Erasmus University Rotterdam

Master Thesis Quantitative Finance

Individual Pensions with Collective Risk-Sharing

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Contents

Abstract

We study the impact of intergenerational risk-sharing on the pension results of an individual pension contract. In this contract equity risk is shared across different generations by means of a collective buffer. This buffer enables investing more in risky assets, without pension plan participants being exposed to high equity risk. The individual pension contract with buffer offers participants of the pension fund the possibility to choose (and change) their own level of risk-aversion. We examine the effect of varying investment policies and the additional effect of the collective buffer on the pension results. Monte Carlo simulations lead to pension distributions for the different designs of the pension contracts. We find in general that the buffer lowers but stabilizes the pension outcomes on average. However, when we can accurately predict the stock return distribution and we apply this distribution in the way we construct the contracts, some individual pension contracts with a collective buffer outperform the individual contracts without a buffer. We perform the analysis both for fixed and variable pension contracts. Compared to fixed pensions, variable pensions lead on average to higher pension outcomes.

1 Introduction

The Netherlands is widely regarded for having one of the strongest and most sophisticated occupational pension system in the world. However, after economically bad years the weaknesses of the Dutch pension system became apparent. As a result, during recent years, many work has been done on investigating a new optimal common pension design. This has finally resulted in an announced shift to another commonly used pension system. This shift is included in a paragraph of the last Dutch government policy accord ('regeerakkoord') released on the 10th of October 2017. The new contract, called the '4C-pension contract', will be introduced in 2020.

The 4C-contract is an extension of the so-called Individual Defined Benefits (IDC) contract. An IDC-contract is characterized by the fact that every participant has its own accrued pension capital, and this pension capital is invested according to an individual-specific life-cycle asset mix. Hence, all the investment risks are borne by the participants in this contract. There is no solidarity element included. The 4C-contract differs in this way from the IDC-contract. Although the 4C-contract is also still mainly an individual contract, there is also a risk-sharing component included by means of a collective buffer for equity returns. This collective buffer is designed to distribute extreme equity returns across different periods. In times of high equity returns, a part of the returns flows into the collective buffer. On the other hand, in times of low equity returns, a part of the negative returns flows from the collective buffer to the individual pension capital of the participants. In this way the stock returns, and therefore the portfolio returns on the individual pension capital of the participants should be stabilized. In this thesis we will scrutinize the effect of the inclusion of the collective buffer in the Individual Defined Benefits contract. Therefore, the research question is as follows:

> Do individual pension contracts with a risk-sharing component lead to better pension outcomes than the pension outcomes in a Defined-Contribution contract?

In this thesis we perform a simulation study to answer this question. In this way we obtain distributions for pension outcomes of the several contracts that we examine. With these distribution we can measure the performance of the pension contracts. The word 'better' in the main question means that we want high median outcomes of the contracts, but also favorable pension outcomes in worse economic scenarios since the contracts need to provide stable pension outcomes.

We test a large number of variants of both the IDC-contract and the 4C-contract. The general conclusion we find is that the pension outcomes are lower for the IDC-contracts with a collective buffer (the 4C-contracts), but the spread between the outcomes in this contract is also lower.

As we mentioned, in an IDC-framework the individual participants are able to choose an individual life-cycle asset mix. This asset mix is applied to their individual pension accrual. In this thesis we determine asset mixes for different types of investors. We examine the effect of varying the asset mixes on the pension outcomes. We find that it is rewarding to choose a risk-seeking investment style, especially when we incorporate the stabilizing collective buffer in the pension contracts.

In an usual IDC-framework the participant has a fixed pension contract. This means that he has to buy an annuity at his retirement age for the pension capital he has accrued. Consequently, he locks his future pension incomes. In the Netherlands, there is also a possibility to choose for a variable pension contract since 2015. In contrast to the fixed pension contract, the participant withdraws only a part of his pension capital each year. The remaining of the pension capital will be re-invested. In this way the participant does not lock all his future pension pay-offs and can consequently have gains of equity premiums, also during retirement. We find that it is favorable to opt for a variable pension, especially when we choose the 4C-contract. The collective buffer lowers the equity risk during retirement in this variable contract, which stabilizes the pension outcomes in turn.

Furthermore, we investigate how to design the optimal buffer. In an optimal world, there is always some capital in the buffer. Along these lines, the buffer prevents the individual pension capitals from large losses by means of very negative stock returns. On the other hand, the buffer may not contain too much capital. This would imply that the participants are more harmed by the buffer than that they have gains from it, since they are only contributing to the buffer.

We find that if we can accurately predict the distribution of the real stock returns, we can clearly take advantage of the buffer. When the boundaries of the buffer are set in such a way that the buffer affects the stocks returns of the portfolios in 90% of the times on average, we find favorable results for the 4C-contract. These specific 4C-contracts outperform the IDC-contracts, regarding higher median pension results and also higher 5^{th} percentile results.

In general, when it is not possible to accurately predict the distribution of the stock returns, including a buffer mainly lowers the pension incomes. Consequently, we can generally conclude that the solidarity aspects the buffer should provide negatively affects the pension outcomes of the IDC-contract.

For the simulations of the pension results, we need to construct life-cycle asset allocations for the individuals suiting their risk-profiles. In many papers about life-cycle investment optimization problems, only a stock index and a risk-free asset are incorporated in the asset mix. Sangvinatsos and Wachter (2005) show for a portfolio with both long-term and short-term bonds, that the allocation on long-term bonds increases with the investment horizon. We apply this idea in the way we construct our portfolios. Therefore, we also incorporate a 5-year government bond in our asset mixes, besides the 'usual' stock index and the nominal cash free account.

For solving the optimal asset allocation problems we use the framework of Koijen et al. (2009) (hereafter: KNW). In this technical framework we analyze the life-cycle asset mixes of investors with different risk-profiles. The algorithm we use can determine the optimal asset mixes for n risky assets. Hence, this method suits our asset mix including both a risky bond and a stock-index. In the KNW-method, the investor obtains utility from both his terminal wealth and consumption during his work life. This terminal wealth and consumption are both incorporated in the value functions that have to be optimized in the algorithm. In this algorithm the Endogenous Grid-Point method of Carroll (2006) is applied in order to optimize the value functions for a large number of grid-points. The value functions in this algorithm are optimized with respect to the asset allocations and the consumption strategy of the investor. We follow Brandt et al. (2005) by approximating the conditional expectations in this algorithm by cross-sectional regressions. Applying this approximation saves us a lot of computation time. The algorithm finally results in life-cycle asset allocations that we use for both a fixed pension investor and a variable pension investor. We find that the riskier the investor, the more the asset mix is dominated by stocks. The 5-year-bond allocation is increasing in time, and the cash account is only part of the portfolios in the last couple of years before retirement.

As input for both the asset allocation optimizations and the pension models, we need to simulate a large amount of paths of a financial market. Each of these paths includes simulations of a stock index, an inflation rate, an instantaneous short rate and bond yields.

For the specification of the financial market processes, we use the financial market specification of Koijen et al. (2009) and Sangvinatsos and Wachter (2005). For the underlying process of these simulated financial market processes, we apply the three-factor model of Hamilton and Wu (2012) (hereafter: 'HW'). For the estimation of the structural form parameters of the financial market processes, we apply the method of HW. They introduce the Maximum-Chi-Square Estimation (MSCE) technique to extract the structural-form parameters from the estimated reduced-form parameters of the model. In this technique, we split the original highly dimensional MLE problem in smaller optimization problems. This trick enables us to find relative fast the desired structural form parameters. Therefore, we circumvent the usual complicated high-dimensional maximum likelihood optimization of non-linear functions to extract the structural form parameters from the reduced form parameters of the financial market models (Sangvinatsos and Wachter (2005), Koijen et al. (2009), Dai and Singleton (2000)).

The 4C-contract is also investigated by van Riel (2016). In this paper he sketched a theoretical framework of the pension contract and included the results of a numerical analysis. The technical analysis is omitted for the reader. This paper forms a good 'manual' for the new pension system, and present the theoretical reasoning behind the characteristics of the model. Also some anomalies are sketched.

In contrast to the theoretical exploration of van Riel (2016), we approach the new pension framework in a more technical way. This master thesis can be seen as the underlying framework of the results that are presented in the theoretical exploration paper. For practitioners this yields useful insights. Moreover, this master thesis extends the exploration papers of van Riel (2016) in the sense that we incorporate multiple risky assets in our asset mixes, instead of only a 'return portfolio' and a 'matching portfolio'. In the framework we construct, we can quite easily extend the asset mixes by more assets.

The entire simulation study is performed in MATLAB. The code for the dynamic programming analysis I have made without any other sources. Also the pension fund model I coded myself. For the estimation of the affine term structure model I used parts of existing code (Hamilton, 2012). Also the GMM-estimation I performed myself.

The content of this thesis is as follows. We start with outlining the current Dutch pension system. We explain here the current DB-system, its anomalies, and what the requirements are for a new system. Thereafter we explain our pension fund-model, and how the pension capital of the individual participants develops over time in both the IDC- and the 4C-framework. After that, we outline the specification of the financial market, the corresponding estimation techniques, and the estimation results of this financial market. Then we discuss how we obtained the data we need for estimating the structural form parameters of the financial market, and the summary statistics of these data. After that the technical dynamic programming analysis of KNW will be explained. Thereafter, we present the pension results that we obtained, followed by the conclusions of our analysis. We end with discussing some shortcomings of the analysis, and finally some recommendations for further research.

2 Setting

In this section we first give an introduction of the Dutch pension system, and thereafter we present the motive for a new pension system. Then we will explain the Defined Benefits contract (DB), the Defined Contributions (DC) and the 4C-contract. We will finally outline the strengths and weaknesses of both a fixed and a variable pension contract.

There are three main pillars in the Dutch pension system. The pillars are designed to have a guaranteed pension income and an additional pension income for people who have worked during their working life.

Pillar 1. The first pillar provides a basic income during retirement for every Dutch individual. This pillar is called the AOW (Dutch: 'Algemene Ouderdomswet'). The system is designed to prevent poverty among older individuals. The AOW is financed according to a Pay-As-You-Go-Principle (PAYG).[1] The amount of AOW-pension that a citizen receives, depends on the number of years he has lived in the Netherlands between the age of 15 and 68, and on his marital status.

Pillar 2. This pillar contains a supplementary pension for employees. Both employees and employers have to accrue some additional capital to ensure an supplementary income above the AOW-income from pillar one for the employees. The accrued capital is managed by pension funds and insurance companies. Every year employees have to pay a certain premium of their wage to the pension fund. Asset managers invest the accrued pension capital in order to let the participants maintain their purchasing power during retirement.

Pillar 3. The third pillar is a voluntary one. If an individual wants to accrue more pension than the sum of the first two pillars, he could buy an extra pension product (for example an annuity). People who do not work for a company with a collective pension, like independent entrepreneurs, use this form of pensions. Insurance companies provide these types of products.

2.1 The Defined Benefits contract and its anomalies

This thesis has to do with the supplementary pension scheme for employees (so the second of the three pillars we described). There are several types of supplementary pension contracts offered. The most popular contract worldwide is the Defined Benefits-contract (DB). In a DB-system every active participant pays the same percentage of his or her salary to the pension funds each year. In the Netherlands, this percentage is set in such a way, that the yearly expected pension payments to the participants are about 70% of their average wage. The pension fund invests all the accrued pension capital collectively in this DB-setting. In this way investment risk is shared by all the participants of the fund.

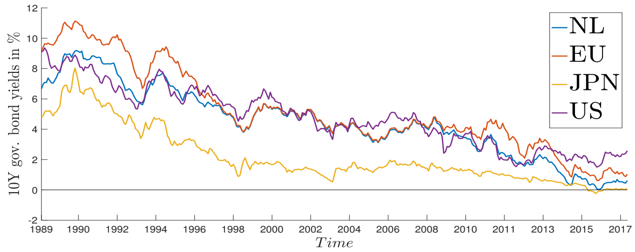
There are several disadvantages of this DB-system. Figure 1 shows the first problem: the low current level of interest rates. The calculations of the appropriate premiums in the past are based on higher interest rates than the low current interest rate. Interest rate risk is the largest risk that the pension fund faces. The interest rate determines how much the future payments to the participants can be discounted, in order to determine the liabilities of the pension fund. With a low interest rate, these future payments are barely discounted. Consequently, the liabilities of the fund can tremendously increase [2]. This leads in turn to a lower funding ratio. The funding ratio represents the ratio of the assets over the liabilities

[1] A PAYG-scheme is characterized by a working generation that contributes part of their wagers, through taxation, to retirees. The working generations trust the system to provide for them when they are retired

[2] As example: The nominal pay-offs of 1 euro in the coming ten years imply a liability of 8.53 euros now, when we calculate with an interest rate of 3.0%. The same ten nominal pay-offs imply a liability of 9.47 euros when we have a 1% interest rate: a liability that is 11% higher

of the fund. It is the key statistic of a pension fund.

Figure 1: Globally we have decreasing trend in the 10-year government bond yields



Another problem is about the actuarial value of the pension accrual. Each year both young and old participants have to pay an equal premium to the pension fund, but the accrued euro of a young person can have benefits of equity premiums for a longer period. This implies that the actuarial value of 1 accrued euro of a young employee is higher than 1 accrued euro of a participant that will almost retire.

When market conditions are bad (e.g. low interest rates) and the funding ratio consequently decreases, the DB-premium in some year can be collectively increased in order to maintain equal pension rights for the participants and let the funding ratio increase. In this way, the younger participant is affected more by the increased premium, because his extra contributed premium has a higher actuarial value.

Also the aging of the population is a problem. The premiums the participants of some age-cohort X have paid 40 years ago, are based on a life-expectancy that was used in that time. However, nowadays the participants of age-cohort X have a higher life-expectation than 40 years ago. This means the liabilities that have to be saved for this age-cohort X have increased, compared to the premiums they accrued. As a result of that the funding ratio lowers. New generations have to contribute to 'repairing' this decreased funding ratio.

Another solution for the increased liabilities is cutting the pension rights of the participants in the fund. Both increasing the premiums and cutting the pension rights lead to lower pension pay-offs of the participants.

In the DB-contract both employers and employees have to contribute to recover the funding ratios. Employers do not want to take the risks belonging to this changing funding ratio, because this could strongly impact the results of the company. The employer prefers that the employees bear the risks where the pension fund is subject to, and does not want to be 'risk sponsors' for their employees.

Goudswaard et al. (2016) describe the DB-system as an 'one-size-fits-all'-approach, in which the accrued pension capital of all participants is investment according to the same asset mix. Moreover, the premiums the participants have to contribute are equal. As a result, the DB-system is a very collective system in which solidarity to other participants is crucial. Negative market developments and an aging society have led to a demand for a new system. This new pension system has to provide solutions for the shortcomings of the common DB-system.

2.2 Requirements for a new system

The discussion of a new optimal pension design was set up by Boelaars et al. (2014). They created several conditions that the new system has to meet. They conclude that the new schedule has to meet four main requirements: *i*) sustainability, *ii*) creating transparency, *iii*) providing flexibility and customization (tailor-made schedules) and *iv*) providing solidarity to other participants. We will briefly elaborate on these requirements.

1) Sustainability

The contract has to be sustainable. This means we want stable pension outcomes for affordable premiums under varying economic circumstances. Furthermore, the participants need an optimal spread of consumption. E.g. this means they want to consume a larger proportion of their wealth when they are young, because their wage is lower in this period so a high pension accrual is unfavorable during this period. Providing sustainability is the main target of the new pension schedule.

2) Creating transparency

Enforcing confidence in the new system by incorporating more transparency about individual property rights. Participants need insight in how much pension capital is reserved for them, and how this capital changes on a yearly basis. This prevents the participants from feeling that there is no money reserved for them when they retire. In the current DB-system there have been many cuts in the past few years, and for the participants it is difficult to understand why these cuts occur (a lack of clarity).

3) Providing customization and flexibility

The system has to provide more flexibility and customization. Customization means that the pension schedules suit the several phases of the life-cycles of the participants. Flexibility means that participants can opt for changing their risk-profile or pay-off scheme when macro-economic conditions change. This can improve their pension outcomes.

4) Providing a collective approach

A sustainable pension system has to provide solidarity between participants. This aspect can be incorporated by means of obligating the participants to share risks. In a collective approach risks can be shared in many ways. Sharing risks will lead to more stable pension outcomes.

2.3 Defined Contribution -contracts

Some occupational pension funds already offer another contract than the common DB-contract: the Defined Contribution contract (DC). Each year all the participants pay a defined amount of premium in this contract. In contrast to the DB-system, these premiums are not equal for the active participants. The premium for an individual participant can depend on age for example. Another difference with the DB-system is that all risks of changing pension incomes are borne by the plan participants, instead of the employers. E.g. bad investment results don't have to be compensated by the employer in this way.

Table 1 shows the premiums that the participants have to pay over their wage in a DC contract. The premiums are determined such that, given the long term interest rates, the expected pension incomes of a DB-contract equal the expected pension incomes in a DC-contract. E.g., if the participant wants to accrue each year 1.875% of his income in a DB-contract, he has accrued 75% of his average income after 40 years. The lower (higher) this DB-accrual percentage of 1.875%, the lower (higher) the premium c^X that the participant of age-cohort X has to contribute in an IDC-contract in order to receive an equal expected amount of pension income. The dynamics of changing DB-premiums is beyond the scope of this thesis.[3]

In the different DC-premiums we see the concept of the actuarial value of the premiums: younger participants contribute lower premiums than older participants. Furthermore, we see that when we have a higher long term interest rate, we can contribute a lower premium in order to receive an equal pension outcome. We explained this concept in section 2.1.

[3]For a more detailed explanation of how the corresponding premiums are determined for these DB-accruals, I refer to appendix F. For now it is important to note that the DB premium and the DB-accrual per year are different concepts here

Table 1: The different age-depending premiums in percentages for DC-models, linked to accruals of DB-models. The premiums differ per long-term interest rate that is used, because with a higher interest rate, the participant have to accrue a lower premium in order to receive the same pension pay-off (as explained in section 2.1). The 'DB-accrual'-column represents the amount of premium that has to be accrued for each year during retirement. The corresponding premium for this 'DB-accrual' is much higher than this 'DB-accrual' itself, because the accrual has to be made for the entire retirement period.

Long-term interest rate	DB-accrual	Age								
		25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-68
4%	1.875	4.8	5.9	7.1	8.7	10.6	13.0	16.0	19.9	23.3
	1.788	4.6	5.6	6.8	8.3	10.1	12.4	15.3	19.0	22.2
	1.701	4.4	5.3	6.5	7.9	9.6	11.8	14.5	18.1	21.1
3%	1.875	7.8	9.0	10.5	12.2	14.2	16.5	19.4	23.0	26.0
	1.788	7.4	8.6	10.0	11.6	13.5	15.8	18.5	21.9	24.8
	1.701	7.0	8.2	9.5	11.0	12.8	15.0	17.6	20.8	23.6

There are several variants of the DC-contracts. We describe two important variants. In the first variant, the Collective Defined Benefit (CDC) contract, assets of participants are collectively owned and invested. The advantage of an CDC-contract is that it has, similar to the DB contract, an intergenerational-risk-sharing component included. Therefore, shocks can be smoothed out over a longer period.

Another DC-contract is the Individual Defined Contributions contract (IDC) . Every participant has its own accrued pension capital in this contract. This capital is invested according to an individual-specific life-cycle asset mix. Individuals can choose the level of risk-aversion that is attained in the asset mixes. Therefore, there is a customization aspect in this pension contract. A disadvantage of the IDC-contract is the lack of solidarity in the contract.

2.4 The 4C-contract

In the contract we investigate in this thesis, we want to use the right elements of the different contracts in order to compose an appropriate pension contracts. On the one hand, a customized contract with actuarial fair premiums is desired, like in the IDC contract. On the other hand, a solidarity element (sharing risks) has to be satisfied, like in the DB-contract. Combining these two requirements leads to the 4C-contract. This contract implies a pension accrual according to the DC-premiums for all the active participants. Providing solidarity to fellow-participants is incorporated in this IDC-contract by means of a collective buffer. Hereby excess (negative) returns flows into (out) a collective buffer when the equity returns are above (below) a maximum (minimum) threshold value. The collective buffer stabilizes the equity returns and therefore the portfolio returns, as the equity returns typically have the highest volatility within the portfolios of the participants.

2.5 Fixed and variable pension contracts

We have now explained the motivation for a new pension system, and we continue with the IDC-contract in the remaining of the thesis. Again, within this IDC- contract the pension capital will be accrued according to a individual-specific life-cycle asset allocation. When a participant turns into his retirement-age, he usually buys an annuity with the accrued individual pension capital. He fixes his pension pay-offs in this sense. Therefore it is called a 'fixed' contract. In contrast to the fixed contract, there is also a possibility to choose for varying pension pay-offs. We will describe in this subsection the difference between a fixed and a variable pension, and the corresponding risks of both types of pension contracts.

First we describe the fixed pension contract and its risks. Again, the pay-offs during retirement are fixed in a IDC-contract, because the participant buys an annuity when he retires. The value of this annuity (so the height of the fixed pay-offs during retirement) strongly depends on the interest rates at the conversion moment. This bears extra risks. When the interest rates are low, the participants receive lower cash-flows in the future in exchange for their accrued pension capital. To reduce this interest rate risk, the interest rate is hedged by pension funds by means of interest rate swaps. Hedging the interest rate brings extra costs.

Another disadvantage of the fixed DC-contract concerns the fact that the pay-offs are fixed. In this way participants do not benefit from risk premiums during retirement. However, they (hopefully) live for a really long period, so they can have lots of investment returns on their accrued pension capital.

As a result of these two disadvantages of the fixed pension contract, the variable pension contract has been designed. Participants in some Dutch pension funds can already opt for such a variable pension. The first advantage of a variable contract concerns the fact that the pension capital can be invested in risky assets, also after the retirement age. Consequently, the participant can have gains of equity premiums, also during retirement. The second advantage the variable contract offers has to do with the lower dependence of economic situation at the conversion moment. Each year the participant receives a pay-off based on several factors, like the long-term interest rate at that moment. This long-term interest rate is re-evaluated each year. In this way the risk of a low interest rate at the conversion time is lowered, because each year the retiree withdraws only a small proportion of his pension accrual based on this unfavorable interest rate. The remaining of his pension capital will be re-invested. We will clarify the risks of the fixed and variable contracts some more in section 3.2.

Regarding the requirements introduced by Boelaars et al. (2016), it seems that both the fixed and variable contract perform better than the DB-contract regarding the transparency requirement. For the fixed contract this is trivial (because the pay-offs are locked so it is clear for the participants what they will receive). Within the variable contract, the participants can exactly see how much of their pension accrual is withdrawn as pension pay-off each year, and how much is re-invested. The participants can even choose how this remaining capital is invested, which makes the system both transparent and customized.

3 The pension fund model

In this section we sketch the cash-flows of the pension fund, and how the individual pension capital of the different age-cohorts develops over time. We start by specifying the population of the pension fund and thereafter we discuss the wage process of the participants. After that we discuss the valuation of annuity factors, which are crucial in determining the pay-off scheme of both the fixed and variable pension contract. Then we continue with the investment policies of both the IDC- and 4C-contracts, and explain the dynamics of the collective buffer in this setting. This sections ends with the explanation of the measure we will use to test the performance of the several pension contracts: the certainty equivalent.

The population

We assume that we have a population with an age ranging from $age_{min} = 25$ until $age_{max} = 100$. The size of the different age-cohorts between 25 and 100 is directly linked to the survival probabilities of the age cohorts. We normalize the number of participants aged 25 to 1. This means that we have a probability 1 of reaching age 25. Every year in our simulation model there enters 1 person of age 25 the model. The probability of reaching the age 26 is p_{26}, so there are $1 \times p_{26}$ person aged 26 each year in the model. In general, the number of persons of a fixed age of X years in the model, is defined by

$$N_X = \begin{cases} 1 & \text{if} \quad X = 25, \\ \prod_{i=25}^{X} p_i & \text{if} \quad X = 26, 27, ..., 100. \end{cases} \tag{1}$$

The survival rates are assumed to be constant over time and consequently the number of participants of age 25 until 100 in the fund is constant over time. For the mortality rates we take the average of men and women from the 'AG-2016' mortality tables in the Netherlands. Table 1 in appendix A shows the survival rates that we apply in our fund. Again, these survival probabilities equal the sizes of the different age cohorts in the fund over time.

We mention that in this specification we used the term age-cohort that represent the set of participants that are born in the same year. In our model the cohort aged X actually acts as one person of size N^X. For notational convenience we use the terms 'investor' and 'participant' in the remaining of the text, while these terms actually represent the whole age-cohort.

Wages

We follow Cocco et al. (2005) with specifying the wage process $y(t)$. They use an exogenous income process. This income process is defined by

$$y(t) = \exp\left(g(t) + v_t + \varepsilon_t\right), \tag{2}$$

where t represents the age t of the participant. The function g is a third order polynomial function, given by $g(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2/10 + \alpha_3 t^3/100$. Additional to the deterministic function g, we add a temporary shock ε_t and a consistent shock v_t in the wage specification in (2). The temporary shock is normally distributed with expectation zero and variance σ_ε^2, so $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$. For the persistent shock we have $v_t = v_{t-1} + u_t$, with $u_t \sim \mathcal{N}(0, \sigma_u^2)$. Cocco et al. (2005) estimated the parameters α_i for different types of groups. We use the parameters belonging to an individual with high-school education, but without a college degree. This type of person is the "high-school" individual in Cocco et al. (2005). The corresponding parameters in the function g are set as $\alpha_1 = 0.1682$, $\alpha_2 = -0.0323$, and $\alpha_3 = 0.0020$. The variance of the transient shock equals $\sigma_u^2 = 0.0738$ and the variance of the permanent shocks equals $\sigma_\epsilon^2 = 0.0106$

In figure 10 and figure 11 in Appendix A we show how the permanent (v) and temporary (ε) shocks influence the deterministic income-streams g. These two figures show that the level of income is far from deterministic in our model. As a consequence, the 5% best or 5% worse income streams can deviate about 20% from the average income stream. Of course, this has a large impact on the pension outcomes of the participants.

3.1 Explanation IDC policy

We start by specifying the pension accrual process in an Individual Defined Contribution contract, because the pension accrual in the 4C-contract is almost similar.

Every year t, the investor from a fixed age cohort X has an initial Financial Wealth FW_t^X. When this person is still working (so between age 25 and age 68), he contributes a certain premium c_t^X of his wage W_t^X which is added to his financial wealth. If the person is already retired, his financial wealth will be reduced because he withdraws some pension income (PI_t^X) in that year (in the case of a variable pension). After these two operations (adding premium or distracting pension income), the new financial wealth will be invested in an individual specific portfolio p_t^X. This portfolio yields some return $r_{p,t+1}^X$ one period later, and the remaining financial wealth grows (or decreases) to a new level FW_{t+1}^X. In formula form, this is the same as

$$FW_{t+1}^X = \left(FW_t^X - PI_t^X \mathbb{1}_{68 \leq X \leq T} + \mathbb{1}_{25 \leq X \leq 68} c_t^X W_t^X\right)\left(1 + r_{p,t+1}^X\right), \tag{3}$$

where the indicator function $\mathbb{1}_A$ equals one when the condition A is satisfied and zero else. This formula holds in case of a variable pension. This means that when the retirement age is reached, we do not lock the pension income into an annuity. Instead, we withdraw each year only PI_t^X of the pension capital and the remaining capital we re-invest. In this way, we keep on having gains from the investments.

Recall from section 2.2 that the premiums c_t^X for the participants of age X depend on multiple factors. In table 1 we presented the dynamics of the DC-premiums when we change the DB-accruals or the long-term interest rates. We use the premiums c_t^X corresponding to long-term interest rate that we use (3.0%), and a 1.875% DB-pension accrual. We keep these premiums constant throughout the entire simulation study.

3.2 The valuation of annuity factors

As we mentioned, annuity factors are crucial for the pension incomes in both a fixed and a variable pension contract. For determining these values of the annuities, we take the survival probabilities and long term interest rates into account. The value of an annuity for age cohort X at time t is given by

$$a_t^X = \sum_{s=max(68-x,0)}^{100-x} N_s \times \frac{1}{(1+r_t(s))^s}. \tag{4}$$

This annuity factor determines how many (deferred) nominal annuities the investor with age X can buy at retirement, when we assume that during retirement there is an yearly payment of 1. This means that one age-cohort participant receives 1 unit of money each year during retirement. For the age cohort this 1 unit of money is corrected for the sizes of the age-cohort at that moment, and discounted with future interest rates. The age-cohort accrual pattern is determined in such a way that the living participants in a specific year in the retirement period, receive the one unit of money each year during retirement. Hence, the fund needs to reserve only $N_X \times 1$ instead of 1 unit of money for the age-cohort X. Consequently this implies that within this contract the participants can have mortality gains. In other words, they can have benefits from the accrued pension capital of fellow participants from the same age-cohort that have died. The mortality chances are taken into account by setting the premiums, so if the population becomes older than expected, the individual participants receive less than the expected one unit pension outcomes they accrued.

If we choose a fixed pension at age 68, we can determine the yearly pension outcomes of a participant following the next two equations.

$$FW_t^{68} = a_t^{68} \times PI_t^{68} \Leftrightarrow \tag{5}$$

$$PI_t^{68} = \frac{FW_t^{68}}{a_t^{68}}. \tag{6}$$

In words: the pension capital has to be divided by the value of annuity, and in this way the values of the future pension incomes equal the value of the current financial wealth.

Like we mentioned, the conversion moment is crucial for the pension incomes of a participant in a fixed contract. If the financial wealth of a participant is locked by a high annuity value (high survival probabilities and low interest rates), the pension cash flows will be lower. Choosing for a variable pension reduces the dependence of the conversion moment. With a variable pension the pension income is locked for only one year. Consequently, the economic and demographic conditions are less crucial for the future cash flows. In the variable pension case only a fraction of the pension capital is withdrawn by the retiree each year during retirement. Each year the long-term interest rate and mortality chances are re-evaluated. With these two variables, we can re-calculate each year the value of the annuity factor a_t^X like we did in equation (4). Dividing the remaining pension FW_t^X capital of the individual participant by this annuity factor a_t^X results in the pension income PI_t^X for a participant with a variable pension.

3.3 Investment allocations IDC-contract and 4C-contract

We note that until now we have mainly focused on the IDC-contract in the explanation of the models. The only extension of the existing IDC-contract that we investigate in this thesis is the inclusion of the collective buffer. An important feature of the IDC-contract is that this contract can be tailor-made for the different individual participants. This means that every participant can choose an appropriate investment mix that is applied to his pension capital. The chosen investment mix follows a life-cycle. In our life-cycle asset mixes we include a stock index, a 5-year bond and a nominal cash account. The asset allocations depend on the investment horizon of the investor and the level of risk-aversion of the investor. The longer the period until retirement, the more the asset allocation is dominated by stocks. The part that is invested in fixed income (5-year-bond and the nominal cash account) increases over time. Within the fixed income part of the portfolios, the long term bond dominates the nominal cash account. The nominal cash account is only part of the portfolio in the last couple of years before retirement. In the way we construct the portfolios we can keep having gains of risk premia, while the risk-profile lowers as the investor is nearing his retirement age.

In section 6 we discuss in detail how we determine the life-cycle asset mixes for different types of investors. We investigate how the different investment styles affect the pension incomes. For a fixed pension we investigate three types of investing: risk-seeking ('Aggressive'), risk-neutral ('Neutral') and risk-averse ('Averse'). With a variable pension the investor can choose at retirement age which of the three investment style he wants to apply in his investment mix for the remaining of his life. He can switch his attitude towards risk in this way. He can for example invests aggressive until retirement, and averse after retirement (we call this variable pension investor 'Aggressive-Averse'). This implies that we have $3 \times 3 = 9$ investment styles that the investor can attain in the case of a variable pension.

3.4 The collective buffer

We already explained how the pension capital of the age cohort X develops over time, and that each year t this money is invested in a portfolio p_t^X, with some return $r_{p,t}^X$. In the portfolios, the investor allocates his money to a portfolio of a nominal cash account, a 5-year bond and a stock index. The weights of these three assets are denoted by $\alpha_{Rf,t}^X$, $\alpha_{B5Y,t}^X$ and $\alpha_{St,t}^X$. The three asset weights sum up to one and the asset weights larger or equal than zero. In an usual IDC- contract the return of the portfolio is a linear combination of the asset weights and their corresponding returns. The difference with an IDC-contract and a 4C-contract is that we manipulate the returns on stocks. This manipulation is done by means of the collective buffer. If the return on stocks $r_{St,t}$ is too high (low), the collective buffer will be used. We define the 'new' stock return of the age cohort as

$$\tilde{r}_{St,t} = \begin{cases} r_{St,t} & \text{if} \quad lb \leq r_{St,t} \leq ub \\ ub & \text{if} \quad ub < r_{St,t} \\ lb & \text{if} \quad r_{St,t} < lb \end{cases}, \tag{7}$$

where ub denotes the upper-bound of the collective buffer where the equity returns are cut, and lb the lower-bound. The question rises what appropriate values are for the upper-bound ub and the lower-bound lb. This is one of the questions that we answer in this thesis. On the one hand we want a high value ub such that the buffer does not lower the high stock returns too much. On the other hand, we want to have enough capital in the collective buffer such that the buffer can protect the pension capital against very negative stock returns.

With the defined stock return $\tilde{r}_{St,t}$, we have the following return for the 4C-portfolio $\tilde{p}_t^X$:

$$\tilde{r}_{p,t}^X = \alpha_{St,t}^X \tilde{r}_{St,t} + \alpha_{B5Y,t}^X r_{B5Y,t} + \alpha_{Rf,t}^X r_{Rf,t}. \tag{8}$$

We plug these modified stock returns in the financial wealth process of the IDC-investor that we specified, and obtain the 4C financial wealth process $\tilde{FW}_t^X$ that is specified by

$$\tilde{FW}_{t+1}^X = \left(\tilde{FW}_t^X - PI_t^X \mathbb{1}_{68 \leq X \leq T} + \mathbb{1}_{25 \leq X \leq 68} c_t^X W_t^X\right)\left(1 + \tilde{r}_{p,t+1}^X\right).. \tag{9}$$

The only difference between the financial wealth of the 4C-contract ($\tilde{FW}_t^X$) and the financial wealth of the IDC-contract (FW_t^X) lies in the different portfolio return caused by the modified stock returns.

The capital in the collective buffer changes when the stock return is above upper-bound ub or below lower-bound lb. The excess return above ub flows into the buffer. In the same way the excess negative return below the lower-bound lb flows from the buffer to the individual pension capitals. Consequently, the excess return is measured by the change between the IDC stock return and the 4C-returns: $(r_{St,t} - \tilde{r}_{St,t})$.

We define the change in the collective buffer caused by age cohort X in year t as

$$\Delta CB_t^X = \alpha_{St,t}^X FW_{t-1}^{X-1} \left(r_{St,t} - \tilde{r}_{St,t}\right). \tag{10}$$

In words, the change in the collective buffer is calculated as the product of the pension capital and the excess return on stocks of the portfolio for a given age cohort. Note that when the stock returns is in the range between the the lower-bound and upper-bound, then we have $r_{St,t} = \tilde{r}_{St,t}$. This implies $\Delta CB_t^X = 0$ for all age-cohorts X at time t.

We obtain the total change of all the age cohorts of the collective buffer for moment t by summing over all the age-cohorts in the fund:

$$\Delta CB_t = \sum_{i=25}^{100} \Delta CB_t^i. \tag{11}$$

Finally, the total size of the collective buffer at moment t is measured by the sum of the previous changes, so

$$CB_t = CB_{t-1}\left(1 + r_t^{CB}\right) + \Delta CB_t, \tag{12}$$

where CB_0 is the initialization step. The return of the portfolio in which the capital of the buffer is invested, is denoted by r_t^{CB}. We allocate 40% of the capital to the stock index and 60% to the risk-free cash account, since in a DB-system the assets are usually invested according to this asset mix. Note that the summation in equation (11) runs until the maximum age of 100 in our model. If we invest also after retirement age, the buffer is also active for participants in the draw-down period. With a fixed pension we do not invest after retirement age, so in this case the summation in equation (11) runs until age 68.

Upper- and lower-bounds buffer

We emphasize that we also need to specify an upper-bound UB_t and a lower-bound LB_t for the maximum amount of capital that the buffer may contain. In formulas, we have to satisfy $CB_t \in [LB_t, UB_t]$. The explanation of these bounds has to with the fact that the buffer needs to act as an stabilizing mechanism. E.g., when the buffer is filled with some capital that equals 100% of the accrued pension capital of the participants, then the upper-bound for the stock returns ub is not realistic. There is too much wealth transferred to the future in this situation. This is not fair for the generations who contributed to filling this buffer. Following this reasoning we specify the upper-bound UB_t as follows:

$$UB_t = \sum_{i=25}^{100} \Pi_{max} FW_t^i, \tag{13}$$

where $\Pi_{max} \in \{0.1, 0.2, 0.3\}$. We do not want that the buffer has more capital than some proportion of the sum of all the financial wealth of the participants of the fund.

For the lower-bound we choose $LB_t = 0$. This means that we do not allow for negative amount of capital in the buffer. A negative buffer is a kind of debt that currently active and future generations have to pay back later. Consequently, this implies that future active participants have to pay for bad results in the past (van Riel, 2016). This way of sharing risks is one of the motivations to renounce the current DB-system. We can easily prevent this way of moving risks forward by setting the lower-bound of the buffer, LB_t to 0.

Intergenerational risk-sharing

One requirement for the buffer is that it may not frequently hit its lower-bound LB_t or upper-bound UB_t. We already remarked that the buffer can act as a mechanism to distribute wealth over different periods. When the upper- or lower-bound is hit the buffer will be switched off. When the buffer is switched off, the 4C-contract is (temporarily) an usual IDC-contract, until the buffer will be switched on again. This is not fair. We illustrate this unfair situation with an example. Imagine the situation when the upper-bound is reached at some time t^*, because there is too many capital in the buffer. Consequently, the buffer is switched off. This is beneficial for the active participants of the fund at time t^*. Their additional stock returns above the upper bound lb does not flow into the collective buffer. The additional returns will be added to their own pension capital now. Imagine that we are two years later, at time $t^* + 2$, in a recession and we have to deal with extremely negative stock returns. The buffer will shrink and will be switched on again. Meanwhile, there are some young participants who have entered the fund. These new participants have less advantage in times of extreme positive returns because the buffer is working again. We can apply a similar reasoning in the case the buffer reaches its lower bound. Negative returns are not hedged for the active generations, while this can be the opposite case some years later.

We conclude that we want a working buffer as often as possible for an optimal distribution of wealth over time. Nevertheless, the idea behind intergenerational risk-sharing is that there can be times that some generations suffer more from some bad economic circumstances than other generations. In this way we can argue that a buffer that is switched-off is unfavorable for some active generations, but contributes to the intergenerational risk-sharing element.

The initialization step of the buffer

We keep the idea of an active buffer in mind by setting the initialization of the buffer.

First we run the pension models by starting with an empty buffer, so $CB_0 = 0$. Then we determine the average size of the buffer over all the simulations. This average size is the new initialization of the buffer. It turns out that both the pension outcomes and the number of times that the buffer is full or empty, are barely affected by the two different initialization steps. In order to have equal gains of the buffer of all the generations, we start with a buffer that is filled for 50% of the long-run average of capital in the buffer. We mention that this can't be done in reality because you transfer some of the initial pension capital of the active participants to the buffer. However, in order to examine the isolated impact of the buffer on the pension results this is a favorable choice. Moreover, the transition issues of the pension contracts (so creating a fair initialization mechanism of the buffer) is beyond the scope of this thesis.

The Certainty Equivalent

In both the fixed and variable pension cases, we calculate the pension incomes for the different age cohorts. In the variable pension case the pension incomes differ also each year, because they depend on the investment results during retirement. Consequently, we obtain a stochastic series of pension incomes in the variable pension contract. We want to capture the performance of such a series of stochastic pension incomes in a single statistic. The certainty equivalent is a powerful statistic in order to measure this performance. We define V_t^{68} as the utility value of a pension contract for moment t as

$$V_t^{68} = \sum_{s=0}^{100-68} N_{68+s} \beta^s \frac{\left(PI_{t+s}^{68+s} \,/\, \bar{Y}_{t,68}\right)^{(1-\gamma)}}{1-\gamma}. \tag{14}$$

Here $\bar{Y}_{68,t}$ is the average income of an individual during his working life. The pension incomes are stochastic, which makes the utility level stochastic as well. The terms $PI^{68+s} \,/\, \bar{Y}_{68,t}$ denote the replacement ratios for future pension pay-offs. We follow KNW and choose $\beta = 0.97$, which represents a subjective discount factor. The parameter γ represents the relative risk-aversion.

From the stochastic utility specification in equation (14) we can extract the Certainty Equivalent (CEQ). This is the fixed level of pension income that gives an individual the same satisfaction level according to the utility function as the equivalent stochastic income streams. In formula form this is the same as

$$V^{CEQ} = E[V_{68}] = \sum_{s=0}^{T-68} N_{68+s} \beta^s \frac{\left(CEQ \,/\, \bar{Y}_{t,68}\right)^{(1-\gamma)}}{1-\gamma}. \tag{15}$$

The CEQ is a powerful measure which can catch the welfare of the different pension series in one single statistic. We remark that the CEQ strongly depends on the chosen utility function.

3.5 The simulation of the results

The pension fund model that we construct runs from 2018 until 2093. We present in the results section only the pension results of age-cohort 1993. This age-cohort enters the fund at age 25 and all the participants of this age-cohort will be death in 2093, in our model. Therefore, this age-cohort runs through the entire simulation and starts with a pension capital of zero.

For the initialization of the pension capital of the participants of all the other age-cohorts, we use a risk-neutral investor that has an IDC-contract. This investor has some specific individual life-cycle asset allocations. We run 500 simulations of the wealth equations over time following this life-cycle mix. We take the average of these 500 wealth levels for each year, such that we obtain an average wealth of the participants from age 25 until 68. This is the initialization of the pension capital for the age-cohorts that are already active in the fund, so the age-cohorts 1950 until 1992. We assume that the participants that are already retired do not participate in the fund anymore for this simulation study, so they do not contribute to the buffer.

For each simulation, we obtain for each pension contract exactly one certainty equivalent. We assume that if we have for example a risk-neutral investor with an IDC-contract, that all the participants apply this investment style in the pension fund throughout the entire simulation of this contract. Note that for each pension contract we specified, we run N simulations yielding N certainty equivalents for the age-cohort 1993. In this way we obtain distributions of the pension outcomes of the different pension contracts. We can compare the distributions of these Certainty Equivalents among the different contracts to measure the performance of the different contracts.

4 Financial market

For the simulation study we simulate $N = 500$ different paths of the financial market as input. The paths have length $T = 75$, the duration of our pension model. Per path we need to simulate multiple processes, e.g. a stock index, a short-term nominal interest rate, bond returns and an inflation rate. In this section we first outline the dynamics of the processes that describe the financial market. After that, we explain the underlying affine term structure model of Hamilton and Wu (2012) (HW) that drives the financial market processes. Finally, we describe the estimation techniques we apply for estimating the structural form parameters that describe the financial market.

4.1 Financial market processes

We adapt the financial market model that is described in Koijen et al. (2009), but we use a three-factor model as underlying process of the financial market simulations, instead of the two-factor model in KNW. This underlying process is the underlying latent process of the bond yields of different maturities. By choosing a term-structure of the yields with three terms we follow Dai and Singleton (2000), Duffee (2002) and Sangvinatsos and Wachter

(2005) .

We implement the model of HW for the latent factors that drive the yields. The three factors in the HW-paper can be interpreted as the level, slope and curvature of the term structure of the yield curve. The three underlying latent factors F_t follow the stochastic differential equation given by

$$dF_t = -\kappa F_t dt + \Sigma_F dZ, \tag{16}$$

where κ is 3×3 matrix[4] and dZ is a 5×1 vector of standard Brownian motions under the physical measure. Furthermore, Σ_F is specified as $[I_3\ 0_{3\times 2}]$.

We assume the instantaneous risk-free rate r_t to be an affine function of the three underlying factors F_t. Therefore, this process is described by

$$r_t = \delta_0 + \delta_1^T F_t. \tag{17}$$

with $\delta_0 \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}^{3\times 1}$.

We assume that we have a time-varying market price of risk. This price of risk is linear in the underlying factors F_t, so

$$\Lambda_t = \Lambda_0 + \Lambda_1 F_t, \tag{18}$$

where $\Lambda_0 \in \mathbb{R}^{3\times 1}$ and $\Lambda_1 \in \mathbb{R}^{3\times 3}$. When $\Lambda_1 = 0_{3\times 3}$, the risk premia are constant and we obtain the multifactor version of Vasicek (1977) .

We denote $P(t, t+n)$ as the nominal price of a bond at time t with maturity $t+n$ and a nominal pay-off of 1. This nominal bond has an exponential affine structure in the state variables,

$$P(t, t+n) = \exp\bigl(A_n + B_n' F_t\bigr), \tag{19}$$

where $B_n \in \mathbb{R}^{3\times 1}$ represent the factor loadings of the yields with maturity n. It follows that the corresponding yield is given by

$$y_t^n = a_n + b_n F_t, \text{ with } a_n = -\frac{A_n}{n},\ b_n = -\frac{B_n}{n}, \tag{20}$$

By applying Ito's lemma and using the theorem of no-arbitrage pricing we can derive that the bond price dynamics are given by

$$\frac{dP}{P} = \left(r_t + B_n' \Sigma_F \Lambda_t\right) dt + B_n' \Sigma_F\ dZ, \tag{21}$$

where dZ represents the same 5×1 standard normal Brownian motion as in equation (16). This formula implies that we have a time-varying risk premium in bonds, since Λ_t is not

[4] The discrete-time specification of equation (16) is $F_{t+1} = c + \rho F_t + \Sigma_F w_{t+1}$, where we have $c = 0$, $\rho = \exp(-\kappa\ dt)$ and w_{t+1} is a 5×1-vector of independent standard normal variables. This is exactly the discrete time specification of HW, the factor model that we plug into the KNW model under the physical measure. The only difference is that the HW-discrete time specification has 3 factors instead of two in the original KNW-specification, so we have an extra risk factor compared to the KNW-model

constant. Further we define an expected inflation which is also affine in the state variables F_t:

$$\pi_t = \xi_0 + \xi_1' F_t \tag{22}$$

with $\xi_1 \in \mathbb{R}^{3\times 1}$. The realized inflation is obtained by the expected inflation and an additional stochastic term:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi' dZ, \text{ with } \sigma_\Pi \in \mathbb{R}^{5\times 1}, \tag{23}$$

with π_t the instantaneous expected inflation. The dynamics of the stock prices is given by

$$\frac{dS_t}{S_t} = (r_t + \eta_s)dt + \sigma_S' dZ \text{ with } \sigma_S \in \mathbb{R}^{5\times 1}, \tag{24}$$

where η_S represents the equity risk premium. We follow Sangvinatsos and Wachter (2005) and Koijen et al. (2009) assume that the equity risk premium is constant over time in our model. With this specification of the financial market, we have five independent risk drivers. We assume that the volatility matrix stacking Σ_F, σ_S and σ_π is lower triangular.

$$\begin{pmatrix} \Sigma_F \\ \sigma_S' \\ \sigma_\Pi' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sigma_{S_1} & \sigma_{S_2} & \sigma_{S_3} & \sigma_{S_4} & 0 \\ \sigma_{\pi_1} & \sigma_{\pi_2} & \sigma_{\pi_3} & \sigma_{\pi_4} & \sigma_{\pi_5} \end{pmatrix} \tag{25}$$

4.2 Affine Term Structure Model

In this subsection we further investigate the Affine term structure model of HW. We recall from equation (16) that we have three latent variables F_t which are driven by the Gaussian vector auto-regression model. This continuous time process is defined under the physical measure. We can write the discrete-time specification of the continuous time stochastic differential equation in 16 is given by

$$F_{t+1} = c + \rho F_t + \Sigma u_{t+1},$$

with $c = 0$, $\rho = \exp(-\kappa\ dt)$, $\Sigma = I_3$ and u_{t+1} a (3×1) vector of independent standard normal variables. This specification implies that $F_{t+1}|F_t, F_{t-1}, \cdots, F_1 \sim N(\mu_t, \Sigma\Sigma)$, where

$$\mu_t = c + \rho F_t. \tag{26}$$

When we take r_t as the nominal risk-free rate and when F_t contained all the information that is relevant to the investor, the price of a discounted asset has to be a function of the current state vector $P_t(F_t)$. Consequently, risk-neutral investors want to pay

$$P_t(F_t) = \exp(-r_t) E_t\left(P_{t+1}(F_{t+1})\right). \tag{27}$$

For risk-averse investors, we have that

$$P_t(F_t) = E_t\left(P_{t+1}(F_{t+1}) M_{t,t+1}\right), \tag{28}$$

where $M_{t,t+1}$ represents the pricing kernel. In this model the pricing kernel equals

$$M_{t,t+1} = \exp\left[-r_t - (1/2)\Lambda_t'\Lambda_t - \Lambda_t u_{t+1}\right]. \tag{29}$$

When we rewrite the conditional expectations in 28 and 29 as integrals, we derive that the expected price for a risk-neutral investor is μ_t. For a risk-averse investor this expected price equals

$$\mu_t^Q = \mu_t - \Sigma\Lambda_t. \tag{30}$$

As a result we see that risk-averse investors have another conditional mean of F_{t+1}, with this specification of the pricing kernel. This implies a difference between the market valuation and the risk-neutral valuation of assets. We can interpret Λ_t as the market price of risk.

Rearranging equations (18), (26) and 30 gives that the conditional mean under the risk-neutral measure Q is given by

$$\mu_t^Q = c^Q + \rho^Q F_t, \tag{31}$$

where

$$c^Q = c - \Sigma\Lambda_t \tag{32}$$

$$\rho^Q = \rho - \Sigma\Lambda_t. \tag{33}$$

We can conclude that risk-averse investors characterize the VAR of the factors by the Q-measure VAR, given by

$$F_{t+1} = c^Q + \rho^Q F_t + \Sigma u_{t+1}^Q. \tag{34}$$

Here is u_{t+1}^Q a vector of independent standard normal variables under the Q-measure.

We want to translate this term structure to the term structure of the yields, in order to estimate the model. Ang and Piazzesi (2003) show that we can write the yield of a risk-free n-period pure discounted bond as

$$y_t^n = a_n + b_n' F_t, \tag{35}$$

where

$$b_n = \frac{1}{n}\left[I_3 + (\rho^Q) + \ldots + (\rho^Q)^{n-1}\right]\delta_1 \tag{36}$$

$$a_n = \delta_0 + \left(b_1' + 2b_2' + \ldots + (n-1)b_{n-1}'\right)c^Q/n \tag{37}$$

$$- (b_1'\Sigma\Sigma' b_1 + 2^2 b_2'\Sigma\Sigma' b_2 + \ldots + (n-1)^2 b_n'\Sigma\Sigma' b_n)/2n. \tag{38}$$

When we know the values of c^Q, ρ^Q, δ_0 and δ_1, then we can iteratively predict the coefficients b_i and a_i for the different maturities $i = 1, \ldots, n$. In this way we can predict the yields of different maturities when we know the factors F_t in equation (34). We remark that the specification of a_n and b_n is important in the estimation of the structural-form parameters.

4.3 Estimation techniques

In the affine Gaussian term structure models we often have to deal with translating the parameters of a simple VAR model (the reduced-form parameters) to the parameters that describe the financial market (the structural-form parameters). Sangvinatsos and Wachter (2005) describe an affine term structure model (ATSM) where Quasi Maximum Likelihood estimation is applied to extract the structural form parameters from the reduced-form parameters. This technique brings a lot of problems. The ATSM-model has a difficult non-linear maximum likelihood surface. This makes finding a global maximum a difficult task. Furthermore, the models include many parameters. Finding good starting values for the optimization turns out to be very hard, which makes convergence of the parameters even harder. HW introduced a method that enables us to circumvent these highly dimensional complex optimization problems. In order to do this, they first derive a VAR model of observed yields. With this VAR-specification, we can link the reduced form parameters of the VAR-model to the structural form parameters c^Q, ρ, ρ^Q, δ_0 and δ_1 by splitting the optimization into smaller problems. These 'smaller' problems can be estimated with the so-called Minimum Chi-Square Estimation (MSCE). We apply this method in order to avoid the difficult QMLE technique. In the appendix we have a technical explanation of this estimation method.

Given the estimated structural form parameters that describe the underlying latent factors of the economy, we continue with estimating the structural form parameters of the stock and inflation processes. Shen (2015) derived GMM-conditions for estimating these structural form parameters. We apply perform this GMM-estimation too. This technique is also presented in appendix B1.

5 Data and estimation results

5.1 Data

For calibrating the financial market we use U.S. data. For the estimation of the term structure of the yields we use data obtained from from Gürkaynak et al. (2007).[5] This dataset includes yields that are measured on a daily basis data from June 1961 until October 2017. For the estimation we use government bond yields for four different maturities: 1-year, 2-year and 3-year and 6-year government bond yields. We pick every last business day of the month. For the inflation rates we use monthly data from the Federal Reserve Bank of St. Louis of 'consumer price indices for all urban customers, and on all items, with seasonal adjustment'. For this proxy of the inflation rate process, we use data from January 1982 until September 2017.

[5] `https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html`.

For the stock index we use the market proxy of the Fama-French 3-factor model, obtained from the data library of French (2013) [6]. This dataset includes monthly observations, and we use the observations from January 1982 September 2017. For the mortality (or survival) tables we use the AG-2016 tables from the Netherlands.

Summary statistics

We see in our dataset the average yields increasing when the maturity increases. The volatility of the bonds increase as well as the maturity of the bonds increases. This has to with the increasing risk-premia when the maturities increase. The monthly stock return is really high, but the stock returns are also very volatile.

Table 2: The annualized sample means of the annualized yields of bonds of different maturities, and their corresponding standard deviation. The stock returns and inflations are on a monthly basis.

	mean	std
3 month yield	4.03%	3.16%
3 year yield	4.91%	3.25%
5 year yield	5.32%	3.62%
10 year yield	5.98%	3.85%
monthly stock return	0.68%	4.34%
monthly inflation	0.22%	0.26%

Table 3: The sample correlations between annualized bond returns of different maturities, monthly stock returns and the monthly inflation of the data-set

	3 month	3 year	5 year	10 year	Stock	Inflation
3 month bond return	1					
3 year bond return	0.972	1				
5 year bond return	0.955	0.996	1			
10 year bond return	0.920	0.977	0.991	1		
monthly stock return	0.290	0.287	0.283	0.278	1	
monthly inflation	0.039	0.038	0.032	0.021	-0.020	1

The correlation table shows that the correlation between yields with different maturities is really high. On the other hand, the correlation between the stock returns and yields is low. The higher the maturity, the lower the correlation between the stock and the bond. The correlation between the inflation and the assets is negligible small.

[6] `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

Table 4: The estimated parameters of the latent factor model in panel A), and the estimated parameters from the financial market processes in panel B). The standard errors of the estimation are displayed between the round brackets

PANEL A: Estimated latent factor parameters process							
P- and Q-parameters F_t process				implied λ representations			
c^Q	0.0462	0.0428	0.0949	λ	-0.0462	-0.0428	-0.0949
	(0.0272)	(0.0840)	(0.1462)				
ρ^Q	0.9983			Λ	-0.0375	0.0390	0.0181
	(0.0019)				0.0210	-0.0623	0.0472
	0.0181	0.9618			-0.0096	-0.0117	0.0070
	(0.0158)	(0.0134)					
	-0.0149	0.0913	0.9177				
	(0.0637)	(0.0363)	(0.0190)				
ρ	0.9608	0.0390	0.0181				
	(0.0219)	(0.0195)	(0.0356)				
	0.0391	0.8995	0.0472				
	(0.0448)	(0.0378)	(0.0451)				
	-0.0245	0.0796	0.9247				
	(0.0419)	(0.0232)	(0.0420)				
Σ_E	7.18E-05						
	(2.45E-06)						

PANEL B: Financial market processes					
Nominal short rate $r_t + \delta_0 + \delta_1' F_t$					
δ_0	4.97%				
	(1.33%)				
δ_1	0.14%	0.14%	0.53%		
	(0.15%)	(0.23%)	(0.10%)		
Exptected inflation rate $\pi_t = \xi_0 + \xi_1' F_t$					
ξ_0	3.25%				
	(0.20%)				
ξ_1	0.05%	0.28%	-0.07		
	(0.01%)	(0.13%)	(0.12%)		
Stock return process $\frac{dS_t}{S_t} = (r_t + \eta_s)dt + \sigma_S' dZ$					
η_S	4.38%				
	(0.52%)				
σ_S	-0.46%	3.10%	-3.39%	14.47%	
	(0.72%)	(0.40%)	(0.38%)	(0.20%)	
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi' dZ$					
σ_π	0.59%	-0.06%	0.03%	0.01%	0.03%
	(0.04%)	(0.02%)	(0.01%)	(0.03%)	(0.37%)

5.2 Estimation results

Table 4 presents the estimated structural form parameters of the financial market, and their corresponding standard errors. We see both the results of the dynamics of F_t (following from the HW-estimation in panel A) and the financial market processes (from the GMM-estimation in panel B).

We note that the estimated expected short term interest δ_0 equals 4.97% based on the historical data. For the simulations we lower this constant to $\delta_0 = 3.0\%$, which is a more reliable long-term interest rate for the future. Moreover, the premiums the participants have to contribute are also based on a 3.0% long-term interest rate. Further we see that the expected yearly inflation rate ξ_o equals 3.25%. We choose $\xi_0 = 2.0\%$. This is currently a more realistic long-term inflation rate than the quite high inflation rate that we estimated (based on the data). The equity premium from the market proxy of the Fama-French 3-factor model is estimated as 4.38%. We apply this equity premium for the stock process simulation in the future, as it is a realistic equity premium for our simulation process.

Figure 12 and 13 show that the factor representation does a good job in estimating the short-term (1-year) yield. Also the 5-year bond is estimated quite accurately, although the estimation errors are larger than the errors in the estimation of the short-term rate.

Figure 15 shows for different maturities the factor loadings B of the factors F_t. We see that the different factors loadings have a comparable shape as the factors loadings in the Nelson-Siegel model.

We mention that the elements of σ_S, the volatility of the stock return process, are really high compared to the volatility parameters of the other processes. In this way, we add a lot of noise to the stock process.

Simulating the financial market

After we have estimated the financial market parameters, we are able to simulate the financial market. Therefore we have to draw $N \times T$ times a 5×1 vector of independent standard normal variables. These vectors are the input for the financial market processes we specified. The other financial market processes (e.g. stock returns, bond returns, inflation) follow by plugging in these random innovations in the formulas that specify the financial market in section 4.1.

6 Dynamic portfolio analysis

In the previous sections we explained how we specified the pension model and how we obtained the simulated processes of the financial market. In this section we explain how we can use these simulated financial market processes for determining the optimal life-cycle asset allocations. First we give a brief sketch of the relevant history of life-cycle portfolio construction problems.

6.1 Brief history of dynamic portfolio theory

When an investor wants to make life-cycle investment decisions, he needs a dynamic strategy to manage his portfolio in the best way. In this setting dynamic means that the investor has to re-balance his portfolio in every period, in response to a changing value of his portfolio. In such problems, the investor has an attitude towards risk which has to be incorporated in his portfolio optimization.

Many researchers consider the paper of Markowitz (1952) as the starting point of modern finance. In this paper a static portfolio optimization is described for a mean-variance (MV) myopic investor. Although this paper gives many useful insights, it has also its shortcomings: the investor in this paper is time-independent and not able to capture changes in the value of the portfolio in his investment decision. However, many institutional investors think and act not only myopically. They are also interested in the long-term behavior of their portfolio. Consequently, they need a model for multi-periods. As a result, researchers started to analyze optimal portfolio decisions for long-horizon investors. Pioneer work on this topic was done by Latane and Tuttle (1967), Mossin (1968) and Merton (1969). In these papers the focus is on the allocation between stocks and a risk-free rate (cash), with i.i.d stock returns. There is no stochastic interest rate included in these models. Merton (1971) was the first one who captured a stochastic interest rate in his model. He showed that this leads to a 'hedge' demand, in addition to the myopic demand. Long-horizon investors wish to hedge changes in the investment opportunity set. Depending on the level of risk-aversion, the investor may wish more or less wealth when investment opportunities deteriorate or improve.

During the 90s, a lot of research was done on the predictability of asset returns. As shown by Fama and French (1988) and (Campbell and Shiller, 1988), the price-dividend ratio predicts excess stock returns with a negative sign. This implies that we can predict the stock market based on the price-dividend ratio and hedge time-variation in expected stock returns. Brennan et al. (1997) analyzed numerically an optimization problem for an investor who can invest in stocks, bonds and cash. It turned out that the equity premiums were predictable by the interest rate and the dividend yield. In this way, the time-varying stock returns can be hedged. Using this hedging strategy leads to significant better portfolio returns in out-of-sample simulations. Campbell and Viceira (2002) showed that when investors have relative risk-aversion greater than one, hedging demand leads to an increasing allocation to stocks when the investment horizon increases.

Sangvinatsos and Wachter (2005) investigated whether the same mechanism holds for bond returns of different maturities. In their paper they showed that long-term investors that are not restricted by portfolio constraints can realize large economic gains by both timing bond markets and hedging time variation in bond risk premia.

For our analysis we incorporate the findings of the papers described above. In other words, we want to determine a life-cycle investment strategy in which we can anticipate on both time-varying stock and bond returns. On top of that, we want to perform this analysis for investors with varying levels of risk-aversion. The investors do not only care about their pension capital, but they want to consume also during their working life. All these requirement are captured in the optimization problem described by KNW.

6.2 The optimization problem

KNW introduce a numerical optimization method in a model with multiple assets and multiple state variables included. In this subsection we outline this model. We remark that in appendix C there is a detailed explanation of this model. Also the solution method that we outline in this subsection, is clarified in more detail in this appendix. We note that we consider the optimization procedure as an exogenous part of the thesis. We use this part only to determine the asset mixes for an IDC-investor. We apply these mixes as input for the several pension models. The effect of the buffer is not incorporated in the analysis of the optimal asset allocations in this section. The impact of the buffer is only incorporated in the pension models.

We consider an investor who starts working at $t = 0$ and retires at $t = \tau$. This investor obtains utility from real consumption $C_t \,/\, \Pi_t$ and the real terminal wealth $W_\tau \,/\, \Pi_\tau$. This person has an Constant Relative Risk Aversion utility index given by

$$\max_{(C_t, x_t) \in K_t} \mathbb{E}_0 \left(\sum_{t=0}^{\tau-1} \frac{\beta^t}{1-\gamma} \left(\frac{C_t}{\Pi_t} \right)^{1-\gamma} + \frac{\varphi \beta^\tau}{1-\gamma} \left(\frac{W_\tau}{\Pi_\tau} \right)^{1-\gamma} \right). \tag{39}$$

Here we have C_t as the consumption rate of the individual during his working life. Further φ reflects the value of an annuity that starts at time τ. The higher φ, the more the investor wants to save for his retirement. Again, $\beta = 0.97$ denotes the subjective discount factor. The set K_t contains the values that C_t and x_t can attain. Note that we have to optimize a conditional expectation over a summation that includes stochastic variables until time T. We obtain these stochastic variables by simulating from the financial market model that we have introduced in section 4. We simulate $N = 500$ paths of this financial market of length $T = 75$.

We summarize the set of investment opportunities in the vector of state variables X_t. We use a stochastic nominal cash account as risk-free rate R_t^f. The $n \times 1$ vector of asset is denoted by R_t. We denote Y_{t+1} as the labor income of the investor. We do not use superscripts X for referring to the specific age cohort here, for notational convenience. We remark that we specify only once the optimal asset allocation for the entire period from age 25 until age 68 (or age 88 with a variable contract). We ignore in this sense which age-cohort corresponds to this asset mix, because the entire asset mix we determine is applied to all the age-cohorts.

The wealth dynamics of the individual are given by

$$W_{t+1} = (W_t - C_t)\left(x_t'(R_{t+1} - \iota R_t^f) + R_t^f\right) + Y_{t+1}, \tag{40}$$

where ι is a vector of ones. This means that we invest at time t a fraction x_t in the vector of risky assets. In our model the risky assets consist of a 5-year bond and a stock index. The remaining of the financial wealth we put on a nominal cash account. We work in real terms, so we first introduce lowercase letters to indicate real values of the variables:

$$c_t = \frac{C_t}{\Pi_t},\ w_t = \frac{W_t}{\Pi_t},\ r_t = \frac{R_t \Pi_{t-1}}{\Pi_t},\ r_t^f = \frac{R_{t-1}^f \Pi_{t-1}}{\Pi_t},\ y_t = \frac{Y_t}{\Pi_t}. \tag{41}$$

We mention that we work with real returns, real wealths and real incomes throughout the entire simulation study. We apply these real exogenous variables also in the pension model, because this enables us to take the purchasing power of the participants into account. We assume that the labor incomes grow with the same rate as the inflation rates. The budget equation (40) can then be rewritten as

$$w_{t+1} = (w_t - c_t)\left(x_t'(r_{t+1} - \iota r_{t+1}^f) + r_{t+1}^f\right) + y_{t+1}. \tag{42}$$

Now we have a system with state variables (X_t, y_t, w_t) and control variables (c_t, x_t). We want to optimize the value function (39) with respect to the control variables. We define the set of constraints $K_t = K(w_t)$. There are some constraints that have to be satisfied. The set of constraints can be stated in a formal mathematical way as

$$K(w_t) = \{(c_t, x_t) : c \leq w_t, \iota' x_t \leq 1, x_t \geq 0\}. \tag{43}$$

This means that the investor can not consume more than its own wealth. On top of that, the investor has borrowing and short-sales constraints. With these constraints the investor can not default.

We aim to determine the optimal investment and consumption strategy that satisfies the constraints. Therefore, we first define the value functions for the different periods. In the terminal period T we consume all wealth, so $c_\tau = w_\tau$. We do not have to choose an asset allocation here, because the model ends. The value function at time τ is given by

$$J_\tau(w_\tau, X_\tau, y_\tau) = \frac{\varphi w_\tau^{1-\gamma}}{1-\gamma}. \tag{44}$$

The Bellman equation for $t = 1, \ldots, \tau - 1$ is given by

$$J_t(w_t, X_t, y_t) = \max_{(C_t, x_t) \in K_t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \left(J_{t+1}(w_{t+1}, X_{t+1}, y_{t+1}) \right) \right). \tag{45}$$

6.3 Solution method

Life-cycle problems have analytical solution in only some specific cases. In most of the situations we have to rely on numerical methods. In these methods, the motivation is to exchange an intractable general problem for a tractable specific alternative that does not change the solution too much. This is also the reasoning behind the solution method we apply on this problem.

For optimizing the value function in (45) we use the simulation-based technique developed by KNW. In the KNW paper, two methods are combined: the method of (Brandt et al., 2005) and Carroll (2006). In the KNW-method that follows from these two papers, the optimal portfolio can be solved for multiple assets and multiple exogenous variables. We will outline this optimization method here. Some details are omitted in this section, because otherwise the text would be destroyed by technical details.

In the KNW-method we first specify a grid of $M = 50$ points of wealth after consumption a_t,

$$a_t = w_t - c_t. \tag{46}$$

This trick enables us to specify the optimal consumption c_t in an analytical way. As a consequence we save a lot of computation time, because we need to determine only the asset allocations x_t in our numerical approach. We come back to the analytical solution of c_t later.

First, we want to maximize the Bellman equations in (45) with respect to x_t. The most important thing to note in this algorithm is that we make a grid of M points for the wealth after consumption for each of the $(\tau - 1)$ points in time (note that at time τ there is no consumption and asset allocation to choose). This results in $(\tau - 1) \times M$ grid-points. For each of these grid-points we will determine the optimal asset allocation $x_{t,i}$ and consumption policy $c_{\,t,i}$ for $t = 1, \dots, \tau - 1$ and $i = 1, \dots, M$.

Given this background information, we continue with the algorithm. First we normalize equations (44) and (45) for income to get rid of one variable. The first order derivative that we have to solve for each grid-point to obtain the optimal asset allocation then reads

$$0 = \mathbb{E}_t \left(\beta \exp((1-\gamma)u_{t+1}) c_{t+1}^{-\gamma} (r_{t+1} - r_{t+1}^f) \right) + \lambda - \mu\iota. \tag{47}$$

Here we have λ as a vector that corresponds to the short-sales constraints of the Lagrange multiplier, and μ a Lagrange multiplier that belongs to the borrowing constraint. First we want to solve this equation with respect to x_t.

In this algorithm we work backwards, so we know the values of both the asset allocation and the optimal consumption at moment $t+1$, for all the M points in the grid. We know this optimal consumption and asset allocation for the different endogenous grid-points of $a_{t+1,i}$ with $i = 1, \dots, M$. We have that $w_{t+1,i} = a_{t+1,i} + c_{t+1,i}$, so consequently we have pairs of

$(w_{t+1,i},\ c_{t+1,i})$ with $i = 1,\dots,M$. Between these points we can apply linear interpolation to find other combinations of wealth and consumption. In this way we can approximate the optimal combinations of wealth and consumption for moment $t+1$. For each pair of the grid-point we have also determined the optimal asset allocation, so actually we have the sets $(w_{t+1,i}, x_{t+1,i}, c_{t+1,i})$ for $i = 1,\dots,M$.

We continue with the algorithm. Recall that we have simulated N paths of T periods of exogenous variables. Using these paths we can make a cross-sectional regression of a vector of length N corresponding to time t , and a vector of length N at time $t+1$. For the vector at time $t+1$ we can construct

$$z_{t+1} := \beta \exp((1-\gamma)u_{t+1})c_{t+1}^{-\gamma}(r_{t+1} - r_{t+1}^{f}),$$

term in the expectation operator in equation (47). For the vector at time t we make a simple linear function of the state variables. One element of this vector is the linear function corresponding to one path i, $f(X_{t,i})$. When we plug in all the N paths we obtain a vector $f(X_t)$. If we pick one asset s, and we execute the cross-sectional regression we obtain the approximation

$$\mathbb{E}_t\left(\beta \exp((1-\gamma)u_{t+1})c_{t+1}^{-\gamma}(r_{s,t+1} - r_{t+1}^{f})\right) \simeq \theta_s(x,a_t)' f(X_{t,i}). \tag{48}$$

Here a_t is one of the M fixed grid-points for wealth after consumption. Note that $\theta_s(x,a_t)$ is dependent of the asset allocation that the investor chooses, x. Now we narrow the optimization with respect to x.

The next trick is that we can accurately approximate the projection coefficients $\theta_s(x,a_t)$ in (48) by a simple linear function of the portfolio weights, $g(x)$. The approximation of θ_s is then given by

$$\theta_s(x,a_t) \simeq \Psi_s(a_t)g(x). \tag{49}$$

With this approximation we have expressed the projection coefficients θ_S as a linear function of a constant coefficient matrix Ψ_s and a function that only depends on the asset weights, $g(x)$. Plugging in the approximation (49) in (48) we obtain the following approximation of the conditional expectation:

$$\mathbb{E}_t\left(\beta \exp((1-\gamma)u_{t+1})c_{t+1}^{-\gamma}(r_{s,t+1} - r_{t+1}^{f})\right) \simeq g(x)'\Psi_s(a_t)' f(X_{t,i}). \tag{50}$$

Then we can plug in approximation (50) in (47) and this gives

$$0 \simeq g(x)'\Psi(a_t)' f(X_{t,i}) + \lambda - \mu\iota. \tag{51}$$

This is the function we want to maximize for the asset allocation x, given that we are in grid-point a_t. We apply standard numerical techniques in this problem, and choose multiple random starting points to reduce the chance that we reach a local maximum, and a wrong asset allocation x. With this algorithm we can find relative fast the optimal asset allocation x_t. The gain that we have is that we can derive an analytical solution for the optimal consumption for the corresponding asset allocation x_t. This optimal consumption strategy is given by

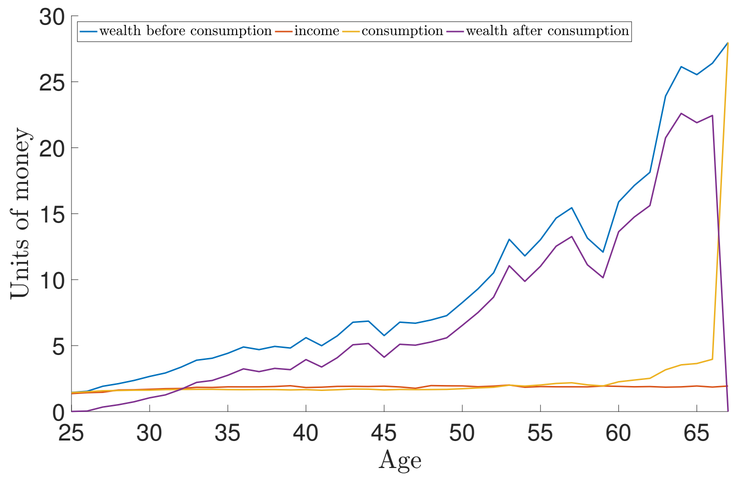
$$c_t^* = \left(\mathbb{E}_t\left(\beta \exp((1-\gamma)u_{t+1})c_{t+1}^{-\gamma} r_{t+1}^p\right)\right)^{-\frac{1}{\gamma}}. \tag{52}$$

Here r_{t+1}^p is the return of the optimal portfolio. Now we have determined both the optimal allocation and the optimal consumption policy for the grid-point a_t in path j.

The output of the grid

By executing this recursive process we obtain for each of the $(\tau-1)\times M$ endogenous grid-points the optimal consumption policy, and the optimal asset allocation. We have to repeat the analysis for each path again. The simulations of the 500 scenarios help us to optimize the control variables for the grid-points of only one path. When we have obtained the entire grid for one path, we can simulate forward through the grid by linear interpolation. We illustrate this procedure in figure 2.

Figure 2: The dynamic programming analysis results in optimal consumption decisions and an optimal asset allocation for all the $M\times(\tau-1)$ grid-points. Following this grid we obtain a wealth and consumption processes. Here the wealth-, income- and consumption process are shown for a risk-neutral investor ($\gamma = 5$) with a fixed pension.



Remember that we have specified the sets $(w_{t,i}, c_{t,i}, x_{t,i})$ for the different grid-points $a_{t,i}$ and $1\le i\le M$, $1\le t\le(\tau-1)$. With these sets we can run through the grid.

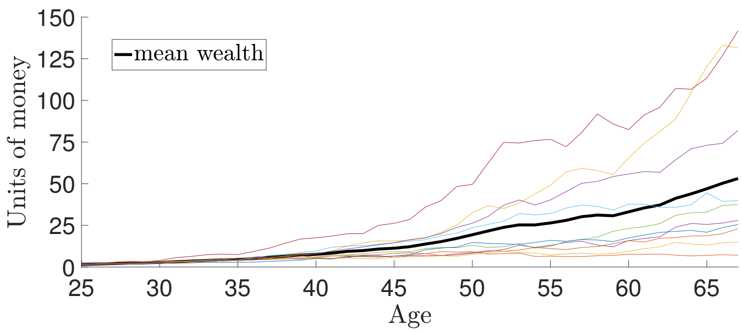
We start at age 25, and in this year we obtain our first income y_1. This income is the wealth before consumption (w_1^*) in the initialization step. Because we have specified the grid, we can determine the corresponding optimal consumption c_1^* and the corresponding optimal asset mix x_1^* for this w_1^*, by linearly interpolating between the sets $(w_{1,j}, c_{1,j}, x_{1,j})$

for $j = 1, \ldots, M$. We apply the appropriate consumption strategy c_1^*, and this results in a wealth after consumption a_1^*. This amount will be invested according to the asset mix x_1^*. One period later, the wealth after consumption has yielded some return $r_{p,2}^*$, so the wealth after consumption in period 1 grows to $a_1^*(1+r_{p,2}^*)$. We add the the income of period 2, and this gives us the new wealth before consumption, $w_2^* = a_1^*(1 + r_{p,2}^*) + y_2$. Again we determine with linear interpolation the optimal corresponding consumption c_2^* and the optimal corresponding asset mix x_2^*. We obtain a new wealth after consumption a_2^*. This wealth after consumption is invested again with the right asset mix x_2^*, etcetera. We repeat this process until we are at time τ. Here we have that the wealth before consumption equals the consumption. This fits our terminal condition $c_\tau = w_\tau$.

We remark that the optimization of equation (51) with respect to x_t is dependent of $f(X_{i,t})$. This function is different for the different paths. Therefore, we have to determine a grid for each separate path. Hence, each path has a different grid in this algorithm. This makes the algorithm quite time-consuming. To deal with this computational issue, we determine the optimal asset mix for the pension fund for only ten paths.

Figure 3 depicts a plot of the ten wealth paths over time, and also the average of these wealths. We see that there is a large variation in these different wealths. This is due to the varying stock returns and therefore the varying portfolio returns. On average we see that the pension accrual is slowly moving upwards.

Figure 3: Ten simulated paths of the wealth process for the risk-neutral investor with $\gamma = 5$. We plot the average of these wealth processes.



The only thing we extract as input for the pension model is the underlying asset mixes that drive the wealth development. We have ten simulated simulated wealth paths, and ten corresponding paths of the asset allocations. For practical purposes we take the average of the ten paths of the asset allocations to determine the asset mix that we apply in the pension fund model. Because the average of the ten asset allocations is also a little volatile, we take a polynomial fit through the average of the ten asset mixes. This polynomial fit is our final asset mix of the investor. The results of these estimated optimal asset allocations are presented in appendix D for the investors with different levels of risk aversion γ.

Figure 4: The average of the ten life-cycle asset allocations for the risk-neutral investor ($\gamma = 5$).

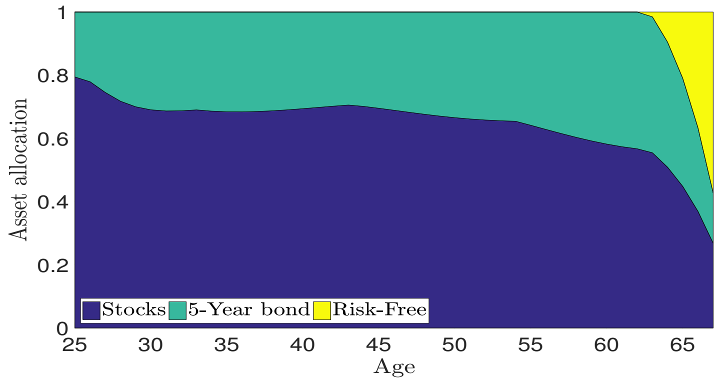


Figure 4 shows the average life-cycle asset allocation for the risk-neutral that we extract from the simulation. The life-cycle mixes are inherently linked to the attitude towards risk. We have three levels of relative risk-aversion. We have $\gamma = 2$ (risk-seeking), $\gamma = 5$ (risk-neutral) and $\gamma = 10$ (risk-averse). These specific parameters of γ lead to the desired life-cycle asset allocations incorporating the corresponding levels of risk.

We see that for all the three levels of risk aversion, the allocation to equity is decreasing with age. This is a well-known result of the literature. If an investor has risk aversion greater than one, hedging demand dictate that their allocation to stock decreases with the horizon (Campbell and Viceira, 2002). Furthermore, the portfolios display the findings of Sangvinatsos and Wachter (2005). In the fixed income part of the portfolios, there is only an allocation to the nominal cash account in the last couple of years before retirement. The longer the horizon, the more of the fixed income portfolio is allocated the 5-year bonds.

7 Results

In this section we present the results of our analysis in order to answer the research question. Before we present the results, we remark that we perform a simulation study. In such a simulation study it is possible to have unrealistic high stock returns for many consecutive years. Consequently, the pension outcomes can be quite extreme, especially within the IDC-contracts. Again, despite these unrealistic outcomes the simulation study is useful to compare the different pension contracts and to investigate the impact of the collective buffer on these contracts.

The amount of times that the collective buffer affects the stock returns of the portfolios, depends on the real stock returns and the boundaries we choose. We have chosen the boundaries ub and lb of the collective buffer based on the distribution of the simulated real stock returns. In this way we know how many times the buffer is used on average, because the real stock returns are above (below) the upper-bound (lower-bound) according to the distribution. Table 7 depicts the amount of times that the buffer is used according to the distribution of the simulated real stock returns, and the corresponding bounds.

Table 5: These percentages indicate how many times the buffer affects the real stock returns of the individual pension capital on average, according to the real stock return distribution. For example, the '30%' chance on a working buffer implies that 15% of the real simulated stock returns are lower than $lb = -10.6\%$ and 15% of the real stock returns are higher than $ub = 19.9\%$.

$\mathbb{P}$ (working buffer)	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
lower-bound	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9	4.9
upper-bound	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6	4.9

Because in reality we do not know the distribution of the real stock returns in the future, we introduce also another set of boundaries. The upper-bounds (lower-bounds) we use are 3%, 6% and 9% above (below) the median result of the median of the simulated real stock returns. For investigating the working of the buffer, we combine the 3 upper-boundaries with the 3 lower-boundaries, such that we obtain 9 extra contracts. In this way we can incorporate that we do not know the distribution of the real stock results in the future, and we can measure the impact of setting statistically incorrect boundaries on the pension outcomes.

In table 6 the terminology that we use in this result section are clarified for the convenience of the reader.

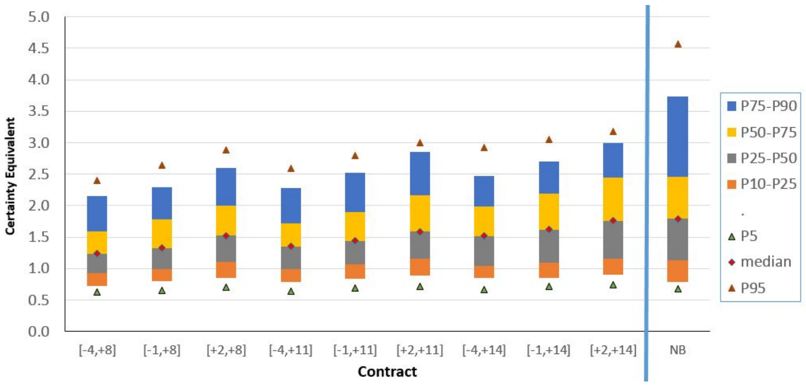
Table 6: The relevant terminology for the description of the result and the meaning of this terminology that we use in describing the results of the simulation study

Definition	meaning
Ag	The fixed pension contracts for the aggressive (Ag), risk-neutral (Ne) and risk-averse investor (Av)
Ag-Ag	The variable pension contract for an investor with aggressive (Ag) investment until retirement, and Aggressive investments after retirement. The same holds for other combinations of aggressive (Ag), risk-neutral (Ne) and averse (Av).
Distribution boundaries	The upper-boundaries ub and lower-boundaries lb that we specified in table 7, based on the distribution of the real stock returns that we have derived. In this way we know how many times the buffer is used in average, because the real stock returns are outside the boundaries. For example: 10% means that 5% of all the simulated real stock returns are below the corresponding lower-bound in table 7 and 5% above the upper-bound.
Alternative boundaries	The set of upper-and lower-bounds we derived by combining three different upper-bounds with three different lower-bounds. For example: $[-1, +11\%]$ means that we have $lb = -1\%$ and $ub = +11\%$
P5	The 5^{th} percentile of the certainty equivalents of a pension contract. This represents how well the pension scheme performs under bad circumstances. We try to find a contract with a favorable P5, as the pensions need to be stable. We denote the other percentiles similar to P5: For example, the 95^{th} percentile is P95, etc.
Π_{max}	The maximum percentage of the total accrued pension capital that the buffer may contain.
buffer full	Buffer empty represents the time average of time that the buffer is switched off, because there is either no capital in the buffer ($CB = 0$). Buffer full represents the average time that there is too much capital in the buffer, so CB equals Π_{max} of the total accrued pension capital of the active participants in the fund.
NB	An IDC-contract, so we have no buffer in this contract (NB)

7.1 General findings collective buffer

The general finding of our analysis is that applying a buffer in the pension contracts reduces the medians of the several pension contracts, compared to the IDC-contracts without a buffer. This is especially the conclusion we find in the case of the alternative boundaries, where we assume that we do not know the distribution of the real stock returns. On the other hand, the contracts without a buffer have a much larger spread in the pension results than the contracts with a buffer. Figure 5 shows these findings for an risk-neutral investor with a fixed pension. We see that the large spread for this IDC-contract (No Buffer, NB) is especially caused by a large difference between the median and the P95 result. The difference between the P5-results of the pension-incomes of the IDC-contract and the 4C is not significant.

Figure 5: The distributions of the pension outcomes for a fixed pension for the aggressive investor ($\gamma = 2$). Below the $x-$axis the different upper- and lower-bounds of the collective buffer of the 4C-contracts are displayed. On the right-hand-side of the blue solid line are the pension results of the IDC contract ('no buffer')

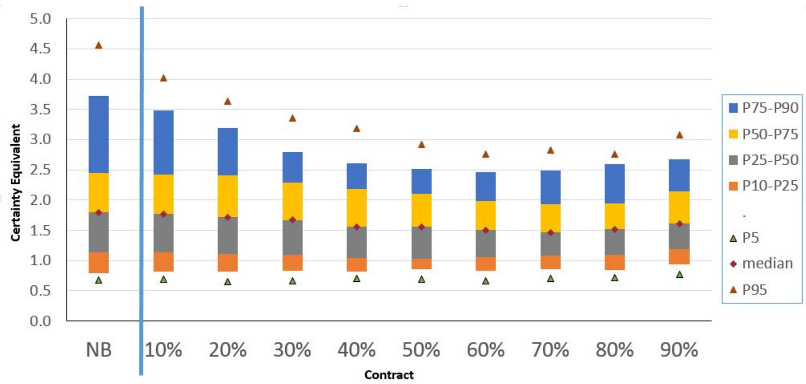


Furthermore figure 5 depicts a pattern that holds for all the 4C-contracts: the higher the lower-bound, the higher the medians. The negative stock returns are more often hedged by the collective buffer in this way. This is beneficial for the pension results. With a similar reasoning we can conclude that the pension outcomes are higher for a higher upper-bound of the buffer.

If we apply the 'distribution boundaries' that we derived in the 4C-contracts (see table 7), we see another pattern. We see that the model without a buffer still has the highest median pension income. The more the average buffer activity increases, the lower the pension results seems to be. However, if we have a 90% activity of the buffer, the pension results are improving compared to the models with the lower buffer activity. Moreover, the P5-outcome of the '90%'-model is 0.78, which is a significant improvement compared to the 0.68 $P5$-outcome of the IDC-contract of the aggressive investor. Still, we have a lower median here compared to the IDC-contract (1.77 vs 1.61).

The IDC contracts have in general much higher P95 outcomes, compared to the 4C

Figure 6: The pension results for a fixed pension for the Ag-investor ($\gamma = 2$) and the distribution boundaries corresponding to table 7. The percentages indicate how many times the buffer affects the real stock returns of the individual pension capital on average, according to the real stock return distribution.



contracts. In this sense it seems to be favorable to not use a collective buffer. However, our pension system has to provide solidarity as well and this feature is ignored in the individual set-up of an IDC-contract.

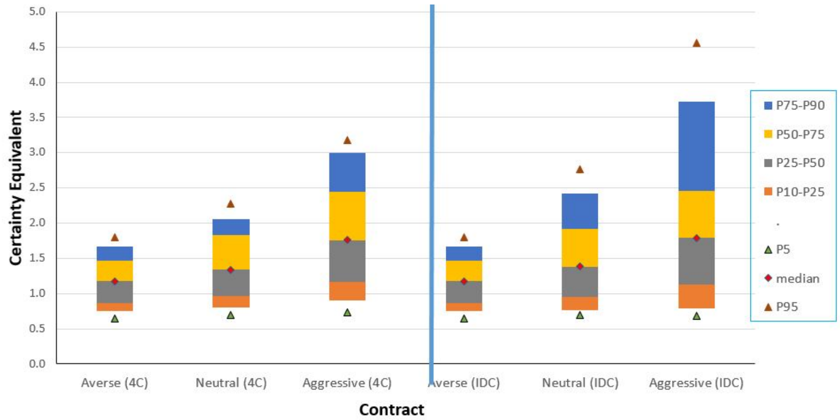
We further note that the life-cycle-portfolio is constructed such that the investor's exposure to risk reduces as he gets older. With a fixed pension, this means that the hedging demand of the investor is already incorporated in the way the life-cycle-portfolio is constructed. Therefore, we can interpret the hedge that the collective buffer provides in a fixed pension contract as an 'additional' hedge.

In contrast to the fixed contracts, the variable pension asset mixes have a longer investment horizon. Therefore, its exposure towards risk is at the conversion moment higher than in the case of a fixed pension. In this way we can argue that there is in a variable pension contract more demand for the 'additional' hedge provided by the buffer, compared to the fixed contracts. Consequently, it is more interesting to investigate how the buffer affects the variable pension contracts. Before we investigate this impact on variable pension outcomes, we scrutinize the consequences of choosing different investment styles in the different pensions contracts.

7.2 Risk-seeking investors rewarded

Figure 7 depicts that within the fixed pension scheme we have higher pension incomes if we attain a risk-seeking attitude. We see again that applying an IDC-contract yields higher pension outcomes. We can argue that it is rewarding to invest risky until retirement age. The buffer can support investing in a risky way, because the buffer can distribute transfer wealth across different periods.

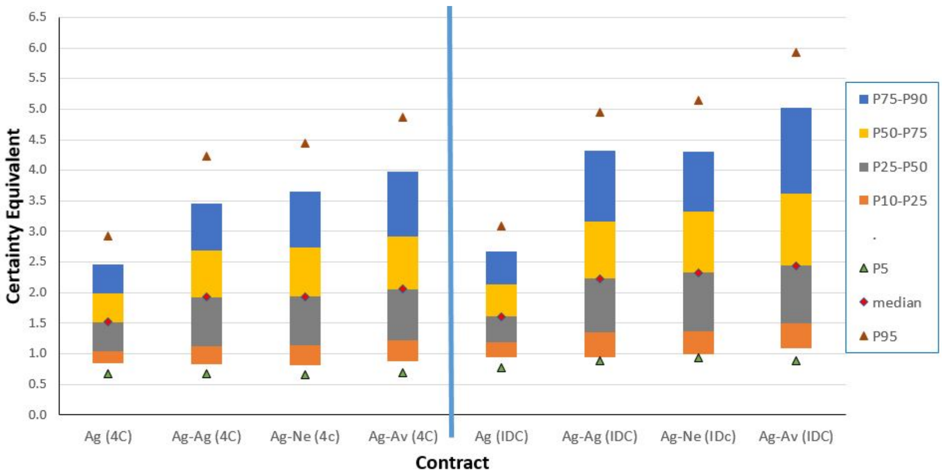
Figure 7: The pension outcomes for a fixed pension 4C-contract (with bounds $lb = -1\%$ and $ub = +8\%$) are displayed on the left-hand-side of the blue solid line. On the right-hand-side of the blue line we have the outcomes of the IDC-contracts.



7.3 Variable pension outcomes

In this subsection we answer the question whether it is favorable to choose for a variable pension contract, instead of a fixed contract. Figure 8 shows the pension outcomes for both a fixed pension (Ag-) and variable pension (Ag-Ag-, Ag-Ne- and Ag-Av-) investor. The variable pension outcomes clearly dominate the fixed pension outcomes in this figure. The medians of the fixed pension contracts are significant lower, compared to the variable pension outcomes. Again we see a large spread between the pension outcomes in the IDC-contracts.

Figure 8: The pension outcomes for the aggressive (Ag) investor with a fixed pension and a variable pension (Ag-Ag, Ag-Ne and Ag-Av). On the left side of the vertical line we have the models with a buffer, and on the right-hand-side we have the models without a buffer. We fix the buffer boundaries here: $lb = -4\%$ and $ub = +14\%$.



We remark that some investment styles are not realistic from a practical point of view. For example, appendix D shows that within the investment mix of the Aggressive-Aggressive

(Ag-Ag) investor, the pension capital is almost entirely allocated to stocks, even when the investor is very old. This is for a pension fund, that has to provide stable supplementary pension outcomes, undesirable. On the other hand, if the solidarity requirement is met (the buffer is not too often full or empty), the models can still be considered.

Considering the results of the variable pension contracts, we see that investing also after retirement yields higher pension results on average. For all types of investors (Ag-Ag, Ag-Ne, and Ag-Av) it pays off to invest after retirement age. Again, the spreads between P5 and P95 increase, but this is mainly due to the high P95-outcomes.

We see that investing aggressive until retirement with investing risk-averse after retirement (Ag-Av) results in a 24% higher median pension outcome, compared to the median of the fixed pension outcomes of the aggressive investor (Ag). The P5-values of the variable 4C-contracts are in general comparable to the P5-values of the fixed IDC-contract when we apply the bounds $ub = +14\%$ and $lb = -4\%$. Therefore it is rewarding to opt for a variable pension with a collective buffer.

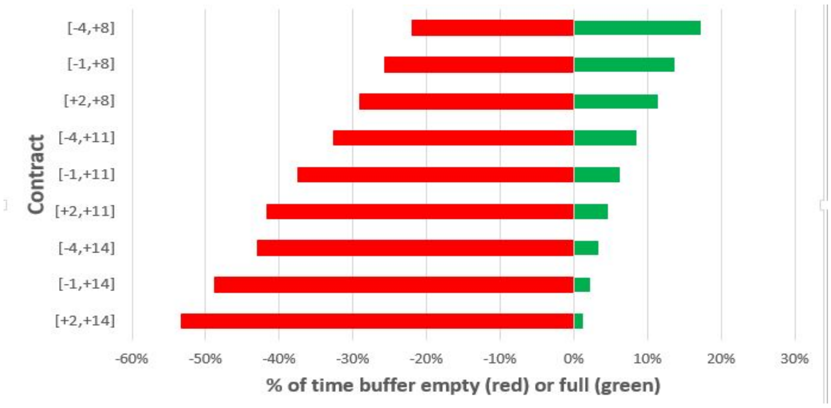
7.4 Bounds and working buffer

Figure 6 shows that the higher the upper-bound ub of the buffer, the higher the pension incomes. In this case, there flows more stock return to the individual pension accrual of the participants. Following this reasoning, it seems to be favorable to choose the upper-bound as high as possible. However, there is a drawback of this higher upper-bound in the 4C-contract. Figure 9 shows this drawback: The lower-bound is hit on average more often when we let the upper-bound increase. In view of solidarity purposes it does not seem to be fair to put the upper-bound too high. In this case different age cohorts have unequal gains from the buffer. Again, it may be possible that the buffer is switched off for some times, such that there is some intergenerational risk-sharing. However, the times that the buffer is full or empty have to be minimized for solidarity purposes.

7.5 Varying the size of the buffer

We also investigate the maximum size of the buffer. With the maximum size we mean the maximum amount of capital that the buffer may contain. We consider respectively 10%, 20% and 30% of the accrued pension capital as the maximum size of the buffer. After setting these maximum sizes, we check for different investment styles what the impact is of varying the maximum size of the buffer. Table 7 shows that an increasing maximum size of the buffer causes a decrease in the median results. With a collective buffer that may contain more capital, there can flow more stock returns to the buffer, which can be unfavorable for the pension results of the participants. We note that for the solidarity aspect, the increase of the buffer is favorable. The amount of times that the buffer is either full or empty decreases when we increase the buffer size. This is beneficial for the solidarity aspect of the pension

Figure 9: The average percentage of time that the buffer is empty (red bars) and full (green bars) for an Ag-Av investor. The lower the amount of times that the buffer is switched off, the more solidarity the buffer can provide to the participants, and the less capital is transfered from the one generation to the other.



contract.

Table 7: Results for varying size of the buffer, where Π_{max} represents the fraction of the total accrued pension capital that the buffer can contain. The last column is the benchmark model, the Aggressive-Neutral investor from an IDC-model (so no buffer, "NB"). Here the boundaries are $ub = +11\%$ and $lb = -1\%$

investment style	Ag-Ag			Ag-Ne			Ag-Av			Ag-Ag
Π_{max}	10%	20%	30%	10%	20%	30%	10%	20%	30%	NB
P5	0.72	0.66	0.65	0.71	0.70	0.69	0.72	0.70	0.70	0.58
P50	2.21	1.96	1.90	2.07	1.85	1.84	1.98	1.83	1.80	2.35
P95	5.81	4.85	4.37	5.57	4.60	4.07	5.26	4.41	3.96	8.06
buffer empty	44%	40%	39%	43%	39%	38%	43%	39%	37%	
buffer full	28%	15%	8%	26%	12%	6%	25%	12%	5%	
.										

Table 8: Results for varying size of the buffer, where Π_{max} represents the fraction of the total accrued pension capital that the buffer can contain. The last column is the benchmark model, the Agressive-Aggressive investor from an IDC-model (so no buffer, "NB"). Here we apply the distribution boundaries, and we use the boundaries corresponding to the '90%' active buffer from table 7: $lb = +2.9\%$ and $ub = +6.6\%$

investment style	Ag-Ag			Ag-Ne			Ag-Av			Ag-Ag
Π_{max}	10%	20%	30%	10%	20%	30%	10%	20%	30%	NB
P5	0.87	0.86	0.86	0.82	0.80	0.88	0.83	0.80	0.87	0.58
P50	2.63	2.42	2.33	2.42	2.20	2.17	2.34	2.16	2.12	2.35
P95	7.10	5.73	5.18	6.08	5.19	4.69	5.83	4.73	4.45	8.06
buffer empty	35%	30%	27%	34%	29%	26%	33%	29%	26%	
buffer full	36%	23%	15%	34%	21%	14%	33%	19%	13%	

Boundaries according to distribution of stock returns

We further find that when we apply the bounds we determined according the distribution of the real stock returns, that we can have wealth gains of the buffer. Table 8 shows for the aggressive investor with a variable pension, that the '90%- active buffer' does a good job. The 'buffer full' or 'buffer empty' score is relative low compared to the other models, which is favorable for solidarity purposes. Also the P5-outcomes of the several 4C-contracts are significant higher, compared to the IDC-contract in table 8. Even some median outcomes of the 4C-contracts are higher than the median outcome of the IDC model that had the highest median outcome in our analysis (Ag-Ag investor)

We note that we already know the distribution of the real stock returns in this example. The 4C contracts with the 'distribution boundaries' clearly outperform the models with the 'alternative boundaries' $ub = +11\%$ and $lb = -1\%$ in table 7 . The $90\%-$ active buffer, has boundaries $lb = +2.9\%$, and $ub = +6.6\%$. Therefore the buffer often affects the real stock returns in the portfolios of the participants. We can argue that if we can accurately predict at least the median of the real stock return distribution, and choose a small 'tube' around this median where the buffer does not affect the stock returns, we can obtain favorable pension results. The buffer provides both more stability and higher pension outcomes in this way.

8 Conclusion

In this master thesis we investigate whether individual pension contracts with a collective risk sharing component (4C-contract) lead to better pension results than individual defined benefits (IDC) contracts. We find several conclusions regarding this question. The main conclusion is that the pension outcomes in the 4C-contract are in general lower than the pension outcomes of the IDC-contract, due to the hedging mechanism of the collective buffer. However, the pension incomes of the 4C-contract are more stable. This stabilization is mainly caused by lower P95-outcomes of the 4C-contract. The P5-outcomes of the two contracts are comparable.

We examine the presence of several requirements in the pension contracts. The new Dutch pension contract appears to meet four important pillars: *i*) sustainability, *ii*) transparency, *iii*) customization and flexibility and *iv*) solidarity to your fellow participants. Regarding these requirements, the 4C-contract seems to be a good alternative to the current common Defined Benefits system.

The 4C-system supports the possibility to choose a variable pension. Introducing the collective buffer enables individual participants to take advantage of equity premium after retirement age, without being exposed to high equity risk. We find for both the 4C- and the IDC-contract, that the mortality risk and interest rate risk are reduced when the participant opts for a variable pension. In the commonly used DB-contract and in the fixed IDC-contract these risks can strongly influence the pension results. Additionally, the 4C contract has a strong customization aspect, because participants can opt for a changing risk-profile in their asset allocation. On top of that, the contract supports transparency, because the participants can easily see how their capital develops over time, and how it is invested. Finally the solidarity requirement is met by means of the collective buffer that can share equity risk amongst different generations.

We find that it pays off if an investor has a low risk-aversion ($\gamma = 2$) until retirement in the fixed pension contract without a buffer. The median income is 34% percent higher compared to the median income of a risk-neutral investor, and the P5-outcomes of the contracts are approximately the same.

A variable pension contract is rewarding compared to a fixed pension contract. Choosing a variable 4C- contract supports investing after retirement age. E.g. the variable contracts of an aggressive investor until retirement outperform the fixed pension contract without a buffer when we look at the medians. The P5-outcomes are comparable.

Regarding the upper-and lower-bounds of the buffer, we see an evident trade-off between these bounds and the pension results. The higher the upper-bound or lower-bound, the more stock returns flow into the pension capital of the participants. The drawback of favorable bounds of the buffer, with respect to pension outcomes, is that the buffer is switched off more frequently. This results in an unequal distribution of the gains that different age-cohort have of the buffer. We allow for intergenerational risk-sharing to some extent, but we want to minimize this form of intergenerational risk-sharing in order to have equal gains for all the participants as much as possible.

We also investigate the impact of a varying the maximum amount of capital that the buffer may contain. We conclude that a buffer that may contain a higher maximum amount of capital is favorable for the solidarity element of the pension contract, since the buffer is less often full or empty. On the other hand, it lowers the pension incomes since there can flow more capital to the buffer, instead of to the individual pension capital of the participants.

Finally we find that if we can accurately predict the distribution of the stock returns, we can have higher pension results for the 4C-investor. If we know the real stock return distribution and we adapt the upper- and lower-boundaries of the buffer to this distribution, then we can obtain better pension results than the results of the IDC-contracts. Especially when the buffer affects 90% of the times the real stock returns on average, the 4C-contracts can outperform the IDC-contracts, both with respect to solidarity purposes and the height of the pension incomes.

9 Discussion

In this section we discuss the shortcomings of the analysis. Thereafter, we present some interesting possibilities where this analysis can be used and continued.

The most important shortcoming is that the model is really simplified. This makes the analysis especially useful to test the sensitivity of the pension results with respect to changing some parameters (e.g. the buffer size), but not to interpret the heights of the pension outcomes.

Also some asset mixes are not very realistic for a supplementary pension contract (e.g. the asset mix of the aggressive-aggressive investor). On the other hand, the isolated function of the buffer is better to understand when we incorporate extreme asset mixes.

Another shortcoming is that we work in our scenario-sets with constant parameters. We already saw with the development of yields over the last decades that this is not a very realistic assumption. The parameters are estimated based on historic data, and this is not a guarantee that in the future these parameters are the same. Other things that we did not incorporate had to do with the possible misspecification of the model, or the correctness of the utility functions. In the end, all outcomes are highly depending on the asset allocations and these allocations depend on the chosen utility functions.

We have only one type of participant (individual with 'high-school education, but without a college degree') with one type of deterministic income process g in the entire pension fund. In a pension fund there is a large diversity of participants with many different characteristics. The size in our model of every age cohort is directly linked to survival rates. This is not a realistic assumption. For the working of the pension contracts, realistic incomes and populations are of course really important. Furthermore, we have assumed that the income-streams are inflation-linked, and this assumption is also questionable.

We also assume in the different pension contracts that all the age-cohort invest according to one level of risk-aversion. The idea of a 4C-system is that each individual can choose for its own level of risk-aversion. For the several contracts and understanding the isolated effect of the buffer it is useful to choose only one investment strategy (like 'variable pension Agressive-Agressive').

In this model we have only a small number of assets. In a real pension fund, the asset manager will invest in thousands of assets. For the understanding of the model it is insightful to have a low number of assets. The asset allocation within these asset categories we chose can be more specified by asset managers.

In reality, pension funds invest a lot in financial derivatives like Interest Rate Swaps to hedge interest rate risk. In a Defined Benefit framework, the interest rate risk is the largest risk because it determines the liabilities of the pension fund. In a 4C-framework the interest rate is important since the interest rate strongly impacts the value of an annuity factor, and consequently the heights of the pension outcomes.

We can further think about how to weight the different contributions to the collective buffer. An aggressive investor intensively uses the buffer because he invests more in stocks than a risk-neutral investor. Do both investors deserve it to have the same upper- and lower-bounds for the buffer, or is it more fair to distinguish several boundaries for different types of investors? And how do we distribute the returns on the capital in the buffer? Do we allocate more of the returns on the buffer to people who have contributed more to the buffer, or does the intergenerational risk-sharing comes here also into play?

Furthermore we assume that the model will be the same for 75 years. It is possible that a certain pension contract could have been the best choice according to a simulation study we do now. If the collective buffer is empty for three consecutive years, and people lose a large part of their pension accrual through a recession, the system will be under pressure. It can be a good choice to change the boundaries we initially chose in such a case. We have not included this option.

In this study we only investigated the buffer that is applied to stock returns. It is of course also interesting to see how the buffer affects the entire portfolio returns.

Incorporating the collective buffer in the value functions in the dynamic programming analysis in section 6 can also be interesting. It can be that the asset mixes for the IDC-investor change in this framework if we put the collective buffer in the budget equations of the individual investor. This is especially interesting from a theoretical point of view.

The 4C-contract is about to be introduced as the new pension system in the Netherlands. By now there are still many points of discussion. Those points are mainly about the transition of the system, and the feasibility. Of course, it is a hard administrative task to move millions of people to another contract. Like we mentioned, there are already exploration papers like the one written by (Goudswaard et al., 2016). The contribution of this thesis to these exploration papers is that this paper serves as a technical framework for the new pension contract. We actually sketch one complete 'supply chain' of the pension contract. First we estimated the parameters of the financial market with a feasible optimization method. Then we estimated the optimal life-cycle mixes for different types of investors, with multiple assets. Finally we used these asset mixes in the pension model, which gave useful insights for the working of the buffer. The technical set-up we outlined in this thesis is a good starting point to perform further research on both the individual defined contributions contract and the 4C-contract. The shortcomings of this study we discussed, are relative easy to capture in this framework. This gives the possibility to test quite easily some effects of the model, like we did in this study by varying the designs of the buffer. In conclusion, this thesis can, together with the existing theoretical exploration papers, form a good starting point for further research in order to create a sustainable pension contract for the future.

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Appendices

A Pension fund characteristics

Table 9: The survival tables that we apply in our model, from the AG-2016 ('Actuarieel Genootschap', which means actuarial society)

age X	$p_{X,man}$	$p_{X,woman}$	$p_{X,average}$	age X	$p_{X,man}$	$p_{X,woman}$	$p_{X,average}$
25	99.32%	99.96%	99.64%	64	89.08%	98.66%	93.87%
26	99.27%	99.96%	99.62%	65	88.06%	98.55%	93.31%
27	99.23%	99.96%	99.59%	66	86.94%	98.43%	92.68%
28	99.19%	99.95%	99.57%	67	85.73%	98.30%	92.01%
29	99.14%	99.95%	99.54%	68	84.40%	98.14%	91.27%
30	99.09%	99.94%	99.52%	69	82.96%	97.94%	90.45%
31	99.04%	99.94%	99.49%	70	81.39%	97.75%	89.57%
32	98.99%	99.94%	99.46%	71	79.67%	97.52%	88.59%
33	98.93%	99.93%	99.43%	72	77.79%	97.27%	87.53%
34	98.87%	99.92%	99.40%	73	75.75%	97.00%	86.38%
35	98.81%	99.91%	99.36%	74	73.54%	96.66%	85.10%
36	98.74%	99.91%	99.32%	75	71.14%	96.28%	83.71%
37	98.67%	99.89%	99.28%	76	68.52%	95.83%	82.18%
38	98.59%	99.88%	99.24%	77	65.69%	95.29%	80.49%
39	98.51%	99.87%	99.19%	78	62.63%	94.66%	78.64%
40	98.42%	99.86%	99.14%	79	59.37%	93.97%	76.67%
41	98.32%	99.84%	99.08%	80	55.90%	93.11%	74.50%
42	98.21%	99.82%	99.02%	81	52.20%	92.07%	72.13%
43	98.09%	99.80%	98.95%	82	48.36%	90.93%	69.64%
44	97.96%	99.78%	98.87%	83	44.36%	89.63%	66.99%
45	97.81%	99.75%	98.78%	84	40.25%	88.19%	64.22%
46	97.65%	99.72%	98.69%	85	36.11%	86.56%	61.33%
47	97.47%	99.68%	98.57%	86	31.94%	84.59%	58.27%
48	97.27%	99.65%	98.46%	87	27.84%	82.36%	55.10%
49	97.05%	99.62%	98.33%	88	23.85%	80.08%	51.96%
50	96.80%	99.59%	98.19%	89	20.07%	77.61%	48.84%
51	96.53%	99.54%	98.03%	90	16.54%	74.66%	45.60%
52	96.22%	99.49%	97.86%	91	13.37%	71.71%	42.54%
53	95.89%	99.45%	97.67%	92	10.56%	68.73%	39.64%
54	95.52%	99.40%	97.46%	93	8.13%	65.45%	36.79%
55	95.11%	99.35%	97.23%	94	6.11%	62.04%	34.07%
56	94.66%	99.30%	96.98%	95	4.46%	58.55%	31.50%
57	94.16%	99.24%	96.70%	96	3.16%	55.02%	29.09%
58	93.62%	99.17%	96.39%	97	2.18%	51.50%	26.84%
59	93.02%	99.10%	96.06%	98	1.45%	48.03%	24.74%
60	92.37%	99.02%	95.69%	99	0.94%	44.67%	22.81%
61	91.66%	98.92%	95.29%	100	0.00%	0.00%	0.00%
62	90.88%	98.84%	94.86%				
63	90.02%	98.76%	94.39%				

Figure 10: A plot of one deterministic income path $\exp(g_t)$ throughout the working age of a participant. For every simulated scenario, we extend this deterministic path with a permanent shock (the blue line), and additionally a transitory shock (results in the black line).

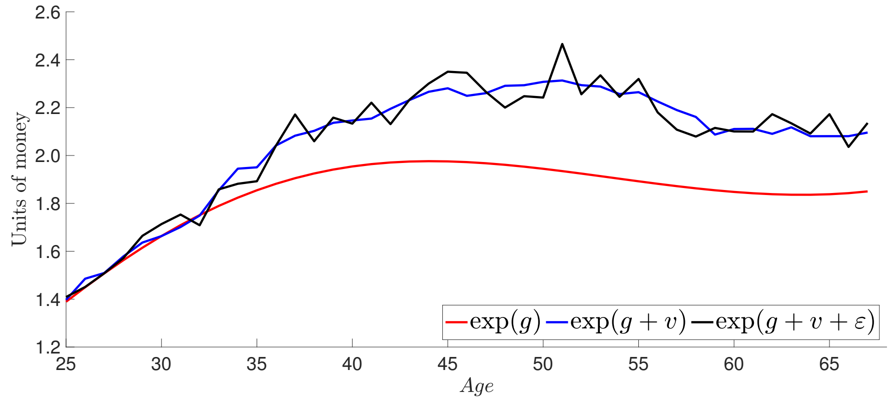
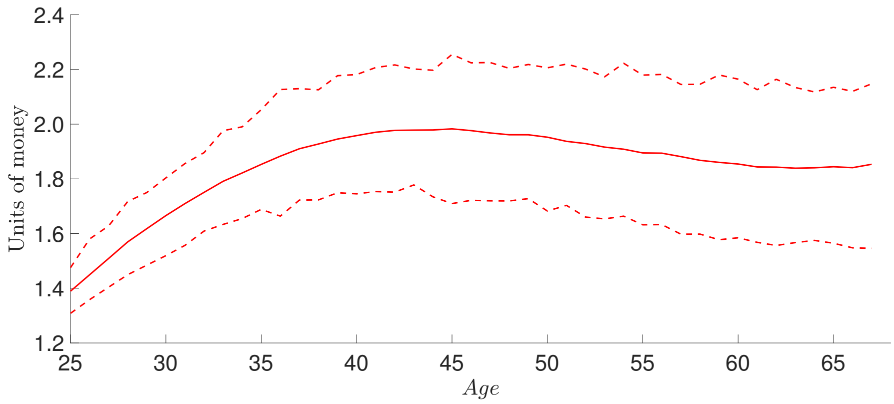


Figure 11: A plot of the average income of the participants (the red solid line) and the P5- and P95-scenarios (the two outer dashed lines). We see that the P5 and P95 incomes differ a lot. Since the incomes are the input for the pension capital, these varying incomes strongly affect the pension outcomes.



B Financial market

B.1 Estimation method

We describe in this appendix how we estimate the structural form parameters that describe the financial market. We apply for the term structure of money the methods introduced by Hamilton and Wu (2012). First we recall where we ended in section 4, and then we derive the VAR-model to estimate the reduced form parameters that we estimate. After that, we link the structural form parameters that describe the term structure model to the reduced form parameters of the VAR-model. Recall that we could interpret the latent factors as the underlying factors of the yield curve. The yield for maturity n at time t is given by

$$y_t^n = a_n + b_n' F_t, \tag{B.1}$$

where Ang and Piazzesi (2003) showed that a_n and b_n are given as

$$\begin{aligned} b_n &= \frac{1}{n}\left[I_3 + (\rho^Q) + \ldots + (\rho^Q)^{n-1}\right]\delta_1 \\ a_n &= \delta_0 + \left(b_1' + 2b_2' + \ldots + (n-1)b_{n-1}'\right)c^Q/n \\ &\quad - (b_1'\Sigma\Sigma' b_1 + 2^2 b_2'\Sigma\Sigma' b_2 + \ldots + (n-1)^2 b_n'\Sigma\Sigma' b_n)/2n. \end{aligned} \tag{B.2}$$

Furthermore, the discrete time specification under the physical measure we specified is given by

$$F_{t+1} = c + \rho F_t + \Sigma_F u_{t+1}.$$

The discrete time specification under the Q- VAR-measure we specified is given by

$$F_{t+1} = c^Q + \rho^Q F_t + \Sigma_F u_{t+1}^Q \tag{B.3}$$

Derivation of the VAR-model

Suppose that we want to describe N_d yields. If N_d is greater or equal than the number of factors M, we can predict the values of one of the yields y_{nt} as exact linear combination of the others. In practice, this fit is never perfect. Ang and Piazessi suppose that the fit holds exactly for N_l, yields and that the other $N_e = N_d - N_l$ yields are observed with errors. Consequently, we can specify the problem in the following way:

$$\begin{bmatrix} \underset{(N_L\times 1)}{Y_t^1} \\ \underset{N_e\times 1}{Y_t^2} \end{bmatrix} = \begin{bmatrix} \underset{(N_L\times 1)}{A_1} \\ \underset{(N_e\times 1}{A_2} \end{bmatrix} + \begin{bmatrix} \underset{(N_L\times M)}{B_1} \\ \underset{N_e\times M}{B_2} \end{bmatrix} F_t + \begin{bmatrix} \underset{N_L\times N_e}{0} \\ \underset{N_e\times N_e}{\Sigma_E} \end{bmatrix} \underset{(N_e\times 1)}{u_t^e}. \tag{B.4}$$

By rewriting this equation we can derive the following specification of the latent factors:

$$F_t = B_1^{-1}(Y_t^1 - A_1). \tag{B.5}$$

In our estimation procedure we have $M = 3$ factors. We choose $N_l = 3$ as the number of yields for which the fit is perfect, and $N_e = 1$ which is measured with error. We use the bond yields with a 12-,24, and 60-months maturity. The 36-months yield is measured with error. For the latent factor specification we then obtain

$$\begin{bmatrix} y_t^{12} \\ y_t^{24} \\ y_t^{60} \\ y_t^{36} \end{bmatrix} = \begin{bmatrix} a_t^{12} \\ a_t^{24} \\ a_t^{60} \\ a_t^{36} \end{bmatrix} + \begin{bmatrix} b_t^{12} \\ b_t^{24} \\ b_t^{60} \\ b_t^{36} \end{bmatrix} F_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Sigma_E \end{bmatrix} u_t^e, \tag{B.6}$$

where we have that u_t^e is a standard normal variable. We use the specification with the risk-averse Q parameters in this system, and we use for normalization of the model $\Sigma_F = I_3$, $\delta_1 \geq 0$, $c = 0$ and ρ^Q lower-triangular. In this case we have 23 unknown structural-form parameters to estimate: 3 in c^Q, 6 in ρ^Q, 9 in ρ and 1 in δ_0, 3 in δ_1, 1 in Σ_E. We can collect these parameters in 23×1 the vector θ. One way to solve this problem is to perform a maximum likelihood maximization. The disadvantage of this problem is that we have a large vector of parameters to estimate, which makes it very hard to find a global maximum. In the analysis of Hamilton and Wu (2012), applying this MLE method was not succesful: Only one of the 100 estimation experiments reached the true global MLE. Therefore, we use another method.

If we multiply the discrete time-specification of the factors under the physical measure by B_1, and add A_1 to this system, together with $c = 0$ and $\Sigma_F = I_3$, we obtain

$$Y_{t+1}^1 = A_1 + BF_t \tag{B.7}$$

$$= A_1^* + \phi_{11}^* Y_{1,t} + u_{1t}^* \tag{B.8}$$

where $A_1^* = A_1 - B_1\rho B_1^{-1}A_1$, $\phi_{11}^* = B_1\rho B_1^{-1}A_1$, and $\Omega_1 = B_1 B_1^T = u_{1t}^* u_{1t}^*$. For Y_t^2 we can derive in a similar way that

$$Y_t^2 = A_2^* + \phi_{21}^* Y_{1t} + u_{2t}^*, \tag{B.9}$$

where $A_2^* = A_2 - B_2 B_1^{-1} A_1$ and $\phi_{21}^* = B_2 B_1^{-1}$, and finally $\Omega_2^* = \Sigma_E \Sigma_E = u_{2t}^* u_{2t}^*$. We see that we have obtained a VAR model where Y_t^1 and Y_t^2 are both dependent of Y_t^1. The reduced-form parameter vector of this model can be specified by

$$\pi = \{\text{vec}\left[(A_1^*, \phi_{11}^*)'\right], \text{vech}(\Omega_1^*), \text{vec}\left[(A_2^*, \phi_{21}^*)'\right], \Omega_2^*\}, \tag{B.10}$$

where vec(X) gives the matrix vectorization of the matrix X, and Vech(X) does the same but only for the elements below and including the diagnonal. The reduced-form parameters can be estimated using normal OLS in equation (B.8) and (B.9), with

$$\Omega_1^* = \frac{1}{T}\sum_{t=1}^{T}\left(Y_{t+1}^1 - A_1^* - \phi_{11}^* Y_t^1\right)\left(Y_{t+1}^t - A_1^* - \phi_{11}^* Y_t^1\right)^T \tag{B.11}$$

$$\Omega_2^* = \frac{1}{N}\sum_{t=1}^{T}(Y_t^2 - A_2^* - \phi_{21}^* Y_t^1)^2. \tag{B.12}$$

From the reduced form parameters we specified, we want to extract our structural-form parameters. This can be done with the following steps.

1. Estimate Σ_e analytically via $\Omega_2^* = \Sigma_e^2 \Delta t$.

2. Estimates of the 6 parameters in ρ^Q and 3 parameters in δ_1 can be done by numerically solving the relations

$$
\begin{aligned}
[B_1(\hat{\rho}^Q, \hat{\delta}_1)][B_1(\hat{\rho}^Q, \hat{\delta}_1)]^T &= \hat{\Omega}_1^* \\
[B_1(\hat{\rho}^Q, \hat{\delta}_1)][B_2(\hat{\rho}^Q, \hat{\delta}_1)]^T &= \phi_{21}^* \hat{\Omega}_1^*.
\end{aligned}
$$

   We can solve this system of equation by vectorizing both the left-hand-side and right hand side of the system, so $\hat{\pi}_2 = ([\mathrm{vech}(\hat{\Omega}_1^*]', [\mathrm{vec}(\phi_{11}^* \Omega_1^*]')$, and $g_2(\rho^Q, \delta_1) = ([\mathrm{vech}(B_1 B_1^T)]^T, [\mathrm{vech}(B_1 B_2^T)]^T)$. Then we can solve solve the parameters of ρ^Q and δ_1 by numerical minimization of $[\hat{\pi}_2 - g_2(\rho^Q, \delta_1)]^T [\hat{\pi}_2 - g_2(\rho^Q, \delta_1)]$.

3. Estimates of the 9 unknown parameters in ρ can be obtained analytically by calculating

$$
\rho = \hat{B}_1^{-1} \phi_{11}^* \hat{B}_1.
$$

4. We can numerically solve the remaining unknowns δ_0, c^Q from the equations in A_1^* and A_2^*.

$$
\left(I_3 - \hat{B}_1 \hat{\rho} \hat{B}_1^{-1}\right) A_1(\delta_0, c^Q, \hat{\rho}^Q, \hat{\delta_1}) = A_1^* \tag{B.13}
$$

$$
A_2(\delta_0, c^Q, \hat{\rho}^Q, \hat{\delta_1}) - \hat{B}_2 \hat{B}_1^{-1} A_1(\delta_0, c^Q, \hat{\rho}^Q, \hat{\delta_1}) = A_2^* \tag{B.14}
$$

   Again, this optimization is done by vectorizing the equations and numerical optimization.

These optimization methods are much simpler from a computational point of view, because the several optimizations includes only a small number of variables.

Now we define θ_B as the vector of all the the 23 structure parameters of the term structure model stacked. We want to test the hypothesis that $\pi = g(\theta_B)$, using the Wald statistic $T([\hat{\pi} - g(\theta_B)]' \hat{R} [\hat{\pi} - g(\theta_B)]$. Here $\hat{R}$ is an consistent estimate of the information matrix, $R = T^{-1} \mathbb{E}[\frac{\partial \mathbf{L}(\pi; X)}{\partial \pi \partial \pi'}],$. and $\mathbf{L}(\pi; x)$ the log likelihood of the entire sample. Note that we can write B.4 as two independent blocks. For i=1,2 we have

$$
Y_t^i = \Pi_i' x_t + u_{it}^*. \tag{B.15}
$$

Then the information matrix R for the system of the reduced form parameters is given by

$$
\hat{R} = \begin{pmatrix} \hat{R_1} & 0 \\ 0 & \hat{R_2}. \end{pmatrix} \tag{B.16}
$$

As shown in Magnus and Neudecker (1988), we have

$$\hat{R}_i = \begin{pmatrix} (\Omega_i^*)^{-1} \otimes \sum x_t x_t' & \\ & \frac{1}{2} D_{q_i} \left((\Omega_i^*)^{-1} \right) \otimes (\Omega_i^*)^{-1} \right) D_{q_i}, \end{pmatrix} \tag{B.17}$$

where D_{q_i} is the $(q_i)^2 \times q_i(q_i+1)/2$ duplication matrix satisfying $D_{q_i}\text{vech}(\Omega_i) = \text{vec}(\Omega_i)$, and q_i the dimension of Ω_i. The Minimum Chi-Square Estimator $\hat{\theta}_B$ is given by

$$\underset{\theta_B}{min}\; T([\hat{\pi} - g(\theta_B)]' \hat{R}[\hat{\pi} - g(\theta_B)]. \tag{B.18}$$

This gives us the desired estimate of the vector of estimated parameters describing the affine term structure model, and the corresponding standard errors. Hamilton and Wu prove that the asymptotic distribution is given by

$$\sqrt{T} \left(\hat{\theta}_B - \theta_B \right) \to N(0, [\Gamma' R \Gamma]^{-1}), \tag{B.19}$$

where we use $\Gamma = \frac{\partial g}{\partial \theta}$ as the numerical gradient of g in the estimated $\hat{\theta}_B$.

Estimation stock and inflation process parameters

With the estimated parameters from the term structure model, we continue with estimating the parameters that describe the stock price dynamics and the inflation rate dynamics. We apply the General Method of Moment in this part. The moment conditons follow from Shen (2015). We have $\Delta t = \frac{1}{12}$ because we work with monthly data and estimate the data on a yearly basis.For the inflation process we have the following discretization process:

$$\Delta\pi_{t+\Delta t} = (\xi_0 + \xi_1' F_t)\, \Delta t + \sqrt{\Delta t} \sigma_\pi^T\; z_{t+\Delta t}. \tag{B.20}$$

Since we work with monthly data and we estimate the data on a yearly basis, we have $\Delta t = \frac{1}{12}$. For the inflation process we have the following 4 equations, resulting in 8 moment conditions.

$$\begin{aligned} \mathbb{E}\left[\Delta\pi_{t+\Delta t} - (\xi_0 + \xi_1' F_t)\Delta t\right] =& 0 \\ \mathbb{E}\left[X_t \left(\Delta\pi_{t+\Delta t} - (\xi_0 + \xi_1' F_t)\Delta t\right)\right] =& 0 \\ \mathbb{E}\left[\left(\Delta\pi_{t+\Delta t} - (\xi_0 + \xi_1' F_t)\Delta t\right)^2 - (\sigma_\pi' \sigma_\pi)\Delta t\right] =& 0 \\ \mathbb{E}\left[\left(\Delta X_{t+\Delta t} - (\rho - I_3) X_t\right)\right) \left(\pi_{t+\Delta t} - (\xi_0 + \xi_1' F_t)\Delta t\right)\right] - \Sigma_F' \sigma_\pi \Delta t =& 0. \end{aligned}$$

We do the same trick for the stock process. First we define the discretization of $r_{t+1} = \frac{s_{t+1} - s_t}{s_t}$, and as parametrization for the stock return process we obtain

$$r_{t+\Delta t} = (\delta_0 + \delta_1' F_t + \eta_S)\, \Delta t + \sqrt{\Delta t} \sigma_S' z_{t+\Delta t}. \tag{B.21}$$

Now we can derive the following 4 equations that give 6 moment conditions for the stock price process.

$$\begin{aligned}
\mathbb{E}\left[r_{t+\Delta t} - (\delta_0 + \delta_1' F_t + \eta_S)\Delta t\right] =& 0 \\
\mathbb{E}\left[(\Delta X_{t+\Delta t} - (\rho - I_3)X_t))\left(r_{t+\Delta t} - (\delta_0 + \delta_1' F_t + \eta_S)\Delta t\right) - \Sigma_F' \sigma_S \Delta t\right] =& 0. \\
\mathbb{E}\left[\left(r_{t+\Delta t} - (\delta_0 + \delta_1' F_t + \eta_S)\Delta t\right)^2 - \Sigma_F' \sigma_\pi \Delta t\right] =& 0. \\
\mathbb{E}\left[\left(r_{t+\Delta t} - (\delta_0 + \delta_1' F_t + \eta_S)\Delta t\right)^2 \left(\pi_{t+\Delta t} - (\xi_0 + \xi_1' F_t)\Delta t\right) - \Delta t \sigma_S^T \sigma_\pi)\right] =& 0
\end{aligned}$$

We denote θ_π as the vector of 9 unknown parameters in the system of moment conditions belonging to π and θ_S the vector of 5 unknown parameters belonging to the stock return process. Then we have $\hat{\theta}_i = \frac{1}{T}\sum f_t(\theta_i)$, where $f_t(\theta_i)$ is the vector of errors of the moment conditions. The matrix S is the sample covariance matrix of $f_t(\theta_i)$. Then we have under efficient GMM that the estimate θ_i is asymptotically normal distributed as

$$\sqrt{T}(\hat{\theta}_{GMM} - \theta_{GMM}) \sim N\left(0, \left(dS^{-1}d\right)^{-1}\right), \tag{B.22}$$

where d is the numerical gradient evaluated in $\hat{\theta}_{GMM}$.

B.2 Data description and estimation output

Figure 12: The estimated 1-year nominal bond yield with the three-factor model, and the observed 1-year nominal bond yield over time

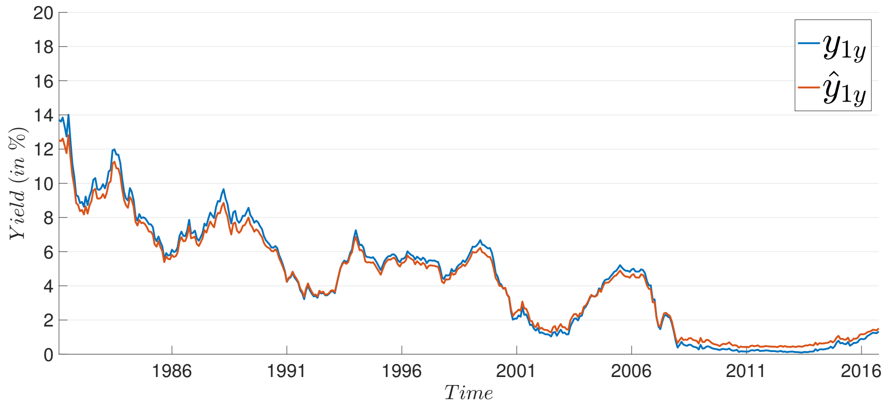


Figure 13: The estimated nominal 5-year nominal bond yields with the three-factor model, and the observed 5-year nominal bond yields over time

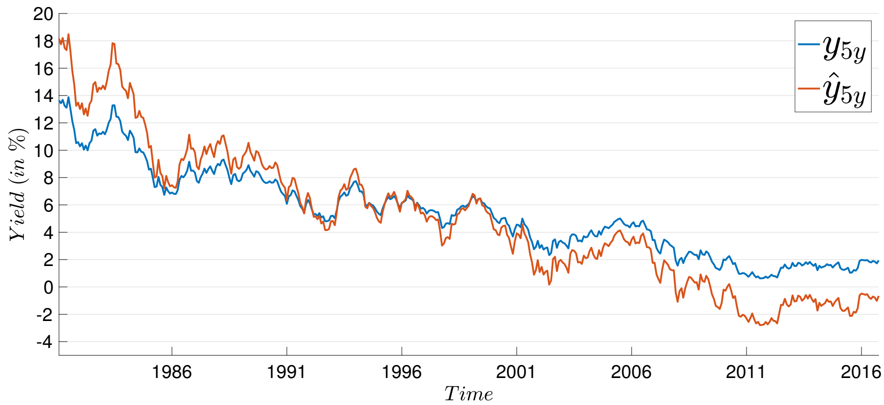


Figure 14: The simulated paths of the real asset returns for one scenario-set, for the instantaneous risk-free rates (r_f), the stock return (r_{St}), the bond yields with a 1-year maturity (r_{1y}) and the bond yield with a 5-year maturity (r_{5y}). The horizon is 75 years, since this is the duration or our model.

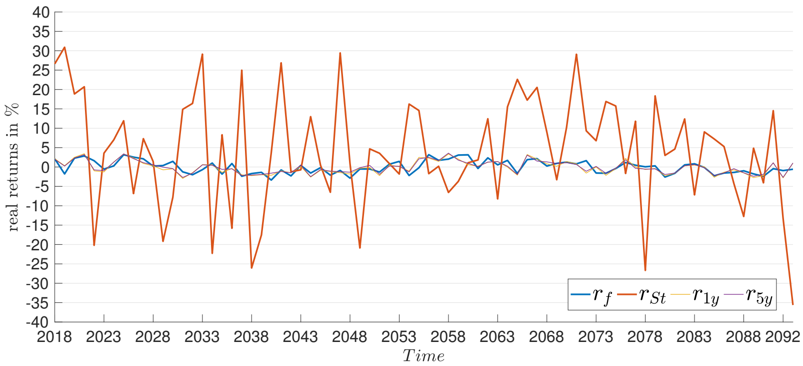


Figure 15: The loadings B of the three factors of the affine term structure model of Hamilton and Wu, for different maturities. The three factors in the HW-model can be interpreted as the level, slope and curvature of the yield curve from the Nelson-Siegel model.

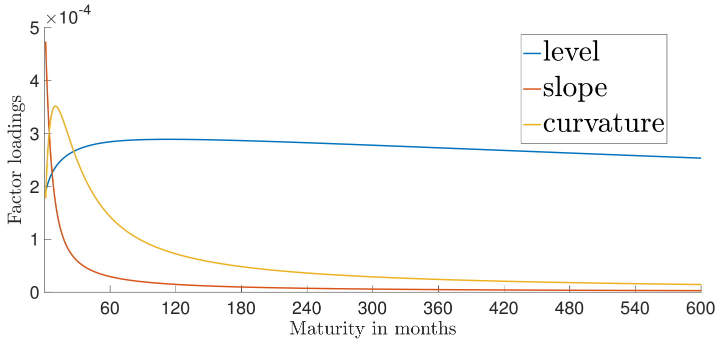


Table 3: The distribution of the simulated $N \times T$ real stock returns from our financial market.

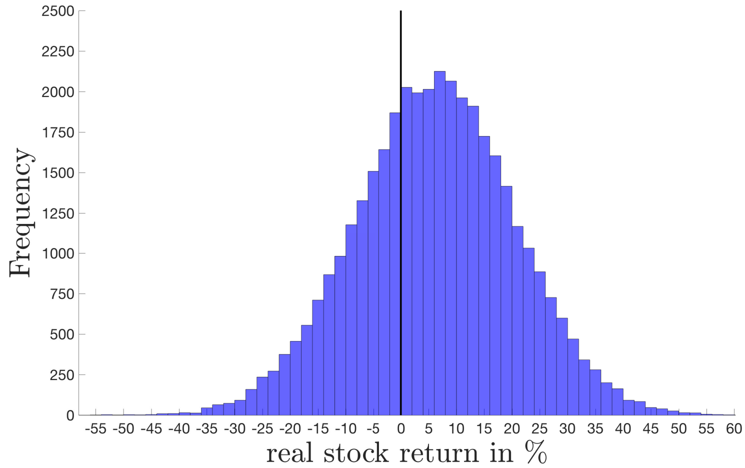


Table 10: The specific upper- and lower-bounds that lead to a different amount of times that the buffer affects the stock returns of the portfolios on average, based on the distribution of the real stock returns shown in the histogram in figure **??**. The '30%' bounds imply that 15% of the real stock returns are lower than $lb = -10.6\%$ and 15% of the real stock returns are higher than $ub = 19.9\%$. Consequently the buffer is used in 30% of the times with these specific bounds throughout our simulation study.

$\mathbb{P}$ (working buffer)	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
lower-bound	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9	4.9
upper-bound	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6	4.9

C Dynamic stochastic optimization procedure

We describe here the stochastic optimization procedure of Koijen et al. (2009) in more detail. We first note that the financial wealth of the individual investor is described by the process

$$w_{t+1} = (w_t - c_t)\left(x_t'(r_{t+1} - \iota r^f_{t+1}) + r^f_{t+1}\right) + y_{t+1}. \tag{C.1}$$

The investor has to meet the following constraints.

$$K(w_t) = \{(c_t, x_t) : c \leq w_t, \iota' x_t \leq 1, x_t \geq 0\}. \tag{C.2}$$

In this way the investor can not default, and he has borrowing and short-sales constraints.

The investor wants to maximize value functions by setting the optimal consumption and investment strategy in each period. The value function in the terminal period is given by

$$J_T(w_\tau, X_\tau, y_\tau) = \frac{\varphi w_\tau^{1-\gamma}}{1-\gamma}. \tag{C.3}$$

The value function for the other moments in time is given by

$$J_t(w_t, X_t, y_t) = \max_{(C_t, x_t)\in K_t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} + \beta\mathbb{E}_t\left(J_{t+1}(w_{t+1}, X_{t+1}, y_{t+1})\right)\right). \tag{C.4}$$

Reducing number of state variables

We first note that our problem is homogeneous in (c_t, w_t, y_t). [7] We see that the value function in (C.4) is homogeneous of order $(1-\gamma)$. The constraints on the asset weights and on consumption in equation (C.2) are homogeneous of order 0 or order 1. Because of these homogeneities we can reduce the number of variables. We recall that the income process is given by

$$y_t = \exp(g_t + v_t + \epsilon_t),$$

where we had that $v_{t+1} = v_t + u_{t+1}$. We can now, following Cocco et al. (2005) normalize our variables by the income process, because this does not affect the optimization over the control variables x_t and c_t. Furthermore, because of the homogeneity, we can also scale up our variables with $\exp(-v_t)$. This results in the following variables:

$$w_t^\nu = w_t\exp(-v_t),\ c_t^\nu = c_t\exp(-v_t),\ y_t^\nu = \exp(g_t + \epsilon_t). \tag{C.5}$$

Substituting these variables in our budget equation (C.1), gives us the new dynamic budget constraint equation:

$$w_{t+1}^\nu = (w_t^\nu - c_t^\nu)\left(x_t'(r_{t+1} - \iota r^f_{t+1}) + r^f_{t+1}\right)\exp(-u_{t+1}) + y_{t+1}^\nu. \tag{C.6}$$

[7] A function f is homogeneous of order k if $f(\alpha x, \alpha y) = \alpha^k f(x,y)$ for some $\alpha \in R$

Re-writing equation (C.4) finally gives us the Bellman equation reduced by one variable:

$$J_t^{\nu}(w_t, X_t) = \max_{(C_t, x_t)\in K_t} \left(\frac{c_t^{\nu 1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \left(\exp\left((1-\gamma)u_{t+1}\right) J_{t+1}^{\nu}(w_{t+1}^{\nu}, X_{t+1}) \right) \right), \qquad \text{(C.7)}$$

with the terminal condition

$$J_t^{\nu}(w_t, X_t) = \frac{\varphi w_{\tau}^{\nu 1-\gamma}}{1-\gamma}. \qquad \text{(C.8)}$$

By means of the normalization we have only two state variables left. We can recover the original state variables easily by multiplying the 'new' variables by $\exp(vt)$.

We want to optimize the Bellman equations in equation (C.7). The only tools we have are our control variables; the consumption c_t and the investment mix x_t for each time t. The problem we encounter is that we have to optimally choose our control variables now, for a result in the (far) future. The solution is backward-recursion.

Endogenous Grid-point Method (EGM)

In the optimization problem we have two types of variables. The first type consists of the exogenous variables, captured in X_t. These exogenous variables are the simulated asset returns and state variables. Further we have the endogenous variable w^{ν}. This variable is endogenous because it depends on decisions we have made until time t. We can deal with endogenous variables by constructing a grid for the values that this variable can attain.

First we create a new variable called 'wealth after consumption',

$$a_t = w_t - c_t.$$

For a_t we define a grid of M points. The grid is constructed in such a way that all the possible values that this variable can attain at time t are in the range of the grid. For every gridpoint we want to determine the asset allocation x_t. The advantage of this method is that we can determine c_t analytically, which saves a lot of computation time.

For the algorithm we use cross-sectional regressions of N simulated paths of the state variables of length T. We define the realized wealth of simulation i at time t as $\omega_{i,t}$. The realized state variables from the simulations are denoted as $X_{i,t}$ with $1 \leq i \leq M$ and $1 \leq t \leq \tau$. The wealth-after-consumption grid-points are denoted by a_j^{ν}, with $1 \leq j \leq M$. We have for each simulation τ points in time, and for each point in time for the specific simulation M grid-points for the possible values for wealth after consumption. This results in in $(\tau - 1) \times M \times N$ grid-points. We omit from now on the superscripts ν for notational convenience.

We specify the process recursively backwards by dynamic programming. The optimization procedure starts at moment $t = \tau$, and we want to optimize the problem at all the grid-points. The idea is that for values between these grid-points, we can interpolate to have optimal solutions.

Time τ: At terminal time T the investor retires. He wants to consume all wealth in this year, so $c_\tau = w_\tau$, and the value function equals

$$J_\tau(w_\tau, X_\tau) = \phi \frac{w_\tau^{1-\gamma}}{1-\gamma}. \tag{C.9}$$

Time $\tau - 1$: Here we have to make our first decision. We assume here that we are in a certain grid-point j, and we are in path i. We first substitute equation (C.9) in equation (C.7), which gives us the following optimization problem:

$$\max_{(c_{\tau-1}, x_{\tau-1}) \in K_{\tau-1}} \left(\frac{c_{\tau-1}^{1-\gamma}}{1-\gamma} + \beta\varphi \mathbb{E}_{\tau-1} \left[\frac{w_\tau^{1-\gamma}}{1-\gamma} \exp\left((1-\gamma)u_\tau\right) |X_{\tau-1} = X_{\tau-1}(\omega_{i,\tau-1}), a_{\tau-1} = a_{\tau-1}(j) \right] \right) \tag{C.10}$$

We first want to optimize (C.10) with respect to the asset allocation. We have to take into account that we have liquidity constraints that have to be satisfied by the asset allocation. The first order condition of (C.10) with respect to x_t reads as

$$\mathbb{E}_{T-1} \left[\beta\phi w_T^{-\gamma} \exp\left((1-\gamma)u_T\right)(r_T - r_T^f) | X_{T-1}, a_{t-1} \right] + \lambda - \mu\iota = 0. \tag{C.11}$$

Here, λ and μ are non-negative Kuhn-tucker multipliers corresponding to the Lagrange optimization. Now comes the trick. The conditional expectation $C.11$ can be approximated by a polynomial function of the asset allocation x_t. We have a vector of N simulated values of the term inside the conditional expectation at moment τ, and N simulated values of the state variables for moment $\tau - 1$. With this information we can do the following regression.

$$\mathbb{E}_{\tau-1}\left(\beta\phi w_\tau^{-\gamma} \exp\left((1-\gamma)u_\tau\right)\left(r_{\tau,s} - r_\tau^f\right) | X_{\tau-1}, a_{\tau-1}\right) \approx \theta_S(x, a_{\tau-1})' f(X_{\tau-1}). \tag{C.12}$$

The function $f(X_{\tau-1})$ is a simple linear function in the state variables. Note that θ_S is a function which depends on the asset weights x. We can estimate x by first constructing test-portfolios for x. We create h test-portfolios for $x_{\tau-1}$. Note that the function inside the conditional expectation is dependent of w_T, and this w_T depends on the asset weights we choose. By running h regressions for the different test-portfolios, we obtain h different functions $\theta_S(x_i, a_{t-1})$, for $i = 1, \ldots, h$. With these functions we can estimate Ψ in the following equation:

$$\theta_S(x_i, a_{\tau-1}) \approx g(x_i)' \Psi_S \text{ for } i = 1, \ldots, h. \tag{C.13}$$

Using this linearization in x we can now write equation (C.11) as

$$g(x)\Psi f(X_{\tau-1,i}) + \lambda - \mu\iota = 0. \tag{C.14}$$

where $g(x)$ is a simple linear polynomial function in the asset weights. Numerically solving this equation with respect to x gives us the optimal asset allocation at time $\tau - 1$ in path i and grid-point j. For each path this asset allocations will gives us a return on the portfolio $r_{\tau,i}^p$, and also a certain wealth $w_{\tau,i}^*$, given that we started in grid-point j. Using this information we can just take the first-order derivative of C.11 w.r.t. $c_{\tau-1}$, and solve for $c_{\tau-1}$. The optimal consumption that follows is given by

$$c^*_{\tau-1} = \left(\mathbb{E}_{\tau-1}(\beta\phi w_\tau^{*-\gamma} r^p_\tau | X_{\tau-1}, a_{\tau-1}\right)^{-\frac{1}{\gamma}}. \tag{C.15}$$

The last step is to construct the endogenous grid for the 'cash-on-hand' $w_{\tau-1}$, given by

$$w_{\tau-1}(\omega_{\tau-1,i}) = c^*(\omega_{\tau-1,i}, j) + a_{\tau-1}(j). \tag{C.16}$$

For every trajectory we have constructed an endogenous grid for the wealth after consumption. Consequently, for all the wealth levels we know the corresponding optimal investment strategies and the optimal consumption policies.

Time $t = \tau - 2, \cdots, 1$ We will now explain how we optimize the consumption choice and the investment policy for the other moments in time. Assume that we are at time t. We know the endogenous grid for wealth at time $t+1$, and the corresponding consumption policies, so we have the combinations $(w_{t+1,j}, c_{t+1,j}, x_{t+1,j})$ for $j = 1\ldots, M$. We want to optimize the following value function:

$$J_t(w_t, X_t) = \max_{(C_t, x_t)\in K_t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \beta\mathbb{E}_t\left(\exp((1-\gamma)u_{t+1})J_{t+1}(w_{t+1}, X_{t+1})\right)\right). \tag{C.17}$$

Taking the first order derivatives with respect to c_t and w_t, and using the chain rule gives the following first order conditions:

$$0 = \mathbb{E}_t\left(\beta\exp((1-\gamma)u_{t+1}\frac{\partial J_{t+1}(w_{t+1}, X_{t+1})}{\partial w_t}(r_{t+1} - r^f_t)\right) + \lambda - \mu\iota \tag{C.18}$$

$$c_t^{-\gamma} = \mathbb{E}_t\left(\beta\exp((1-\gamma)u_{t+1}\frac{\partial J_{t+1}(w_{t+1}, X_{t+1})}{\partial w_t}r^p_{t+1}\right) \tag{C.19}$$

The Kuhn-Tucker multipliers λ and μ have to satisfy

$$\lambda_s x_{st} = 0 \ (s = 1, \cdots, n) \text{ and } \mu(x'\iota - 1) = 0.$$

with n the amount of risky assets. When we take the partial derivative of C.17 w.r.t to w_t we obtain

$$\frac{\partial J_t(w_t, X_t)}{\partial w_t} = \mathbb{E}_t\left(\beta\exp((1-\gamma)u_{t+1}\frac{\partial J_{t+1}(w_{t+1}, X_{t+1})}{\partial w_t}(r^p_t)\right), \tag{C.20}$$

so we can see that

$$c^{*-\gamma} = \frac{\partial J_t(w_t, X_t)}{\partial w_t}. \tag{C.21}$$

The first order conditions in equations (C.18) and (C.19) then read

$$0 = \mathbb{E}_t\left(\beta\exp((1-\gamma)u_{t+1})c^{-\gamma}_{t+1}(r_{t+1} - r^f_{t+1})\right) + \lambda - \mu\iota \tag{C.22}$$

$$c_t^{-\gamma} = \mathbb{E}_t\left(\beta\exp((1-\gamma)u_{t+1})c^{-\gamma}_{t+1}r^p_{t+1}\right). \tag{C.23}$$

In equation (C.22) we determine the optimal investment policy in the same way as we did in period $\tau - 1$: we are in a certain grid-point j, and try to linearize the conditional expectation in x_t, by cross sectional regressions. Therefore, we make test portfolios for x_t. For a fixed test portfolio, we obtain corresponding wealth levels one period later for all the N paths, so w_{t+1}. With these wealth levels we can interpolate the corresponding consumption levels c_{t+1}. When we have determined the optimal consumption levels at $t+1$ we can solve equation (C.23) for the optimal x_t. With the optimal x_t we can calculate the return of the portfolio r^p_{t+1} one period later, and substitute this in equation (C.23). Again, we can determine in this way the consumption policy analytically. The optimal consumption equals

$$c_t^* = \left(\mathbb{E}_t\left(\beta \text{exp}((1-\gamma)u_{t+1})c_{t+1}^{-\gamma} r_{t+1}^p\right)\right)^{-\frac{1}{\gamma}}. \tag{C.24}$$

Again, we complete our analysis by calculating the endogenous 'cash-on-hand':

$$w_t(\omega_{it}, j) = c_t^*(\omega_{i,t}, j) + a_t(j). \tag{C.25}$$

By defining the grid, we have to take into account that we can not consume more then our wealth. As a consequence, we always include $a_t(1) = 0$ as the smallest grid-point. Further we note that the wealth and consumption have a linear relationship, except for very small values of wealth. Therefore, we follow Carroll (2006) and choose a triple-exponential grid. With this specification of the grid, the density of the grid-points near zero is much higher.

After we have constructed the grids, we can run forward from $t = 0$ to $t = \tau$ through the grids with linear interpolation, in order to determine the optimal consumption and investment policies for the investor. We have outlined this procedure section 6.

D Life-cycle asset allocations

Figure 16: The asset allocations for the investor with different levels of risk-aversion, with a fixed pension contract.

Figure 16A: Risk-seeking investor ($\gamma = 2$)

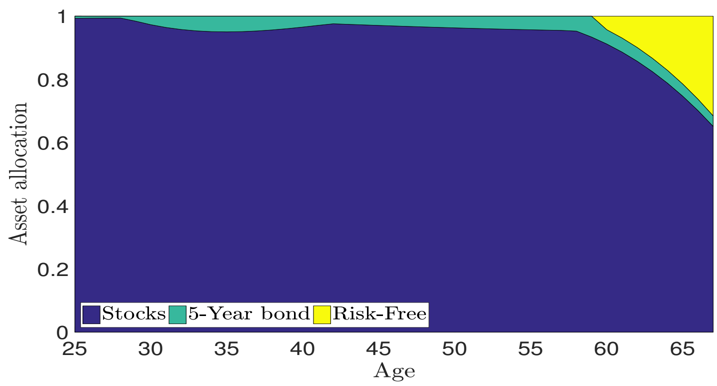


Figure 16B: Risk-neutral investor ($\gamma = 5$)

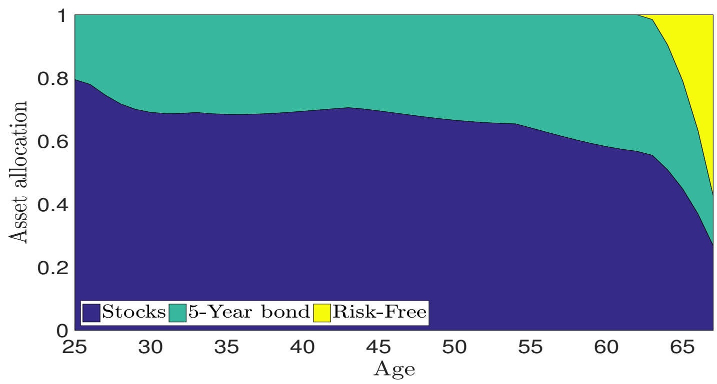


Figure 16C: Risk-averse investor ($\gamma = 10$)

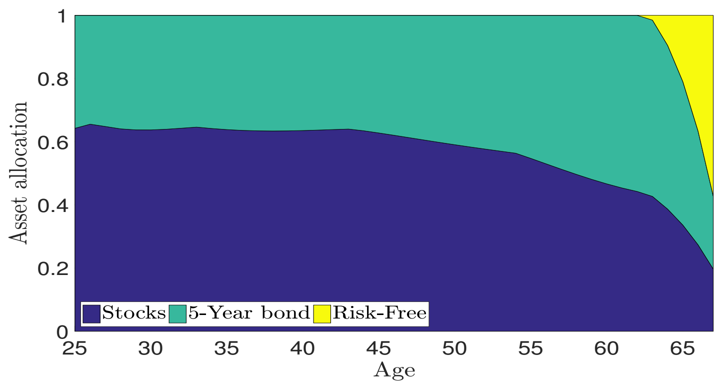


Figure 17: The asset allocations for the investor with different levels of risk-aversion, with a variable pension contract.

Figure 17A: Risk-seeking investor with $\gamma = 2$

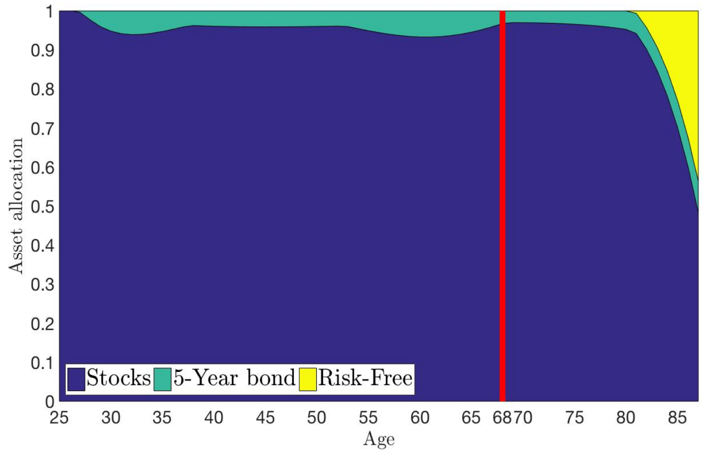


Figure 17B: Risk-neutral investor with $\gamma = 5$

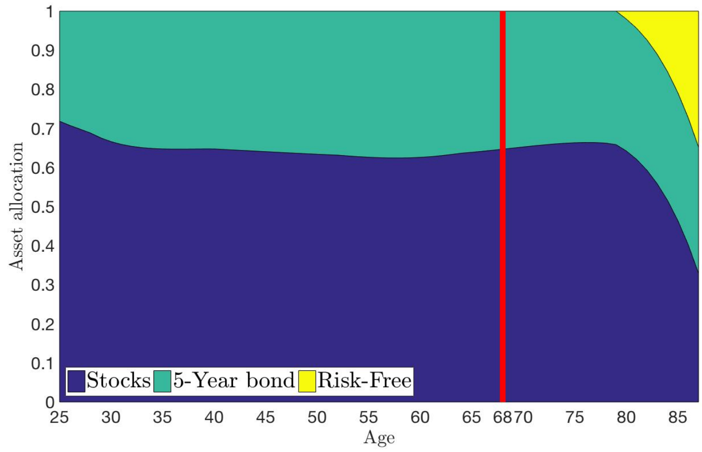
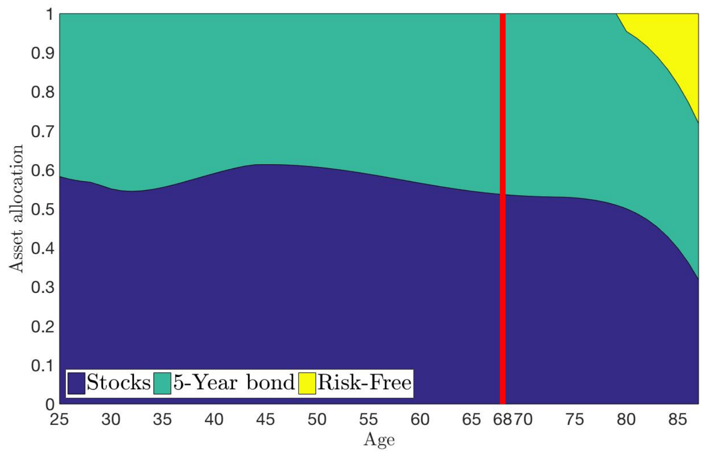


Figure 17C: Risk-averse investor with $\gamma = 10$



E Pension results

Table 6: The relevant terminology for the description of the result and the meaning of this terminology that we use in describing the results of the simulation study

Definition	meaning
Ag	The fixed pension contracts for the aggressive (Ag), risk-neutral (Ne) and risk-averse investor (Av)
Ag-Ag	The variable pension contract for an investor with aggressive (Ag) investment until retirement, and Aggressive investments after retirement. The same holds for other combinations of aggressive (Ag), risk-neutral (Ne) and averse (Av).
.	.
Distribution boundaries	The upper-boundaries ub and lower-boundaries lb that we specified in table 7, based on the distribution of the real stock returns that we have derived. In this way we know how many times the buffer is used in average, because the real stock returns are outside the boundaries. For example: 10% means that 5% of all the simulated real stock returns are below the corresponding lower-bound in table 7 and 5% above the upper-bound.
Alternative boundaries	The set of upper-and lower-bounds we derived by combining three different upper-bounds with three different lower-bounds. For example: $[-1, +11\%]$ means that we have $lb = -1\%$ and $ub = +11\%$
P5	The 5^{th} percentile of the certainty equivalents of a pension contract. This represents how well the pension scheme performs under bad circumstances. We try to find a contract with a favorable P5, as the pensions need to be stable. We denote the other percentiles similar to P5: For example, the 95^{th} percentile is P95, etc.
Π_{max}	The maximum percentage of the total accrued pension capital that the buffer may contain.
buffer full	Buffer empty represents the time average of time that the buffer is switched off, because there is either no capital in the buffer ($CB = 0$). Buffer full represents the average time that there is too much capital in the buffer, so CB equals Π_{max} of the total accrued pension capital of the active participants in the fund.
NB	An IDC-contract, so we have no buffer in this contract (NB)

E.1 Results based on distribution boundaries

Tables pension outcomes

Table 11: The pension results for the fixed and variable contracts for the investor who invests aggressive ($\gamma = 2$) until retirement age with the 'distribution' boundaries.

Investment strategy	buffer active	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
	lb	-	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9
	ub	-	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6
Aggressive	P5	0.68	0.69	0.65	0.66	0.70	0.70	0.67	0.71	0.71	0.78
	P50	1.79	1.77	1.72	1.67	1.56	1.56	1.50	1.46	1.52	1.61
	P95	4.56	4.01	3.63	3.36	3.19	2.93	2.76	2.82	2.76	3.09
	buffer empty		56%	51%	49%	47%	45%	41%	38%	33%	26%
	buffer full		0%	1%	1%	1%	3%	4%	6%	9%	14%
	.										
Aggressive-Aggressive	P5	0.61	0.64	0.64	0.68	0.67	0.69	0.70	0.69	0.75	0.89
	P50	2.50	2.45	2.46	2.31	2.18	2.18	2.06	2.01	2.01	2.44
	P95	8.15	7.49	6.65	5.85	5.53	5.02	4.95	4.89	5.13	5.92
	buffer empty		43%	44%	45%	44%	42%	38%	35%	31%	24%
	buffer full		1%	1%	2%	3%	4%	6%	10%	13%	19%
	.										
Aggressive-Neutral	P5	0.63	0.63	0.63	0.63	0.67	0.68	0.65	0.72	0.79	0.94
	P50	2.31	2.29	2.23	2.20	2.06	2.02	1.96	1.93	1.92	2.33
	P95	7.70	6.75	6.19	5.28	4.82	4.62	4.17	4.31	4.62	5.14
	buffer empty		43%	44%	44%	44%	42%	38%	34%	29%	22%
	buffer full		0%	1%	1%	2%	3%	4%	7%	11%	17%
	.										
Aggressive- Averse	P5	0.61	0.61	0.61	0.61	0.64	0.68	0.65	0.73	0.76	0.89
	P50	2.26	2.23	2.26	2.11	2.00	1.96	1.88	1.90	1.90	2.23
	P95	7.35	6.21	5.74	5.07	4.76	4.42	4.03	4.07	4.18	4.94
	buffer empty		43%	44%	44%	44%	42%	37%	34%	29%	22%
	buffer full		0%	0%	1%	1%	2%	4%	6%	10%	16%

Table 12: The pension results for the fixed and variable contracts for the investor who invests risk-neutral ($\gamma = 5$) until retirement age. After retirement age the investor has several investment strategies.

Investment strategy	time active	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
	lb	-	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9
	ub	-	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6
Neutral	P5	0.69	0.69	0.69	0.66	0.69	0.70	0.65	0.66	0.67	0.73
	P50	1.38	1.35	1.34	1.31	1.27	1.25	1.22	1.24	1.22	1.28
	P95	2.77	2.65	2.49	2.24	2.15	2.05	2.04	1.91	1.95	2.02
	buffer empty		56%	51%	48%	46%	44%	40%	36%	30%	23%
	buffer full		0%	0%	0%	0%	1%	1%	2%	4%	8%
	.										
Neutral-Agressive	P5	0.51	0.48	0.55	0.57	0.63	0.65	0.63	0.62	0.65	0.69
	P50	1.80	1.85	1.84	1.76	1.71	1.65	1.62	1.58	1.61	1.84
	P95	4.67	4.50	3.98	3.87	3.47	3.32	3.18	3.03	3.16	3.58
	buffer empty		43%	44%	44%	44%	41%	37%	34%	29%	22%
	buffer full		0%	1%	1%	2%	3%	4%	6%	10%	16%
	.										
Neutral-Neutral	P5	0.53	0.59	0.61	0.60	0.60	0.62	0.61	0.61	0.65	0.74
	P50	1.78	1.83	1.72	1.72	1.63	1.62	1.56	1.53	1.59	1.69
	P95	4.22	3.74	3.89	3.71	3.41	3.17	3.09	2.84	3.08	3.34
	buffer empty		43%	44%	44%	43%	41%	37%	33%	27%	20%
	buffer full		0%	0%	0%	1%	2%	2%	4%	7%	14%
	.										
Neutral- Averse	P5	0.52	0.54	0.57	0.58	0.60	0.56	0.61	0.61	0.65	0.70
	P50	1.74	1.70	1.70	1.64	1.58	1.55	1.51	1.50	1.54	1.68
	P95	4.19	3.90	3.73	3.45	3.24	3.15	3.08	2.98	3.01	3.14
	buffer empty		43%	44%	44%	43%	41%	36%	32%	27%	19%
	buffer full		0%	0%	0%	1%	1%	2%	4%	6%	13%

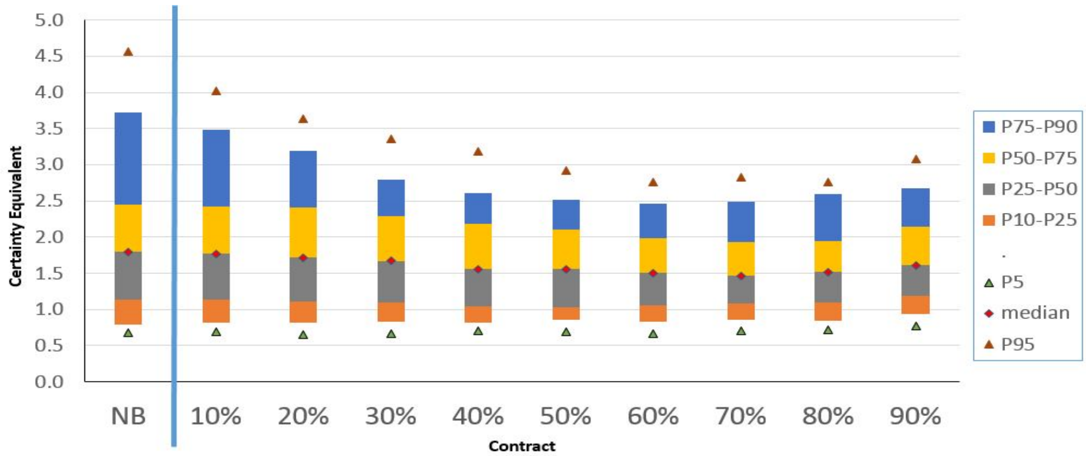
Table 13: The pension results of the pension results of the risk-averse ($\gamma = 10$) investor, with a fixed contract ('Averse') and a variable contract ('Averse-Agressive', 'Averse-Neutral' and 'Averse-Averse')

Investment strategy	time active	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
	lb	-	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9
	ub	-	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6
Averse	P5	0.66	0.66	0.66	0.65	0.67	0.67	0.66	0.65	0.65	0.72
	P50	1.27	1.26	1.24	1.21	1.20	1.18	1.16	1.15	1.18	1.19
	P95	2.54	2.40	2.24	2.09	2.01	1.90	1.91	1.85	1.80	1.86
	buffer empty		56%	50%	48%	46%	44%	40%	35%	29%	22%
	buffer full		0%	0%	0%	0%	1%	1%	2%	3%	7%
	.										
Averse-Agressive	P5	0.52	0.53	0.56	0.53	0.51	0.32	0.46	0.51	0.59	0.68
	P50	1.68	1.57	1.70	1.64	1.49	1.51	1.49	1.43	1.48	1.62
	P95	4.31	4.05	3.81	3.65	3.56	2.93	3.10	3.07	3.04	3.39
	buffer empty		43%	44%	44%	44%	41%	37%	34%	28%	21%
	buffer full		0%	0%	1%	1%	2%	4%	6%	9%	15%
	.										
Averse-Neutral	P5	0.53	0.51	0.53	0.53	0.55	0.58	0.62	0.58	0.68	0.67
	P50	1.62	1.57	1.59	1.50	1.44	1.43	1.40	1.42	1.42	1.54
	P95	3.80	3.56	3.39	3.23	3.12	3.08	2.90	2.71	2.80	2.82
	buffer empty		43%	44%	44%	43%	41%	36%	33%	27%	19%
	buffer full		0%	0%	0%	1%	1%	2%	4%	7%	13%
Averse- Averse	P5	0.47	0.52	0.55	0.57	0.54	0.57	0.48	0.58	0.60	0.62
	P50	1.57	1.56	1.55	1.56	1.46	1.43	1.34	1.40	1.42	1.56
	P95	3.56	3.47	3.52	3.16	2.87	2.90	2.56	2.55	2.68	2.82
	buffer empty		43%	44%	44%	43%	41%	36%	32%	26%	18%
	buffer full	0%	0%	0%	0%		1%	2%	3%	6%	11%

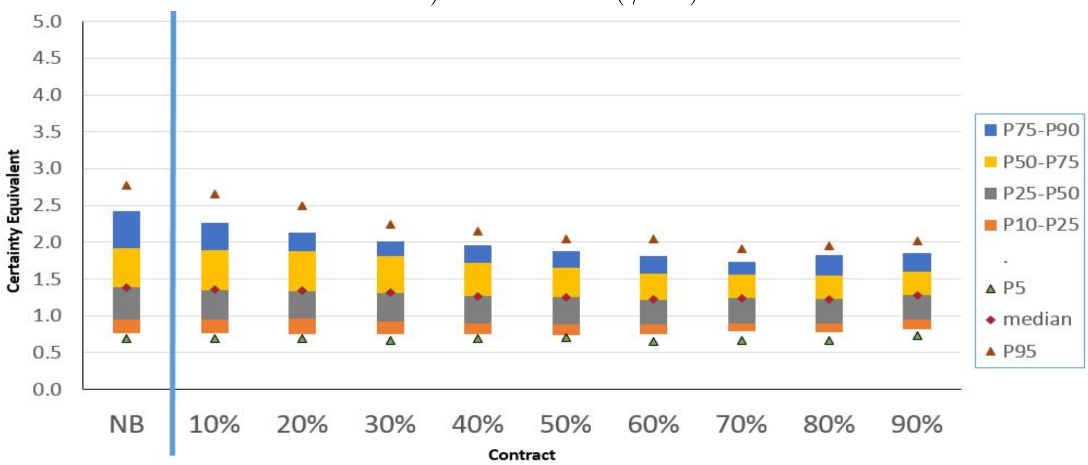
Distribution pension outcomes

Figure 18: Distributions of the certainty equivalents in the case of a fixed pension, for the different investment styles

18A) Fixed aggressive ($\gamma = 2$).



18B) Fixed neutral ($\gamma = 5$).



18C) Fixed averse ($\gamma = 10$) .

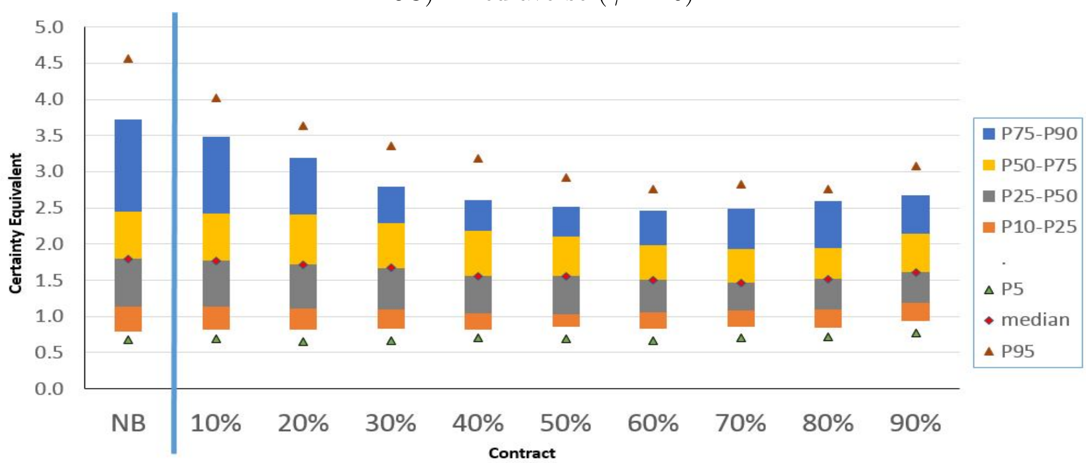
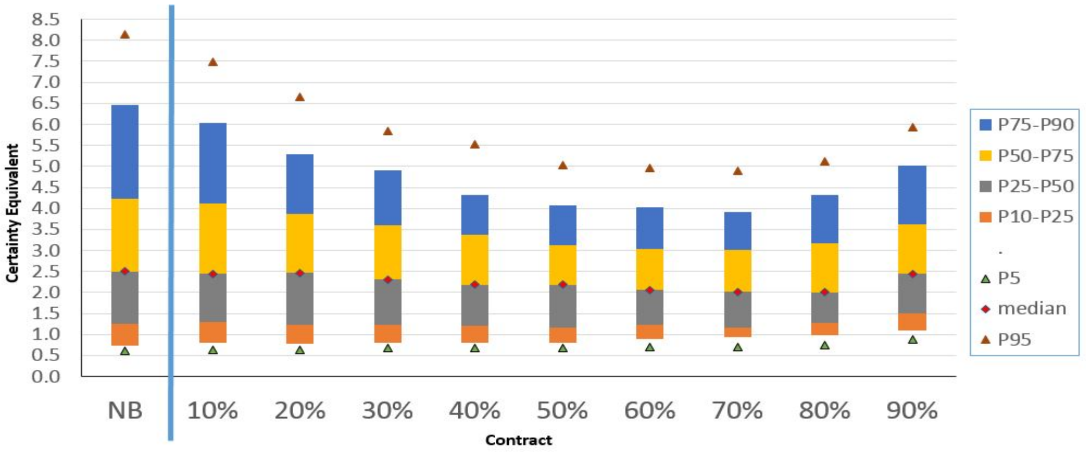
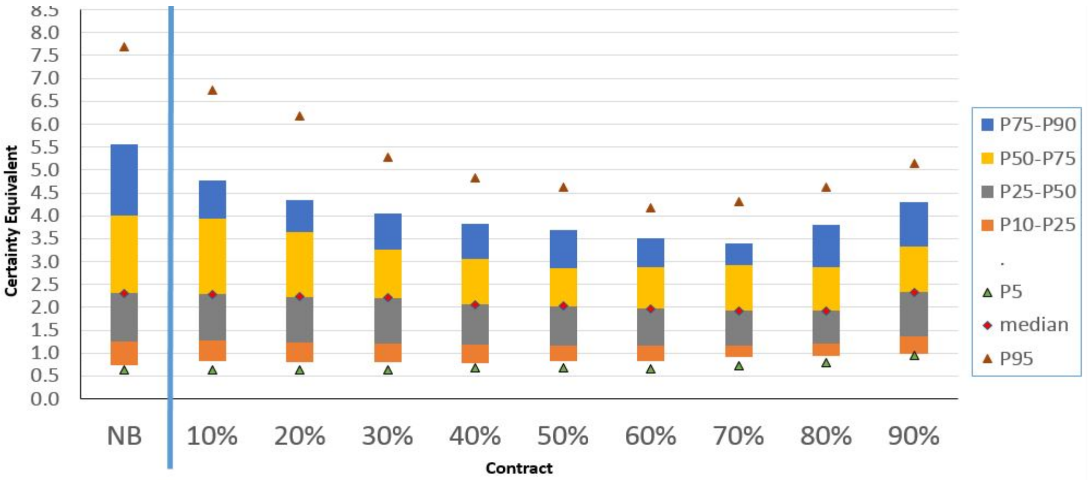


Figure 19: Distributions of certainty equivalents for the aggressive (Ag) investor until retirement with a variable pension.

19A) Variable pension with aggressive investing after retirement (Ag-Ag).



19B) Variable pension with risk-neutral investing after retirement (Ag-Ne).



19C) Variable pension with risk-averse investing after retirement (Ag-Av).

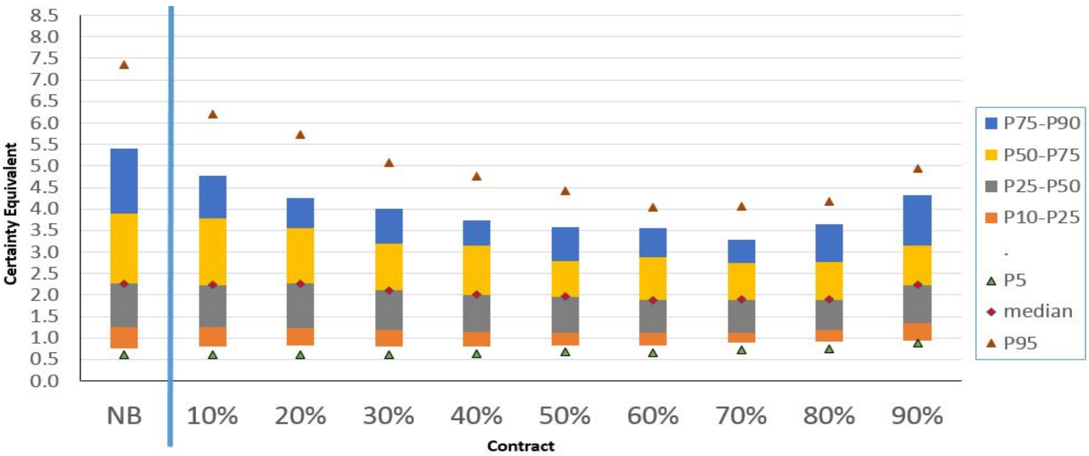
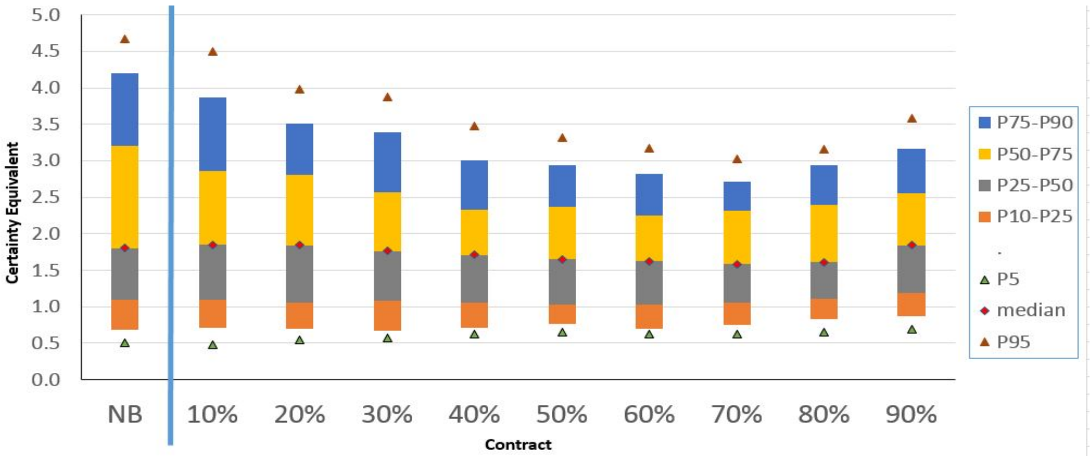
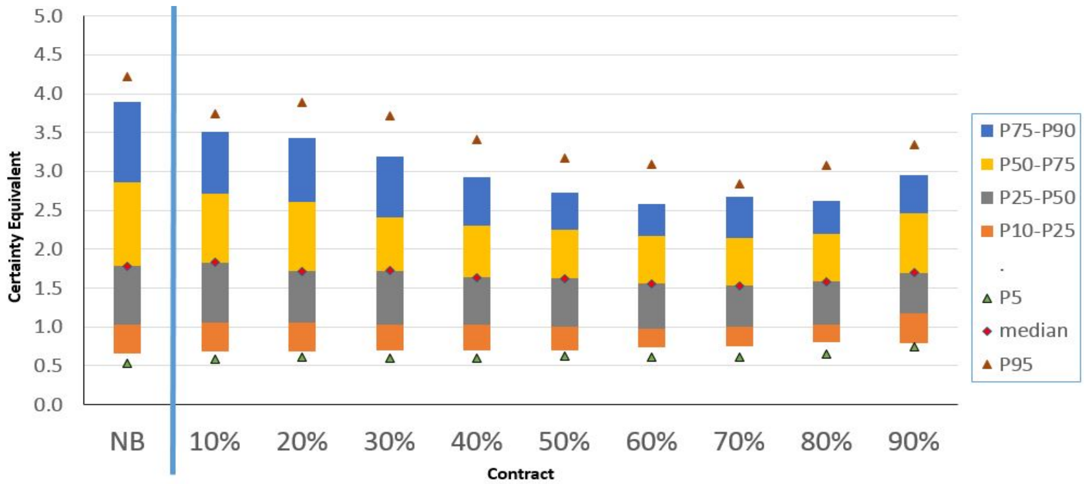


Figure 20: Distributions of certainty equivalents with a variable pension contract for the risk-neutral (Ne) investor until retirement with a variable pension.

20A) Variable pension with aggressive investing after retirement (Ne-Ag).



20B) Variable pension with risk-neutral investing after retirement (Ne-Ne).



20C) Variable pension with risk-averse investing after retirement (Ne-Av).

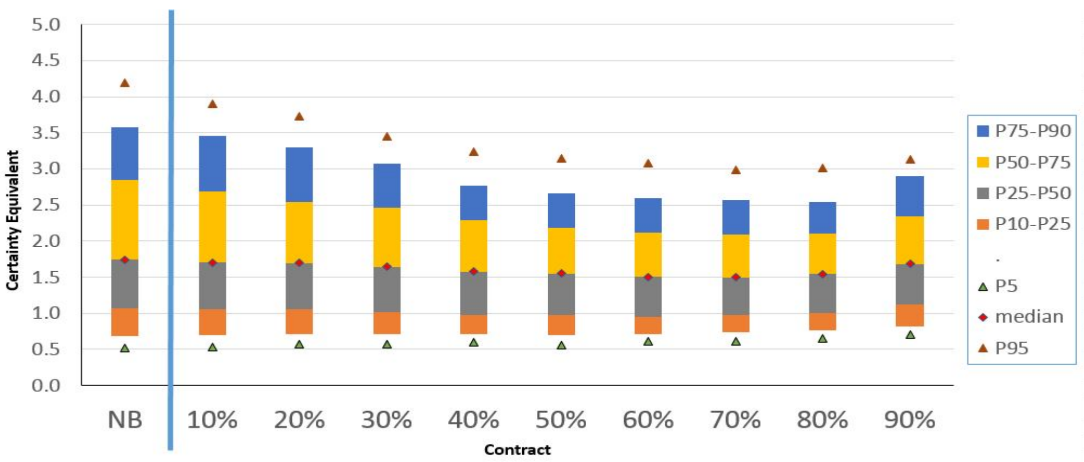
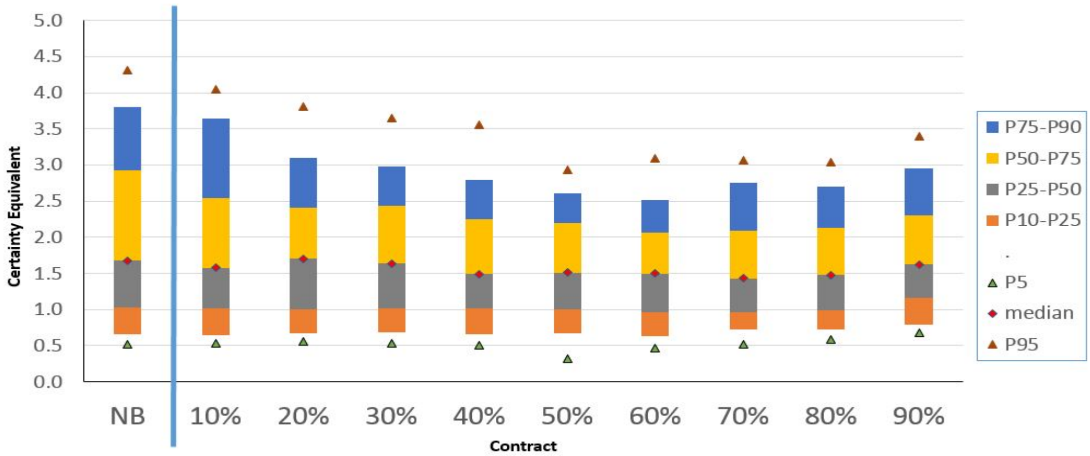
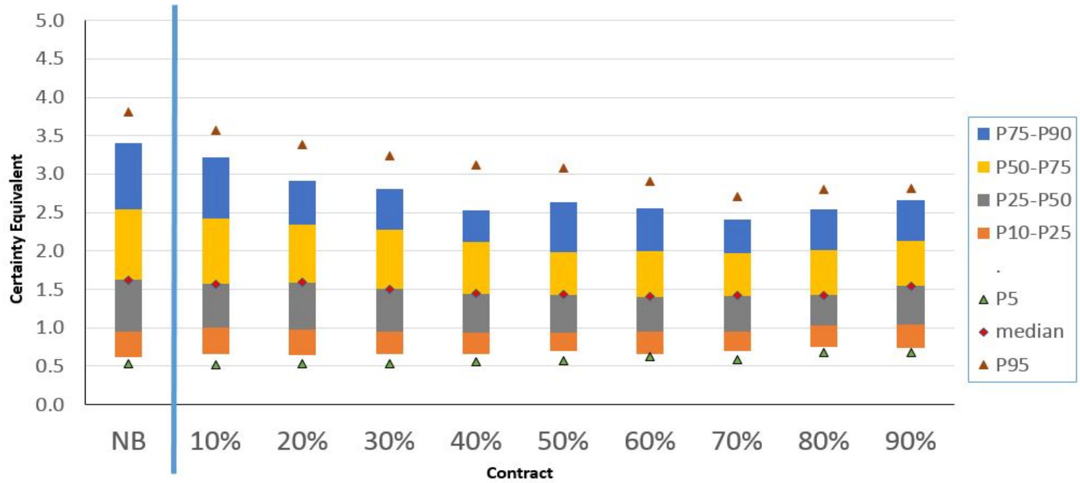


Figure 21: Distributions of certainty equivalents with a variable pension contract for the risk-averse (Av) investor until retirement with a variable pension.

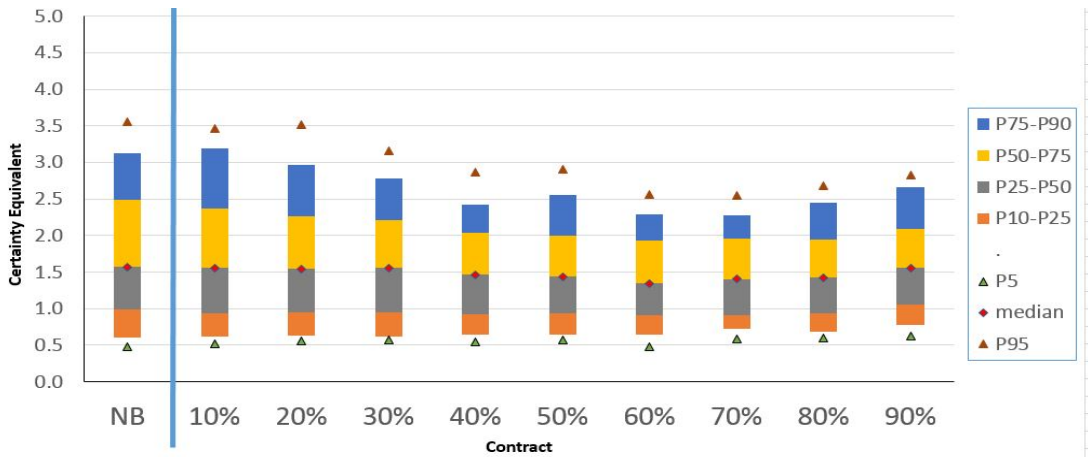
21A) Variable pension with aggressive investing after retirement (Av-Ag).



21B) Variable pension with risk-neutral investing after retirement (Av-Ne).



21C) Variable pension with risk-averse investing after retirement (Av-Av).



Buffer full or empty

Figure 22: Stacked charts of average part of times that the buffer is empty (red) and the part of the time that the buffer is full (green) for the aggressive (Ag) investor until retirement, with a fixed pension, and for the variable pensions with aggressive investing until retirement (Ag-Ag, Ag-Ne and Ag-Av).

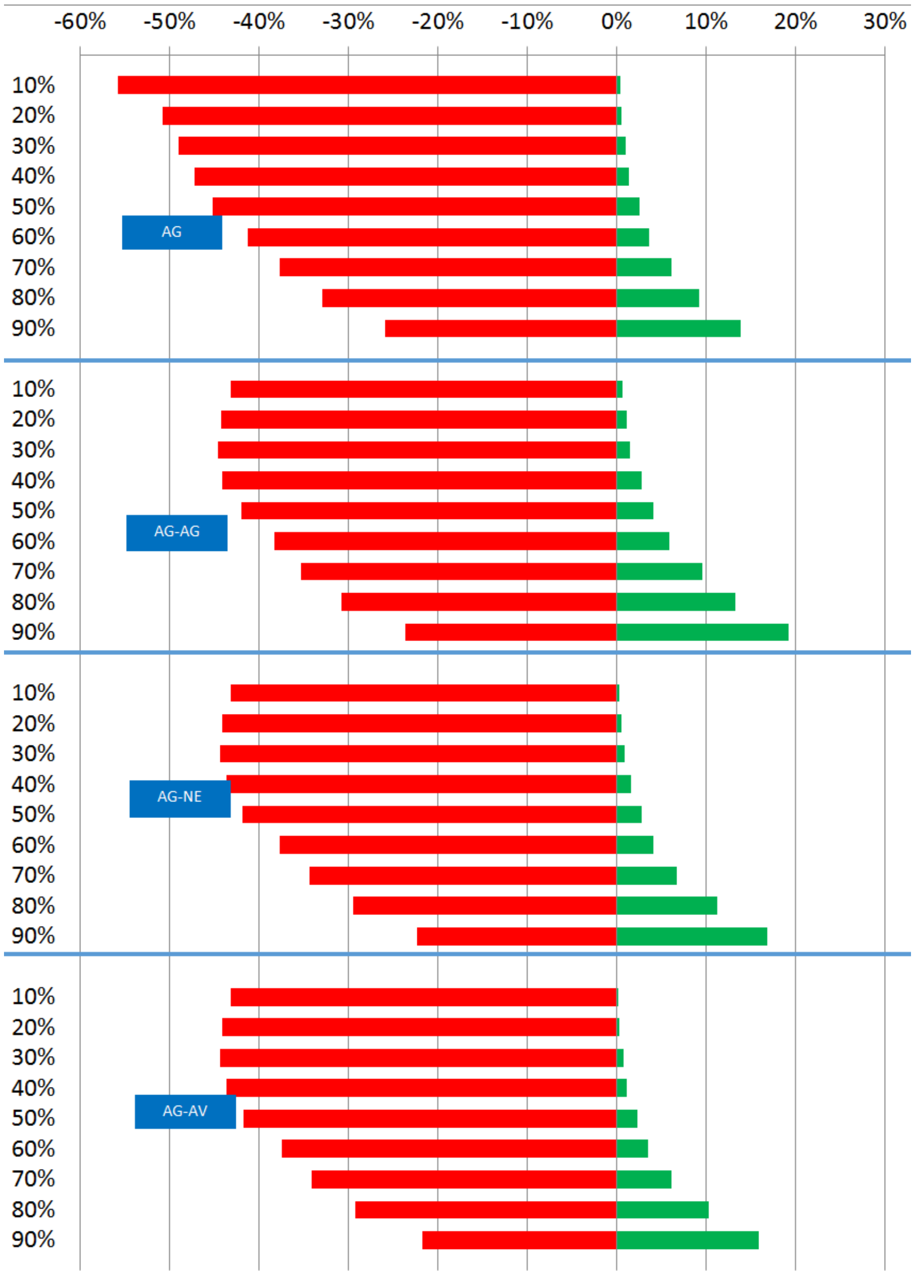


Figure 23: Stacked charts of the average percentage of times that the buffer is empty (red) and the part of the time that the buffer is full (green) for the risk-neutral (Ne) investor until retirement with a fixed pension, and for the variable pensions with risk-neutral investing until retirement (Ne-Ag, Ne-Ne and Ne-Av).

Figure 24: Stacked charts of the part of the time that the buffer is empty (red) and the part of the time that the buffer is full (green) for the averse (Av) investor until retirement with a fixed pension, and for the variable pensions with aggressive investing until retirement (Av-Ag, Av-Ne and Av-Av).

Varying maximum amount of capital allowed in buffer

Table 14: The pension results for a varying buffer size (Π_{max} equals respectively 10%, 20% and 30% of the total accrued pension capital in the fund). Here we have an Aggressive-Aggressive investor

Π_{max}	time active	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
	lb	-	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9
	ub	-	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6
10%	P5	0.58	0.58	0.63	0.62	0.63	0.67	0.67	0.68	0.72	0.87
	P50	2.35	2.30	2.32	2.26	2.27	2.22	2.25	2.18	2.36	2.63
	P95	8.06	7.21	6.73	6.40	6.49	6.08	5.67	5.81	6.12	7.10
	buffer empty		56%	51%	49%	49%	48%	47%	44%	41%	35%
	buffer full		8%	13%	17%	20%	23%	25%	28%	31%	36%
20%	P5	0.58	0.58	0.63	0.62	0.61	0.67	0.66	0.67	0.71	0.86
	P50	2.35	2.29	2.32	2.24	2.13	2.10	2.07	2.02	2.13	2.42
	P95	8.06	6.91	6.39	6.01	5.13	5.03	4.87	4.92	4.92	5.73
	buffer empty		56%	51%	49%	49%	47%	44%	41%	37%	30%
	buffer full		2%	4%	5%	7%	8%	12%	14%	18%	23%
30%	P5	0.58	0.58	0.63	0.62	0.61	0.67	0.65	0.66	0.70	0.86
	P50	2.35	2.29	2.32	2.23	2.13	2.10	1.95	1.94	1.97	2.33
	P95	8.06	6.91	6.39	5.68	4.95	4.89	4.50	4.44	4.60	5.18
	buffer empty		56%	51%	49%	48%	46%	42%	39%	34%	27%
	buffer full		1%	1%	1%	2%	4%	5%	7%	10%	15%

Table 15: The pension results for a varying buffer size (Π_{max} respectively 10%, 20% and 30% of the total accrued pension capital in the fund). Here we have an Aggressive-Neutral investor.

Π_{max}	time active	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
	lb	-	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9
	ub	-	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6
10%	P5	0.63	0.63	0.62	0.63	0.66	0.70	0.71	0.73	0.71	0.82
	P50	2.29	2.23	2.23	2.18	2.06	2.02	2.10	2.06	2.21	2.42
	P95	7.34	6.36	5.93	5.96	5.68	5.51	5.30	5.38	5.56	6.08
	buffer empty		56%	51%	49%	49%	48%	46%	43%	40%	34%
	buffer full		7%	11%	14%	17%	20%	23%	26%	29%	34%
20%	P5	0.63	0.63	0.62	0.63	0.64	0.68	0.65	0.72	0.76	0.80
	P50	2.29	2.23	2.20	2.11	2.04	1.96	1.90	1.91	1.93	2.20
	P95	7.34	6.36	5.73	5.34	5.06	4.74	4.60	4.54	4.51	5.19
	buffer empty		56%	51%	49%	48%	46%	43%	40%	36%	29%
	buffer full		1%	2%	4%	5%	7%	10%	12%	15%	21%
30%	P5	0.63	0.63	0.62	0.63	0.64	0.68	0.63	0.71	0.75	0.88
	P50	2.29	2.23	2.20	2.08	2.03	1.96	1.88	1.85	1.87	2.17
	P95	7.34	6.36	5.64	5.22	4.68	4.49	4.07	4.15	4.13	4.69
	buffer empty		56%	51%	49%	47%	45%	42%	38%	33%	26%
	buffer full		0%	0%	1%	1%	2%	3%	6%	9%	14%
	14%										

Table 16: The pension results for a varying buffer size (Π_{max} equals respectively 10%, 20% and 30% of the total accrued pension capital in the fund). Here we have an Aggressive-Averse investor.

Π_{max}	time active	0%	10%	20%	30%	40%	50%	60%	70%	80%	90%
	lb	-	-19.7	-14.2	-10.6	-7.6	-5.1	-2.9	-0.8	1.1	2.9
	ub	-	28.8	23.4	19.9	17.1	14.7	12.5	10.5	8.5	6.6
10%	P5	0.61	0.60	0.60	0.61	0.63	0.70	0.64	0.73	0.72	0.83
	P50	2.26	2.20	2.20	2.09	2.00	2.01	2.04	2.03	2.09	2.34
	P95	7.08	6.19	5.94	5.54	5.64	5.16	5.19	5.21	5.21	5.83
	buffer empty		56%	51%	49%	49%	48%	46%	43%	39%	33%
	buffer full		6%	10%	13%	17%	19%	22%	24%	28%	33%
20%	P5	0.61	0.60	0.60	0.61	0.63	0.67	0.62	0.71	0.73	0.80
	P50	2.26	2.20	2.17	2.07	1.99	1.93	1.88	1.89	1.93	2.16
	P95	7.08	6.14	5.47	5.07	4.87	4.57	4.36	4.15	4.38	4.73
	buffer empty		56%	51%	49%	48%	46%	43%	39%	35%	29%
	buffer full		1%	2%	3%	4%	6%	8%	12%	15%	19%
30%	P5	0.61	0.60	0.60	0.61	0.63	0.68	0.62	0.71	0.72	0.87
	P50	2.26	2.20	2.17	2.02	1.96	1.91	1.87	1.83	1.83	2.12
	P95	7.08	6.14	5.45	4.91	4.60	4.42	3.89	3.95	3.93	4.45
	buffer empty		56%	51%	49%	47%	45%	41%	38%	33%	26%
	buffer full		0%	0%	1%	1%	2%	3%	5%	8%	13%

E.2 Pension results alternative boundaries

Tables pension outcomes

Table 17: In this section we reply the analysis we did in appendix E1, except that we combine the upper-bounds +8%, +11%, and +14 with the lower-bounds −4%, −1% and +1% to examine what the effect is of not knowing exactly the distribution of the real stock returns, and consequently setting unfavorable bounds.

Investment strategy	*ub*	+8%			+11%			+14%			No buffer
	lb	-4%	-1%	+2%	-4%	-1%	+2%	-4%	-1%	+2%	
Aggressive	P5	0.62	0.65	0.71	0.64	0.70	0.72	0.67	0.71	0.74	0.68
	P50	1.24	1.33	1.53	1.35	1.44	1.59	1.52	1.62	1.76	1.79
	P95	2.40	2.65	2.88	2.59	2.80	3.00	2.92	3.05	3.18	4.56
	buffer empty	22%	26%	29%	33%	37%	42%	43%	49%	53%	
	buffer full	17%	14%	11%	8%	6%	5%	3%	2%	1%	
	.										
Aggressive-Aggressive	P5	0.60	0.69	0.83	0.63	0.74	0.76	0.69	0.77	0.78	0.58
	P50	1.61	1.82	2.21	1.75	1.94	2.27	2.06	2.18	2.57	2.50
	P95	4.44	4.82	5.39	4.34	4.77	5.13	4.86	5.42	5.77	8.15
	buffer empty	20%	24%	27%	30%	35%	39%	40%	46%	51%	
	buffer full	23%	18%	16%	13%	10%	7%	5%	3%	2%	
	.										
Aggressive-Neutral	P5	0.64	0.66	0.78	0.67	0.70	0.80	0.66	0.71	0.65	0.64
	P50	1.51	1.74	2.04	1.71	1.89	2.10	1.93	2.04	2.32	2.29
	P95	3.95	4.06	4.47	3.90	4.22	4.48	4.44	4.97	5.22	7.45
	buffer empty	18%	22%	26%	29%	34%	38%	39%	46%	51%	
	buffer full	21%	16%	14%	10%	7%	5%	4%	2%	1%	
	.										
Aggressive- Averse	P5	0.65	0.64	0.78	0.62	0.71	0.77	0.68	0.69	0.68	0.61
	P50	1.46	1.72	1.94	1.69	1.91	2.08	1.93	2.01	2.26	2.26
	P95	3.60	3.91	4.44	3.67	4.11	4.31	4.23	4.40	5.13	7.19
	buffer empty	18%	22%	25%	28%	34%	38%	40%	46%	51%	
	buffer full	19%	16%	13%	9%	6%	5%	3%	2%	1%	

Table 18: The distribution of the pension results of the risk-neutral ($\gamma = 5$) investor, with a fixed contract ('Neutral') and a variable contract ('Neutral-Agressive', 'Neutral-Neutral' and 'Neutral-Averse')

Investment strategy	*ub*	+8%			+11%			+14%			No buffer
	lb	-4%	-1%	+2%	-4%	-1%	+2%	-4%	-1%	+2%	
Neutral	P5	0,63	0,68	0,67	0,65	0,65	0,68	0,69	0,66	0,69	0,69
	P50	1,04	1,13	1,22	1,14	1,20	1,28	1,23	1,30	1,34	1,38
	P95	1,68	1,84	1,96	1,80	1,90	2,08	2,02	2,16	2,27	2,77
	buffer empty	19%	23%	26%	30%	35%	40%	42%	48%	52%	
	buffer full	12%	8%	7%	4%	3%	2%	1%	1%	0%	
	.										
Neutral-Aggressive	P5	0,51	0,62	0,68	0,60	0,68	0,62	0,61	0,68	0,67	0,53
	P50	1,37	1,44	1,67	1,48	1,54	1,69	1,64	1,74	1,82	1,82
	P95	2,90	3,09	3,57	3,31	3,38	3,57	3,39	3,76	3,86	4,93
	buffer empty	18%	22%	25%	28%	34%	38%	39%	45%	50%	
	buffer full	20%	16%	14%	9%	7%	5%	4%	2%	1%	
	.										
Neutral-Neutral	P5	0,55	0,56	0,70	0,56	0,63	0,63	0,61	0,66	0,65	0,57
	P50	1,31	1,42	1,58	1,37	1,50	1,68	1,52	1,61	1,73	1,72
	P95	2,56	2,89	3,12	2,69	3,05	3,08	3,18	3,10	3,42	4,23
	buffer empty	16%	20%	23%	27%	32%	37%	39%	45%	50%	
	buffer full	18%	12%	10%	7%	4%	3%	2%	1%	1%	
	.										
Neutral- Averse	P5	0,53	0,60	0,69	0,57	0,60	0,60	0,55	0,63	0,61	0,58
	P50	1,27	1,41	1,57	1,37	1,47	1,69	1,50	1,61	1,74	1,74
	P95	2,45	2,67	2,76	2,67	2,93	3,04	3,05	3,19	3,29	4,15
	buffer empty	15%	19%	23%	26%	32%	37%	38%	45%	50%	
	buffer full	16%	12%	9%	6%	4%	3%	2%	1%	1%	

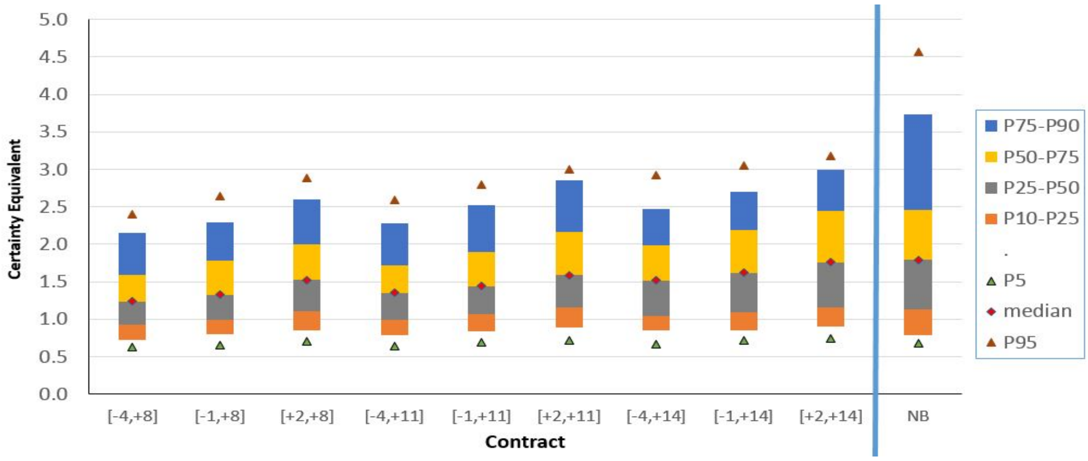
Table 19: The pension results for the averse ($\gamma = 10$) investor, both for the fixed ('Averse') and the variable pensions ('Averse-Aggressive','Averse-Neutral','Averse-Averse').

Investment strategy	UB	+8%			+11%			+14%			No buffer
	LB	-4%	-1%	+2%	-4%	-1%	+2%	-4%	-1%	+2%	
Averse	P5	0.62	0.64	0.66	0.64	0.64	0.65	0.67	0.64	0.66	0.66
	P50	1.01	1.06	1.16	1.09	1.12	1.19	1.16	1.22	1.28	1.27
	P95	1.61	1.67	1.77	1.73	1.83	1.92	1.92	1.99	2.05	2.54
	buffer empty	18%	22%	25%	30%	35%	40%	41%	47%	52%	
	buffer full	10%	7%	5%	3%	2%	1%	1%	0%	0%	
	.										
Averse-Aggressive	P5	0.54	0.54	0.61	0.52	0.56	0.61	0.57	0.47	0.59	0.44
	P50	1.22	1.31	1.51	1.40	1.41	1.58	1.45	1.51	1.69	1.62
	P95	2.63	2.78	2.85	2.88	2.86	3.48	3.09	3.36	3.53	4.20
	buffer empty	17%	21%	24%	28%	33%	38%	39%	45%	50%	
	buffer full	19%	16%	13%	8%	6%	4%	3%	2%	1%	
	.										
Averse-Neutral	P5	0.53	0.55	0.60	0.53	0.50	0.63	0.40	0.61	0.45	0.53
	P50	1.23	1.25	1.47	1.28	1.39	1.54	1.42	1.48	1.57	1.58
	P95	2.41	2.51	2.73	2.56	2.65	2.76	2.96	3.02	3.23	3.68
	buffer empty	15%	19%	23%	26%	32%	37%	38%	45%	50%	
	buffer full	16%	12%	9%	6%	4%	3%	2%	1%	1%	
	.										
Averse- Averse	P5	0.47	0.54	0.60	0.53	0.48	0.55	0.57	0.60	0.60	0.52
	P50	1.21	1.30	1.46	1.24	1.37	1.49	1.40	1.44	1.56	1.58
	P95	2.49	2.44	2.73	2.54	2.62	3.05	2.67	2.78	2.98	3.56
	buffer empty	14%	19%	22%	26%	32%	37%	38%	45%	50%	
	buffer full	14%	11%	8%	5%	3%	2%	1%	1%	0%	

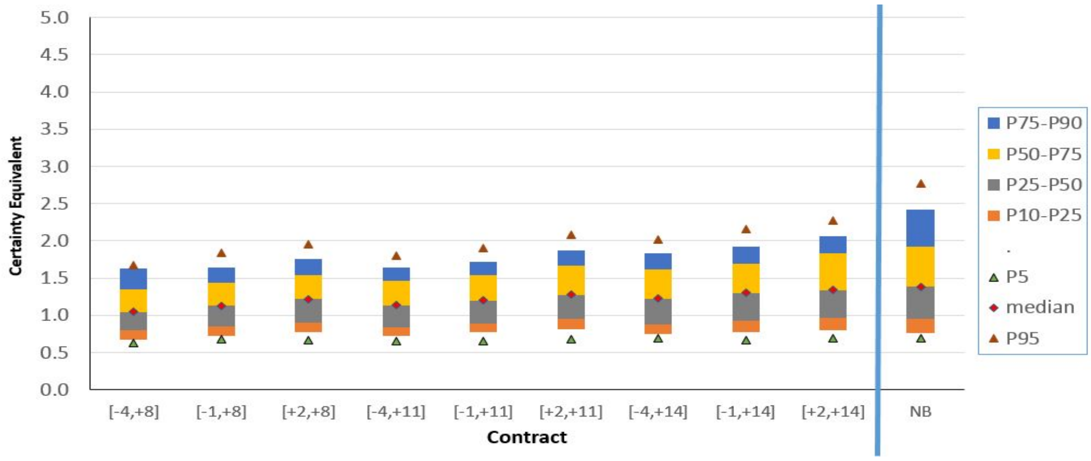
The distributions of the pension outcomes

Figure 25: Distributions of the certainty equivalents in the case of a fixed pension contract, for different investment styles.

25A) Fixed aggressive ($\gamma = 2$)



25B) Fixed neutral ($\gamma = 5$)



25C) Fixed averse ($\gamma = 10$)

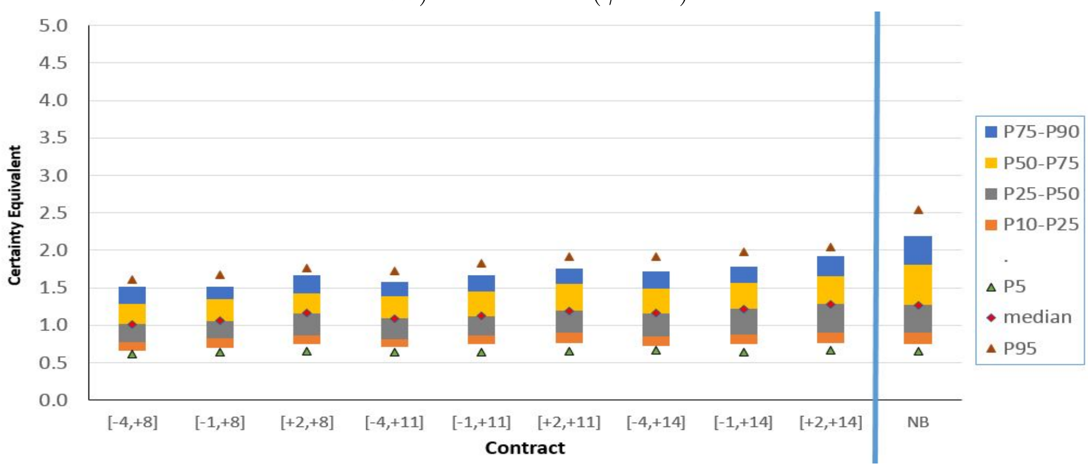
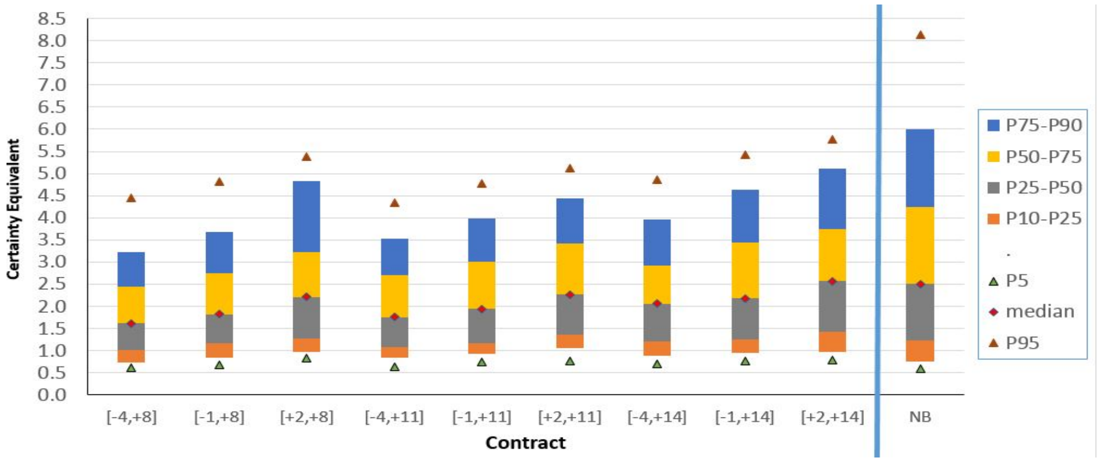
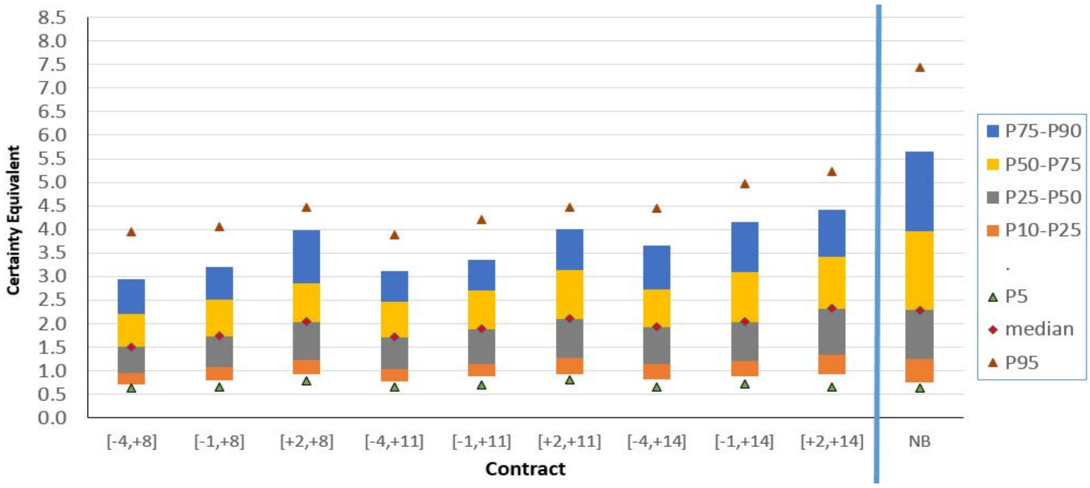


Figure 26: Distributions of the certainty equivalents for a variable pension contract and an investor that invests aggressive (Ag) until retirement.

26A) Variable pension with aggressive investing after retirement (Ag-Ag)



26B) Variable pension with risk-neutral investing after retirement (Ag-Ne)



26C) Variable pension with risk-averse investing after retirement (Ag-Av)

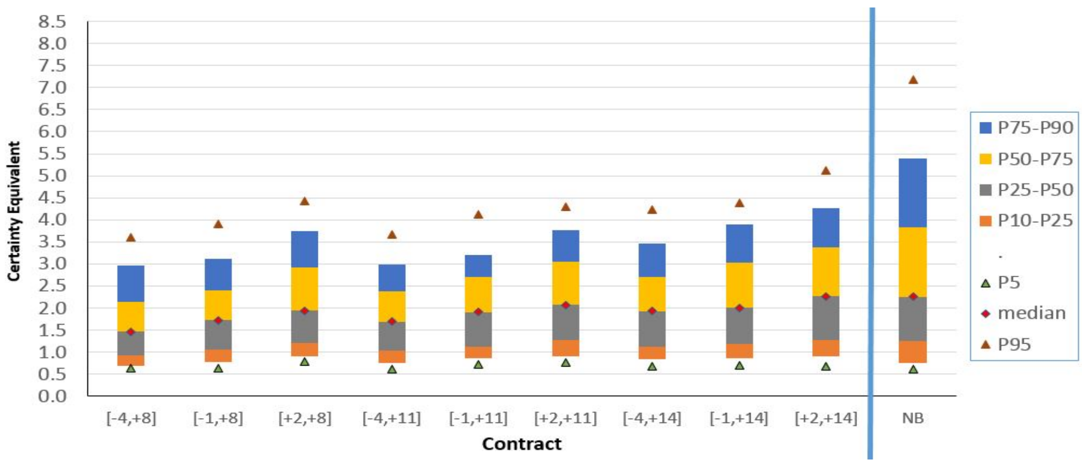
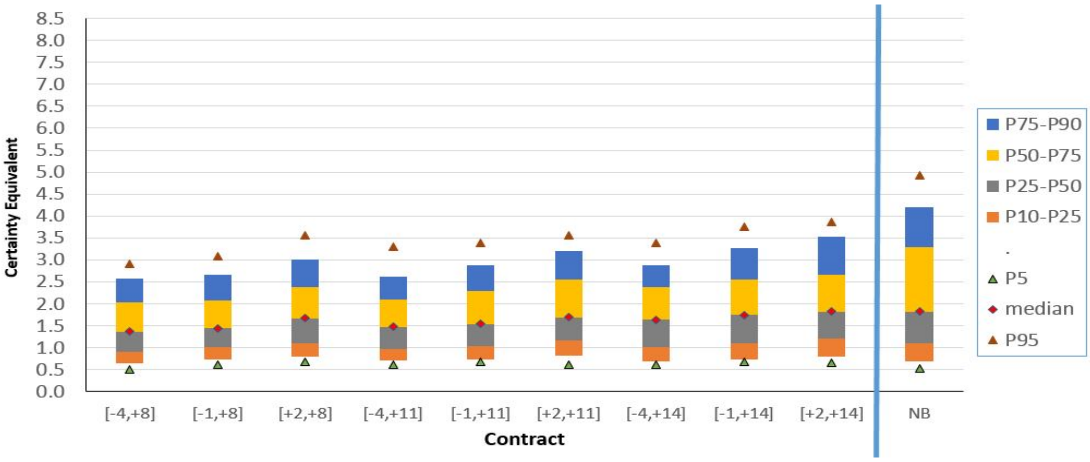
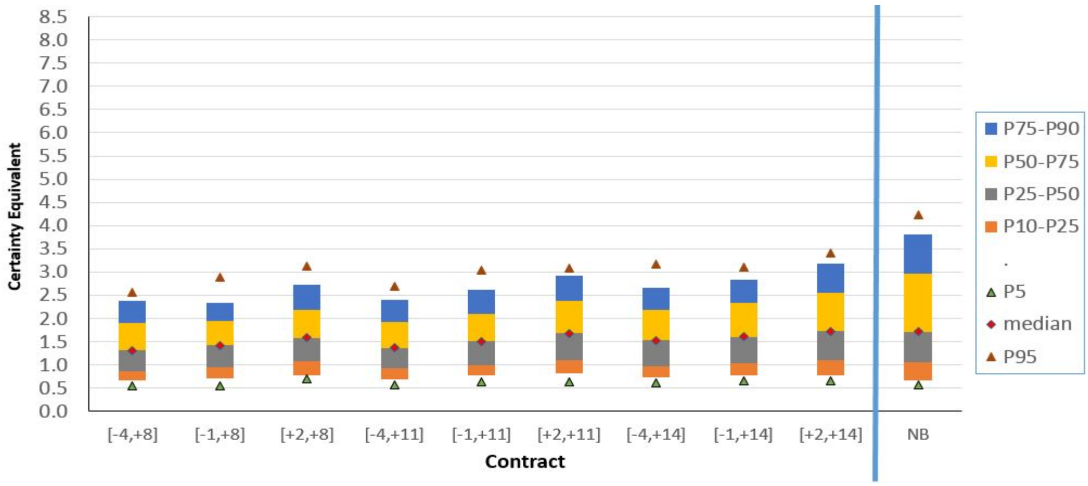


Figure 27: Distributions of the certainty equivalents for a variable pension contract and an investor that invests risk-neutral (Ne) until retirement.

27A) Variable pension with aggressive investing after retirement (Ne-Ag)



27B) Variable pension with risk-neutral investing after retirement (Ne-Ne)



27C) Variable pension with risk-averse investing after retirement (Ne-Av)

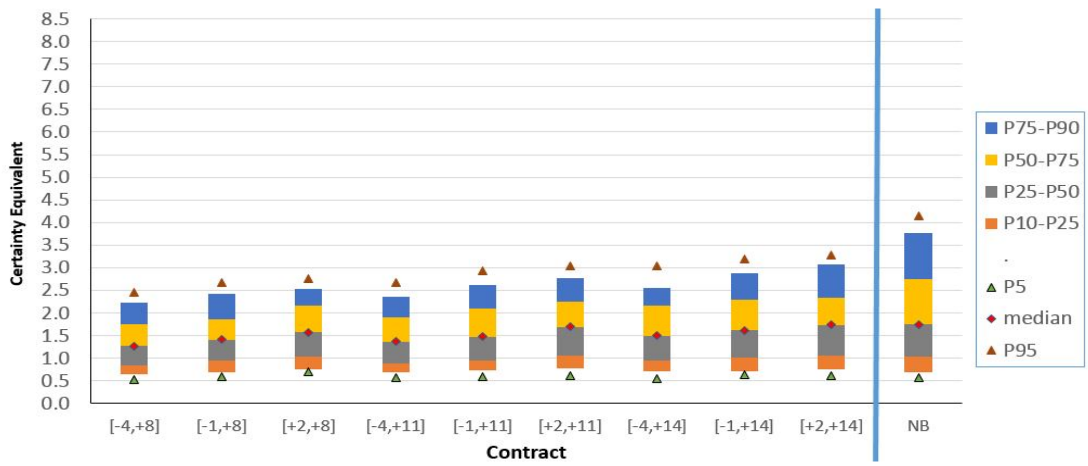
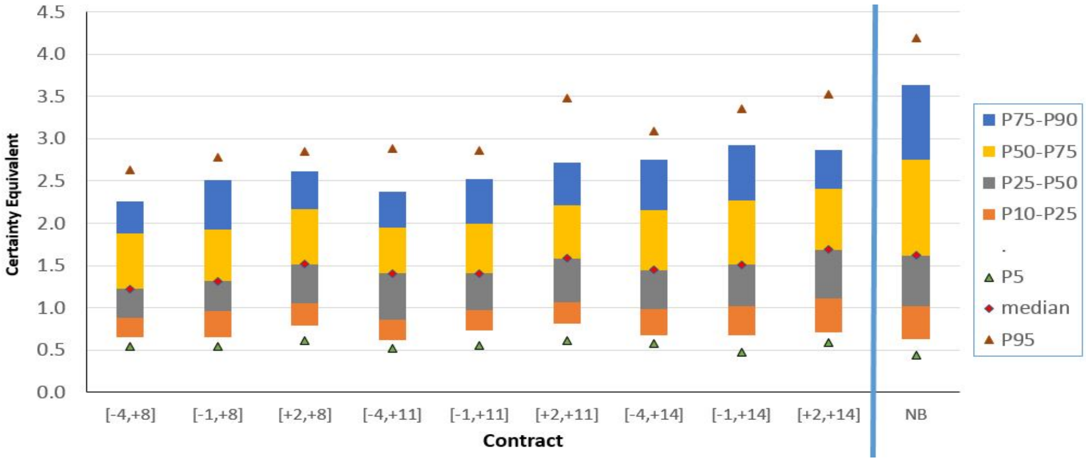
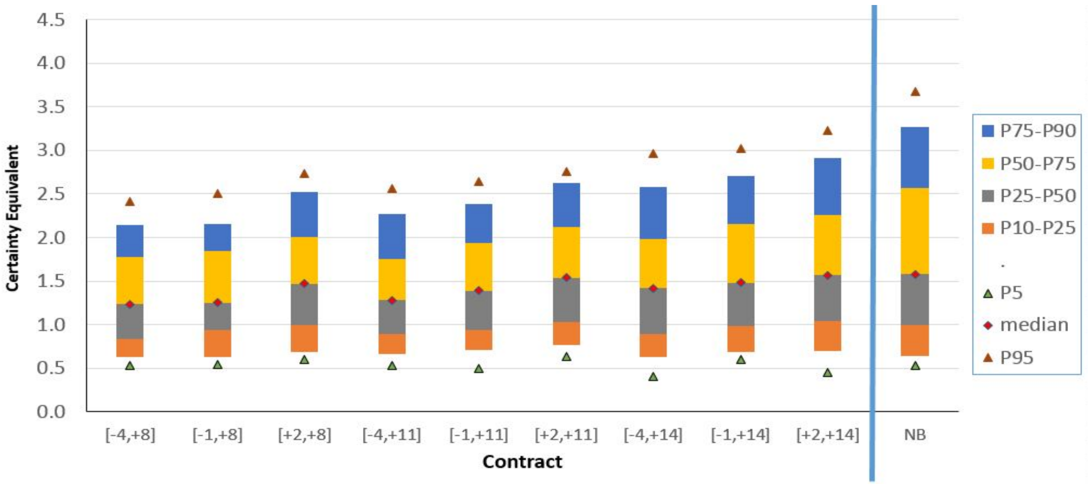


Figure 28: Distributions of the certainty equivalents for a variable pension contract and an investor that invests risk-averse (Av) until retirement.

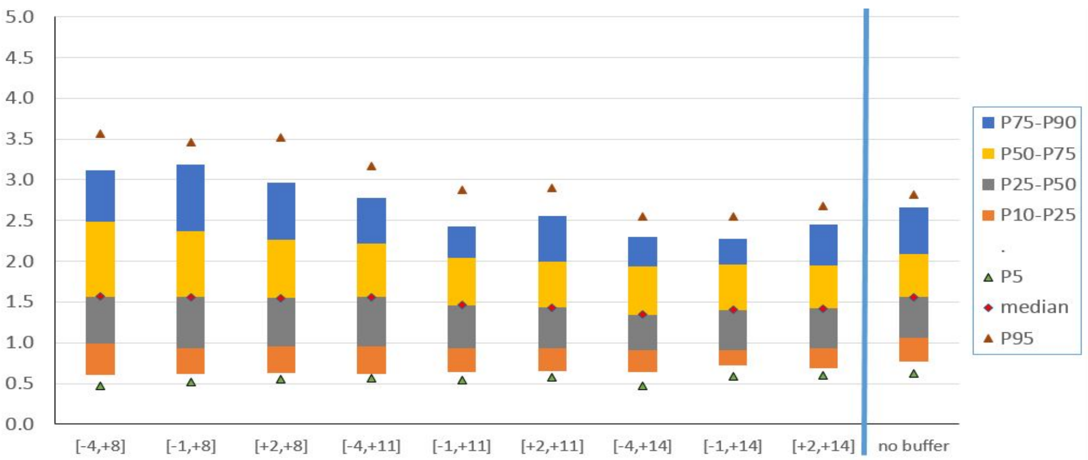
28A) Variable pension with aggressive investing after retirement (Av-Ag)



28B) Variable pension with risk-neutral investing after retirement (Av-Ne)



28C) Variable pension with risk-averse investing after retirement (Av-Av)



Filling buffer

Figure 29: Stacked charts of the part of the time that the buffer is empty (red) and the part of the time that the buffer is full (green) for the aggressive (Ag) investor until retirement with a fixed pension, and for the variable pensions with aggressive investing until retirement (Ag-Ag, Ag-Ne and Ag-Av).

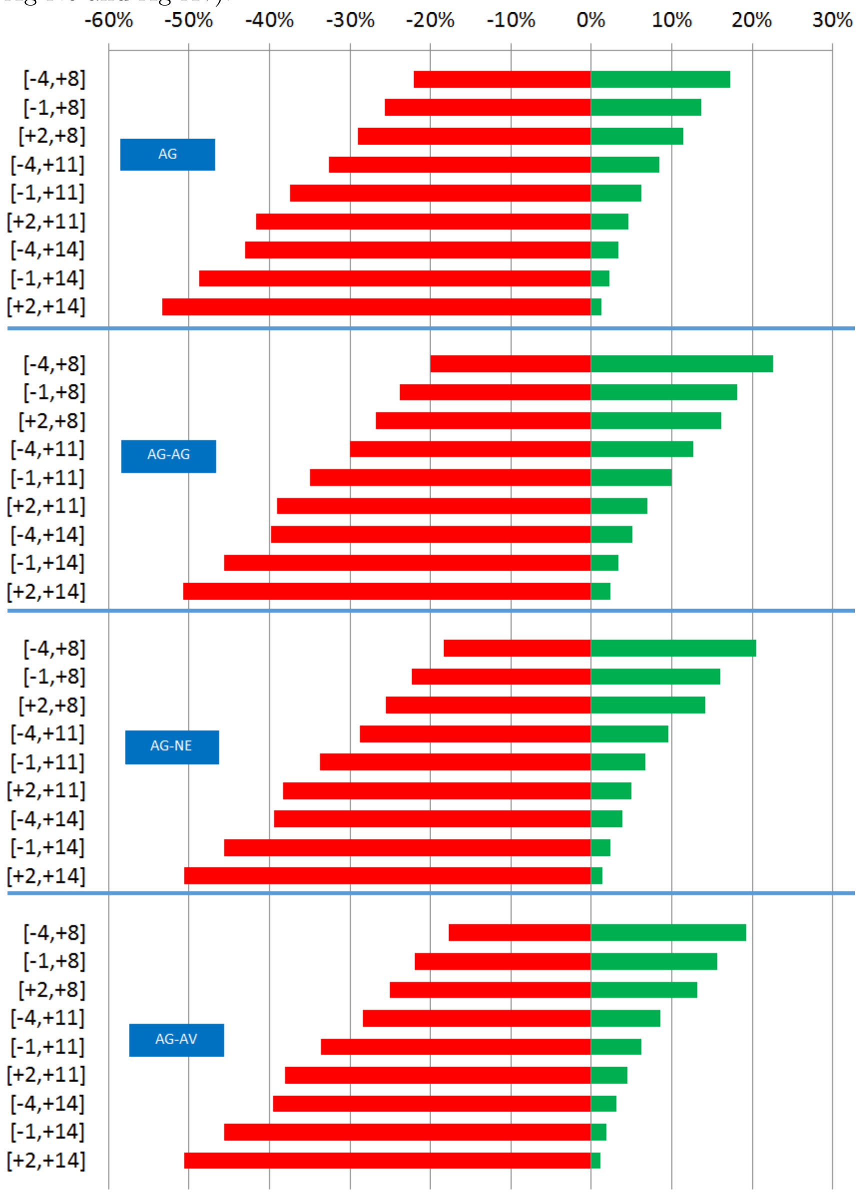


Figure 30: Stacked charts of the part of the time that the buffer is empty (red) and the part of the time that the buffer is full (green) for the risk-neutral (Ne) investor until retirement with a fixed pension, and for the variable pensions with risk-neutral investing until retirement (Ne-Ag, Ne-Ne and Ne-Av).

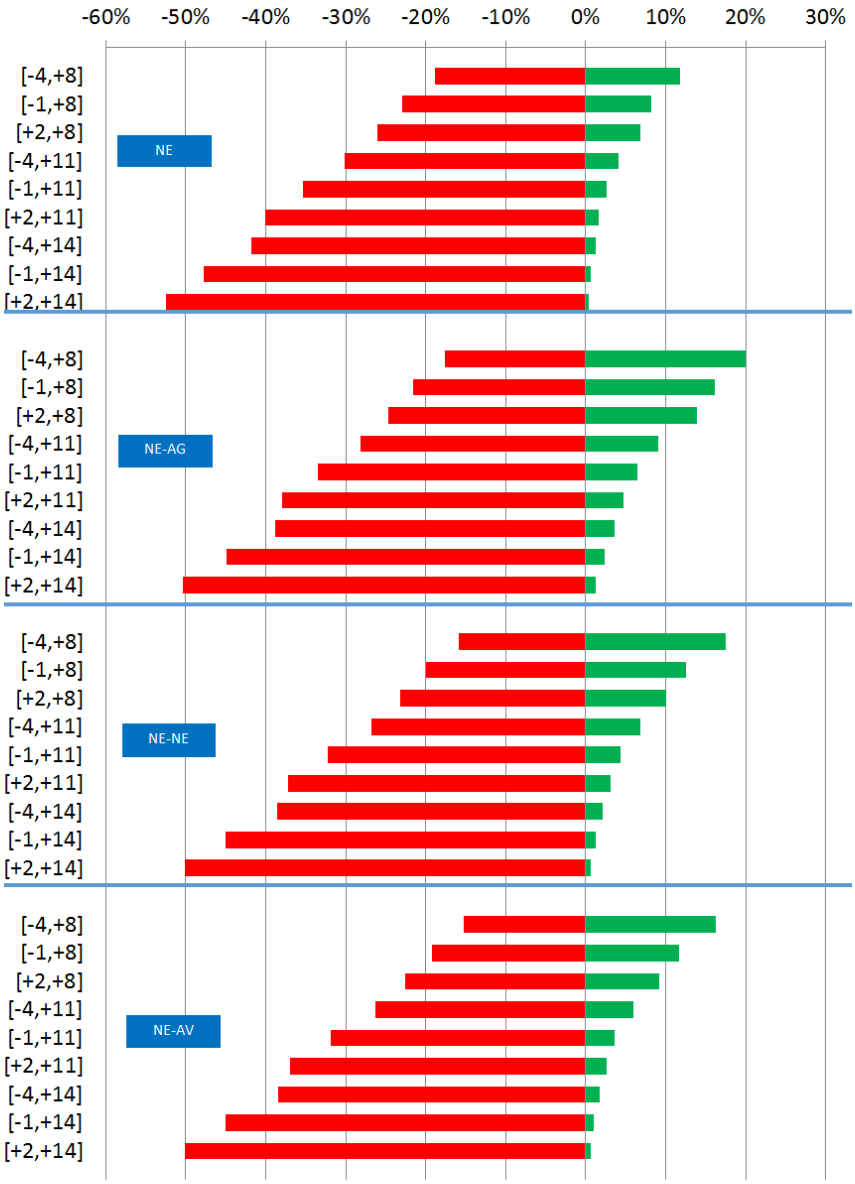
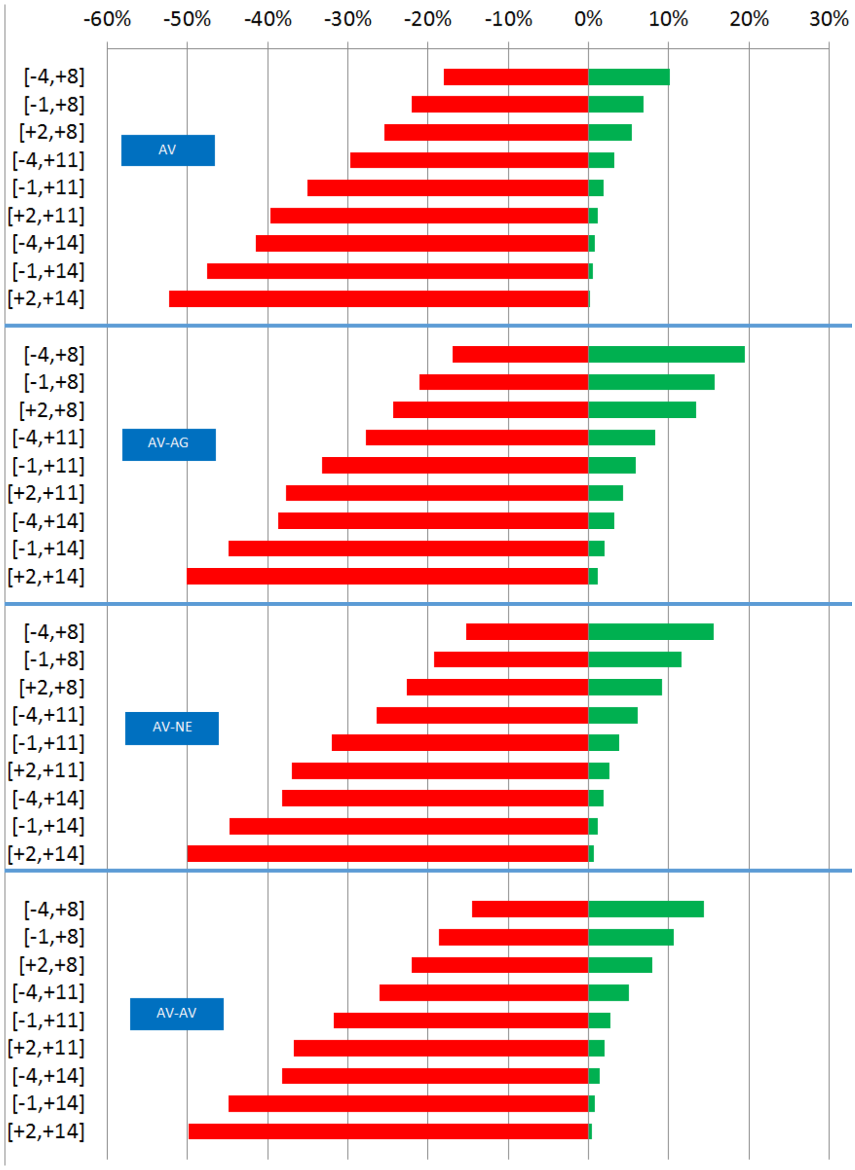


Figure 31: Stacked charts of the part of the time that the buffer is empty (red) and the part of the time that the buffer is full (green) for the averse (Av) investor until retirement with a fixed pension, and for the variable pensions with aggressive investing until retirement (Av-Ag, Av-Ne and Av-Av).



Varying maximum amount of capital allowed in buffer

Table 20: The pension results for a varying buffer size (Π_{max} equals respectively 10%, 20% and 30% of the total accrued pension capital in the fund). Here we have an investor who invests aggressive until retirement age, and aggressive after retirement age.

Π_{max}	ub	+8%			+11%			+14%			NB
	lb	-4%	-1%	+2%	-4%	-1%	+2%	-4%	-1%	+2%	
10%	P5	0.65	0.68	0.79	0.65	0.72	0.72	0.64	0.69	0.70	0.58
	P50	1.87	2.11	2.38	2.06	2.21	2.46	2.17	2.20	2.50	2.35
	P95	5.39	6.06	6.32	5.65	5.81	6.28	5.81	6.30	6.45	8.06
	buffer empty	32%	35%	38%	40%	44%	48%	47%	52%	57%	
	buffer full	41%	37%	34%	33%	28%	25%	24%	21%	18%	
	.										
20%	P5	0.59	0.68	0.79	0.61	0.66	0.76	0.64	0.67	0.70	0.58
	P50	1.65	1.84	2.21	1.76	1.96	2.29	2.05	2.17	2.52	2.35
	P95	4.35	4.66	5.40	4.65	4.85	5.67	4.88	5.34	5.84	8.06
	buffer empty	27%	31%	33%	36%	40%	44%	45%	50%	55%	
	buffer full	27%	23%	20%	18%	15%	13%	10%	8%	6%	
	.										
30%	P5	0.59	0.68	0.79	0.59	0.65	0.75	0.64	0.66	0.69	0.58
	P50	1.53	1.72	2.06	1.73	1.90	2.17	1.99	2.14	2.50	2.35
	P95	3.87	4.30	4.75	4.19	4.37	4.78	4.66	4.77	5.51	8.06
	buffer empty	24%	28%	31%	34%	39%	43%	44%	49%	54%	
	buffer full	18%	15%	13%	10%	8%	6%	4%	3%	2%	

Table 21: The pension results for a varying buffer size (Π_{max} equals respectively 10%, 20% and 30% of the total accrued pension capital in the fund). Here we have an investor who invests aggressive until retirement age, and risk-neutral after retirement age.

Π_{max}	ub	+8%			+11%			+14%			NB
	lb	-4%	-1%	+2%	-4%	-1%	+2%	-4%	-1%	+2%	
10%	P5	0.63	0.67	0.75	0.64	0.71	0.83	0.67	0.71	0.68	0.63
	P50	1.80	1.93	2.20	1.87	2.07	2.26	2.01	2.05	2.30	2.29
	P95	4.76	5.09	5.61	4.83	5.57	5.83	5.37	5.80	5.92	7.34
	buffer empty	31%	34%	37%	39%	43%	47%	46%	52%	56%	
	buffer full	39%	35%	32%	30%	26%	23%	22%	18%	16%	
20%	P5	0.61	0.65	0.75	0.61	0.70	0.82	0.65	0.71	0.65	0.63
	P50	1.53	1.75	2.06	1.70	1.85	2.10	1.93	2.04	2.23	2.29
	P95	4.06	4.47	4.74	4.07	4.60	4.69	4.52	4.85	5.27	7.34
	buffer empty	25%	29%	32%	35%	39%	43%	44%	50%	54%	
	buffer full	24%	21%	18%	15%	12%	10%	8%	6%	5%	
30%	P5	0.59	0.64	0.75	0.61	0.69	0.79	0.65	0.71	0.65	0.63
	P50	1.50	1.65	1.93	1.65	1.84	2.03	1.90	2.03	2.23	2.29
	P95	3.60	3.75	4.32	3.64	4.07	4.15	4.28	4.49	4.96	7.34
	buffer empty	23%	26%	29%	33%	38%	42%	43%	49%	53%	
	buffer full	17%	13%	11%	8%	6%	4%	3%	2%	1%	

Table 22: The pension results for a varying buffer size (Π_{max} equals respectively 10%, 20% and 30% of the total accrued pension capital in the fund). Here we have an Aggressive-Averse investor.

Π_{max}	*ub*	+8%			+11%			+14%			NB
	lb	-4%	-1%	+2%	-4%	-1%	+2%	-4%	-1%	+2%	
10%	P5	0.65	0.66	0.74	0.64	0.72	0.81	0.65	0.69	0.63	0.61
	P50	1.75	1.90	2.17	1.81	1.98	2.22	2.00	2.07	2.25	2.26
	P95	4.57	5.15	5.47	4.62	5.26	5.52	5.25	5.35	5.68	7.08
	buffer empty	31%	34%	36%	39%	43%	46%	46%	52%	56%	
	buffer full	38%	34%	31%	29%	25%	22%	21%	17%	15%	
	.										
20%	P5	0.59	0.62	0.72	0.62	0.70	0.78	0.63	0.69	0.62	0.61
	P50	1.49	1.73	1.99	1.66	1.83	2.02	1.90	2.03	2.22	2.26
	P95	3.70	4.16	4.55	3.96	4.41	4.66	4.32	4.87	4.91	7.08
	buffer empty	25%	28%	31%	34%	39%	43%	44%	49%	54%	
	buffer full	24%	20%	17%	14%	12%	10%	7%	6%	4%	
	.										
30%	P5	0.61	0.62	0.72	0.61	0.70	0.76	0.63	0.69	0.62	0.61
	P50	1.44	1.65	1.88	1.64	1.80	2.00	1.89	2.01	2.19	2.26
	P95	3.41	3.66	4.27	3.56	3.96	4.19	4.18	4.40	4.84	7.08
	buffer empty	22%	26%	29%	32%	37%	42%	43%	49%	53%	
	buffer full	16%	12%	10%	7%	5%	4%	3%	2%	1%	

F Explanation DB model

In this appendix, we briefly outline the cash-flows in the common Defined Benefits model. This gives insight in the premium policy, and the lack of customization in the contract. We follow the model introduced by Veldhuijzen (2014).

Pension policy

We expect that age cohort X receives every year during retirement a fixed percentage c of his current income (we specified $c = 1.875\%$ earlier but denote it as c for notational convenience). We have to take into account the probability that the person is alive during retirement. We use N_x again in this context. Each payment of $c\%$ in the future has to be discounted to the current period, in the same way as we did with an annuity. This gives the next expression for the accrued pension (AP) for an age cohort x at time t:

$$AP_t^x = \sum_{i=67-x}^{100-x} \frac{c\, y_t^X N_{x+i}}{(1 + r_t(i))^i}, \tag{F.1}$$

where y_t^X denotes the wage of age-cohort X at time t. This is called the actuarial value of the accrued pension for age-cohort X. It now easily follows that the actuarial value of the total accrued pensions at time t is given by

$$AP_t = \sum_{x=25}^{68} AP_t^x. \tag{F.2}$$

If the contribution rate C_t is such that only the accrued pensions have to be covered, we can determine the contribution rate in a DB-policy as

$$C_t = \frac{AP_t}{y_t}, \tag{F.3}$$

where y_t is the sum of the wages of all the age-cohorts in year t. We can multiply this C_t with some constant $\theta > 1$ such that not only the accrued pensions are covered, but also costs of the execution of the system. We ignore this for now and choose $\theta = 1$.

Indexation

The idea of indexation is that we have to adjust the pension payments with the inflation rates to preserve purchasing power. Giving indexation is not always possible. It depends on the position of the pension fund. When the pension fund has a funding ratio (the rate of assets over liabilities) which is too, the fund can not obey the indexation demand in that year. The accrued pensions have to be adjusted in this case. Every year, the recovery indexation and cuts that need to be made depend on the funding ratio of the year before. We use a simplified indexation policy. If the funding ratio is above a certain threshold, we have 100 % indexation, and if the funding ratio is below a certain threshold we can not give indexation. Between these thresholds we use a linear indexation. We use as upper-bound 130% and lower-bound 110%. This gives the following equation for the indexation.

$$i_t = min[max\left[\frac{F_t^{policy} - 110}{130\% - 110\%}, 0\right], 1] \times \pi_t \tag{F.4}$$

When the indexation can not be entirely provided by the pension fund, the missed indexation has to be provided later. If the pension fund is healthy this can only be done, so if the funding ratio is above the upper threshold. If the pensions are not indexed for several years, the compensation can be large. The fund spends only a fraction of the extra money it has above the upper funding ratio. We define the recovery indexation at moment t as i_t^r and it is described by

$$i_t^r = max\left[\frac{1}{10}\left(\frac{F_t^{policy}}{130} - 1\right), 0\right]. \tag{F.5}$$

This has to be done only if the fund was not able to give the indexation obligation in an earlier stage.

Cuts

Cuts need to be made if the capacity to recover is not enough to obtain a policy funding ratio equal to the Required Funding ratio. We use 105% in this example as the required funding ratio. Pension funds need to announce cuts if their policy funding ratio, plus the capacity for recovery, is below the required funding ratio (RF). So in 10 years the fund has to be at least above the lower-bound, otherwise they need to cut. We make a difference between small cuts and big cuts. Small cuts is the $1/10-$th fraction of the difference between the minimum required funding ratio and the actual funding ratio. If the funding ratio is for 5 consecutive years below the RF, the fund has to do a large cut such that it is back to a 'healthy' situation. The cuts have to be increased and the cut can be spread over 10 years. We assume with the large cut that this is done in once, for convenience. We can describe the big cut as bc_t and the formula for this big cut is given by

$$bc_t = \frac{F_t^{end}}{105\%} - 1$$

This 105 % depends on the level of risky investments. The riskier the investments, the larger the minimum required funding ratio. The total amount of cuts equals the sum of the small cuts and the big cuts.

Liabilities

The pension rights, or financial wealth in our case, develop sequentially as follows. Every year the participants accrue a percentage c of their current wage that will be received each year during retirement. In year t, the pension rights are the rights from year $t-1$, plus the newly accrued pension. This new amount has to be adjusted for indexations and/or cuts. For retirees the same principle holds, but they do not accrue new pension rights of course. We can for age cohort x describe this pension rights for moment t as

$$PR_t^X = \left(PR_{t-1}^{x-1} + cy_t^x N_x \mathbb{1}_{25 \leq x < 68}\right)(1 + i_t)(1 + i_t^r)(1 + c_t). \tag{F.6}$$

Where $\mathbb{1}$ denotes the indicator function to mention that there is no new pension accrual after retirement. This system has to be initialized of course.

Now we are able to determine the liabilities of the pension fund for different periods. These liabilities are the sum of the of the liabilities for the different age cohorts. If the age cohort is retired, this is an annuity of the pension rights accrued so far, starting from $68 - x$ years from now. The liabilities can be calculated by the current value of that annuitization times the probability of being alive (the size of the cohort group in our case). The discount factor is the estimated term structure at time t. The liabilities for the participants with age x can be defined by

$$L_t^x = \sum_{i=max(68-x,1)}^{100-x} \frac{PR_t^x N_{x+i}}{(1 + r_t(i))^i}. \tag{F.7}$$

Finally, the total liabilities at time t for the pension funds follows from summing over all the age cohorts.

$$L_t^{begin} = \sum_{x=25}^{100} L_t^x. \tag{F.8}$$

Annual funding ratio process

We now determine how the funding ratio over time will develop. First the pension benefits are paid to the retired participants. Also the contributions are added. Our level of assets changes with

$$A_t^{begin} = A_{t-1}^{end} + \text{contributions}_t - \text{payments}_t. \tag{F.9}$$

The contributions are specified in equation (F.2). The payments are conditioned on the fact that the participants are alive. We link the pension incomes to the payments of the pension fund below and sum over the retirement ages. We then have

$$\text{payments}_t = \sum_{j=67}^{100} PI_t^j = \sum_{i=67}^{100} PR_t^i N_i \tag{F.10}$$

The annuitizations are locked for the pension fund. The liabilities are defined by F.8. This results in an initial value of our funding ratio, namely

$$F_t^{begin} = \frac{A_t^{begin}}{L_t^{begin}}. \tag{F.11}$$

In the second step we go one year further, so the investments on the assets have rendered returns, so the assets will increase, say with r_i^t, then

$$A_t^{end} = A_t^{begin}(1 + r_t^i). \tag{F.12}$$

There is only one investment mix for all the assets in the fund. The liabilities grow with the short term interest rate. In this way we obtain the liabilities for time t,

$$L_t^{end} = L_t\left(1 + r_t(1)\right). \tag{F.13}$$

Finally

$$F_t^{end} = \frac{A_t^{end}}{L_t^{end}} \tag{F.14}$$

This is a procedure that works recursive trough the time. It is important to note that in this DB-contract, the contribution rate C_t is equal for all the participants. Therefore, the contribution rate can be collectively increased in order to keep the funding ratio at the required level. Like we mentioned in section 2, this can be harmful for the incomes of new entering participants (young employees), as they have to contribute to bad results from the past.