F_\epsilon$ in Equation (2) are all Gaussian distributions, the latent variable $\mathbf{X}$ is multivariate Gaussian, implying a Gaussian copula for which a closed-form expression is available.

[2] Note that the employed standardized t distribution also has a variance equal to one.

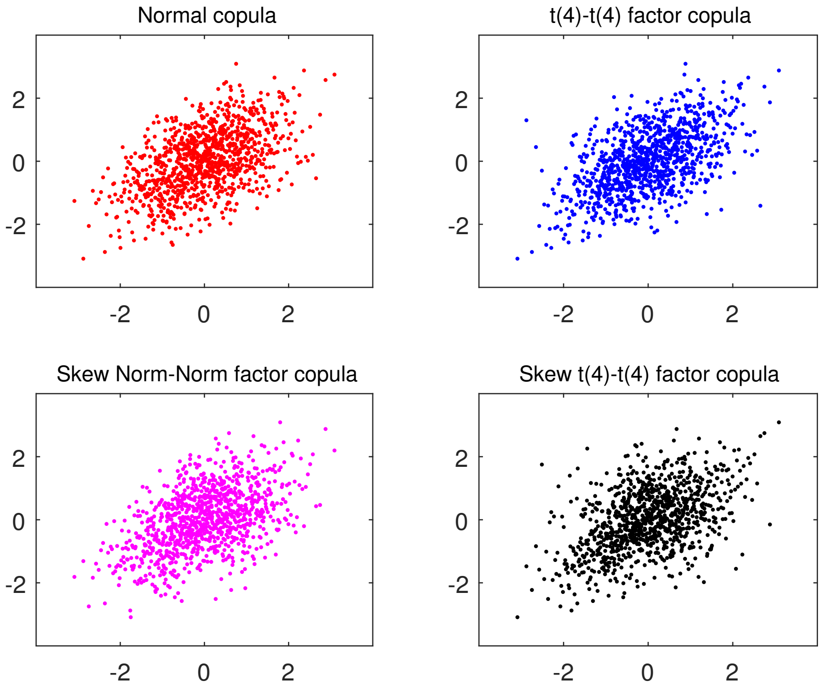

Figure 1: Scatterplots from four bivariate distributions, all with $N(0,1)$ margins and linear correlation of 0.5, constructed using four different factor copulas.

random draws, where the marginal distributions are set to $N(0,1)$. In all cases we set $\sigma_z^2 = \sigma_\varepsilon^2 = 1$, resulting in a linear correlation of 1/2 between the two variables of each of the bivariate distributions. The first copula is generated from a factor structure with $F_z = F_\varepsilon = N(0,1)$, implying a Gaussian copula, that does not allow for asymmetric dependence, with crashes being more correlated than booms, or tail dependence. For the second copula we consider $F_z = F_\varepsilon = t(4)$, again generating a symmetric copula, but now with positive tail dependence. The third copula sets F_z = Skew $t(\infty, -0.25)$, corresponding to a skewed Normal distribution, and $F_\varepsilon = N(0,1)$. The resulting skewed Normal-Normal factor copula exhibits asymmetric dependence, but imposes zero tail dependence. Lastly, we consider the Skew *t*-*t* factor copula as discussed in Section 2.2, with F_z = Skew $t(4, -0.25)$ and $F_\varepsilon = t(4)$. This copula generates both asymmetric dependence and positive tail dependence. Note that the Skew *t*-*t* factor copula nests the other three factor copulas. More specifically, the Normal factor copula is obtained by setting $\lambda = 0$ and $\nu \to \infty$, which corresponds to a Gaussian copula with correlation matrix implied by the factor structure. Setting $\lambda = 0$ and $\nu \not\to \infty$ yields the factor *t*-*t* copula, while the Skew *t*-*t* factor copula becomes the skewed Normal-Normal factor copula if $\lambda \neq 0$ and $\nu \to \infty$.

Figure 1 illustrates the difference in dependence structure of the variables across the four factor copulas mentioned above. In the figure we see that tail events tend to be uncorrelated across the two variables when the distributions in the factor structure are Normal or skewed Normal. For the second copula we observe several draws in the joint positive and negative quadrant, where the degrees of freedom is set to 4, indicating positive tail dependence. The lower two panels of Figure 1 show that if the skewness parameter is negative, the clustering of observations is stronger in the joint lower tail than in the joint upper tail. That is, we observe asymmetric dependence.

To further illustrate the tail dependence properties of the Skew *t*-*t* factor copula we implement a crash dependence measure proposed by Oh and Patton (2017). Conditional on observing j out of N stocks crashing, this measure examines the expected number $(\kappa^q(j))$ and the expected proportion $(\pi^q(j))$ of remaining $(N-j)$ stocks that will crash. If we define the number of crashes as $N_q^* \equiv \sum_{i=1}^N \mathbb{1}\{U_i \leq q\}$, then

$$\kappa^q(j) = E[N_q^* | N_q^* \geq j] - j \text{ and } \pi^q(j) \equiv \frac{\kappa^q(j)}{N-j}.$$

Here, we define a "crash" as a realization in the lower 1/66 tail, corresponding to a once-in-a-quarter event

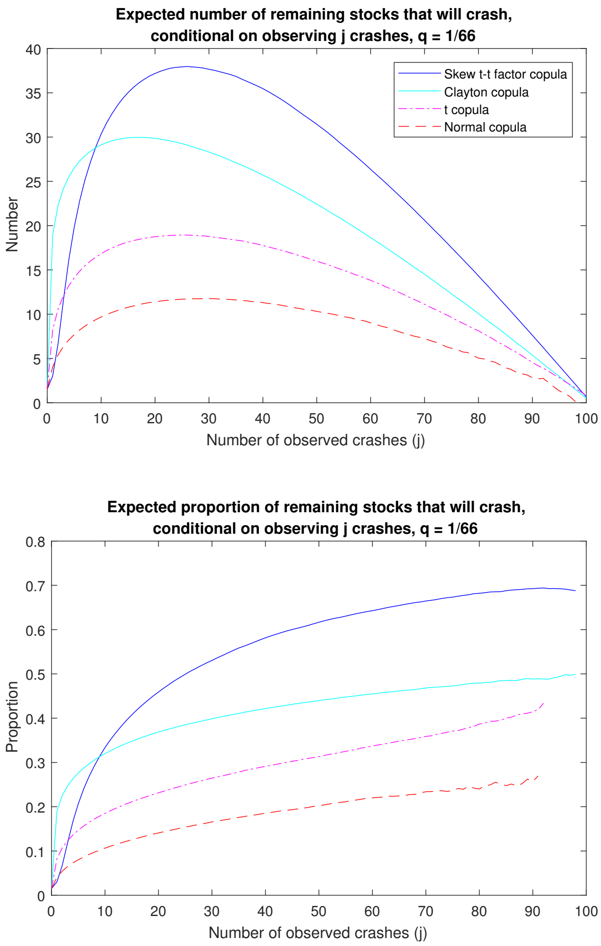

Figure 2: The expected number (upper panel) and proportion (lower panel) of remaining stocks that will crash, conditional on observing j out of 101 stocks crashing, under four different copulas. A "crash" is defined as a stock return in the lower 1/66 tail.

for daily asset returns. Figure 2 is obtained using 10 million simulations and illustrates the difference in implied crash dependence between the Skew t-t factor copula and three well-known, existing copulas in a multivariate framework, with $N = 101$ variables. Besides the Skew t-t copula we consider the Clayton, the Student's $t(4)$ and the Normal copula. The copula parameters are again chosen such that the implied linear correlation across the variables is 1/2, where we use 4 degrees of freedom for the Skew t-t factor copula. In the upper panel of Figure 2 we see that for each copula the expected number of remaining variables that will crash initially increases as we condition on more variables having crashed. Around $j = 25$ the conditional expected number of crashes peaks for all copulas, where the Skew t-t factor copula predicts that around 38 other variables will crash. This expected number of crashes around $j = 25$ is substantially larger than the expected number under the other three copulas. For example, the Normal copula only expects around 11 other variables to crash at this point. If we condition on more variables having crashed there will be fewer variables left to crash. As a result, for all copulas the conditional expected number of remaining variables that will crash decreases to zero after the peak. Looking at the lower panel of Figure 2 we see that under each copula the expected proportion of remaining stocks that will crash generally increases up to $j = 100$[3]. The features of dependence illustrated by this figure are unique to high-dimensional applications, providing further motivation for the class of flexible, parsimonious factor models we consider.

2.4 Simulated method of moments

Similar to Chen and Fan (2006) and Rémillard (2010), among others, we consider a class of data-generating processes (DGPs) that allows each variable to have time-varying conditional mean and conditional variance, governed by parametric models. Further, we assume each variable to have an unknown marginal distribution, estimated using the empirical distribution function (EDF). The conditional copula of the data is assumed to be constant and belong to a parametric family with unknown parameter. More specifically, we consider the DGP

$$[y_{1t}, \dots, y_{Nt}]' \equiv Y_t = \boldsymbol{\mu}_t(\boldsymbol{\theta}_m) + \boldsymbol{\sigma}_t(\boldsymbol{\theta}_m)\boldsymbol{\eta}_t, \qquad (5)$$

where

$$\boldsymbol{\mu}_t(\boldsymbol{\theta}_m) \equiv [\mu_{1t}(\boldsymbol{\theta}_m), \dots, \mu_{Nt}(\boldsymbol{\theta}_m)]',$$
$$\boldsymbol{\sigma}_t(\boldsymbol{\theta}_m) \equiv \operatorname{diag}\{\sigma_{1t}(\boldsymbol{\theta}_m), \dots, \sigma_{Nt}(\boldsymbol{\theta}_m)\},$$

such that $\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_T$ is iid and

$$[\eta_{1t}, \dots, \eta_{Nt}]' \equiv \boldsymbol{\eta}_t \sim \text{iid } \mathbf{F}_\eta \qquad (6)$$
$$= \mathbf{C}(F_1(\boldsymbol{\eta}_1), \dots, F_N(\boldsymbol{\eta}_N); \boldsymbol{\theta}_c).$$

Here, $\boldsymbol{\theta}_m$ is the vector of all marginal parameters, governing the dynamics of the variables. We assume $\boldsymbol{\theta}_m$ to be (consistently) estimable in a stage prior to the copula estimation. $\boldsymbol{\theta}_c$ is the (unknown) vector of factor copula parameters that parameterizes the copula. $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are $\mathcal{F}_{t-1}$-measureable and independent of $\boldsymbol{\eta}_t$, where $\mathcal{F}_{t-1}$ is the sigma-field that contains information generated by $\{Y_{t-1}, Y_{t-2}, \dots\}$.

To estimate the parameter vector $\boldsymbol{\theta}_c$ we make use of a simulation-based estimation method proposed by Oh and Patton (2013). This method is similar to the simulated method of moments (SMM), where the

[3]Note that for the Normal copula a certain degree of simulation error is present in this plot for j $\geq$ 80, since joint 1/66 tail crashes are rare under this copula.

"moments" that are used are functions of rank statistics, or measures of concordance. These measures are "pure" dependence measures, in the sense that they are invariant to (monotone) transformations of the random variable, or to changes in the marginal distribution. The dependence measures we consider are (i) Spearman's rank correlation and (ii) quantile dependence, that are both functions only of the copula (see Oh and Patton (2013)). For each pair $(\boldsymbol{\eta}_i, \boldsymbol{\eta}_j)$ these two measures are respectively defined as

$$\begin{aligned}\rho^{ij} &\equiv 12\mathrm{E}[F_i(\boldsymbol{\eta}_i)F_j(\boldsymbol{\eta}_j)] - 3 \\ &= 12 \iint uvdC_{ij}(u,v) - 3, \end{aligned} \tag{7}$$

and

$$\lambda^{ij} \equiv \begin{cases} \Pr[F_i(\boldsymbol{\eta}_i) \leq q | F_j(\boldsymbol{\eta}_j) \leq q] = \frac{C_{ij}(q,q)}{q} \\ \qquad q \in (0, 0.5] \\ \Pr[F_i(\boldsymbol{\eta}_i) > q | F_j(\boldsymbol{\eta}_j) > q] = \frac{1-2q+C_{ij}(q,q)}{1-q} \\ \qquad q \in (0.5, 1), \end{cases} \tag{8}$$

where C_{ij} is the copula of the pair $(\boldsymbol{\eta}_i, \boldsymbol{\eta}_j)$. Let $\hat{F}_i$ be the empirical distribution function of $\hat{\boldsymbol{\eta}}_i$. Then the sample counterparts of these measures are defined as

$$\hat{\rho}^{ij} \equiv \frac{12}{T} \sum_{t=1}^{T} \hat{F}_i(\hat{\eta}_{it})\hat{F}_j(\hat{\eta}_{jt}) - 3, \tag{9}$$

and

$$\hat{\lambda}^{ij} \equiv \begin{cases} \frac{1}{Tq} \sum_{t=1}^{T} \mathbb{1}\{\hat{F}_i(\hat{\eta}_{it}) \leq q | \hat{F}_j(\hat{\eta}_{jt}) \leq q] \\ \qquad q \in (0, 0.5] \\ \frac{1}{T(1-q)} \sum_{t=1}^{T} \mathbb{1}\{\hat{F}_i(\hat{\eta}_{it}) > q | \hat{F}_j(\hat{\eta}_{jt}) > q] \\ \qquad q \in (0.5, 1). \end{cases} \tag{10}$$

where $\mathbb{1}$ denotes the indicator function. For the implementation of the SMM estimator we will use the Spearman's rank correlation and the 0.05, 0.10, 0.90, and 0.95 quantile dependence as the five measures of dependence for each pair of variables. The preliminary study of Oh and Patton (2017) on the estimation accuracy and identification of different dependence measures provides motivation for this choice.

We need to compare non-parametric estimators of bivariate dependence measures of $(\boldsymbol{\eta}_i, \boldsymbol{\eta}_j)$, $i, j \in \{1, \ldots, N\}$, with model-based bivariate dependence measures. As stated before, most factor copulas do not have a closed-form, and hence we rely on simulation to evaluate them - from simulated data we can extract dependence properties of the copula. More specifically, the estimation method of Oh and Patton (2013) estimates $\boldsymbol{\theta}_c$ based on the standardized residuals $\{\hat{\boldsymbol{\eta}}_t \equiv \boldsymbol{\sigma}_t^{-1}(\hat{\boldsymbol{\theta}}_m)[Y_t - \boldsymbol{\mu}_t(\hat{\boldsymbol{\theta}}_m)]\}_{t=1}^T$ and simulations from the copula $\mathbf{C}_x$ implied by Equation (3). Let $\hat{\mathbf{m}}_T = (\hat{m}_{1T}, \ldots, \hat{m}_{MT})$ be the M-dimensional vector of dependence measures computed using the data $\{\hat{\boldsymbol{\eta}}_t\}_{t=1}^T$ and let $\tilde{\mathbf{m}}_S(\boldsymbol{\theta}) = (\tilde{m}_{1S}, \ldots, \tilde{m}_{MS})$ be the corresponding M-dimensional vector of dependence measures computed using the simulations $\{\mathbf{X}_s\}_{s=1}^S$. If $\boldsymbol{\Theta}$ is the parameter space of the copula parameters, the SMM estimation method is based on searching across $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ to make the difference between $\hat{\mathbf{m}}_T$ and $\tilde{\mathbf{m}}_S(\boldsymbol{\theta})$ as small as possible. If we define this difference as

$$\mathbf{g}_{T,S}(\boldsymbol{\theta}) \equiv \hat{\mathbf{m}}_T - \tilde{\mathbf{m}}_S(\boldsymbol{\theta}),$$

then the SMM estimator is defined as

$$\hat{\boldsymbol{\theta}}_{T,S} \equiv \underset{\boldsymbol{\theta} \in \Theta}{\arg\min} \; Q_{T,S}(\boldsymbol{\theta}), \tag{11}$$

where

$$Q_{T,S}(\boldsymbol{\theta}) \equiv \mathbf{g}'_{T,S}(\boldsymbol{\theta})\hat{\mathbf{W}}_T\mathbf{g}_{T,S}(\boldsymbol{\theta}). \tag{12}$$

Here, $\hat{\mathbf{W}}_T$ is a positive definite weight matrix that may depend on the data. For our analysis we consider the identity weight matrix[4] - that is, $\hat{\mathbf{W}}_T = \mathbf{I}_M$ - which yields

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{T,S} &= \underset{\boldsymbol{\theta} \in \Theta}{\arg\min} \; \mathbf{g}'_{T,S}(\boldsymbol{\theta})\mathbf{g}_{T,S}(\boldsymbol{\theta}) \\ &= \underset{\boldsymbol{\theta} \in \Theta}{\arg\min} \sum_{l=1}^{M} (\hat{m}_{lT} - \tilde{m}_{lS}(\boldsymbol{\theta}))^2. \end{aligned} \tag{13}$$

As the copula parameter estimates are obtained using the standardized residuals estimated in a first step, assumptions are needed to control the estimation error the may arise from this first step. Oh and Patton (2013) show that, under regularity conditions, the SMM estimator is consistent and asymptotically normal:

$$\frac{1}{\sqrt{1/T + 1/S}} \left(\hat{\boldsymbol{\theta}}_{T,S} - \boldsymbol{\theta}_c\right) \xrightarrow{d} N(0, \boldsymbol{\Omega}_0) \text{ as } T, S \to \infty, \tag{14}$$

where

$$\boldsymbol{\Omega}_0 = (\mathbf{G}'_0\mathbf{G}_0)^{-1}\mathbf{G}'_0\boldsymbol{\Sigma}_0\mathbf{G}_0(\mathbf{G}'_0\mathbf{G}_0)^{-1}. \tag{15}$$

$\boldsymbol{\Sigma}_0$ is the asymptotic variance of $\hat{\mathbf{m}}_T$, estimated using the following iid bootstrap:

[4]The simulation study of Oh and Patton (2017) shows that results based on an efficient weight matrix, depending on the data, are generally comparable to those based on the identity weight matrix, while coverage rates are worse than based on the identity weight matrix.

(i) sample with replacement from the standardized residuals $\{\hat{\boldsymbol{\eta}}_t\}_{t=1}^T$ to obtain B bootstrap samples, $\{\hat{\boldsymbol{\eta}}_t^{(b)}\}_{t=1}^T$, $b = 1, \ldots, B$;

(ii) use $\{\hat{\boldsymbol{\eta}}_t^{(b)}\}_{t=1}^T$ to compute the sample moments, denoted as $\hat{\mathbf{m}}_T^{(b)}$;

(iii) calculate the sample covariance matrix of $\hat{\mathbf{m}}_T^{(b)}$ across the bootstrap replications and scale it by the sample size T.

To estimate $\mathbf{G}_0$, the derivative of $\mathbf{g}_0(\boldsymbol{\theta}) \equiv p-\lim_{T,S\to\infty}\mathbf{g}_{T,S}(\boldsymbol{\theta})$ at $\boldsymbol{\theta}_c$, a numerical derivative of $\mathbf{g}_{T,S}(\boldsymbol{\theta})$ at $\hat{\boldsymbol{\theta}}_{T,S}$ is used. Let p be the number of parameters in $\boldsymbol{\theta}$, and let $\mathbf{e}_k$ denote the kth unit vector with p elements. If $\varepsilon_{T,S}$ is the step size, the kth column of the two-sided numerical derivative estimator of $\mathbf{G}_0$ is given by

$$\hat{\mathbf{G}}_{T,S,k} = \frac{\mathbf{g}_{T,S}(\hat{\boldsymbol{\theta}}_{T,S} + \mathbf{e}_k\varepsilon_{T,S}) - \mathbf{g}_{T,S}(\hat{\boldsymbol{\theta}}_{T,S} - \mathbf{e}_k\varepsilon_{T,S})}{2\varepsilon_{T,S}} \qquad k = 1, \ldots, p.$$

Given $\hat{\mathbf{G}}_{T,S}$ and $\hat{\boldsymbol{\Sigma}}_{T,B}$ we can use the expression in Equation (15) to estimate $\boldsymbol{\Omega}_0$.

Lastly, we consider a simple test of the over-identifying restrictions, as employed by Oh and Patton (2013). This test - the J-test - can be used as a specification test of the factor model. Given that the number of moments (M) is greater than the number of copula parameters (p), the test statistic is given by

$$\begin{aligned} J_{T,S} &\equiv \min(T,S)\mathbf{g}_{T,S}(\hat{\boldsymbol{\theta}}_{T,S})'\mathbf{g}_{T,S}(\hat{\boldsymbol{\theta}}_{T,S}) \\ &\xrightarrow{d} \mathbf{u}'\mathbf{A}_0'\mathbf{A}_0\mathbf{u} \text{ as } T,S\to\infty, \end{aligned} \tag{16}$$

where $\mathbf{u} \sim N(0,\mathbf{I})$, $\mathbf{A}_0 \equiv \boldsymbol{\Sigma}_0^{1/2}\mathbf{R}_0$, and $\mathbf{R}_0 \equiv \mathbf{I} - \boldsymbol{\Sigma}_0^{-1/2}\mathbf{G}_0(\mathbf{G}_0'\mathbf{G}_0)^{-1}\mathbf{G}_0'\boldsymbol{\Sigma}_0^{1/2}$. This test static has a sample-specific limiting distribution and hence we use simulation to obtained critical values:

(i) compute $\hat{\mathbf{R}}$ using $\hat{\mathbf{G}}_{T,S}$ and $\hat{\boldsymbol{\Sigma}}_{T,B}$;

(ii) simulate $\mathbf{u}^{(k)} \sim$ iid $N(0,\mathbf{I})$, for $k = 1, \ldots, K$, where K is large;

(iii) for each simulation, compute $J_{T,S}^{(k)} = \mathbf{u}^{(k)\prime}\hat{\mathbf{R}}'\hat{\boldsymbol{\Sigma}}_{T,B}^{1/2\prime}\hat{\boldsymbol{\Sigma}}_{T,B}^{1/2}\hat{\mathbf{R}}\mathbf{u}^{(k)}$;

(iv) use the sample $(1-\alpha)$ quantile of $\{J_{T,S}^{(k)}\}_{k=1}^K$ as the critical value for the J-test statistic.

In the Section 5 we will use this simulation-based estimation method in an empirical application involving 101 variables. Oh and Patton (2013) considered applications involving only up to ten variables, and showed that their asymptotic theory provides a good approximation for the finite-sample behaviour of these applications. To examine whether this asymptotic theory still holds in higher dimensions, in Section 4 we provide results of a simulation study of the SMM estimator.

3 Markov-Switching Copula Models

In the previous section we assumed the dependence structure of the variables to be constant over time. However, there is ample evidence that this assumption might be too restrictive. For example, Engle (2002) and Cappiello et al. (2006) demonstrated that the correlations between financial assets, and hence the dependence, change through time. To incorporate this time-varying dependence[5] into the factor copulas, in this section we consider Markov-switching copula models, inspired by the Markov switching or regime switching model of Hamilton (1989). This modeling approach is suitable for multivariate applications and assumes different copulas, or at least different copula parameters, in different regimes. We now present the concept of Markov-switching in the context of factor copulas.

3.1 Markov-switching and log-likelihood

Following the notation introduced in Section 2.1, let $f_1, \ldots, f_N$ be the densities of the marginal distributions of $\mathbf{Y}$. Similar to Hamilton (1989) we assume that $Y_t = (y_{1t}, \ldots, y_{Nt})$ depends on a latent state variable S_t, which indicates the current regime of the economy. The regime is allowed to affect the dependence structure, but not the marginal distributions. The density of the data, conditional on being in regime j, is then given by

$$\begin{aligned} &f^{(j)}(Y_t;\boldsymbol{\theta}_m,\boldsymbol{\delta}|\mathcal{F}_{t-1}, S_t = j) \\ &\quad = \mathbf{c}^{(j)}(u_{1t},\ldots,u_{Nt};\boldsymbol{\theta}_c^{(j)})\prod_{i=1}^N f_i(y_{it};\boldsymbol{\theta}_{m,i}), \end{aligned} \tag{17}$$

where $\mathbf{c}^{(j)}$ is the copula density conditional on the regime, with parameter vector $\boldsymbol{\theta}_c^{(j)}$. The vector $\boldsymbol{\theta}_m = (\boldsymbol{\theta}_{m,1}, \ldots, \boldsymbol{\theta}_{m,N})$ collects the parameters of each of the marginal densities f_i. The information on the full process $\mathbf{Y}$ (Y_i) up to time t-1 is contained

[5] See Manner and Reznikova (2012) for a survey on time-varying copulas.

in the sigma-field $\mathcal{F}_{t-1}$ ($\mathcal{F}_{i,t-1}$). The probability integral transform is given by $u_{it} = F_i(y_{it}|\mathcal{F}_{i,t-1};\boldsymbol{\theta}_{m,i})$, such that the variable u_{it} follows a $Unif(0,1)$ distribution. Assuming we have two states or regimes, $S_t \in (1,2)$ is the latent state variable for the regime. We assume this unobserved variable to be generated from a first-order Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}. \tag{18}$$

Here, p_{ij} is the probability of moving to state j at time t+1, given that we were in state i at time t. Note that $\mathbf{P}$ is completely parameterized by p_{11} and p_{22}. As the state variable follows a first-order Markov chain, the current value of the state variable depends on its immediate past value. As a results, one dependence structure might prevail for a period of time and will be replaced by another dependence structure whenever a switching occurs, allowing us to capture dynamic patterns in the dependence of the observed variables.

Although for most factor copulas the density function is not available in closed form, in Section 3.2 we provide a procedure to obtain the factor copula density through numerical integration. This allows us to specify and evaluate the likelihood function of the Markov-switching factor copula based models. Specifically, the log-likelihood function of the Markov-switching model is given by

$$L(\mathbf{Y};\boldsymbol{\theta}_m,\boldsymbol{\delta}) = \sum_{t=1}^{T} \log f(Y_t;\boldsymbol{\theta}_m,\boldsymbol{\delta}|\mathcal{F}_{t-1}). \tag{19}$$

The density of data and therefore the log-likelihood depends on the hidden state. Integration over the possible states gives the unconditional density

$$\begin{aligned} f(Y_t;&\boldsymbol{\theta}_m,\boldsymbol{\delta}|\mathcal{F}_{t-1}) \\ &= \sum_{j=1}^{2} \xi_{j,t|t-1} f^{(j)}(Y_t;\boldsymbol{\theta}_m,\boldsymbol{\delta}|\mathcal{F}_{t-1}, S_t = j). \end{aligned} \tag{20}$$

Here, $\xi_{j,t|t-1} = \Pr(S_t = j|\mathcal{F}_{t-1};\boldsymbol{\theta}_m,\boldsymbol{\delta})$ is the probability of being in regime j at time t given all observations up to t-1, also referred to as a predicted probability. Note that we can thus make an inference about the unobserved value of S_t based on the observed data Y_t. The parameter vector $\boldsymbol{\delta} = (\boldsymbol{\theta}_c^{(1)}, \boldsymbol{\theta}_c^{(2)}, p_{11}, p_{22})$ contains the copula parameters over each regime, as well as the parameters of the Markov transition probability matrix. The likelihood function must be evaluated unconditionally to the unobservable regime variable S_t. For this, following Chollete et al. (2009) we decompose the likelihood function specified in Equation (19) into two parts:

$$L(\mathbf{Y};\boldsymbol{\theta}_m,\boldsymbol{\delta}) = L_m(\mathbf{Y};\boldsymbol{\theta}_m) + L_c(\mathbf{Y};\boldsymbol{\delta},\boldsymbol{\theta}_m), \tag{21}$$

where

$$L_m(\mathbf{Y};\boldsymbol{\theta}_m) = \sum_{t=1}^{T}\sum_{i=1}^{N} \log\, f_i(y_{it}|\mathcal{F}_{i,t-1};\boldsymbol{\theta}_{m,i}) \tag{22}$$

$$L_c(\mathbf{Y};\boldsymbol{\delta},\boldsymbol{\theta}_m) = \sum_{t=1}^{T} \log(\mathbf{1}'(\xi_{t|t-1} \odot \varphi_t)), \tag{23}$$

with

$$\varphi_t = \begin{pmatrix} \mathbf{c}^{(1)}(u_{1t},\ldots,u_{Nt};\boldsymbol{\theta}_c^{(1)}) \\ \mathbf{c}^{(2)}(u_{1t},\ldots,u_{Nt};\boldsymbol{\theta}_c^{(2)}) \end{pmatrix}. \tag{24}$$

Here, $\xi_{t|t-1}$ is the (2×1) vector of predicted probabilities, $u_{it} = F_i(y_{it}|\mathcal{F}_{i,t-1};\boldsymbol{\theta}_{m,i})$, $\mathbf{1}$ is a (2×1) vector of ones, and $\odot$ denotes the Hadamard product, or element-by-element product. The likelihood function allows for decomposition as the marginal distributions are not affected by the latent state variable. The likelihood of the marginal distributions, L_m, is a function of $\boldsymbol{\theta}_m$ only. The copula likelihood, L_c, depends both directly on $\boldsymbol{\delta}$ and indirectly on $\boldsymbol{\theta}_m$, through the distribution function F_i. Namely, F_i transforms observations into uniform [0,1] variables that are the input of the copula density.

The decomposed likelihood function allows for a two-stage estimation procedure, whose properties have been studied by Newey and McFadden (1994). Among others, Patton (2006), Chollete et al. (2009) and Garcia and Tsafack (2011) already applied this method in a similar context. The two-stage estimation is especially useful in our case, since it makes estimation in high-dimensional models feasible. In a first step we estimate the parameters of the marginal distributions, $\boldsymbol{\theta}_m$, assuming no contemporaneous correlation between the different series. Note that we can split L_m further, resulting in one term for each of the margins, and hence we can estimate the parameters of each of the marginal distributions as

$$\hat{\boldsymbol{\theta}}_{m,i} = \underset{\boldsymbol{\theta}_{m,i}\in\Theta_m}{\arg\max} \sum_{t=1}^{T} \log\, f_i(y_{it}|\mathcal{F}_{i,t-1};\boldsymbol{\theta}_{m,i}). \tag{25}$$

Given $\hat{\boldsymbol{\theta}}_m = (\hat{\boldsymbol{\theta}}_{m,1},...,\hat{\boldsymbol{\theta}}_{m,N})$, in the second stage we estimate the copula parameters $\boldsymbol{\delta}$ as

$$\hat{\boldsymbol{\delta}} = \underset{\boldsymbol{\delta}\in\Theta}{\arg\max}\, L_c(\mathbf{Y};\boldsymbol{\delta},\hat{\boldsymbol{\theta}}_m). \tag{26}$$

Conditional on having consistently estimated the marginal models, we consider an EM algorithm in order to estimate the copula parameters. Since the Markov chain S_t is not observable, we need to use the filter of Hamilton (1989). Specifically, the filtered system obeys

$$\xi_{t|t} = \frac{\xi_{t|t-1} \odot \varphi_t}{\mathbf{1}'(\xi_{t|t-1} \odot \varphi_t)} \tag{27}$$

$$\xi_{t+1|t} = \mathbf{P}'\xi_{t|t}, \tag{28}$$

where $\xi_{t|t}$ is the (2×1) vector containing the probabilities of being in each regime at time t, conditional on the observations up to time t. As discussed by Chollete et al. (2009), Equation (27) consists of a Bayesian updating of the probability of being in either regime given present time observations (φ_t), while Equation (28) corresponds to one forward iteration of the Markov chain. Given a starting value $\xi_{1|0}$, copula parameter values $\boldsymbol{\theta}_c^{(1)}$, $\boldsymbol{\theta}_c^{(2)}$, and transition probabilities p_{11} and p_{22}, we iterate over both equations. Here, we set $\xi_{1|0}$ to be the vector of stationary transition probabilities of the Markov chain[6], as suggested by Hamilton (1994). Finally, the value of the copula likelihood is obtained using Equation (23).
Proof: see Chollete et al. (2009), Garcia and Tsafack (2011).

3.2 Obtaining the factor copula density

Again using the notation introduced in Section 2.1, let $\mathbf{c}$ denote the copula density function corresponding to the copula $\mathbf{C}_x$ implied by Equation (3) with $K = 1$ common factor. To obtain the copula density we use the numerical integration procedure proposed by Oh and Patton (2013), "integrating out" the latent common factor Z. They show that copula density is given by

$$\mathbf{c}(u_1, \ldots, u_N) = \frac{\mathbf{F}_x(G_1^{-1}(u_1), \ldots, G_N^{-1}(u_N))}{\prod_{i=1}^N g_i(G_i^{-1}(u_i))}, \tag{29}$$

where $\mathbf{F}_x(x_1, \ldots, x_N)$ is the joint density of $\mathbf{X}$, $g_i(x_i)$ is the marginal density of X_i, and $G_i^{-1}(x_i)$ is the inverse marginal distribution of X_i. To obtain $\mathbf{F}_x(x_1, \ldots, x_N)$, $g_i(x_i)$ and $G_i^{-1}(x_i)$, we first note that due to the independence of Z and ε_i we have

$$f_{x_i|z}(x_i|z) = f_\varepsilon(x_i - \beta_i z)$$
$$F_{x_i|z}(x_i|z) = F_\varepsilon(x_i - \beta_i z)$$
$$\mathbf{F}_{x|z}(x_1, \ldots, x_N) = \prod_{i=1}^N f_\varepsilon(x_i - \beta_i z).$$

Integration over the common factor gives

$$\begin{aligned} g_i(x_i) &= \int_{-\infty}^{\infty} f_{x_i|z}(x_i|z) f_z(z) dz \\ &= \int_{-\infty}^{\infty} f_\varepsilon(x_i - \beta_i z) f_z(z) dz, \end{aligned} \tag{30}$$

$$\begin{aligned} G_i(x_i) &= \int_{-\infty}^{\infty} \Pr[X_i \leq x_i | Z = z] f_z(z) dz \\ &= \int_{-\infty}^{\infty} F_\varepsilon(x_i - \beta_i z) f_z(z) dz, \end{aligned} \tag{31}$$

and similarly

$$\begin{aligned} \mathbf{F}_x(x_1, \ldots, x_N) &= \int_{-\infty}^{\infty} \mathbf{F}_{x|z}(x_1, \ldots, x_N|z) f_z(z) dz \\ &= \int_{-\infty}^{\infty} \prod_{i=1}^N f_\varepsilon(x_i - \beta_i z) f_z(z) dz. \end{aligned} \tag{32}$$

The above integrals are unbounded and we use the probability integral transformation of Z, $u \equiv F_z(z)$, to convert the domain of the integrals to the interval [0, 1]. For example:

$$\begin{aligned} g_i(x_i) &= \int_{-\infty}^{\infty} f_\varepsilon(x_i - \beta_i z) f_z(z) dz \\ &= \int_0^1 f_\varepsilon(x_i - \beta_i F_z^{-1}(u)) du, \end{aligned} \tag{33}$$

and likewise for $G_i(x_i)$ and $\mathbf{f}_x(x_1, \ldots, x_N)$. We then approximate these integrals using Gauss-Legendre quadrature with 150 nodes, as suggested by Oh and Patton (2013). Finally, to obtain $G_i^{-1}(x_i)$, which is a function of both x_i and the factor loading β_i, we use the following procedure:

(i) create a grid of 100 points for x_i in the interval $[x_{i,min}; x_{i,max}]$ and 50 points for β_i in the interval $[\beta_{i,min}; \beta_{i,max}]$ and evaluate G_i at each of these points;

(ii) use two-dimensional linear interpolation to obtain $G_i^{-1}(x_i)$ given x_i and β_i.

[6] For a two-state Markov chain these probabilities are given by $(1 - p_{22})/(2 - p_{11} - p_{22})$ and $(1 - p_{11})/(2 - p_{11} - p_{22})$ for the first and second state, respectively.

Table 1: Simulation results equidependence model - iid data

	Normal	Factor t - t		Factor Skew t - t		
	β	β	ν^{-1}	β	ν^{-1}	λ
True Value	1.00	1.00	0.25	1.00	0.25	-0.50
Bias	-0.0008	-0.0083	0.0130	0.0088	-0.0080	-0.0295
Std	0.0231	0.0421	0.0698	0.0463	0.0748	0.0936
Median	1.0009	0.9857	0.2672	1.0078	0.2457	-0.5146
90%	1.0276	1.0471	0.3471	1.0734	0.3263	-0.4327
10%	0.9699	0.9413	0.1723	0.9513	0.1482	-0.6245
90-10 Diff	0.0577	0.1058	0.1748	0.1221	0.1781	0.1918

NOTES: This table presents simulation results from 100 simulations of the Normal copula, the t-t factor copula and the Skew t-t factor copula, under the assumption of equidependence. The copula models are estimated by SMM, as described in Section 2.4, and the data are assumed to be iid with standard Normal marginal distributions. Results are reported for dimension $N = 100$, the sample size is $T = 1000$ and $S = 25 \times T$ simulations are used in the computation of the dependence measures to be matched in the SMM estimation. The first row reports the average difference between the estimated parameters and their true values, while the second row reports the standard deviation in the estimated parameters. The third, fourth, and fifth row respectively report the 50^{th}, 90^{th}, and 10^{th} percentiles of the distribution of the estimated parameters. The last row reports the difference between the 90^{th} and 10^{th} percentiles.

4 Simulation Study SSM Estimator

Prior to the empirical analysis of the next section we perform a simulation study to examine the finite-sample properties of the SMM estimator discussed in Section 2.4. We consider three different factor copulas, all of them form specified by Equation (2) and (3). In the first case we assume a single common factor, with the same loading on the common factor for each variable, corresponding to an "equidependence" model:

$$X_i = \beta Z + \varepsilon_i,\ i = 1, \dots, N \tag{34}$$
$$[X_1, \dots, X_N]' \equiv \mathbf{X} \sim F_x = \mathbf{C}_x(G_1, \dots, G_N).$$

For the simulations we focus on $Z \sim$ Skew $t(\nu, \lambda)$, corresponding to the skewed t distribution of Hansen (1994), and $\varepsilon_i \sim$ iid $t(\nu)$, where t denotes the standardized t distribution. We assume Z to be independent of ε_i for each i and set $\beta = 1$ for all models. The first model we consider is a simple Normal copula - obtained by setting $\nu \to \infty$ and $\lambda = 0$. In the second model we set $\nu = 4$ and $\lambda = 0$ (t-t factor copula), and in the third case we set $\nu = 4$ and $\lambda = -0.5$ (Skew t-t factor copula). We estimate the inverse degree of freedom parameter ν^{-1} rather than ν, such that its parameter space is [0, 0.5) instead of (2, ∞].

Next, we consider two extensions or generalizations of this equidependence model. First, we examine a "block equidependence" model, where the variables are grouped into M groups and where we assume that all variables within each group have the same loading on the common factor Z. In this case, the components of $\mathbf{X}$ are defined as

$$X_i = \beta_{G(i)} Z + \varepsilon_i,\ i = 1, \dots, N, \tag{35}$$

where $G(i)$ is the group for variable i. We consider $M = 3$ groups and set $[\beta_1, \beta_2, \beta_3] = [0.25,\ 0.5,\ 0.75]$. Lastly, we perform simulations for a multi-factor model specification, including a common factor Z_0 and industry factors $Z_{G(i)}$:

$$X_i = \beta Z_0 + \gamma_{G(i)} Z_{G(i)} + \varepsilon_i,\ i = 1, \dots, N. \tag{36}$$

In this case we set $[\beta, \gamma_1, \gamma_2, \gamma_3] = [1,\ 0.25,\ 0.5,\ 0.75]$, and we focus on $Z_0 \sim$ Skew $t(\nu, \lambda)$, $Z_G \sim$ iid $t(\nu)$ for $G = 1, \dots, M$, and $\varepsilon_i \sim$ iid $t(\nu)$, where Z_0 and Z_G are assumed to be independent for each G.

For the simulations we first assume that the data are iid with standard Normal marginal distributions. For the equidependence model we also consider a second case, where we assume that each marginal distribution of the variables of interest follows an AR(1)-GARCH(1,1) process:

$$\begin{aligned} y_{it} &= \phi_0 + \phi_1 y_{i,t-1} + \sigma_{it}\eta_{it},\ t = 1, \dots, T \\ \sigma_{it}^2 &= \omega + \gamma\sigma_{i,t-1}^2 + \alpha\sigma_{i,t-1}^2\eta_{i,t-1}^2 \\ \boldsymbol{\eta}_t &\equiv [\eta_{1t}, \dots, \eta_{Nt}] \sim \text{iid } \mathbf{F}_\eta = \mathbf{C}(\Phi, \dots, \Phi). \end{aligned} \tag{37}$$

Here, $\mathbf{C}_x$ is the factor copula implied by the factor structure of Equation (34) and Φ is the standard Normal distribution. Following Oh and Patton (2017), we set the AR-GARCH parameters $[\phi_0, \phi_1, \omega, \gamma, \alpha]$ = [0.01, 0.05, 0.05, 0.85, 0.10]. We first estimate the parameters of the margins via quasi-maximum

Table 2: Simulation results block equidependence and multi-factor model - factor Skew t-t

	β_1	β_2	β_3	ν^{-1}	λ	β	γ_1	γ_2	γ_3	ν^{-1}	λ
True Value	0.25	0.5	0.75	0.25	-0.5	1	0.25	0.5	0.75	0.25	-0.5
Bias	0.0009	0.0038	0.0086	-0.0164	-0.0684	0.0371	-0.0033	0.0314	0.1485	-0.0424	-0.1742
Std	0.0218	0.0302	0.0453	0.0904	0.1583	0.0748	0.4795	0.4405	0.2470	0.0845	0.1797
Median	0.2493	0.4977	0.7537	0.2318	-0.5312	1.0360	0.3325	0.6476	0.8883	0.2015	-0.6472
90%	0.2778	0.5432	0.8231	0.3477	-0.4178	1.1054	0.7834	0.9672	1.1808	0.3123	-0.4566
10%	0.2243	0.4719	0.7074	0.1284	-0.7795	0.9579	-0.4913	-0.2939	0.6189	0.0891	-0.9534
Diff	0.0535	0.0713	0.1157	0.2193	0.3616	0.1475	1.2748	1.2611	0.5619	0.2232	0.4968

NOTES: This table presents simulation results from 100 simulations of Skew t-t factor copula, under the assumption of block equidependence (left) and for a multi-factor specification (right). The copula models are estimated by SMM, as described in Section 2.4, and the data are assumed to be iid with standard Normal marginal distributions. Results are reported for dimension is $N = 20$, where the variables are divided into three groups of decreasing size. The sample size is $T = 1000$ and $S = 25 \times T$ simulations are used in the computation of the dependence measures to be matched in the SMM estimation. The first row reports the average difference between the estimated parameters and their true values, while the second row reports the standard deviation in the estimated parameters. The third, fourth, and fifth row respectively report the 50^{th}, 90^{th}, and 10^{th} percentiles of the distribution of the estimated parameters. The last row reports the difference between the 90^{th} and 10^{th} percentiles.

likelihood, after which we estimate the standardized residuals η_{it}. These estimates are then used to estimate the factor copula parameters. We consider a time series of length $T = 1000$ and use $S = 25 \times T$ simulations for the computation of the dependence measures in the SMM estimation procedure, where we use the dependence measure introduced in Section 2.4. Appendix B elaborates on the moments used in the SMM estimation for each of the considered model specification, as discussed by Oh and Patton (2017). Lastly, each scenario is repeated 100 times.

All results of the simulation study can be found in Appendix C. Since the simulation results are largely comparable across the two scenarios discussed above, in this section we will not touch upon the results for the AR-GARCH margins. Instead, we cover the simulation results for the iid data and state that an equivalent conclusion is reached for the AR-GARCH data. Table 1 presents the simulation results for the equidependence model with idd data for $N = 100$ variables. The table shows that, on average, the estimated parameters are close to the true values - for all copulas parameters the average bias is small relative to the standard deviation. Besides, the median of the parameter estimates is generally centered on the true value of the parameter. Looking at Table C.1 in Appendix C, we notice is that the accuracy of the estimates, in terms of the standard deviation and the 90-10 percentile difference, improves as the dimension grows. This is intuitive: as the dimension grows the information available increases, while the number of parameters to be estimated does not change. Lastly, we note that the estimator accuracy decreases when more parameters are added to the model, which is as expected.

Table 2 reports simulation results for the block equidependence (left) and the multi-factor (right) model specification, obtained with the Skew t-t factor copula for $N = 20$ variables. For the block equidependence model we see that the results are similar to the results for the equidependence model. That is, the average bias is small relative to the standard deviation and the median of the simulated distribution is close to the true values. Further, by comparing Table C.3 and C.4 in Appendix C we again conclude that the copula parameters are generally more precisely estimated as the dimension grows. Within the multi-factor model we see that the SMM estimator is relatively accurate for the common factor loading β and the "shape" parameter ν^{-1}. However, in this model the parameters $\gamma_1, \gamma_2, \gamma_3$, corresponding to the group factors, and the shape parameter λ are not well estimated by the SMM estimator. More specifically, especially for the group parameters we find relatively large standard deviations compared to the standard deviations of the copula parameter estimates within the other model specifications. The same holds true for the 90-10 percentile difference.

The results of this preliminary study are similar to the results of the simulation study performed by Oh and Patton (2017), verifying that for the equidependence and the block equidependence model the SMM estimator continues to have satisfactory properties - for example, finite-sample bias is small - in high-dimensional applications. On the other hand, we conclude that for multi-factor specifications, for which Oh and Patton (2017) do not present simulation results, the performance of the estimator is poor. The simulation study provides motivation for the choices that will be made in our empirical analysis and validates the use of the SMM estimator in this analysis.

5 Empirical Study S&P 100 Returns

In the empirical research we study the dependence between 101 of the constituents of the S&P 100 equity index as of September 1, 2017. We use daily (log-)return data over the period July 2015 - May 2018, a total of $T = 717$ trading days. Appendix D contains a full list of stocks we consider in our study[7], together with their three-digit Standard Industrial Classification (SIC) codes. We will use these codes in Section 5.2 and 5.3.

We filter each of the individual return series by taking into account the time variation in conditional means and variances of these series. For this, we employ a AR-GJR-GARCH(1,1) model[8], as proposed by Glosten et al. (1993), to model each series:

$$\begin{aligned} r_{it} &= \phi_{0i} + \phi_{1i} r_{i,t-1} + \varepsilon_{it},\ \varepsilon_{it} = \sigma_{it}\eta_{it},\ t = 1, \dots, T \\ \sigma_{it}^2 &= \omega_i + \beta_i \sigma_{i,t-1}^2 + \alpha_i \varepsilon_{i,t-1}^2 \\ &\qquad + \gamma_i \varepsilon_{i,t-1}^2 \mathbb{1}\{\varepsilon_{i,t-1} \leq 0\}. \end{aligned} \tag{38}$$

where r_{it} is the return on one of the 101 firms. Due to the term $\gamma_i \varepsilon_{i,t-1}^2 \mathbb{1}\{\varepsilon_{i,t-1} \leq 0\}$, the model allows for an asymmetric impact of shock $\varepsilon_{i,t-1}$ on the volatility. Typically, negative shocks have a stronger effect on the volatility than positive shocks and therefore $\gamma_i > 0$ - commonly referred to as leverage effect. We estimate the distribution of the standardized residuals η_{it} using the EDF, allowing us to (nonparametrically) capture possible skewness and excess kurtosis in the residuals of each return series. The parameters of the mean and variance model are estimated using quasi-maximum likelihood, and from the estimated model we obtain the estimated standardized residuals that are used in the estimation of the dependence structure.

The upper part of Table 3 reports summary statistics of the return series. We see that most series - more than 75% - have a mean return slightly above zero. The skewness is negative on average, varying from -0.72 to -0.01 across the interquartile range. The average kurtosis is well above the normal value of three and takes on a wide range of values for the different return series. The heterogeneity in the skewness and kurtosis coefficients validates the use of the EDF for the distributions of the asset returns. Table 3 also presents summary statistics for the estimated AR(1)-GJR-GARCH(1,1) parameters. We have, consistent with numerous other studies, a small negative AR(1) coefficient for more than half of the stocks. Looking at the GJR-GARCH parameters, the parameter corresponding to the lagged volatility, β, takes on a considerable positive value for most stocks, which indicates strong persistence in conditional variance or volatility. The asymmetry parameter, γ, is positive for almost all stocks, supporting the existence of a leverage effect in the conditional volatility of the return series. The lower part of Table 3 presents summary statistic for measures of dependence between each pair of standardized residuals. The linear correlation ρ and the rank correlation ρ_s measure the strength and sign of the dependence. The average lower and upper 1% tail dependence, $(\tau_{0.99}+\tau_{0.01})/2$, measures the strength of the dependence in the tails, while the difference in upper and lower 10% tail dependence, $\tau_{0.90} - \tau_{0.10}$, provides an indication of the symmetry of the dependence in the tails. The summary statistics for the two correlation measures are similar, where the rank correlation is, on average, slightly higher than the linear correlation - 0.34 versus 0.30. Across the pair of stocks the linear correlation ranges from 0.22 (25^{th} percentile) to 0.37 (75^{th} percentile), while these percentiles take on values of 0.27 and 0.40, respectively, for the rank correlation. Hence, there is moderate heterogeneity in the correlation coefficients. The average lower and upper 1% tail dependence is 0.08 on average and the 75^{th} percentile is 0.14, indicating that the returns display considerable dependence in the tails. The difference in upper and lower 10% tail dependence is negative on average and the corresponding 75^{th} percentile takes on a value of -0.03. Thus, this difference is negative for over 75% of the pairs of stocks, which indicates the presence of asymmetric dependence between the stock returns. To facilitate comparison between the estimation results presented in the following sections and the results presented by Oh and Patton (2017), it is worth noting the returns of the S&P 100 constituents over the period April 2008 to December 2010, as studied by Oh and Patton (2017), displayed considerably higher linear and rank correlations. Besides, the returns exhibited stronger asymmetric dependence, while tail dependence was found to be weaker for these returns, compared to the returns we consider.

[7] Booking Holdings Inc. is not included in our analysis due to missing values in the return series.

[8] Oh and Patton (2017), who also studied the S&P 100 constituents, state that from a wide range of GARCH models this model is preferred for almost all stocks according to the BIC.

Table 3: Statistics daily returns S&P 100 constituents

	Cross-sectional distribution					
	Mean	5%	25%	Median	75%	95%
Mean	0.0003	-0.0006	0.0001	0.0003	0.0006	0.0011
Std dev	0.0141	0.0099	0.0120	0.0139	0.0155	0.0194
Skewness	-0.2421	-1.4546	-0.7243	-0.3978	-0.0135	0.5913
Kurtosis	12.9276	4.6143	6.3130	8.2334	10.7258	20.9926
ϕ_0	0.0004	-0.0005	0.0001	0.0004	0.0007	0.0012
ϕ_1	-0.0040	-0.0723	-0.0314	-0.0066	0.0326	0.0575
$\omega \times 1000$	0.0167	0.0004	0.0012	0.0043	0.0226	0.0654
β	0.8247	0.3465	0.7402	0.9244	0.9695	0.9979
α	0.0499	0.0000	0.0000	0.0185	0.0739	0.2347
γ	0.0960	-0.0271	0.0130	0.0587	0.1440	0.2973
ρ	0.3023	0.1284	0.2244	0.2921	0.3672	0.5056
ρ_s	0.3358	0.1585	0.2651	0.3305	0.4005	0.5277
$(\tau_{0.99}+\tau_{0.01})/2$	0.0750	0.0000	0.0000	0.0698	0.1397	0.2095
$(\tau_{0.90}-\tau_{0.10})$	-0.0660	-0.1676	-0.1117	-0.0698	-0.0279	0.0419

NOTES: This table reports some statistics of the daily equity returns of the S&P 100 constituents used in the empirical analysis, over the period July 7, 2015 to May 9, 2018. The top part of the table reports summary statistics of the daily returns, while the middle part reports the estimated AR(1)-GJR-GARCH(1,1) parameters for these return series. The lower part of the table reports the linear correlation, Spearman's rank correlation, average 1% upper and lower tail dependence, and the difference between the upper and lower 10% tail dependence, computed using the standardized residuals from the estimated AR-GJR-GARCH models. The mean and quantiles of the cross-sectional distributions are reported, for the top and middle part of the table across the $N = 101$ marginal distributions and for the lower part across the $N(N-1)/2 = 5050$ distinct pairs of stocks.

In the simulation study of the previous section we investigated the finite-sample properties of the SMM estimator of Oh and Patton (2013). The results of this study showed that in high-dimensional applications the SMM estimator has satisfactory properties for equidependence and block equidependence specifications, while these satisfactory properties disappear when we add additional factors to the specification. Following these results, in our empirical study we focus on equidependence and block equidependence copula specifications, and do not consider multi-factor specifications. Section 5.1 and Section 5.2 contain results for the equidependence and block equidependence specifications, respectively, obtained with the SMM estimation method of Oh and Patton (2013). Section 5.3 presents results for these specifications within the context of Markov-switching, obtained using maximum likelihood estimation.

5.1 Results equidependence copula specifications

We first focus on an equidependence model, as specified in Equation (34), where a single common factor with the same loading for each asset is assumed. We consider four factor copulas, generated by a linear latent factor structure and characterized by the assumed distributions for the common factor and the idiosyncratic variables: t-Normal, Skew t-Normal, t-t, and Skew t-t. Besides, for comparative purposes we consider three well-known copulas: the Clayton copula, the Normal copula, and the Student's t copula. Note that for the latter two copulas "equidependence" is achieved by imposing equal correlation across assets (equicorrelation), which is implicit in the construction of the Clayton copula. Each copula is used to model the dependence structure of the 101 return series of the S&P 100 constituents.

Table 4 presents the empirical results obtained for the equidependence specifications. Besides the estimated copula parameters, we report the corresponding value of the SMM objective function specified in Equation (12), together with the p-value from the J-test. The reported standard errors are obtained using $B = 1000$ bootstrap samples for the estimation of $\hat{\boldsymbol{\Sigma}}_{T,B}$ and step size $\varepsilon_{T,S} = 0.1$ to compute $\hat{\mathbf{G}}_{T,S}$[9]. The table shows that the estimates for the loading on the common factor, β, range from 0.72

[9] This step size is suggested by Oh and Patton (2017), who performed an elaborated simulation study on the 95% confidence interval coverage rates for different choices of $\varepsilon_{T,S}$.

Table 4: Estimation results daily S&P 100 stock returns - equidependence

	β		ν^{-1}		λ			
	Est	Std Err	Est	Std Err	Est	Std Err	Q_{SMM}	p-Value
Clayton†	0.5142	0.0426	—	—	—	—	0.0222	0.0000
Normal	0.7834	0.0299	—	—	—	—	0.0075	0.0029
Student's t	0.7259	0.0155	0.1192	0.0271	—	—	0.0046	0.0010
Factor t - N	0.7748	0.0322	0.1833	0.0380	—	—	0.0053	0.0065
Factor Skew t - N	0.7699	0.0291	0.1251	0.0388	-0.1833	0.0590	0.0004	0.0051
Factor t - t	0.7183	0.0304	0.2113	0.0535	—	—	0.0046	0.0119
Factor Skew t - t	0.7254	0.0297	0.2170	0.0562	-0.1762	0.0506	0.0003	0.0156

NOTES: This table reports estimation results for the dependence structure of the 101 return series of the S&P 100 constituents over the period July 2015 to May 2018 using various copula models, under the assumption of equidependence. Estimates and asymptotic standard errors for the copula model parameters are reported, as well as the value of the SMM objective function Q_{SMM} evaluated at the estimated parameters and the p-value from the J-test of the over-identifying restrictions.
†Note that the Clayton copula parameter does not correspond to β, but for simplicity it is reported in this column.

to 0.78, implying a correlation coefficient of around 0.36 on average across the assets[10]. The estimated inverse degrees of freedom parameter, ν^{-1}, is around 1/8 for the Student's t copula and the factor Skew t-Normal copula, and around 1/5 for the other three factor copulas. Using the critical values for the distribution of the squared t-statistic reported in Oh and Patton (2017), we conclude that ν^{-1}, the tail dependence parameter, is significant at the 5% significance level for all models in which it is estimated. The last copula parameter, λ, is also significant at the 5% level and estimated to be around -0.18 for the two models that allow for asymmetric dependence between the returns. These results provide evidence of a non-normal dependence structure between the asset returns, where stocks tend to crash simultaneously, and where the correlation between returns tends to be larger during crashes than during booms. This emphasizes the need for a fat-tailed and asymmetric implied common factor. A similar conclusion was reached by Oh and Patton (2017), who found higher loadings on the common factor and evidence of a more left-skewed and less fat-tailed common factor. This is in line with the comparison made in the previous section.

In Figure 3 we again consider the crash dependence measure of expected proportion of $(N - j)$ stocks that will crash, conditional on observing j out of N stocks crashing $(\pi^q(j))$, as introduced in Section 2.3. This measure offers us more insight into our high-dimensional analysis, involving $N = 101$ stocks. We define a "crash" as a stock return in the lower 1/22 and 1/66 tail, respectively corresponding

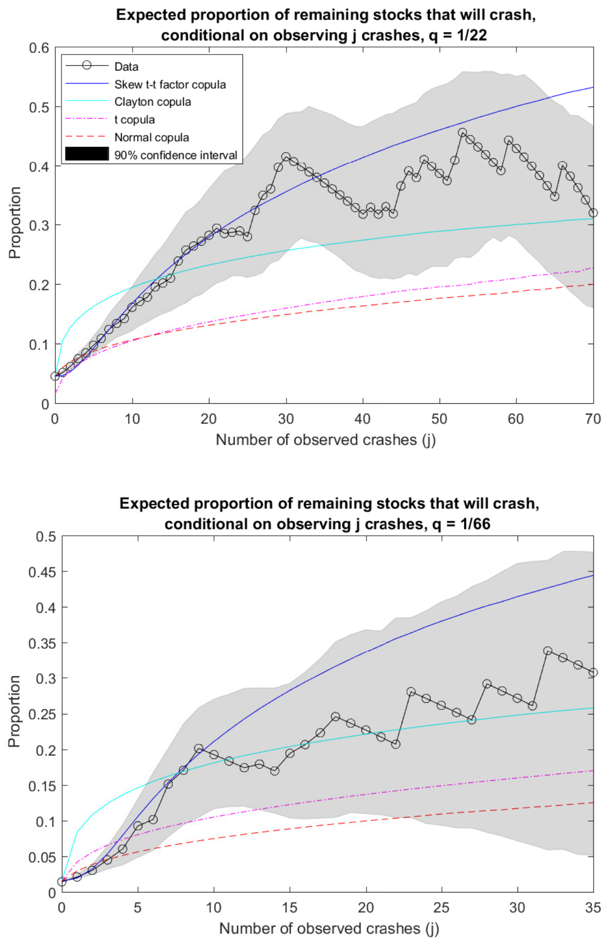

Figure 3: The expected proportion of remaining stocks that will crash, conditional on observing j out of 101 stocks crashing, under four different estimated copula models, as reported in Table 4. A "crash" is defined as a stock return in the lower 1/22 (upper panel) and 1/66 (lower panel) tail. Note that in both plots the horizontal axes are truncated due to the scarcity of joint crashes.

[10]The relationship between the factor loading β and the correlation coefficient ρ is given by $\rho = \beta^2/(1 + \beta^2)$.

to a once-in-a-month and a once-in-a-quarter event for daily returns. For the confidence intervals of these estimates we sample with replacement from the standardized residuals to obtain 1000 bootstrap samples, and for each bootstrap sample we determine the expected proportion of crashes for each j. Pointwise (in j) 90% bootstrap confidence intervals are then obtained using the 0.05 and 0.95 quantiles of the simulated distribution of $\pi^q(j)$. We again use 10 million simulations and consider the Clayton, Normal, and Student's t copula, and the Skew t-t factor copula, with copula parameters set to the values reported in Table 4. In the upper part of the Figure 3, corresponding to once-in-a-month crashes, the Skew t-t factor copula adequately models the tail events up to around $j = 60$, after which the Clayton copula appears to provide a better fit. Both the Normal and the Student's t copula seem to underestimate the proportion of remaining stocks that will crash. The lower part of the figure shows that for once-in-a-quarter events the Skew t-t factor copula again seems appropriate for modeling (moderate) tail events, up to around $j = 10$, while for larger values of j the empirical plot follows that of the Clayton copula. Note that the Clayton copula imposes zero upper tail dependence, but has the ability to capture lower tail dependence. The estimated Clayton parameter yields a lower tail dependence coefficient of around 0.26, while the Normal copula imposes zero tail dependence and the Student's t copula only has a lower tail dependence coefficient of around 0.06 for the copula parameter estimates presented in Table 4 above[11]. Thus, it appears that a copula that allows for greater lower tail dependence, such as the Clayton copula or the Skew t-t factor copula, is needed to adequately model joint crashes of the data. However, for extreme tail events there is limited information, resulting in wide confidence intervals, which makes it hard to make precise statements about the relative fit of the copulas.

In terms of the value of the SMM objective function, Q_{SMM}, reported in the second-to-last column of Table 4, we clearly observe that the two models that allow for asymmetric dependence between the returns perform substantially better than the models that impose symmetry. However, the p-values in the last column of this table indicate that at the 5% significance level all models are rejected by the J-test of the over-identifying restrictions. Following this, we conclude that in order to adequately describe the dependence structure of the asset returns we need to consider a skewed common factor, as stated before, while the assumption of a single factor loading renders the considered models inadequate according to the specification test. The summary statistics reported in Table 3 already called this assumption into question, and consequently in the next section we relax the assumption of equidependence. Note that, although the equidependence model is a restrictive model, the results reported and discussed in this section motivate the use of factor copulas in modeling the dependence structure of assets, indicating their importance and their advantage over well-known models on which previous literature in the field of copula models has mainly focused.

5.2 Results block equidependence copula specifications

We now drop the assumption of a single loading on the common factor across assets - all copula models based on this assumption were rejected - and generalize the equidependence model to a more realistic model specification. Specifically, we consider a block equidependence model, as specified in Equation (35). Contrary to the equidependence model, the block equidependence model allows for heterogeneous dependence between asset returns. We form seven groups of stocks corresponding to the seven different industries the stocks belong to, based on the first digit of the SIC code of each stock (see Appendix D). Stocks within the same group or industry are imposed to have the same factor loading, while stocks across industries are permitted to have different loadings on the common factor. As a result, by considering block equidependence specifications the flexibility of the model greatly increases compared to equidependence specifications.

Table 5 presents the estimation results for the copula models based on block equidependence. Note that the t-Normal and the Skew t-Normal factor copula are not reported here. Instead, following the results discussed in the previous section we focus on the t-t and the Skew t-t factor copula, that allow for a fat-tailed distribution for the idiosyncratic variables. Besides, since the Clayton copula imposes equidependence by construction it is not comparable to the other models when considering block equidependence specifications, and is thus not reported in

[11] If α is the Clayton parameter, ρ the correlation coefficient of the Student's t copula, and $\mathrm{t}_{\nu+1}(x)$ the Student's t distribution function with $\nu+1$ degrees of freedom, the lower tail dependence coefficients for the Clayton and the Student's t copula are respectively given by

$$\tau^L = 2^{-1/\alpha} \text{ and } \tau^L = 2\ \mathrm{t}_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right).$$

Table 5: Estimation results daily S&P 100 stock returns - block equidependence

	Normal		Student's t		Factor t - t		Factor Skew t - t	
	Est	Std Err	Est	Std Err	Est	Std Err	Est	Std Err
ν^{-1}	—	—	0.0858	0.0328	0.1513	0.0639	0.2216	0.0537
λ	—	—	—	—	—	—	-0.1777	0.0469
β_1	0.7798	0.0469	0.7467	0.0445	0.7277	0.0457	0.7317	0.0481
β_2	0.7070	0.0329	0.6373	0.0307	0.6487	0.0316	0.6491	0.0331
β_3	0.8682	0.0396	0.8217	0.0336	0.8767	0.0370	0.7775	0.0379
β_4	0.6463	0.0320	0.5866	0.0292	0.6430	0.0327	0.5928	0.0322
β_5	0.6388	0.0338	0.6186	0.0314	0.6842	0.0366	0.5919	0.0360
β_6	1.0033	0.0467	1.0562	0.0462	0.9208	0.0430	0.9238	0.0456
β_7	0.8665	0.0404	0.8679	0.0377	0.7682	0.0384	0.8201	0.0441
Q_{SMM}	0.0577		0.0388		0.0384		0.0034	
p-Value	0.0016		0.0047		0.0107		0.0832	

NOTES: This table reports estimation results for the dependence structure of the 101 return series of the S&P 100 constituents over the period July 2015 to May 2018 using various copula models, under the assumption of block equidependence. Estimates and asymptotic standard errors for the copula model parameters are reported, as well as the value of the SMM objective function Q_{SMM} evaluated at the estimated parameters and the p-value from the J-test of the over-identifying restrictions.

Table 5 either. For the Normal and Student's t copula we achieve "block equidependence" by specifying the correlation matrix based on the discussion of Creal and Tsay (2015). The degrees of freedom parameter, ν^{-1}, is estimated to be around 1/12 for the Student's t copula and 1/7 for the t-t factor copula. Both these parameter estimates are smaller and less significant than the corresponding estimate for the equidependence model. On the other hand, for the Skew t-t factor copula the estimated degrees of freedom parameter slightly increases and becomes more significant compared to the equidependence model. Similarly, the estimated asymmetry parameter, λ, is more significantly negative in the model based on block equidependence than in the restrictive model based on equidependence. This implies that we find stronger evidence of tail dependence and asymmetric dependence, when focusing on the model that allows for these two forms of non-normal dependence - the Skew t-t factor copula model. Hence, by considering industry-specific loadings on the common factor instead of a single loading for all assets the market-wide common factor is found to be more fat-tailed and left-skewed.

According to the value of the SMM objective function, also reported in Table 5, the Skew t-t factor copula model again provides a considerably better fit than the other models. Focusing on this copula, that seems to be a good choice for modeling the dependence structure of the S&P 100 constituents, we see that the coefficients on the common factor range from 0.59 for SIC group 4 (transportation, communications) and 5 (trade) to 0.92 for SIC group 6 (finance, insurance). This industry-specific dependence is in line with our preliminary study on the dependence across the pairs of stocks, reported in the first part of Section 5, where we found moderate heterogeneity in the correlation coefficients. Carefully comparing these loadings on the common factor to the values found by Oh and Patton (2017), who focused on a multi-factor model including industry factors, we see that the estimated loadings in our model are considerably smaller. Furthermore, the implied common factor found by Oh and Patton (2017) is again more asymmetric and less fat-tailed than in our analysis. Considering the comparison made in the first part of Section 5, this is as expected.

Table E.1 in Appendix E and Table 6 below respectively presents the intra- and inter-industry rank correlations and tail dependence coefficients implied by the estimated Skew t-t factor copula model[12] reported in Table 5. Table E.1 shows that the model implies rank correlations ranging from 0.27 (for stocks within SIC groups 4 and 5, and between stocks of these groups) to 0.44 (for stocks in SIC group 6), illustrating the heterogeneous dependence capture by this model. The coefficients reported in Table 6 reveal the substantial degree of asymmetry

[12] Rank correlations are based on 50,000 simulations from the estimated model, as a closed-form expression of the rank correlation is not available in this case. Upper and lower tail dependence coefficients are based on Proposition 1 and 2.

Table 6: Rank correlation and tail dependence implied by block equidependence model

	SIC 1	SIC 2	SIC 3	SIC 4	SIC 5	SIC 6	SIC 7
SIC 1	0.36\0.07	0.04	0.07	0.03	0.03	0.07	0.07
SIC 2	0.25	0.25\0.04	0.04	0.03	0.03	0.04	0.04
SIC 3	0.36	0.25	0.43\0.09	0.03	0.03	0.09	0.09
SIC 4	0.18	0.18	0.18	0.18\0.03	0.03	0.03	0.02
SIC 5	0.18	0.18	0.18	0.18	0.18\0.03	0.03	0.03
SIC 6	0.36	0.25	0.43	0.18	0.18	0.62\0.18	0.12
SIC 7	0.36	0.25	0.43	0.18	0.18	0.49	0.49\0.12

NOTES: This table reports dependence measures implied by the estimated Skew *t*-*t* factor copula model as reported in Table 5, under the assumption of block equidependence. The diagonal elements correspond to instra-industry dependence and the off-diagonal elements to inter-industry dependence. We report the upper tail dependence coefficients (upper triangle) and lower tail dependence coefficients (lower triangle) based on Proposition 1 and 2 and the estimated copula model parameters.

that this copula model captures: lower tail dependence coefficients take on an average value of 0.28 and range from 0.18 to 0.62, while the upper tail dependence averages 0.06 and ranges from 0.03 to 0.18.

Finally, we examine the specification of the copula models considered in this section. The p-value from the J-test of the over-identifying restrictions is reported in the last row of Table 5 and shows that the Normal, the Student's t, and the factor t-t copula models are all rejected at the 5% significance level. However, for the Skew t-t factor copula model we have a p-value of 0.08, indicating that this model passes the J-test at this significance level. Hence, by relaxing the restrictive assumption of equidependence and allowing for heterogeneous dependence across industries the goodness of fit improves significantly. The results suggest that a factor copula model with heterogeneous, industry-specific loadings on the common factor, that allows for tail dependence and asymmetric dependence, is needed to capture the dependence between the S&P 100 returns.

5.3 Results Markov-switching copula specifications

Whereas in the previous sections the dependence structure was assumed to be constant over time, in this section we consider a different approach to modeling the dependence of the S&P 100 asset returns. More specifically, we examine the equidependence and block equidependence models in the context of Markov-switching, where the parameters characterizing the dependence structure are permitted to differ across regimes over which can be switched through time. As we have seen, a fat-tailed and asymmetric market-wide common factor is needed to model the dependence structure of the stock returns. Therefore, we solely focus on the Skew t-t factor copula. For comparative purposes we consider both an equidependence and a block equidependence specification, for both two regimes and a single regime. Note that the latter simply corresponds to estimation of the Skew t-t factor copula as examined in the previous sections, where in this case maximum likelihood estimation is used instead of the simulation-based estimation method.

The estimation results for the Markov-switching factor copula models can be found in Table 7. The reported standard errors are obtained via numerical estimation of the second order derivatives of the copula log-likelihood function[13]. Moving from SMM to MLE yields more efficient parameter estimates and it is thus interesting to compare the results presented in the right-hand side of the table to the results in Table 4 and 5. We see that the inverse degrees of freedom parameter, ν^{-1}, is estimated to be around 1/4 by MLE for both equidependence and block equidependence, whereas SMM estimates this parameter to be around 1/5. Besides, for both specifications MLE yields an estimated asymmetry parameter, λ, of around -0.13, which is smaller than SMM estimates this parameter to be. Lastly, MLE yields lower industry-specific loadings on the common factor for each industry, except for the finance and insurance industry. Thus, the dependence measures used in the simulation-based estimation method of Oh and Patton (2013) appear to better matched for larger loadings on the common factor, compared to maxi-

[13]Due to time limitations we resort to this method, instead of working with the sandwich form variance-covariance matrix as discussed by Chollete et al. (2009). As a result, the reported standard errors do not take into account any estimation error that may result from the estimation of the marginal distribution parameters. The standard errors are reported for indicative purposes only and should be interpreted with caution.

Table 7: Estimation results daily S&P 100 stock returns - Markov-switching

	REGIME 1		REGIME 2		NO REGIME SHIFTS	
	Est	Std Err	Est	Std Err	Est	Std Err
Panel A: Equidependence						
ν^{-1}	0.2096	0.0140	0.2368	0.0162	0.2529	0.0046
λ	-0.0660	0.0347	-0.1442	0.1043	-0.1275	0.0172
β	1.1415	0.0259	0.3375	0.0306	0.7225	0.0075
p_{11}	0.9663	0.0239				
p_{22}			0.9623	0.0237		
Panel B: Block Equidependence						
ν^{-1}	0.1934	0.0117	0.2587	0.0159	0.2540	0.0047
λ	-0.0878	0.0325	-0.1446	0.0696	-0.1344	0.0169
β_1	0.8470	0.1212	0.1579	0.0328	0.4818	0.0373
β_2	1.0187	0.0410	0.1251	0.0167	0.5776	0.0151
β_3	1.1821	0.0440	0.3026	0.0311	0.8321	0.0153
β_4	1.0109	0.0526	0.0878	0.0183	0.4591	0.0178
β_5	0.8936	0.0664	0.1123	0.0202	0.5125	0.0237
β_6	1.4102	0.0488	0.7338	0.0533	0.9575	0.0163
β_7	1.2100	0.0607	0.2577	0.0322	0.8190	0.0222
p_{11}	0.9696	0.0238				
p_{22}			0.9720	0.0239		

	Equidependence		Block Equidependence	
	One regime	Two regimes	One regime	Two regimes
L_c	15421.2	16837.7	15740.0	17428.4
AIC	-30836.3	-33659.4	-31461.9	-34816.8
BIC	-30822.6	-33622.8	-31420.8	-34725.3

NOTES: This table reports estimation results for the dependence structure of the 101 return series of the S&P 100 constituents over the period July 2015 to May 2018 using a Skew *t-t* Markov-switching factor copula model. Results are reported for equidependence and block equidependence specifications, for both two regimes and a single regime. Estimates and standard errors for the model parameters are reported, where standard errors are based on the inverse Hessian of the copula log-likelihood function. The value of the copula log-likelihood at the estimated model parameters is reported, as well as the corresponding AIC and BIC values.

mum likelihood estimation, which is generally more efficient.

However, of particular interest is the comparison between the right-hand side of Table 7, where a single regime is assumed, and the left-hand side of the table, where switches between two regimes are allowed. According to both the AIC and BIC the Markov-switching Skew *t-t* factor copula models outperform the models where a single regime is imposed, providing evidence of a nonconstant dependence structure of the S&P 100 returns. Furthermore, the Skew *t-t* factor copula model with industry-specific loadings and regime switches is the best performing model in terms of these information criteria. Focusing on this model, we notice that the estimated transition probability parameters, p_{11} and p_{22}, are close to one, indicating the persistence of both regimes. The estimated inverse degree of freedom parameter implies around 5 degrees of freedom for the first regime and around 4 degrees of freedom for the second regime. The asymmetry parameter is estimated to be around -0.09 for regime 1 and -0.15 for regime 2. It is worth noting that, although the standard errors are not asymptotically correct, we may expect that any neglected estimation error arising from the estimation of the marginal distribution parameters has a similar impact on each of the reported standard errors, as we assume the marginal distributions to be regime-free. Hence, the standard errors indicate that the estimated asymmetry parameter (inverse degree of freedom parameter) for the second regime need not be more significantly negative (positive) than

Table 8: Tail dependence implied by block equidependence Markov-switching model

	SIC 1	SIC 2	SIC 3	SIC 4	SIC 5	SIC 6	SIC 7
				REGIME 1			
SIC 1	0.41\0.19	0.19	0.19	0.19	0.19	0.19	0.19
SIC 2	0.41	0.65\0.38	0.38	0.37	0.24	0.38	0.38
SIC 3	0.41	0.65	0.80\0.57	0.37	0.24	0.57	0.57
SIC 4	0.41	0.64	0.64	0.64\0.37	0.24	0.37	0.37
SIC 5	0.41	0.48	0.48	0.48	0.48\0.24	0.24	0.24
SIC 6	0.41	0.65	0.80	0.64	0.48	0.91\0.77	0.60
SIC 7	0.41	0.65	0.80	0.64	0.48	0.86	0.82\0.60
				REGIME 2			
SIC 1	0.00\0.00	0.00	0.00	0.00	0.00	0.00	0.00
SIC 2	0.00	0.00\0.00	0.0	0.00	0.00	0.00	0.00
SIC 3	0.00	0.00	0.02\0.00	0.00	0.00	0.00	0.00
SIC 4	0.00	0.00	0.00	0.00\0.00	0.00	0.00	0.00
SIC 5	0.00	0.00	0.00	0.00	0.00\0.00	0.00	0.00
SIC 6	0.00	0.00	0.02	0.00	0.00	0.36\0.12	0.00
SIC 7	0.00	0.00	0.01	0.00	0.00	0.01	0.01\0.00

NOTES: This table reports dependence measures implied by the estimated Skew *t*-*t* Markov-switching factor copula model as reported in Table 7, under the assumption of block equidependence. The diagonal elements correspond to instra-industry dependence and the off-diagonal elements to inter-industry dependence. Conditional on the first regime (top panel) and the second regime (lower panel), we report the upper tail dependence coefficients (upper triangle) and lower tail dependence coefficients (lower triangle) based on Proposition 2 and 3 and the estimated copula Markov-switching model parameters.

for the first regime. This makes it difficult to compare the extent to which the the common factor is fat-tailed and left-skewed for both regimes. The estimated loadings on the market-wide common factor, β_i, differ considerably across the two regimes: for the first regime the estimates average 1.08 and range from 0.85 for SIC group 1 (mining, construction) to 1.41 for SIC group 6, while for the second regime the coefficients average 0.25 and range from 0.09 for SIC group 4 to 0.73, again for SIC group 6. On the one hand, similar to the previous section this indicates the varying degrees of inter-industry dependence, while on the other hand it provides an indication of the difference in dependence across the regimes.

We examine the dependence structure implied by the estimated Skew *t*-*t* factor copula Markov-switching model by again considering the intra- and inter-industry rank correlations and tail dependence coefficients implied by the model. The implied rank correlations can be found in Table E.2 in Appendix E and show that, similar to the previous section, the considered model captures heterogeneous dependence across the industries, where the dependence is markedly stronger in the first regime than in the second regime. Table 8 presents the implied upper and lower tail dependence coefficients conditional on the regime and reveals a substantial difference in tail dependence across the two regimes. For the first regime we find relatively large tail dependence coefficients, compared to the values reported in Table 6. The lower tail dependence coefficients are now 0.59 on average and range from 0.41 to 0.91, and the upper tail dependence averages 0.35 and ranges from 0.19 to 0.77. On the other hand, conditional on the second regime the implied tail dependence coefficients are considerably lower than those reported in Table 6. Namely, apart from the tail dependence within the finance and insurance industry, across and within most industries the lower and upper tail dependence coefficients are zero to two decimal places, averaging to 0.02 and 0.00, respectively. It is especially apparent for the first regime that these results are in line with the negative estimate for the asymmetry parameter. The distinct difference in rank correlations and tail dependence across the two regimes indicates the time-varying nature of the dependence structure of the 101 daily asset returns.

The results reported in this section provide evidence of the presence of different regimes. The first regime is characterized by large tail dependence and asymmetric dependence across the asset returns, while the second regime corresponds to periods of substantially less dependence between the assets. In the previous section we stated that a factor copula model with industry-specific loadings on the common factor, that allows for both tail dependence and asymmetric dependence, is needed to fit the dependence of the S&P 100 returns. While this model is capable of capturing important non-normal features of the dependence structure, following the results presented in this section we conclude that imposing constant dependence over time is too restrictive, and the dependence structure of the asset returns can be more adequately modeled by considering regime shifts.

6 Conclusion

In this paper we study the dependence structure of a large set of economic variables. To this end, copula models are attractive and hence widely used, gaining popularity over traditional correlation-based approaches. However, while there are numerous bivariate copula specifications available, there is a relative scarcity of copula models suitable for high-dimensional applications. We build on the research conducted by Oh and Patton (2017) and employ their proposed factor copula model. On the one hand, the factor structure that generates the factor copulas provides dimension reduction in the component of the joint distribution that is difficult to estimate in high dimensions - the copula, - facilitating the study of high-dimensional problems. On the other hand, the factor structure allows us to easily account for different forms of non-normal dependence across the variables.

We apply the factor copula model to daily returns of the S&P 100 constituents over the period July 2015 - May 2018. We consider a single market-wide common factor and show that industry-specific loadings are needed to adequately capture heterogeneous dependence. Further, we find evidence of a fat-tailed and left-skewed common factor, implying tail dependence and asymmetric dependence between the returns. We propose an extension of this model to a Markov-switching factor copula model, relaxing the assumption of constant dependence over time, which allows us to capture more complex dynamics of the dependence structure. We show that the proposed Markov-switching model is preferred to the factor copula model for modeling the dependence structure of the asset returns, in terms of two frequently used information criteria. Moreover, we demonstrate the presence of two persistent regimes, where the dependence structure of the returns is characterized by relatively large tail dependence and asymmetric dependence during the first regime, while the second regime corresponds to periods of considerably less dependence. These results again stress the importance of permitting the implied common factor to be fat-tailed and asymmetric, as also shown by Oh and Patton (2017). However, our high-dimensional Markov-switching analysis, empirically showing the presence of regime shifts in the dependence structure of 101 of the S&P 100 constituents, is novel.

7 Discussion and Further Research

Our study was bounded by time and computational capabilities, giving rise to several limitations. First, we assume two regimes that may characterize the dependence structure at any point in time. Allowing for a larger number of regimes makes estimation of the model more computationally demanding, and will likely decrease estimation accuracy. If such problems can be overcome, by tightly parameterizing the relation between the regimes (Calvet and Fisher (2004)) or with prior Bayesian information (Sims and Zha (2006)), for example, this would be an interesting avenue for future research.

Second, the standard errors of the Markov-switching factor copula model parameters are estimated using the second order derivatives of the copula log-likelihood function, underestimating the asymptotic standard errors. This makes it difficult to make precise statements about the (relative) significance of the estimated Markov-switching model parameters. Obtaining the robust standard errors of the model parameters is computationally expensive, as shown by Chollete et al. (2009), among others, incorporating general theorems of Newey and McFadden (1994). However, once the asymptotically correct standard errors are obtained a more adequate comparison between the estimated Markov-switching parameters and the implied market-wide common factor across the different regimes will be possible.

Lastly, due to the poor performance of the SMM estimator for the multi-factor specifications, contrary to Oh and Patton (2017) we did not study factor structures with more than one factor. If the parameters of a multi-factor copula model can be accurately estimated one would expect the goodness of fit to improve when additional factors are added to the single-factor copula model. Within the context of Markov-switching, where maximum likelihood estimation is used, it would be interesting to consider multiple common factors or the inclusion of industry-specific factors. We leave this extension for future work.

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Appendix A: Tail dependence properties

Proposition 1. *Tail dependence one-factor copula — Consider the factor copula generated by Equation* (3) *with* $K = 1$ *factor. Assume* F_Z*, and* F_ε *have regularly varying tails with common tail index* $\alpha > 0$*, that is,*

$$p\text{–}\lim_{s\to\infty}[Z < -s] \to A_z^L s^{-\alpha} \text{ and } p\text{–}\lim_{s\to\infty}[\varepsilon_i < -s] \to A_\varepsilon^L s^{-\alpha}$$
$$p\text{–}\lim_{s\to\infty}[Z > s] \to A_z^U s^{-\alpha} \text{ and } p\text{–}\lim_{s\to\infty}[\varepsilon_i > s] \to A_\varepsilon^U s^{-\alpha},$$

where lower tail coefficients A_z^L, A_ε^L *and upper tail coefficients* A_z^U, A_ε^U *are positive constants. Then,*
(a) if $\beta_i, \beta_j > 0$

$$\tau_{ij}^L = \frac{\min\ (\beta_i,\ \beta_j)^\alpha A_z^L}{\min\ (\beta_i,\ \beta_j)^\alpha A_z^L + A_\varepsilon^L}$$
$$\tau_{ij}^U = \frac{\min\ (\beta_i,\ \beta_j)^\alpha A_z^U}{\min\ (\beta_i,\ \beta_j)^\alpha A_z^U + A_\varepsilon^U};$$

(b) if $\beta_i, \beta_j < 0$

$$\tau_{ij}^L = \frac{\min\ (|\beta_i|,\ |\beta_j|)^\alpha A_z^U}{\min\ (|\beta_i|,\ |\beta_j|)^\alpha A_z^U + A_\varepsilon^L}$$
$$\tau_{ij}^U = \frac{\min\ (|\beta_j|,\ |\beta_j|)^\alpha A_z^L}{\min\ (|\beta_i|,\ |\beta_j|)^\alpha A_z^L + A_\varepsilon^U};$$

(c) if $\beta_i\beta_j = 0$ *or (d) if* $\beta_i\beta_j < 0$*, both the lower and the upper tail dependence coefficients are zero.*

Proposition 2. *Tail coefficients one-factor Skew t-t copula — Consider the factor copula generated by Equation* (3) *with* $K = 1$ *factor. If* $F_Z = \text{Skew } t(\nu, \lambda)$ *and* $F_\varepsilon = t(\nu)$*, then the tail indices of* Z *and* ε_i *equal* ν*, and the tail coefficients are given by*

$$A_Z^L = \frac{bc}{v}\left(\frac{b^2}{(v-2)(1-\lambda)^2}\right)^{-(v+1)/2}$$
$$A_Z^U = \frac{bc}{v}\left(\frac{b^2}{(v-2)(1+\lambda)^2}\right)^{-(v+1)/2}$$
$$A_\varepsilon^L = A_\varepsilon^U = \frac{c}{v}\left(\frac{1}{v-2}\right)^{-(v+1)/2},$$

where $a = 4\lambda c(\nu-2)/(\nu-1)$*,* $b = \sqrt{1+3\lambda^2-a^2}$*,* $c = \Gamma(\frac{\nu+1}{2})/(\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)})$ *and* Γ *denotes the gamma function.*

Appendix B: Moments SMM estimation

Let δ_{ij} denote one of the dependence measure (i.e., rank correlation or 0.05, 0.10, 0.90, or 0.95 quantile dependence) between variables i and j, and define the "pair-wise dependence matrix":

$$D = \begin{bmatrix} 1 & \delta_{12} & \dots & \delta_{1N} \\ \delta_{12} & 1 & \dots & \delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1N} & \delta_{2N} & \dots & 1 \end{bmatrix}. \tag{B.1}$$

For the equidependence model, as specificied in Equation (34), we use as "moments" the average of the five dependence measures across all pairs of variables, reducing the number of moments to match from $5N(N-1)/2$ to just 5:

$$\bar{\delta} \equiv \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\delta}_{ij} \tag{B.2}$$

For the block equidependence and the multi-factor model, as specified in Equation (35) and (36), respectively, we exploit the fact that (i) all variables in the same group exhibit equidependence, and (ii) any pair of variables (i, j) in groups (r, s) has the same dependence as any other pair (i', j') in the same two groups (r, s). This allows us to average all intra- and intergroup dependence measures. Consider the following general design, where we have N variables, M groups, and k_m variables in group m, where $\sum_{m=1}^{M} k_m = N$. Then decompose the $(N \times N)$ matrix D into sub-matrices according to the groups:

$$\underset{(N\times N)}{D} = \begin{bmatrix} D_{11} & D'_{12} & \dots & D'_{1M} \\ D_{12} & D_{22} & \dots & D'_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1M} & D_{2M} & \dots & D_{MM} \end{bmatrix}, \text{ where } D_{ij} \text{ is } (k_i \times k_j). \tag{B.3}$$

Then create the matrix of average values from each of these sub-matrices, taking into account the fact that the diagonal blocks are symmetric:

$$\underset{(M\times M)}{D^*} = \begin{bmatrix} \delta^*_{11} & \delta^*_{12} & \dots & \delta^*_{1M} \\ \delta^*_{12} & \delta^*_{22} & \dots & \delta^*_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^*_{1M} & \delta^*_{2M} & \dots & \delta^*_{MM} \end{bmatrix}, \tag{B.4}$$

where $\delta^*_{ss} \equiv \frac{2}{k_s(k_s-1)} \sum\sum \hat{\delta}_{ij}$, the average of all upper triangle values in D_{ss}, and $\delta^*_{rs} \equiv \frac{1}{k_r k_s} \sum\sum \hat{\delta}_{ij}$, the average of all elements in D_{rs}, for $r \neq s$. Finally, create the vector of average measures $[\bar{\delta}^*_1, \dots, \bar{\delta}^*_M]$, where

$$\bar{\delta}^*_i \equiv \frac{1}{M} \sum_{j=1}^{M} \delta^*_{ij}. \tag{B.5}$$

This gives a total of M moments for each dependence measure, so 5M in total.

Appendix C: Simulation study SMM estimator

Table C.1: Simulation results equidependence model - iid data

	Normal	Factor t-t		Factor Skew t-t		
	β	β	ν^{-1}	β	ν^{-1}	λ
True Value	1.00	1.00	0.25	1.00	0.25	-0.50
			$N = 3$			
Bias	-0.0062	-0.0129	0.0349	0.0165	-0.0143	-0.0772
Std	0.0496	0.0620	0.1076	0.0653	0.1273	0.1543
Median	0.9996	0.9901	0.2888	1.0101	0.2172	-0.5559
90%	1.0556	1.0655	0.4257	1.0827	0.4438	-0.4090
10%	0.9297	0.9046	0.1406	0.9322	0.1014	-0.8119
90-10 Diff	0.1260	0.1608	0.2851	0.1504	0.3424	0.4029
			$N = 10$			
Bias	-0.0018	-0.0116	0.0196	0.0091	-0.0240	-0.0549
Std	0.0283	0.0489	0.0925	0.0568	0.1042	0.1441
Median	0.9965	0.9900	0.2632	1.0016	0.2213	-0.5221
90%	1.0329	1.0499	0.3988	1.0790	0.3548	-0.4139
10%	0.9670	0.9192	0.1718	0.9421	0.0934	-0.7619
90-10 Diff	0.0659	0.1306	0.2270	0.1369	0.2614	0.3480
			$N = 100$			
Bias	-0.0008	-0.0083	0.0130	0.0088	-0.0080	-0.0295
Std	0.0231	0.0421	0.0698	0.0463	0.0748	0.0936
Median	1.0009	0.9857	0.2672	1.0078	0.2457	-0.5146
90%	1.0276	1.0471	0.3471	1.0734	0.3263	-0.4327
10%	0.9699	0.9413	0.1723	0.9513	0.1482	-0.6245
90-10 Diff	0.0577	0.1058	0.1748	0.1221	0.1781	0.1918

NOTES: This table presents simulation results from 100 simulations of the Normal copula, the t-t factor copula and the Skew t-t factor copula, under the assumption of equidependence. The copulas are estimated by SMM, as described in Section 2.4, and the data are assumed to be iid with standard Normal marginal distributions. Results are reported for dimensions $N = 3, 10$, and 100, the sample size is $T = 1000$ and $S = 25 \times T$ simulations are used in the computation of the dependence measures to be matched in the SMM estimation. For each panel, the first row reports the average difference between the estimated parameters and their true values, while the second row reports the standard deviation in the estimated parameters. The third, fourth, and fifth row respectively report the 50^{th}, 90^{th}, and 10^{th} percentiles of the distribution of the estimated parameters. The last row reports the difference between the 90^{th} and 10^{th} percentiles.

Table C.2: Simulation results equidependence model - AR-GARCH data

	Normal	Factor t-t		Factor Skew t-t		
	β	β	ν^{-1}	β	ν^{-1}	λ
True Value	1.00	1.00	0.25	1.00	0.25	-0.50
			$N = 3$			
Bias	-0.0021	-0.0149	-0.0181	0.0043	-0.0485	-0.0299
Std	0.0463	0.0670	0.1248	0.0653	0.1100	0.1817
Median	1.0020	0.9904	0.2082	1.0034	0.2101	-0.4947
90%	1.0525	1.0728	0.4367	1.0892	0.3318	-0.3244
10%	0.9373	0.9036	0.0929	0.9315	0.0482	-0.7660
90-10 Diff	0.1152	0.1691	0.3438	0.1577	0.2836	0.4416
			$N = 10$			
Bias	-0.0084	-0.0021	-0.0111	-0.0032	-0.0558	-0.0137
Std	0.0330	0.0456	0.0828	0.0427	0.0926	0.1163
Median	0.9927	1.0034	0.2544	0.9993	0.1931	-0.4873
90%	1.0280	1.0547	0.3454	1.0496	0.3098	-0.4023
10%	0.9524	0.9437	0.1237	0.9406	0.0602	-0.6510
90-10 Diff	0.0756	0.1110	0.2217	0.1090	0.2496	0.2487
			$N = 100$			
Bias	-0.0040	-0.0138	-0.0209	-0.0051	-0.0220	0.0388
Std	0.0265	0.0406	0.0818	0.0423	0.0674	0.0894
Median	0.9936	0.9856	0.2276	0.9974	0.2234	-0.4679
90%	1.0265	1.0341	0.3474	1.0425	0.3073	-0.3785
10%	0.9704	0.9320	0.1284	0.9382	0.1375	-0.5441
90-10 Diff	0.0561	0.1021	0.2190	0.1044	0.1698	0.1656

NOTES: This table presents simulation results from 100 simulations of the Normal copula, the t-t factor copula and the Skew t-t factor copula, under the assumption of equidependence. The copulas are estimated by SMM, as described in Section 2.4, and the marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes, as discussed in Section 4. Results are reported for dimensions $N = 3, 10$, and 100, the sample size is $T = 1000$ and $S = 25 \times T$ simulations are used in the computation of the dependence measures to be matched in the SMM estimation. For each panel, the first row reports the average difference between the estimated parameters and their true values, while the second row reports the standard deviation in the estimated parameters. The third, fourth, and fifth row respectively report the 50^{th}, 90^{th}, and 10^{th} percentiles of the distribution of the estimated parameters. The last row reports the difference between the 90^{th} and 10^{th} percentiles.

Table C.3: Simulation results block equidependence and multi-factor model - $N = 10$

	β_1	β_2	β_3	ν^{-1}	λ	β	γ_1	γ_2	γ_3	ν^{-1}	λ
True Value	0.25	0.5	0.75	0.25	-0.5	1	0.25	0.5	0.75	0.25	-0.5
						Normal					
Bias	-0.0009	0.0018	-0.0022	—	—	0.0241	-0.1147	-0.0328	0.0667	—	—
Std	0.0199	0.0343	0.0495	—	—	0.0625	0.4368	0.5207	0.4859	—	—
Median	0.2460	0.4974	0.7518	—	—	1.0168	0.1563	0.5579	0.8457	—	—
90%	0.2748	0.5459	0.8119	—	—	1.0986	0.6800	1.0905	1.5031	—	—
10%	0.2278	0.4582	0.6891	—	—	0.9488	-0.4482	-0.4119	0.2153	—	—
Diff	0.0471	0.0877	0.1228	—	—	0.1498	1.1282	1.5025	1.2879	—	—
						Factor t - t					
Bias	-0.0046	-0.0083	-0.0040	-0.0023	—	0.0548	-0.0278	0.0883	-0.0852	-0.0224	—
Std	0.0264	0.0423	0.0633	0.1024	—	0.0695	0.5575	0.5536	0.5340	0.0982	—
Median	0.2443	0.4941	0.7529	0.2622	—	1.0493	0.2064	0.6597	0.9511	0.2257	—
90%	0.2805	0.5423	0.8245	0.3663	—	1.1488	0.9860	1.2591	1.6267	0.3532	—
10%	0.2136	0.4297	0.6598	0.0975	—	0.9742	-0.4829	-0.3063	0.1454	0.1010	—
Diff	0.0670	0.1125	0.1646	0.2688	—	0.1745	1.4689	1.5654	1.4813	0.2522	—
						Factor Skew t - t					
Bias	0.0017	0.0093	0.0075	-0.0216	-0.0509	0.0494	-0.0136	0.0350	0.1420	-0.0292	-0.1522
Std	0.0242	0.0372	0.0653	0.0923	0.1463	0.0846	0.4951	0.4918	0.4749	0.1009	0.1597
Median	0.2523	0.5112	0.7496	0.2114	-0.5267	1.0354	0.2768	0.6434	0.9605	0.2180	-0.6362
90%	0.2827	0.5527	0.8409	0.3487	-0.3867	1.1327	0.8210	1.1095	1.4458	0.3428	-0.4569
10%	0.2213	0.4683	0.6749	0.1330	-0.7758	0.9702	-0.4213	-0.1962	0.1085	0.0984	-0.8936
Diff	0.0614	0.0844	0.1660	0.2156	0.3891	0.1625	1.2423	1.3057	1.3373	0.2444	0.4367

NOTES: This table presents simulation results from 100 simulations of Skew t-t factor copula, under the assumption of block equidependence (left) and for a multi-factor specification (right). The copula models are estimated by SMM, as described in Section 2.4, and the data are assumed to be iid with standard Normal margins. We consider dimension $N = 10$, where the variables are divided into three groups of decreasing size. The sample size is $T = 1000$ and $S = 25 \times T$ simulations are used in the computation of the dependence measures to be matched in the SMM estimation. For each panel, the first row reports the average difference between the estimated parameters and their true values, while the second row reports the standard deviation in the estimated parameters. The third, fourth, and fifth row respectively report the 50^{th}, 90^{th}, and 10^{th} percentiles of the distribution of the estimated parameters. The last row reports the difference between the 90^{th} and 10^{th} percentiles.

Table C.4: Simulation results block equidependence and multi-factor model - $N = 20$

	β_1	β_2	β_3	ν^{-1}	λ	β	γ_1	γ_2	γ_3	ν^{-1}	λ
True Value	0.25	0.5	0.75	0.25	-0.5	1	0.25	0.5	0.75	0.25	-0.5
						Normal					
Bias	0.0020	0.0008	0.0007	—	—	0.0169	-0.1408	-0.0871	0.0879	—	—
Std	0.0131	0.0187	0.0312	—	—	0.0637	0.4878	0.5157	0.3938	—	—
Median	0.2514	0.5007	0.7490	—	—	1.0087	0.0832	0.4421	0.8396	—	—
90%	0.2684	0.5227	0.7918	—	—	1.1041	0.7073	1.0485	1.3472	—	—
10%	0.2365	0.4768	0.7112	—	—	0.9458	-0.5882	-0.3943	0.4268	—	—
Diff	0.0319	0.0458	0.0805	—	—	0.1583	1.2955	1.4429	0.9204	—	—
						Factor t - t					
Bias	-0.0012	-0.0007	0.0099	-0.0111	—	0.0443	0.0099	0.1491	0.1910	0.0026	—
Std	0.0228	0.0356	0.0518	0.1013	—	0.0788	0.5533	0.5096	0.5060	0.0769	—
Median	0.2492	0.5028	0.7568	0.2359	—	1.0446	0.3093	0.7281	1.0146	0.2488	—
90%	0.2808	0.5483	0.8295	0.3818	—	1.1417	1.0385	1.3125	1.5633	0.3362	—
10%	0.2193	0.4497	0.7034	0.0938	—	0.9395	-0.5118	-0.0826	0.1959	0.1465	—
Diff	0.0615	0.0986	0.1261	0.2879	—	0.2022	1.5503	1.3951	1.3674	0.1897	—
						Factor Skew t - t					
Bias	0.0009	0.0038	0.0086	-0.0164	-0.0684	0.0371	-0.0033	0.0314	0.1485	-0.0424	-0.1742
Std	0.0218	0.0302	0.0453	0.0904	0.1583	0.0748	0.4795	0.4405	0.2470	0.0845	0.1797
Median	0.2493	0.4977	0.7537	0.2318	-0.5312	1.0360	0.3325	0.6476	0.8883	0.2015	-0.6472
90%	0.2778	0.5432	0.8231	0.3477	-0.4178	1.1054	0.7834	0.9672	1.1808	0.3123	-0.4566
10%	0.2243	0.4719	0.7074	0.1284	-0.7795	0.9579	-0.4913	-0.2939	0.6189	0.0891	-0.9534
Diff	0.0535	0.0713	0.1157	0.2193	0.3616	0.1475	1.2748	1.2611	0.5619	0.2232	0.4968

NOTES: This table presents simulation results from 100 simulations of Skew t-t factor copula, under the assumption of block equidependence (left) and for a multi-factor specification (right). The copula models are estimated by SMM, as described in Section 2.4, and the data are assumed to be iid with standard Normal margins. We consider dimension $N = 20$, where the variables are divided into three groups of decreasing size. The sample size is $T = 1000$ and $S = 25 \times T$ simulations are used in the computation of the dependence measures to be matched in the SMM estimation. For each panel, the first row reports the average difference between the estimated parameters and their true values, while the second row reports the standard deviation in the estimated parameters. The third, fourth, and fifth row respectively report the 50^{th}, 90^{th}, and 10^{th} percentiles of the distribution of the estimated parameters. The last row reports the difference between the 90^{th} and 10^{th} percentiles.

Appendix D: Stocks empirical analysis

Table D.1: List of the 101 stocks used in empirical analysis

Ticker	Name	SIC	Ticker	Name	SIC	Ticker	Name	SIC
AAPL	Apple	357	EXC	Exelon	493	MRK	Merck	283
ABBV	AbbVie	283	F	Ford Motor	371	MS	Morgan Stanley	671
ABT	AbbottLab	283	FB	Facebook	737	MSFT	Microsoft	737
ACN	Accenture	739	FDX	FedEx	451	NEE	NextEra Energy	491
AGN	Allergan	283	FOX	Fox (B)	484	NKE	Nike	302
AIG	American Int	633	FOXA	Fox (A)	484	ORCL	Oracle	737
ALL	Allstate	633	GD	GeneralDynam	373	OXY	OccidentalPetrol	131
AMGN	Amgen	283	GE	General Elec	351	PEP	Pepsico	208
AMZN	Amazon.com	737	GILD	GileadScience	283	PFE	Pfizer	283
AXP	AmericanExpr	671	GM	General Motors	373	PG	Procter&Gamble	284
BA	Boeing	372	GOOG	Alphabet (B)	737	PM	Phillip Morris	211
BAC	Bank of Am	602	GOOGL	Alphabet (A)	737	PYPL	PayPal	739
BIIB	Biogen	284	GS	GoldmanSachs	621	QCOM	Qualcomm	366
BK	Bank of NY	602	HAL	Halliburton	138	RTN	Raytheon	381
BLK	BlackRock	621	HD	Home Depot	525	SBUX	Starbucks	581
BMY	Bristol-Myers	283	HON	Honeywell	372	SLB	Schlumberger	138
BRK/B	BerkshHath (B)	633	IBM	IBM	357	SO	Southern Co	491
C	Citigroup	602	INTC	Intel	367	SPG	Simon Property	680
CAT	Caterpillar	353	JNJ	Johnson&J.	283	T	AT&T	481
CELG	Celgene	283	JPM	JP Morgan	672	TGT	Target	533
CHTR	Charter Comm	484	KHC	Kraft Heinz	203	TWX	Time Warner	737
CL	Colgate	284	KMI	Kinder Morgan	492	TXN	Texas Inst	367
CMCSA	Comcast	484	KO	Coca-Cola	208	UNH	UnitedHealth	632
COF	Capital One	614	LLY	Eli Lilly	283	UNP	Union Pacific	401
COP	ConocoPhillips	291	LMT	Lock'd Martin	376	UPS	United Parcel	451
COST	Costco	533	LOW	Lowe's	521	USB	US Bancorp	602
CSCO	Cisco	367	MA	MasterCard	615	UTX	United Tech	372
CVS	CVS Health	591	MCD	McDonald's	581	V	Visa	739
CVX	Chevron	291	MDLZ	Mondelez Int.	200	VZ	Verizon Comm	481
DHR	Danaher	382	MDT	Medtronic	384	WBA	Walgreens	591
DIS	Walt Disney	484	MET	Metlife	671	WFC	Wells Fargo	602
DUK	Duke Energy	493	MMM	3M	384	WMT	Walmart	533
DWDP	DowDuPont	282	MO	Altria Group	211	XOM	Exxon Mobil	291
EMR	EmersonElec	360	MON	Monsanto	287			

	Description	Num		Description	Num
SIC 1	Mining, Construct	3	SIC 5	Trade	9
SIC 2	Manuf; Food, Furn	25	SIC 6	Finance, Insurance	18
SIC 3	Manuf; Electr, Mach	21	SIC 7	Services	10
SIC 4	Transprt, Comm	15	ALL		101

NOTES: The upper part of this table presents the ticket symbols, names, and three-digit SIC codes of the 101 stocks used in the empirical analysis reported in Section 5. The lower part of this table provides a description of the industries corresponding to each one-digit SIC group, based on the first digit of the SIC code, and reports the number of stocks in each group.

Appendix E: Rank correlations

Table E.1: Rank correlation implied by block equidependence model

	SIC 1	SIC 2	SIC 3	SIC 4	SIC 5	SIC 6	SIC 7
SIC 1	0.34						
SIC 2	0.32	0.30					
SIC 3	0.36	0.33	0.37				
SIC 4	0.30	0.28	0.31	0.27			
SIC 5	0.30	0.28	0.31	0.27	0.27		
SIC 6	0.39	0.36	0.40	0.34	0.34	0.44	
SIC 7	0.37	0.34	0.38	0.32	0.32	0.41	0.39

NOTES: This table reports dependence measures implied by the estimated Skew *t*-*t* factor copula model as reported in Table 5, where block equidependence is assumed. The diagonal elements correspond to instra-industry dependence and the off-diagonal elements to inter-industry dependence. We report the Spearman's rank correlation coefficients based on 50,000 simulations from the estimated factor copula model.

Table E.2: Rank correlation implied by block equidependence Markov-switching model

	SIC 1	SIC 2	SIC 3	SIC 4	SIC 5	SIC 6	SIC 7
				REGIME 1			
SIC 1	0.41						
SIC 2	0.45	0.49					
SIC 3	0.47	0.52	0.55				
SIC 4	0.44	0.49	0.52	0.48			
SIC 5	0.42	0.46	0.49	0.46	0.43		
SIC 6	0.50	0.55	0.59	0.55	0.52	0.63	
SIC 7	0.48	0.52	0.56	0.52	0.49	0.59	0.56
				REGIME 2			
SIC 1	0.03						
SIC 2	0.03	0.02					
SIC 3	0.06	0.05	0.10				
SIC 4	0.02	0.02	0.03	0.01			
SIC 5	0.02	0.02	0.04	0.01	0.02		
SIC 6	0.10	0.08	0.19	0.06	0.08	0.35	
SIC 7	0.05	0.04	0.09	0.03	0.04	0.16	0.08

NOTES: This table reports dependence measures implied by the estimated Skew *t*-*t* Markov-switching factor copula model as reported in Table 7, where block equidependence is assumed. The diagonal elements correspond to instra-industry dependence and the off-diagonal elements to inter-industry dependence. Conditional on the first regime (top panel) and the second regime (lower panel), we report the Spearman's rank correlation coefficients based on 50,000 simulations from the estimated factor copula model.