Modeling Dependence and Risk in High Dimensions with Static and Dynamic Factor Copulas

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Abstract

In this study we[1] examine multiple models for the dependence structure (copulas) of economic variables based on a latent factor structure. We differentiate between three types of copulas, allowing for a degree of time-varying dependence in increasing order: static, block-dynamic and GAS-dynamic copulas. After an extensive simulation study, the models are applied to daily returns on 50 constituents of the S&P 100 index. We compare various static and dynamic dependence models by their ability to create risk forecasts for the Value at Risk (VaR) and Expected Shortfall (ES). We find significant evidence of asymmetric dependence and tail dependence, implying that dependence is stronger during crises than in booms. We also show that factor copula models provide superior estimates of some risk measures, with the Skew $t-t$ factor copula being most consistent.

Keywords: Copulas; GAS dynamics; Value at Risk; Expected Shortfall; Correlation; (Time-varying) Dependence; Tail dependence; Risk.

[1] Throughout this report I use 'we' to refer to myself: Ridouan Mhand Yamna

Contents

1 Introduction

The financial crisis of 2007 has left a dent in the trust and beliefs of investors. This involves investors being more cautious with the risk they take. This change in behavior was due to the fact that, before the crisis, there was less reason to fear the risk of a total collapse. It was a commonplace for investors that if one diversified their portfolio enough, risk reduction would be adequate. However, this only holds if the correlation between assets is constant. In reality, correlations between assets can shift dramatically (even from negative to positive). Hence, investors were surprised to the extent to which assets that had previously behaved mostly independently suddenly moved together during the crisis. Many models for the dependence structure used during the crisis were inadequate. For example, correlation-based models failed to incorporate the dependence between the variables in the tails, since they rely on the assumption of multivariate Gaussianity. However, the validity of dependence models are of great importance for portfolio managers, since they are interested in quantifying their portfolio risk. This brings us to the following research question: *how can we adequately model the dependence structure between a set of financial assets, considering our goal is to quantify portfolio risk* ?

In recent years, existing dependence models have been improved and extended in order to capture this tail dependence and other desired features, such as asymmetric dependence. Copulas, or dependence structures, show great potential for capturing the mentioned desired features. Traditionally, we have a multivariate distribution with an implicit dependence structure. However, a copula allows us to split a multivariate distribution into marginal distributions and a dependence structure. The existing copulas suffer from either the curse of dimensionality or are just too simplistic to model dependence. Models that were developed for low dimensional problems are usually not applicable for problems of high dimension. The reason behind this is that generalizations beyond the bivariate model do not exist or are too restrictive (e.g., Archimedean copulas only have one/two free parameters regardless of the dimension size). In high-dimensional applications, we are interested in a model that is flexible and parsimonious. Oh and Patton (2017) propose a copula based on a latent factor structure, called factor copula model. The use of factors makes this model extremely attractive for high-dimensional applications, particularly for those involving more than 50 variables. This makes the factor copula an interesting tool for portfolio managers as portfolio's may contain a large set of financial assets. The factor copula also allows for a great amount of flexibility, since it offers the possibility to capture positive and negative dependence, asymmetric dependence and non-zero tail dependence in the form of common factors. In this paper we will use a selection of widely used copulas: the Normal, Clayton, Student's t and the proposed factor copulas. The Clayton copula allows for tail dependence and some asymmetry, but only has one parameter to characterize the dependence between all variables, which is a restriction for higher-dimension applications. The Normal copula introduced by Li (2000) is easy to implement, but has several limitations in the sense that it assumes zero tail dependence and symmetric dependence. Demarta and McNeil (2005) proposed the Student's t copula, which allows for positive tail dependence, but (like the Normal copula) imposes the unrealistic assumption of symmetric dependence. The various copulas are included as they can capture some, but not all, of the desired features that are evident in financial time series. Also, they may serve as a useful benchmark for our factor copulas.

This article makes three primary contributions. First, we introduce a set of block-dynamic factor copula. It is dynamic in the sense that we split our dataset into multiple sub-periods and estimate the parameters for each sub-period. This allows our model to capture a small amount of time-varying dependence. Oh and Patton (2018) proposed a different model for capturing time-varying dependence in high dimensions. They combine the "GAS" model of Creal et al. (2011,2013) and the factor copula model of Oh and Patton (2017). We also examine this GAS-dynamic factor copula model in our paper.

After an extensive simulation study, we study the dependence structure of 50 constituent firms of the Standard and Poor's 100 index, using daily log return data over the period 2015–2018. We will mainly examine the factor copula in more depth. For example, we will examine whether different time periods influence the accuracy of the estimation method significantly and hence, examine whether this dependence structure is vastly applicable in the financial world.

Our final contribution consists of comparing various copulas in terms of their ability to create risk forecasts for the Value at Risk (VaR) and Expected Shortfall (ES). Many Financial industry regulations (e.g. Basel II/III and Solvency II) use the VaR to measure the risk exposure of financial institutions. In words, VaR is the minimum return that could occur over a given holding period with a specified confidence level. Nevertheless, VaR has several theoretical disadvantages. Artzner et al. (1997,1999) have shown that the VaR ignores the loss beyond the VaR level, but it is also not sub-additive. This equals loss of important information and practical inefficiency for portfolio and risk managers. In order to deal with this, Artzner et al. (1999) introduced a new measure of financial risk referred to as the expected shortfall. The ES measures the expected loss exceeding VaR.

The remainder of the article is structured as follows. Section 2 presents general copula theory, the class of (dynamic) factor copulas, simulation-based estimation method, likelihood-based estimation method, Value at Risk and Expected Shortfall. Section 3 presents a simulation study for copula parameter estimation. Section 4 describes the data we use in an empirical study of daily returns on 50 constituents of the S&P 100 equity index over the period 2015–2018. Section 5 presents the results of the empirical study. Technical details and additional results are presented in the Appendix.

2 Methodology

Let $\mathbf{Y_t}$ = $(Y_{1t},...,Y_{Nt})$ for $t \in$ {1,...,T} be a strictly stationary time series, which represents daily observations of the log-returns of N stocks over time t. We assume that the dynamics of $\mathbf{Y}$ are given by

$$\mathbf{Y}_t = \boldsymbol{\mu}_t(\boldsymbol{\phi}) + \boldsymbol{\sigma}_t(\boldsymbol{\phi})\boldsymbol{\eta}_t, \tag{1}$$

where

$$\boldsymbol{\mu}_t(\boldsymbol{\phi}) \equiv [\mu_{1t}(\boldsymbol{\phi}),...,\mu_{Nt}(\boldsymbol{\phi})]'$$
$$\boldsymbol{\sigma}_t(\boldsymbol{\phi}) \equiv \text{diag}\{\sigma_{1t}(\boldsymbol{\phi}),...,\sigma_{Nt}(\boldsymbol{\phi})\}$$
$$\boldsymbol{\eta}_t(\boldsymbol{\phi}) \equiv [\eta_{1t},...,\eta_{Nt}]' \sim \text{iid } \mathbf{F}_{\boldsymbol{\eta}} = \mathbf{C}(F_1,...,F_N;\theta),$$

where $\boldsymbol{\eta}_t$ are the standardized innovations. We assume that $\boldsymbol{\mu_t}$ and $\boldsymbol{\sigma_t}$ are measurable with respect to ϑ_{t-1}, the information about the return process available up to time $t-1$. The $r \times 1$ vector $\boldsymbol{\phi}$ contains parameters which govern the dynamics of the variables, whereas θ is a $p \times 1$ vector of parameters for copula $\mathbf{C}$. In the following sections we will go in more detail concerning copulas and their estimation.

2.1 General Copula Theory

A copula is a function that represents a multivariate distribution by linking univariate distribution functions together. If all variables are continuously distributed, then their copula is simply a multivariate distribution function with standard uniform marginal distributions or *margins*. Consider a vector of random variables, $\boldsymbol{\eta} \equiv [\eta_1,...,\eta_N]$ with joint distribution $\mathbf{F}$ and margins $F_1,...,F_N$. We can map the joint distribution function from the individual distribution functions by making use of Sklar's (1959) theorem, which gives:

$$\mathbf{F}(\eta) \equiv \mathbf{C}_\eta(F_1(\eta_1),...,F_N(\eta_N)), \quad \forall \boldsymbol{\eta} \in \mathbb{R}^N \tag{2}$$

Here $\mathbf{C}_\eta$ denotes the copula of η and contains all the dependence information between the η_i's. Furthermore, it can also be shown that $\mathbf{U} = [U_1,...,U_N]' \sim \mathbf{C}_\eta$, where $U_i \equiv F_i(\eta_i) \sim Unif(0,1)$.

2.2 Factor Copula

The factor copula is based on latent variables, which are simulated and from which the desired copula is created. A latent factor structure is composed out of $N+K$ latent variables and is defined by:

$$\begin{aligned}
\text{Let} \quad X_i &= \sum_{k=1}^{K} \beta_{ik} Z_k + \varepsilon_i, \quad i = 1,2,...,N \\
\text{so} \quad [X_1,...,X_N]' &\equiv \mathbf{X} = \mathbf{BZ} + \boldsymbol{\varepsilon} \\
\text{where} \quad \varepsilon_i &\sim \text{iid } F_\varepsilon(\gamma_\varepsilon), \\
Z_k &\sim \text{inid } F_{zk}(\gamma_k),\ Z_k \perp\!\!\!\perp \varepsilon_i\ \forall i,k \\
\text{Then } \mathbf{X} &\sim \mathbb{F}_x = \mathbf{C}(G_1(\theta),....,G_N(\theta);\theta),
\end{aligned} \tag{3}$$

where $\theta = [vec(\mathbf{B})', \gamma_\varepsilon', \gamma_1', \gamma_k']'$. The latent variables in this factor structure are denoted as $\mathbf{X}$, where its copula is used to model the copula of the observable variables $\mathbf{Y}$. In this paper, we only consider $K = 1$, implying a one-factor copula.

2.3 Block-Dynamic Factor Copula

For the block-dynamic factor copula, β is time-varying, such that we have the set $\{\beta_h\}_{h=1}^H$. Here H denotes the number of periods of length T/H. This gives us the following one-factor model:

$$X_{ij} = \beta_{ih} Z + \varepsilon_i, \quad i = 1,2,...,N, \quad h = 1,2,...,H \tag{4}$$

$$j \in \{\frac{hT-T+H}{H}, \frac{hT-T+2H}{H}, ..., \frac{hT}{H}\}$$

The parameters of the block-dynamic and static factor copula will be estimated by means of an alternative of the simulated method of moments (SMM) proposed by Oh and Patton (2013). Section 2.6 will go in further detail concerning this estimation method.

2.4 GAS-Dynamic Factor Copula

The Generalized Autoregressive Score (GAS) is a method of Creal et al (2013) used for the modeling of time-varying parameters. Oh and Patton (2018) use this model to allow the factor copula to capture time-variation in the factor loadings. The model is given by

$$X_{it} = \psi_{g(i),t} Z_t + \varepsilon_{it}, \quad i = 1,2,...,N \tag{5}$$
$$\log \psi_{g,t} = \omega_g + \beta_{GAS} \log \psi_{g,t-1} + \alpha_{GAS} s_{g,t-1},$$
$$g = 1,2,...,G$$

where $g(i)$ denotes the group that corresponds to variable i, and G is the total number of groups. $G = 1$ corresponds to an equidependence structure, where the pair-wise dependence between each of the variables are identical. In our empirical application for the GAS-dynamic factor copula, we also consider grouping variables into $G = 6$ industries and allowing a different factor loading for each variable ($G = N$), corresponding to the block-equidependence and heterogeneous dependence case, respectively. Here $s_{g,t} = \partial \log \mathbf{c}(\boldsymbol{\eta}; \boldsymbol{\psi}, \gamma_\varepsilon, \gamma_k)/\partial \psi_{g,t}$ and $\boldsymbol{\psi} = [\psi_{1,t}, ..., \psi_{G,t}]$, with $\mathbf{c}(\boldsymbol{\eta}; \boldsymbol{\psi}, \gamma_\varepsilon, \gamma_k)$ denoting the copula density of $\mathbf{X}$. In the Appendix, we show how to obtain the copula density.

2.5 Factor Copula examples

The class of factor copulas is known for its flexibility. We illustrate this flexibility in Figure 1, where we take 1000 random draws from four bivariate distributions. We constructed these distributions by using four different factor copulas. In all four cases, the margins F_i are set to N(0,1), and the latent variables are set to have unit variance. We generate the first copula from a factor structure with $F_z = F_\varepsilon = N$(0,1), implying a Normal copula. The second copula sets $F_z = F_\varepsilon = t(4)$, which generates asymmetric copula with positive tail dependence. The third copula sets $F_\varepsilon = N(0,1)$ and F_z = Skew $t(\infty, -0.25)$, implying a skewed Normal distribution. This copula exhibits asymmetric dependence, such that crashes are more correlated than booms, but zero tail dependence. The fourth copula sets $F_\varepsilon = t(4)$ and F_z = Skew $t(4,-0.25)$, generating positive tail dependence and asymmetric dependence. For all cases, we use linear correlation of 0.5 by setting all off-diagonal elements of the correlation matrix to 0.5.

Figure 1 shows that there is zero tail dependence when the factor structure is based on a Normal or skewed Normal distribution. However, when the degrees of freedom is set to 4, we observe multiple draws in the joint upper and lower tails. The lower panels of Figure 1 show draws with a negative skewness parameter λ. Here, we observe stronger clustering of observations in the joint lower tail compared with the joint upper tail.

In our study, we are particularly interested in the differences between copulas in high-dimensional applications. Therefore, we consider a multivariate approach related to our study of systemic risk. We use a measure based on Hartmann, Straetmans, and de Vries (2006) and Geluk, de Haan, and de Vries (2007). Figure 2 illustrates, conditional on observing j out of 50 stocks crashing, the expected number and proportion of the remaining $(50-j)$ stocks that will crash. Define

$$N_q^* \equiv \sum_{i=1}^{N} \mathbb{1}\{U_i \leq q\},$$
$$\kappa^q(j) = E[N_q^* | N_q^* \geq j] - j, \tag{6}$$

and

$$\pi^q(j) \equiv \frac{\kappa^q(j)}{N-j}.$$

In this example, a crash is defined as a realization in the lower 1/66 tail. We consider four copulas: the Normal, Student's $t(4)$, Clayton copula and the Skew $t(4)$-$t(4)$ factor copula, such that the parameters imply a linear correlation of 0.5. The upper panel indicates that conditioning on more variables crashing, initially increases the expected number of other variables that will crash, $\kappa^q(j)$. Its peak is around j=15, where the Skew t (4)-t (4) factor copula predicts that around another 16 variables will

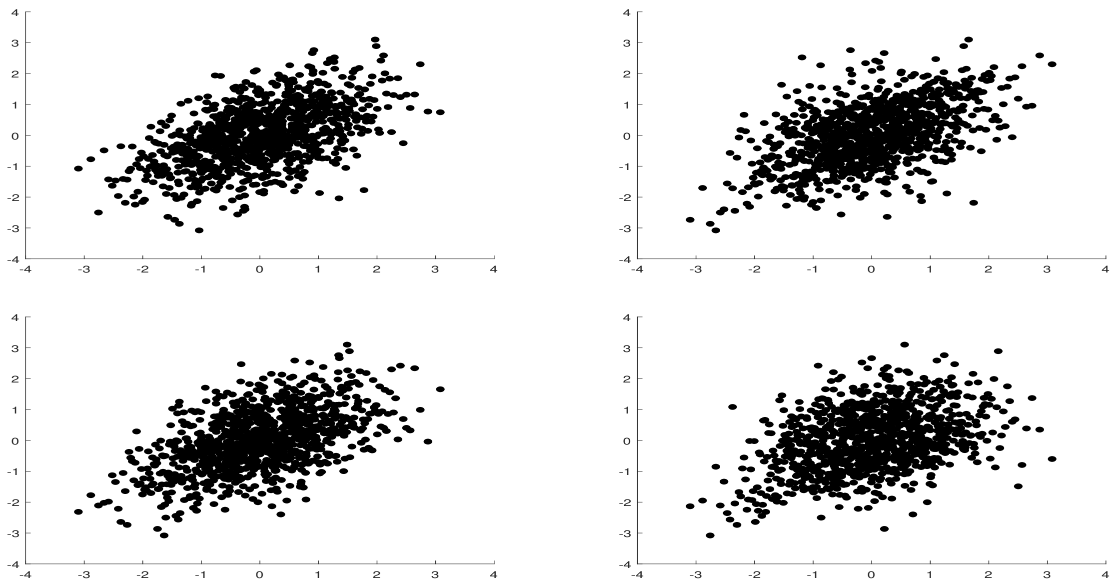

Figure 1: Scatterplots from four bivariate distribution, constructed by using four different factor copulas. Note: all distributions have $N(0,1)$ margins and linear correlation of 0.5.

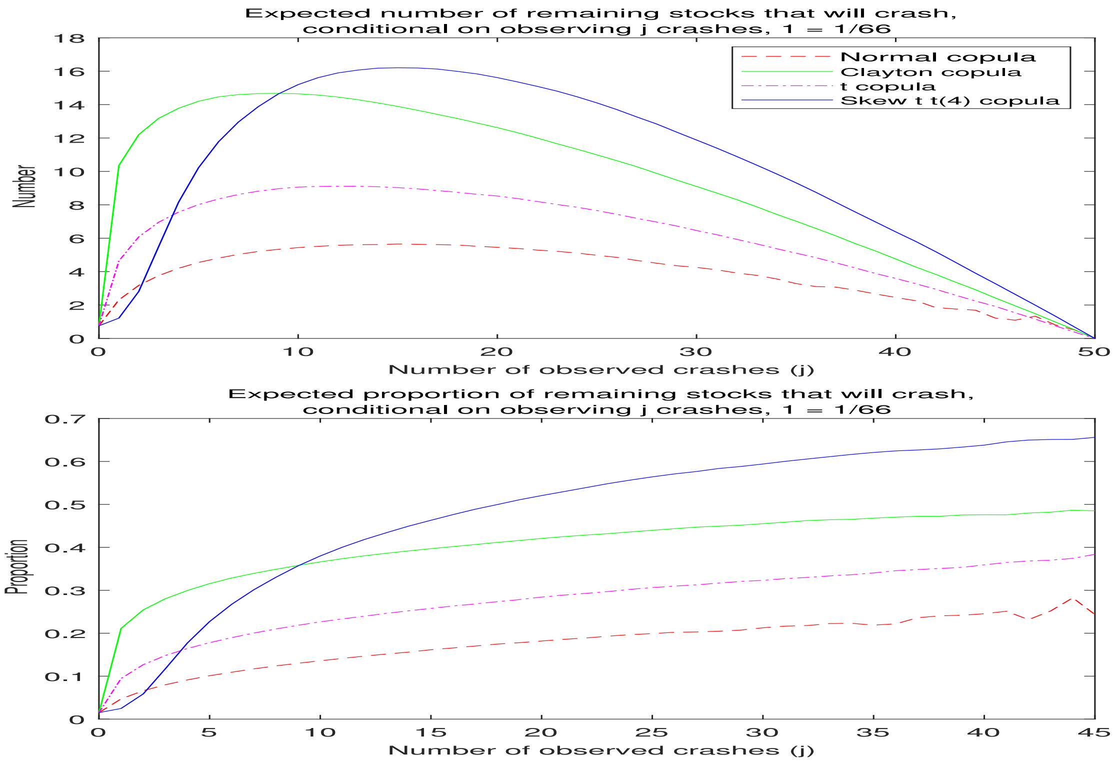

Figure 2: The upper panel shows the expected number and the lower panel shows the proportion of the remaining (50-j) stocks that will crash, conditional on observing j out of 50 stocks crashing. Note: we define a crash as returns in the lower 1/66 tail.

crash. Under the Normal copula, we only expect around 5 more variables to crash. The plot converges to zero, since conditioning on observing more crashes leaves fewer variables able to crash. The lower panel of Figure 2 shows that the expected proportion of remaining stocks that will crash, $\pi^q(j)$, generally increases all the way to $j = 47$. The importance of this figure is that it illustrates some of the relevant dependence features, which are unique to high-dimension applications. It provides a motivation for a class of flexible, parsimonious models such as the factor copula.

2.6 Simulation-based Estimation of Factor Copulas

Factor copulas do not generally have a closed-form density, which makes maximum likelihood estimation for these copulas difficult. Hence, we employ the simulation-based estimator proposed in Oh and Patton (2013), which is similar to the simulated method of moments. However, they use rank statistics to replace moments. They also verify that the estimator, and corresponding asymptotic distribution, has adequate finite-sample properties. The intuition behind this method is that the copula parameters are estimated by minimizing the difference in dependence measures between residuals and copula simulations. In this paper, we will use the Spearman's rank correlation (denoted as ρ_s) and lower and upper tail dependence (denoted as τ^L and τ^U respectively). Spearman's rank correlation is just Pearson's correlation applied to the ranks of the random variables instead of to the random variables themselves. The definition of tail dependence for two variables X_i, X_j with marginal distributions G_i, G_j is given by:

$$\tau_{ij}^L \equiv \lim_{q\to 0} \frac{\Pr[X_i \le G_i^{-1}(q), X_j \le G_j^{-1}(q)]}{q} \qquad (7)$$
$$\tau_{ij}^U \equiv \lim_{q\to 1} \frac{\Pr[X_i > G_i^{-1}(q), X_j > G_j^{-1}(q)]}{1-q}$$

Or equivalently

$$\tau_{ij}^L \equiv \lim_{q\to 0} \frac{\Pr[U_i \le q, U_j \le q]}{q} \qquad (8)$$
$$\tau_{ij}^U \equiv \lim_{q\to 1} \frac{\Pr[U_i > q, U_j > q]}{1-q},$$

where $U_i = G_i(X_i) \sim Unif(0,1)$. Lower tail dependence τ_{ij}^L measures the probability of one variable lying below its q quantile, for q converging to zero, given the probability of the other variable lying below its q quantile. Upper tail dependence τ_{ij}^U is defined analogously. In the Appendix, we go into further detail on the dependence measures used in our study.

The SMM estimator is defined as:

$$\hat{\boldsymbol{\theta}}_{T,S} \equiv \arg\min_{\theta\in\Theta} Q_{T,S}(\boldsymbol{\theta}), \qquad (9)$$

where

$$Q_{T,S}(\boldsymbol{\theta}) \equiv \mathbf{g}_{T,S}^{'}(\boldsymbol{\theta})\hat{\mathbf{W}}_T \mathbf{g}_{T,S}(\boldsymbol{\theta}),$$
$$\mathbf{g}_{T,S}(\boldsymbol{\theta}) \equiv \hat{\mathbf{m}}_T - \tilde{\mathbf{m}}_S(\boldsymbol{\theta})$$

Here $\tilde{\mathbf{m}}_S(\boldsymbol{\theta})$ is an $m \times 1$ vector of dependence measures computed using S simulations $\{X_s\}$. We define $\hat{\mathbf{m}}_T$ to also be a vector of dependence measures, but which are computed using the standardized residuals $\{\eta_t\}$. $\hat{\mathbf{W}}_T$ is an m-dimensional identity matrix.

Oh and Patton (2013) showed that if $S/T \to \infty$ as $T \to \infty$, SMM gives consistent and asymptotically normal estimates:

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_{T,S} - \boldsymbol{\theta}_0) \xrightarrow{d} N(0, \boldsymbol{\Omega}_0) \quad as \quad T, S \to \infty,$$

where

$$\boldsymbol{\Omega}_0 = (\mathbf{G}'_0\mathbf{W}_0\mathbf{G}_0)^{-1}\mathbf{G}'_0\mathbf{W}_0\boldsymbol{\Sigma}_0\mathbf{W}_0\mathbf{G}_0(\mathbf{G}'_0\mathbf{W}_0\mathbf{G}_0)^{-1}$$

and

$$\boldsymbol{\Sigma}_0 = \text{avar}[\hat{\mathbf{m}}_T], \mathbf{G}_0 = \nabla_\theta \mathbf{g}_0(\boldsymbol{\theta}_0)$$
$$\mathbf{g}_0(\boldsymbol{\theta}_0) = \text{p-lim}_{T,S\to\infty} \mathbf{g}_{T,S}(\boldsymbol{\theta})$$

To test for correct specification, Oh and Patton (2013) presented the distribution of a test for over-identifying restrictions. We use this so-called "J"-test in our empirical application. We provide a more detailed explanation of the J-test in the Appendix. Furthermore, we undertake a simulation study to verify the asymptotic properties of the SMM estimator.

2.7 Likelihood-based Estimation of GAS-Dynamic Factor Copulas

For the GAS-dynamic factor copula, we do not use SMM-type estimation, since it fails to incorporate the GAS dynamics. Rather, we use a maximum likelihood approach. We already established that factor copulas lack a closed-form likelihood, leaving likelihood approximation via standard numerical integration as our best choice. We provide a detailed explanation on how to obtain the likelihood for factor copulas in the Appendix.

Using this estimation method for the heterogeneous dependence case would result in quite a computational challenge, since it involves numerically searching over more than 50 parameters. We overcome this challenge by implementing an idea from the DCC model of Engle (2002). It involves the use of a "variance targeting" method to replace the constant ω_i in Equation 5 with a certain sample dependence measure transformation. We redefine the evolution of ψ_{it} in Equation 5 to

$$\log \psi_{g,t} = E[\log \psi_{g,t}](1-\beta_{GAS}) + \beta_{GAS} \log \psi_{g,t-1} + \alpha_{GAS} s_{g,t-1}$$

by using some results discussed in Creal et al. (2013). These results include that $E_{t-1}[s_{it}] = 0$, implying that $E[\log \psi_{g,t}] = \omega_g/(1-\beta_{GAS})$. We use sample rank correlations to estimate $E[\log \psi_{g,t}]$, removing the necessity of numerical optimization for the incept parameters ω_g. For the exact details of this estimation method, we refer to Oh and Patton (2018).

2.8 VaR and ES as Risk Measures

After the copula parameters are estimated, we can forecast the VaR and the ES. Mathematically, the VaR is an extreme quantile of the P&L distribution for a given portfolio over a prescribed holding period, given by

$$Y_q = \inf\{y \in \mathbb{R} : F_Y(y) \geq q\}, \qquad \text{with} \qquad 0 < q < 1.$$

Here q is the probability of a return being lower than the VaR. The performance of the VaR as a risk measure depends on the econometric model used and the forecast method. We will use simulation and bootstrapping methods to simulate one-day ahead returns. We will use the simulated returns to construct an equally-weighted portfolio and estimate the 1-day ahead VaR by means of taking an empirical quantile of the simulated sample. This process is summarized by a simple algorithm, which was constructed by Campagne (2014):

1. Draw a vector of standard uniformly distributed random variables from the considered copulas, using their estimated parameters.

2. Use the inverse of the marginal distributions on the standard uniformly distributed variables to simulate the standardized returns. In this case the normal distribution.

   $$e_i = \Phi^{-1}(u_i)$$

   where i represents the financial asset. Φ represent the cumulative distribution functions of the Normal distribution.

3. We simulate the returns by:

   $$y_{i,t+1} = \mu_{i,t+1} + \sigma_{i,t+1}\eta_{i,t+1}.$$

   Where , $\mu_{i,t+1}$ and $\sigma_{i,t+1}$ are obtained from the AR-GJR-GARCH process by forecasting one step ahead recursively, by means of a moving window. $\eta_{i,t+1}$ is equal to the simulated standardized return e_i.

Once the returns are simulated, we construct an equally-weighted portfolio of the 50 assets contained in the S&P100. The empirical quantile we use to estimate the VaR is based on 1000 bootstraps.

The VaR is regularly used in practice, but has some serious disadvantages which we discussed in the introduction. Artzner et al. (1997) proposed the *expected shortfall* (ES) as an alternative risk measure. The ES is defined as the expected size of a loss that exceeds VaR:

$$S_q = E[Y|Y < y_q]$$

Again, we use similar bootstrapping methods to estimate ES.

2.9 VaR Backtesting

In our research we make use of backtesting methods to test and verify whether our VaR model is correct, in the sense of unconditional coverage and independence of violations.

2.9.1 Expected Exceptions

we calculate the number of exceptions for the constructed portfolio, consisting of 50 S&P 100 stocks. To backtest the method on a historical series $y_1,...,y_m$, where $m >> n$, we calculate the conditional quantile $\hat{y}_q^t$ on days $t \in T = \{n,...,m-1\}$ using a moving window of $n = 100$ trading days (equivalent of five months) each time. On each day $t \in T$ we fit a new AR(1)-GJR-GARCH(1,1) model to estimate the conditional variance and mean.

We compare $\hat{y}_q^t$ with Y_{t+1} for various confidence levels $q \in \{0.90, 0.95, 0.99\}$. An exception corresponds to $Y_{t+1} < \hat{y}_q^t$. We can construct a binomial test for the quantile estimates based on the total number of exceptions. We have that the indicator for an exception at time $t \in T$ is Bernoulli distributed, with probability of success, or VaR violation, equal to $1-q$, such that

$$I_t : 1_{\{Y_{t+1}<y_q^t\}} \sim \mathrm{Be}(1-q),$$

where I_t and I_s are independent $\forall t,s \in T : t \neq s$. It follows that

$$\sum_{t\in T} I_t \sim B(\mathrm{card}(T), 1-q). \tag{10}$$

It holds that the total number of exceptions is binomially distributed under the model. The null hypothesis is defined as that the model correctly estimates the conditional quantiles. We test the null against the alternative that the method has a systematic estimation error, and gives too few or too many exceptions, by performing a two-sided binomial test.

2.9.2 Independence

A correct VaR model should have independent VaR violations, in the sense that violations should be spread out over the sample instead of in clusters.
We test independence against the specific alternative of a first-order Markov chain:

$$\prod = \begin{pmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{pmatrix},$$

where $\pi_{ij} = P[I_{t+1} = j | I_t = i]$.
The maximum likelihood estimate of $\prod$ is given by

$$\hat{\prod}_1 = \begin{pmatrix} \frac{T_{00}}{T_{00}+T_{01}} & \frac{T_{01}}{T_{00}+T_{01}} \\ \frac{T_{10}}{T_{10}+T_{11}} & \frac{T_{11}}{T_{10}+T_{11}} \end{pmatrix},$$

where T_{ij} is the number of observations (out of $T-1$) such that $I_{t+1} = j$ and $I_t = i$.
Under the null hypothesis of independence, $\pi_{01} = \pi_{11} \equiv \pi_2$ and

$$\mathscr{L}(\pi_2; I_T, I_{T-1}, ..., I_1) = (1-\pi_2)^{(T_{00}+T_{10})} \pi_2^{(T_{01}+T_{11})},$$

where the maximum likelihood estimate of π_2 is given by

$$\hat{\pi}_2 = \frac{T_{01}+T_{11}}{T_{00}+T_{10}+T_{01}+T_{11}}.$$

We compute the likelihood ratio test of correct unconditional coverage by

$$LR_{ind} = -2\log\left(\frac{\mathscr{L}(\hat{\pi}_2; I_T, I_{T-1}, ..., I_1)}{\mathscr{L}(\hat{\prod}_1; I_T, I_{T-1}, ..., I_1)}\right) \overset{asy}{\sim} \chi^2(1).$$

2.10 ES Backtesting

To test whether our ES model performs adequately, we use a test described in McNeil and Frey (2000). The intuition behind this test is to check whether the ES at time t S_q^t is equivalent to the mean of returns Y_{t+1} exceeding VaR. Whenever a VaR violation occurs, we take the difference between Y_{t+1} and S_q^t. We define iid residuals

$$R_{t+1} = \frac{Y_{t+1} - S_q^t}{\sigma_{t+1}}$$

where $E[R_{t+1} | Y_{t+1} < y_q^t] = 0$. We backtest on the days in set T, which was previously defined, by forming empirical residuals on days with VaR violation. These residuals are known as *exceedance residuals* and are defined as

$$\{r_{t+1} : t \in T, y_{t+1} < \hat{y}_q^t\}, \quad \text{where} \quad r_{t+1} = \frac{y_{t+1} - \hat{S}_q^t}{\hat{\sigma}_{t+1}}.$$

These exceedance residuals should be approximately iid with mean zero and unit variance, under the null of correct estimation of volatility dynamics. We conduct a one-sided bootstrap test which makes no assumption about the underlying distribution of the residuals (see Efron and Tibshirani (1993)) against the alternative hypothesis that the residuals have mean greater than zero, which corresponds to systematically underestimated expected shortfall.

3 Simulation Study

This section presents an analysis of the finite sample properties of SMM estimation for factor copulas. We first consider an "equidependence" model, which is similar to the equicorrelation model of Engle and Kelly (2012). This model assumes a single common factor and requires that all stocks have the same coefficient on the common factor. For this simulation, we consider three different factor copulas, all of them governed by:

$$X_i = \beta_i Z + \varepsilon_i, \quad i = 1,2,...,N$$
$$Z \sim Skew \quad t(v,\lambda)$$
$$\varepsilon_i \sim \text{iid} \quad t(v),$$
$$[X_1,...,X_N]' \sim \mathbb{F}_x = \mathbf{C}(G_1(\theta),....,G_N(\theta);\theta), \tag{11}$$

Table 1: Simulation results for equidependence factor copula models T = 1000

	Normal	Factor t-t		Factor Skew t-t		
	β	β	ν^{-1}	β	ν^{-1}	λ
True value	1.00	1.00	0.25	1.00	0.25	-0.25
			$N = 3$			
Bias	-0.0163	-0.0018	-0.0183	0.0126	-0.0197	-0.0317
Std	0.1041	0.1089	0.0967	0.1301	0.1107	0.1483
Median	0.9947	0.9959	0.2351	1.0091	0.2339	-0.2612
90%	1.1071	1.1461	0.3658	1.1821	0.3651	-0.1841
10%	0.8490	0.8695	0.0994	0.8655	0.0650	-0.3705
90-10 Diff	0.2581	0.2766	0.2664	0.3166	0.3001	0.1864
			$N = 10$			
Bias	-0.0115	-0.0029	-0.0133	-0.0040	-0.0161	-0.0097
Std	0.0647	0.0699	-0.0608	0.0840	0.0790	0.0705
Median	0.9902	0.9922	0.2363	0.9904	0.2392	-0.2504
90%	1.0699	1.0975	0.3094	1.1087	0.3411	-0.2030
10%	0.8963	0.9102	0.1706	0.9001	0.1335	-0.3106
90-10 Diff	0.1736	0.1873	0.1388	0.2086	0.2076	0.1076
			$N = 50$			
Bias	-0.0079	-0.0012	-0.0142	0.0021	-0.0141	-0.0051
Std	0.0373	0.0762	0.0689	0.0840	0.0790	0.0379
Median	0.9945	0.9986	0.2358	0.9951	0.2392	-0.2515
90%	1.0372	1.0724	0.3261	1.1034	0.3217	-0.2292
10%	0.9395	0.9249	0.1661	0.9022	0.1335	-0.2783
90-10 Diff	0.0977	0.1475	0.1600	0.2086	0.2012	0.0491

Table 1 presents the equidependence results from 100 trials of three different factor copulas, the Normal copula, the t-t factor copula and the Skew t-t factor copula. We estimate the copulas by means of SMM. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes. Problems of dimension $N = 3$, $N = 10$ and $N = 50$ are considered, the sample size is $T = 1000$. The number of simulations used is $S = 25 \times T$. The first row of each panel gives an overview of the average difference between the estimated parameter and its true value. The second row presents the standard deviation in the estimated parameters. The third, fourth and fifth rows present the 50^{th}, 90^{th} and 10^{th} percentiles of the distribution of estimated parameters, and the final row presents the difference between the 90^{th} and 10^{th} percentiles.

We set $N = 3$, $N = 10$ and $N = 50$ to match number of series we use in our empirical application. We assume that the common factor Z follows the skewed t distribution of Hansen (1994). We set $\beta = 1$, suggesting that the common factor accounts for one-half of the variance of each X_i, or equivalently a rank correlation of around 0.5. The first model we consider is the Normal copula, for which we set $\nu \to \infty$ and $\lambda = 0$. For the second model we set $\nu = 4$ and $\lambda = 0$, corresponding to a symmetric factor copula that generates tail dependence. For the third and final model we set $\nu = 4$ and $\lambda = -0.25$, yielding a factor copula that generates tail dependence and asymmetric dependence, in the sense that the lower tails of the factor copula are more dependent than the upper tails. We estimate ν^{-1} instead of ν, so that normality is nested at $\nu^{-1} \to 0$ rather than $\nu \to \infty$.

We also consider a less restrictive scenario, which is known as the "block-equidependence" case. This case allows each X_i to have a different coefficient on the common factor Z. We use $N = 50$ and assume that the all 50 variables can be grouped *ex ante* into 5 groups, such that $[\beta_1, \beta_2, \beta_3, \beta_4, \beta_5] = [0.5, 0.75, 1, 1.25, 1.5]$ and all variables in each group have identical β_i.

We do not perform a simulation study for the GAS-dynamic factor copula, since we primarily used the Matlab code provided by Oh and Patton (2018)[2] and only made some slight adjustments to the code.

[2]The code for the GAS-dynamic copula can be found at Andrew Patton's Matlab code page. Link: http://public.econ.duke.edu/ ap172/code.html

Table 2: Simulation results for block-equidependence factor copula models T = 1000 and N = 50

	ν^{-1}	λ	β_1	β_2	β_3	β_4	β_5
True value	0.25	-0.25	0.50	0.75	1.00	1.25	1.50
			Normal				
Bias	-	-	-0.0053	-0.0026	-0.0097	-0.0031	0.0107
Std	-	-	0.0167	0.0290	0.0207	0.0483	
Median	-	-	0.4895	0.7491	1.0024	1.2312	1.4911
90%	-	-	0.5258	0.7803	1.0395	1.3214	1.5510
10%	-	-	0.4699	0.7086	0.9452	1.2078	1.4113
90-10 Diff	-	-	0.0559	0.0717	0.0943	0.1136	0.1397
			Factor $t-t$				
Bias	-0.0109	-	-0.0253	0.0047	0.0051	0.0335	-0.0337
Std	0.0883	-	0.0481	0.0543	0.0422	0.0604	0.0701
Median	0.2543	-	0.4795	0.7336	0.9896	1.2513	1.4654
90%	0.3662	-	0.5485	0.8064	1.0520	1.3503	1.5934
10%	0.1842	-	0.4533	0.7222	0.9268	1.1661	1.4261
90-10 Diff	0.1820	-	0.0952	0.0838	0.1234	0.1842	0.1673
			Factor Skew $t-t$				
Bias	-0.0102	0.0183	0.0032	0.0058	0.0025	-0.0057	0.0028
Std	0.0965	0.0527	0.0415	0.0502	0.0649	0.0644	0.0831
Median	0.2583	-0.2592	0.5022	0.7531	1.0081	1.2528	1.5239
90%	0.3467	-0.1836	0.5431	0.8099	1.0860	1.3915	1.6201
10%	0.1498	-0.2906	0.4552	0.7131	0.9323	1.1359	1.3946
90-10 Diff	0.1969	0.1070	0.0879	0.0968	0.1537	0.2556	0.2255

Table 2 presents the block-equidependence results from 100 trials of three different factor copulas, the Normal copula, the t-t factor copula and the Skew t-t factor copula. We estimate all copulas in this table by means of SMM. The margins of the data are assumed to follow AR(1)-GARCH(1,1) processes. Only dimension $N = 50$ is considered, the sample size is $T = 1000$. The number of simulations used is $S = 25 \times T$. The first row of each panel presents the average difference between the estimated parameter and it true value. The second row presents the standard deviation in the estimated parameters. The third, fourth and fifth rows present the 50^{th}, 90^{th} and 10^{th} percentiles of the distribution of estimated parameters, and the final two presents the difference between the 90^{th} and 10^{th} percentiles.

Hence, for the sake of space and computation time, we refer to Oh and Patton (2018) for the simulation results.

We assume that the marginal distributions of the variables follow an AR(1)-GARCH(1,1) process:

$$y_{it} = \phi_{0i} + \phi_{1i} + \varepsilon_{it}, \quad t = 1, 2, ..., T \tag{12}$$
$$\sigma_{it}^2 = \omega_i + \beta_i^G \sigma_{i,t-1}^2 + \alpha_i \varepsilon_{i,t-1}^2,$$

where the G in β^G stands for "GARCH". We set the corresponding AR(1)-GARCH(1,1) parameters as [ϕ_0, ϕ_1, ω, β, α] = [0.01, 0.05, 0.05, 0.85, 0.10], roughly matching the parameter values for daily equity returns. We estimate these parameters by means of QML and consequently obtain the estimated standardized residuals:

$$\hat{\eta}_{it} = \frac{y_{it} - \hat{\phi}_0 - \hat{\phi}_1 y_{i,t-1}}{\hat{\sigma}_{it}} \tag{13}$$

We set the length of the time series to $T = 1000$, approximately matching 4 years of daily return data. We use $S = 25 * T$ simulations for dependence measure computation. We repeat this procedure 100 times for each scenario.

The results for the equidependence model simulation shown in Table 1 suggest that the estimates are centered around the true values. We can verify this since there exists a small average estimated bias for all parameters and the median of the simulated distribution is roughly identical to the true values.
The table also reveals, through the standard deviation and 90-10 percentile difference, that adding

more parameters to the models leads to greater estimation error.
Moreover, increasing the dimension size results in more precisely estimated copula model parameters. This can be intuitively explained by the fact that increasing the dimension size in an equidependence setting increases the amount of information at our disposal on the unknown parameters, while the number of parameters to be estimated remains unchanged.

Table 2 presents the results for the simulation under block-equidependence for $N = 50$. This table reveals that the estimated parameters are centered around their true values. However, adding the "shape" parameters λ and ν^{-1} deteriorates the accuracy. The simulation results for the equidependence case and the block-equidependence case are roughly in line with those reported by Oh and Patton (2013).

4 Data used in Empirical Study

We apply the discussed copulas to an empirical problem. We study the dependence between 50 stocks from various industries that were constituents of the S&P100 as of September 1, 2017, from 6 July 2015 to 9 May 2018 (T=717). The tickers, names and SIC industry codes of the stocks we use in our empirical study are listed in Table 3.
We assume that the marginal distributions of the variables used in our empirical study follow an AR(1)-GJR-GARCH(1,1) process:

$$y_{it} = \phi_{0i} + \phi_{1i} + \varepsilon_{it} \tag{14}$$
$$\sigma_{it}^2 = \omega_i + \beta_i^G \sigma_{i,t-1}^2 + \alpha_i \varepsilon_{i,t-1}^2 + \gamma_i \varepsilon_{i,t-1}^2 \mathbb{1}\{\varepsilon_{i,t-1} \leq 0\},$$

We will filter out time-dependence from $\mathbf{Y_1},..,\mathbf{Y_T}$ and calculate the corresponding residuals $\eta_\mathbf{1},..,\eta_\mathbf{T}$, which are approximately iid. We estimate the AR(1)(-GJR)-GARCH(1,1) parameters by means of Quasi-Maximum Likelihood (QML), and we estimate the distribution of the standardized residuals using the empirical distribution function (EDF). This allows us to capture skewness and leptokurtosis in the residuals in an non-parametric fashion.

Table 4 present some summary statistics of the daily return data used for our empirical analysis. The top panel gives an overview of sample moments for all stocks used in our analysis. The means and standard deviations are approximately identical to those reported in other studies which use daily log returns. The use of nonparametric estimation (EDF) in our analysis is motivated by the fact that the coefficients

Table 3: Stocks used in the empirical analysis

Ticker	Name	Ticker	Name	Ticker	Name
AAPL (3)	Apple	BIIB (2)	Biogen Inc.	DIS (7)	Walt Disney
ABBV (2)	AbbVie Inc.	BK (6)	Bank of New York	DUK (4)	Duke
ABT (2)	Abbott lab.	BKNG (6)	Booking Holdings	DWDP (2)	DowDuPont
ACN (7)	Accenture	BLK (62)	BlackRock	EMR (3)	Emersont
AGN (2)	Allergan	BMY (2)	Bristol-Myers	EXC (4)	Exelon
AIG (6)	Am. Inter.Group	BRK-B (6)	Berkshire Hath	F (3)	Ford
ALL (6)	Allstate Corp	C (6)	Citi Group	FB (7)	Facebook, Inc.
AMGN (2)	Amgen Inc.	CAT (3)	Caterpillar	FOX (4)	FOX
AMZN (7)	Amazon.com	CELG (2)	Celgene Corp.	FOXA (4)	FOXA
AXP (6)	American Ex	CHTR (4)	Charter Com	GD (3)	GeneralDynam
BA (3)	Boeing	CL (2)	Colgate	GE (3)	General Elec
BAC (6)	Bank of Am	CMCSA (4)	Comcast	CVS (5)	CVS
CVX (2)	Chevron	COF (6)	Capital One Fin. Corp.	DHR (3)	Danaher Corp.
COST (5)	Costco	COP (2)	Conocophillips	GILD (2)	GileadScience
FDX (4)	Fedex	CSCO (3)	Cisco	GM (3)	General Motors
GOOG (7)	Google Inc	GOOGL (7)	Alphabet Inc.	GS (6)	GoldmanSachs
HAL (1)	Halliburton	HD (5)	Home Depot		
SIC	Description		SIC	Description	
1	Mining, construct.		5	Trade	
2	Manuf: food, furn		6	Finance, Ins	
3	Manuf: elec, mach		7	Services	
4	Transprt, comm’s				

Table 3 presents the ticker symbols and names of the 50 stocks used in the empirical analysis of this report. The SIC industry codes are reported in parentheses.

Table 4: Statistical cross-sectional distribution properties daily log returns

Daily log returns: $T = 716$						
	Mean	5%	25%	Median	75%	95%
Mean	0.0004	-0.0007	0.0001	0.0004	0.0007	0.0012
Std. dev.	0.015	0.0106	0.0133	0.0148	0.0167	0.0205
Skewness	-0.1052	-1.7175	-0.7362	-0.4042	-0.0718	1.1548
Kurtosis	16.33	4.6347	6.1648	8.2229	10.902	24.137
ϕ_0	0.0004	-0.0006	0.0001	0.0005	0.0007	0.0015
ϕ_1	0.0004	-0.0624	-0.0245	-0.0003	0.0365	0.0565
$\omega \times 1000$	0.0171	0.0004	0.0012	0.0064	0.0199	0.0551
β^G	0.8353	0.5643	0.7669	0.9012	0.9697	0.9979
α	0.0428	0	0	0.0162	0.0674	0.2292
γ	0.0925	-0.0066	0.0137	0.0583	0.1521	0.2229
ρ	0.3176	0.126	0.2289	0.304	0.3951	0.5325
ρ_s	0.3549	0.1613	0.2806	0.3497	0.4249	0.5583
$(\tau_{0.99} + \tau_{0.01})/2$	0.0967	0.0223	0.0223	0.0922	0.162	0.2318
$(\tau_{0.90} - \tau_{0.10})$	-0.0906	-0.1927	-0.1369	-0.095	-0.0531	0.0168

Table 4 presents some summary statistics of the daily equity returns data, of the 50 used S&P100 constituents over the period 6 July 2015 to 9 May 2018. The top panel gives an overview of some unconditional moments for the daily log returns. The second panel shows the estimated AR(1)–GJR-GARCH(1,1) model parameters. Values for linear correlation, rank correlation, average 1% upper and lower tail dependence, and the difference between the 10% tail dependence measures are based on the standardized residuals and are presented in the lower panel. The columns represent the mean and quantiles from the cross-sectional distribution of the measures listed in the rows. The top two panels give summaries across the N = 50 margins, while the lower panel presents a summary across the N(N −1)/2= 1225 distinct pairs of stocks.

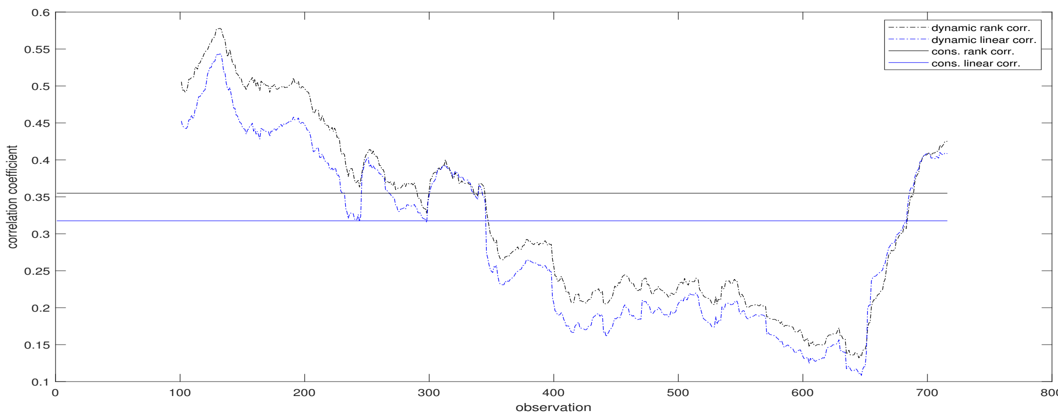

Figure 3: Linear and rank correlation coefficients averaged over all stocks using both the full sample window (giving a constant coefficient) and a moving window of length 100.

for skewness and kurtosis indicate strong heterogeneity in the shape of the distribution of the daily equity returns. The second panel of Table 4 gives an overview of the AR(1)-GJR-GARCH(1,1) model parameters. Our parameter estimates are comparable with those reported in other studies. The parameters indicate volatility clustering. The asymmetry parameter, γ, is positive for almost all stocks, supporting the widespread finding of a “leverage effect” in the conditional volatility of equity returns. The lower panel of Table 4 gives a summarized description for four dependence measures between pairs of standardized residuals: linear correlation (ρ), rank correlation (ρ_s), average upper and lower 1% tail dependence $(\tau_{0.99} + \tau_{0.01})/2)$, and the difference in upper and lower 10% tail dependence $(\tau_{0.90} - \tau_{0.10})$. The sign and strength of dependence is measured by means of linear and rank correlation. The tail dependence statistics measure the strength and symmetry of dependence. The rank correlation for daily returns varies from 0.28 to 0.42 from the 25th and 75th percentiles of the cross-sectional distribution. This suggests the presence of weak heterogeneity in the correlation coefficients. The 1% tail dependence measure is 0.10 on average, and varies from 0.02 to 0.16 across the interquartile range. The difference in the 10% tail dependence measures is -0.09 on average. Also, this measure is negative for over 75% of the pairs of stocks, suggesting the presence of asymmetric dependence. Figure 3 shows linear and rank correlation coefficients averaged over all stocks for each time period, verifying our need for a time-varying *or* dynamic model for the dependence structure.

5 Results Empirical Study

In this section, we present the results of our empirical study[3] on the dependence structure of the 50 stock returns We use three existing non-factor copulas: Clayton, Normal and Student's t copula. We also consider four static factor copulas: t-Normal, Skew t-Normal, t-t, Skew t-t. Finally, we also examine the block-dynamic and GAS-dynamic factor copulas with a Skew $t-t$ specification. All models are estimated using the SMM method described in Section 2.6, except for the GAS-dynamic factor copula, which is estimated via the Maximum Likelihood approach described in Section 2.7. For each model estimated by SMM, we present the value of the SMM objective function Q_{SMM} at the estimated parameters. We also present the J-test of over-identifying restrictions, along with their critical values. Standard errors are based on 1000 bootstraps to estimate $\Sigma_{T,S}$, and step size ε_T = 0.1 to compute $\hat{G}$. The rank dependence measures that are used in the SMM estimation of this model are the same as described in our simulation study.

The parameter estimates for the Factor GAS-dynamic copulas is based on maximizing the likelihood function for each $t \in T$. The standard errors are based on $B = 1000$ bootstraps of a set of parameters that we estimate (see Radovanov and Marcikic (2014)): $s^*(\hat{\zeta}) = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\zeta}^*{}_b - \overline{\zeta^*})^2}$, where $\zeta = [\omega_1, \ldots, \omega_G, \alpha_{GAS}, \beta_{GAS}, v_z^{-1}, v_\varepsilon^{-1}, \lambda_z]$.

For the block-equidependence case, we consider $G = 6$ corresponding to six different industries, according to the SIC codes 2 to 7 ($g = SIC - 1$). We neglect SIC code 1 (Mining), because we only have one stock that operates in that industry. This leaves us with only 49 stocks for the block-equidependence application.

5.1 Copula Parameter Estimates

Table 5 suggests that β for the static model and daily returns varies from 0.65 to 1.00, which implies a correlation coefficient that varies from 0.32 to 0.5. The estimated inverse degrees of freedom parameter lies around 0.15, corresponding to around seven degrees of freedom. The standard errors on v^{-1} reveal that this parameter is significant in all models used, at the 5% level.[4] The parameter for asymmetry (λ) is significantly negative for the Factor Skew $t-N$ model with a t-statistic of 2.09, implying that the dependence structure between the 50 stock returns is significantly asymmetric.

The values of the objective function Q_{SMM} indicate that allowing for asymmetry improves the performance in terms of dependence modeling. The Factor Skew $t-t$ model outperforms all other static models. The last column shows that none of the static models pass the J specification test. This might be due to the restrictive assumption of equidependence and the use of one common factor.

Table 6 presents parameter estimation results for the block-equidependence case, for the Normal, Factor $t-t$ and Factor Skew $t-t$ Copula. This table suggests that the market-wide factor is less negatively skewed for the block-equidependence case than for the equidependence case, due to a smaller λ (in absolute value). We also find that the Factor Skew $t-t$ Copula is preferred in terms of objective function value. Moreover, it is the only copula of the three that does not reject the null hypothesis of correct specification. The coefficients on the market factor β_g for this copula range from 0.75 (Trade) to 1.14 (Services), indicating the varying degree of inter-industry dependence.

Table 7 presents results for the dynamic factor copula, for which we only consider the Skew $t-t$ case

[3] In the empirical study we consider 20 trials, each corresponding to different starting points. This is necessary to make sure we find the global minimum of objective function Q_{SMM}, instead of a local minimum.

[4] Zero tail dependence corresponds to inverse degrees of freedom equal to zero. The squared t-statistic no longer has an asymptotic χ_1^2 distribution under the null, but is distributed as an equal-weighted mixture of a χ_1^2 and χ_0^2; see Gourieroux and Monfort (1996). The 90% and 95% critical values for this distribution are 1.64 and 2.71, corresponding to t-statistics of 1.28 and 1.65.

[5] Table 11 in the Appendix provides a small simulation of what happens to copula parameter estimation for a small sample size, which is the case for the block-dynamic copula.

Table 5: Static Copula parameter estimation results under equidependence for daily returns of the 50 S&P100 constituents

	β		V^{-1}		λ		Q_{SMM}	J-test
Daily log returns: $T = 716$	Est	Std Err	Est	Std Err	Est	Std Err		
Clayton*	0.4662	(0,0694)	-		-		0.0251	R
Normal	0.8283	(0,0565)	-		-		0.0080	R
Student's t*	0.3769	(0,0523)	0.0936	(0,0350)	-		0.0064	R
Factor $t-N$	0.6587	(0,0418)	0.1541	(0,0429)	-		0.0054	R
Factor Skew $t-N$	0.8116	(0,0573)	0.1236	(0,0704)	-0.2489	(0,1186)	0.0006	R
Factor $t-t$	0.7409	(0,1299)	0.2510	(0,1346)	-		0.0050	R
Factor Skew $t-t$	1.0289	(0,2027)	0.2187	(0,0933)	-0.2071	(0,2627)	0.0004	R

*Table 5 presents estimation results for various static copula models under equidependence applied to 50 daily stock returns over the period 6 July 2015 to 9 May 2018. This table contains copula parameter estimates and their asymptotic standard errors. The SMM objective function value Q_{SMM} at the estimated parameters and whether we fail to reject (A_p) or reject (R) the null of the overidentifying restriction test for a p% significance level is also presented. *Note that the parameter of the Clayton and Student's t copula is not β but we report it in this column for simplicity.*

Table 6: Static Copula parameter estimation results under block-equidependence for daily returns of 49 S&P100 constituents from six different industries

	Normal		Factor $t-t$		Factor Skew $t-t$	
Daily log returns: $T = 716$	Est	Std Err	Est	Std Err	Est	Std Err
v^{-1}	-		0.2840	(0.0342)	0.2142	(0.0827)
λ	-		-		-0.1739	(0.1531)
β_1	0.8725	(0.0796)	0.8179	(0.1413)	1.0562	(0.1924)
β_2	0.8832	(0.0912)	0.7468	(0.0959)	1.1220	(0.1848)
β_3	0.6960	(0.1252)	0.6118	(0.0813)	0.9012	(0.1203)
β_4	0.6116	(0.1128)	0.5577	(0.0703)	0.7556	(0.0966)
β_5	0.8099	(0.1390)	0.6361	(0.1095)	0.9702	(0.1436)
β_6	0.9358	(0.1337)	0.8766	(0.1139)	1.1430	(0.1599)
Q_{SMM}	0.0419		0.0269		0.0039	
J-test	R		R		A_{10}	

Table 6 presents estimation results under block-equidependence for daily returns of 49 S&P100 constituents, from six different industries, over the period 6 July 2015 to 9 May 2018. This table contains copula parameter estimates and their asymptotic standard errors. The SMM objective function value Q_{SMM} at the estimated parameters and whether we fail to reject (A_p) or reject (R) the null of the overidentifying restriction test for a p% significance level is also presented.

due to its performance in a static environment [5]. Notice that all dynamic factor copulas outperform the static copulas in terms of objective function value Q_{SMM}. However, this improvement in fit does come costly in terms of accuracy. The standard errors are relatively large for all models, due to the fact that we us H times less information to estimate each β_h compared to the static model. Furthermore, the sheer increase in the number of parameters deteriorates the accuracy even more. The dynamic nature of this copula is especially costly for the Factor Dynamic(5) Skew $t-t$, which has uninterpretable coefficients as a result of the extremely high standard errors. A possible explanation can be the minimal variation between β_1 to β_5, resulting in a close to singular parameter covariance matrix Ω_0.

Lastly, Table 8 presents the parameter estimation results for the Factor GAS-dynamic Skew $t-t$ Copula. The bootstrap standard errors are reported in parentheses. We find that the inverse degrees of freedom is significantly greater than zero, implying a non-Normal factor copula. We further notice that the asymmetry parameter λ for the common factor slightly negative, indicating greater dependence for joint downward moves in equity returns. The model under heterogeneous dependence gives superior results in terms of log-likelihood value. Surprisingly, the equidependence case gives better results than the block-equidependence case. This may be due to the fact that our use of industries to form G groups is

Table 7: Factor Dynamic(H) Skew $t-t$ Copula parameter estimation results for daily returns of the 50 S&P100 constituents

Factor Dynamic	(10) Skew $t-t$		(5) Skew $t-t$		(2) Skew $t-t$	
Daily log returns: $T=716$	Est	Std Err	Est	Std Err	Est	Std Err
ν^{-1}	0.0872	(0,0291)	0.0117	(0.0577)	0.1779	(0.2053)
λ	-0.3198	(0.1255)	-0.3648	(0.9051)	-0.2368	(0.8691)
β_1	1.1334	(0.3065)	0.0836	(1.5895)	0.7990	(0.7314)
β_2	0.1349	(0.2910)	1.0604	(0.6645)	1.1346	(0.2100)
β_3	0.6775	(0.2325)	0.7886	(1.0311)	-	
β_4	0.9245	(0.5042)	0.9052	(1.6116)	-	
β_5	0.8865	(0.8014)	0.9060	(1.3657)	-	
β_6	1.0884	(0,3343)	-		-	
β_7	1.0370	(0,5919)	-		-	
β_8	0.6958	(0,7706)	-		-	
β_9	1.0881	(0.8595)	-		-	
β_{10}	0.8998	(0.3833)	-		-	
Q_{SMM}	0.0004		0.0003		0.0004	
J-test	A_5		R		R	

Table 7 presents estimation results for the Factor Dynamic(H) Skew $t-t$ copula models applied to 50 daily stock returns over the period 6 July 2015 to 9 May 2018.This table contains copula parameter estimates and their asymptotic standard errors. The SMM objective function value Q_{SMM} at the estimated parameters and whether we fail to reject (A_p) or reject (R) the null of the overidentifying restriction test for a p% significance level is also presented.

Table 8: Factor GAS-dynamic Skew $t-t$ Copula parameter estimation results the equidependence, block-equidependence and heterogeneous dependence case using daily returns of the 50 S&P100 constituents

	Equidep.		Block-equidep.		Heterog. dep.	
Daily log returns: $T=716$	Est	Std Err	Est	Std Err	Est	Std Err
$\omega_{1\to G}$	-0.0292	(0,0047)	-		-	
α_{GAS}	0.0245	(0.0021)	0.0705	(0.0109)	0.2000	(0.0232)
β_{GAS}	0.8856	(0.0074)	0.8748	(0.0103)	0.7948	(0.0263)
ν_z^{-1}	0.2241	(0.0403)	0.2241	(0.0531)	0.2191	(0.0577)
ν_ε^{-1}	0.2418	(0.0105)	0.2377	(0.0119)	0.2248	(0.0148)
λ	-0.0480	(0.0484)	-0.0694	(0.0796)	-0.0676	(0.0628)
$\log LL$	8.9872		8.9639		9.0211	

Table 8 presents estimation results for the Factor GAS-dynamic Skew $t-t$ copula models applied to 50 daily stock returns over the period 6 July 2015 to 9 May 2018. We consider the equidependence, block-equidependence and heterogeneous dependence cases. This table presents copula parameter estimates and standard errors. $\log LL$ is the log-likelihood at the estimated parameters. The intercept parameters ω_g for the block-equidependence and heterogeneous dependence are not reported to conserve space.

inferior to alternative grouping schemes.
Figure 4 and Figure 5 illustrate the movement of the common factor loading ψ_{gt} over time for the equidependence and block-equidependence case, respectively. Figure 5 shows the varying degree of inter-industry dependence, and also the strong time-varying dependence, with the Trading industry being the most stable and the Services industry the least stable in terms of ψ_{gt}.

5.2 Risk Forecasts

In this section, we discuss the results of our risk models as described in Section 2.8. We used bootstrapping methods and the copula parameter estimates of Section 5.1 to estimate the VaR and ES, for which the backtesting results are presented in Tables 9 and 10. Table 9 shows theoretically expected number of VaR violations and number of actual violations, using daily log returns and normal innovations, obtained using various static and dynamic

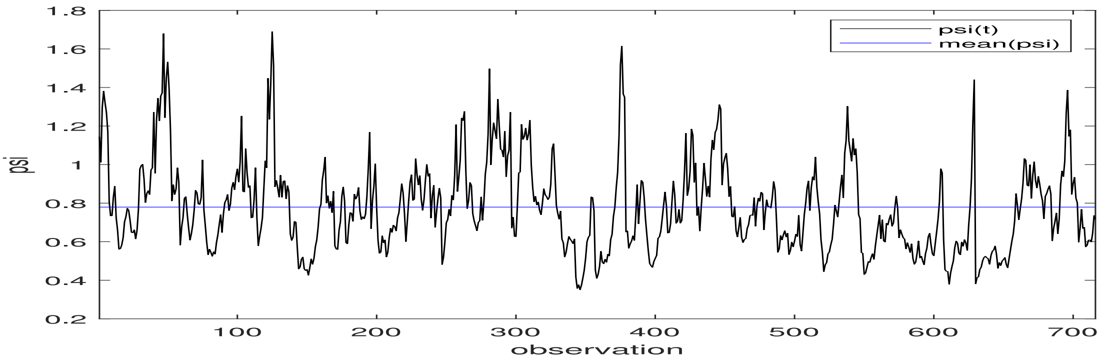

Figure 4: Equidependence GAS Dynamic Factor Skew $t-t$ coefficients on the common factor, ψ_{gt}, over the full sample, along with its average. For the equidependence case we have $G=1$.

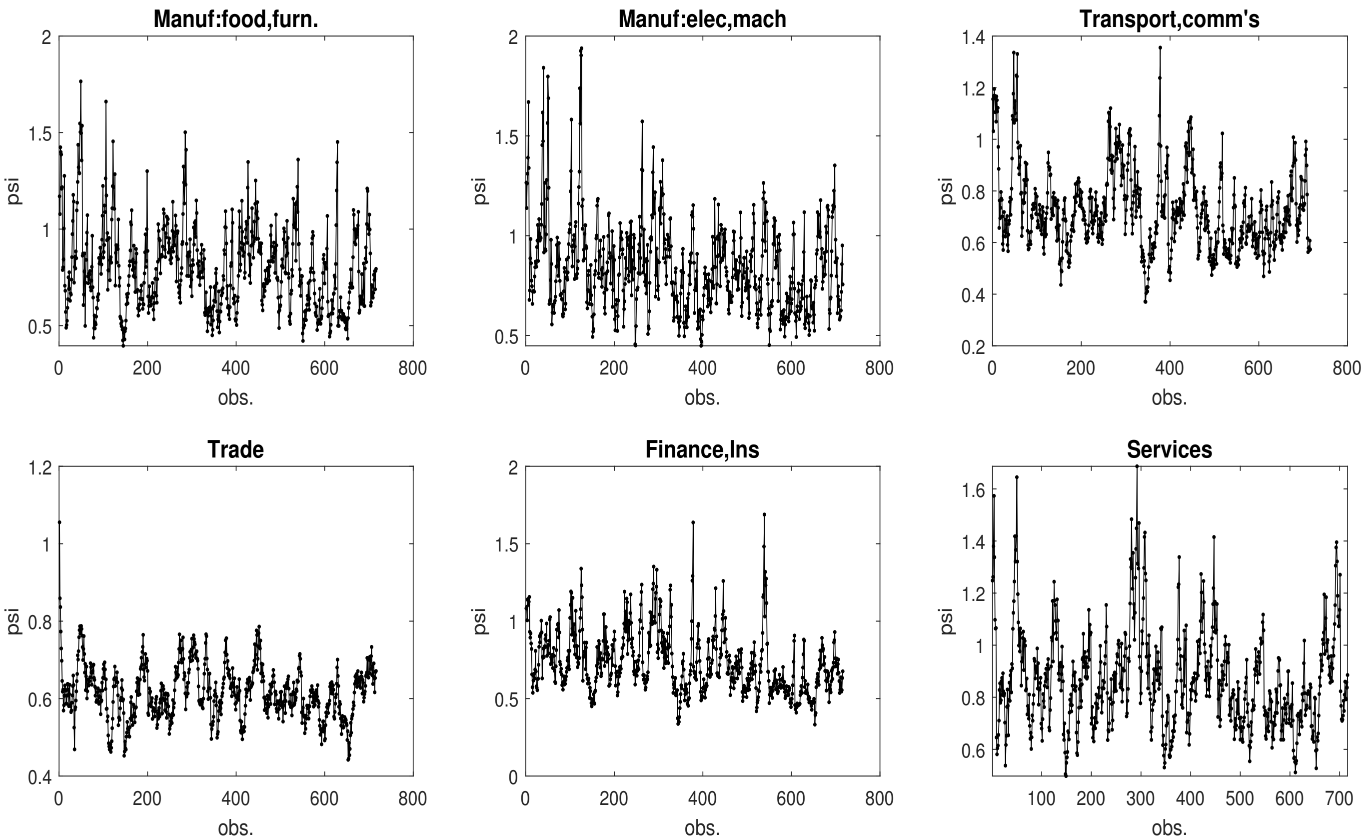

Figure 5: Block-equidependence GAS Dynamic Factor Skew $t-t$ coefficients on the common factor, ψ_{gt}, over the full sample. We distinguish the stocks in six different industries corresponding to their SIC code, such that $G=6$.

copulas. Furthermore, it shows the p-values of the exceptions and independence test mentioned in Section 2.9. This table suggests that different copulas are preferred for different quantiles. For the 0.90 quantile, the Factor GAS-Dynamic (hetero) copula outperforms all other copulas in terms of number of exceptions. However, in terms of independence of violations, the GAS-Dynamic copula under equidependence performs better than the heterogeneous case. The Clayton and Factor $t-t$ copula are closest to the expected number of exceptions, outperforming the other models. Finally, for the 0.99 quantile, the Factor Dynamic(2) Skew $t-t$, the Factor Skew $t-t$ (block-equi) and the Factor Skew $t-t$ copula give exactly the expected number of violations, outperforming all other models in terms of VaR violation modeling.

Table 10 shows the p-values for the for the one-sided Exceedance Residual ES test mentioned in Section 2.10, using daily log returns and normal innovations, obtained using various static and dynamic copulas. The null hypothesis of this test is that the volatility dynamics and expected shortfall are modeled adequately. Table 10 suggests that factor copulas outperform non-factor copulas. The Factor Skew $t-t$ (equi. and block-equi.) and the Factor Dynamic(10) Skew $t-t$ copula are the most consistent and vastly applicable models, in the sense that they perform relatively well over all quantiles. Other models only perform well over one or two quantiles. For instance, the Factor Skew $t-N$ model performs relatively well for the 0.90 quantile, but belongs to the weaker models for the 0.95 and 0.99 quantiles.

Figure 6 shows 95% Value at Risk and Expected Shortfall estimates using historical return series, the Normal Copula, the Dynamic(10) Factor Skew $t-t$ Copula and the Factor GAS-Dynamic (hetero) Skew $t-t$ copula. It provides an illustration of the differences between some copulas. The VaR and ES estimates based on historical simulation are constant, and thus not applicable to heteroscedastic financial time series. We notice that the risk measure estimates of the Factor Dynamic(10) Skew $t-t$ and the Normal copula are relatively constant compared to the Factor GAS-Dynamic (hetero) Skew $t-t$ copula. Estimating the coefficient on the common factor for every $t \in T$ allows the model to capture the volatility dynamics more precisely, resulting in more dynamic risk measure estimates. Figures 7 and 8 in the Appendix show additional plots of risk measure estimates for the other discussed copulas.

6 Conclusion

This article tries to compare various copulas (factor and non-factor) based on their ability to model dependence and risk in high-dimensional applications. The factor copulas presented in this article generally lack a closed-form likelihood. We verify that simulation-based methods are adequate in estimating copula parameter estimates and testing specifications in problems involving up to 50 variables. We

Table 9: VaR summary statistics

	0.90 Quantile		*0.95 Quantile*		*0.99 Quantile*	
	Exceptions	$(P_{exc}\|\|P_{ind})$	Exceptions	$(P_{exc}\|\|P_{ind})$	Exceptions	$(P_{exc}\|\|P_{ind})$
Expected	62		31		6	
Clayton	43	(0.0110 ‖ 0.0080)	22	(0.1150 ‖ 0.2290)	8	(0.4140 ‖ 0.6460)
Normal	39	(0.0020 ‖ 0.0100)	23	(0.1650 ‖ 0.0010)	8	(0.4140 ‖ 0.6460)
Student's t	40	(0.0030 ‖ 0.0130)	24	(0.2300 ‖ 0.0120)	10	(0.1490 ‖ 0.1470)
Factor $t-N$	42	(0.0070 ‖ 0.0230)	24	(0.2300 ‖ 0.0120)	8	(0.4140 ‖ 0.6460)
Factor Skew $t-N$	41	(0.0050 ‖ 0.0170)	18	(0.0160 ‖ 0.0130)	7	(0.6820 ‖ 0.6880)
Factor $t-t$	43	(0.0110 ‖ 0.0080)	28	(0.7110 ‖ 0.0010)	8	(0.4140 ‖ 0.6460)
Factor Skew $t-t$	40	(0.0030‖ 0.0130)	22	(0.1150 ‖ 0.0060)	6	(**1.0000** ‖ 0.7310)
Normal (block-equi)	39	(0.0020 ‖ 0.0100)	23	(0.1650 ‖ 0.0080)	8	(0.4140 ‖ 0.6460)
Factor $t-t$ (block-equi)	47	(0.0510 ‖ 0.0260)	28	(**0.7110** ‖ 0.0380)	8	(0.4140 ‖ 0.6460)
Factor Skew $t-t$ (block-equi)	41	(0.0050 ‖ 0.0170)	24	(0.2300 ‖ 0.0120)	6	(**1.0000** ‖ 0.7310)
Factor Dyn.(2) Skew $t-t$	39	(0.0020 ‖ 0.0020)	24	(0.2300 ‖ 0.0120)	6	(**1.0000** ‖ 0.7310)
Factor Dyn.(5) Skew $t-t$	58	(0.6870 ‖ 0.0000)	50	(0.1150 ‖ 0.0010)	43	(0.0000 ‖ 0.0000)
Factor Dyn.(10) Skew $t-t$	44	(0.0180 ‖ 0.0110)	34	(0.5180 ‖ 0.0090)	19	(0.0000 ‖ 0.1290)
Factor GAS-Dynamic (equi)	65	(0.6380 ‖ **0.0980**)	40	(0.0950 ‖ 0.0490)	15	(0.0020 ‖ 0.0040)
Factor GAS-Dynamic (block-equi)	42	(0.0070 ‖ 0.0230)	25	(0.3100 ‖ 0.0160)	5	(0.8390 ‖ **0.7750**)
Factor GAS-Dynamic (hetero)	62	(**0.9460** ‖ 0.0330)	41	(0.0640 ‖ **0.0640**)	12	(0.0380 ‖ 0.0180)

Table 9 shows theoretically expected number of violations and number of violations (using daily log returns and normal innovations) obtained using the Clayton, Normal, t, Factor (Skew) $t-N$ Factor (Skew) $t-t$, Factor Dynamic(H) Skew $t-t$ and GAS Factor Dynamic Skew $t-t$ for our portfolio of 50 constituents of the S&P100. A moving window of 100 trading days is used for backtesting. The p-values for the number of exceptions P_{exc} and the p-values for independence P_{ind} are given in brackets $(P_{exc}||P_{ind})$. Note: the bold p-values correspond to the best VaR model per quantile, in terms of number of exceptions or independence.

Table 10: ES summary statistics

	0.90 Quantile	*0.95 Quantile*	*0.99 Quantile*
	P-value	P-value	P-value
Clayton	0.0700	0.0640	**0.2810**
Normal	0.0040	0.0080	0.0010
Student's t	0.0010	0.0100	0.0062
Factor $t-N$	0.0050	0.0570	0.1230
Factor Skew $t-N$	0.1820	0.0010	0.0000
Factor $t-t$	0.0110	0.1650	0.0580
Factor Skew $t-t$	0.1010	0.2450	0.1610
Normal (block-equi)	0.0090	0.0060	0.0000
Factor $t-t$ (block-equi)	0.0680	0.1440	0.2670
Factor Skew $t-t$ (block-equi)	**0.2190**	**0.2990**	0.1580
Factor Dynamic(2) Skew $t-t$	0.0340	0.1570	0.0590
Factor Dynamic(5) Skew $t-t$	0.0890	0.1400	0.0380
Factor Dynamic(10) Skew $t-t$	0.1290	0.2030	0.2330
Factor GAS-Dynamic (equi.)	0.0000	0.0000	0.0440
Factor GAS-Dynamic (block-equi.)	0.1130	0.1850	0.0000
Factor GAS-Dynamic (hetero)	0.0000	0.0070	0.0580

Table 10 shows the p-values for the for the one-sided Exceedance Residual ES test (using daily log returns and normal innovations) obtained using the Clayton, Normal, t , Factor (Skew) $t-N$ Factor (Skew) $t-t$, Factor block-Dynamic(H) Skew $t-t$ and Factor GAS-Dynamic copulas for our portfolio of 50 constituents of the S&P100. A moving window of 100 trading days is used for backtesting. Note: the bold p-values correspond to the best ES shortfall model per quantile.

also introduce a Factor block-Dynamic Copula and adopt the Factor GAS-Dynamic Copula of Oh and Patton (2018).
We apply the factor copulas of Oh and Patton (2017) and our proposed dynamic factor copulas to study daily returns on 50 constituents of the S&P100 index over the period 2015-2018. We find significant evidence of a skewed, fat-tailed common factor, implying an asymmetric dependence structure. We also consider an application to the estimation of the Value at Risk and Expected Shortfall, and we show that the risk modeling performances vary over different quantiles. We find that the factor copulas outperform the non-factor copulas in terms of risk measure estimation, with the Factor Skew $t-t$ Copula under block-equidependence giving the most consistent and on average superior estimates of the expected shortfall.

7 Discussion and Future Research

The fact that this study was bounded by time and computational capabilities gives rise to multiple limitations that we will discuss in this section. This research was mainly restricted by the heavy computational burden of the factor copula parameters estimation and simulation. The copula based on a latent factor structure is still a relatively new concept, leaving room for improvement in parsimony in the future. We had to make several concessions in order to retain feasible computation times. The simulation study resulted in a crashed laptop, which left no choice but to give up on parallel computing and to reduce the number of trials for the empirical study. Also, the comparison of the static copula with the dynamic copula was solely based on their ability to create risk forecasts. This was due to the fact that we used different estimation methods for the two. A possible solution would be to use the likelihood approach for both copulas.
Moreover, it is difficult to make strong conclusions based on the VaR, since we need to take both number and independence of violations into account.

We advise readers and potential implementers to not fully rely on the results, because they may not be resilient for changing conditions. The results presented in this paper are highly susceptible to changes in sample period and size. Our sample period did not contain extreme booms or crashes. We defend our choice by arguing that a 'correct' dependence model should perform well in all volatility circumstances, including and excluding crises and booms.

In this report, we captured time-varying dependence

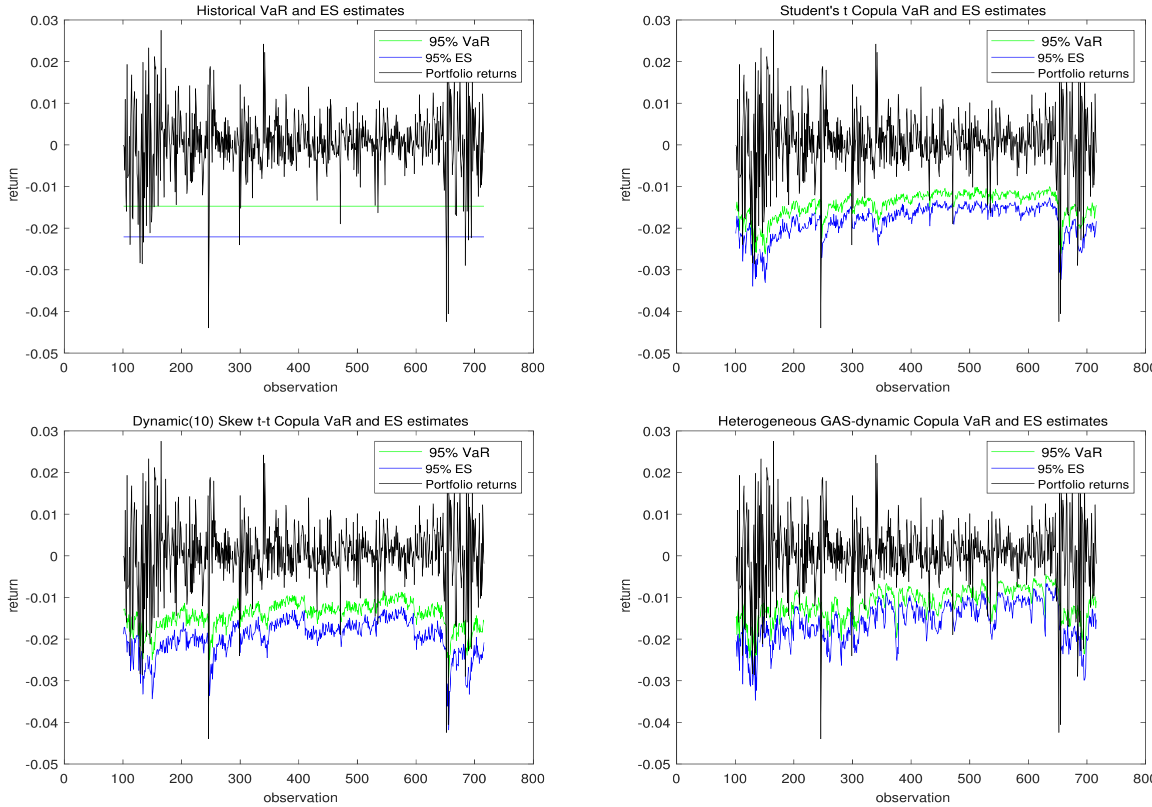

Figure 6: 95% Value at Risk and Expected Shortfall estimates using historical return series, the Normal Copula, the Factor Dynamic (10) Skew $t-t$ Copula and the Factor GAS-Dynamic Skew $t-t$ Copula under heterogeneous dependence. The return series are obtained from an equally weighted portfolio of 50 S&P100 constituents

by dividing the dataset in H sections and estimating β for each section. We also considered the GAS factor copula of Oh and Patton (2018) for time-varying dependence. For future research, this can be improved by extending the factor copula of Oh and Patton (2017) to capture conditional time-varying dependence. The coefficient on the common factor could then be conditioned on the state of the economy (Hectic and Quiet), and perform posterior estimation by Bayesian inference and MCMC.

Acknowledgements

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Appendix A1: Dependence Measures and Moment Calculation

In this section, we go in further detail on the dependence measures used and how we compute the "moments" for the equidependence and block-equidependence case. We use a total of five dependence measures in our SMM procedure: Spearman's rank correlation, and the 0.05, 0.10, 0.90, 0.95 quantile dependence measures. The choice of these dependence measures is based on preliminary studies of estimation accuracy and identification in Oh and Patton (2017). We let δ_{ij} denote one of the dependence measures we use between economic variables i and j. Then, we can define the "pair-wise dependence matrix" as:

$$D = \begin{bmatrix} 1 & \delta_{12} & \dots & \delta_{1N} \\ \delta_{12} & 1 & \dots & \delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1N} & \delta_{2N} & \dots & 1 \end{bmatrix}$$

For the "moments" in the equidependence model, we consider the average of the five dependence measures across all pairs. This way we reduce the number of moments to match from 5N(N −1)/2 to 5:

$$\overline{\delta} = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\delta_{ij}} \tag{15}$$

In the case of block-equidependence we can't take the average of the five dependence measures across all pairs, since we consider G groups. All variables contained in the same group $g = 1,2,..,G$ exhibit equidependence. Moreover, arbitrary pairs of variables (i, j) and (i', j') in groups (k,s) have identical dependence. These facts allows for intra- and intergroup dependence measure averaging. We have N variables, G groups, and r_G variables per group. We divide the previous defined matrix D into sub-matrices, corresponding to the groups:

$$D_(N \times N) = \begin{bmatrix} D_{11} & D'_{12} & \dots & D'_{1G} \\ D_{12} & D_{22} & \dots & D'_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1G} & D_{2G} & \dots & D_{GG} \end{bmatrix}, \text{where} \quad D_{ij} = (r_i \times r_j).$$

We compute the average values for each sub-matrix and assign them to a new matrix:

$$D^*_(G \times G) = \begin{bmatrix} \delta^*_{11} & \delta^*_{12} & \dots & \delta^*_{1G} \\ \delta^*_{12} & \delta^*_{22} & \dots & \delta^*_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^*_{1G} & \delta^*_{2G} & \dots & \delta^*_{GG} \end{bmatrix},$$

where $\delta^*_{ss} \equiv \frac{2}{r_s(r_s-1)} \sum\sum \hat{\delta_{ij}}$ denotes the average of all upper triangle values in D_{ss}, and $\delta^*_{ks} \equiv \frac{1}{r_k r_s} \sum\sum \hat{\delta_{ij}}$ denotes the average of all elements in matrix $D_{ks}, k \neq s$.

Now, we can construct a vector of average dependence measures $[\overline{\delta^*_1}, ..., \overline{\delta^*_G}]$, where

$$\overline{\delta^*_i} \equiv \frac{1}{G} \sum_{j=1}^{G} \delta^*_{ij}. \tag{16}$$

This results in a total of G moments for each dependence measure, corresponding to $5G$ in total overall.

Appendix A2: Test for over-identifying restrictions (J-test)

The J-test is a specification test for over-identifying restrictions. It can be used when the number of moments used in estimation is greater than the number of copula parameters, and if some additional assumptions mentioned in Oh and Patton (2013) hold. It follows that:

$$J_{T,S} \equiv \min(T,S)\mathbf{g}_{T,S}(\hat{\boldsymbol{\theta}}_{T,S})'\hat{\mathbf{W}}\mathbf{g}_{T,S}(\hat{\boldsymbol{\theta}}_{T,S}) \xrightarrow{\mathrm{d}} \mathbf{u}'\mathbf{A}_0'\mathbf{A}_0\mathbf{u} \quad as \quad T,S \to \infty, \tag{17}$$

where $\mathbf{u} \sim N(0,\mathbf{I})$, $\mathbf{A}_0 \equiv \mathbf{W}_0^{1/2}\boldsymbol{\Sigma}_0^{1/2}\mathbf{R}_0$ and $\mathbf{R}_0 \equiv \mathbf{I} - \Sigma_0^{-1/2}\mathbf{G}_0(\mathbf{G}_0'\mathbf{W}_0\mathbf{G}_0)^{-1}\mathbf{G}_0'\mathbf{W}_0\boldsymbol{\Sigma}_0^{1/2}$.

Since we use an identity weight matrix, the above test statistics has a sample-specific limiting distribution. We obtain the critical values via simulation:

1. Compute $\hat{\mathbf{R}}$ by using $\hat{\mathbf{G}}_{T,S}$, $\hat{\mathbf{W}}_T$ and $\hat{\boldsymbol{\Sigma}}_{T,B}$;
2. Simulate $\mathbf{u}^{(k)} \sim \text{iid} \quad N(0,\mathbf{I}),$ for $k = 1,2,...,K,$ with K large;
3. Compute $J_{T,S}^{(k)} = \mathbf{u}^{(k)\prime}\hat{\mathbf{R}}'\hat{\boldsymbol{\Sigma}}_{T,B}^{1/2\prime}\hat{\mathbf{W}}_T\hat{\boldsymbol{\Sigma}}_{T,B}^{1/2}\hat{\mathbf{R}}'\mathbf{u}^{(k)}$;
4. We set the critical value to the (1-α) quantile of $\{J_{T,S}^{(k)}\}_{k=1}^{K}$, where α denotes the significance level used.

Appendix B: Obtaining the Factor Copula Likelihood

To obtain the likelihood by numerical integration, we first need to obtain the copula density of $\mathbf{X_t}$.

$$c_t(u_1,...,u_N) = \frac{f_{xt}\left(G_{1t}^{-1}(u_1),...,G_{1t}^{-1}(u_N)\right)}{g_{1t}\left(G_{1t}^{-1}(u_1)\right),...,g_{Nt}\left(G_{1t}^{-1}(u_N)\right)} \tag{18}$$

where $f_{xt}(x_1,...,x_N)$ is the joint density of $\mathbf{X_t}$, $g_{it}(x_i)$ is the marginal density of X_i, and $c_t(u_1,...,u_N)$ is the ‘ copula density. We need all functions used on the right-hand side to construct the copula density.

The independence of the common factor Z and idiosyncratic variables ε_i implies that:

$$f_{\mathbf{X}|Z,t}(x_1,...,x_N|z) = \prod_{i=1}^{N} f_{\varepsilon_i}(x_i - \psi_{it}z)$$

By using these conditional distributions, we can obtain the marginal by one dimensional integration:

$$g_{i,t} = (x_i) = \int_{-\infty}^{\infty} f_{X_i,Z,t}(x_i,z)dz = \int_{-\infty}^{\infty} f_{X_i|Z,t}(x_i|z)f_{Z,t}(z)dz = \int_{-\infty}^{\infty} f_{\varepsilon_i}(x_i - \psi_{it}z)f_{Z,t}(z)dz \tag{19}$$

and equivalently

$$G_{it}(x_i) = \int_{-\infty}^{\infty} F_{\varepsilon_i}(x_i - \psi_{it}z)f_{Z,t}(z)dz$$

$$f_{xt}(x_1,...,x_N) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} f_{\varepsilon_i}(x_i - \psi_{it}z)f_{Z,t}(z)dz$$

We apply a variable transformation, $U \equiv F_{Z,t}(z)$, to get bounded integrals:

$$g_{it}(x_i) = \int_0^1 f_{\varepsilon_i}\left(x_i - \psi_{it}F_{Z,t}^{-1}(u)\right)du$$

$$G_{it}(x_i) = \int_0^1 F_{\varepsilon_i}\left(x_i - \psi_{it}F_{Z,t}^{-1}(u)\right)du$$

$$f_{xt}(x_1,...,x_N) = \int_0^1 \prod_{i=1}^{N} f_{\varepsilon_i}\left(x_i - \psi_{it}F_{Z,t}^{-1}(u)\right)du$$

This implies that we only need to compute one-dimensional[6] integrals to construct the factor copula density. For the integration we use Gauss-Legendre quadratore, described by Judd (1998), using 50 nodes. The number of nodes is based on a small simulation study performed by Oh and Patton (2018).

[6]This only applies for a one factor copula model. For a factor copula with L common factors, a L-dimensional integral needs to be solved.

Appendix C: VaR and ES Estimates

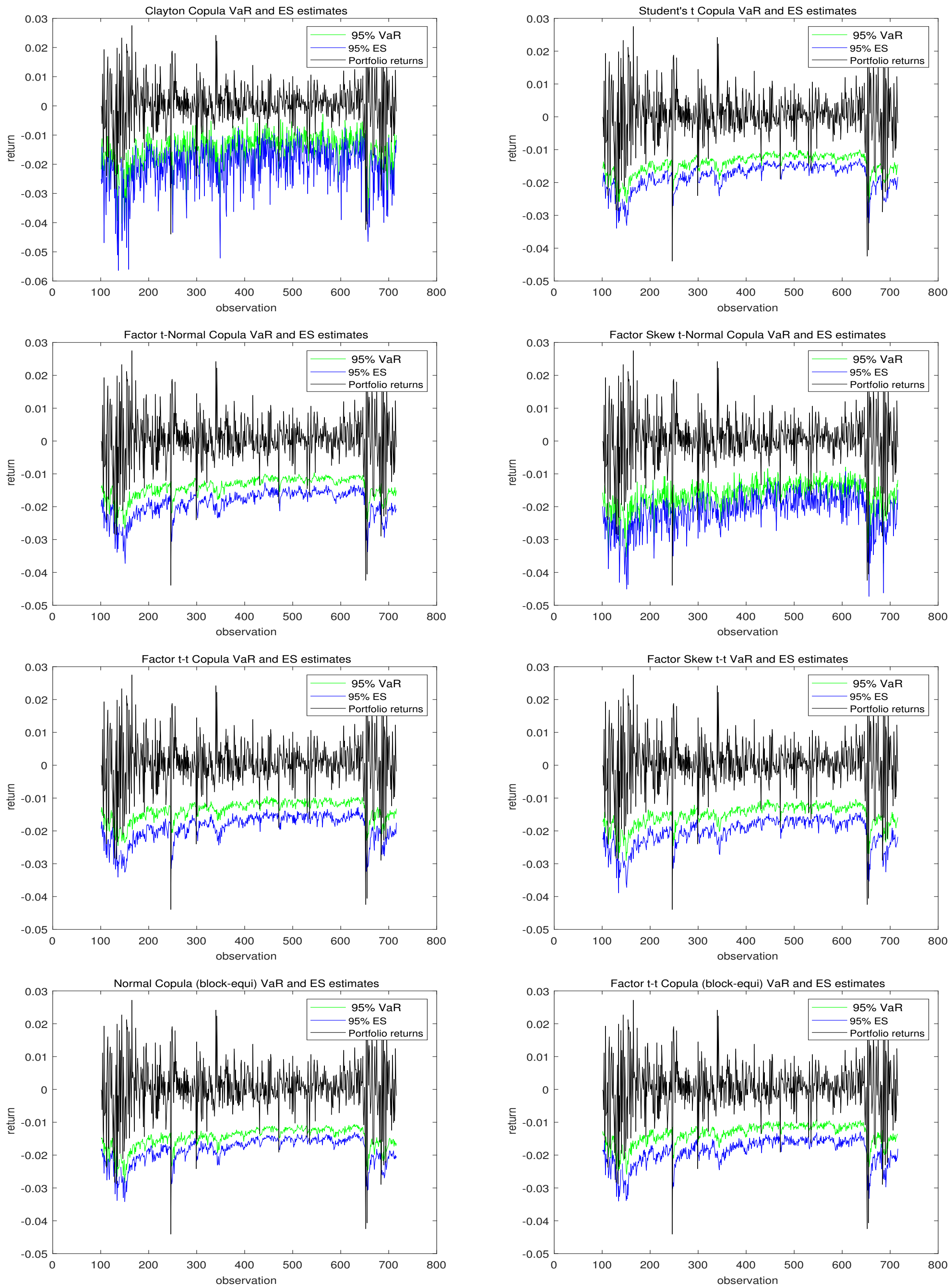

Figure 7: 95% Value at Risk and Expected Shortfall estimates for the various (Factor) Copulas used in our Empirical study. The return series are obtained from an equally weighted portfolio of 50 S&P100 constituents

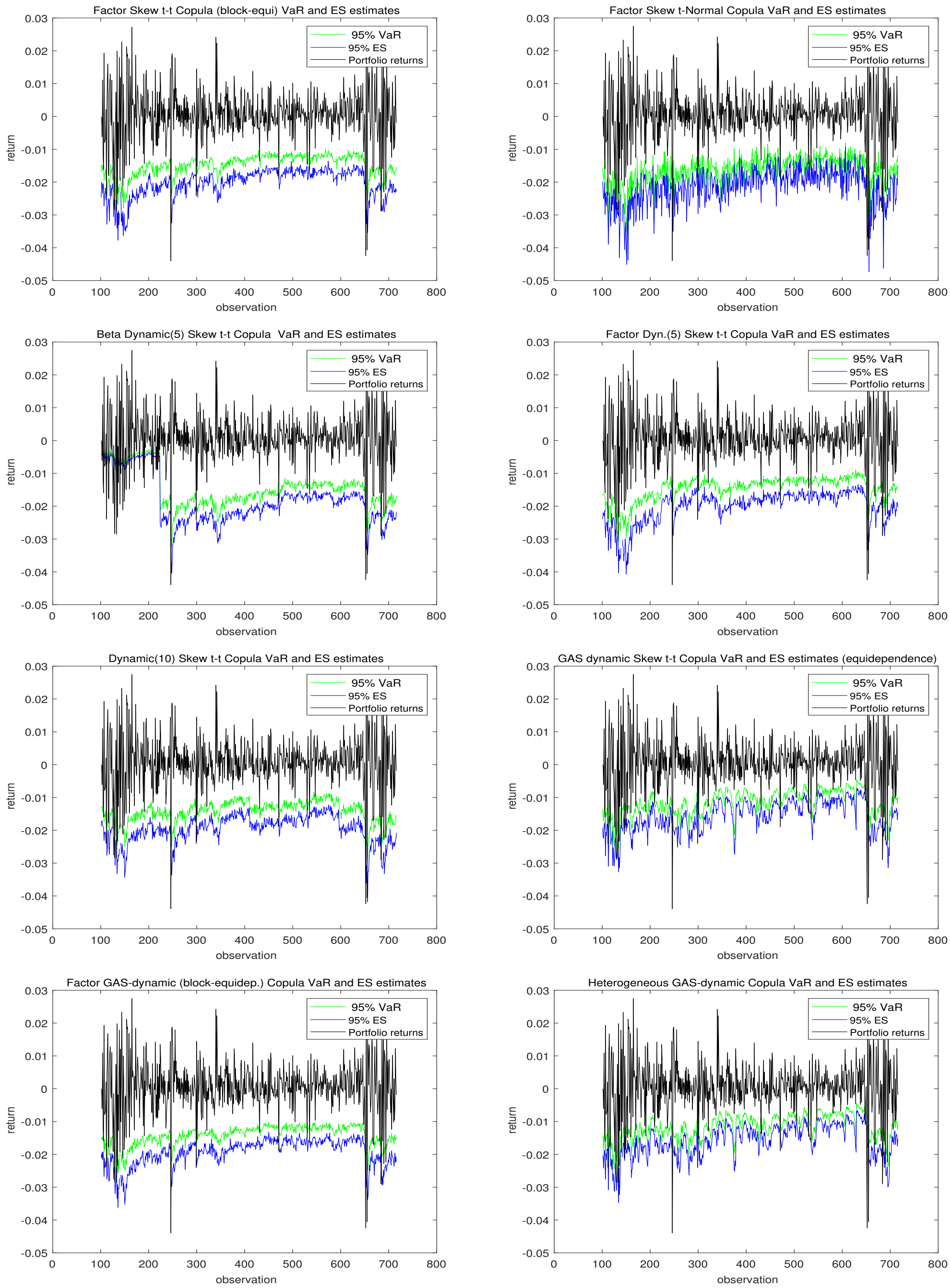

Figure 8: 95% Value at Risk and Expected Shortfall estimates for the various (Factor) Copulas used in our Empirical study. The return series are obtained from an equally weighted portfolio of 50 S&P100 constituents

Appendix D: Additional Results

Table 11: Simulation results for factor copula models T = 100

	Normal	Factor t-t		Factor Skew t-t		
	β	β	ν^{-1}	β	ν^{-1}	λ
True value	1.00	1.00	0.25	1.00	-0.25	-0.25
			$N=3$			
Bias	-0,0139	-0,0023	-0,0153	0,0032	0,0085	-0,0641
Std	0.2037	0.1589	0.2882	0.4191	0.1589	0.5952
Median	0.9589	0.9261	0.2532	0.8946	0.3035	-0.2822
90%	1.2014	1.3480	0.4792	1.5404	0.4476	0.6777
10%	0.8026	0.7206	0.0454	0.4875	0.0259	-0.9036
90-10 Diff	0.3988	0.6274	0.4338	1.0529	0.4217	1.5813
			$N=10$			
Bias	-0,0148	-0,0056	-0,0052	0,0582	0,3586	0,0088
Std	0.1349	0.2091	0.1556	0.2496	0.1264	0.4831
Median	0.9730	0.9774	0.2283	1.0070	0.3669	-0.0198
90%	1.0946	1.2049	0.4835	1.2677	0.4128	0.9630
10%	0.8551	0.7767	0.0422	0.7011	0.0787	-0.3365
90-10 Diff	0.2396	0.4282	0.4413	0.5666	0.3341	1.2995
			$N=50$			
Bias	-0.0115	-0.0029	-0.0133	-0.0040	-0.0161	-0.0097
Std	0.0647	0.0699	-0.0608	0.0840	0.0790	0.0705
Median	0.9902	0.9922	0.2363	0.9904	0.2392	-0.2504
90%	1.0699	1.0975	0.3094	1.1087	0.3411	-0.2030
10%	0.8963	0.9102	0.1706	0.9001	0.1335	-0.3106
90-10 Diff	0.1736	0.1873	0.1388	0.2086	0.2076	0.1076

Table 11 gives an overview of the simulation results from 100 trials for three different factor copulas, the Normal copula, the t-t factor copula and the Skew t-t factor copula. The copula parameters are estimated by means of SMM. The marginal distributions of the data are assumed to follow AR(1)-GARCH(1,1) processes. Problems of dimension $N=3$, $N=10$ and $N=50$ are considered, the sample size for is $T=100$. The number of simulations used is $S=25\times T$. The first row of each panel presents the average difference between the estimated parameter and its true value. The second row presents the standard deviation in the estimated parameters. The third, fourth and fifth rows present the 50^{th}, 90^{th} and 10^{th} percentiles of the distribution of estimated parameters, and the final two presents the difference between the 90^{th} and 10^{th} percentiles.