Assessing The Economic Value of Dynamic Conditional Correlations in a Multi-Asset Framework

Bachelor Thesis Double Degree Programme Econometrics & Economics

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June 30, 2018

Abstract

In this study the economic value of dynamic conditional correlations in a multivariate framework is evaluated with an economic loss function. Data on 65 assets in seven different asset classes over the period 2003-2018 is employed to construct mean-variance optimised portfolios based on predetermined expected returns and covariance estimates from different models. In addition to traditional multivariate GARCH models used by Engle and Colacito (2006), a Block Dynamic Equicorrelation and a Dynamic Factor GARCH model are used to benefit from high model parsimonity and a Combination model is used to exploit model complementarities. Using the testing procedure of Barendse and Patton (2018) with 1000 expected return simulations, the Scalar BEKK model outperforms all other models. On average, traditional multivariate GARCH models yield required returns that are respectively 2 and 3.3 times higher compared to required returns from portfolios constructed by the Block Dynamic Equicorrelation and the Dynamic Factor GARCH model.

1 Introduction

One of the key challenges that every individual investor faces is how to allocate wealth over a large set of assets. Based on the advice of DeMiguel et al. (2007), the first rule of thumb every beginner learns is 'not to put all your eggs in a single basket'. In other words, never invest all your capital in a single financial asset, but instead divide it over multiple assets. This rule of thumb is not only relevant to beginners, but also remains at the centre of principles for professional asset managers. Therefore, it is of great importance for asset managers that they have a well-established rule to optimally construct their portfolios.

The foundations for modern portfolio theory were laid by Markowitz (1952), from which the Capital Asset Pricing Model, as proposed by Sharpe (1964) and Lintner (1965), later followed. In the latter model, a rational investor constructs a portfolio of financial assets, including a risk-free asset, by optimising a mean-variance trade-off. In this framework, expected return is estimated by a function of past returns and anticipated variance is given by the covariance matrix of historical returns of this portfolio. As such, investors with different expected returns forecasts and different covariance forecasts hold different portfolios. To accurately estimate the standard deviation of the optimal portfolio, a proper estimator of the variances and correlations of all available assets is needed. However, poor portfolio performance could also be due to a poor estimate of the expected return. Merton (1980) finds that there is a considerable amount of noise in estimates for the expected return. Hence, it is essential to be able to disentangle the loss of performance from wrongly estimating the expected return on the one side and the covariance structure on the other side.

The paper of Bollerslev et al. (1988) marked the start of a stream within the literature which investigated the application of the time-varying covariances in portfolio theory. Ever since, many models for estimating time-varying covariances and assessing its value have been introduced. One straightforward way to estimate multivariate time-series models is the orthogonal generalised autoregressive conditional heteroscedasticiy (GARCH) (Alexander, 2000), which constructs the covariance matrix by making estimates for univariate GARCH models for linear combinations of asset returns. The vec representation of a multivariate GARCH model as presented by Bollerslev et al. (1988), in which the conditional covariance matrix is vectorised and estimated accordingly, poses an alternative to estimate the covariance structure in a Vector Autoregressive (VAR) setting. As discussed by Engle (2002), the Baba, Engle, Kraft and Kroner (BEKK) representation of the multivariate GARCH can provide useful restrictions to the vec representation. A more recent type of multivariate GARCH models is the Dynamic Conditional Correlation (DCC) model, introduced by Engle (2002). The DCC first estimates volatilities for each asset, then computes the standardised residuals and finally estimates the covariances using Maximum Likelihood estimation. Cappiello et al. (2006) introduce a variation on the DCC model which allows for an asymmetric effect of the correlation when both returns are either increasing or decreasing.

In assessing the contribution to economic performance in a multivariate framework, the academic literature first focused static covariance matrices. In this setting, Elton and Gruber (1973) were the first to use ex-post mean return for comparison and many other authors followed (e.g. Cumby et al., 1994; Fleming et al., 2001). However, using ex-post mean returns does not isolate the performance of the covariance matrix, as ex-post mean returns are not the same as expected returns. Other approaches focused on minimum variance portfolios or portfolios that have minimum variance compared to a benchmark (e.g., Chan et al., 1998), but this approach effectively made the unattractive assumption that all assets have the same expected return. Later, Fleming et al. (2001, 2003) and Engle and Colacito (2006) introduced methods to evaluate the economic performance of time-varying covariances.

Based on the work of Diebold and Mariano (2002), Engle and Colacito (2006) develop tests for the relative performance of two covariance matrices. The work of Chopra and Ziemba (1993) proves why it is important to isolate expected return estimation, since the authors argue that estimating the expected return correctly is about ten times as important as estimating the covariance structure correctly. Therefore, it is essential to create an expected return-free framework to prevent the noise from expected return estimations from blurring the results of covariance value. For this reason, the tests are applied to a number of predetermined time-invariant vectors of expected returns. Engle and Colacito (2006) develop a metric that gives an economic value to correct covariance information. They find that correct dynamic correlations are worth about 6% of the required return and correctly forecasting dynamic correlations is particularly important when the assets are highly correlated.

Building on previously mentioned literature, this study aims to answer the following research question:

"To what extent can the economic value of covariance estimation be improved with more parsimonious multivariate GARCH models compared to traditional multivariate GARCH models in a multi-asset allocation framework?"

With this research question, I aim to fill several gaps in the existing literature. The major innovation lies in the extension to a multivariate framework with a large number of assets. This extension results in two additional complications compared to the bivariate setting of Engle and Colacito (2006), which I first replicate. Firstly, the number of parameters in traditional multivariate GARCH models grows quickly with the number of assets, which means that the results are subject to substantial estimation error. To reduce the estimation error, I use the Block Dynamic Equicorrelation model of Engle and Kelly (2012) and the Dynamic Factor GARCH model of Santos and Moura (2014), which aim to increase the model parsimonity. Furthermore, I use the Combination model of Caldeira et al. (2017) to benefit from complementarities of several other models. Secondly, as Engle and Colacito (2006) notice, it is cumbersome to carry out their test in a multivariate setting, as the parameter space grows substantially. The solution of this issue lies in the implementation of a novel testing procedure by Barendse and Patton (2018).

In the bivariate setting, the results are largely in line with Engle and Colacito (2006), except that the Scalar BEKK model outperforms the Diagonal BEKK model. Furthermore, due to the use of the Composite Likelihood estimator – typically used in large systems – in the DCC-type models, these models perform significantly worse than the BEKK-type models. In the multivariate setting, a clear block structure between asset classes in the sample correlation matrix is visible, presenting tentative evidence that the assumption of constant correlations within each asset class holds. However, the results of the statistical tests show that the Block Dynamic Equicorrelation model performs significantly worse than traditional multivariate GARCH models and the Dynamic Factor GARCH model underperforms all models including the Block Dynamic Equicorrelation model. Due to the use of weights based on the Sharpe ratio, the performance of the Combination estimator is negatively influenced by the performance of the Block Dynamic Equicorrelation and the Dynamic Factor GARCH models, resulting in significant underperformance compared to the traditional multivariate GARCH models. As a conclusion, investors should be cautious with implementing more parsimonious models in their portfolio construction routine, as it could lead to target returns that are 2 or even 3.3 times lower compared to the case where a traditional multivariate GARCH model is used.

The remainder of this paper is structured as follows. A literature review on the influence of multivariate GARCH models in portfolio theory is carried out in Section 2. Section 3 discusses the multivariate GARCH models, corresponding estimation techniques and statistical testing procedures to assess model performance. In Section 4, a description about the employed data is given. Section 5 presents estimation and testing results. Finally, Sections 6 and 7 discuss the findings, conclude the paper and provide suggestions for further research.

2 Literature Review

Markowitz (1952) is seen as the father of modern portfolio theory. His original paper on the subject outlined modern portfolio theory for the first time. He formulated the portfolio construction problem as a choice of the mean and variance of a portfolio of assets and he proved that an investor, holding variance constant, should maximise expected return or, equivalently, for a given expected return, the investor should minimise portfolio variance, which later was deemed the fundamental theorem of mean-variance portfolio theory. These two principles led to the formulation of an efficient frontier of portfolios from which the investor could choose depending on his individual risk-return preferences. The important notion was that an investor could not select assets only based on the characteristics that are unique to that security, but the investor also would have to consider how each asset co-moves with other assets.

In reality, there are more moments than the mean and variance of a security that an investor can consider to get a better view on the return distribution. Various researches investigated alternative portfolio theories that included more moments such as skewness (see Lee, 1977; Kraus and Litzenberger, 1976) or were accurate for more realistic representations of the return distribution (see Fama, 1965; Elton and Gruber, 1974). Nevertheless, mean-variance theory has remained the cornerstone of modern portfolio theory, since there is no convincing evidence that adding more moments improves the desirability of the portfolio selected and both the implications and use of mean-variance theory have an intuitive appeal.

The single-period framework of Markowitz (1952) was later extended by various researchers to a multi-period problem. Papers by Fama (1970), Hakansson (1975, 1974), Merton (1990) and Mossin (1968) have all analysed this problem under various assumptions. However, they all found that under several sets of assumptions, the multi-period problem can be solved as a sequence of single-period problems. The optimal portfolio would be different from the one in a single-period setting, since the utility function in the multi-period setting takes into account the utility in multiple future time periods. The difference arises from the assumption that in a single-period setting the investor maximises his utility for only one period.

Another stream withing the academic literature has been concerned with the study of the separation theorem. This theorem states that if an investor has access to a risk-free asset, the choice of the optimum portfolio of risky assets is unambiguous and independent of the investor's risk return preferences. It has two important implications. Firstly, the theorem restates the portfolio construction problem to a problem of finding the tangency portfolio to the efficient frontier which can be connected by a line through the risk-free asset in a mean-volatility diagram. Consequently, the tangency portfolio is the portfolio that maximises the ratio of expected return minus the risk-free rate to the standard deviation. Secondly, it leads to the mutual fund theorem, which states that all investors can create their desired portfolio out of two mutual funds: the risk-free asset and the tangency portfolio. Even though there is abundant theoretical research on the separation theorem, a reconciliation between and theory and practice still lacks.

Significant effort has been invested in research on how the effect of transaction costs and parameter uncertainty can affect portfolio rebalancing. Goldsmith (1976) was one of the first to implement transaction costs in the framework of portfolio construction and both Magill and Constantinides (1976) and Davis and Norman (1990) implemented transaction costs in the work of Merton (1990) and concluded that the introduction of transaction costs leads to certain no-trade intervals. Gârleanu and Pedersen (2013) show that the assumption of quadratic transaction costs is more tractable and they provide closed-form expressions for the optimal portfolio policies. From the viewpoint of parameter uncertainty, attempts have been carried to characterise the importance of parameter uncertainty out with many approaches. In addition to robust optimisation techniques (e.g. Garlappi et al., 2006) and shrinkage approaches (Ledoit and Wolf, 2008), examples include Bayesian approaches with different priors such as diffuse priors (Klein and Bawa, 1976), priors based on asset pricing models (Pástor, 2000) and priors based on economic objectives (Tu and Zhou, 2010). Finally, DeMiguel et al. (2015) combined both quadratic transaction costs and parameter uncertainty in a portfolio construction setting.

After mean-variance portfolio theory was introduced, a significant part of the academic world focused on estimating the inputs: expected returns and covariances. Before the introduction of multivariate GARCH models, index models were the principal tool for estimating covariance matrices. This was first discussed by Markowitz, but developed and popularised by Sharpe (1963). Important aspects for this model were that the number of estimates was low, the type of inputs needed were easy for an analyst to understand and the accuracy of portfolio optimisation was high. Elton and Gruber (1973) show that using an index model to estimate the covariance structure is more accurate than using the regular covariance estimator.

Multivariate GARCH models have become more common to be implemented as covariance estimators in the academic literature. There are many multivariate GARCH models, each with their own advantages and disadvantages. In many cases, the design of a multivariate GARCH model boils down to the flexibility-parsimonity trade-off. On the one hand, it is desirable that a model is able to capture the correlation dynamics in a setting with many different assets. On the other hand, the number of parameters increases quickly with the complexity of the model, potentially leading to instable or unfeasible estimation and difficult interpretation of the parameter estimates. In addition, all models must guarantee in some way that the covariance matrix is positive definite. For some models, a set of conditions is specified to guarantee positive definiteness, while in other models positive definiteness is inherent to the model specification. Multivariate GARCH models can be divided into five categories: 1) direct estimation models, 2) factor models, 3) modelling of conditional variances and correlations, 4) semi- and non-parametric models, and 5) multivariate stochastic volatility models. The first four models are briefly described and the reader is referred to Chib et al. (2008) for a elaborate review of multivariate stochastic volatility models.

The first multivariate GARCH model is the so-called vec model introduced by Bollerslev et al. (1988), belongs to the first class where the conditional covariance matrix is directly estimated, and is a relatively straightforward generalisation of the univariate GARCH model. In this model, every conditional variance and conditional covariance parameter, organised in a vectorised matrix, is a function of all lagged conditional variances and conditional covariances, as well as lagged squared and cross-product returns. The advantage of this model is that it is very flexible. However, due to the large amount of parameters, estimation is difficult and, perhaps even more concerning, there exist only sufficient and quite restrictive conditions to ensure positive definiteness of the conditional covariance matrix. In the same paper, Bollerslev et al. (1988) also specify a more parsimonious version of the vec model, where the parameter matrices are assumed to be diagonal, where no interaction between the conditional covariances and conditional variances is allowed and each equation is estimated separately. Another restricted version was introduced by Engle and Kroner (1995), which is commonly referred to as the Baba-Engle-Kraft-Kroner (BEKK) representation. This model has the benefit that it is positive definite by construction, but interpretation of the parameters is not easy. Similar to the vec model, different variations on the BEKK representation exist, where the parameter matrices are reduced to either diagonal matrices (Diagonal BEKK) or scalars (Scalar BEKK). Even though estimation of the BEKK model is less difficult than estimation of a vec model, it still requires some heavy computations.

The second class of multivariate GARCH models consists of factor models. This class aims at increasing the model parsimonity and hence targets easier estimation. Engle et al. (1990) were the first to introduce a factor GARCH model. In this model, the observations are generated by a number of underlying factors, which follow a GARCH-type process themselves. This approach greatly reduces the dimensionality when the number of factors is small compared to the amount of assets in the investment universe. In their paper, Engle et al. (1990) choose a two-factor model, where the factors are the risk-free asset and value-weighted stock index returns. Another type of factor model, the Generalised Orthogonal GARCH model, was introduced by Van der Weide (2002) as an extension to the Orthogonal GARCH model of Alexander (2000). In this model, it is assumed that the returns are generated by a number of uncorrelated, conditionally heteroscedastic factors, which unconditional variance is restricted to be the identity matrix. The matrix that transforms the factors to the returns is assumed to be invertible, but does not need to be orthogonal. A drawback of the Generalised Orthogonal GARCH approach is that it requires at least as many factors as assets. Other related factor models are the Full Factor GARCH model of Vrontos et al. (2003) and the Generalised Orthogonal Factor GARCH model by Lanne and Saikkonen (2007). Later, Santos and Moura (2014) used the concept of a factor GARCH model and introduced dynamic factor loads, which are estimated via the Kalman Filter.

Thirdly, the type of models where the variances and correlations are estimated based on a decomposition of the conditional covariance matrix is perhaps the most commonly used type of multivariate GARCH models. The simplest multivariate correlation model that is nested in the other conditional correlation models, is the Constant Conditional Correlation (CCC) model by Bollerslev (1990). In this model, the conditional covariance matrix is decomposed into a time-varying diagonal matrix with standard deviations as entries and a time-invariant conditional correlation matrix. Consequently, univariate GARCH(p, q) models are estimated for the individual returns, where the parameter matrices for the conditional covariances are assumed to be diagonal. Later, Jeantheau (1998) relaxed the assumption of diagonal parameter matrices in the CCC model, such that past squared returns and variances of other assets could influence the conditional covariance of a particular asset. Since the correlation matrix is time-invariant, estimation via a Maximum Likelihood technique is relatively easy, as it needs only one matrix inversion per iteration. Even though the CCC model is attractive to use, empirical studies have shown that the assumption of constant correlation is too restrictive. At the cost of easy estimation, Tse and Tsui (2002) introduced the Varying Correlation (VC) model, in which the correlation matrix is assumed to be a function of the correlations in the previous period and a set of estimated correlations.

Engle (2002) introduced the Dynamic Conditional Correlation model, in which the correlation matrix is obtained by rescaling a matrix process that is similar to the VC model of Tse and Tsui (2002), but differs in the fact that it uses standardised errors of univariate GARCH-type processes instead of a sample estimate of the correlations. Even though this model allows for relatively easy estimation, as the loglikelihood can be split into a variance and correlation part, the drawback of this model is that it imposes the same dynamic structure on all assets. Billio et al. (2006) attempted to overcome this problem in their Quadratic Flexible DCC GARCH model by imposing a BEKK structure on the conditional correlations and adding extra restrictions to ensure covariance stationarity. In addition, due to the increased number of parameters, this model is unfeasible in large systems. Other variations to the DCC model are the Asymmetric DCC model by Cappiello et al. (2006), which allows for different sensitivities to increases or decreases in correlations, the cDCC model by Aielli (2013), which is more feasible with large systems, the Smooth Transition Conditional Correlation model by Silvennoinen and Teräsvirta (2005), where the conditional correlation matrix varies smoothly between two extreme states, and the Regime Switching Dynamic Correlation model by Pelletier (2006). Lastly, Engle and Kelly (2012) introduce the (Block) Dynamic Equicorrelation model, which allows for assumptions on equal correlations within a predetermined block of assets.

The last class of multivariate GARCH models contains the semi- and non-parametric models. The advantage of this approach is that it does not impose a particular structure on the data, which might potentially be misspecified. In addition, most semi-parametric models succeed in retaining the interpretability of the model parameters and non-parametric models are robust to distributional misspecification. However, as pointed out by Stone (1982), non-parametric models suffer from the 'curse of dimensionality': due to the lack of data in all directions of the multidimensional space, the performance of the local smoothing estimator deteriorates quickly as the dimension of the conditioning variable increases. In the semi-parametric model of Bauwens et al. (2007), the data is generated by a parametric multivariate GARCH model and the error distribution is unspecified and estimated non-parametrically. However, an approach like this might make it hard to verify whether the correct multivariate GARCH model is used, since the non-parametric error distribution might absorb some of the misspecification. Long and Ullah (2005) introduce an approach similar to the approach of Bauwens et al. (2007) in the sense that the model is based on some parametric multivariate GARCH model. After estimating the parametric model, the estimated standardised residuals are extracted. When the model is not correctly specified, these residuals may have some structure in them, and non-parametric estimation is applied to extract this information. In the Semi-Parametric Conditional Correlation GARCH model of Hafner et al. (2006), the conditional variances are modelled parametrically by a univariate GARCH model. The conditional correlations are then estimated using a transformed Nadaraya-Watson estimator.

Engle and Colacito (2006) analyse the economic value of correct conditional covariance information. In their paper, they compare a Scalar BEKK, Diagonal BEKK, DCC, Asymmetric DCC, Orthogonal GARCH models and the conventional covariance estimator in a setting with two assets. In the case where the two assets have a high sample correlation, the required return could be 2% to 5% lower when using the conventional covariance estimator compared to the case where a multivariate GARCH model is used. These values are even higher in the case where the sample correlation is closer to 0. Results from a Diebold-Mariano-type test on a quadratic economic loss function show that the conventional covariance estimator performs significantly worse compared to the multivariate GARCH models. Therefore, Engle and Colacito (2006) show that there is significant added value in using multivariate GARCH models over the conventional covariance estimator.

The last element in portfolio theory is the evaluation of the portfolio that is constructed by the various models. Not only do evaluation models follow naturally from portfolio construction theory, it is only by employing evaluation measures that a portfolio manager can assess which technique adds the most value. Surprisingly, one of the earliest studies on performance evaluation was done before the seminal work of Markowitz. It was Cowles (1933) who compared the average performance of a set of managed portfolios to a passive portfolio. His conclusion was that the managed portfolios underperformed the passive benchmark. Although Cowles (1933) was ahead of his time, he only considered return as a measure for performance and thereby ignoring any measure of risk.

Mean-variance performance measures were introduced much later. Sharpe (1966) introduced a performance measure defined as the ratio of the average return of the constructed portfolio minus the return of the risk-free asset to the standard deviation of the constructed portfolio. This ratio has become very popular among practitioners and is now commonly known as the Sharpe Ratio. Treynor (1965) proposed to evaluate the performance of a portfolio manager by calculating a ratio where average return of the constructed portfolio minus the risk-free rate is divided by the beta of the constructed portfolio. Unlike the Sharpe ratio, the Treynor ratio incorporates the view that risk is inherent to the entire market and hence must be removed. Next, Jensen (1968, 1969) introduces a measure to determine the abnormal return of the constructed portfolio over the theoretical expected return.

In addition to the previously mentioned performance measures, which are popular among practitioners to compare performance, researchers often use economic loss functions with a statistical testing procedure for inference. Fleming et al. (2001, 2003) and Marquering and Verbeek (2004) analyse the economic value of volatility timing for a mean-variance investor with a quadratic utility loss function for different risk aversion parameters. Engle and Colacito (2006) use tests inspired by Diebold and Mariano (2002) to test whether performance of various bivariate multivariate GARCH models is statistically significant from zero, where they use the difference in squared portfolio returns as the loss function. Finally, DeMiguel et al. (2007) uses differences in certainty equivalent returns of portfolio construction strategies for mean-variance investors to make inferences about performance.

3 Methodology

This section first gives an introduction into the classical asset allocation problem and then describe the statistical tests in order to assess the value of correct conditional covariances. Next, the multivariate GARCH models and corresponding estimation techniques is further explained. This paper distinguishes between two settings: a bivariate and a multivariate setting. In the bivariate setting, the same models as in Engle and Colacito (2006) are used: the Diagonal BEKK, Scalar BEKK, Orthogonal GARCH, DCC and Asymmetric DCC models. Since estimation of these models gets cumbersome in a large multivariate framework and the previously mentioned models cannot allow for any structure in the correlation matrix, two other models that are designed to solve this issue are introduced: the Dynamic Equicorrelation model and the Dynamic Factor GARCH Model. For the latter model, Kalman Filter estimation with diffuse priors is used, while the Composite Likelihood estimator is used for all DCC-type models. Finally, this paper uses a Combination estimator introduced by Caldeira et al. (2017) that combines the estimates for the conditional covariance matrix of all previously mentioned models.

3.1 Classical Asset Allocation Problem

The classical asset allocation problem is concerned with minimising the variance of a portfolio of assets subject to a required return constraint. This problem can be formulated as follows:

$$\min_{\boldsymbol{w}_t} \quad \boldsymbol{w}_t' \boldsymbol{H}_t \boldsymbol{w}_t, \qquad s.t. \quad \boldsymbol{w}_t' \boldsymbol{\mu} = \mu_0 \tag{1}$$

where $\boldsymbol{w}_t$ is the $N \times 1$ vector of portfolio weights for time t chosen at time $t-1$, $\boldsymbol{H}_t$ is the estimated conditional covariance matrix of a vector of excess returns for time t by some model, $\boldsymbol{\mu}$ is the assumed vector of excess returns with respect to the risk-free rate, and $\mu_0 \geq 0$ is the required return. Then, the solution to the problem in (1) is $\boldsymbol{w}_t = \frac{\boldsymbol{H}_t^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}'\boldsymbol{H}_t^{-1}\boldsymbol{\mu}}\mu_0$. Generally, $\sum_{i=1}^{N} w_{i,t}$ does not need to be equal to 1, $w_{i,t}$ is the weight on asset i for time t and $1-\sum_{i=1}^{N} w_{i,t}$ is the weight attached to the risk-free asset.

The above problem constitutes the classical portfolio construction problem, in which an optimal weight is attached to the market portfolio – also known as the tangency portfolio – and the rest of the capital is allocated to the risk-free asset. Then, the optimal risk-return trade-off is represented by the set of linear combinations of the risk-free asset and the market portfolio. Engle and Colacito (2006) prove that the variance obtained from an incorrect estimate of the covariance matrix is always greater than the variance obtained from the true covariance matrix. To do so, they introduce the true covariance matrix $\boldsymbol{\Omega}_t$ and the vector of optimal weights from this covariance matrix $\boldsymbol{w}_t^* = \frac{\boldsymbol{\Omega}_t^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\mu}}\mu_0$. This results in two conditional standard deviations which can be compared:

$$\frac{\sigma_t}{\mu_0} = \frac{\sqrt{E_{t-1}[\boldsymbol{w}_t'(\boldsymbol{r}_t - E_{t-1}\boldsymbol{r}_t)]^2}}{\mu_0} = \frac{\sqrt{\boldsymbol{w}_t'\boldsymbol{\Omega}_t\boldsymbol{w}_t}}{\mu_0} = \frac{\sqrt{\boldsymbol{\mu}'\boldsymbol{H}_t^{-1}\boldsymbol{\Omega}_t\boldsymbol{H}_t^{-1}\boldsymbol{\mu}}}{\boldsymbol{\mu}'\boldsymbol{H}_t^{-1}\boldsymbol{\mu}} \tag{2}$$

$$\frac{\sigma_t^*}{\mu_0} = \frac{\sqrt{\boldsymbol{w}_t^{*\prime}\boldsymbol{\Omega}_t\boldsymbol{w}_t^*}}{\mu_0} = \sqrt{\frac{1}{\boldsymbol{\mu}'\boldsymbol{\Omega}_t^{-1}\boldsymbol{\mu}}} \tag{3}$$

where $E_t[\cdot]$ is the conventional expectation operator. Engle and Colacito (2006) prove that the portfolio standard deviation defined in Equation 2, σ_t, is greater than the realised standard deviation defined in Equation 3, σ_t^*, and $\sigma_t = \sigma_t^*$ when $\boldsymbol{\mu}$ is an eigenvector of $\boldsymbol{\Omega}_t\boldsymbol{H}_t$.

3.2 Statistical Testing Procedures

3.2.1 Testing in the bivariate setting

Engle and Colacito (2006) develop a testing procedure to test the equality of two models, which is adopted in this research. Define $\{\boldsymbol{H}_t^j\}_{j=1}^2$ as two different time-series of covariance matrices obtained from two different models and $\{\boldsymbol{\mu}^k\}_{k=1}^K$ a set of hypothetical vectors of expected returns divided by the required excess return μ_0. In each time period, a set of portfolio weights, $\boldsymbol{w}_t^{j,k}$, can be calculated from the estimated covariance matrix and an expected return. Denote the portfolio return as:

$$\pi_t^{j,k} = \boldsymbol{w}_t^{j,k\prime}(\boldsymbol{r}_t - \overline{\boldsymbol{r}}) \tag{4}$$

where $\boldsymbol{w}_t^{j,k} = \frac{(\boldsymbol{H}_t^j)^{-1}\boldsymbol{\mu}^k}{(\boldsymbol{\mu}^k)'(\boldsymbol{H}_t^j)^{-1}(\boldsymbol{\mu}^k)}$. Consequently, the squared portfolio return is calculated and let the difference of the squared returns be denoted as u_t^k. Then, u_t^k is defined as:

$$u_t^k = (\pi_t^{1,k})^2 - (\pi_t^{2,k})^2, \qquad t = 1, \ldots, T \tag{5}$$

In the test of equality between the two models, the null hypothesis is that the mean of u is 0 for all k. Ignoring parameter uncertainty, the Diebold-Mariano test would examine each u time-series separately by regressing on a constant and using Newey and West (1986) standard errors. In general, the covariance matrix should correct the standard t-test for heteroscedasticity, autocorrelation and non-normality, the null hypothesis of equal variance is simply a test that the mean of u equals 0.

However, the test is less powerful due to the amount of heteroscedasticity present. Hence, dividing u by its standard deviation could improve the efficiency of the estimation. A natural adjustment, v_t^k, is obtained by dividing by the geometric mean of the two variance estimators, since the true covariance matrix is not observed and two estimators are compared. Since this correction factor is known and positive, the null hypothesis is not changed. Then, v_t^k is obtained as follows:

$$v_t^k = u_t^k[2(\boldsymbol{\mu}^{k\prime}(\boldsymbol{H}_t^1)^{-1}\boldsymbol{\mu}^k)(\boldsymbol{\mu}^{k\prime}(\boldsymbol{H}_t^2)^{-1}\boldsymbol{\mu}^k)]^{1/2} \tag{6}$$

Since it could be the case that the above adjustment does not improve the sampling properties of the test such that v is not an independent, identically distributed series, the test must also be carried out with a Heteroscedasticity and Autocorrelation Consistent (HAC) covariance matrix of Newey and West (1986), which is the case for u_t^k. Therefore, the Diebold-Mariano test must be carried out for the v series as well as for the u series for all k. Define $\boldsymbol{U}_t = (u_t^1, \ldots, u_t^k)'$ and $\boldsymbol{V}_t = (v_t^1, \ldots, v_t^k)'$. Then, a Generalised Method of Moments (GMM) with a vector HAC covariance matrix estimation is applied to jointly estimate:

$$\boldsymbol{U}_t = \beta_u \boldsymbol{\iota} + \boldsymbol{\varepsilon}_{u,t}, \qquad \boldsymbol{V}_t = \beta_v \boldsymbol{\iota} + \boldsymbol{\varepsilon}_{v,t} \tag{7}$$

where $\boldsymbol{\iota}$ is a $k \times 1$ vector of ones and β_u and β_v are scalars. Under the null hypothesis, β_u and β_v are both equal to 0.

3.2.2 Testing in the multivariate setting

Barendse and Patton (2018) extend the DM test to evaluate the null hypothesis of equal expected loss over a large Euclidean parameter space instead of one element from this space. They show that their approach can increase power over local tests and provide robustness against rejections of the null hypothesis at specific points in the parameter space. Barendse and Patton (2018) develop their procedure for an out-of-sample testing exercise, but, since Engle and Colacito (2006) use an in-sample testing method, this test is also performed in-sample. Possible consequences of this approach are further discussed in Section 7.

Let $L(W_t; \boldsymbol{\mu}) = u_t(\boldsymbol{\mu}) = (\pi_t^1(\boldsymbol{\mu}))^2 - (\pi_t^2(\boldsymbol{\mu}))^2$ be the loss difference function, where $\pi_t^j(\boldsymbol{\mu})$ is defined as in Equation 4 and $\boldsymbol{\mu} \in M$, $M = \{\boldsymbol{\mu} : \mu_1, \ldots, \mu_N \in [0,1]\}$ is a $N \times 1$ vector of expected returns. Then, the null hypothesis is as follows: $H_0 : E[L(W_t; \boldsymbol{\mu})] = 0, \forall \boldsymbol{\mu} \in M$.

Next, two robust test statistics are introduced to test this hypothesis: $\sup t_{m,n} = \sup_{\boldsymbol{\mu} \in \boldsymbol{M}} t^2_{m,n}(\boldsymbol{\mu})$ and ave $t_{m,n} = \int_M t^2_{m,n}(\boldsymbol{\mu}) dJ(\boldsymbol{\mu})$. In this study, we choose the weighting function J over M to be uniform on the interval [0,1] and independent for all elements of $\boldsymbol{\mu}$. The choice of this interval is based on the fact that in the bivariate setting, the set of expected return combinations is also between 0 and 1. Furthermore, $t^2_{m,n}$ are individual Diebold and Mariano (2002) test statistics at $\boldsymbol{\mu}$:

$$t^2_{m,n}(\boldsymbol{\mu}) = \sqrt{n} \frac{\bar{Z}_{m,n}(\boldsymbol{\mu})}{\hat{\sigma}_{m,n}(\boldsymbol{\mu}, \boldsymbol{\mu}')} \tag{8}$$

where $\bar{Z}_{m,n}(\boldsymbol{\mu}) = \frac{1}{n} \sum_{t=1}^{T} Z_t(\boldsymbol{\mu})$, $Z_t(\boldsymbol{\mu}) = L(W_t; \boldsymbol{\mu})$ and $\hat{\sigma}_{m,n}(\boldsymbol{\mu}, \boldsymbol{\mu}')$ is the HAC estimator of $\sigma_m(\boldsymbol{\mu}, \boldsymbol{\mu}') = Cov(\sqrt{n}\bar{Z}_{m,n}(\boldsymbol{\mu}), \sqrt{n}\bar{Z}_{m,n}(\boldsymbol{\mu}'))$. Barendse and Patton (2018) show how to obtain critical values using a a simulation procedure similar to the dependent wild bootstrap procedure introduced by Shao (2010), which preserves the dependence structure of the data.

3.3 Correlation Models

3.3.1 Models for the bivariate framework

The first conditional correlation model that is used is the Baba, Engle, Kraft and Kroner (BEKK) representation of the vec model introduced by Engle and Kroner (1995). Both Engle and Kroner (1995) and Engle (2002) discuss the BEKK representation extensively. The general representation uses the following specification for the conditional covariance matrix, $\boldsymbol{H}_t$:

$$\boldsymbol{H}_t = \boldsymbol{\Omega} + \boldsymbol{A}(\boldsymbol{r}_{t-1}\boldsymbol{r}'_{t-1})\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{H}_{t-1}\boldsymbol{B}' \tag{9}$$

This study employs two special representations of $\boldsymbol{A}$ and $\boldsymbol{B}$: one where both matrices are diagonal (Diagonal BEKK) and one where both matrices are reduced to scalars (Scalar BEKK).

DCC-type methods use a multi-step approach to estimating the conditional covariance matrix, $\boldsymbol{H}_t$ and decomposes $\boldsymbol{H}_t$ as:

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t \tag{10}$$

where $\boldsymbol{D}_t = \text{diag}\{\sqrt{h_{i,t}}\}$, $h_{i,t}$ are univariate conditional volatilities estimated in a GARCH-type setting during the first step, and $\boldsymbol{R}_t$ is the time-varying conditional correlation matrix. In the second step, a GARCH$(1,1)$ model is estimated for the conditional correlations, $\boldsymbol{R}_t$, where $\varepsilon_{i,t}$ are the standardised residuals from the univariate GARCH estimations in the first step, defined as $\varepsilon_{i,t} = y_{i,t}/\sqrt{h_{i,t}}$ and $\boldsymbol{S}$ is the unconditional correlation matrix of the standardised residuals:

$$\boldsymbol{R}_t = \boldsymbol{Q}_t^{*-\frac{1}{2}} \boldsymbol{Q}_t \boldsymbol{Q}_t^{*-\frac{1}{2}}, \qquad \boldsymbol{Q}_t = \boldsymbol{S}(1 - \alpha - \beta) + \alpha(\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}'_{t-1}) + \beta \boldsymbol{Q}_{t-1} \tag{11}$$

where $\boldsymbol{Q}_t^*$ is a diagonal matrix where its entries are the elements on the diagonal of $\boldsymbol{Q}_t$. As long as $\alpha + \beta < 1$, the model is mean-reverting. In the case where $\alpha + \beta = 1$, the model ends up being a exponential smoothing model where $\boldsymbol{Q}_t = (1-\lambda)(\varepsilon_{t-1}\varepsilon_{t-1}') + \lambda \boldsymbol{Q}_{t-1}$. As Engle (2002) indicates, this version of the DCC model can be extended to a version where the parameters are matrices instead of scalars. This version is referred to as the full DCC model and can be written as follows:

$$\boldsymbol{Q}_t = \boldsymbol{S} \circ (\boldsymbol{\iota\iota}' - \boldsymbol{A} - \boldsymbol{B}) + \boldsymbol{A} \circ (\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}') + \boldsymbol{A} \circ \boldsymbol{Q}_{t-1} \tag{12}$$

where $\boldsymbol{\iota}$ is a vector of ones and $\circ$ denotes the standard Hadamard product, which is performed by element-wise multiplication. Moreover, Ding and Engle (2001) show that if $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{S} \circ (\boldsymbol{\iota\iota}' - \boldsymbol{A} - \boldsymbol{B})$ are positive semidefinite, then $\boldsymbol{Q}$ will also be positive semidefinite.

Inspired by asymmetric univariate GARCH models, Cappiello et al. (2006) introduced the Asymmetric DCC (ADCC) model with scalar and matrix parameters, which is obtained by modifying $\boldsymbol{Q}_t$:

$$\boldsymbol{Q}_t = \boldsymbol{S}(1-\alpha-\beta-\gamma) + \alpha(\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}') + \beta\boldsymbol{Q}_{t-1} + \gamma(\eta_{t-1}\eta_{t-1}') \tag{13}$$

$$\boldsymbol{Q}_t = (\boldsymbol{S} - \boldsymbol{A}'\boldsymbol{S}\boldsymbol{A} - \boldsymbol{B}'\boldsymbol{S}\boldsymbol{B} - \boldsymbol{G}'\boldsymbol{N}\boldsymbol{G}) + \boldsymbol{A}'(\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}')\boldsymbol{A} + \boldsymbol{B}'\boldsymbol{Q}_{t-1}\boldsymbol{B} + \boldsymbol{G}'(\eta_{t-1}\eta_{t-1}')\boldsymbol{G} \tag{14}$$

where $\eta_t = I[\varepsilon_t < 0]\varepsilon_t$ in Equation 13, $\eta_t = \boldsymbol{I}[\boldsymbol{\varepsilon}_t < 0] \circ \boldsymbol{\varepsilon}_t$ in Equation 14 and $I[\cdot]$ is the standard indicator function.

All DCC-type models are estimated with Composite Likelihood estimation, on which is elaborated in Section 3.4.1. Even though BEKK-type models can be estimated with Composite Likelihood, this study employs the regular Maximum Likelihood estimation routine. Details for all previously mentioned models in a bivariate setting can be found in Appendix A.

3.3.2 Additional models for the multivariate setting

In addition to the models used by Engle and Colacito (2006), this paper includes other models that have been introduced after Engle and Colacito (2006) published their paper in the multivariate setting. Firstly, the Dynamic Factor multivariate GARCH, referred to as DFGARCH, model by Santos and Moura (2014) is employed, which reduces the dimensionality of the problem. They specify a factor model with observable factors and time-varying factor loadings for the asset returns:

$$y_{i,t} = \boldsymbol{f}_t\boldsymbol{\beta}_{i,t} + \varepsilon_{i,t} \tag{15}$$

where $\varepsilon_{i,t} \sim N(0, h_{i,t})$. Next, Santos and Moura (2014) estimate two different state-space equations with a Diffuse Kalman Filter for the unobservable factor loadings: a Random Walk and a Learning model, where $\boldsymbol{B}_i$ can be interpreted as the long-run mean of the factor loadings, $\boldsymbol{\beta}_{i,t}$, and $\phi \in (-1, 1)$. As an extension, a VAR(1) specification is added:

$$\boldsymbol{\beta}_{i,t} = \boldsymbol{\beta}_{i,t-1} + \boldsymbol{u}_{i,t} \tag{16}$$

$$\boldsymbol{\beta}_{i,t} = (1-\phi)\boldsymbol{B}_i + \phi\boldsymbol{\beta}_{i,t-1} + \boldsymbol{u}_{i,t} \tag{17}$$

$$\boldsymbol{\beta}_{i,t} = (\mathbf{I} - \boldsymbol{\Phi})\boldsymbol{\alpha}_i + \boldsymbol{\Phi}\boldsymbol{\beta}_{i,t-1} + \boldsymbol{u}_{i,t} \tag{18}$$

where $\boldsymbol{u}_{i,t} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{u_i})$. Then, the conditional covariance matrix of the asset returns can be calculated as follows:

$$\boldsymbol{H}_t = \boldsymbol{\beta}_t \boldsymbol{\Omega}_t \boldsymbol{\beta}_t' + \boldsymbol{\Xi}_t \tag{19}$$

where $\boldsymbol{\Omega}_t$ is the conditional covariance matrix of the factors and $\boldsymbol{\Xi}_t$ is a diagonal covariance matrix of the residuals from the factor model in Equation 15. Finally, $\boldsymbol{\Omega}_t$ is estimated with a DCC model.

The DFGARCH model extends previous econometric specifications in two directions. Firstly, it achieves greater flexibility by allowing alternative econometric specifications for the common factors and individual assets. In particular, the model allows for a parsimonious multivariate specification for the covariances among factors based on a conditional correlation model instead of modelling the conditional covariance matrix for all assets, which could reduce the effect of parameter uncertainty. Secondly, the factor loads are time-varying, since the factors are observed.

The second model that is employed is the Dynamic Equicorrelation (DECO) model as proposed by Engle and Kelly (2012). The main assumption in this model is that in every time period, all pairwise correlations are equal, which results in easy estimation of large covariance matrices. It uses a similar estimation method as the DCC model, but uses a different specification for the correlation matrix, $\boldsymbol{R}_t$, which is called the equicorrelation matrix:

$$\boldsymbol{R}_t^{DECO} = (1-\rho_t)\boldsymbol{I}_n + \rho_t \boldsymbol{J}_n \tag{20}$$

where ρ_t is the equicorrelation, $\boldsymbol{I}_n$ is the n-dimensional identity matrix, and $\boldsymbol{J}_n$ is a $n \times n$ matrix of ones. Consequently, the DECO model sets ρ_t equal to the average pairwise DCC correlation:

$$\rho_t = \frac{\boldsymbol{\iota}' \boldsymbol{R_t^{DCC}} \boldsymbol{\iota} - n}{n(n-1)} = \sum_{i<j} \frac{2q_{i,j,t}^{cDCC}}{n(n-1)\sqrt{q_{i,i,t}^{cDCC} q_{j,j,t}^{cDCC}}} \tag{21}$$

where $q_{i,j,t}$ is the i,j-th element of the conditional covariance matrix in the cDCC model specified by Aielli (2013).

However, there is reason to make less restrictive assumptions on the conditional correlation matrix structure, since it is not likely that all assets have the same correlation. In fact, it is likely that the correlation matrix has a particular block structure across asset classes. In this case, the Block DECO (BDECO) model can be applied, such that a more flexible model than the DECO model is obtained. Correspondingly, an I-block equicorrelation matrix, R_t^{BDECO} is formed:

$$\boldsymbol{R}_t^{BDECO} = \begin{bmatrix} (1-\rho_{1,1,t})\boldsymbol{I}_{n_1} & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & 0 & (1-\rho_{I,I,t})\boldsymbol{I}_{n_I} \end{bmatrix} + \begin{bmatrix} \rho_{1,1,t}\boldsymbol{J}_{n_1} & \rho_{1,2,t}\boldsymbol{J}_{n_1\times n_2} & \cdots \\ \rho_{2,1,t}\boldsymbol{J}_{n_2\times n_1} & \ddots & \\ \vdots & & \rho_{I,I,t}\boldsymbol{J}_{n_I} \end{bmatrix} \tag{22}$$

where $\rho_{l,m,t} = \rho_{m,l,t}$ for all $l, m \in \{1, \ldots, I\}$.

Block DECO specifies that, conditional on the past, each variable is standard normally distributed and correlations taking the structure in Equation 22. The return vector $\boldsymbol{r}_t$ is partitioned into I subvectors, where each subvector $\boldsymbol{r}_m$ contains n_m returns. This structure allows the correlation matrix to have I unique diagonal processes and $I(I-1)/2$ distinct off-diagonal blocks. Blocks on the diagonal have equicorrelations $\rho_{l,l,t}$ where off-diagonal blocks have equicorrelation $\rho_{l,m,t}$ as follows:

$$\rho_{l,l,t} = \frac{1}{n_l(n_l-1)} \sum_{i\in l, j\in l, i\neq j} \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}, \qquad \rho_{l,m,t} = \frac{1}{n_l n_m} \sum_{i\in l, j\in m} \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}} \tag{23}$$

where $q_{i,j,t}$ is the i,j-th element in $\boldsymbol{Q}_t$ of Equation 11. Even though the Block DECO correlations are calculated as the average DCC correlation within each block, the DCC-parameters, α and β, are the same for each block, since Equation 11 remains the underlying model.

The last approach is based on the work of Caldeira et al. (2017) and makes convex combinations of the M previously mentioned models into one estimator for the conditional covariance matrix, $\boldsymbol{H}_t^{Comb}$, as follows:

$$\boldsymbol{H}_t^{Comb} = \lambda_{1,t}\boldsymbol{H}_t^1 + \cdots + \lambda_{M,t}\boldsymbol{H}_t^M \tag{24}$$

where $\sum_{m=1}^{M} \lambda_{m,t} = 1$ and $\lambda_{m,t} \leq 0$, $\forall\, m$. Moreover, similar to Caldeira et al. (2017), $\lambda_{m,t}$ are chosen based on historical Sharpe ratios of portfolios constructed by the individual models. Details on the exact calculation of $\lambda_{m,t}$ can be found in Appendix B. Caldeira et al. (2017) find that their model outperforms all other individual candidate models, which is due to the fact that the model exploits complementarities of all individual correlation models.

3.4 Estimation techniques

3.4.1 Composite Likelihood

Engle and Colacito (2006) estimate the multivariate GARCH models with an estimator that is subject to the constraint that the long-run covariance matrix is the sample covariance matrix. This approach differs from (Quasi-)Maximum Likelihood only in finite samples (see Engle and Mezrich, 1996). Maximum Likelihood estimation has been the common estimation method for multivariate GARCH models. Engle (2002) formulates the DCC model as the following specification:

$$\begin{aligned}
&\boldsymbol{r}_{t|t-1} \sim N(\mathbf{0}, \boldsymbol{D}_t\boldsymbol{R}_t\boldsymbol{D}_t), \qquad \boldsymbol{D}_t^2 = \text{diag}\{\boldsymbol{\omega}_i\} + \text{diag}\{\boldsymbol{\kappa}_i\} \circ \boldsymbol{r}_{t-1}\boldsymbol{r}_{t-1}' + \text{diag}\{\boldsymbol{\lambda}_i\} \circ \boldsymbol{D}_{t-1}^2 \\
&\boldsymbol{\varepsilon}_t = \boldsymbol{D}_t^{-1}\boldsymbol{r}_t, \qquad \boldsymbol{Q}_t = \boldsymbol{S} \circ (\boldsymbol{\iota\iota}' - \boldsymbol{A} - \boldsymbol{B}) + \boldsymbol{A} \circ (\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}') + \boldsymbol{A} \circ \boldsymbol{Q}_{t-1} \\
&\boldsymbol{R}_t = \text{diag}\{\boldsymbol{Q}_t\}^{-1}\boldsymbol{Q}_t\text{diag}\{\boldsymbol{Q}_t\}^{-1}
\end{aligned} \tag{25}$$

Consequently, the loglikelihood can be written as follows, where $|\cdot|$ is the determinant operator:

$$\begin{aligned}
L &= -\frac{1}{2}\sum_{t=1}^{T}(n\log 2\pi + \log|\boldsymbol{H}_t| + \boldsymbol{r}_t'\boldsymbol{H}_t^{-1}\boldsymbol{r}_t) \\
&= -\frac{1}{2}\sum_{t=1}^{T}(n\log 2\pi + 2\log|\boldsymbol{D}_t| + \boldsymbol{r}_t'\boldsymbol{D}_t^{-2}\boldsymbol{r}_t - \boldsymbol{\varepsilon}_t'\boldsymbol{\varepsilon}_t + \log|\boldsymbol{R}_t| + \boldsymbol{\varepsilon}_t'\boldsymbol{R}_t^{-1}\boldsymbol{\varepsilon}_t)
\end{aligned} \tag{26}$$

Now, let the parameters in $\boldsymbol{D}$ be denoted by $\boldsymbol{\theta}$ and the remaining parameters in $\boldsymbol{R}$ be denoted by $\boldsymbol{\phi}$. Then, the loglikelihood can be split into a part with only volatility parameters, $L_V(\boldsymbol{\theta})$, and a part with both volatility and correlation parameters, $L_C(\boldsymbol{\theta}, \boldsymbol{\phi})$:

$$L(\boldsymbol{\theta}, \boldsymbol{\phi}) = L_V(\boldsymbol{\theta}) + L_C(\boldsymbol{\theta}, \boldsymbol{\phi}) \tag{27}$$

$$L_V(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{t=1}^{T}(n\log 2\pi + 2\log|\boldsymbol{D}_t| + \boldsymbol{r}_t'\boldsymbol{D}_t^{-2}\boldsymbol{r}_t) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{t=1}^{T}(n\log 2\pi + \log h_{i,t} + \frac{r_{i,t}^2}{h_{i,t}}) \tag{28}$$

$$L_C(\boldsymbol{\theta}, \boldsymbol{\phi}) = \frac{1}{2}\sum_{t=1}^{T}(\log|\boldsymbol{R}_t| + \boldsymbol{\varepsilon}_t'\boldsymbol{R}_t^{-1}\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_t'\boldsymbol{\varepsilon}_t) \tag{29}$$

Newey and McFadden (1994) give sufficient conditions for the consistency and asymptotic normality of the estimators for $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ in a two-step maximisation algorithm. In the first step, the volatility part in Equation 28 is the sum of the individual univariate GARCH likelihoods, which are jointly maximised by separately maximising each term: $\hat{\boldsymbol{\theta}} = \arg\max\{L_V(\boldsymbol{\theta})\}$. In the second step, the second part of the likelihood is used to estimate the correlation parameters, where the estimates of the volatility parameters are from the first step: $\hat{\boldsymbol{\phi}} = \arg\max_{\phi}\{L_C(\hat{\boldsymbol{\theta}}, \boldsymbol{\phi})\}$.

However, Engle et al. (2009) show that computational challenges arise when fitting a large-dimensional model and propose a Composite Likelihood (CL) estimator in the second step to ease the computational burden of estimating the correlation parameters. The idea is to first introduce a data array $\boldsymbol{Y}_t = \{\boldsymbol{Y}_{1,t}, \dots, \boldsymbol{Y}_{N,t}\}$ which is composed out of the returns $\boldsymbol{r}_t$ according to some deterministic selection matrix $\boldsymbol{S}_j$, such that the submodel $\boldsymbol{Y}_{j,t}$ is defined as $\boldsymbol{Y}_{j,t} = \boldsymbol{S}_j\boldsymbol{r}_t$.[1] Let

[1]In this research, a pairwise selection like in Engle et al. (2009) is used

$\mathcal{F}_{t-1}$ denote the information set containing all information up to and including time $t-1$, then $E(\boldsymbol{Y}_{j,t}|\mathcal{F}_{t-1}) = \mathbf{0}$ and $\text{Var}(\boldsymbol{Y}_{j,t}|\mathcal{F}_{t-1}) = \boldsymbol{H}_{j,t} = \boldsymbol{S}_j\boldsymbol{H}_t\boldsymbol{S}'_j$, where $\boldsymbol{H}_t$ is as defined in Equation 10. Then, a valid quasi-loglikelihood function for $\boldsymbol{\psi}$ off the j-th submodel can be formulated as:

$$L_j(\boldsymbol{\psi}) = \sum_{t=1}^{T} l_{j,t}(\boldsymbol{\psi}), \quad \text{where } l_{j,t}(\boldsymbol{\psi}) = \log f(\boldsymbol{Y}_{j,t};\boldsymbol{\psi}) = -\frac{1}{2}\log|\boldsymbol{H}_{j,t}| - \frac{1}{2}\boldsymbol{Y}'_{j,t}\boldsymbol{H}_{j,t}\boldsymbol{Y}_{j,t} \tag{30}$$

More information about $\boldsymbol{\psi}$ can be obtained by averaging the same operation of many submodels and summing over all time periods $t = 1, \ldots, T$, the Composite Likelihood function is obtained:

$$CL(\boldsymbol{\psi}) = \frac{1}{N}\sum_{t=1}^{T}\sum_{j=1}^{N}\log L_{j,t}(\boldsymbol{\psi}) \tag{31}$$

Finally, the parameters $\boldsymbol{\psi}$ are found by maximising $CL(\boldsymbol{\psi})$ with a numerical maximisation routine. For fixed N, as T approaches infinity, $\hat{\boldsymbol{\psi}}$ has well-known asymptotic properties, since the Composite Likelihood estimator is a particular case of a quasi-likelihood. For further details about the CL estimator in a multivariate GARCH setting, refer to Engle et al. (2009).

3.4.2 Exact initial Kalman Filter

In this research, Kalman Filter estimation is used to estimate the factor loads in the DFGARCH model. The Kalman filter is an estimation algorithm that tracks a time-series – in this case the asset returns – and simultaneously estimates the parameters of the underlying model. Equations 15 and 16 together form our observation process and our state space process, respectively, and are repeated:

$$y_{i,t} = \boldsymbol{f}_t\boldsymbol{\beta}_{i,t} + \varepsilon_{i,t}, \;\; \varepsilon_{i,t} \sim N(0, h_i) \tag{32}$$

$$\boldsymbol{\beta}_{i,t} = \boldsymbol{F}\boldsymbol{\beta}_{i,t-1} + \boldsymbol{S}\boldsymbol{u}_{i,t}, \;\; \boldsymbol{u}_{i,t} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{u_i}) \tag{33}$$

where $\boldsymbol{f}_t$ are the time-varying factors, $\boldsymbol{\beta}_{i,t}$ represents the time-varying factor loadings, which are treated as unobserved states. Transition matrix $\boldsymbol{F}$ makes sure that the states evolve according to the Random Walk, Learning and VAR models specified in Equations 16 - 18 and $\boldsymbol{S}$ is a selection matrix containing zeros and ones. See Appendix C for detailed specifications of $\boldsymbol{F}$, $\boldsymbol{\beta}_{i,t}$, $\boldsymbol{f}_t$ and $\boldsymbol{S}$ and refer to Adrian and Franzoni (2009) for a complete derivation of the specification of these matrices.

Let $\boldsymbol{\beta}_{i,t|t-1}$ be the expectation of $\boldsymbol{\beta}_{i,t}$ conditional on $y_{i,1}, \ldots, y_{i,t-1}$ with mean square error matrix $\boldsymbol{P}_{t|t-1}$. When observation $y_{i,t}$ is observed, the prediction error, $v_{i,t}$, the covariance matrix of the prediction error, Δ_t, and the Kalman gain, $\boldsymbol{K}_t$ can be calculated:

$$v_{i,t} = y_{i,t} - \boldsymbol{f}_t'\boldsymbol{\beta}_{i,t|t-1}, \qquad \Delta_t = \boldsymbol{f}_t'\boldsymbol{P}_{t|t-1}\boldsymbol{f}_t + h_{i,t}, \qquad \boldsymbol{K}_t = \boldsymbol{P}_{t|t-1}\boldsymbol{f}_t'\Delta_t^{-1} \tag{34}$$

Then, after observing $y_{i,t}$ and for given values of $\boldsymbol{\beta}_{i,t|t-1}$ and $\boldsymbol{P}_{t|t-1}$, a more accurate inference of $\boldsymbol{\beta}_{i,t|t}$ and $\boldsymbol{P}_{t|t}$ can be made:

$$\boldsymbol{\beta}_{i,t|t} = \boldsymbol{\beta}_{i,t|t-1} + \boldsymbol{K}_t v_{i,t}, \qquad \boldsymbol{P}_{t|t} = \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{f}_t'\Delta_t^{-1}\boldsymbol{f}_t\boldsymbol{P}_{t|t-1} \tag{35}$$

Lastly, an estimate of the state vector and its covariance matrix in period $t+1$ conditional on $y_{i,1}, \cdots, y_{i,t}$ is given by the prediction step:

$$\boldsymbol{\beta}_{i,t+1|t} = \boldsymbol{F}\boldsymbol{\beta}_{i,t|t}, \qquad \boldsymbol{P}_{i,t+1|t} = \boldsymbol{F}\boldsymbol{P}_{i,t|t}\boldsymbol{F}' + \boldsymbol{S}\boldsymbol{\Sigma}_{u_i}\boldsymbol{S}' \tag{36}$$

Usually, the Kalman Filter computations are carried out recursively for $t = 1, \cdots, T$ for a given time-series $y_{i,1}, \cdots, y_{i,T}$. However, since the equation for $\boldsymbol{\beta_{i,t}}$ is non-stationary, the initialisation is implemented using the exact initial Kalman Filter introduced by Koopman (1997), which uses diffuse priors. Details of the exact initial Kalman Filter can be found in Appendix D.

The parameters in the covariance matrix $\boldsymbol{\Sigma}_{u_i}$ are treated as unknown coefficients and are collected in the parameter vector $\boldsymbol{\psi}$. Estimation of $\boldsymbol{\psi}$ is done by executing the numerical quasi-Newton maximisation algorithm on the loglikelihood function, $l(\boldsymbol{\psi})$, which is constructed by the prediction error decomposition:

$$l(\boldsymbol{\psi}) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log|\Delta_t| - \frac{1}{2}\sum_{t=1}^{T} v_t'\Delta_t^{-1}v_t \tag{37}$$

4 Data

For the purpose of performing analysis in a bivariate setting, identical data as Engle and Colacito (2006) used is employed. This data set consists of daily closing prices for the S&P 500 Index and the Dow Jones Index are obtained from Yahoo! Finance. The data range in this case is 8/26/1988 - 8/26/2003 (2635 observations). Consequently, returns are calculated as the differences in the natural logarithm of the prices. Panel A of Table 1 displays summary statistics for the returns of these two assets.

In practice asset managers can choose divide their capital over more asset classes. Therefore it is interesting to extend the bivariate setting of only the S&P 500 Index and the Dow Jones Index to a multivariate framework, where an investment universe of multiple assets is constructed. In this multivariate setting, two variations on Engle and Colacito (2006) with regards to the data are used.

Table 1: Summary statistics for returns in the bivariate and multivariate settings

	Panel A: Bivariate setting		Panel B: Multivariate setting						
	S&P 500	Dow Jones	Alt. Assets	Equities	Currencies	Credit	Commodities	Real Estate	Gov. bonds
Mean	7.53	9.43	4.80	6.95	-0.51	6.18	-1.38	9.33	5.46
Variance	1.25	1.20	0.98	1.58	0.51	0.52	1.48	1.07	0.53
Kurtosis	6.46	7.30							
Correlation	0.939								

In Panel A, summary statistics for the returns of the S&P 500 Index and the Dow Jones Index for the period 8/29/1988 - 8/26/2003 are presented. Panel B displays summary statistics for the returns of each asset class used in the multivariate setting for the period 4/1/2003 - 4/30/2018. The mean returns have been annualised with a factor of 252.

Firstly, the data range is changed to the period 4/1/2003 - 4/30/2018 (3771 observations). In this way, models can be estimated over more periods of different volatility and correlation levels that smoothly transition into each other. Secondly, 65 indices from seven different asset classes are used to capture different mean-variance profiles. Based on the work of Doeswijk et al. (2014), the seven selected asset classes are (1) Equities, (2) Government bonds, (3), Corporate bonds – for brevity referred to as Credit, (4) Commodities, (5) Currencies, (6) Real estate, and (7) Alternative assets. For asset classes (1) and (2), various indices that differ in geographical exposure on country level are used, as constructed by FTSE All-World and DataStream, respectively. In asset class (3), next to a distinction between investment grade and high-yield corporate bonds, indices for European, American and Emerging Market Debt are selected. In order to get a diversified representation of the Commodities asset class, multiple product-specific indices as constructed by Bloomberg are used. The DXY Index is considered as a representative index for the Currencies asset class, as it reflects the value of the U.S. Dollar against a basket of various different currencies. For asset class (6), a global property index constructed by S&P, which is considered as representative for the entire asset class, is used. Lastly, one Private Equity index and one Hedge Fund index are used to form the Alternative Asset class. All price data is converted to Euro and, with the exception of the Government bond indices, for which data is obtained from DataStream, all data is obtained from Bloomberg. For a list of Bloomberg Tickers and DataStream RICs, refer to Figure E1 in Appendix E. Returns are again obtained as the differences in the natural logarithm of the asset prices. Panel B of Table 1 displays summary statistics for the returns of the seven asset classes.

From Panel A in Table 1 it is clear that the S&P 500 Index and the Dow Jones Index have similar mean-variance characteristics and are in line with the findings of Engle and Colacito (2006). Contrary to the standard rule of thumb, the S&P 500 Index has a higher variance compared to the Dow Jones Index, while having a lower mean return. The sample correlation is extremely high:

0.939. Engle and Colacito (2006) argue that the importance of correct estimation of correlations is higher in this setting compared to a setting with a low correlation.

Panel B shows that the different asset classes indeed have different mean-variance characteristics over the entire sample period. For example, Real estate has the highest mean return, but has a lower standard deviation than Equities and Commodities. Currencies have the lowest variance accompanied by a poor mean return and Government bonds have the best mean-variance ratio. Figure F1 in Appendix F shows the complete correlation matrix of all indices over the entire sample period. In this picture, a clear block-structure is visible between the various asset classes. While the correlations within each asset class are quite positive, correlations across asset classes can become quite negative (e.g. -0.43 for the pair Italian Equities - German Government Bonds). During this sample period, the correlation between the S&P 500 Index and the Dow Jones Index was 0.980.

Lastly, in order to implement the Dynamic Factor GARCH model, the factor returns of the 3-factor model as proposed by Fama and French (1993) are used. This approach is similar to the approach of Santos and Moura (2014). All factor returns are obtained from the web page of Kenneth French.[2] See Fama and French (1993) for details regarding the construction of these factor portfolios.

5 Results

5.1 Bivariate setting

The price of the S&P 500 Index, the Dow Jones Index and the correlations estimated by the Asymmetric DCC model over the sample period are shown in Figure G1 in Appendix G. The correlation between the two assets has been close to 1 over the majority of the sample period, which indicates that the potential loss of efficiency between conditional and unconditional estimators could be considerable. Starting in 1998, the correlation dropped to a value slightly below 0.6. However, this brief decrease is no reason to cast doubt on the results. These outliers might even provide useful information due to increased variability in the correlation regimes.

Table 2 shows the estimates for the parameters of the Scalar BEKK, Diagonal BEKK, DCC, Asymmetric DCC, Orthogonal GARCH models as described in Section 3. Due to slight differences in estimation techniques for different models, the parameters differ slightly compared to the estimates that Engle and Colacito (2006) report. A particularly notable difference is in the correlation parameters for the Asymmetric DCC model. While Engle and Colacito (2006) reported values of 0.053, 0.938 and 0.004 for θ_1, θ_2 and θ_3, respectively, values of 0.031, 0.948 and 0.034 are found here.

[2]The web page can be found at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

Table 2: Parameter Estimates for the bivariate setting

	S&P 500 variance parameters				Dow Jones variance parameters				Correlation parameters			
Models	ω_1	α_1	β_1	γ_1	ω_2	α_2	β_2	γ_2	ω_{12}	θ_1	θ_2	θ_3
SBEKK	.073	.222	.973		.072	.222	.973		.029			
	(.015)	(.022)	(.005)		(.015)	(.022)	(.005)		(.006)			
DBEKK		.222	.973			.227	.971					
		(.019)	(.005)			(.022)	(.006)					
OG.	.005	.068	.931		.003	.098	.877					
	(.001)	(.006)	(.006)		(.001)	(.008)	(.011)					
DCC	.005	.067	.931		.009	.084	.912			.042	.954	
	(.003)	(.016)	(.015)		(.004)	(.019)	(.017)			(.008)	(.010)	
ADCC	.012	.002	.923	.138	.016	.018	.913	.119		.031	.948	.034
	(.004)	(.002)	(.016)	(.033)	(.005)	(.009)	(.016)	(.030)		(.031)	(.016)	(.036)

This table shows the parameter estimates and corresponding standard deviations in brackets of the Scalar BEKK (SBEKK), Diagonal BEKK (DBEKK) Orthogonal GARCH (OG.), DCC and Asymmetric DCC (ADCC) models. The sample period is 4/1/2003-4/30/2018. Returns have been multiplied by 100 to improve numerical performance of the estimation routines.

In addition, the standard errors reported in Table 2 are considerably higher than those reported in Engle and Colacito (2006), which can be explained by subtle differences in estimation methods. Finally, note that the covariance stationarity condition $1 - \alpha - \beta > 0,\ \alpha, \beta > 0$ is not satisfied for the Scalar BEKK and Diagonal BEKK model.

Table 3 reports the in-sample volatility ratios of all correlation models. These ratios are obtained by first calculating for each model the sample standard deviation of the portfolio returns for time $t = 1, \ldots, T$ as defined in Equation 4, after which the standard deviation of the model with the lowest standard deviation is normalised to 100 and the other ratios are adjusted accordingly. A value such as 105 can be interpreted as follows: an investor would get a 5% higher required return when using the better covariance model than might have been required had he or she used the worse covariance matrix. Hence, if an investor has a target return of 10% by using the wrong covariance

Table 3: Comparison of volatility ratios

$\boldsymbol{\mu}_{SP}$	$\boldsymbol{\mu}_{Dow}$	Scalar BEKK	Diagonal BEKK	DCC	OGARCH	ADCC	Constant
1.00	0.00	100.473	100.525	101.637	101.303	101.002	100.000
0.99	0.16	100.265	100.327	101.337	100.783	100.506	100.000
0.95	0.31	100.000	100.063	100.883	100.273	100.117	100.479
0.89	0.45	100.000	100.049	100.569	100.548	100.495	102.585
0.81	0.59	100.000	100.053	100.461	102.120	101.029	106.041
0.71	0.71	100.000	100.020	100.312	101.274	100.228	100.085
0.59	0.81	100.000	100.056	100.696	101.262	100.464	104.825
0.45	0.89	100.000	100.022	100.196	102.694	100.533	104.985
0.31	0.95	100.000	100.011	100.484	102.744	101.291	103.770
0.16	0.99	100.000	100.009	100.687	102.616	101.649	102.794
0.00	1.00	100.000	100.008	100.831	102.500	101.791	102.070

This table presents the normalised standard deviations of minimum variance portfolios subject to a required return of 1. Each row of the table shows the results of the pair of expected returns of the corresponding two columns. The lowest standard deviation is normalised to 100.

matrix, this investor could have required a return of 10.5% had he known the better covariance matrix. Contrary to Engle and Colacito (2006), who find that the Diagonal BEKK model is usually the best model, Table 3 shows that the Scalar BEKK model is usually the best model. In addition, the constant estimator is considerably worse than other models except for the cases where the S&P 500 Index has a much larger expected return relative to the Dow Jones Index or when the expected returns are equal. The latter phenomenon was also found by Engle and Colacito (2006), who gave a mathematical foundation for it and called it the 'costless mistake'.

The results from the weighted Diebold-Mariano joint test in Table 4 confirm the findings from Table 3. With the exception of the Scalar BEKK model, the implications of the results are similar to the implications of the results of Engle and Colacito (2006), even though the magnitude of the test statistics does not correspond. The Scalar BEKK model outperforms all other models here, but is only significantly better than the DCC and Constant estimator at a 5% significance level. Furthermore, the constant estimator is significantly worse than all models except the ADCC model. Since the ratio of mean returns in the sample is about 1.25, the univariate weighted Diebold-Mariano test is performed for $\boldsymbol{\mu} = [0.59; 0.81]$ and similar results as in the joint test are obtained.

5.2 Multivariate setting

In the multivariate setting, the Block Dynamic Equicorrelation (BDECO), Random Walk Dynamic Factor GARCH (DFGARCH)[3] and Combination (Comb) models are added to the set of models analysed in the bivariate setting, where the Orthogonal GARCH model is discarded due to its poor performance. The BDECO model is estimated based on the assumption of equal correlation within each of the seven asset classes, such that a 7-Block Dynamic Equicorrelation matrix is obtained. For the Combination model, scaling parameters of $\delta = 0.9$ and $\eta = 2$ are used.[4] Next, $K = 1000$ vectors for $\boldsymbol{\mu}$ are generated where each element in $\boldsymbol{\mu}$ is obtained from a uniform distribution over the unit interval. Like in the bivariate case, the target return μ_0 is set to 1.

Table 5 shows the summary statistics of the volatility ratios as defined in Section 5.1. Surprisingly, the Scalar BEKK model seems to outperform all other models, as it has on average the lowest volatility ratio and the lowest maximum volatility ratio. In addition, the Scalar BEKK model is in 98.1% of all draws of $\boldsymbol{\mu}$ the best model in the sense that the portfolio returns attain the lowest volatility. The Diagonal BEKK, DCC and Asymmetric DCC model follow relatively closely,

[3]Due to the size of the systems, the estimates of the Learning and VAR DFGARCH models are instable and hence not trustworthy. Results for these models are therefore not reported.

[4]A robustness check has been carried out over the sets $\Delta = [1, 0.9, 0.8, 0.7]$ and $\mathcal{H} = [0, 2, 6, 8]$ for δ and η respectively. The results were similar to the findings of Caldeira et al. (2017).

Table 4: Diebold-Mariano Test statistics

	SBEKK	DBEKK	DCC	OGARCH	ADCC	Constant
Joint Test						
SBEKK		0.069	2.210*	1.561	1.668	2.383*
DBEKK	-0.069		2.543*	1.607	2.122*	2.406*
DCC	-2.210	-2.543		0.737	0.788	1.990*
OGARCH	-1.561	-1.607	-0.737		-1.126	3.160*
ADCC	-1.668	-2.122	-0.788	1.126		1.931
Constant	-2.383	-2.406	-1.990	-3.160	-1.931	
$\boldsymbol{\mu}$ = [.59, .81]						
SBEKK		0.986	2.415*	2.741*	1.496	4.252*
DBEKK	-0.986		2.297*	2.643*	1.514	4.356*
DCC	-2.415	-2.297		1.429	0.186	4.191*
OGARCH	-2.741	-2.643	-1.429		-1.268	3.133*
ADCC	-1.496	-1.514	-0.186	1.268		4.300*
Constant	-4.252	-4.356	-4.191	-3.133	-4.300	

The top panel reports the t-statistics for the Diebold-Mariano test for the joint weighted test, where all assumed vectors of expected returns are taken into account. The bottom panel shows the same test for the vector of expected returns $\boldsymbol{\mu} = [0.59, 0.81]$. A positive number means that the model in the corresponding row outperforms the model in the corresponding column. * denotes that the t-statistic is significant at a 5% significance level.

although the required return is on average 5 to 6% higher than for the Scalar BEKK model and are 0.1%, 1.5% and 0.3% of the cases the best model, respectively.

Contrary to expectations, the two more parsimonious models, the Block Dynamic Equicorrelation and the Dynamic Factor GARCH models, perform worse than the conventional multivariate GARCH models. As a matter of fact, the average volatility ratio are a factor 2 and 3.3 larger for the BDECO and DFGARCH models, respectively. Thus, if an investor has a required return of 10% when using the BDECO and DFGARCH models, this target return would be approximately 20.4% and 32.8% when using the best performing model, respectively. This is an early indication that the restrictions or assumptions imposed on the models do not hold in this framework.

Even though Caldeira et al. (2017) find similar perofrmance for the BDECO model, the results for the Combination estimator do not correspond. On average, the volatility of the portfolio created by the Combination estimator is 15% higher than the portfolio created by the best performing model. In addition, the Combination estimator is never the best performing model.

A more detailed insight into model performance can be obtained from considering Table 6, where the entries are the percentage of draws for $\boldsymbol{\mu}$ in which the model in the corresponding row has a lower volatility ratio than the model in the corresponding column. The Scalar BEKK outperforms all models close to 100% of the cases and the BDECO and DFGARCH models perform worse than all other models in all cases, where the DFGARCH model is the worst of these two parsimonious models. Finally, the DCC model and Asymmetric DCC model have a similar, but slightly worse portfolio volatilities than the Diagonal BEKK model, while the DCC model outperforms its Asymmetric counterpart. As expected from the results in Table 5, the Combination estimator performs worse than the traditional multivariate GARCH models in most cases. The Scalar BEKK model outperforms the Comb model in 100% of the cases and this percentage is around 5% for the Diagonal BEKK, DCC and Asymmetric DCC models. These results are confirmed by the pairwise scatter plots of the portfolio volatilities, which can be found in Appendix H.

The inferences from the previous paragraph on Table 6 can be tested formally with the sup $t_{m,n}$ and ave $t_{m,n}$ statistics described in Section 3. The results are shown in Table 7, where the sup $t_{m,n}$ statistics are in the upper triangle and the ave $t_{m,n}$ statistics are in the lower panel. Unlike the t-statistics in the bivariate case, the statistics in Table 7 do not indicate which of the two models is better, but only test the equality in portfolio variance of the two respective models. Since all statistics are significantly different from 0 at a 1% significance level, one can conclude that all models have a significantly different performance over the set of expected return vectors M.

Combining these results with the insights from Table 6, the Scalar BEKK model produces significantly better conditional covariance estimators than all other models considered. One has to be careful about making conclusive statements about the runner-up models, as the results in Table 5 and 6 do not indicate that one of the DBEKK, DCC or ADCC models is clearly better. For example, the DCC model has the lowest volatility ratio more often than the Diagonal BEKK model, but the average volatility ratio is higher and it has a higher portfolio volatility than the Diagonal BEKK model in over half of the cases. However, there seems to be enough evidence that the DCC model is better at capturing correlation dynamics than the ADCC model. Furthermore, there is abundant evidence to conclude that the BDECO and the DFGARCH models are inferior to all other models. Finally, the Combination estimator is significantly better than the BDECO and DFGARCH models, but significantly worse for the traditional multivariate GARCH models.

Even though the two more parsimonious models perform worse than the traditional multivariate

Table 5: Volatility ratio summary statistics

	SBEKK	DBEKK	DCC	ADCC	BDECO	DFGARCH	Comb
Average	100.03	105.11	105.72	106.48	204.47	328.97	115.14
Supremum	104.20	111.64	112.66	114.46	352.04	473.79	141.96
Infimum	100.00	100.00	100.00	100.00	141.48	217.26	100.63
Lowest ratio (% of all $\boldsymbol{\mu}$ draws)	98.1	0.1	1.5	0.3	0.0	0.0	0.0

This table displays summary statistics for the $K = 1000$ volatility ratios as defined in the bivariate case. In particular, the average, supremum and infimum of the volatility ratios and the % of draws that the respective model has the lowest ratio are displayed. The models are the Scalar BEKK (SBEKK), Diagonal BEKK (DBEKK), DCC, Asymmetric DCC (ADCC) Block Dynamic Equicorrelation (BDECO), Dynamic Factor GARCH (DFGARCH) and the Combination models (Comb).

Table 6: Model comparison of volatility ratio

	SBEKK	DBEKK	DCC	ADCC	BDECO	DFGARCH	Comb
SBEKK		99.9	98.2	98.7	100.0	100.0	100.0
DBEKK	0.1		59.4	68.6	100.0	100.0	95.4
DCC	1.8	40.6		93.4	100.0	100.0	96.2
ADCC	1.3	31.4	6.6		100.0	100.0	95.1
BDECO	0.0	0.0	0.0	0.0		100.0	0.0
DFGARCH	0.0	0.0	0.0	0.0	0.0		0.0
Comb	0.0	4.6	3.8	4.9	100.0	100.0	

This table displays the % of draws for $\boldsymbol{\mu} \in M$ that the model in the row has a lower volatility ratio than the model in the column. The models are the Scalar BEKK (SBEKK), Diagonal BEKK (DBEKK), DCC, Asymmetric DCC (ADCC) Block Dynamic Equicorrelation (BDECO), Dynamic Factor GARCH (DFGARCH) and the Combination models.

GARCH models, the more parsimonious versions in the latter class outperform the more flexible counterparts. Table 7 confirms earlier findings that the Scalar BEKK model is significantly better than the Diagonal BEKK model and that the DCC model is significantly better than the Asymmetric DCC model. Hence, increasing model parsimonity can benefit the performance of the models to some extent, but it can come at the cost of flexibility.

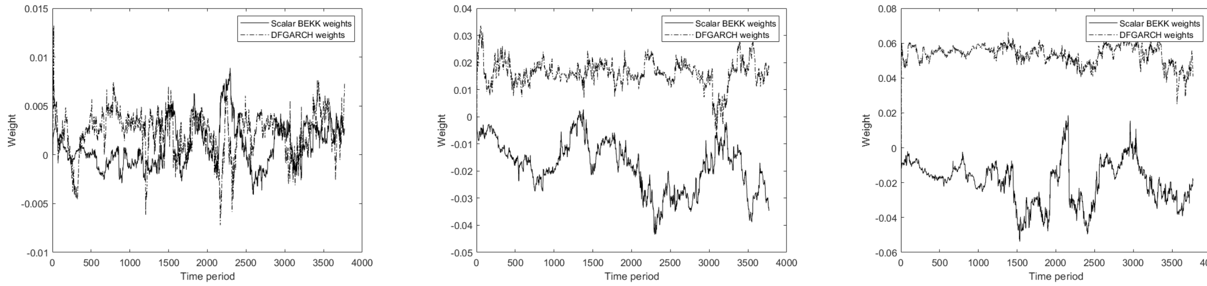
It is interesting further investigate the differences between the DFGARCH, BDECO and Combination models on the one side and the traditional multivariate GARCH models on the other side. Figure 1 displays the average weights, as defined in Equation 4, over the $K = 1000$ vectors of generated expected returns given to three assets from the Equity, Credit and Government Bond asset classes by the DFGARCH and Scalar BEKK models. Graphs for other asset classes can be found in Appendix I. The two models give similar weights to the Hong Kong Index in the Equity asset class, but the weights differ substantially in the specific examples for the Credit and and Government Bond asset classes. Therefore, the DFGARCH model might give more weight to assets that do not perform well, which in turn negatively affects the performance of the DFGARCH model.

Additionally, Figure 2 displays the correlations between the indices in the Alternative Asset

Table 7: Robust test statistics

	SBEKK	DBEKK	DCC	ADCC	DECO	DFGARCH	Comb
SBEKK		55.1*	43.3*	33.9*	760.9*	651.6*	228.1*
DBEKK	10.1*		60.2*	57.4*	707.3*	647.4*	178.8*
DCC	2.9*	8.1*		39.4*	553.7*	567.5*	219.2*
ADCC	3.3*	6.4*	6.4*		559.7*	561.2*	197.9*
DECO	366.2*	324.4*	248.5*	253.4*		609.4*	696.0*
DFGARCH	521.0*	479.1*	368.5*	379.3*	375.3*		669.2*
Comb	64.3*	24.4*	51.4*	45.9*	344.6*	541.1*	

This table displays the sup $t_{m,n}$ (upper triangle) and ave $t_{m,n}$ (lower triangle) statistics for the Scalar BEKK (SBEKK), Diagonal BEKK (DBEKK), DCC, Asymmetric DCC (ADCC) Block Dynamic Equicorrelation (BDECO), Dynamic Factor GARCH (DFGARCH) and the Combination models. * denotes that the two models in the corresponding row and column are significantly different from each other at a 1% significance level.



(a) Equities: Hong Kong Index (b) Credit: U.S. High Yield Index (c) Government Bonds: Germany

Figure 1: Weights given to particular assets by the SBEKK and DFGARCH models.

and Credit asset classes as estimated with the DCC model and the correlation between the two asset classes as estimated by the Block Dynamic Equicorrelation model. Even though the BDECO correlation seems to represent the average pairwise DCC correlation, the dispersion around this average is substantial. Around time period 2150, the BDECO correlation is about 0.18, while the DCC correlations range from 0.9 to almost -0.6. This makes the assumption of equal correlation within each asset class questionable.

Finally, the weights, $\lambda_{t,m}$, given to each model m in the Combination estimator over time are presented in Figure 3. As expected the Combination estimator gives most of the time more weight to the best performing models, such as the SBEKK and DCC models. Interestingly, the weight given to the ADCC model is relatively large in the first third of the sample period compared to the rest of the sample period. The most important result from this figure is that sometimes considerable weights are allocated to the BDECO and DFGARCH models, despite their poor individual performance. This might also explain the poor performance of the Combination estimator.

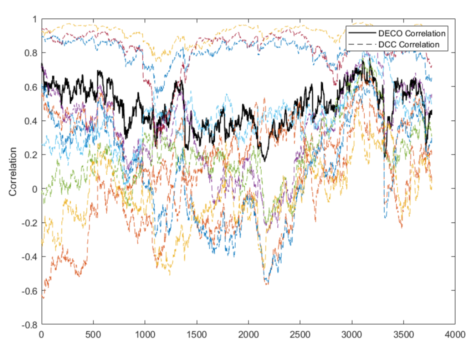


Figure 2: Correlations between Alt. Assets and Credit asset classes for DCC (dashed) and DECO models (solid)

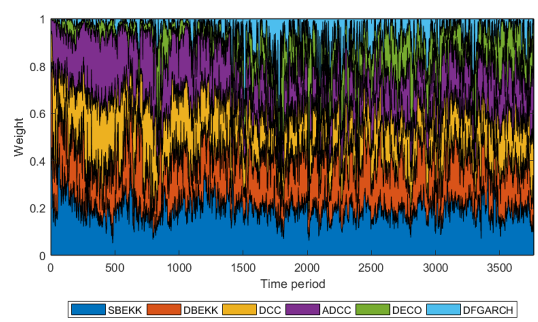


Figure 3: Model weightings in the Combination estimator

6 Conclusion

In this paper, the economic importance of dynamic conditional covariances is investigated. Specifically, to what extent recently developed parsimonious multivariate GARCH models can improve the economic value of conditional correlation in a multi-asset framework. A multi-asset universe is constructed out of 65 assets from seven asset classes, being equities, government bonds, corporate bonds, currencies, commodities, real estate and alternative assets. The summary statistics for the returns over the period 4/1/2003 - 4/30/2018, show that many different mean-variance profiles are represented. In addition, the correlation table shows clearly that the asset classes form blocks in which correlations are similar. To test the economic value of conditional correlations, the bivariate framework of Engle and Colacito (2006) is extended to a multivariate setting. In this framework, Engle and Colacito (2006) use a predetermined set of expected returns for every asset to prevent any noise from estimating the expected returns and test whether the economic loss difference of two models equals zero by using a test statistic based on the work of Diebold and Mariano (2002). Since the number of expected return combinations gets large for increasing number of assets, it becomes unfeasible to reliably test, present and interpret the results of all tests for every set of expected returns. Instead, the testing procedure of Barendse and Patton (2018) is used, which allows for testing over a bounded Euclidean parameter space.

In the bivariate case, price data for the S&P 500 Index and the Dow Jones Index for the period 2/4/1993 - 7/22/2003 is used to estimate five multivariate GARCH models: Scalar BEKK, Diagonal BEKK, DCC, Asymmetric DCC and Orthogonal GARCH. While Engle and Colacito (2006) use a Maximum Likelihood estimator for the DCC-type models, this study uses the Composite Likelihood estimator to retain comparative power between the bivariate and multivariate frameworks. In environments with a small number of assets, the Maximum Likelihood estimator is preferred, but the Composite Likelihood is proven to have better properties in the a multivariate setting. Except for the fact that the Scalar BEKK model outperforms the Diagonal BEKK model in this study, the results are largely in line with the findings of Engle and Colacito (2006). The DCC-type models perform significantly worse compared the BEKK-type models at a 5% significance level, which is likely due to slight differences in estimation techniques. Lastly, the Constant estimator is significantly outperformed by all models, except for the Asymmetric DCC model.

In addition to the five models used in the bivariate case, two parsimonious models, the Block Dynamic Equicorrelation and the Dynamic Factor GARCH models, and a Combination estimator are introduced in the multivariate case. The performance of both parsimonious models, measured by the volatility ratio and statistically tested with the procedure of Barendse and Patton (2018), is significantly worse compared to the traditional multivariate GARCH models. In the case of the Block Dynamic Equicorrelation model, this could be an indication that the assumption of constant correlations within the blocks of asset classes is too restrictive. In the case of the Dynamic Factor GARCH model, it is likely that the factors are not able to capture the dynamics in the assets, but an inaccurate Kalman estimation procedure could also provide an explanation. The model with the best performance is the Scalar BEKK model, like in the bivariate setting. Even though the differences seem small with other traditional correlation models, they are significantly different from zero. In fact, an investor could see the required return decrease by on average 5% and in worst cases by even 10% when switching from the a traditional multivariate GARCH model to the Scalar BEKK model. Due to the use of weights based on the Sharpe ratio, the performance of the Combination estimator is negatively influenced by the performance of the Block Dynamic Equicorrelation and the Dynamic Factor GARCH models.

This research shows that the economic value of correct conditional covariance information is not improved by using more parsimonious models. The implementation of the Block Dynamic Equicorrelation or Dynamic Factor GARCH model can lead to a required return that is 2 or even 3.3 times lower compared to a traditional multivariate GARCH model. Hence, portfolio managers and other practitioners should be cautious when implementing models that aim to increase the model parsimonity in a multi-asset framework.

7 Discussion & Further Research

The fact that this study was bounded by time and computational capabilities gives rise to several limitations that are discussed in detail in this section. Firstly, the statistical tests in both the bivariate and the multivariate case are performed only in-sample. In an in-sample setting, for example when using a Likelihood-Ratio test, more flexible models have an advantage over more parsimonious models, as the number of parameters matters less. Engle and Colacito (2006) devote minimal attention to the out-of-sample evaluation and potential problems that arise from the nestedness of the models. Typically, one would expect that more flexible models have a benefit in-sample and more parsimonious models benefit more in an out-of-sample setting. However, the simpler versions of the traditional multivariate GARCH models outperform their more flexible counterparts in this in-sample study and therefore function as counterexample to the previously stated ‘rule’. As the test of Barendse and Patton (2018) has more power in an out-of-sample setting, further research could use a similar approach to evaluate the out-of-sample performance.

Secondly, other assumptions for the Block Dynamic Equicorrelation and the Combination models can be made. The Block Dynamic Equicorrelation model is estimated based on the assumption that correlations of assets within the same asset class have the same correlation with other asset classes. This choice is defended by the apparent block structure in the sample correlation matrix. However, further research could attempt to analyse other assumptions on the correlation structure, such as constant correlation across assets with similar geographical location within asset classes, to further improve model performance. In addition, other criteria than the Sharpe ratio, such as equal-weighting or the Treynor ratio, could be used to determine the weights given to the individual models in the Combination model.

Thirdly, one could argue that there is a mismatch between the constructed investment universe and the choice of factor model, which might lead to weak estimation results in the Kalman Filter. The investment universe consists of many different asset classes, whereas the factor returns are based on equity portfolios. Since investment managers typically have access to a global universe consisting of various asset classes, it is of most interest to them to study the economic value of conditional covariances in a setting with many different asset classes. Further research could attempt to either study alternative factors or investigate settings where the three-factor model of Fama and French (1993) is appropriate, such as an investment universe consisting of equity industries.

Finally, several concessions had to be made in order to retain feasible computation times and parameter stability. One such measure has been the assumption of homoscedasticity of the errors in the observation equation in the Dynamic Factor GARCH model as this increased computation times rather drastically and without the bounds of possibility for this research. Conventional Maximum Likelihood estimation for the state-space parameters is unfeasible to estimate the Learning and VAR variants for the Random Walk in the state space equation. A variant with an optimised, time-varying variance of $\varepsilon_{i,t}$ would be a possible topic for further research. More importantly, this study used the conventionally employed Maximum Likelihood. Approximations of the conventional Kalman Filter, like the approximation similar to the one of Doz et al. (2011), who introduce a two-step Generalised Least Squares approach, or a multi-step estimator, where first an AR(1) model is estimated with the assumption that ε_t are independent and identically distributed and then a GARCH model is estimated, could enable analysis of other state space equation specifications.

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Appendix A: Correlation models in a bivariate setting

Bivariate Mean-reverting DCC (DCC-MR)

The DCC-MR follows the following process:

$$\boldsymbol{y}_t = \boldsymbol{H}_t^1 \boldsymbol{\xi}_t$$

$$\boldsymbol{H}_t = \begin{bmatrix} h_{1,t} & \rho_t \sqrt{h_{1,t} h_{2,t}} \\ \rho_t \sqrt{h_{1,t} h_{2,t}} & h_{2,t} \end{bmatrix}$$

where the conditional variances estimated by a GARCH-type model are specified as

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1,t-1}, \qquad h_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_2 h_{2,t-1}$$

and $\rho_t = h_{12,t} / \sqrt{h_{1,t}^* h_{2,t}^*}$ is calculated by

$$\begin{aligned} h_{1,t}^* &= (1 - \theta_1 - \theta_2) + \theta_1 \varepsilon_{1,t-1}^2 + \theta_2 h_{1,t-1}^* \\ h_{2,t}^* &= (1 - \theta_1 - \theta_2) + \theta_1 \varepsilon_{2,t-1}^2 + \theta_2 h_{2,t-1}^* \\ h_{12,t} &= \phi_{12} \cdot (1 - \theta_1 - \theta_2) + \theta_1 \varepsilon_{1,t-1} \varepsilon_{2,t-1} + \theta_2 h_{12,t-1} \end{aligned}$$

where ϕ_{12} is the average sample correlation of the returns and $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are the standardised residuals.

Bivariate Asymmetric DCC

The Asymmetric DCC model gives higher tail dependence for the upper tail of the multi-period joint density compared to the symmetric DCC. The Asymmetric DCC follows the following process:

$$\boldsymbol{y}_t = \boldsymbol{H}_t^1 \boldsymbol{\xi}_t$$

$$\boldsymbol{H}_t = \begin{bmatrix} h_{1,t} & \rho_t \sqrt{h_{1,t} h_{2,t}} \\ \rho_t \sqrt{h_{1,t} h_{2,t}} & h_{2,t} \end{bmatrix}$$

There are two important difference to the mean-reverting DCC model. Firstly, the univariate conditional volatilities are estimated with the GJR-GARCH(p, q) model of Glosten et al. (1993), which allows for asymmetric effects of positive and negative returns on the conditional volatility, instead of a regular GARCH(p, q) model. This yields:

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1,t-1} + \gamma_1 d_{1,t-1} y_{1,t-1}^2, \qquad h_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_2 h_{2,t-1} + \gamma_2 d_{2,t-1} y_{2,t-1}^2$$

Secondly, the specification of $\rho_t = h_{12,t}/\sqrt{h^*_{1,t}h^*_{2,t}}$ changes to:

$$h^*_{1,t} = (1 - \theta_1 - \theta_2 - \frac{\theta_3}{2}) + \theta_1\varepsilon^2_{1,t-1} + \theta_2 h^*_{1,t-1} + \theta_3 d_{1,t-1}\varepsilon^2_{1,t-1}$$

$$h^*_{2,t} = (1 - \theta_1 - \theta_2 - \frac{\theta_3}{2}) + \theta_1\varepsilon^2_{2,t-1} + \theta_2 h^*_{2,t-1} + \theta_3 d_{2,t-1}\varepsilon^2_{2,t-1}$$

$$h_{12,t} = \phi_{12} \cdot (1 - \theta_1 - \theta_2) - \phi_3\theta_3 + \theta_1\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \theta_2 h_{12,t-1} + \theta_3(d_{1,t-1}\varepsilon^2_{1,t-1})(d_{2,t-1}\varepsilon^2_{2,t-1})$$

where $d_{1,t}$ and $d_{2,t}$ are dummies that assume value 1 whenever $y_{1,t}$ and / or $y_{2,t}$ are negative, respectively, $\theta_3/2$ relies on the assumption that ε_1 and ε_2 have a symmetric distribution, ϕ_{12} and ϕ_3 are the average correlation of returns and the average of the asymmetric component $(d_{1,t-1}\varepsilon^2_{1,t-1})(d_{2,t-1}\varepsilon^2_{2,t-1})$.

Bivariate scalar BEKK

A scalar BEKK model is specified as:

$$\boldsymbol{y}_t = \boldsymbol{H}^1_t \boldsymbol{\xi}_t$$

$$\boldsymbol{H}_t = \begin{bmatrix} h_{1,t} & h_{12,t} \\ h_{12,t} & h_{2,t} \end{bmatrix}$$

where

$$h_{1,t} = \omega_1 + \alpha y^2_{1,t-1} + \beta h_{1,t-1}, \qquad h_{2,t} = \omega_2 + \alpha y^2_{2,t-1} + \beta h_{2,t-1}, \qquad h_{12,t} = \omega_1 2 + \alpha y^2_{1,t-1} y^2_{2,t-1} + \beta h_{12,t-1}$$

Bivariate scalar BEKK with Variance Targeting

When variance targeting is introduced, the entries of $\boldsymbol{H}_t$ in the bivariate scalar BEKK model should be changed to

$$h_{1,t} = \phi_1(1 - \alpha^2_1 - \beta^2_1) + \alpha^2_1 y^2_{1,t-1} + \beta^2_1 h_{1,t-1}$$

$$h_{2,t} = \phi_2(1 - \alpha^2_2 - \beta^2_2) + \alpha^2_2 y^2_{2,t-1} + \beta^2_2 h_{2,t-1}$$

$$h_{12,t} = \phi_1 2(1 - \alpha_1\alpha_2 - \beta_1\beta_2) + \alpha_1\alpha_2 y^2_{1,t-1} y^2_{2,t-1} + \beta_1\beta_2 h_{12,t-1}$$

where

$$\phi_i = \frac{1}{T}\sum_{t=1}^{T} y^2_{i,t}, \;\; i \in \{1, 2\}, \qquad \phi_{12} = \frac{1}{T}\sum_{t=1}^{T} y_{1,t}y_{2,t}$$

Orthogonal GARCH with ARCH Estimates of Components

The conditional covariance matrix in the Orthogonoal GARCH model is constructed as

$$\boldsymbol{H}_t = \begin{bmatrix} h_{1,t} & \omega_{12} \cdot h_{1,t} \\ \omega_{12} \cdot h_{1,t} & \omega^2_{12} \cdot h_{1,t} + \tilde{h}_{2,t} \end{bmatrix}$$

where $h_{1,t}$ and $\tilde{h}_{2,t}$ are extracted from two univariate GARCH estimations: one for $y_{1,t}$ and one for $y_{2,t}$ with $y_{1,t}$ includes as a regressor in the mean equation:

$$y_{1,t} = \sqrt{h_{1,t}}\varepsilon_t,$$
$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1,t-1}$$

and

$$y_{2,t} = \omega_{12} \cdot y_{1,t} + \sqrt{\tilde{h}_{2,t}}\varepsilon_t,$$
$$\tilde{h}_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_1 \tilde{h}_{2,t-1}$$

Appendix B: Calculation of $\lambda_{m,t}$ in the Combination estimator

First, two performance metrics are calculated, namely the average portfolio return of the portfolio $\hat{\mu}_{m,t}$ and its variance $\hat{\sigma}_{m,t}$ as follows:

$$\hat{\mu}_{m,t} = \frac{1}{t-1}\sum_{k=2}^{t} \delta^{t-k}\pi_k^m, \qquad \hat{\sigma}_{m,t} = \frac{1}{t-1}\sum_{k=2}^{t} \delta^{t-k}(\pi_k^m - \hat{\mu}_{m,t})^2 \tag{38}$$

where π_k^m is the portfolio return at time $k = 1, \ldots, t$ for model $m = 1, \ldots, M$ defined as in Equation 4 and δ is a discount parameter, as introduced by Stock and Watson (2004).

Then, $\lambda_{m,t}$ is calculated as follows, where η is a scaling parameter that gives more weight to portfolios with a higher ratio of $\hat{\mu}_{m,t}$ to $\hat{\sigma}_{m,t}$:

$$\lambda_{m,t} = \frac{(\hat{\mu}_{m,t}/\hat{\sigma}_{m,t})^\eta}{\sum_{m=1}^{M}(\hat{\mu}_{m,t}/\hat{\sigma}_{m,t})^\eta} \tag{39}$$

Appendix C: Specification of state-space matrices for the Kalman Filter

Let $\tilde{\boldsymbol{\beta}}_{i,t}$ and $\tilde{\boldsymbol{f}}_t$ be the transformed factor loadings and factor returns that are used in the Kalman Filter.

For the Random Walk, the transformed matrices are specified as follows, where $\boldsymbol{I}_n$ is the regular $n \times n$ identity matrix:

$$\tilde{\boldsymbol{\beta}}_{i,t} = \boldsymbol{\beta}_{i,t}, \qquad \tilde{\boldsymbol{f}}_t = \boldsymbol{f}_t, \qquad \boldsymbol{F} = \boldsymbol{I}_3, \qquad \boldsymbol{S} = \boldsymbol{I}_3 \tag{40}$$

For the Learning model, the transformed matrices are specified as follows, where $F^i_{1,-} = 1 - F^i_1$:

$$\tilde{\boldsymbol{\beta}}_{i,t} = \begin{bmatrix} B^i_1 \\ B^i_2 \\ B^i_3 \\ \beta^i_{t,1} \\ \beta^i_{t,2} \\ \beta^i_{t,3} \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ F^i_{1,-} & 0 & 0 & F^i_1 & 0 & 0 \\ 0 & F^i_{1,-} & 0 & 0 & F^i_1 & 0 \\ 0 & 0 & F^i_{1,-} & 0 & 0 & F^i_1 \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{f}'_t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f^i_{t,1} \\ f^i_{t,2} \\ f^i_{t,3} \end{bmatrix} \tag{41}$$

For the Learning model, the transformed matrices are specified as follows, where $F^i_{j,-} = 1 - F^i_j$:

$$\tilde{\boldsymbol{\beta}}_{i,t} = \begin{bmatrix} B^i_1 \\ B^i_2 \\ B^i_3 \\ \beta^i_{t,1} \\ \beta^i_{t,2} \\ \beta^i_{t,3} \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ F^i_{1,-} & 0 & 0 & F^i_1 & 0 & 0 \\ 0 & F^i_{2,-} & 0 & 0 & F^i_2 & 0 \\ 0 & 0 & F^i_{3,-} & 0 & 0 & F^i_3 \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{f}'_t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f^i_{t,1} \\ f^i_{t,2} \\ f^i_{t,3} \end{bmatrix} \tag{42}$$

Appendix D: Exact initial Kalman Filter

In this method, $\boldsymbol{\beta}_1$ and $\boldsymbol{P}_1$ are initialised with diffuse priors. This method is based around the decomposition of the initial state vector, $\boldsymbol{\beta}_1$ mean square error matrix $\boldsymbol{P}_{t|t-1}$ and the covariance matrix of the prediction error, Δ_t:

$$\boldsymbol{\beta}_1 = A\boldsymbol{\delta} + \boldsymbol{R}_0\boldsymbol{u}_0, \qquad \boldsymbol{u}_0 \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_u) \tag{43}$$

$$\boldsymbol{P}_{t|t-1} = \kappa\boldsymbol{P}_{\infty,t|t-1} + \boldsymbol{P}_{*,t|t-1} + \mathcal{O}(\kappa^{-1}), \qquad \boldsymbol{\Delta}_{t|t-1} = \kappa\boldsymbol{\Delta}_{\infty,t|t-1} + \boldsymbol{\Delta}_{*,t|t-1} + \mathcal{O}(\kappa^{-1}) \tag{44}$$

where $\boldsymbol{\Sigma}_u$ is treated as known, certain restrictions hold for $\boldsymbol{A}$ and $\boldsymbol{R}_0$, $\boldsymbol{\delta} \sim N(\boldsymbol{0}, \kappa\boldsymbol{I})$, $\boldsymbol{P}_{\infty,t|t-1}$ and $\boldsymbol{P}_{*,t|t-1}$ do not depend on κ and $\kappa \to \infty$. Koopman (1997) show that the influence of $\boldsymbol{P}_{\infty,t|t-1}$ disappears after d iterations and derives the value for d for some state-space models. The value of d for the models described in Equations 16 - 18 can be determined analytically, but doing so is out of the scope of this research. In addition, knowing the exact value for d is not necessary, since Koopman and Durbin (2003) shows that after d iterations, the exact initial state filtering equations given below collapse into the regular updating equations of the Kalman Filter given by Equations (35) - (36):

$$\boldsymbol{\beta}_{i,t+1|t} = \boldsymbol{F}\boldsymbol{\beta}_{t|t} + \boldsymbol{K}_{\infty,t}v_t \tag{45}$$

$$\boldsymbol{P}_{\infty,t+1|t} = \boldsymbol{F}\boldsymbol{P}_{\infty,t|t}\boldsymbol{L}'_{\infty,t}, \qquad \boldsymbol{P}_{*,t+1|t} = \boldsymbol{F}\boldsymbol{P}_{*,t|t}\boldsymbol{L}'_{\infty,t} - \boldsymbol{K}_{\infty,t}\boldsymbol{\Delta}_{\infty,t|t-1}\boldsymbol{K}'_{\infty,t} + \boldsymbol{S}\boldsymbol{\Sigma}_{u_i}\boldsymbol{S}' \tag{46}$$

where

$$\begin{aligned} \boldsymbol{K}_{\infty,t} &= \boldsymbol{F}\boldsymbol{P}_{*,t|t}\boldsymbol{f}'_t\boldsymbol{\Delta}^{-1}_{\infty,t|t-1}, & \boldsymbol{K}_{*,t} &= (\boldsymbol{F}\boldsymbol{P}_{*,t|t}\boldsymbol{f}'_t + \boldsymbol{K}_{\infty,t}\boldsymbol{\Delta}_{*,t})\boldsymbol{\Delta}^{-1}_{\infty,t} \\ v_{i,t} &= y_{i,t} - \boldsymbol{f}'_t\boldsymbol{\beta}_{i,t|t-1}, & \boldsymbol{L}_{\infty,t} &= \boldsymbol{F} - \boldsymbol{K}_{\infty,t}\boldsymbol{f}_t \\ \boldsymbol{\Delta}_{\infty,t} &= \boldsymbol{f}_t\boldsymbol{P}_{\infty,t|t}\boldsymbol{f}'_t, & \boldsymbol{\Delta}_{*,t} &= \boldsymbol{f}_t\boldsymbol{P}_{*,t|t}\boldsymbol{f}_t + h_{i,t} \end{aligned} \tag{47}$$

with initialisations $\boldsymbol{x}_1 = \boldsymbol{\delta}\boldsymbol{A}$, $\boldsymbol{P}_{\infty,1} = \boldsymbol{I}_3$ and $\boldsymbol{P}_{*,1} = \boldsymbol{0}$.

Appendix E: Complete list of tickers used

Alternative Assets	Equities	Currencies	Credit	Commodities	Real Estate	Government Bonds
HFRXGL Index	FTE1JN Index	DXY Index	LF98TRUU Index	BCOMAGTR Index	SPBMGPTU Index	.TRXVAUGOV10D
LPX50TR Index	FTE1IS Index		LP01TREU Index	BCOMENTR Index		.TRXVOEGOV10E
	FTE1HK Index		LFGUSTRUU Index	BCOMGRTR Index		.TRXVBGGOV10E
	FTE1GR Index		LGPETREU Index	BCOMINTR Index		.TRXVCNGOV10D
	FTE1DE Index		EMUSTRUU Index	BCOMLITR Index		.TRXVCZGOV10D
	FTE1CZ Index			BCOMPETR Index		.TRXVDKGOV10D
	FTE1CA Index			BCOMPRTR Index		.TRXVFNGOV10E
	FTE1BR Index			BCOMSOTR Index		.TRXVFRGOV10E
	FTE1IT Index			SPGSCITR Index		.TRXVBDGOV10E
	FTE1IE Index					.TRXVGRGOV10E
	FTE1GE Index					.TRXVIRGOV10E
	FTE1FR Index					.TRXVITGOV10E
	FTE1FI Index					.TRXVJPGOV10D
	FTE1BE Index					.TRXVNLGOV10E
	FTE1AT Index					.TRXVNZGOV10D
	FTE1AU Index					.TRXVNWGOV10D
	FTE1IN Index					.TRXVSAGOV10D
	FTE1ID Index					.TRXVESGOV10E
	FTE1EG Index					.TRXVSDGOV10D
	FTE1CO Index					.TRXVSWGOV10D
	FTE1CH Index					.TRXVUKGOV10D
	FTE1CL Index					.TRXVUSGOV10U
	SPX Index					
	INDU Index					
	SX5E Index					

Figure E1: Overview of the assets that are used in this research. The column for Government Bonds contains DataStream RICs and the other columns contain Bloomberg Tickers.

Appendix F: Sample correlation matrix in multi-variate setting

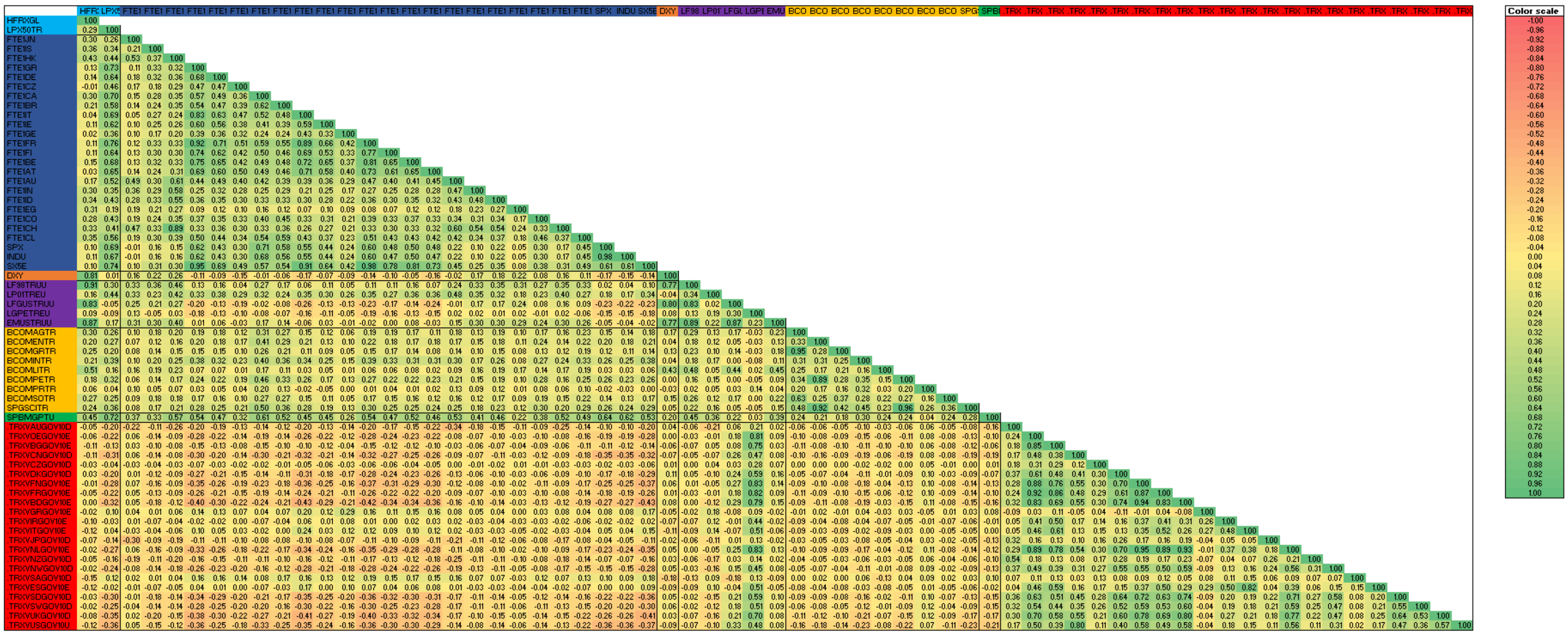


Figure F1: Sample correlations between all indices

Appendix G: Correlations from the ADCC model in the bivariate case

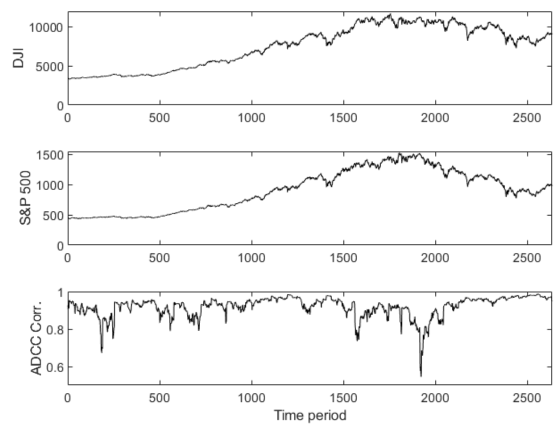


Figure G1: Time-series plot of the S&P 500, Dow Jones Index and the correlation estimated by the ADCC model.

Appendix H: Volatility ratio scatter plots

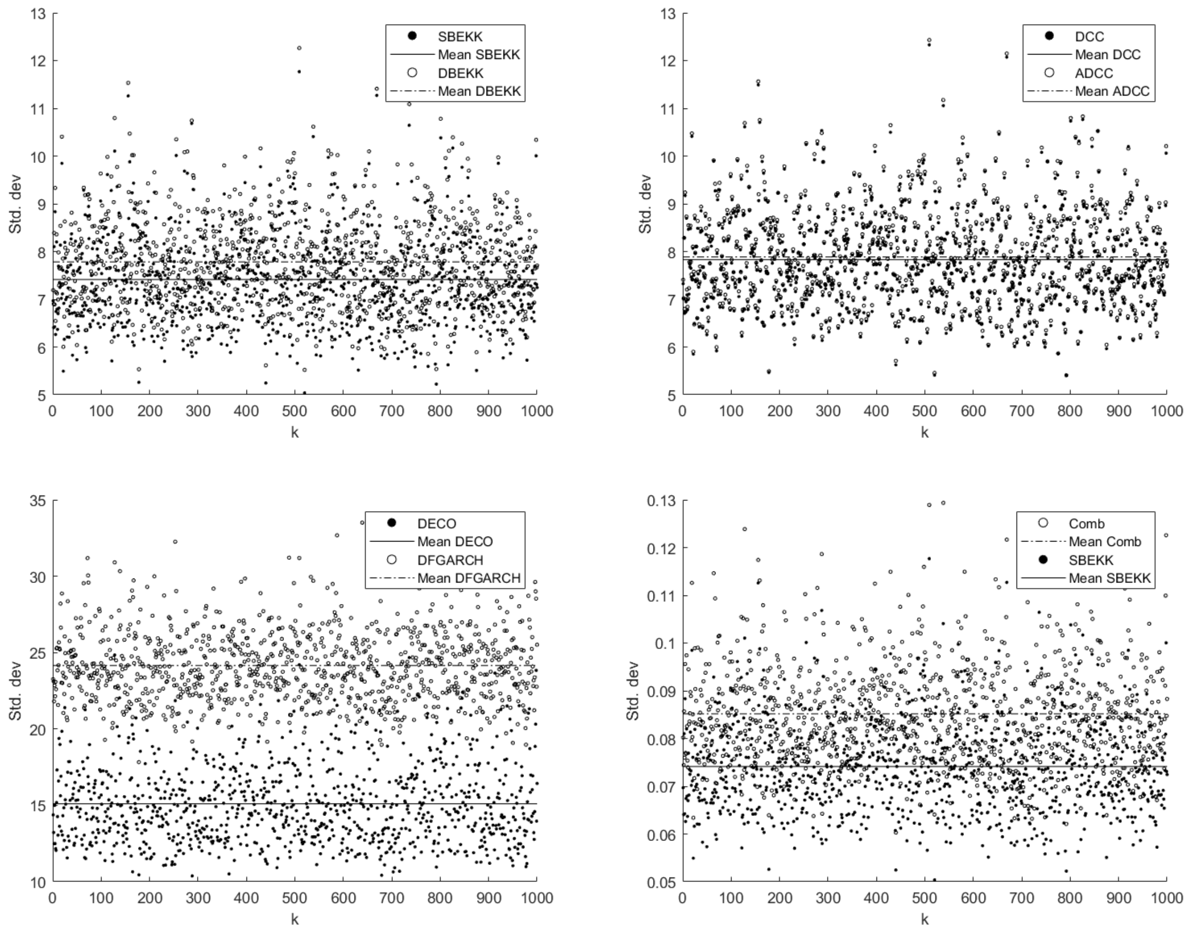
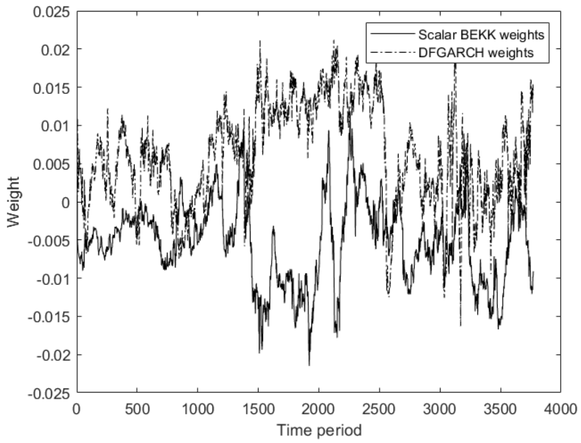
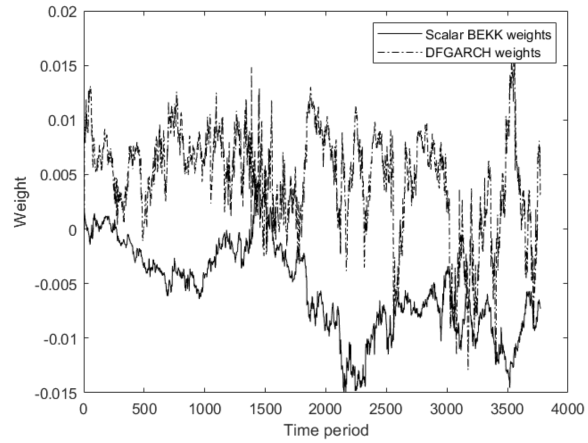


Figure H1: Scatter plots of volatility ratios for all draws, $k = 1, \ldots, 1000$ of $\boldsymbol{\mu}$ for the Scalar BEKK (SBEKK), Diagonal BEKK (DBEKK), DCC, Asymmetric DCC (ADCC), Dynamic Equicorrelation (DECO), Dynamic Factor GARCH (DFGARCH) and Combination (Comb) models.

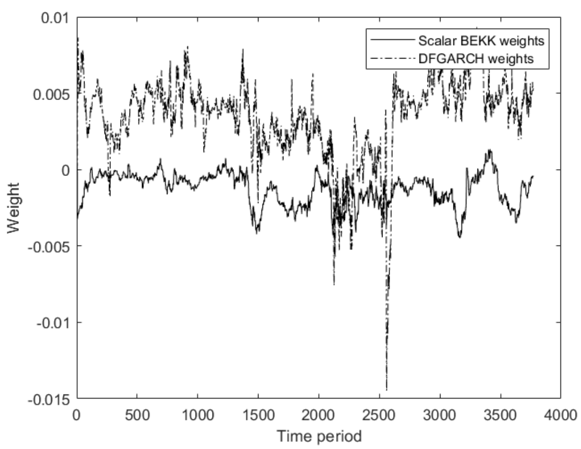
Appendix I: Weights for other asset classes



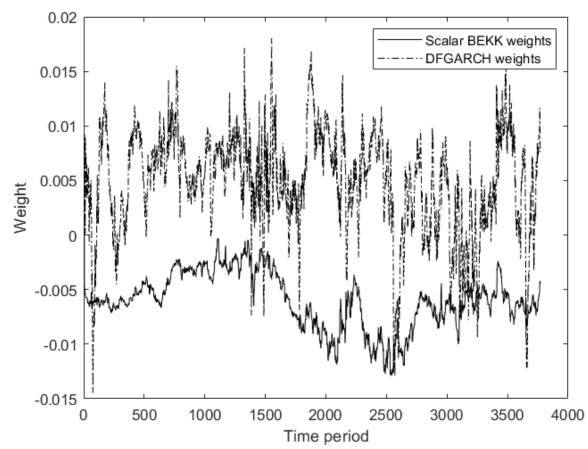
(a) Currencies: U.S. Dollar Index



(b) Real Estate: Global Property Index



(c) Commodities: Precious Metals Index



(d) Alternative Assets: Private Equity Index

Figure I1: These figures present the weights given to particular assets by the SBEKK and DF-GARCH models.