ERASMUS UNIVERSITY

Erasmus School of Economics

Replicating and extending "Testing and Valuing Dynamic Correlations for Asset Allocation"

Bachelor Thesis Econometrics & Operations Research
Based on the work of Engle and Colacito (2006)

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Abstract

A new approach to evaluate different conditional variance information specifications is presented by Engle and Colacito (2006) using an economic loss function. They choose portfolio weights according to the popular minimum variance criterion (see Markowitz (1952)). They test for relative performance of different covariance specifications based on the work of Diebold and Mariano. To extend this research, we look at an alternative weighting criterion which minimizes expected shortfall instead of portfolio variance. We repeat the same tests with the alternative weights and find that for both methods the Diagonal BEKK conditional covariance specification performs best.

1 Introduction

Covariance matrix forecasts are an important tool in financial risk- and portfolio management. Covariance estimates are used to construct portfolio weights to optimally allocate capital over different assets. Today there is a wide variety of time series models for covariance matrices. The most commonly used conditional variance models are GARCH specifications. But explicit comparisons between these models have been hindered by the wide variety of metrics to use in forming the comparisons. Most conditional volatility comparisons just consider the mean squared error between future realized volatility and model forecasts. However, because the realized volatility has a skewed distribution, the mean is only one way to measure the centre. The mean is also only a statistical measure without an economic meaning.

There have been several approaches to develop economic loss functions in the univariate case. West and Cho (1995) used a mean variance utility maximizer choose between a risky and a riskless asset and argued that the best model would be the one with the highest utility for its investor. Engle et al. (1996) let investors use different volatility models to price options and see which strategy ends up with positive profits. By taking long/ short positions on strategies with higher/ lower volatilities, over a long period the best volatility forecast should take money from the inferior ones.

In the classical asset allocation framework, investors aim to choose portfolio weights which minimize variance subject to a required return constraint. Different volatility and return forecasts will lead to different weights which result in different portfolios. Low portfolio returns could be the result of failure of the volatility or the return forecast, so it is important to distinguish between these cases. Because it is impossible to know the true expected returns, we are left with the question of how to compare covariance matrices. Elton and Gruber (1973) examined the problem of comparing the effectiveness of different static covariance matrices in asset allocation. They were the first to use ex-post means in the comparison. Subsequently Cumby et al. (1994) and Fleming et al. (2001) followed this direction. Because expected returns are not the same as realized mean returns, this method does not avoid the problem.

This paper is a natural extension to univariate forecast evaluation. We look at an asset allocation perspective to measure the value of covariance information introduced by Engle and Colacito (2006). They have shown that correctly specified covariance matrices lead to minimization of realized volatility, for any vector of expected returns. The increase in required return that could be achieved for a fixed volatility level will be used to measure the value of correct covariance information. This is an economic value expressed in normalized volatility ratios used to compare different covariace model specifications. Also, tests based on the work of Diebold and Mariano (2002) are used for examining the relative performance of covariance matrices. In these tests, the difference in squared excess returns is used to test the equality of two models. A case study using time series data of two financial assets is performed to estimate the models with and apply the proposed tests to.

We find that the Diagonal BEKK model specification scores best at both the Diebold and Mariano Tests and the volatility ratios. The difference in volatility ratio between the Diagonal BEKK specification and an unconditional constant volatility model shows that for these assets a 4.7 higher required return could have been achieved using this model specification. According to the DM tests, the Diagonal BEKK model outperforms almost all models significantly.

The mean variance approach for asset allocation is only one way to construct portfolio weights. Another increasingly popular optimization criterion for weights is Value at Risk and Conditional Value at Risk (Powell and Allen (2009)). CVaR, also called Mean Excess Loss, Mean Shortfall, or Tail VaR, is considered to be a more consistent measure of risk than VaR (Rockafellar et al. (2000)). As a matter of fact, optimizing weights for CVaR has attractive mathematical features (Rockafellar et al. (2000)), in terms of convexity. Rockafellar et al. (2000) show how the optimization for CVaR can be formulated as a linear optimization problem. To extend the research of Engle and Colacito (2006) we implement this formulation to construct an alternative set of portfolio weights for all models. The aim of this extension is to examine the value of correct covariance information in a CVaR optimization context. Using the new portfolio weights we gain new in sample portfolio returns. These returns are then used to perform the same comparison tests. We then compare and interpret the new test results with the results of the original asset allocation framework.

We see that for the alternative weights, again the Diagonal BEKK specification outperforms the other models when looking at the volatility ratios. The Diagonal BEKK also performs better than the other models according to the Diebold and Mariano test. However, these differeces do not seem to be significant.

This paper is organized as follows. Section 2 contains the techniques and models we use to do our research with. We analyze two aspects of multivariate systems. First, we provide a utility and expected return-free framework to assess the importance of volatility and correlation timing. Engle and Colacito (2006) have done this by developing a metric which gives an economic value to correct covariance information. Secondly, they offer a comparison between the relative performance of alternative methods of dynamic covariance modeling. Section 3 describes the data sets that are used for empirical analysis. In section 4, the results of our analysis are displayed and interpreted. Section 5 extends our research by looking at an alternative weighting criterion and compares the results acquired by these new weights with the results of the weights which resulted from the original asset allocation problem. Finally we conclude our research in section 6.

2 Methodology

2.1 Asset Allocation Problem

For the modelling of the different conditional variance models, we define the asset returns $r_{i,t}$ as described in (1). Here $r_{i,t}$ denotes the log return of asset i on time t. We use settlement price data to compute the log returns of assets in the following way:

$$r_{i,t} = ln(P_{i,t}/P_{i,t-1}) \tag{1}$$

We formulate the variance minimization problem subject to a required return in the following way:

$$\begin{array}{ll} \min_{\mathbf{w}_t} & \mathbf{w}_t'\mathbf{H}_t\mathbf{w}_t \\ \text{subject to} & \mathbf{w}_t'\boldsymbol{\mu} = \mu_0 \end{array} \tag{2}$$

Let n denote the number of different assets to allocate our capital to, excluding the risk free asset. Here $\mathbf{w}_t$ is the $n \times 1$ vector of portfolio weights for time t chosen at time $t-1$, $\mathbf{H}_t$ is the $n \times n$ conditional covariance matrix of a vector of excess returns for time t. Here, $\boldsymbol{\mu} = [\mu_{SP}; \mu_{DOW}]$ is the assumed vector of excess returns with respect to the risk-free asset, and $\mu_0 > 0$ is the required return. The solution to the optimization problem is

$$\mathbf{w}_t = \frac{\mathbf{H}_t^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{H}_t^{-1}\boldsymbol{\mu}}\mu_0 \tag{3}$$

The derivation of the expression in (3) can be found in the appendix section 8.1. The elements of the weight vector $\mathbf{w}_t = (w_{1,t}, \ldots, w_{n,t})'$, where weight $w_{i,t}$ is the share on asset i at time t, generally do not add up to 1. This means that we denote $1 - \sum_{i=1}^{n} w_{i,t}$ as the share in a risk-free asset.

2.2 Estimators

This paper focuses on four conditional covariance models. We look at the Orthogonal GARCH, Diagonal BEKK, DCC and Asymmetric DCC volatility models. Orthogonal generalized autoregressive conditional heteroscedasticity (GARCH) (Alexander (2000)) models are a simple approach for estimating multivariate models. This procedure relies on the construction of unconditionally uncorrelated linear combinations of the series of returns. First, a univariate GARCH model is estimated for the first asset (SP500). Then a second univariate GARCH model is estimated for the second asset (Dow), but with the returns of the first asset (SP500) taken as a regressor in the mean specification.

Then we can define the conditional covariance estimates $\mathbf{H}_\mathrm{t}$ as follows.

$$\mathbf{H_t} = \begin{bmatrix} h_{1,t} & \omega_1 h_{1,t} \\ \omega_1 h_{1,t} & \omega_{12}^2 h_{1,t} + \tilde{h}_{2,t}. \end{bmatrix} \tag{4}$$

Here $h_{1,t}$ and $\tilde{h}_{2,t}$ are the conditional covariance series of the two GARCH models, ω_1 is the intercept of the first variance specification and ω_{12} is the coefficient of the SP500 return regressor in the mean specification of the second GARCH model.

We also estimate the BEKK representation as discussed by Engle and Kroner (1995) and Engle (2002). This model can provide the constraint that the long-run covariance matrix is the sample covariance matrix. The first order case of this model can be written as described in (5). In this paper we consider a special case where $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices. We refer to this model as the Diagonal Bekk model.

$$\mathbf{H}_t = \mathbf{\Omega} + \mathbf{A}(\mathbf{r}_{t-1}\mathbf{r}'_{t-1})\mathbf{A}' + \mathbf{B}\mathbf{H}_{t-1}\mathbf{B}'. \tag{5}$$

Furthermore we look at the DCC model. This is a multivariate GARCH model that is particularly convenient for big systems. This method first estimates volatility's and standard deviations for each asset. Then it estimates covariances between these using a ML criterion and one of several models for the correlations.

$$\mathbf{y_t} = \mathbf{H_t^{1/2}} \boldsymbol{\xi_t} \tag{6}$$

$$\mathbf{H_t} = \begin{bmatrix} h_{1,t} & \rho_t \sqrt{h_{1,t} h_{2,t}} \\ \rho_t \sqrt{h_{1,t} h_{2,t}} & h_{2,t} \end{bmatrix}, \tag{7}$$

where

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1.t-1}, \tag{8}$$

and

$$h_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_2 h_{2.t-1}, \tag{9}$$

Besides the standard DCC model with mean reversion (DCC-MR) this paper also uses asymmetric DCC (Asy-DCC) introduced by Cappiello et al. (2006). Asy-DCC contains an additional term that allows correlation to increase more when both returns are falling than when they are both rising. Here the conditional variance terms contain additional dummie terms.

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1.t-1} + \gamma_1 d_{1,t-1} y_{1,t-1}^2, \tag{10}$$

and

$$h_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_1 h_{2.t-1} \gamma_2 d_{2,t-1} y_{2,t-1}^2, \tag{11}$$

Lastly, we look at a constant volatility model. Here we model $\mathbf{H}_\mathrm{t}$ as the sample covariance matrix of the return data.

$$\mathbf{H}_\mathrm{t} = \Sigma_{\mathrm{SP,DOW}} = \mathrm{Cov}(r_{SP}, r_{DOW}) \tag{12}$$

Note that the two expressions on the right side of (12) have no subscript t and therefore it is an unconditional variance model.

2.3 Tests

To test the equality of two models we propose a test based on the work of Diebold and Mariano (2002). For two different time series of covariance matrices $\{\mathbf{H}_t^j\}_{j=1}^2$, a set $\{\boldsymbol{\mu}^k\}_{k=1}^K$ of hypothesized vectors of expected returns divided by the required excess return μ_0 and portfolio weights $\mathbf{w}_t^{j,k}$ we can denote portfolio return by

$$\pi_t^{j,k} = (\mathbf{w}_t^{j,k})'(\mathbf{r}_t - \bar{\mathbf{r}}), \tag{13}$$

We can now construct the difference in squared return from the first and second portfolio as follows

$$u_t^k = (\pi_t^{1,k})^2 - (\pi_t^{2,k})^2, \qquad t = 1\ ,\dots,\ T. \tag{14}$$

Dividing u by its standard deviation improves the efficiency of the mean estimation. This results in

$$v_t^k = u_t^k[2(\boldsymbol{\mu}^{k\prime}(\mathbf{H}_t^1)^{-1}\boldsymbol{\mu}^k)(\boldsymbol{\mu}^{k\prime}(\mathbf{H}_t^2)^{-1}\boldsymbol{\mu}^k)]^{1/2}. \tag{15}$$

We can test whether covariance methods 1 and 2 are equal by doing a joint test for all k. Define

$$\mathbf{U}_t = (u_t^1, \dots, u_t^k)' \tag{16}$$

and

$$\mathbf{V}_t = (v_t^1, \dots, v_t^k)'; \tag{17}$$

then using GMM with a Newey-West vector Heteroscedasticity and Autocorrelation Consistent (HAC) covariance matrix to estimate

$$\mathbf{U}_t = \beta_u \boldsymbol{\iota} + \epsilon_{u,t} \tag{18}$$

and

$$\mathbf{V}_t = \beta_v \boldsymbol{\iota} + \epsilon_{v,t} \tag{19}$$

The null of this test is $H_0 : \beta_u$ and $\beta_v = 0$. This null hypothesis states that there is no difference between the models. The alternative hypothesis states that the model in the row is better than the one in the column.

We do not analyze the results of all vectors of $\boldsymbol{\mu}$, instead we look at the expected return vector that comes closest to the true unconditional averages of stocks and bonds in the sample that we consider. To choose the expected return vector we look at the observed ratio of mean returns, which is $\frac{9.43}{7.53} = 1.2520$ (using our sample statistics in table 1). The $[\boldsymbol{\mu}_{SP}; \boldsymbol{\mu}_{DOW}]$ vector that comes closest to this ratio is [0.59;0.81]. The bottom panel reports the results of the joint Diebold-Mariano test. In this test all of the assumed vectors of expected returns are taken into account. Just like with the univariate test; a positive number means that the row is better than the column.

2.4 Volatility ratios

After we have estimated the coefficients of the models we can estimate covariance matrices and use them to construct optimal weights according using (3). Having estimated these weights, we can calculate the portfolio returns according to (13). Engle and Colacito (2006) show that for a portfolio constructed to optimize (2) with standard deviation σ_t^*, and for an arbitrary portfolio with standard deviation σ_t, the following inequality holds:

$$\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(\sigma_t^*)^2] \leq \mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(\sigma_t)^2] \tag{20}$$

Therefore, (20) offers a strategy to compare the covariance matrices $\mathbf{H}_t$ based on the idea of choosing covariance estimates that minimize portfolio variance. By computing the standard deviations of these $\pi_t^{j,k}$'s we can therefore compare the resulting volatility's of the different models. The lowest standard deviation will be normalized to 100.
Let J denote the number of different models we consider in our analysis. Then the volatility ratio's $\mathrm{VR}_{i,k}$ can be expressed as in (21).

$$\mathrm{VR}_{i,k} = \frac{\sigma(\pi^{i,k})}{\min[\sigma(\pi^{1,k}), \ldots, \sigma(\pi^{J,k})]} \tag{21}$$

Note that the $\pi^{i,k}$ are the portfolio return vectors as defined in (13). Here $\mathrm{VR}_{i,k}$ denotes the volatility ratio of model i and for the k'th expected return vector.

3 Data

To estimate the proposed models described in section 2 we use price data from Yahoo! finance for SP500 and Dow Jones Industrials indices. The samples of both time series range from 2/4/1993 to 7/22/2003. Table 1 reports some summary statistics of the log returns.

Table 1: Sample Statistics (SP500 and Dow)

	SP500	Dow Jones
Mean	7.53	9.43
Variance	1.25	1.20
Kurtosis	3.47	4.31
Skewness	-0.11	-0.26
Correlation	0.939	

Note that the mean returns have been annualized

The sample mean and standard deviation seem similar, and the average correlation is high. Figure 1 shows the graphs of the time series of these assets. We standardize both time series with respect to their starting values at time 2/4/1993 to make them visually more comparable. The steep drop in 1998 was caused by frantic selling which pounded Wall Street Monday, which sent the Dow industrials 512 points lower. We can link the drop in prices between 2001 and 2002 to the terrorist attack on the World Trade Centre on 9/11. We can also see the stock market crash in 2002.

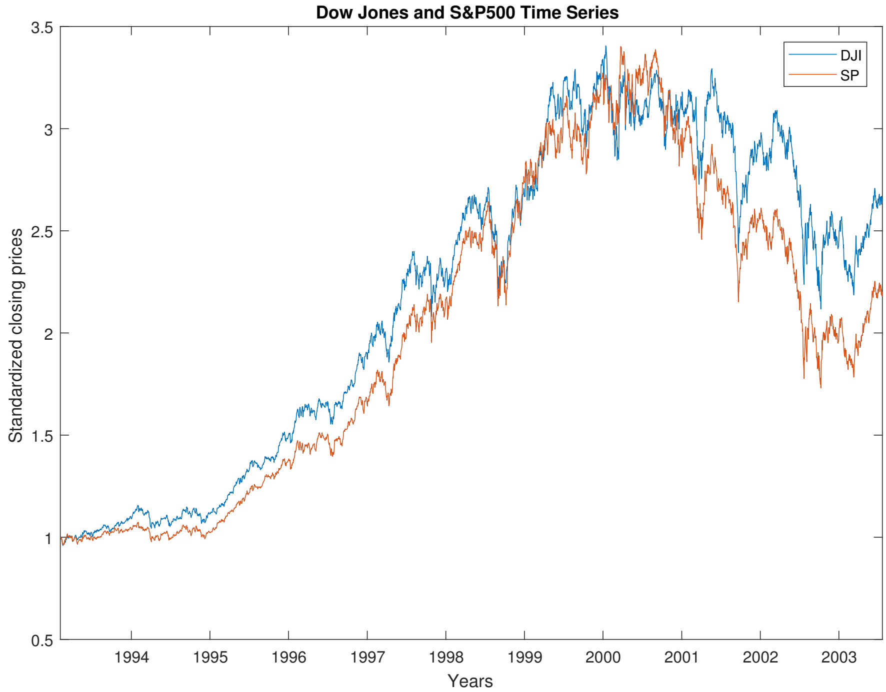

Figure 1: SP500 Futures and Dow Jones Index

Figure 2 displays the dynamic correlations estimated using the Diagonal BEKK model. In figure 2 we can again see the high correlation between the two time series. Engle and Colacito (2006) show that highly correlated assets are interesting in this analysis because they can demonstrate bigger differences in efficiency between conditional and unconditional estimators.

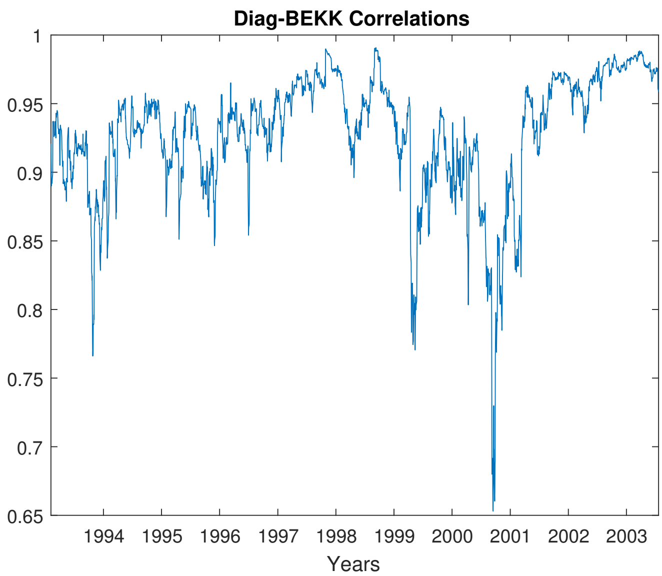

Figure 2: Dynamic correlations computed with the Diagonal BEKK model

4 Results

Table 2 displays the estimated parameters of the different models. We can see that the variance parameters of the Orthogonal GARCH and DCC-MR are very close to being equal.

Table 2: Parameter Estimates (SP500 and Dow)

	S&P500 variance parameters				Dow Jones variance parameters				Correlation parameters			
Models	ω_1	α_1	β_1	γ_1	ω_2	α_2	β_2	γ_2	ω_{12}	θ_1	θ_2	θ_3
Diagonal BEKK		0.222	0.972			0.227	0.971					
		(0.020)	(0.005)			(0.023)	(0.006)					
Orthogonal GARCH	0.005	0.068	0.931		0.003	0.092	0.887		0.952			.
	(0.001)	(0.006)	(0.006)		(0.001)	(0.007)	(0.010)		(0.006)			
DCC-MR	0.005	0.067	0.931		0.009	0.084	0.912			0.042	0.954	
	(0.003)	(0.016)	(0.015)		(0.004)	(0.019)	(0.017)			(0.008)	(0.011)	
Asy-DCC	0.012	0.002	0.923	0.138	0.016	0.018	0.913	0.119				
	(0.004)	(0.005)	(0.018)	(0.032)	(0.005)	(0.009)	(0.017)	(0.030)		(0.030)	(0.017)	(0.035)

The numbers in parentheses are standard errors.

After estimating the parameters of the models, we made forecasts $\mathbf{H}_t$ of conditional covariance matrices. We use these volatility forecasts to construct minimum variance portfolios and calculate the portfolio returns. Taking the standard deviations of these return vectors and normalizing for the smallest standard deviation results in the volatility ratios in table 3.

Table 3: Comparison of Volatilities (SP500 and Dow)

μ_{SP}	μ_{DOW}	Diagonal BEKK	DCC-MR	Orthogonal GARCH	Asy-DCC	Constant
1.00	0	100.540	101.644	100.944	101.015	100.000
0.99	0.16	100.356	101.364	100.358	100.537	100.000
0.95	0.31	100.319	101.143	100.000	100.375	100.684
0.89	0.45	100.049	100.567	100.000	100.492	102.516
0.81	0.59	100.000	100.405	101.757	100.972	105.937
0.71	0.71	100.000	100.294	101.189	100.195	100.046
0.59	0.81	100.000	100.633	100.829	100.406	104.770
0.45	0.89	100.000	100.168	101.957	100.504	104.969
0.31	0.95	100.000	100.467	102.354	101.273	103.774
0.16	0.99	100.000	100.672	102.515	101.635	102.806
0	1	100.000	100.815	102.522	101.779	102.087

The volatility ratios in table 3 do not vary much between the different conditional variance estimators. A number like 102.516 for the constant model and $\boldsymbol{\mu} = [.89, .45]'$ can be interpreted as a 2.516% higher required return than might have been required had we known the true covariance matrix. The Diagonal BEKK specification usually performs best. We can see that the constant model produces the highest volatility ratios, indicating that higher returns could be realized using conditional variance models. The Diagonal BEKK also seems to perform best for the expected returns that are most likely to be the true ones.

Table 4: Diebold and Mariano Test (SP500 and Dow)

	Diagonal BEKK	DCC-MR	Orthogonal GARCH	Asy-DCC	Constant
$\boldsymbol{\mu} = [.59, .81]$					
Diagonal BEKK		2.291	2.481	1.506	4.302
		(-1.599)	(2.784)	(1.207)	(3.821)
DCC-MR	-2.291		0.792	0.192	4.133
	(-1.599)		(0.336)	(-0.465)	(3.250)
Orthogonal GARCH	-2.481	-0.792		-0.784	3.417
	(-2.784)	(-0.336)		(-0.919)	(3.132)
Asy-DCC	-1.506	-0.192	0.784		4.238
	(-1.207)	(0.465)	(0.919)		(3.866)
Constant	-4.302	-4.133	-3.417	-4.238	
	(-3.821)	(-3.250)	(-3.132)	(-3.866)	
Joint test					
Diagonal BEKK		3.038	2.341	2.081	2.376
DCC-MR	-3.038		0.199	0.759	1.962
Orthogonal GARCH	-2.341	-0.199		-0.936	4.677
Asy-DCC	-2.081	-0.759	0.936		1.897
Constant	-2.376	-1.962	-4.677	-1.897	

The top panel of table 4 displays the t-values of the univariate Diebold-Mariano test. Here we use the expected return vector $\mu = [0.59; 0.81]$. The numbers in parentheses refer to the unweighted version of the test. The bottom panel reports the results of the joint Diebold-Mariano test. Positive numbers indicate that the model in the row outperforms the model in the column.

We can see from the t-values of the joint tests in table 4 that the Diagonal BEKK specification outperforms all other models at a 5% significance level. The negative signs and magnitude of the t-statistics of the Diagonal BEKK column provide evidence for the model's good performance. These results confirm the findings of the volatility ratios found in table 3. The two different DCC specifications do not seem to be significantly different. The large negative t-values in the rows of the constant model support the findings of the volatility ratio performance of the constant model in table 3 as well.

5 Extension

Besides variance minimization, another popular optimization criterion is expected shortfall. Expected shortfall is a risk measure which is defined as the average of all losses which are greater or equal than VaR, in other words the average loss in the worst $(1 - \beta)\%$ cases, where β is the confidence level. Expected shortfall is also called Conditional Value at Risk (CVaR).

$$ES_t^{\beta} = \mathbb{E}[r_t | r_t < VaR_t^{\beta}] \tag{22}$$

To extend Engle and Colacito (2006) we look at how the results from minimizing expected shortfall compare to the results obtained by minimizing variance. To optimize Conditional Value at Risk we use a method proposed by Rockafellar et al. (2000). They show that expected shortfall minimization has attractive mathematical features compared to for example minimizing Value at Risk. Optimizing for Conditional Value at Risk is a convex problem, which is useful for optimization (Boyd and Vandenberghe (2004)).

The optimal weights are approximated using Monte Carlo simulated random return series. For each time t we simulate Q random return vectors $\mathbf{r}^*_{t,q}$. Here $\mathbf{r}^*_{t,q}$ is q'th simulated random return vector for time t.

$$\mathbf{r}^*_{t,q} = (r^*_{1,t,q}, \dots, r^*_{n,t,q}) \qquad t = 1, \dots, T \text{ and } q = 1, \dots, Q \tag{23}$$

For the generation of random elements in $\mathbf{r}^*_{t,q}$ we use Monte Carlo simulation. Let $h_{i,t}$ denote the conditional variance of asset $i \in \{1, 2\}$ at time t. Then by generating random normal shocks $\epsilon^*_{i,t,q}$ we can generate the q'th random return at time t of asset i as follows:

$$r^*_{i,t,q} = \sqrt{h_{i,t}}\epsilon^*_{i,t,q} \tag{24}$$

After we have simulated Q random return vectors for time t, we use them in the linear optimization problem presented by Rockafellar et al. (2000) in (25) to gain the optimal weights for time t. We solve (25) for each time t to find the optimal weights for time t.

$$\begin{array}{lll} \text{minimize}_{(\alpha_t, \mathbf{w}_t)} & \alpha_t + \frac{1}{Q(1-\beta)} \sum_{q=1}^{Q} u_{t,q} & \\ \text{subject to} & u_{t,q} \geq 0 & q = 1, \dots, Q \\ & \mathbf{w}_t^T \mathbf{r}^*_{t,q} + \alpha + u_{t,q} \geq 0 & q = 1, \dots, Q \\ & w_{i,t} \geq 0 & i = 1, \dots, n \\ & \sum_{i=1}^{n} w_{i,t} = 1 & \\ & \mathbf{w}_t^T \mathbf{y}_k \geq \mu_0 & \end{array} \tag{25}$$

Here the decision vector $\mathbf{w}_t$ represents a portfolio of financial instruments in the sense that $\mathbf{w}_t = (w_{1,t}, \dots, w_{n,t})'$, with $w_{i,t}$ being the position in instrument i at time t. Here α_t is the unconditional Value at Risk of the portfolio, and β is the confidence level for which the Conditional Value at Risk will be determined. $u_{t,q}$ is constructed as an auxiliary variable. Q is the number of simulated return vectors used for the simulation at time t. The computation of the optimal weights at time t is time consuming, and our data set consists of more than 2000 time observations. Besides that, we estimate all weights for all times t for 5 different models. That's why, for computational purposes, I set Q equal to 50. Furthermore we denote μ_0 as the required return in the same way as described earlier in this paper. The third restriction avoids short positions in all assets. The fourth restriction makes sure all capital is invested. This is a small difference compared to the original asset allocation problem, where the excess capital was invested in a riskless asset. The last restriction makes sure that the resulting portfolio return meets the required return μ_0. The optimization problem described above is a linear optimization problem which can be solved for $(\alpha_t, \mathbf{w}_t)$. After we have constructed the new weights we construct volatility ratios to compare the relative performance of the CVaR method with the models of the paper. This is done the same way as explained in section 2.4.

Table 5: Comparison of Volatilities (SP500 and Dow)

μ_0	Diagonal BEKK	DCC-MR	Orthogonal GARCH	Asy-DCC	Constant
1	100.000	100.164	100.191	100.173	100.171

If we look at the results of the new volatility ratio's constructed by weights which minimize expected shortfall, we see again that the Diagonal BEKK specification scores best. The Orthogonal GARCH model performs worst, and is shortly followed by the Asymmetric DCC model. The new weights are not optimized for volatility minimization, so perhaps it would be more interesting to construct new ratios which examine realized expected shortfall instead of return volatility's. If we define ES^*_t as the resulting expected shortfall of the optimal solution of 25, and ES_t as the resulting expected shortfall of an arbitrary portfolio, then the following inequality holds:

$$\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T} ES^*_t] \leq \mathbb{E}[\frac{1}{T}\sum_{t=1}^{T} ES_t] \tag{26}$$

Therefore, 26 offers a strategy to compare the covariance matrices $\mathbf{H}_t$ based on their resulting expected shortfall values. If we take the mean of the realized expected shortfalls at all times t and again normalize the lowest expected shortfall to 100, we get the results displayed in table 6.

Table 6: Comparison of Expected shortfall (SP500 and Dow)

Diagonal BEKK	DCC-MR	Orthogonal GARCH	Asy-DCC	Constant
100.343	100.983	102.971	100.000	107.056

At this relative performance comparison an interesting finding is that the Asymmetric DCC model outperforms the Diagonal BEKK specification. This suggests that for the purpose of minimizing shortfall, the Asy-DCC model would be most suitable. Again, we see that the constant volatility model performs worst.

We run the same univariate Diebold & Mariano test again and see in table 7 that the Diagonal BEKK outperforms the other models, but not at a significant level. As a matter of fact, all models do not significantly outperform or are not outperformed by any other model.

Table 7: Diebold and Mariano Test(SP500 and Dow) for CVaR weights

	Diagonal BEKK	DCC-MR	Orthogonal GARCH	Asy-DCC	Constant
Diagonal BEKK		0.628	0.492	0.652	0.581
DCC-MR	-0.628		0.167	0.390	-0.327
Orthogonal GARCH	-0.492	-0.167		0.153	-0.426
Asy-DCC	-0.652	-0.390	-0.153		-0.446
Constant	-0.581	0.327	0.426	0.446	

6 Conclusion

In this paper we showed an innovative approach to evaluate correct covariance information in a multivariate framework. Portfolio weights were constructed using a variance minimization criterion. We looked at different model specifications to estimate $\mathbf{H}_\mathrm{t}$ series of condition covariance matrices. We evaluated the relative performance of the models by constructing volatility ratios of the portfolio returns. These ratio's can be interpreted as the additional return an informed portfolio manager could have achieved using correct covariance information. The differences in these ratio's are not big; they differ on average just a few basis points in annualized terms. However, we do see that the models using time-varying information perform better than the models using constant estimators. The Diagonal BEKK scored best for most of the expected return combinations. This suggests that the inclusion of the diagonal structure conditions for the $\mathbf{A}$ and $\mathbf{B}$ in (5) are valuable.

Furthermore Diebold-Mariano tests were performed to test the equality of the different model specifications. We performed the univariate test using the expected returns vector which comes closest to the true return vector. The results for these tests showed that Diagonal BEKK performs better than the other estimators, but the difference is not always significant. At the joint test, which included all expected return specifications, the Diagonal BEKK model significantly outperformed all other model specifications.

To extend Engle and Colacito (2006) we examined how choosing a different weighting criterion would affect the ratio and test results. We alternatively computed our portfolio weights according to an expected shortfall minimization criterion. Again we observed the good performance of the Diagonal BEKK model.

Because variance minimization and expected shortfall minimization are such popular weighing criteria, knowing that the Diagonal BEKK performs so well is valuable. On the other hand, in this paper we limit ourselves to a combination of highly correlated assets. Engle and Colacito (2006) showed that for less correlated assets the Asymmetric DCC model showed the best results. So we must be careful not to draw generalized conclusions.

Also, in the expected shortfall extension we restrict ourselves to a normal distribution in the return shocks. It would be interesting to examine different shock densities like the skewed t distribution.

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8 Appendix

8.1 Derivation solution asset allocation problem

$$\min_{\mathbf{w}_t} \mathbf{w}_t' \mathbf{H}_t \mathbf{w}_t \quad \text{s.t. } \mathbf{w}_t' \boldsymbol{\mu} = \mu_0$$

Proof: From the Lagrangean $\frac{\boldsymbol{w}_t' \mathbf{H}_t \boldsymbol{w}_t}{2} - \lambda_1(\boldsymbol{w}_t' \boldsymbol{\mu} - \mu_0)$ we obtain the first order conditions

$$\mathbf{H}_t \boldsymbol{w}_t - \lambda_1 \boldsymbol{\mu} = 0,$$

$$\boldsymbol{w}_t' \boldsymbol{\mu} = \mu_0.$$

The first condition gives $\boldsymbol{w}_t = \lambda_1 \mathbf{H}_t^{-1} \boldsymbol{\mu}$. Substituting in the second condition (written as $\boldsymbol{\mu}' \boldsymbol{w}_t = \mu_0$) gives

$$\lambda_1 = \frac{1}{\boldsymbol{\mu}' \mathbf{H}_t^{-1} \boldsymbol{\mu}} \mu_0$$

Substituting this in $\mathbf{w}_t = \lambda_1 \mathbf{H}_t^{-1} \mu$ produces the expression for the minimum variance portfolio weights

$$\mathbf{w}_t = \frac{\mathbf{H}_t^{-1} \boldsymbol{\mu}}{\boldsymbol{\mu}' \mathbf{H}_t^{-1} \boldsymbol{\mu}} \mu_0$$

8.2 Orthogonal GARCH with ARCH Estimates of Components

We start with making a univariate GARCH estimation for $y_{1,t}$ as follows:

$$y_{1,t} = \sqrt{h_{1,t}} \epsilon_t,$$

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1.t-1}.$$

For the second GARCH model we include $y_{2,t}$ as a regressor in the mean specification:

$$h_{2,t} = \omega_{12} \cdot y_{1,t} + \sqrt{\tilde{h}_{2,t}} \epsilon_t,$$

$$\tilde{h}_2, t = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_2 \tilde{h}_{2,t-1}$$

This leaves us with constructing the conditional covariance matrix as follows:

$$\mathbf{H_t} = \begin{bmatrix} h_{1,t} & \omega_1 h_{1,t} \\ \omega_1 h_{1,t} & \omega_{12}^2 h_{1,t} + \tilde{h}_{2,t}. \end{bmatrix}$$

8.3 DCC-MR

DCC-MR follows the process:

$$\mathbf{y_t} = \mathbf{H_t^{1/2}} \boldsymbol{\xi_t}$$

$$\mathbf{H_t} = \begin{bmatrix} h_{1,t} & \rho_t \sqrt{h_{1,t} h_{2,t}} \\ \rho_t \sqrt{h_{1,t} h_{2,t}} & h_{2,t} \end{bmatrix},$$

where the conditional variances are specified as:

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1.t-1},$$

and

$$h_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_2 h_{2.t-1},$$

and $\rho_t = h_{12,t}/\sqrt{h_{1,t}^* h_{2,t}^*}$ comes from:

$$h_{1,t}^* = (1 - \theta_1 - \theta_2) + \theta_1 \epsilon_{1,t-1}^2 + \theta_2 h_{1,t-1}^*,$$

$$h_{2,t}^* = (1 - \theta_1 - \theta_2) + \theta_1 \epsilon_{2,t-1}^2 + \theta_2 h_{2,t-1}^*,$$

and

$$h_{12,t} = \phi_{12}(1 - \theta_1 - \theta_2) + \theta_1 \epsilon_{1,t-1}\epsilon_{2,t-1} + \theta_2 h_{12,t-1},$$

with ϕ_{12} equal to the average sample correlation of returns. Also, $\epsilon_{1,t} = y_{1,t}/\sqrt{h_{1,t}}$ and $\epsilon_{2,t} = y_{2,t}/\sqrt{h_{2,t}}$.

8.4 ASYMMETRIC DCC

Asymmetric DCC follows the following process:

$$\mathbf{y_t} = \mathbf{H_t}^{1/2}\boldsymbol{\xi}_t$$

$$\mathbf{H_t} = \begin{bmatrix} h_{1,t} & \rho_t\sqrt{h_{1,t}h_{2,t}} \\ \rho_t\sqrt{h_{1,t}h_{2,t}} & h_{2,t} \end{bmatrix},$$

where the conditional variances are specified as:

$$h_{1,t} = \omega_1 + \alpha_1 y_{1,t-1}^2 + \beta_1 h_{1.t-1} + \gamma_1 d_{1,t-1} y_{1,t-1}^2,$$

and

$$h_{2,t} = \omega_2 + \alpha_2 y_{2,t-1}^2 + \beta_1 h_{2.t-1} \gamma_2 d_{2,t-1} y_{2,t-1}^2,$$

and $\rho_t = h_{12,t}/\sqrt{h_{1,t}^* h_{2,t}^*}$ comes from:

$$h_{1,t}^* = (1 - \theta_1 - \theta_2 - \frac{\theta_3}{2}) + \theta_1 \epsilon_{1,t-1}^2 + \theta_2 h_{1,t-1}^* + \theta_3 d_{1,t-1} y_{1,t-1}^2,$$

$$h_{2,t}^* = (1 - \theta_1 - \theta_2 - \frac{\theta_3}{2}) + \theta_1 \epsilon_{2,t-1}^2 + \theta_2 h_{2,t-1}^* + \theta_3 d_{2,t-1} y_{2,t-1}^2,$$

and

$$h_{12,t} = \phi_{12} \cdot (1 - \theta_1 - \theta_2) - \phi_3\theta_3 + \theta_1 \epsilon_{1,t-1}\epsilon_{2,t-1} + \theta_2 h_{12,t-1} + \theta_3 (d_{1,t-1}\epsilon_{1,t-1})(d_{2,t-1}\epsilon_{2,t-1}),$$

The variables $d_{1,t}$ and $d_{2,t}$ are dummies for $y_{1,t}$ and $y_{2,t}$ and are 1 when these variables are negative and 0 otherwise. The coefficient $\frac{\theta_3}{2}$ relies on the assumption that ϵ_1 and ϵ_2 have a symmetric distribution. The parameters ϕ_{12} and ϕ_3 are the average correlation of returns and the average asymmetric components $(d_{1,t-1}\epsilon_{1,t-1})$ and $(d_{2,t-1}\epsilon_{2,t-1})$, and $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are defined as before.