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Valuation of Hybrid Capital Instruments using Lévy Processes

Master thesis in Econometrics & Management Science
(Quantitative Finance)

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Abstract

This master thesis introduces a pricing procedure for hybrid capital instruments issued by insurance companies when the underlying interest rate process is modeled by $\alpha-$stable Lévy processes, as several empirical researches have shown that models based on normal distribution might be improper for financial modeling. Increments of these stochastic processes are independent and follow an $\alpha-$stable distribution. This distribution is a generalization of normal distribution, only with heavier tails and infinite variance. To obtain the instrument prices, I solve a partial integro-differential equation (PIDE), which is the generalization of the Black-Scholes PDE for Lévy processes. This PIDE is solved by a finite difference method. To improve the stability and the precision of the standard pricing procedure, I provide my own refinement based on interpolation and extrapolation of instrument prices. In comparison to the Gaussian models, the Lévy models provide lower instrument prices and therefore are closer to the market values and allow more flexibility for the price modeling.

Keywords: Lévy process, $\alpha-$stable distribution, hybrid capital instruments, callable bonds, partial integro-differential equation

Contents

1 Introduction

Funding is an essential part of every company. The funding of a company can be accomplished in several ways, for example by borrowing money from a bank, issuing stocks, or issuing bonds. Bonds issued by a company can have several features. They can pay coupons, have embedded optionality, or even be perpetual. These bonds with embedded exotic optionality are called hybrid capital instruments. After the issuance of such instruments, they are often traded on financial markets. Issuance of hybrid capital instruments is an essential part of the funding of insurance companies. Based on characteristics, the instruments are divided into different groups, called "tiers." According to a new regulation, Solvency II, the tiers have different capacities and therefore the company has to have a strategy on calling the instruments. Moreover, while until recently the regulation required values to be the current market prices, after the Solvency II ratification the instrument prices have to be obtained based on theoretical valuation with fixed credit spread. Thus, theoretical valuation of capital instruments is necessary. Moreover, the proper theoretical valuation is useful also for investors and traders to determine the right non-arbitrage price and therefore help with trading.

The first main contribution of this thesis to existing literature is pricing of hybrid capital instruments implementing an $\alpha-$stable Lévy framework and subsequent comparison to prices obtained by Gaussian models. This $\alpha-$stable process lies in the same set of so-called self-similar processes as the brownian motion (a part of the process is similar to the whole process). The main difference between the $\alpha-$stable process and the Brownian motion is the continuity. While the Brownian motion is continuous, the $\alpha-$stable process is not and allows jumps. The pricing is performed on three theoretical and three real instruments issued by two Dutch insurance companies (NN and AEGON). As the instruments are interest rate derivatives, important part of the thesis is correct interest rate modelling and subsequent calibration. To obtain a fair price of an instrument considering the current market conditions, correct selection of parameters is crucial. How the changing parameters affect the final instrument price is the last part of this thesis and another contribution to the literature. The whole sensitivity analysis of the instruments is performed on one of the theoretical instruments. Moreover, I conduct the price dependence on the changing stability parameter α.

To price the instruments, three types of models are used, the model for yield curve production, the instantaneous interest rate model and the model used to obtain the final price. As the instruments priced in this thesis are bond options, typical option pricing models are used. For Gaussian models, I use the well-known Black-Scholes model proposed by Black and M.Scholes (1973). The $\alpha-$stable model used in this thesis is a generalized Black-Scholes model, proposed by Swishchuk (2008). The instantaneous interest rate model used in this thesis is Hull and White (1990) model. The Hull-White model is a one-factor short-rate model with two unknown parameters. It is so-called arbitrage-free model. The generalized form of this model (used for Lévy models) is proposed by Swishchuk (2008), where the Wiener process is replaced by a Lévy process. Moreover, as the short-rate models are fitted to the initial yield curve, the yield curve construction is needed. This yield curve consist of two elements - risk-free curve and CDS curve. The risk-free curve is modeled by Svensson (1994) model which is used by European Central Bank to produce yield curves on a daily basis. For CDS curve, I use Nelson and Siegel (1987) model that is fitted to current CDS on the market.

Methods used for the instruments pricing are based on the evaluation of the Black-Scholes partial differential equation and generalized partial integro-differential equation using finite volume and finite difference numerical methods. The numerical methods can be found in Kroner (1997), or d'Halluin et al. (2001). While Gaussian models can be priced using a simulation method based on multiple interest rate paths modeling and subsequent backward discounting, replicating this method for Lévy models would cause issues. The main reason, why the simulation method is inappropriate for Lévy processes is the infinite variance of its distribution what causes a substantial increase in the computational time. The thesis explains the valuation procedure using mixed (implicitly-explicit) finite difference method proposed by Duffy (2005). As the basic numerical methods for partial integro-differential equation (PIDE) evaluation do not lead to precise results, I develope an advanced procedure for the PIDE evaluation.

When evaluating the partial integro-differential equation, one has to create a grid, where each grid point represents a concrete interest rate level and a time level. When creating this grid, the lowest and the highest grid points determine the interest rate interval on which the computation is done. During the PIDE evaluation procedure problems arise on the boundaries. To receive the price at some specific time level, we use the known values of an instrument at the next time point. However, to calculate the respective price, we need to know the price of a claim also outside of the grid. To overcome this issue, I propose linear extrapolation on the boundaries to obtain the prices outside of the grid and thereafter use them to calculate the correct prices inside the grid. Moreover, to correctly evaluate the integral in the PIDE, we need to know the prices not only outside of the grid, but also between two grid points. I propose in this thesis cubic spline interpolation to solve this issue.

There have been several researches conducted on callable bond pricing in the past. The first methods were based on a tree framework. The binomial tree approach is used by Kalotay, Williams, and Fabozzis (1993). Later, a trinomial tree approach was proposed by Hull and White (1994a), (1994b) and (1996). With the beginning of the new century, numerical methods of solving PDEs were examined by d'Halluin et al. (2001), where finite volume method is proposed to obtain the prices of the semi-American options on coupon bearing bonds. However, later it was proved that the normality assumption is rather strong. Several papers, e.g. Raible (2000) show, that the Gaussian models are improper for modeling the term structure. Raible proposes rather to use the generalized Lévy processes for modeling the interest rates.

Data needed for this thesis can be divided into 3 categories, concretely the data for the instrument interest rate curve, for the calibration, and for the instrument coupon payment. Firstly, there are two data sets needed for interest rate modeling - risk-free rates and CDS of the corresponding company. Data used for risk-free rate modeling are the Svensson parameters provided daily by the European Central Bank. The used Svensson parameters are from the first 11 trading days in August 2017. Another necessary data set for instrument interest rate modeling is the credit spread of the corresponding instruments issued by AEGON and NN. For the construction and correct future prediction of the CDS, I use daily CDS data between 01/08/2017 and 15/08/2017 for maturities 0.5Y-10Y. Secondly, for the interest rate model calibration, I use interest rate swaption volatility data. The ATM Black volatilities are used from 01/08/2017 until 15/08/2017 for maturities 5Y-15Y and tenors 1Y-15Y. Lastly, there is data needed for the floating coupon prediction of one instrument. I use historical 3m EURIBOR rates from 01/1994 until 08/2017 with monthly periodicity.

The final results obtained using our models exceed the prices observed on the market. While the model price of two of them (both issued by NN) shows significantly higher values for any credible stability parameters, the value of the instrument issued by AEGON, maturing in 2023, deviates only slightly. These price differences can be caused either by the inefficiency of markets or by different interest rate (credit spread) expectations by market participants. The sensitivity analysis shows that our model prices are increasing with the growth of any of the considered parameters (a, σ and α). The price dependence plot on parameter α is S-shaped. While the price for the Gaussian model (model with $\alpha = 2$) is increasing nearly linearly with growing a and σ, the price change for Lévy models is lower for higher parameter values.

The remainder of the paper is organized as follows. Chapter 2 discusses capital instruments priced in this thesis and the data that is needed. Chapter 3 explains the the basics of Lévy models and some typical examples. In the next chapter, I state models and the techniques needed for interest rates modeling. Chapter 5 deals with the models and methods necessary for obtaining the final results, which are stated in chapter 6. Finally, in chapter 7, is conclusion.

2 Instruments and data

This chapter deals with the instruments priced in this thesis and the data needed for the pricing procedure. Section 2.1 provides theoretical foundation of the hybrid capital instruments and also describes the concrete instruments, that are priced. Section 2.2 names all the data used in this thesis.

2.1 Instruments

In this work I price hybrid capital instruments. But what are the hybrid capital instruments? What features do they have? Instruments priced in this work are coupon bearing callable bonds. A callabe bond is a bond with a call option. That means, that an issuer can call the underlying bond before the maturity. The price of the instrument, therefore, depends on the price of the bond and the price of the call option on the bond. The price of the call option on the bond lowers the price of the underlying bond as it protects an issuer before the potential high future cash flows. The bond can, in general, be called at pre-specified dates, called the call dates, when the investor has to pay the call price.

Call dates are typically the same as coupon dates, when the issuer pays the interest on the bond. These interest rates are either fixed, floating or even the combination of the two. A typical feature of floating rate bonds are caps or floors. The cap is the maximum interest that can be paid, while the floor is the minimum value. Moreover, it is usually not possible to call the instrument until some fixed date that is agreed when issuing the bond (first call date). The period from the instrument emission until the first call date is called the lock-up period. This lock-up period can also affect the pricing procedure of instruments, as they have to be discounted from the first call date until the most recent date. However, when pricing capital instruments already placed on the market, it does not affect the pricing procedure most of the time as the lock-up period is already expired.

Maturity factor plays another important role in hybrid capital instruments. There are also perpetual instruments on the market. That means, that there does not exist any fixed, pre-definied, maturity. This attribute is convenient for the issuer, as the perpetual capital instrument is of a higher capital quality, which helps companies to improve their capital adequacy ratios. Another common feature on the market is a so-called notice date. This is a date prior to a potential call date, when the issuing company has to decide, whether it will call the bond at the next call date or will not. The difference between the notice date and the call date is usually not more than several days.

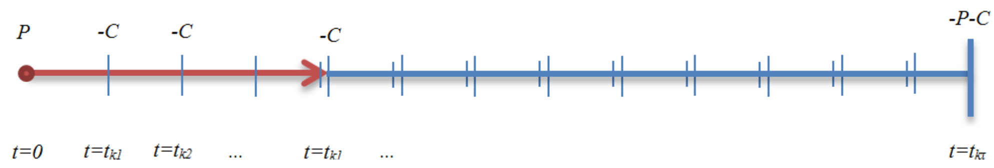


Figure 1: Scheme of the instrument - labels below the figure represent time (coupon dates), labels above represent cashflow from an issuer point of view

In this thesis, I omit the presence of the notice date. Or in the other words, I assume that the notice date and the call date are the same.

The redemption of an instrument is based on several factors such as the capital levels of a company or the existence of a relevant investor that is willing to buy an issued instrument. Theoretically, a company should redeem the instrument whenever its price is above the call price. However, the correct time for redemption can be different in reality. These instruments are usually issued by a company to improve the capital adequacy ratios. For instance, as already mentioned, the perpetual instrument is of higher capital quality and thus the redemption could cause a company issues on some levels of capital adequacy ratio. This issue could be solved by issuing a new instrument with the fair value based on the current status of the market (that could be lower through lower coupon payments). In spite of this fact, there are additional factors that influence companies. First of all a company has to get a regulatory permission and additionally, there are another administration costs that are not considered in this thesis. Thus I assume that there are no administration costs or other legal issues that could affect final decision of a company. Therefore the company redeems the instrument whenever the price at a call date is higher than the call price.

Figure 1 shows a typical callable bond issued by companies. The horizontal line represents a timeline. The point at time $t = 0$ is the issuance date, whilst the time point $t = t_{k\tau}$ is the maturity of the instrument represented by a bold vertical line. The medium length vertical lines represent the coupon dates (identical with the call dates) and the shortest lines represent the notice date at which an issuer has to decide, whether call the instrument or not. As we can see, the shortest lines are not highlighted in the red part of the timeline. This part is the lock-up period of the instrument. If we assume that the instrument is not called until the maturity date, we can represent a cashflow by the labels above the figure. The values reflect the issuers point of view, thus the negative values mean issuers expenses. The coupon payments are represented by letter C and the principal by letter P. The instrument depicted in 1 is paying fixed coupons. However if we represent a coupon at time t_{ki} by $C_{t_{ki}}$, we can describe any instrument with a floating coupon rate.

The whole work deals with callable bonds with fixed finite maturity and coupons without caps or floors. This thesis provides valuation for 3 different instruments from both categories (theoretical and practical). All of these instruments have a different maturity and a different coupon structure. During the work, the instruments are called in the abbreviated forms (stated in the parentheses). For the simplification, the credit spread of theoretical instruments is 300 basis points for every maturity. So-called real instruments priced in this thesis are issued by companies AEGON and NN. One of the differences between a theoretical and a real instrument is the lock-up period. Two out of 3 real instruments have still a valid lock-up period and therefore the first call date is not the same as the next coupon date. Another different feature is, that the time until the next coupon date is not exactly the same as the difference between two coupon payments. The last important difference is the coupon structure, as two of the priced instruments have variable coupons after some reset date.

The first theoretical instrument (**T-1**) is the instrument with an eliminated lock-up period. The coupon payments take place quarterly and its price in this work is determined the day after the last coupon payment, therefore the next coupon is paid in 3 months. Call dates are the same as coupon dates. As all of the other instruments, this callable bond is denominated in euros. The coupon is 1% and the claim matures in 1 year. The second theoretical instrument (**T-3**) is, similarly as T1, with an eliminated lock-up period and with similar coupon structure (3 months before the next coupon and call date). The differences with the first instrument are only in the maturity and the interest of the coupon. The coupon is 2% and the maturity of this callable bond is 3 years. The third theoretical instrument (**T-5**) is the last from the category of theoretical instruments. It is with the longest maturity, 5 years. The coupon sructure is different for this bond than for the previous ones. The coupon is 2%, like the coupon for T-3, but is paid annually. The next coupon payment (and thus call date) is exactly in 1 year.

The first of the real instruments priced in this thesis is a senior note issued by AEGON. In fact, this instrument is not callable, therefore it is a coupon paying bond. The note was issued on December 9, 2016 and pays 1% coupon annually until the maturity (December 8, 2023). Throughout the work, this instrument is called **AEG-23**. The second and third real instruments are instruments issued by Nationale-Nederlanden. One of them is a subordinated note (**NN-44**) issued on April 8, 2014 that bears a coupon of interest of 4.625% annually until the first call date (April 8, 2024). After this date, the interest of a coupon becomes floating and dependent on 3 month EURIBOR rate. Moreover, the coupon is paid quarterly after this reset. The note matures on 8 April 2044. The last considered instrument is again a subordinated note issued by NN group. This instrument is the most recent one, since it was issued on 11 January 2017 (**NN-48**). Similarly, like the previous one, bears a fixed coupon until the first call date (13 January 2028). This coupon is of interest of 4.625% paid annually. Afterwards, the floating coupon is equal to 3 month EURIBOR rate plus 495 basis points paid quarterly until the maturity of this claim.

2.2 Data

To model the interest rate curve, I need the risk-free curve and the spread curve corresponding to the proper company. All of the priced instruments are denominated in the EUR, so the risk-free rate curve is created using the five Svensson model parameters published by European Central Bank daily due to the first 11 trading days in August. Another inportant part is to capture the yield curve of a credit spread for the corresponding instrument. The yield curve is constructed using Nelson-Siegel model. The parameters for this model are estimated in this thesis (for more I refer to section 4.1). To estimate the correct parameters, I use the historical CDS for both companies (AEGON and NN) for maturities 0.5Y-10Y. I used the daily CDS data from 02/05/2017 to 31/07/2017.

The second data category is the data needed for the calibration. Our interest rate models are calibrated to swaptions. The swaption volatilities are used from August 01, 2017 for maturities 5Y-15Y and tenors 1Y-15Y. The last data category is the data needed for the final instrument pricing. To price the real instruments and to compare them to the market prices, I use dataset for the first 11 trading days in August 2017 for all of these instruments. As the coupons for the instruments issued by NN are floating after the lock-up period and are dependent on 3m EURIBOR rates, the future rates have to be predicted. I use the historical rates from 01/1994 until 08/2017 with monthly periodicity for the future prediction of EURIBOR rates. For the respective rates I refer to Table 8 and Table 9.

3 Lévy processes

Best-known models used in financial mathematics for pricing all types of instruments from bonds, stocks to any exotic options follow a premise, that the financial intrument prices are driven by Brownian motion. However, while the increments of Brownian motion follow normal distribution, empirical findings show, that price changes of financial assets show signs of non-normality. To cope with these discrepancies, there have been developed models that follow other processes. Stochastic processes, that are used to describe the behavior of financial markets are called Lévy processes.

Definition 3.1 (Miyahara (2012), p.7). *Suppose that a probability space* (Ω, F, P) *and a filtration* $\{F_t, t \geq 0\}$ *are given. A continuous time stochastic process* $\{Z_t, 0 \leq t \geq T\}$ *defined on the probability space* (Ω, F, P) *is a Lévy process if the following conditions are satisfied.*

1. *(independent increments property) For any* $0 \leq t_0 < t_1 \cdots < t_n \leq T$, $Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \cdots, Z_{t_n} - Z_{t_{n-1}}$ *are independent.*

2. *(stationary increments property) The distribution of* $Z_{t+s} - Z_t$ *is the same for all* t.

3. $Z_0 = 0(P - a.s.)$

4. *(stochastic continuity)*

$$\forall t \geq 0, \forall \epsilon > 0, \lim_{s \to t} P\left(|Z_s - Z_t| > \epsilon\right) = 0. \tag{3.1}$$

5. *(cadlag property) There is a subset* $\Omega_0 \in F, P(\Omega_0) = 1$, *such that, for every* $\omega \in \Omega_0, Z(t,\omega)$ *is right continuous in* t *and has left limits.*

The main part of this work is about the valuation of capital instruments with models based on Lévy processes. In fact, Wiener processes are a subset of a larger set, called Lévy processes. While the Wiener process is a process that follows a normal distribution, the increments of Lévy processes can follow also other distributions such as a Poisson distribution, or other fat-tailed distributions, or even a combination of these distributions. Many times are Lévy processes described as typical Wiener process with jumps at random times. These processes are only a subset of Lévy processes and are so-called Jump-diffusion processes. Another exapmles of processes are α-stable processes, tempered-stable processes, and many more.

As we can see from definition (3.1), Brownian motion (Wiener process) is a specific example of a Lévy process, where the conditional distribution $P\left(W_s - W_t | F_t\right)$ follows a normal distribution with mean 0 and variance $s - t$. During the work I refer to a model that is based on Brownian motion as a Gaussian model. Any other model, that uses Lévy process, I call as a Lévy model. Every Lévy process can be written in a general form. This form is so-called Lévy-Khintchine triplet - $(\sigma, \nu(dy), \gamma)$. The γ in the triplet represents the linear part of the Lévy process (drift), σ stands for the Wiener process and the $\nu(dy)$ – Lévy measure - stands for jumps. The crucial part of the Lévy-Khintchine triplet is the Lévy measure. Lévy measure is a typical mathematical measure. A measure is a rule or formula that assigns a value to any subset of a considered set and allows the comparison of the subsets. This rule is usually characterized by a function (e.g. μ), that satisfies the measure axioms, which are defined and explained in Appendix A1.

Although some measures, such as area or perimeter, are more intuitive, other are not intuitive at all as the measures can be represented by nearly any function. This is also the case of the Lévy measure responsible for jumps, or heavy-tailed distributions. The process considered in this work is so-called α-stable Lévy process. This process can be written in form $(0, \nu(dx), b)$, where $\nu(dx)$ is the measure that generates the process. The measure in this process is defined as follows (Miyahara (2012),p.16)

$$\nu(dx) = \begin{cases} c_1 \frac{1}{|x|^{1+\alpha}}, & x < 0, \\ c_2 \frac{1}{x^{1+\alpha}}, & x > 0 \end{cases} \tag{3.2}$$

with any parameters c_1 and c_2 and $\alpha \in (0, 2]$. The parameters c_1 and c_2 are responsible for asymmetry (symmetry) of the measure. When $c_1 > c_2$, the measure is skewed positively and vice versa. This formula for the $\alpha-$stable Lévy measure is essential for the following part of the work as for the evaluation of the claims when using Lévy processes it is necessary to integrate with respect to this measure.

Definition 3.2 (Cartea and Howison (2009),p.6)**.** *A Lévy process L_t is called an α-stable process if L_0 has independent increments; and $L_t - L_s$ follows an α-stable distribution with parameters $(t-s)^{1-\alpha}, \beta$ and 0 (i.e.$L_t - L_s \sim S_\alpha\left((t-s)^{1-\alpha}, \beta, 0\right)$) for any $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$ (time-homogeneity of the increments).*

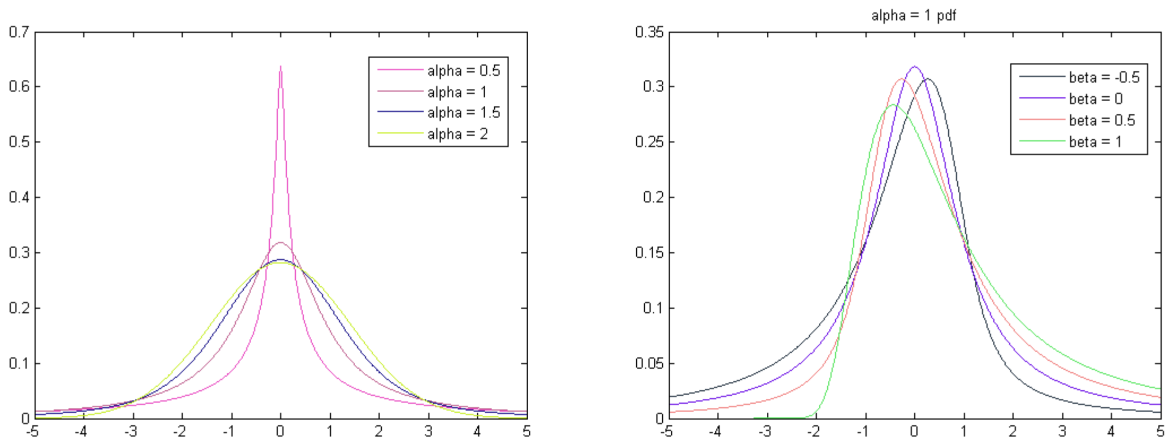
As we can see from definition 3.2, increments of the α-stable process follow an α-stable distribution. The α-stable distribution is a distribution that follows one important feature. Any linear combination of α-stable distributions results into another α-stable distribution. Any stable distribution is uniquely defined by 4 different parameters, α, β, c, μ. For example, if a random variable X follows an α-stable distribution, we can write $X \sim S_\alpha(c, \beta, \mu))$, where α is called a stability parameter, β is responsible for the skewness of the distribution and therefore is called a skewness parameter, c is a scale parameter and μ is the location parameter.

Alpha-stable distributions describe the whole set of the distributions. For the vast majority of the parameters, the distribution function cannot be written in a closed-form and therefore the whole distribution needs to be defined by its characteristic function as follows (Miyahara (2012),p.16)

$$\phi_{stable}(t; \alpha, c, \beta, \mu) = E\left[e^{itX}\right] = \begin{cases} \exp\left(i\mu t - |ct|^\alpha \left(1 - i\beta \operatorname{sign}(t) \tan \frac{\pi\alpha}{2}\right)\right), & \alpha \neq 1, \\ \exp\left(i\mu t - c|t| \left(1 + i\beta \frac{2}{\pi} \operatorname{sign}(t) \log|t|\right)\right), & \alpha = 1. \end{cases}$$

On the other hand, there are several well-know distributions, which pdf can be written in closed-form. One of the best-known distributions is the normal distribution. It is a specific case, when $\alpha = 2$. When $\alpha = 2$, regardless of the skewness parameter, we receive a normal distribution with scale parameter equal to $2c^2$ and location parameter equal to μ. Anther well-known distribution is for example Cauchy distribution ($\alpha = 1$, $\beta = 0$).

Figure 2 shows probability density functions for different $\alpha-$stable distributions. On the left, we can see distribution functions for different stability parameters with other parameters fixed ($\beta = 0$, $c = 0$, $\mu = 0$). A Gaussian distribution is the one with $\alpha = 2$, therefore the yellow curve. The figure shows that a decrease in stability parameter results in heavier tails. The second part of the figure shows, how is the skewness parameter, β, affecting the distribution function. The skewness parameter can be of any value from interval $[-1, 1]$. The figure shows, that while the distribution with negative β leans to the left (is positively skewed), the positive β creates distribution negatively skewed. The other parameter values are fixed ($\alpha = 1, c = 0, \mu = 0$).



(a) Probability density functions for varying α (b) Probability density functions for varying β

Figure 2: Probability density functions for different $\alpha-$ stable processes

One very important feature of an $\alpha-$stable distribution is its variance. Except of the normal distribution, all the other distributions have infinite variance. This variance is responsible for one big difference between the Brownian motion and the $\alpha-$stable Lévy process (any Lévy process in general as well). While the Brownian motion is continuous, the $\alpha-$stable Lévy process is not necessarily, as the infinite variance (or in general any jump process) can cause jumps. The difference can be seen in the 4th point of definition 3.1, which says, that even if the process is not continuous, there is an upper limit for the jump. For the visualization of jumps, see Figure 12.

For the simulation of interest rate paths, it is necessary to be able to generate random numbers from mentioned distributions. The generation of the random numbers is straightforward for distributions, that can be written in analytic form. However, as I already stated, the distribution has a closed-form distribution function only for specific parameters, for example for $\alpha = 2$ (normal distribution). In general, the closed-form distribution function is not available and thus to simulate a random number from a distribution, one has to follow Weron's algorithm introduced in Weron (1996b) and Weron (1996a). The algorithm for random variable $X \sim S_\alpha(1, \beta, 0)$ goes as follows (Weron (1996b), p.8)

1. *Generate a uniformly distributed random variable* V *on* $(-\frac{\pi}{2}, \frac{\pi}{2})$
2. *Generate exponentially distributed random variable* W *with mean 1*
3. *calculate* X *using the following formulas*
   - *for* $\alpha \neq 1,\ X = S_{\alpha,\beta} \cdot \frac{\sin\big(\alpha(V+B_{\alpha,\beta})\big)}{(\cos(V))^{1/\alpha}} \cdot \left(\frac{\cos\big(V-\alpha(V+B_{\alpha,\beta})\big)}{W}\right)^{(1-\alpha)/\alpha}$*,*
   where
   $$B_{\alpha,\beta} = \frac{\arctan\big(\beta \tan \frac{\pi\alpha}{2}\big)}{\alpha}$$
   $$S_{\alpha,\beta} = \big(1 + \beta^2 \tan^2 \tfrac{\pi\alpha}{2}\big)^{1/(2\alpha)};$$
   - *for* $\alpha = 1,\ X = \frac{2}{\pi}\left[\big(\frac{\pi}{2} + \beta V\big) \tan V - \beta \log\left(\frac{W \cos V}{\frac{\pi}{2} + \beta V}\right)\right].$

After obtaining the results for X, we can get any generalized random variable Y ($Y \sim S_\alpha(\sigma, \beta, \mu)$) by simple transformations. The generalized form for Y then follows from

$$Y = \begin{cases} \sigma X + \mu, & \alpha \leq 1 \\ \sigma X + \frac{2}{\pi}\beta\sigma \log \sigma + \mu, & \alpha = 1. \end{cases}$$

4 Interest rate modeling

Essential part of hybrid capital instruments valuation is the interest rate modeling. Section 4.1 shows, how do I construct the yield cruve for the instrument, while section 2.2 provides the short-rate model, namely Hull-White model. The last part of this section, 4.3 describes the methodology of the short-rate model calibration.

4.1 Yield curve preparation

An interest rate curve of an insurance company consists of two elements. The first is the risk-free rate and the second is the credit spread, that is specific for each company (or even differs for different capital instruments). There are some papers dealing with the issue of incorporation a credit spread for the interest rate modeling. For example, Schönbucher (1999) advocates, that there is a negative correlation between these two elements that should be taken into account. Nevertheless, during this work I simply add the spread component to the risk-free interest rate curve and fit the short-rate model to this new interest rate curve, that means that the initial zero curve looks as follows

$$y^{Inst} = y^{rf} + CS, \tag{4.1}$$

where y^{Inst} is the zero curve of the instrument, y^{rf} is the risk-free zero curve and CS is the credit spread curve of the corresponding instrument. Therefore to construct a term structure of an instrument, we have to construct the risk-free curve and the CDS curve first.

I begin with the risk-free curve modeling. As the instruments priced in this thesis are denominated in EUR, the initial risk-free zero curve needs to be fitted to the situation in the Eurozone. The initial zero curve is created using a Svensson (1994) model. This model is the typical model for zero curve modeling used by central banks all over the world. Svensson model parameters are published daily by the European Central Bank.[1] The formula for the Svensson model goes as follows

$$\begin{aligned} y^{rf}(\tau) = \beta_0^1 + \beta_1^1 \left[\frac{1 - \exp\left(-\frac{\tau}{\tau_1^1}\right)}{\frac{\tau}{\tau_1^1}} \right] &+ \beta_2^1 \left[\frac{1 - \exp\left(-\frac{\tau}{\tau_1^1}\right)}{\frac{\tau}{\tau_1^1}} - \exp\left(-\frac{\tau}{\tau_1^1}\right) \right] \\ &+ \beta_3^1 \left[\frac{1 - \exp\left(-\frac{\tau}{\tau_2^1}\right)}{\frac{\tau}{\tau_2^1}} - \exp\left(-\frac{\tau}{\tau_2^1}\right) \right], \end{aligned} \tag{4.2}$$

where $y(\tau)$ is the yield for maturity τ and $\beta_0^1, \beta_1^1, \beta_2^1, \beta_3^1, \tau_1^1, \tau_2^1$ are parameters provided by the ECB. By using the ECB input parameters for the Svensson model, I can determine the risk-free rate for any maturity.

While the Svensson parameters for the risk-free interest rates are provided by the ECB, parameters for the credit spread curve are estimated in this work. CDS of the instruments are quoted for different maturities on the market. The problem, that could arise is with the longest maturity. There are no CDS on the market with maturity longer than 10 years. While many of these instruments are maturing in more than 20 years, the initial yield curve needs to be fitted for longer maturities. To obtain spread values for any maturity, I use Nelson and Siegel (1987) model. I selected the Nelson-Siegel model because of the smaller number of model parameters.

[1] Data available at: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html`

While the Svensson model needs four input parameters, the Nelson-Siegel model needs only three. The Nelson-Siegel model used in this work can be written as follows

$$CS(\tau) = \beta_0^2 + \beta_1^2 \left[\frac{1 - \exp\left(-\frac{\tau}{\tau_1^2}\right)}{\frac{\tau}{\tau_1^2}} \right] + \beta_2^2 \left[\frac{1 - \exp\left(-\frac{\tau}{\tau_1^2}\right)}{\frac{\tau}{\tau_1^2}} - \exp\left(-\frac{\tau}{\tau_1^2}\right) \right], \tag{4.3}$$

where $CS(\tau)$ is the credit spread for maturity τ and $\beta_0^2, \beta_1^2, \beta_2^2, \tau_1^2$ are parameters estimated in this work. When constructing Nelson-Siegel model, parameter τ_1^2 has to be selected by the user and parameters β_0^2, β_1^2 and β_2^2 are estimated by linear regression. Therefore, we can see, that the selection of τ_1^2 is crucial to obtain a good fit for the model. In practice, τ_1^2 is usually selected such that the curvature of the model (the third term in Equation 4.1) peaks between maturity equal to 2 and 3 years, which corresponds to $\tau_1^2 \approx 1.35$.

As the credit spreads are not constant over time (therefore the values differ between trading days), Nelson-Siegel parameters are estimated on a daily basis. The regression is conducted such that the dependent variables are 8 CS values for different maturities τ (0.5Y, 1Y, 2Y, 3Y, 4Y, 5Y, 7Y, 10Y) and the indpendent variables are the corresponding loadings. The detailed figures for each trading day with the residual statistics are stated in Appendix A.6. The overall statistics (over the whole considered period) of the parameters used for yield curves preparation are shown in Table 1. Panel (a) shows the mean and the variance of Svensson parameters for the reference period (first 11 trading days in August). We can see, that the parameters are very stable over time, as the variance is negligible. The second panel shows the same statistics for the estimated Nelson-Siegel parameters for the CDS curve. As the credit spread curve is stated in basis points, the average values for estimated parameters are higher than for the risk-free rates. Alongside the higher means of the parameters, the variance captures bigger deviations between the CDS over time.

The constructed risk-free yield curve and the credit spread curve are on Figure 3. Both subfigures contain in total 11 different curves that correspond to the first 11 trading days in August 2017. As we can see, the risk-free curve is in the short end downward sloping and in negative numbers. The minimum (approximately -0.7%) of all curves is reached between maturities equal to 2 and 3 years. After reaching the minimum, the curves are upward sloping over all of the displayed tenors. The Svensson model predicts low rates in general, as the highest rates are not higher than 1.5%. The second subfigure displays credit spread of Nationale Nederlanden for our subordinated debt securities described in section 2.2. The credit spread between is 40 and 200 basis points. It is increasing and the dispersion between the spread curves is also growing with maturity. Points displayed by dark stars are the data retrieved from Bloomberg, while the other parts of the lines are modeled using the calibrated parameters of the Nelson-Siegel model.

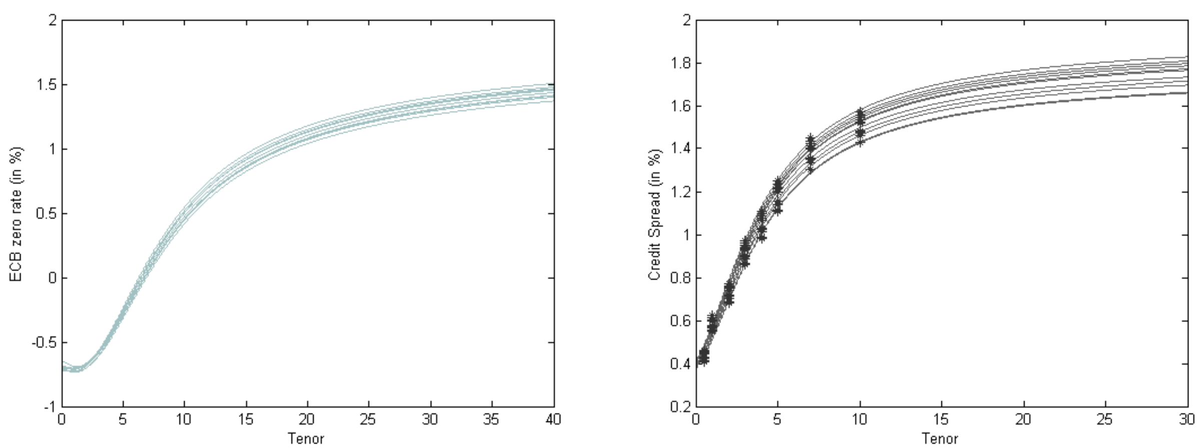
Parameter	β_0^1	β_1^1	β_2^1	β_3^1	τ_1^1	τ_2^1
mean	1.7721	-2.4757	23.3787	-27.5478	1.5621	1.6667
variance	0.0015	0.0012	0.0021	0.0021	0.0013	0.0012

(a) Statistics of the ECB Svensson model parameters for risk-free rate

Parameter	β_0^2	β_1^2	β_2^2
mean	186.7151	-146.8655	-124.5909
variance	36.8308	26.0181	4.4266

(b) Statistics of estimated Nelson-Siegel model parameters for credit spread of NN

Table 1: Statistics parameters for risk-free curve and CDS curve for first 11 trading days in August 2017



(a) Risk-free curve

(b) NN Credit spread curve

Figure 3: The constructed risk-free curve and the credit spread for first 11 trading days in August 2017

4.2 Short-rate model

After the initial term structure is modeled using equation (4.1), We can finally construct a short-rate model used for the instrument price evaluation. Instantaneous interest rates can be modeled using several models, for example Hul-White, Black-Karasinski, Vasicek, etc. The most used models in the literature are mainly first two mentioned models. Both of them have some advantages and also disadvantges. I use the Hull-White model during this thesis. This short-rate model is a generalized Vasicek model, that models interest rate using Wiener process (Gaussian process) with two unknown parameters. It is the crossover between the equilibrium model and a no-arbitrage model. These parameters are the mean reversion rate and the standard deviation of the short-rate.

The Black-Karasinski model is, on the other hand, model that models the log interest rates. The biggest difference is, that while the Hull-White model allows interest rates to be negative, the Black-Karasinski model does not. Therefore, it means that I assume the possibility of negative interest rates. Even though the interest rates in the eurozone are negative, when thinking about the interest rate of a company (that contains also the credit spread, not just the risk-free rates), the interest rates are rather positive (especially during economic conjuncture). Despite this fact, the work is build on the Hull-White model. It is the most popular and most used model for modelling instantaneous interest rates.

The partial differential equation that describes the Hull-White model is defined as follows

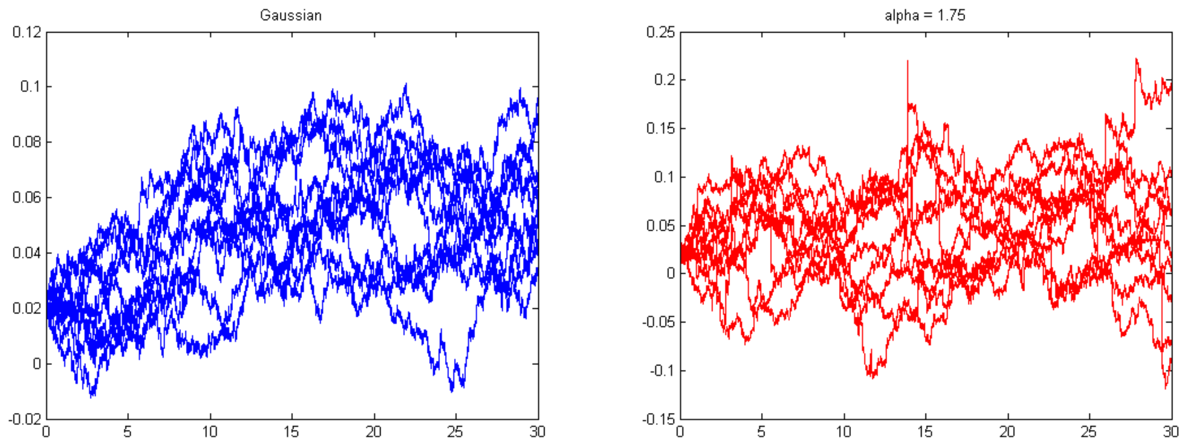
$$dr = (\theta(t) - ar)\, dt + \sigma dW_t, \tag{4.4}$$

where σ is the instantaneous standard deviation of the short rate and a is the mean reversion rate. The last unknown quantity, $\theta(t)$, is a function of time chosen to ensure that the model fits the initial term structure (4.1). Detailed explanation about the $\theta(t)$ construction are provided in Hull (2012). First term of the equation is responsible for the mean reversion fit over time and the second term is volatility, which is shifting up or down the level of r. Swishchuk (2008) proposes a generalized form of Hull-White model that stands as follows

$$dr = (\theta(t) - ar)\, dt + \sigma dL_t, \tag{4.5}$$

where L is the $\alpha-$stable Lévy process introduced in section 3. As we can see, the only difference between two models is the change of the Wiener process for the Lévy process.

The difference between the models in (4.4) and (4.5) can be easily seen on when looking at simulated interest rate paths. The process of simulation needs discretization of the mentioned differential equations. For every small time step a random number is simulated from the respective distribution. Therefore, for Brownian models I simulate a random number from a normal distribution with mean 0



(a) Gaussian model

(b) Alpha stable model ($\alpha = 1.75$)

Figure 4: Interest rate paths for different α

and variance δt and for Lévy model I simulate a number from α-stable distribution with parameters $((\delta t)^{1/\alpha}, \beta, 0)$. Thereafter for each time step $\theta(t)$ is obtained from the initial term structure (4.1). Figure 12 shows 10 simulated interest rate paths using different models. Subfigure (a) describes the Gaussian model and subfigure (b) the paths generated using Lévy model with stability parameter $\alpha = 1.75$. We can see that the jumps generated by Lévy models cause more extreme values than the Gaussian model.

4.3 Calibration

One of the most important parts, when modeling the instantaneous interest rate is to calibrate the parameters a and σ in (4.4) and (4.5) correctly. As the interest rate process is the essential and the same underlying process for all instruments on the market, the parameters are not calibrated to the instruments that we want to price, but rather to more essential instruments on the market. Thus the whole calibration in this work will be done on the at-the-money swaptions. Using this way of calibration, I explicitly assume, that the all of the uncertainty is captured by risk-free interest rates. Therefore the CDS curves are fixed and have no variation, which can be different in reality.

To calibrate the parameters, one has to select the proper "goodness-of-fit" measure. After this measure is selected, the minimization of this measure provides the correct calibrated parameters. The calibration used in this work is the calibration proposed by Hull (2012), which is the minimization of the quadratic difference. Interest rate models consist of two unknown variables, σ and a. Therefore the minimization is with respect to 2 parameters. The corresponding calibration technique can be written as follows

$$\underset{\sigma,a}{min} \sum_{j=1}^{M} \left(P_j^{market} - P_j^{model}\right)^2, \tag{4.6}$$

where P_j^{market} is the market price of calibrated instrument and P_j^{model} is the price obtained by the theoretical no-arbitrage evaluation and M is number of instruments used for calibration. The prices (P_j^{market} and P_j^{model}) reflect market and model prices of any instruments that are used for the model calibration.

In this thesis, the instruments used for the calibration are interest rate swaptions. As all the priced instruments are quoted in the euros, the swaptions used in this work are denominated in euros as well. For the better understanding of the calibration process, I explain the swaption meaning. A swaption is an option on the interest rate swap. Similarly, as with the other options, the swaptions can be of different types, such as European, American, Bermudan, or other exotic options. The swaptions used for the calibration are of European style, therefore the option can be called only at the date of maturity. Similarly as other options, there are call and put swaptions. However, we usually refer to them as receiver and payer swaptions.

The term swaption is closely connected to interest rate swaps. The interest rate swap is an agreement to exchange the future interest payments. The payer swaption is an agreement to pay fixed interests and receive floating interests in exchange. In this thesis I work with a typical example of a payer swap. Lets assume, that L is the notional of the swap and the swap payment starts at time T_n and finishes at time T_N, with fixed rate equal to s_0 and with m swap payments per year. These swap payments are the explained interest rate changes from one participant to another and vice versa.

To price the swaption and therefore to obtain the model price, one has to price the interest rate swap at first. The price of the swap is determined by the price of the fixed leg and by the price of the floating leg. The price of the fixed leg can be obtained as follows

$$s_0 \frac{L}{m} \sum_{i=n+1}^{N} P(0, T_i), \tag{4.7}$$

where $P(0, T_i)$ is a discount factor that discounts future payments at time T_i to present value. The price of the floating leg is

$$L \left(P(0, T_n) - P(0, T_N)\right). \tag{4.8}$$

From the equations above we can get the forward swap rate, that is a fair fixed rate, such that the swap value is at par. This leads to the following equation

$$s_0 = \frac{P(0, T_n) - P(0, T_N)}{\frac{1}{m}\sum_{i=n+1}^{N} P(0, T_i)}. \tag{4.9}$$

This equation provides us the forward swap rate necessary for the correct evaluation of the swap and therefore also of the swaption.

As a payer swaption is a term for the put option on the swap, we can use the traditional formulas for pricing the options to receive the swaption model price. The swaption model price goes after straightforward algebraic operations as follows

$$P^{model} = \frac{L}{m} max\left(s_0 - s_k, 0\right) \sum_{i=n+1}^{N} P(0, T_i), \tag{4.10}$$

where s_k is the strike rate of the swaption determined in advance. To obtain the model price of the swaption, we have to model the interest rates first and afterwards calculate the the corresponding discount factors and the forward swap rate. The discount factors are calculated the same way as I price the instruments. More elaboration on this procedure I state in sections 5.2.3 and 5.3.1.

The both price terms in calibration have to be calculated by different formulas. As the swaptions are quoted on the market in volatilities, the market price has to be calculated. The model for market price calculation of the swaption follows the famous Black-76 formula. This formula explicitly assumes, that the forward prices of any considered claim follows normal distribution. The Black-76 formula for the swaptions takes the following form

$$P^{market} = \frac{L}{m} \sum P\left(0, T_i\right)\left[s_0 N\left(d_1\right) - s_k N\left(d_2\right)\right] \tag{4.11}$$

where

$$d_1 = \frac{ln\left(\frac{s_0}{s_k}\right) + \sigma^2 \frac{T}{2}}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

$P\left(0, T_i\right)$ is the discount factor for time T_i and $N(d)$ is the CDF of normal distribution evaluated in d.

While the Black-76 formula assumes normality of the forward price changes, the model can be used just for the Guassian model calibration. As the vast majority of the work deals with the $\alpha-$stable Lévy processes, the normality assumption is not suitable for the calibration. Thus the model for the swaption market price is adjusted in this work. The interest rate model is assuming, that the forward prices are following $\alpha-$stable distribution and not normal distribution. Therefore the normal distribution function in the Black-76 model is replaced by the distribution function of the respective $\alpha-$stable distribution. The generalized Black-76 formula therefore looks as follows

$$P^{market} = \frac{L}{m} \sum P\left(0, T_i\right)\left[s_0 N_\alpha\left(d_1\right) - s_k N_\alpha\left(d_2\right)\right] \tag{4.12}$$

with N_α being the CDF of the $\alpha-$stable distribution. This generalized Black-76 formula is in this work used without any mathematical derivation as it goes beyond the scope of this thesis. However, the model was tested on the different numerical examples with the $\alpha = 2$ and the traditional Black-76 model and the calibration results were the same for the both models.

5 Pricing methods

In this section I provide the pricing methods for the instruments. In section 5.1 I describe the general pricing scheme, after which, in section 5.2, I describe and compare three different methods that can be used to evaluate the Gaussian models. The subsection 5.3 deals with the pricing method for the Lévy models.

5.1 Pricing scheme

For pricing a callable bond, we can use two different approaches. One is to price the bond and the option separately. Then it is enough to subtract the price of the call option from the price of a plain bond to obtain the final price of the instrument ($X_{callable} = X_{plain} - X_{call}$). In this thesis, I follow the second approach, that is pricing the instrument as a whole. This approach was proposed by d'Halluin et al. (2001), where a callable coupon paying bond is priced. In general, a coupon paying bond is similar to a bond without a coupon, however the final condition is equal to a principal plus coupon ($X(r, T_B; T_B) = P + C$).

d'Halluin et al. (2001) take into account also a difference between notice and call date. This period is a period between deciding to call the bond and the call date. In this work, I assume that the notice date and the call date are the same. When pricing the callable bonds, d'Halluin et al. divide the time frame into smaller parts where every part is the time between two call dates. Then they pick a reference interest rate which indicates, when it is worth to call the bond. If the simulated interest rate is lower than the reference interest rate then the company calls the instrument and vice versa. After dividing the time frame, we can price the instrument using backward recursion. That means that we start from the last call date and discount the price of the instrument from the final condition. Then we compare the interest rate to the reference rate. If it is not worth to call, we can add coupon and continue with discounting the price until the beginning of the time frame.

Suppose, that the instrument matures in τ years and the coupon is paid n times a year and there are m days left until the next coupon date. The P denotes the value of the principal and C the coupon. Moreover denote by t the order of a coupon payment and by X_t the price of the bond at t-th coupon payment. Therefore there are $n \cdot \tau$ coupon payments to be made until the maturity unless the instrument is called.

1. Set the price at the maturity (X_t) equal to $P+C$ (this is at the last coupon date, thus $t = n \cdot \tau$),
2. Evaluate the price of the bond at $t = n \cdot \tau - 1$ (prior coupon date)
3. Compare the discounted price (X_t) and the call price ($P+C$). If $P+C < X_t$ then the instrument should be called and therefore the price at time t is equal to call price ($X_t = P + C$), else the instrument price at time t is equal to a price obtained by discounting (X_t). Set $t = t - 1$ and repeat steps 2 and 3 until $t = 0$.
4. Once there are no more coupon payments, discount the last obtained price from the first coupon date for the remaining m days.

As we can see, the essential part of the work is the correct discounting between two call dates. This is done in two different ways in this thesis. The first I call the Simulation approach. This approach is using the simulated interest rates and subsequent discounting of the instrument prices. The second, PDE approach is based on the evaluation of the Black-Scholes partial differential equation for Gaussian models and the generalized partial integro-differential equation for Lévy models. The next sections elaborate mainly on the second point of the pricing scheme.

5.2 Gaussian models

I describe the partial differential equation used for the hybrid capital instrument evaluation in subsection 5.2.1. To price the instruments, when the interest rates are modeled by Gaussian models, I use three different methods. The first method (PDE-FVM)[1] is slightly modified approach proposed by d'Halluin et al. (2001), that uses finite volume method to solve the Black-Scholes PDE described in subsection 5.2.2. The second method (PDE-FDM)[2] is described in subsection 5.2.3 and is based on the finite difference method. For this method, I slightly modify the pricing scheme. Moreover, it is also the method used for the instrument pricing when the short-rate interest rates are modeled by Lévy processes. Finally, in subsection 5.2.4 I propose the third method (Simulation method) that is based on interest rate simulation and subsequent price discounting,

5.2.1 Black-Scholes PDE

As mentioned in the overview of this section, for evaluation of Gaussian models, I use 2 numerical methods based on partial differential equation. This equation is the famous implementation of the

[1]The abbreviation stands for Partial Differential Equation - Finite Volume Method
[2]The abbreviation stands for Partial Differential Equation - Finite Difference Method

heat equation in finance by Fischer Black and Myron Scholes. For the correct evaluation, I state the equation first. Consider the stochastic differential equation of the interest rate in general form

$$dr(t) = a(r,t)dt + b(r,t)dW(t), \tag{5.1}$$

where $a(r,t)$ and $b(r,t)$ are functions of r and t.Then the Gaussian Bond pricing for One-factor stochastic interest rate model via partial differential equation is

$$\frac{\partial X}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 X}{\partial r^2} + (a - \lambda b)\frac{\partial X}{\partial r} - rX = 0, \tag{5.2}$$

with final condition $X\left(r, T_B; T_B\right) = Z$, where the function λ is often called the market price of risk. The market price of risk is the compensation for undertaken additional risk.

This parameter, λ, is necessary for all equilibrium interest rate models (such as Vasicek, or Cox–Ingersoll–Ross model) to ensure arbitrage-free pricing. On the other hand, no-arbritrage models, such as considered Hull-White model are constructed in such manner, that the market price of risk is already contained in $\theta(t)$. The considered model is a model for coupon free bond. In case, when the instantaneous interest rate is modeled by Hull-White model described by equation 4.4, the parameter $a(t,r)$ is equal to $\theta(t) - a \cdot r$ and $b(t,r) = \sigma$. The partial differential equation for Hull-White model goes then as follows

$$\frac{\partial X}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 X}{\partial r^2} + (\theta(t) - ar)\frac{\partial X}{\partial r} - rX = 0. \tag{5.3}$$

Parameters t and r are independent variables. This equation describes the price behavior of our instrument between two call dates. To obtain the final price, we need to solve this PDE using numerical methods.

5.2.2 Finite volume method

When using numerical methods to evaluate the PDE stated in equation 5.3, a two-dimensional grid has to be created. The position on the grid represents the time and the level of the interest rate. In this thesis, the time level is represented by $n \in [0, N]$ and the interest rate level by $i \in [-I, I]$. Selection of the intervals is crucial. As the Hull-White model is unbounded, the interest rate at a specific time level can be of any value. Moreover, the price of a claim at level i and time n depends the price at both higher and lower price at following time level. Therefore when selecting the grid boundaries, the maximum level of the interest rate (I) should be high enough to take into account also this feature. The length of the time interval is given as the interval between two subsequent call dates. However, it is important to select the time step small enough to obtain accurate results.

I introduce the **PDE-FVM** as the first out of two numerical methods. This method was used by d'Halluin et al. (2001) to price a callable bond and is slightly modified for this thesis. It is used to compute the claim value between two call dates. It uses a discretization technique called the finite volume method to solve the equation (5.3). This finite volume method is based on the decomposition of the problem into so-called control volumes. We can think of the control volume as a small amount of volume around each point in our two-dimensional grid. The method used in this thesis comes from a general version proposed by Kroner (1997). The derivation is stated in Appendix A3 and the final formula looks as follows

$$X_i^n = -\frac{\Delta\tau}{A_i}\left(\sum_{j\in\eta_i}\alpha_{ij}\left(X_j^{n+1} - X_i^{n+1}\right) + \sum_{j\in\eta_i} L_{ij}\cdot V_i X_{ij+\frac{1}{2}}^{n+1} - r_i A_i X_i^{n+1}\right) + X_i^{n+1}. \tag{5.4}$$

where

$$A_i = \frac{r_{i+1} - r_{i-1}}{2}, \qquad V_i = \theta(n) - ar_i,$$

$$\eta_i = \{i+1, i-1\} \qquad \alpha_{ij} = \frac{\sigma^2}{2|r_j - r_i|},$$

$$\Delta\tau = \tau^{n+1} - \tau^n, \qquad L_{ij} = \begin{cases} -1, & \text{if} j = i+1, \\ 1, & \text{if} j = i-1. \end{cases}$$

Moreover, X_i^n represents the price of our instrument at time at node n and interest rate level corresponding to node i. We can see, that V_i is dependent only on Hull-White parameters and the other variables on the interest rate levels. $X_{ij+\frac{1}{2}}^n$ represents the value of our claim between two volumes. The value for $X_{ij+\frac{1}{2}}^n$ can therefore be calculated as follows

$$X_{ij+\frac{1}{2}}^n = \frac{X_i^n + X_j^n}{2}.$$

We see that X_i^n depends on the price of this claim one step ahead in time at three different interest rate levels ($\{i-1, i, i+1\}$). This method is so-called explicit method, as we can obtain the price of a claim (at time n and interest rate level i) explicitly.[3]

As this approach uses PDE to price the instrument only between two call dates, simulated interest rate paths are needed. The prices are evaluated for every simulated path separately. After reaching the next call date, the price of a claim for the interest rate path is determined. It is done by picking the grid position that corresponds to the simulated instantaneous interest rate at that time. If this price is higher than the call price, the price is reset to principal plus coupon. Otherwise the price for the corresponding interest rate path stays unchanged and the pricing scheme continues as described in section 5.1. After obtaining all the prices for all of the interest rate paths at $t = 0$, the final price is equal to the simple average of the individual prices.

[3]For a general overview about different numerical methods, please refer to Appendix A2.

5.2.3 Finite difference method

The second method, how to evaluate Black-Scholes equation when using Gaussian models is **PDE-FDM**. As already mentioned, I use this method also to evaluate PIDEs for Lévy models. The main difference is, that the PDE is evaluated not only between 2 different call dates, but over the whole time period until the maturity of an instrument (this means that the grid is created for the whole maturity and not only for the period between two call dates). Therefore the numerical method has to be more precise and stable, as the errors are accumulating over longer time. Another improvement in comparison to the first method described in 5.2.2 is, that the interest rate paths simulation is not needed. The situation at the call dates is solved naturally. For all the grid points, at which the price of a claim is higher than the call price, is the price set to the value equal to principal plus coupon and thereafter the prices for the next time steps are evaluated.

The discretized partial differential equation can therefore be written as follows

$$\begin{aligned}\frac{X_i^{n+1}-X_i^{n+1}}{\Delta t} = \\ &\gamma\left(-\frac{1}{2}\sigma^2\frac{X_{i+1}^{n+1}-2X_i^{n+1}+X_{i-1}^{n+1}}{\Delta r^2}-(\theta(n+1)-ar_i)\frac{X_{i+1}^{n+1}-X_{i-1}^{n+1}}{2\Delta r}+r_iX_i^{n+1}\right)+\\ &(1-\gamma)\left(-\frac{1}{2}\sigma^2\frac{X_{i+1}^{n}-2X_i^{n}+X_{i-1}^{n}}{\Delta r^2}-(\theta(n)-ar_i)\frac{X_{i+1}^{n}-X_{i-1}^{n}}{2\Delta r}+r_iX_i^{n}\right).\end{aligned} \tag{5.5}$$

The equation (5.5) is generalized version for any finite difference method. By setting different values to the scaling parameter, γ, the formula can represent fully explicit ($\gamma = 1$) or fully implicit method ($\gamma = 0$). When selecting $\gamma = \frac{1}{2}$, the equation is transformed into so-called Crank-Nicolson method. This method belongs to mixed methods and for evaluation of equation, one has to solve system of linear equations at each time step. It is also the one method that is used in this thesis.

We see from the equation (5.5), that the price at time level n and at interest rate level i depends on values at 3 different nodes one step further in time ($n+1$). This holds for any interest rate level, except for $i = 1$ and $i = I$, what corresponds to the price that belongs to the lowest interest rate on the grid and the highest rate on the grid. These two values depend only on 2 values one step further in time. If we consider the price at the highest node (for the highest interest rate on the grid), there are only 2 known values from equation (5.5) as the X_{i+1}^n exceeds the grid borders. The same applies for the lowest node. Moreover, we can see, that also every node on the grid at time $n+1$ affects 3 different nodes at time n (again, except for the highest and the lowest node). Therefore we can solve this system of equations using the tridiagonal algorithm (Thomas algorithm).

By setting $\gamma = \frac{1}{2}$ and after some linear algebra, the tridiagonal system can be created by tridiagonal matrix A, vector of unknown values of a claim at time n (X^n) and vector of known values,

d^{n+1}. If we denote for all $i = 1, 2, ..., I$

$$A_0(i,n) = -\frac{r_i}{2} - \frac{\sigma^2}{2\Delta r} - \frac{1}{\Delta t}, \qquad A_1(i,n) = \frac{\sigma^2}{4\Delta r^2} + \frac{\theta\,(n) - ar_i}{4\Delta r}$$

$$A_{-1}(i,n) = -\frac{\theta(n) - ar_i}{4\Delta r} + \frac{\sigma^2}{4\Delta r^2},$$

$$\begin{aligned} d_i^{n+1} = & \frac{1}{4}\left(-\frac{\sigma^2}{\Delta r^2} - \frac{\theta\,(n+1) - ar_i}{\Delta r}\right) X_{i+1}^{n+1} + \left(\frac{r_i}{2} + \frac{\sigma^2}{2\Delta r} - \frac{1}{\Delta t}\right) X_i^{n+1} \\ & + \frac{1}{4}\left(\frac{\theta(n+1) - ar_i}{\Delta r} - \frac{\sigma^2}{\Delta r^2}\right) X_{i-1}^{n+1}, \end{aligned}$$

the mentioned matrix and vector look as follows

$$A = \begin{pmatrix} A_0(1,n) & A_1(1,n) & 0 & 0 & 0 & 0 & 0 \\ A_{-1}(2,n) & A_0(2,n) & A_1(2,n) & 0 & 0 & 0 & 0 \\ 0 & A_{-1}(3,n) & A_0(3,n) & A_1(3,n) & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_{-1}(I-1,n) & A_0(I-1,n) & A_1(I-1,n) \\ 0 & 0 & 0 & 0 & 0 & A_{-1}(I,n) & A_0(I,n) \end{pmatrix}$$

$$X^n = \begin{pmatrix} X_1^n \\ X_2^n \\ \vdots \\ X_I^n \end{pmatrix}.$$

The price at time level n can be calculated using following equation

$$X^n = A^{-1} d^{n+1}. \tag{5.6}$$

Moreover, at the call dates, the price X^n is compared at each node to the call price. The lower of the two prices is taken as a final price at the node. This whole procedure continues until we reach $n = 0$, when we get the final price of a claim. For more detailed explanation of the method and mathematical derivation, please refer to Appendix A4.

5.2.4 Simulation method

The first pricing method I describe is the Simulation method. The whole approach is based on the simulation of multiple instantaneous interest rate paths. Every interest rate path is simulated by discretizing the Hull-White differential equation. The instantaneous interest rate is therefore, strictly speaking, not continuous, but discrete between two time steps. As the interval between these time steps is extremely small, it is a good approximation of continuous instantaneous interest rate. The fact, that these interest rates are discrete, is used for the evaluation of the price of a claim using backward discounting approach.

The price of an instrument is discounted between two time points by the interest rate generated at that time point. Therefore the price at a call date at time t_C when knowing the price at time t_{C+1} is given by formula

$$X^i_{t_C} = X^i_{t_{C+1}} \exp(-\Delta t \cdot r^i_{t_{C+1-\Delta t}}) \cdot \exp(-\Delta t \cdot r^i_{t_{C+1-2\Delta t}}) \dots \exp(-\Delta t \cdot r^i_{t_C}). \tag{5.7}$$

After obtaining the price at time t_C, we follow the third and fourth point of the pricing scheme in section 5.1. This procedure gives us the price of an instrument for one simulated interest rate path. By simulating many interest rate paths (consider N paths), the expected value gives us the correct arbitrage-free price for the Gaussian models

$$X_0 = \frac{1}{N} \sum_{i=1}^{N} X^i_0. \tag{5.8}$$

5.3 Lévy models

In this section I describe the pricing method for Lévy processes. In subsection 5.3.1 I describe the partial integro-differential equation that needs to be solved in order to obtain instrument prices. In subsection 5.3.2 I propose the numerical method used for evaluation of the equation stated in 5.3.1.

5.3.1 PIDE for Lévy model

To price the instruments when the underlying interest rate is modeled by Lévy process, I use only the last method (PDE-FDM) modified to ensure the accuracy for Lévy models. The other two methods are inappropriate because of the need of the interest rate simulation. The simulation method uses simulated multiple interest rate paths to determine the final price. As the $\alpha-$stable distribution has infinite variance, the expected value of the final price converges significantly slower. This increases the computational time substantially, what makes this method improper. The PDE-FVM also needs the simulated interest rate paths. Moreover, while this method needs also the numerical evaluation, the computational time is even higher.

For pricing the instruments using Lévy models, Swishchuk (2008) shows the corresponding partial integro-differential equation. Let suppose, that $(\sigma_L, \nu(dy), \delta)^4$ is a Lévy process generating triplet already defined in section 2.2. The $\nu(dy)$ represents the Lévy measure stated in equation (3.2). When pricing the instruments using Lévy processes I intend to follow the **PDE-FDM** explained in section 5.2.3. The only difference is in the partial differential equation, which will be changed for the partial integro-differential equation defined in Swishchuk (2008). The corresponding partial integro-differential equation is as follows

$$\begin{aligned}&\frac{\partial X}{\partial t} + \frac{1}{2}b^2\sigma_L^2\frac{\partial^2 X}{\partial r^2} + (a + b\delta - \lambda b\sigma_L)\frac{\partial X}{\partial r}\\ &\quad + \int_{-\infty}^{+\infty}\left[X(t, r + by) - X(t,r) - by\frac{\partial X(t,r)}{\partial r}\right]\nu(dy) - rX = 0,\end{aligned} \tag{5.9}$$

with the final equation $X(r, T_B; T_B) = Z$.

The general partial integro-differential equation has to be adjusted for Hull-White model driven by α-stable Lévy process. Using the fact, that Hull-White interest rate model is arbitrage-free, thus the market price of risk is equal to 0 the adjusted PIDE looks as follows

$$\begin{aligned}&\frac{\partial X}{\partial t} + ((\theta(t) - ar) + \sigma\delta)\frac{\partial X}{\partial r}\\ &\quad + \int_{-\infty}^{+\infty}\left[V_B(t, r + \sigma y) - X(t,r) - \sigma y\frac{\partial X(t,r)}{\partial r}\right]\nu(dy) - rX = 0.\end{aligned} \tag{5.10}$$

We see, that the difference between the PIDE for the Lévy models and the PDE for the Gaussian model described by equation (5.3) is that the PIDE is missing the second order derivative with respect to the interest rate and contains the integral part.

5.3.2 Finite difference method

It is possible to obtain the solution of the PIDEs using numerical approach, which involves discretizing the equations. To solve the PIDE's, there are several other papers in the literature, describing how to solve these types of equations. For example, Duffy (2005) describes different finite difference methods that can be used to obtain results, namely Explicit and Implicit, IMEX, Operator Splitting and

[4]Note, that I use different order of elements in the triplet from Swishchuk, who uses triplet notation in order $(\delta, \sigma_L, \nu(dy))$

Predictor-Corrector method. In this work, I use the Crank-Nicolson method based on Duffy (2005). The procedure of the equation evaulation is basically the same as for partial differential equations. The only difference is the presence of the integral part in the equation. Duffy (2005) proposes the evaluation of the integral at time level, at which the prices are already known.

Crucial part is the evaluation of the integral with respect to α-stale Lévy measure on infinite interval. Assume, that one wants to evaluate integral $\int_{\mathbb{R}} f(x) \cdot g(dx)$, where f is integrand and $g(dx)$ is a measure given by a function. This integration is equivalent to an integral $\int_{\mathbb{R}} f(x) \cdot g(x) dx$. Therefore in our case, we have to evaluate typical Riemann integral of a multiplication of two functions. To cope with the numerical integration over infinite interval, boundaries are introduced. In general, the boundaries have to be selected very carefully to ensure approximately correct evaluation of an integral. The function of the α-stable measure defined as in (3.2) is rapidly decreasing on the tails. This allows me to not consider extreme values for the boundaries.

When using numerical methods, the region of integration is a bounded symmetric interval $[B_l, B_u]$, where B_l is equal to $-B_u$. After selecting lower and upper boundary, the integral can be evaluated. The integral discretization closely follows Cont (2003).

$$
\begin{aligned}
Int_i^n = & \int_{B_l}^{B_u} \left[X\left(t, r + \sigma y\right) - X(t, r) - \sigma y \frac{\partial X(t, r)}{\partial r} \right] \nu(dy) \approx \\
& \sum_{j=K_l}^{K_u} \left[X\left(n+1, r_i + \sigma y_j\right) - X(n+1, r_i) - \sigma y_j \frac{X(n+1, r_{i+1}) - X(n+1, r_{i-1})}{2\Delta r} \right] \nu(y_j) \Delta y,
\end{aligned}
$$

where $K_l = -K_u$ and therefore $j \in \{-K_u, -K_u + 1, ..., -1, 0, 1, ..., K_u\}$. The value of y_0 is equal to 0 and $y_j = j\Delta y$ to hold the symmetricity of the region of integration. The Moreover, I set $\Delta r = \Delta y$ during this thesis. We see that the value of this integral at time n depends only on the claim values that are already known (values at time $n+1$). We therefore do not have any unknown values within the integral that need to be solved during the pricing scheme.

After obtaining the integral value, we can state the final discretization of equation (5.10), that goes as follows

$$
\begin{aligned}
\frac{X_i^{n+1} - X_i^{n+1}}{\Delta t} = & \\
& \gamma \left(-(\theta(n+1) - a r_i + \sigma \delta) \frac{X_{i+1}^{n+1} - X_{i-1}^{n+1}}{2\Delta r} + r_i X_i^{n+1} \right) + \\
& (1 - \delta) \left(-(\theta(n) - a r_i + \sigma \delta) \frac{X_{i+1}^{n} - X_{i-1}^{n}}{2\Delta r} + r_i X_i^{n} \right) + Int_i^n.
\end{aligned}
\tag{5.11}
$$

The equation (5.11) is generalized version for type of finite difference method. Similarly as in section 5.2.3, I set parameter $\gamma = \frac{1}{2}$ to obtain the Crank-Nicolson method. Again, by some linear algebra we can obtain the necessary formulas for our tridiagonal matrix[5], where

$$A_0(i,n) = -\frac{r_i}{2} - \frac{1}{\Delta t}, \qquad A_1(i,n) = \frac{\theta\left(n\right) - ar_i}{4\Delta r}$$

$$A_{-1}(i,n) = -\frac{\theta(n) - ar_i}{4\Delta r},$$

$$\begin{aligned} d_i^{n+1} &= \frac{1}{4}\left(-\frac{\theta\left(n+1\right) - ar_i + \sigma\delta}{\Delta r}\right) X_{i+1}^{n+1} + \left(\frac{r_i}{2} - \frac{1}{\Delta t}\right) X_i^{n+1} \\ &+ \frac{1}{4}\frac{\theta(n+1) - ar_i + \sigma\delta}{\Delta r} X_{i-1}^{n+1} - \Delta t Int_i^n. \end{aligned}$$

the tridiagonal system can be now created by tridiagonal matrix A, vector of unknown values of a claim at time n (X^n) and vector of known values, d^{n+1}. The mentioned matrix and vector look as follows

$$A = \begin{pmatrix} A_0(1,n) & A_1(1,n) & 0 & 0 & 0 & 0 & 0 \\ A_{-1}(2,n) & A_0(2,n) & A_1(2,n) & 0 & 0 & 0 & 0 \\ 0 & A_{-1}(3,n) & A_0(3,n) & A_1(3,n) & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_{-1}(I-1,n) & A_0(I-1,n) & A_1(I-1,n) \\ 0 & 0 & 0 & 0 & 0 & A_{-1}(I,n) & A_0(I,n) \end{pmatrix},$$

$$X^n = \begin{pmatrix} X_1^n \\ X_2^n \\ \vdots \\ X_I^n \end{pmatrix}.$$

The price at time level n can be calculated using following equation

$$X^n = A^{-1} d^{n+1}. \tag{5.12}$$

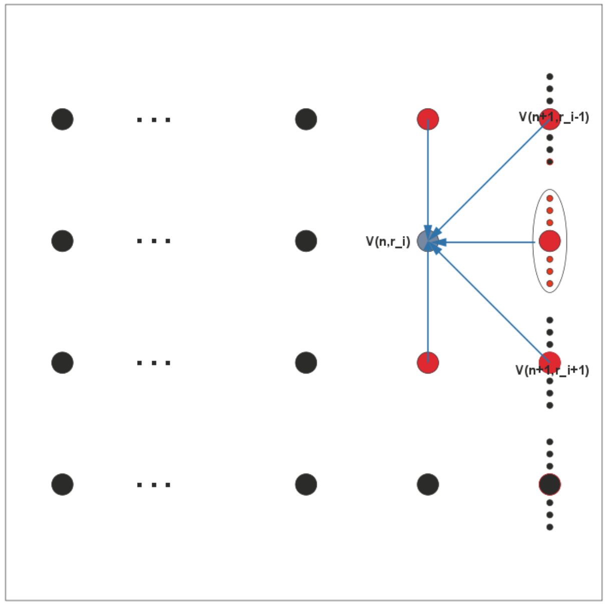
This procedure goes until the next call date, where the price of the instrument at each node is compared to the call price. The lower of the two prices is taken as a final price at the node. This whole procedure then continues until we reach $n = 0$.

[5]The derivation is done similarly as in 5.2.3

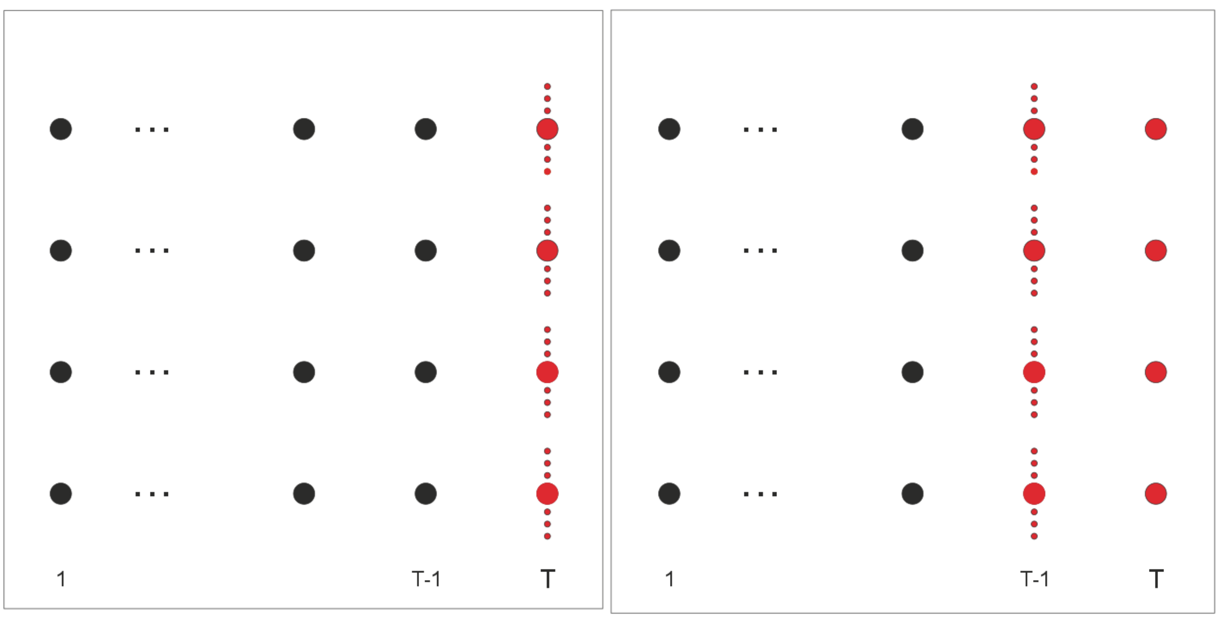
When evaluating the partial integro-differential equation, the $X(t, r + \sigma y)$ plays an important role. As this term is integrated with respect to our measure on a symmetric interval, possible issues can arise on the interest rate grid boundaries. As y in the term takes values on a symmetric interval $[-B_u, B_u]$, the evaluation of this interval on the first and the last interest rate points for a specific time step is not defined (as $X(n+1, 1-\sigma y_i)$ is not defined for nonzero σ and y_i and the same applies for the $X(n+1, r_K + \sigma y_i)$). To solve this issue, I introduce the method based on the extrapolation of the price of the claim. For the purpose of the integral evaluation, I extrapolate the prices at time $t+1$ on the boundaries. Let assume, that we have interest rate grid on interval $[-I, I]$. The extrapolated values create another grid points outside of this interval and therefore to evaluate the prices at time t on the interval $[-I, I]$, I use values at time $t + 1$ on the interval $[-I - 5\sigma\Delta r, I + 5\sigma\Delta r]$.

Another problem occures with the density of the grid points, not only with the missing values on the boundaries. The same issue arises on any interior grid point. For the correct valuation of a claim at grid point (n, r), we need to know the values at all points $(n, r + \sigma y)$, where $y \in [-B_t, B_t]$. As $\Delta r = dy$ and $\sigma < 1$, the value of the claim at any point $(n, r + \sigma y)$ is unknown. To solve this issue, I interpolate the instrument values for respective points using cubic spline. These interpolated values are then used to evaluate the integral in (5.11), what is necessary to price the claim. The interpolated values are only temporary to ensure small computational time.

The whole procedure of the valuation is described on Figure 5. Subfigure 5 (a) shows the scheme of the Crank-Nicolson method for the partial integro-differential equation. In comparison to the PDE approach, we need more values from the following time step. As visualised on the figure, the small dots represent the interpolated and extrapolated values. The number of necessary interpolated values depends on the measure (how quickly are the tails decreasing) and the required accuracy. Subfigure 5 (b) shows the progression of the claim valuation. After knowing the values at a specific time (T), I interpolate and extrapolate the values for the other necessary interest rate levels, thereafter I obtain the prices at time $T - 1$. After obtaining these prices, the procedure is replicated until we obtain the values at time $t = 1$.



(a) PIDE evaluation explanation



(b) The first and the second step of PIDE evaluation

Figure 5: PIDE evaluation explanation

6 Results

I divide the result part of this work into 4 sections. The first section, 6.1, presents the calibration results. In sections 6.2 and 6.3 I demonstrate the prices of theoretical instruments and the real instrument prices, respectively. The final part of this chapter, section 6.4, displays the instrument price depenedence on changing parameters.

6.1 Calibration results

In section 4.3 I state the calibration method for this work. The Hull-White parameters in equations (4.4) and (4.5), a and σ are calibrated to the interest rate swaptions. For this calibration, I use the famous Black-76 formula, which assumes log-normality of the forward prices, what implicates the log-normal volatilities (so-called Black volatilities). While the current interest rates are below zero, the log-normal volatilities are not available for the interest rate swaptions denominated in EUR. The problem with missing swaption data is, however, only local. This means, that when the interest rates are expected to be above zero (and the zero curve for long enough tenors is above zero), the Black volatilities are available.

The only negative effect of negative interest rates for the calibration is, that the swaptions with short enough maturity cannot be priced by the Black-76 formula. Therefore, these swaptions are not included in the calibration procedure.Concretely, I use 72 swaptions with different maturities and different tenors. The maturities of the swaptions are 5, 6, 7, 8, 9, 10, 12 and 15 years and for all of these maturities we have 9 swaptions with different tenors (1Y, 2Y, 3Y, 4Y, 5Y, 7Y, 10Y, 12Y, 15Y). Therefore the respective calibrated parameters are the ones, that minimize the equation (4.6) over all all 72 instruments. The respective volatilities are stated in the Appendix. The whape of the error function is showed on figure 6.

Figure 6 shows the error function from equation (4.6) for different parameters. The left subfigure shows the error function for Gaussian model (or Lévy model with $\alpha = 2$), while subfigure on right side shows the error function for Lévy model with stability parameter equal to 1.7. Both figures are nearly identical. We can see, that the graphs take U shape, where the global minimum is in the middle part and the value is increasing on both sides of the graphs (with increasing parameter a with stable value of σ and vice versa). The minimized parameters for all Lévy models (and Gaussian model as well) are $a = 0.02$ and $\sigma = 0.01$, therefore these are the parameters used for the pricing of

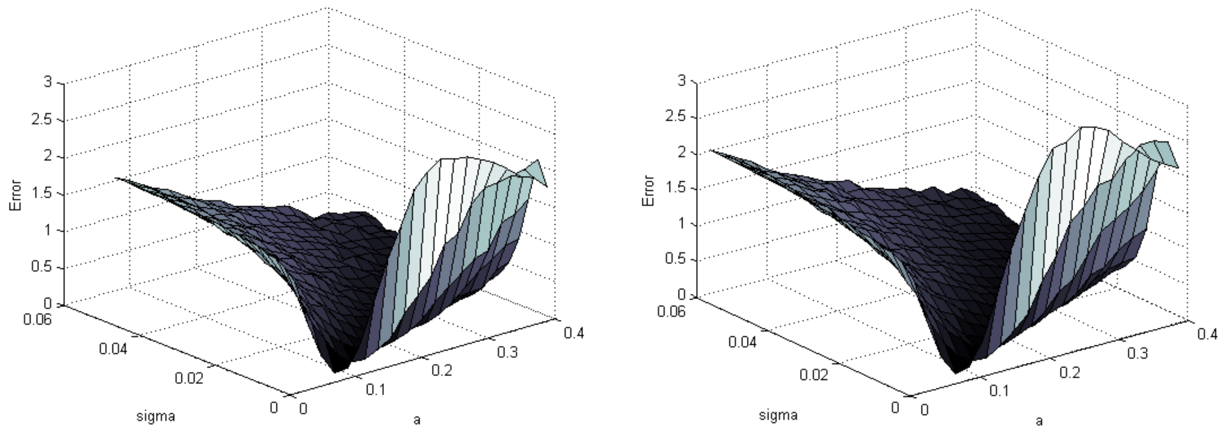


Figure 6: Calibration error for Gaussian model and Lévy model ($\alpha = 1.7$)

the instruments. However, other pairs of parameters minimize the error function similarly well. The errors of more than 10 pairs of parameters are within 15% range from the error of the smallest pair of parameters Thus the investigation of the instrument price dependence on the parameters is crucial and conducted in section 6.4.

6.2 Final prices of theoretical instruments

As mentioned in section 2 this work investigates the price behavior of three theoretical instruments. All of these instruments are priced using Gaussian models by 3 different methods, called PDE-FVM, PDE-FDM and Simulation method (Backward discounting) introduced in section 5.2. The PDE-FDM method is the mixed numerical method, the PDE-FVM is the explicit method, that uses the simulated interest rates as well and the Backward discounting is using only backward discounting to determine the final prices of the instruments. The results for Lévy models are obtained using the finite difference method stated in section 5.3.2. This method is the generalized version of PDE-FDM.

To obtain prices as accurately as possible, the proper interest rate and time steps are necessary. In this thesis, I work with time step equal to 1/1008 of a year (prices are updated 4 times a day). The interest rate step (Δr) is 0.1%. This level can cause some problems, as the initial interest rate level (that is needed to obtain the final prices using PDE and PIDE methods) can be between two subsequent interest levels. In such situations, the final price of the instrument is obtained by linear extrapolation between these two levels. When creating the grid for the numerical methods, the interval for the interest rates is $[-10\%10\%]$. For the PDE-FVM method, I simulate 100 interest rate paths. For the purpose of the Backward discounting approach, 200 interest rate paths are simulated and thereafter the instrument price is obtained. This procedure is repeated 10 times and the average of these 10 prices is the final instrument price.

Table 2 shows prices for our theoretical instruments using different methods. All of the models are calibrated to the same parameter values that were stated in section 6.1. The final prices of all instruments using Gaussian models are between 99% and 100% of the principal. When using Lévy models, we can see also smaller values. The differences between the Gaussian numerical methods are negligible, however steadily growing with longer maturity. This anomaly can be seen when comparing any of 2 numerical methods with the Backward discounting approach. Moreover, to provide similar accuracy with Backward discounting method as the other 2 methods for the instruments with longer maturity needs more interest rate paths and therefore is more time consuming.

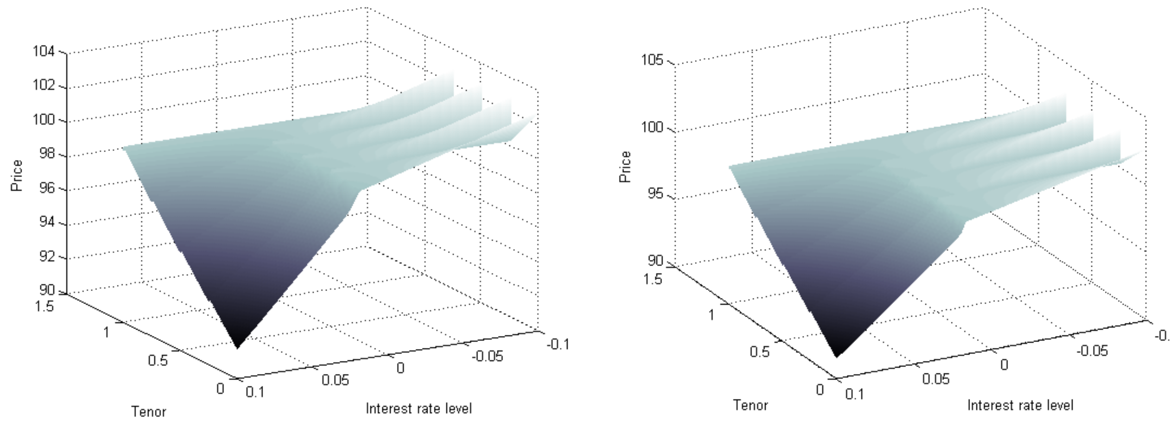
The second part of the table is about the instrument prices obtained for Lévy models. As the Lévy models cannot be approximated by the simulation methods, the only way of pricing the instruments is by the PIDE approach (generalized PDE-FDM), therefore the mixed numerical method. The prices for the theoretical models are obtained for 3 different stability parameters ($\alpha = 1.7, 1.8$ and 1.9). We can see, that the price for the instruments is decreasing with decreasing stability parameter. While the price difference is relatively small for the instruments T-1, and T-3, the price difference for the instrument T-5 is much higher. One of the main reasons, why this is the case, is the frequency of call dates. When the call dates are less frequent, the effect of the call on the price is not that major, while with more frequent call dates the possibility of calling the instrument is influencing the next prices much more.

Prices for different callable bonds						
Type	PDE-FDM	PDE-FVM	Simulation method	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$
T-1 (1% quarterly)	99.0188	99.0137	99.0244	98.6251	98.6621	98.7681
T-3 (2% quarterly)	99.9252	99.9694	99.8930	99.9094	99.9117	99.9162
T-5 (2% annually)	99.8468	99.8926	99.9324	94.4651	97.7153	99.6952

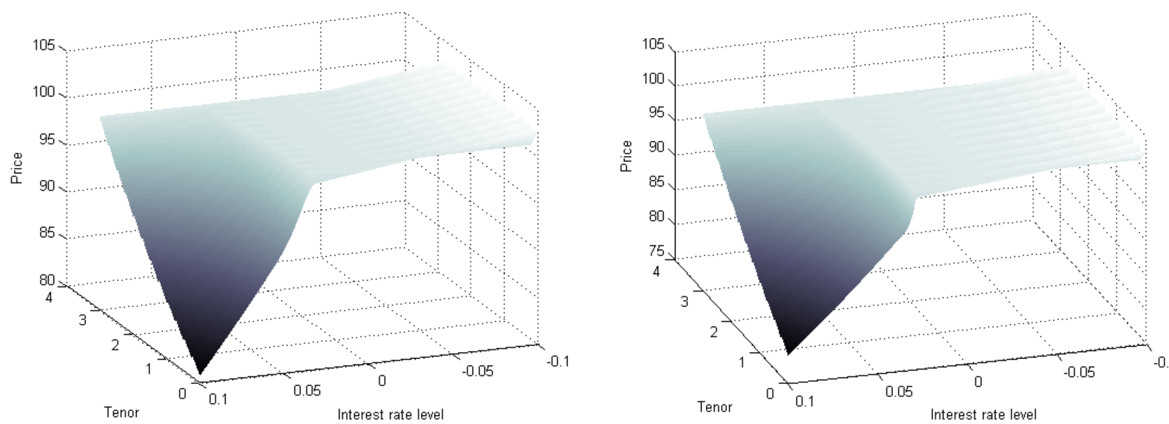
Table 2: Prices for different callable bonds using different models

Figure 7 shows the price level of the theoretical instruments of the Gaussian model and at the Lévy model ($\alpha = 1.7$) under different interest rate levels and tenor levels. We can see, that the price of the instruments is decreasing with decreasing time to maturity and high interest rates (approximately higher than 2%). The lower interest rates cause the price increase in between two call dates. However, at the call date, the price of the instrument falls at the price level equal to principal plus coupon. This attribute causes the wave effect. Small upwards shifts of the price are the effect of the coupon payments.

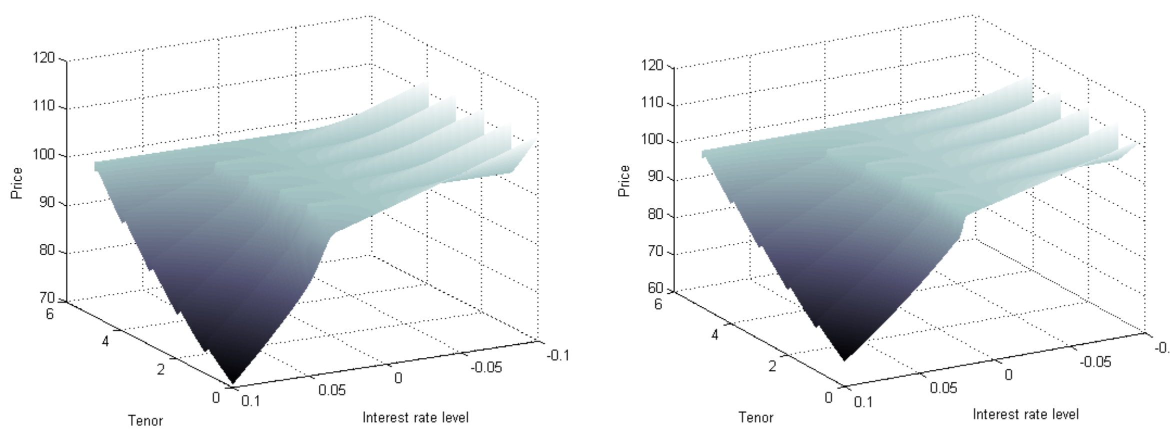
The figure moreover shows the price behavior of the instruments price, when the underlying



(a) T-1 price behavior



(b) T-3 price behavior



(c) T-5 price behavior

Figure 7: Price behavior of theoretical instruments for Gaussian model and Lévy model with stability parameter $\alpha = 1.7$.

interest rate is modeled by $\alpha-$stable process with stability parameter 1.7. The price behavior is very similar to the price behavior of Gaussian model. The only difference is the extremal behavior. The waves on the low interest rate parts of the figures are with slightly smaller amplitude than for Gaussian model. The second difference are the lowest possible values, that the instruments can reach. While for the first instrument (T-1), this difference is minor, for the instrument moderate, but T-5 is this price difference bigger than 10%.

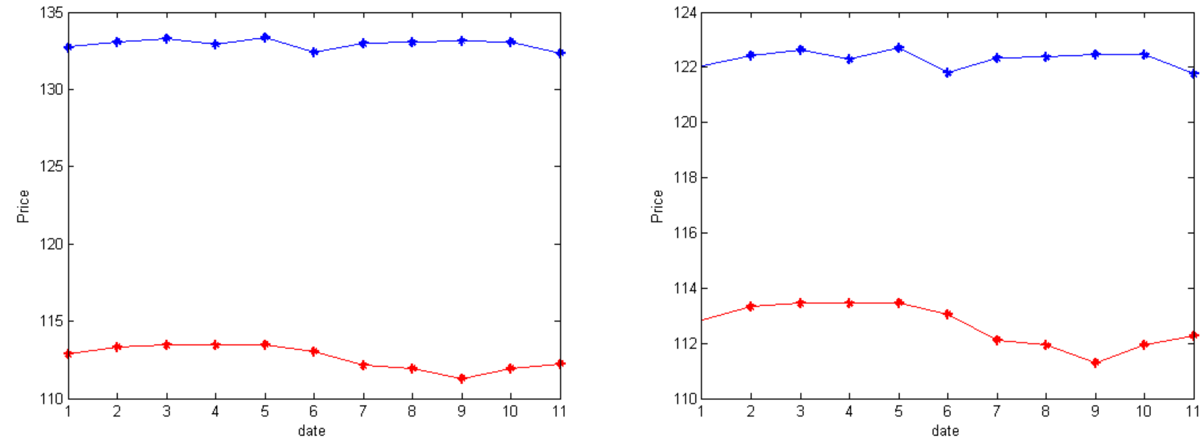
6.3 Final prices of real instruments

To price the real instruments, the interest rate curve is slightly changed. While the interest rate for the theoretical instruments was the zero curve plus 300 bps as a fictional credit spread for the instruments. For real instruments, the final yield curve consists of ECB zero yield plus the respective credit spread for the security and the day. For two instruments, both issued by NN, the future 3m EURIBOR rates need to be estimated as they are the underlying floating As the estimated rates are prone to overfiffting to the selected model, I decided to estimate the future rates using simple average of the historical values. The predicted 3m EURIBOR rates are predicted for X time periods ahead as the simple average of the last X time periods. The historical interest rates are shown in Appendix A.2.

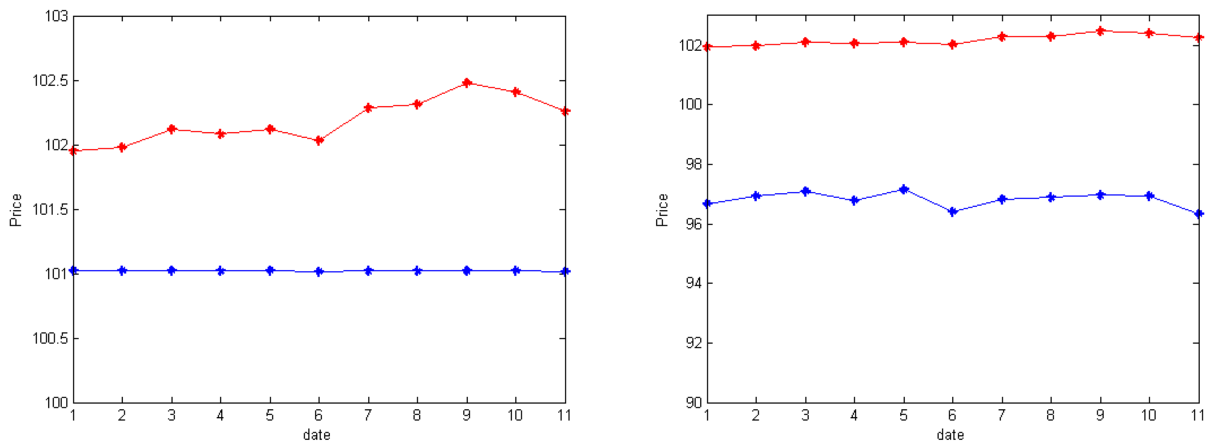
Table 3 shows the obtained model prices for real instruments as of 01/08/2017 for different stability parameters. We can see, that the the prices are increasing with increasing stability parameter. While the prices for AEG-23 approximately correspond with the market prices (101.94 as of August 01), all of the theoretical prices for instruments issued by NN are much higher than the market prices. The market prices do not exceed 112, while the model prices are higher than 120 for all of the considered stability parameters. Similar result can be seen in figure 8. It shows the price behavior (for $\alpha = 2$ and $\alpha = 1.7$) over time in comparison to the market prices over first 11 trading days in August for NN-44 and AEG-23 respectively. We can see, that the model prices are much more stable than the market prices. Slightly bigger price movements can be seen between the fourth and sixth trading day, what is caused by bigger changes in the zero curve and the credit spreads for the instruments.

Prices for real instruments					
Type	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 2$ (Gaussian)	Market price
AEG-23	97.7394	99.1500	101.0752	101.0925	110.679
NN-44	123.1977	124.7314	127.8138	134.1447	112.860
NN-48	122.7263	126.4506	134.1980	141.2399	111.566

Table 3: Model prices for real instruments as of 01 August 2017



(a) NN-44 price for the Gaussian model (left) and the Lévy model with $\alpha = 1.7$ (right)



(b) AEG-23 price for the Gaussian model (left) and the Lévy model with $\alpha = 1.7$ (right)

Figure 8: The actual market price (red) and theoretical price (blue) of NN-44 (up) and AEG-23 (down) over first 11 trading days in August

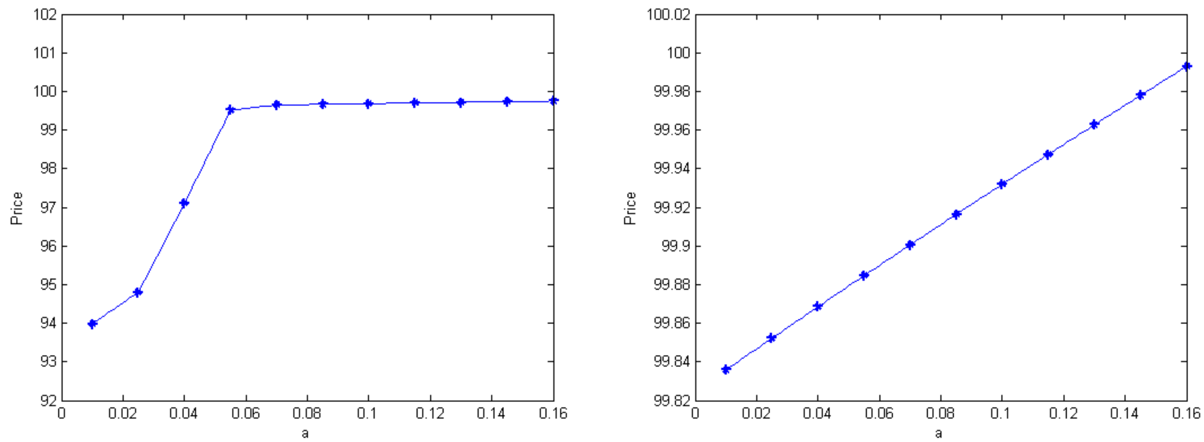
Both subfigures moreover show, that the AEG-23 prices are approximated fairly well. The lowest differences between the model prices and the market prices are around 1 to 1.5% of a principal. Moreover, we can see, that the Gaussian model is the one that prices the instrument the best in comparison to the market data and with decreasing stability parameter the difference is growing. However, the NN-44 model prices are much higher than the prices on the market for all of the considered parameters α. The biggest differences between the real prices and the model prices are for the Gaussian model, while for the model with $\alpha = 1.7$ are the prices the closest to the market data. These prices are still about 8% higher than the ones on the market.

6.4 Sensitivity analysis

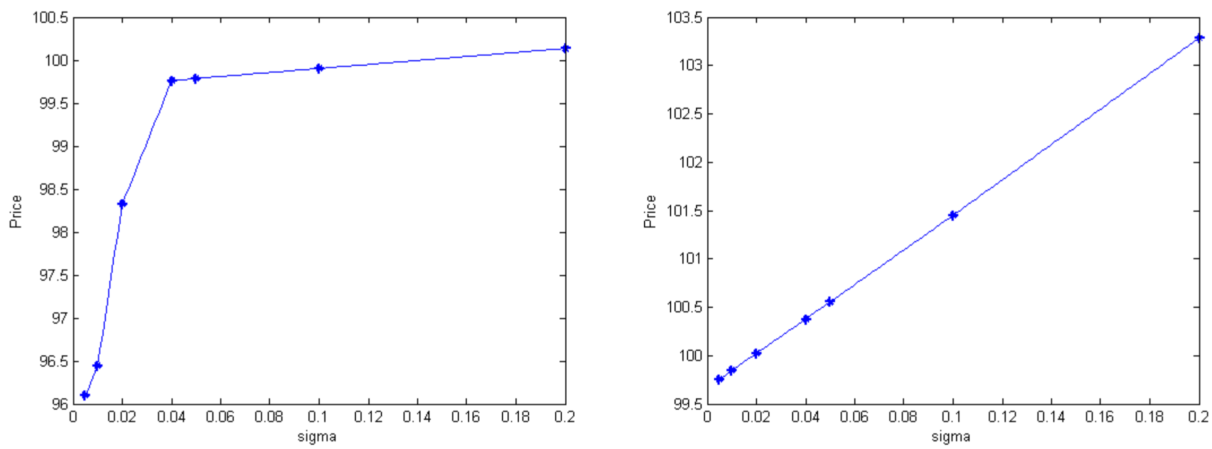
To determine, how do the model prices depend on some parameters, I conduct the sensitivity analysis. The analysis consists on analyzing the price dependence on the interest rate model parameters, a and σ, when the second parameter is stable and calibrated to the swaption data ($a = 0.02$, $\sigma = 0.01$). Moreover, I conduct the analysis on the stability parameter, where I observe the price changes with changing the parameter α while the interest rate model parameters are stable and calibrated to swaptions. The whole sensitivity analysis is conducted on the instrument T-5.

Figure 9 shows the T-5 price dependence on the interest rate parameters a and σ for Lévy model with stability parameter 1.7 and for Gaussian model. The results for both parameters are nearly identical. We can see, that the prices are increasing in both cases. The price of the instrument using Gaussian model is increasing nearly linearly with growing parameter a. However, the price when using Lévy model is increasing for small values of a slowly, for parameter a greater than approximately 0.03 is the price change very fast until $a \approx 0.06$. After reaching this value, the price change of a claim is very steady, nearly negligible. For parameter σ, there is a small deviation from linear dependence, which might be caused by the inaccuracy of the numerical method.

The last of the price dependences examined in this work is the price dependence on the parameter of stability (α), when the interest rate parameters are fixed at calibrated values. The plot describing the price change with growing parameter α is shown on Figure 10 We can see, that the price of the T-5 instrument is approximately 93.5 for $\alpha = 1.5$. For the parameters slightly higher, the final value of the instrument is growing slowly. When α is approximately equal to 1.65, the price starts to grow faster with growing stability parameter until approximately $\alpha = 1.85$, when the price change is lower with the growing α. Therefore the final plot reminds the letter "S".



(a) Price sensitivity on interest rate parameter a with σ fixed at 0.01 - Lévy ($\alpha = 1.7$) left, Gaussian right



(b) Price sensitivity on interest rate parameter σ with a fixed at 0.02 - Lévy ($\alpha = 1.7$) left, Gaussian right

Figure 9: Price sensitivity on interest rate parameters a and σ

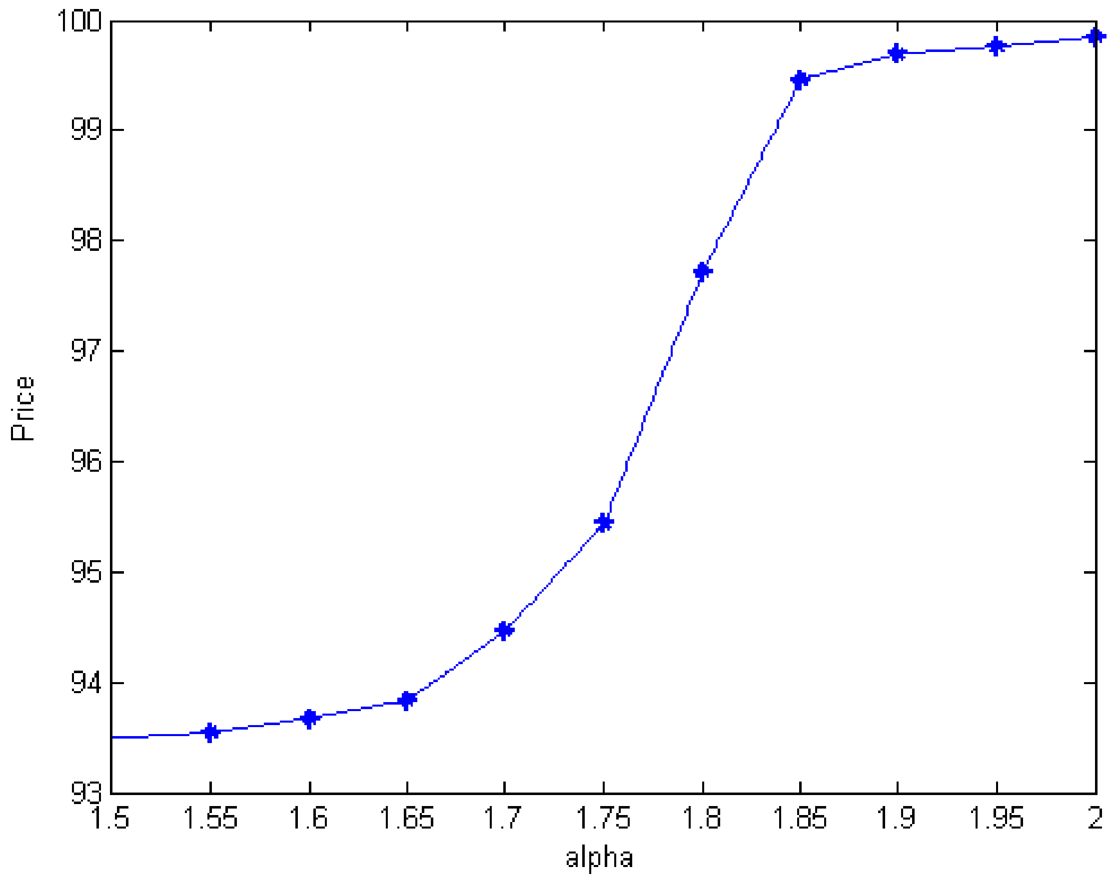


Figure 10: Price sensitivity on stablity parameter α with interest rate parameters fixed at $\sigma = 0.01$ and $a = 0.02$

7 Conclusion

This master's thesis deals with hybrid capital instruments pricing when the underlying interest rate process is modeled by the $\alpha-$stable Lévy process. It examines numerical methods that can be used to price such instruments and reflects the differences between $\alpha-$stable and Gaussian models. It elaborates on the necessity of use of numerical methods, and the issues when the valuation is executed using simulation methods. The thesis moreover deals with the selection and calibration issues of the short-rate interest model that is necessary for the instrument pricing. The interest rate model used in this thesis is a generalized Hull-White model, which calibrated parameters (a and α) were used for the final evaluation of the partial integro-differential equation. The calibration of the thesis was done on interest rate swaptions to capture the underlying market conditions.

The pricing method chosen for this thesis was the generalized form of the famous Black-Scholes equation. This generalized equation is a partial integro-differential equation that needs to be solved in order to obtain the price of the instrument. For the evaluation of these equations I used finite difference numerical method. Moreover, the thesis provides my own refinement to increase the stability of the finite difference approach based on the interpolation and extrapolation of instrument prices. Thereafter, the performance of the finite difference method was compared to methods commonly used to price the instruments when the underlying interest rates are modeled using normal distribution. The differences between the results are negligible.

There are 3 real instruments, issued by Dutch insurance companies, priced in this work. While the model price of two of them shows significantly higher values for any credible stability parameters, the value of the instrument issued by AEGON, maturing in 2023, deviates only slightly. The differences in prices can be caused either by the inefficiency of the markets, of by different interest rate (credit spread) expectation by the market participants. Moreover, the work prices another 3 theoretical instruments and on one of them is conducted sensitivity analysis on different model parameters. The analysis shows, that the model prices increase with the higher values of the considered parameters (a, σ and α). The price dependence plot on parameter α is S-shaped. While the price for the Gaussian models (models with $\alpha = 1$) is increasing nearly linearly with growing a and σ, the price change for Lévy models is lower for higher parameters.

This thesis showed that the exotic bond options can be priced by more versatile models than normal models and therefore capture better the non-normal behavior of the market. The thesis shortly elaborates on the fact that the use of simulation based methods is inappropriate to price the instruments when the underlying interest rate model is modeled by $\alpha-$stable Lévy process. It is recommended to use finite difference numerical method in order to price the market instruments. Moreover, the use of the proposed interpolation and extrapolation refinement leads to a stable and precise method. The sensitivity analysis showed the necessity of correct interest rate parameters calibration as the prices can deviate significantly by different parameter selection.

The work can be extended in several ways. For example, the selection of the proper interest rate model can be crucial to correct instrument valuations. This work uses Hull-White model, that allows negative interest rates. As a credible alternative, one might select CIR model. If the there is a belief of only positive interest rates (that is a credible assumption in case of the private companies), Black-Karasinski model can be selected. We saw, that the final prices are very sensitive to interest rate model parameters. Therefore the second modification of this work can be different calibration technique, or different calibration instruments. One may use the Bachelier model proposed in Bachelier (1964) for the swaption market prices. Moreover we saw, that the prices of real instruments are mispriced from the ones on the market. While the result might indicate the incorrect market prices, it can be caused by the prediction of the EURIBOR rates as well. Therefore, the possible extension of this work can be connected with the sensitivity analysis on the predicted floating coupon rates, whether the model prices can relate more to the ones on the market with different EURIBOR rates. The most interesting improvement can, however, be caused by using non-symmetrical $\alpha-$stable process. The market price changes are known to be skewed and by using different skewness parameters, we can fit the real market behavior better.

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A Appendix

A.1 Measure explanation

The axioms defining a mathematical measure look as follows

1. For any subset E of a set X, $\mu(E) \geq 0$,
2. $\mu(\emptyset) = 0$,
3. For any countable sequence $\{E_i\}_{i=1}^{\infty}$ holds $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$

An intuitive explanation of the first axiom is, that any measure (size) of a subset is nonnegative, the second means, that the measure of a null set is always equal to 0. That means, that, for example the length of a point in 2-dimensional world is always equal to zero. The third axiom speaks about the countability of the subsets, that a measure of the union of some objects is the sum of the measures of the respective objects.

A.2 Overview of numerical methods

To evaluate partial differential equations, we can use three different numerical methods. also need boundary conditions. Assume that we want to evaluate our instrument and we know the value at maturity. Thereafter to evaluate the equation, already known prices are used to evaluate the prices backward in time. If we want to evaluate only one unknown value using more known values, then we speak about explicit method. It means, that we can determine the price directly (explicitly). When we have in our final discretized equation more than 1 unknown variable and one known, then we speak about implicit method. There are also other methods, that use more than one known variable for the evaluation of multiple unknown variables (so-called mixed methods). The difference between the methods is captured in Figure 11. There are also qualitative differences between the methods. While explicit method needs less computational time and is therefore faster, it faces stability issues. On the other hand, it is proven, that the implicit method and Crank-Nicolson methods are stable.

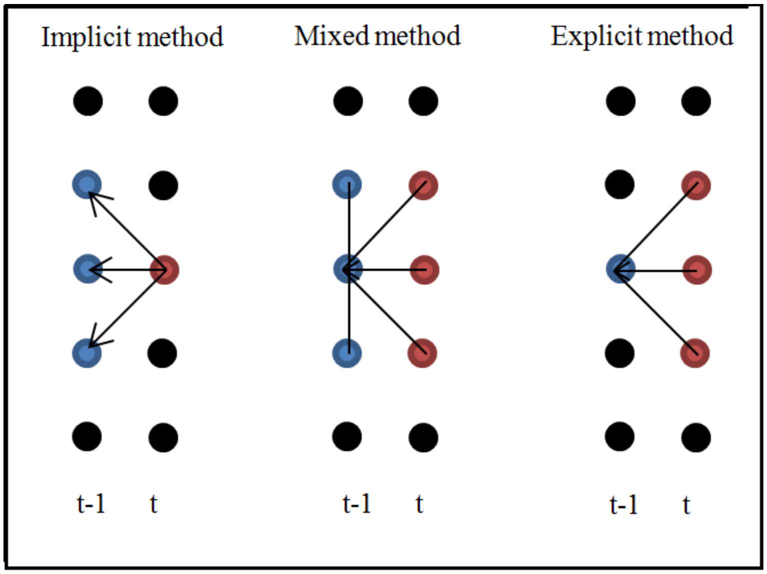


Figure 11: Explanation of numerical methods

A.3 PDE-FVM derivation

I introduce the **PDE-FVM** as the first out of two numerical methods. This method was proposed by d'Halluin et al. (2001) and is slightly modified for this thesis. The discretized formula of the Black-Scholes PDE in d'Halluin et al. (2001) looks as follows

$$\begin{aligned} A_i\left(\frac{X_i^{n+1}-X_i^n}{\Delta\tau}\right) = \\ \gamma\left(\sum_{j\in\eta_i}\alpha_{ij}\left(X_j^{n+1}-X_i^{n+1}\right)+\sum_{j\in\eta_i}L_{ij}\cdot V_i X^{n+1}_{ij+\frac{1}{2}}-r_iA_iX_i^{n+1}\right)+ \\ (1-\gamma)\left(\sum_{j\in\eta_i}\alpha_{ij}\left(X_j^{n}-X_i^{n}\right)+\sum_{j\in\eta_i}L_{ij}\cdot V_i X^{n}_{ij+\frac{1}{2}}-r_iA_iX_i^{n}\right), \end{aligned} \tag{A.1}$$

where

$$A_i = \frac{r_{i+1}-r_{i-1}}{2},$$
$$\eta_i = \{i+1, i-1\}$$
$$\Delta\tau = \tau^{n+1}-\tau^n,$$
$$V_i = \theta(n) - ar_i,$$

$$\alpha_{ij} = \frac{\sigma^2}{2|r_j-r_i|},$$
$$L_{ij} = \begin{cases} -1, & \text{if} j = i+1, \\ 1, & \text{if} j = i-1, \end{cases}$$
$$X^n_{ij+\frac{1}{2}} = \frac{X_i^n+X_j^n}{2}.$$

The parameter γ can be selected arbitrarily by the researcher. Different values of the parameter give different numerical approaches. While, by selecting $\gamma = 0$ we have an implicit method, by selecting $\gamma = 1$ we have explicit method. This method is plausible if the data at time $n+1$ is know before the data at time n, what exactly is our situation when using backward induction. After selecting $\gamma = 1$ and using basic linear algebra we obtain the fully explicit method, where

$$X_i^n = -\frac{\Delta\tau}{A_i}\left(\sum_{j\in\eta_i}\alpha_{ij}\left(X_j^{n+1} - X_i^{n+1}\right) + \sum_{j\in\eta_i}L_{ij}\cdot V_i X_{ij+\frac{1}{2}}^{n+1} - r_i A_i X_i^{n+1}\right) + X_i^{n+1}. \tag{A.2}$$

The equation (A.2) is the final equation we need to solve in order to receive the price between two call dates.

A.4 PDE-FDM derivation

Another possible way how to evaluate the Black-Scholes equation is the finite difference method. Moreover, I calculate the PDE by the mixed, Crank-Nicolson method. For finite difference method and not explicit as in PDE-FVM. To discretize the PDE, we have to find the discretization of all partial derivations. The discretization looks as follows

$$\frac{\partial X}{\partial t} = \frac{X_i^{n+1} - X_i^n}{\Delta t}, \qquad \frac{\partial X}{\partial r} = \frac{X_{i+1}^{n+1} - X_{i-1}^{n+1}}{2\Delta r}, \text{ or } \frac{\partial X}{\partial r} = \frac{X_{i+1}^{n} - X_{i-1}^{n}}{2\Delta r},$$

$$\frac{\partial^2 X}{\partial r^2} = \frac{X_{i+1}^{n+1} - 2X_i^{n+1} + X_{i-1}^{n+1}}{\Delta r^2}, \text{ or } \frac{\partial^2 X}{\partial r^2} = \frac{X_{i+1}^{n} - 2X_i^{n} + X_{i-1}^{n}}{\Delta r^2}.$$

The discretized partial differential equation can therefore be written as follows

$$\begin{aligned}\frac{X_i^{n+1} - X_i^{n+1}}{\Delta t} &= \\ &\gamma\left(-\frac{1}{2}\sigma^2\frac{X_{i+1}^{n+1} - 2X_i^{n+1} + X_{i-1}^{n+1}}{\Delta r^2} - (\theta(n+1) - ar_i)\frac{X_{i+1}^{n+1} - X_{i-1}^{n+1}}{2\Delta r} + r_i X_i^{n+1}\right) + \\ &(1-\gamma)\left(-\frac{1}{2}\sigma^2\frac{X_{i+1}^{n} - 2X_i^{n} + X_{i-1}^{n}}{\Delta r^2} - (\theta(n) - ar_i)\frac{X_{i+1}^{n} - X_{i-1}^{n}}{2\Delta r} + r_i X_i^{n}\right).\end{aligned} \tag{A.3}$$

The equation (A.3) is generalized version for any finite difference method. By setting different values to the scaling parameter, γ, the formula can represent fully explicit ($\gamma = 1$) or fully implicit method ($\gamma = 0$). When selecting $\gamma = \frac{1}{2}$, the equation is transformed into so-called Crank-Nicolson method.

This method belongs to mixed methods and for evaluation of equation, one has to solve system of linear equations at each time step.

As we are moving backward in time, the unknown variables are ones with superscript n and known with superscript $n+1$. By setting $\gamma = \frac{1}{2}$ and rearranging the equation to have all known variables on right hand side and unknown on left hend side, we can get following equation

$$\frac{1}{4}\left(\frac{\sigma^2}{\Delta r^2} + \frac{\theta\left(n\right) - ar_i}{\Delta r}\right) X_{i+1}^n + \left(-\frac{r_i}{2} - \frac{\sigma^2}{2\Delta r} - \frac{1}{\Delta t}\right) X_i^n + \frac{1}{4}\left(-\frac{\theta(n) - ar_i}{\Delta r} + \frac{\sigma^2}{\Delta r^2}\right) X_{i-1}^n =$$
$$\frac{1}{4}\left(-\frac{\sigma^2}{\Delta r^2} - \frac{\theta\left(n+1\right) - ar_i}{\Delta r}\right) X_{i+1}^{n+1} + \left(\frac{r_i}{2} + \frac{\sigma^2}{2\Delta r} - \frac{1}{\Delta t}\right) X_i^{n+1} + \frac{1}{4}\left(\frac{\theta(n+1) - ar_i}{\Delta r} - \frac{\sigma^2}{\Delta r^2}\right) X_{i-1}^{n+1} \tag{A.4}$$

We see from the equation (A.4), that the price at time level n and at interest rate level i depends on values at 3 different nodes one step further in time $(n+1)$. This holds for any interest rate level, except for $i = 1$ and $i = I$, what corresponds to the price that belongs to the lowest interest rate on the grid and the highest rate on the grid. This shows us that we can use a matrix notation in order to describe the discertized PDE. The equation looks as follows

$$X^n = A^{-1} d^{n+1}, \tag{A.5}$$

where

$$A = \begin{pmatrix} A_0(1,n) & A_1(1,n) & 0 & 0 & 0 & 0 & 0 \\ A_{-1}(2,n) & A_0(2,n) & A_1(2,n) & 0 & 0 & 0 & 0 \\ 0 & A_{-1}(3,n) & A_0(3,n) & A_1(3,n) & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_{-1}(I-1,n) & A_0(I-1,n) & A_1(I-1,n) \\ 0 & 0 & 0 & 0 & 0 & A_{-1}(I,n) & A_0(I,n) \end{pmatrix},$$

$$X^n = \begin{pmatrix} X_1^n \\ X_2^n \\ \vdots \\ X_I^n \end{pmatrix}$$

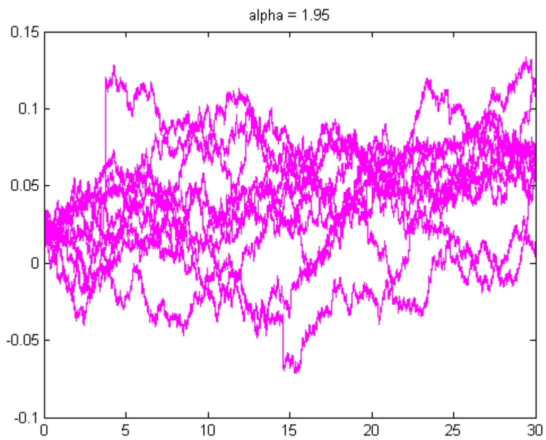
and the values in A are

$$A_0(i,n) = -\frac{r_i}{2} - \frac{\sigma^2}{2\Delta r} - \frac{1}{\Delta t}, \qquad A_1(i,n) = \frac{\sigma^2}{\Delta r^2} + \frac{\theta\left(n\right) - ar_i}{\Delta r}$$

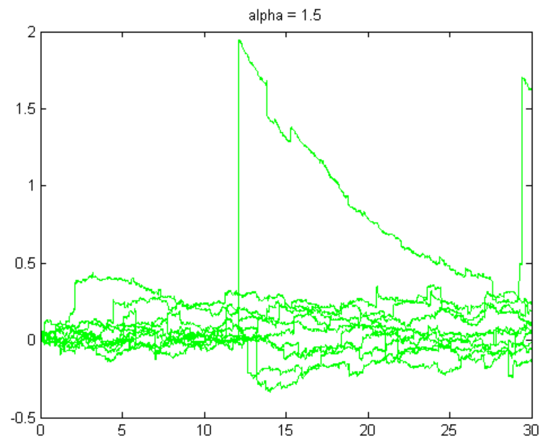
$$A_{-1}(i,n) = -\frac{\theta(n) - ar_i}{\Delta r} + \frac{\sigma^2}{\Delta r^2},$$

$$d_i^{n+1} = \frac{1}{4}\left(-\frac{\sigma^2}{\Delta r^2} - \frac{\theta\,(n+1) - ar_i}{\Delta r}\right) X_{i+1}^{n+1} + \left(\frac{r_i}{2} + \frac{\sigma^2}{2\Delta r} - \frac{1}{\Delta t}\right) X_i^{n+1}$$
$$+ \frac{1}{4}\left(\frac{\theta(n+1) - ar_i}{\Delta r} - \frac{\sigma^2}{\Delta r^2}\right) X_{i-1}^{n+1},$$

A.5 Figures



(a) Alpha stable model ($\alpha = 1.95$)



(b) Alpha stable model ($\alpha = 1.5$)

Figure 12: Interest rate paths for different α

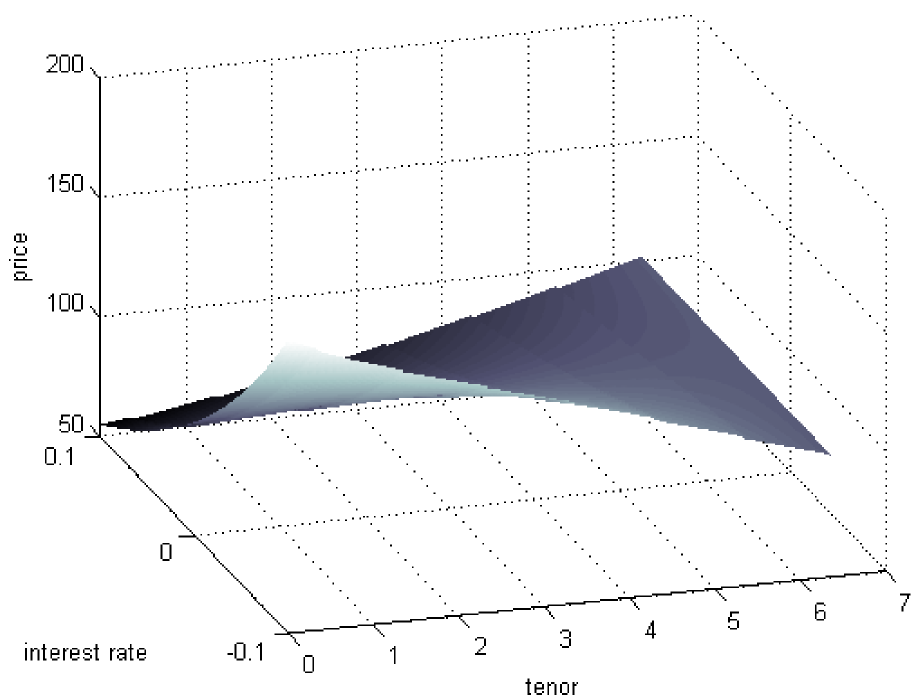


Figure 13: Price behavior of instrument AEG-23 at different interest rate and tenor levels, when the underlying interest rate is modeled by $\alpha-$stable Lévy process with $\alpha = 1.7$

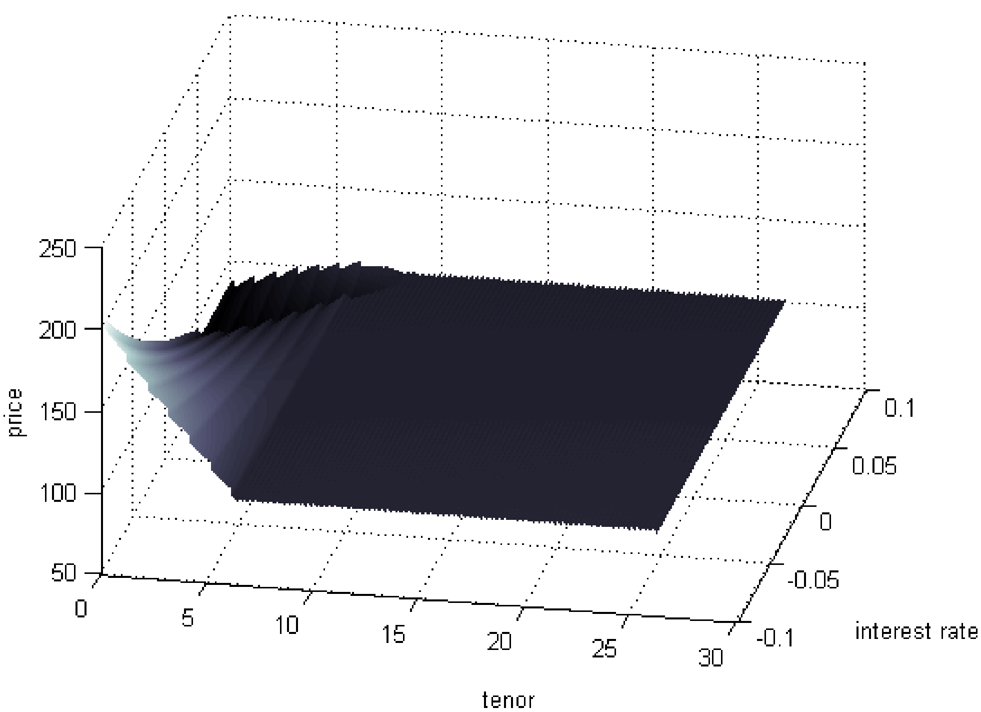


Figure 14: Price behavior of instrument NN-44 at different interest rate and tenor levels, when the underlying interest rate is modeled by $\alpha-$stable Lévy process with $\alpha = 1.7$

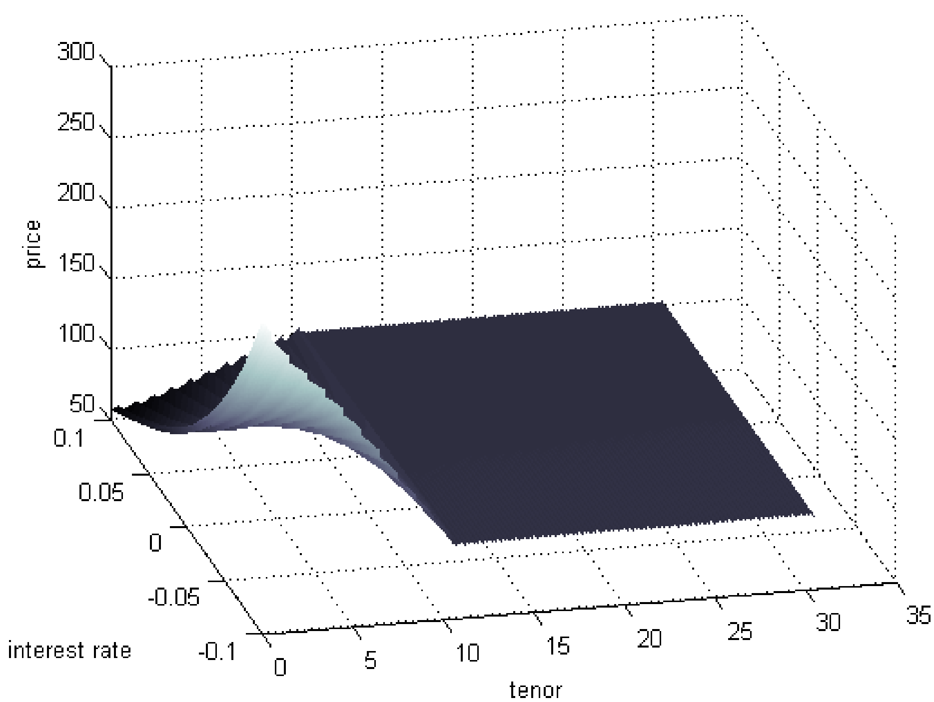
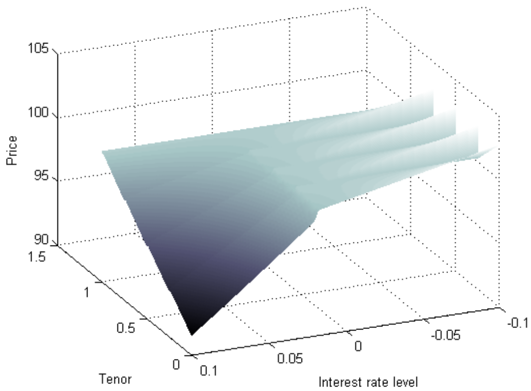
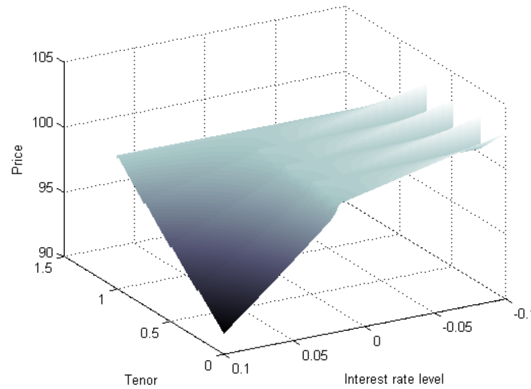


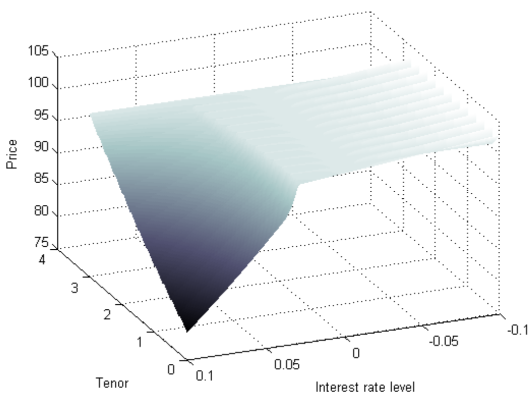
Figure 15: Price behavior of instrument NN-48 at different interest rate and tenor levels, when the underlying interest rate is modeled by $\alpha-$stable Lévy process with $\alpha = 1.7$



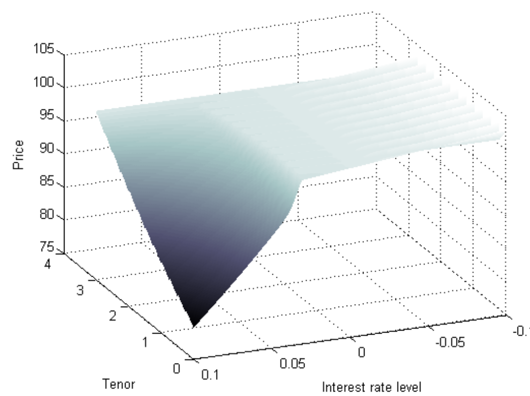
(a) T-1 price behavior with $\alpha = 1.8$



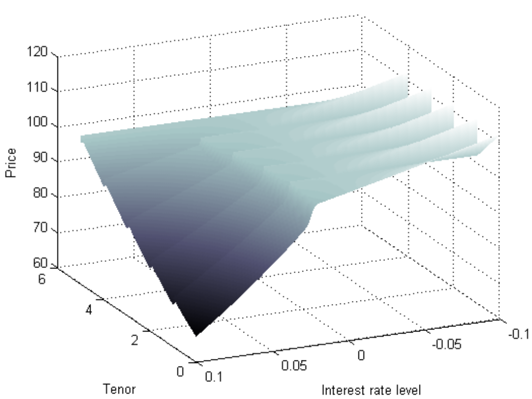
(b) T-1 price behavior with $\alpha = 1.9$



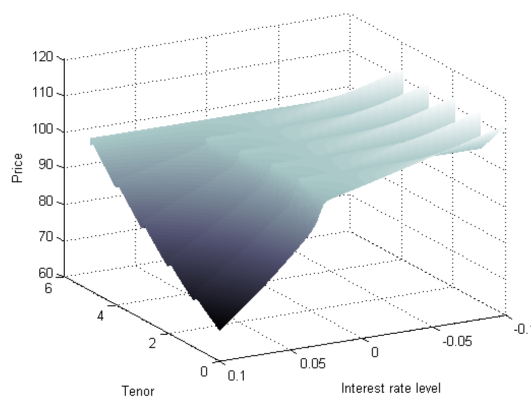
(c) T-3 price behavior with $\alpha = 1.8$



(d) T-3 price behavior with $\alpha = 1.9$



(e) T-5 price behavior with $\alpha = 1.8$



(f) T-5 price behavior with $\alpha = 1.9$

Figure 16: Price behavior of the theoretical instruments at different interest rate and tenor levels, when the underlying interest rate is modeled by $\alpha-$stable Lévy process

A.6 Tables

ECB Svensson model parameters						
Date	β_0	β_1	β_2	β_3	τ_1	τ_2
01/08/2017	1.8346	-2.4753	23.2761	-27.6514	1.5091	1.6125
02/08/2017	1.8072	-2.4992	23.3094	-27.6180	1.4949	1.6043
03/08/2017	1.7827	-2.4977	23.3802	-27.5469	1.5334	1.6395
04/08/2017	1.7983	-2.5273	23.4106	-27.5163	1.5820	1.6850
07/08/2017	1.7930	-2.4970	23.3710	-27.5560	1.5534	1.6588
08/08/2017	1.7926	-2.4966	23.3891	-27.5378	1.5625	1.6678
09/08/2017	1.7443	-2.4523	23.4063	-27.5205	1.5767	1.6830
10/08/2017	1.7298	-2.4358	23.4184	-27.5082	1.5795	1.6859
11/08/2017	1.6990	-2.4103	23.4159	-27.5106	1.5916	1.6955
14/08/2017	1.7446	-2.4499	23.3838	-27.5414	1.6079	1.7081
15/08/2017	1.7673	-2.4913	23.4052	-27.5192	1.5916	1.6935

Table 4: ECB Svensson model parameters for the first 11 trading days in August

NN Nelson-Siegel model parameters					
Date	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$E(\hat{\epsilon}) \cdot 10^3$	$\mathrm{Var}(\hat{\epsilon})$
01/08/2017	153.6006	-138.2805	-165.4838	0.1688	4.5006
02/08/2017	153.4333	-137.6016	-165.8496	0.0711	4.2290
03/08/2017	155.3434	-139.6600	-168.7551	0.3642	4.1809
04/08/2017	152.0455	-136.4193	-159.5262	0.2132	4.2421
07/08/2017	156.4406	-136.0087	-178.9616	-0.0178	4.6741
08/08/2017	155.2234	-138.2591	-161.7605	0.2665	4.7767
09/08/2017	157.3685	-138.2591	-157.0893	0.1510	4.6094
10/08/2017	159.8847	-139.3644	-175.2298	-0.2487	4.7261
11/08/2017	157.1287	-137.5161	-162.7463	0.0888	4.8164
14/08/2017	153.3679	-134.4655	-160.5645	0.2487	11.3601
15/08/2017	154.3508	-135.4758	-166.9918	0.2309	5.0993

Table 5: Estimated NN Nelson-Siegel model parameters and residual statistics for the first 11 trading days in August

AEGON Nelson-Siegel model parameters					
Date	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$E(\hat{\epsilon}) \cdot 10^3$	Var($\hat{\epsilon}$)
01/08/2017	191.9560	-150.8692	-126.9449	-0.0812	5.8847
02/08/2017	183.3487	-143.7914	-125.0672	0.0457	5.4528
03/08/2017	177.5054	-138.9254	-124.5036	0.2081	4.5329
04/08/2017	181.3698	-142.2040	-124.3873	0.1675	5.6314
07/08/2017	177.9303	-139.2067	-124.0953	0.0990	4.6283
08/08/2017	193.0630	-152.0016	-125.2925	-0.3477	3.3724
09/08/2017	188.5161	-148.2554	-124.4103	-0.1370	6.2157
10/08/2017	189.2622	-148.7767	-125.9007	-0.2284	2.5952
11/08/2017	190.7460	-150.0412	-125.8369	-0.1827	3.9003
14/08/2017	185.0856	-147.6074	-118.7671	-0.1269	3.5654
15/08/2017	195.0826	-153.8418	-125.2939	-0.0254	2.9555

Table 6: Estimated AEGON Nelson-Siegel model parameters and residual statistics for the first 11 trading days in August

Term/Tenor	1Y	2Y	3Y	4Y	5Y	7Y	10Y	12Y	15Y
5Y	58.8	53.57	49.18	45.55	42.88	39.66	37.27	36.12	34.27
6Y	50.25	46.62	43.35	40.97	39.2	37.42	35.94	34.79	33.28
7Y	44.16	41.56	39.51	37.92	36.73	35.72	34.96	33.81	32.69
8Y	38.82	37.7	36.48	35.57	35.24	34.13	34.09	32.93	32.19
9Y	35.91	35.48	34.77	34.69	34.24	33.85	33.52	32.78	31.96
10Y	34.21	34.17	34.4	33.92	33.2	33.73	33.07	32.79	31.89
12Y	33.84	33.37	32.54	33.21	33.37	33.26	33.46	32.88	31.95
15Y	34.73	34.7	34.17	33.66	33.19	34.24	33.43	33.76	32.36

Table 7: ATM swaption volatilities due to 01/08/2017 (all values are in %)

3m EURIBOR rates (01/1994-12/2005)					
01/1994-12/1995	01/96-12/97	01/98-12/99	01/00-12/01	01/02-12/03	01/2004-12/2005
6.9100	5.8100	4.2600	3.3431	3.3388	2.0895
6.8600	5.5800	4.2400	3.5368	3.3571	2.0706
6.7500	5.5000	4.1100	3.7470	3.3908	2.0288
6.5700	5.2700	4.0900	3.9253	3.4069	2.0488
6.2400	5.0600	4.0600	4.3620	3.4671	2.0859
6.3000	5.0800	4.0200	4.5017	3.4640	2.1127
6.3400	5.0600	3.9500	4.5829	3.4100	2.1160
6.4300	5.0800	3.9300	4.7771	3.3519	2.1143
6.3800	4.8600	3.9300	4.8528	3.3101	2.1186
6.4300	4.6900	3.8100	5.0413	3.2613	2.1473
6.4000	4.5700	3.6900	5.0920	3.1241	2.1703
6.6900	4.5000	3.3700	4.9392	2.9410	2.1732
6.6600	4.3900	3.1321	4.7707	2.8318	2.1454
6.5900	4.4300	3.0934	4.7558	2.6875	2.1384
7.5800	4.5000	3.0467	4.7086	2.5300	2.1372
7.2900	4.3900	2.6965	4.6820	2.5333	2.1372
7.0400	4.3000	2.5790	4.6367	2.4005	2.1256
7.0800	4.2900	2.6267	4.4536	2.1519	2.1110
6.9200	4.3000	2.6765	4.4671	2.1300	2.1194
6.6600	4.3600	2.6950	4.3535	2.1404	2.1325
6.5000	4.3100	2.7267	3.9829	2.1473	2.1391
6.7500	4.4400	3.3757	3.5999	2.1436	2.1966
6.4500	4.4900	3.4677	3.3857	2.1590	2.3609
6.3200	4.3700	3.4460	3.3449	2.1463	2.4729

Table 8: Historical 3m EURIBOR rates (01/1994-12/2005)

3m EURIBOR rates (01/2006-8/2017)					
01/2006-12/2007	01/08-12/09	01/10-12/11	01/12-12/13	01/14-12/15	01/2016-08/2017
2.5117	4.4815	0.6798	1.2222	0.2920	-0.1461
2.6004	4.3621	0.6617	1.0483	0.2881	-0.1836
2.7226	4.5964	0.6450	0.8585	0.3053	-0.2285
2.7938	4.7835	0.6447	0.7443	0.3297	-0.2492
2.8890	4.8574	0.6865	0.6849	0.3246	-0.2572
2.9857	4.9405	0.7276	0.6589	0.2414	-0.2679
3.1022	4.9610	0.8488	0.4970	0.2050	-0.2945
3.2265	4.9652	0.8955	0.3324	0.1916	-0.2982
3.3354	5.0192	0.8805	0.2463	0.0971	-0.3016
3.5020	5.1131	0.9977	0.2079	0.0826	-0.3090
3.5972	4.2383	1.0420	0.1920	0.0809	-0.3127
3.6842	3.2926	1.0217	0.1855	0.0809	-0.3158
3.7519	2.4565	1.0172	0.2049	0.0627	-0.3255
3.8182	1.9431	1.0867	0.2234	0.0482	-0.3286
3.8909	1.6355	1.1755	0.2061	0.0272	-0.3293
3.9753	1.4223	1.3212	0.2089	0.0047	-0.3304
4.0714	1.2817	1.4251	0.2012	-0.0104	-0.3295
4.1478	1.2279	1.4886	0.2103	-0.0139	-0.3300
4.2162	0.9750	1.5976	0.2214	-0.0187	-0.3304
4.5436	0.8605	1.5521	0.2259	-0.0277	-0.3291
4.7417	0.7721	1.5365	0.2232	-0.0370	
4.6874	0.7375	1.5759	0.2258	-0.0536	
4.6385	0.7162	1.4847	0.2234	-0.0876	
4.8484	0.7120	1.4261	0.2735	-0.1263	

Table 9: Historical 3m EURIBOR rates (01/2006-8/2017)