The influence of demographics on real interest rates and house prices

Master Thesis: Quantitative Finance

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Abstract

This paper complements and extends the research of Lu & Teulings (2016). I investigate the effect of demographics on real interest rates and house prices for six developed countries by using their stylized overlapping generations model. The current demographic structures, which consist i.a. of a baby boom followed by smaller cohorts, are far from a balanced growth path, in which real interest rates would equal population growth. The relatively large cohort causes the population to be biased towards saving. Oversaving results in the real interest rate undershooting the new lower population growth/decline. The low real interest rates are most likely to stay until 2030. This result is backed by observing Japan where the demographic transition took place 15 years earlier. Finally, I find that the growth of a bubble is a natural and efficient response to the asset shortage and low interest rates and consistent with the observed increase in real house prices over the past decades.

1 Introduction

This paper investigates the effect of demographics on real interest rates and house prices in the following six developed countries: France, Germany, Japan, the Netherlands, the United Kingdom and the United States. In recent years, we have witnessed a constant decrease in real and nominal interest rates. One might argue that the falling interest rates are solely caused by the monetary policies applied by Central Banks (Quantitative Easing (QE)). But if we look at the larger picture, we observe that the real interest rates have been declining for some decades now. In addition, De Santis (2016) shows that QE only leads to an average decrease of 0.63% on a 10 year government bond. Therefore, the decline in interest rates cannot be explained by the monetary policies of Central Banks alone.

Lu & Teulings (2016) argue that the change of demographic structure, caused by the baby boom followed by a drop in fertility rates primarily through the introduction of the pill in 1960,[1] resulted simultaneously in the decline of real interest rates and in the rise of house prices over the past decades. The large cohort born before the introduction of the pill and the drop in fertility rate caused by the pill, lead to a disruption in the life-cycle saving patterns of overlapping generations. As a result the current demographic structure is far from a balanced growth path (BGP). The BGP of a model is a trajectory of a model where all the variables grow at a constant rate.

To optimise utility, a cohort saves during the years it is active on the labour market, first to repay the debt built up during the early stage of life and then to accumulate assets to finance their consumption after retirement. This smoothens their consumption during a lifetime. In a balanced growth path (a pyramid in case of a growing population, a pillar in case of a stable population or an inverted pyramid in case of a decreasing population), the asset accumulation of cohorts is smoothly supplied by the other cohorts in different stages of their life cycles. However, because of the present transition in demographics there is an unusually large middle-aged cohort size. They are accumulating assets while the retirees and youngsters are short in supply of assets. Based on these observations, Lu & Teulings (2016) predict that the fall in real interest rates will continue until 2035. At that point, the large cohort will retire and will use its savings causing real interest rates to settle around the new balanced growth path. Samuelson (1958) derived the balanced growth path relation between the growth rate of population g and

[1] The pill is widely used since the late 1960s. Increasing education and career opportunities for women also contribute to the decrease in fertility rates.

the return on capital r in an economy with overlapping generations of workers and retirees. He finds that the market rate of interest in a pure consumption-loan world equals the rate of population growth. If one follows the assumptions of Lu & Teulings (2016) then $r = g$, where g is the growth of the workforce which is equal to the growth of the population in a balanced growth path. Due to the introduction of the pill, the population growth has decreased which in turn, because of the new equilibrium condition, results in a lower return on capital. The path towards this new equilibrium is non-trivial. The extremely large cohort that saves for future consumption after retirement makes our society more focused on savings than it would be in a balanced growth path. As a consequence, the supply of savings is higher than the demand for savings and the supply of assets is lower than the demand for assets, making the economy dynamically inefficient with $r < g$. So the real interest rates will undershoot the new balanced growth path until the large cohort retires.

Lu & Teulings (2016) point out one non-artificial solution that is enforced by the market to solve the asset shortage problem. To save for retirement, retaining cash or putting money on the bank is unattractive so people invest their money in assets such as stocks and real estate. The demand pushes up the prices of these assets and buyers expect that these assets will also carry a positive price in the future and can therefore be used to smooth consumption between generations. So the asset shortage requires a bubble to accommodate for the demand of the large cohort. The supply of land is largely fixed, which is a favourable feature for a bubbly asset. This is exactly what we empirically observe when looking at the house prices and that is why Lu & Teulings (2016) argue that the demographic transition causes the decrease in real interest rates and the rise in house prices. They subsequently argue that the emergence of the housing bubble is a rational response to the demographic transition and low interest rates rather than an irrational phenomenon.

The idea of using demographics to explain interest rates is not new. Carvalho, Ferrero and Nechio (2016) investigate the relation between demographics and real interest rates using a tractable life-cycle model and find a reduction of the equilibrium interest rate by at least one and a half percentage point between 1990 and 2014. A demographic variable, the ratio middle (40-49) to young (20-29), is also a common predictor in literature for securities such as bonds and stocks and is used by Fevero, Gozluklu and Yang (2016) and several others.[2] However, the use of demographics in their manner has a limitation. Although it does capture the ageing of a

[2] Favero, Gozluklu and Tamoni (2011), Geanakoplos, Magill and Quinzii (2004) and Favero and Tamoni (2010).

population, it does not capture the complete distribution of age within the labour force. This distribution, which is more dense towards to older part of the labour force, leads to oversaving and therefore has a further impact on real interest rates. The OLG model does not have this limitation.

Be that as it may, it seems generally accepted that demographic change is an important driver behind the change in the amount of savings, thus behind the supply of capital and the fallen real interest rates. On their turn, real interest rates are at the centre of the secular stagnation debate.[3] However, there are other drivers too; Gordon (2012) argues that technological progress has slowed down. Furthermore, he identifies educational stagnation, rising inequality and increasing consumer and government debts as determinants behind secular stagnation. Rachel & Smith (2015) indicate that the demand for savings/capital has also decreased because of a structural lower investment demand. Ayres & Warr (2009) argue from the supply side. An abundance in fossil fuels since the Industrial Revolution led to enormous growth in productivity of advanced economies. A reduction in availability leads to a slowdown of production growth. Nonetheless, this research is limited to the influence of demographic changes on real interest rates and house prices.

The aim of this research is to complement the research by Lu & Teulings (2016) and extend it to other developed countries. I complement the research by making some small corrections and extensions to their stylized model. This leads to additional and slightly different results. Furthermore, I include the missing derivations, which make the model better reproducible for the selected audience. I also include an improved version of their overlapping generations model (OLG model), based on some suggestions by J. Lu and C. Teulings, such that the simulations are less volatile and converge towards the new BGP.[4]
I will try to predict how the real interest rates of France, Germany, Japan, the Netherlands, the United Kingdom and the United States will develop using their general OLG model specifically calibrated for each separate country. Most of the parameters in this model are set to values that are generally accepted in literature. Chirinko, Fazzari and Meyer (2004) investigate the elasticity of substitution between capital and labour, Nadiri and Prucha (1996) research the depreciation of capital in the US and Havránek (2015) analyses the optimal elasticity of intertemporal substitution in consumption. The remaining two parameters are chosen somewhat arbitrarily by Lu & Teulings (2016) and these parameters, not coincidentally,

[3] See Teulings & Baldwin (2014) for an overview on the subject.

[4] I would like to thank Jason Lu and Coen Teulings for their much appreciated support and suggestions during my research.

determine if the existence of a bubble is efficient. I will fix one of those parameters on the basis of the research by Song, Storesletten and Zilibotti (2011), and optimise the other remaining parameter by using different loss functions. In addition, given the development of the real interest rates and the optimal values of the parameters, I will discuss the efficiency of a bubble and the development of the house prices in each country. In sum, I can confirm the findings of Lu & Teulings (2016) for six developed countries. The demographic transition is able to explain the fall in real interest rates and subsequently the growth of a bubble. The real interest rates will remain low until at least 2030 when the large cohorts convert their savings to consumption. This result is evidenced by the development of real interest rates in Japan. The growth of a bubble is a natural solution enforced by the market to solve the problem of an asset shortage. The development of the real house prices of the six countries indicate that the bubble manifests itself as a housing bubble. The only exception is Germany, where the house prices only started rising in 2010.

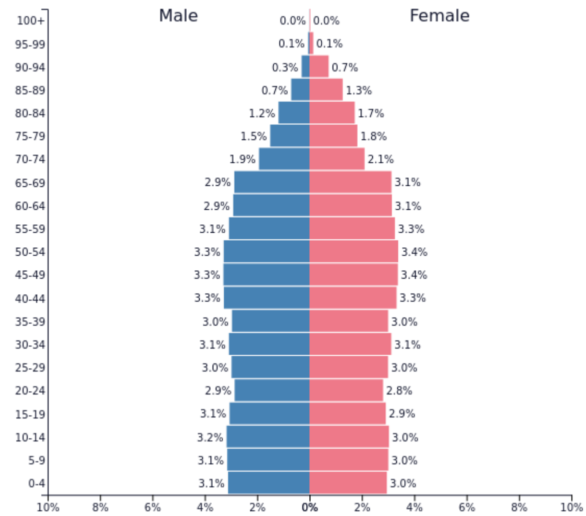
2 Data

The birth control pill was introduced in 1960. Therefore, to examine the resulting changes in demographics and its consequences, I use annual observations from 1960 to 2016 resulting in 57 observations per series. The data is obtained for France (FR), Germany (GR), Japan (JP), the Netherlands (NL), the United Kingdom (UK) and the United States (US). The data sources are described in Appendix A.1. For the six mentioned countries, the following data is considered:

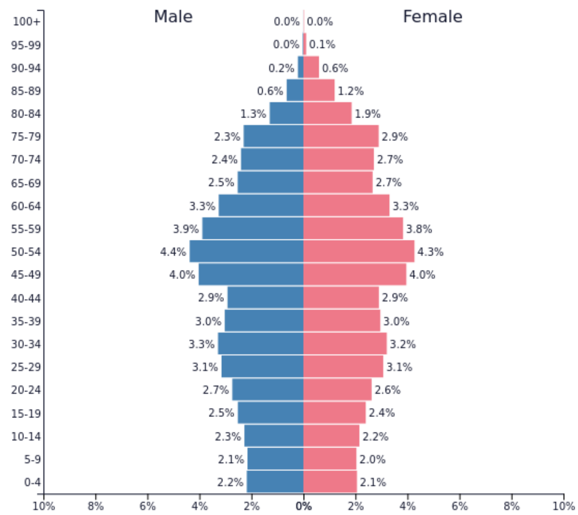
- Demographics: Population data is gathered per cohort (age group). Each cohort consists of five years. The sample contains data from 1960-2016 and predictions up to 2050.
- CPI: Year-on-year inflation rates which are used to correct nominal data.
- Government bond yields: Nominal yields on local government bonds with maturities between 7-10 years. The yields serve as a proxy for interest rate and are corrected for inflation to obtain real interest rates.
- House prices: Annual real house prices from 1960-2016. Only Germany starts at 1962 because there is no reliable data available for the first two years.
- Fertility rates: The total number of children per woman in a lifetime.

The Figures 1 to 6 below illustrate the population structure, the real interest rates and the real house prices for the relevant countries. The figures are arranged in the alphabetical order of the countries. All population distributions in 2016 differ from their historic pyramid and from the usual graph of developing countries. Appendix A.2 gives the historic and future population distributions which are all stable showing that we are at present dealing with a transitional period.

Figure 1a and Figure 1b show the current population distribution of France and Germany. The large cohorts are now aged 45-54. There is a small drop in cohort size in France after the introduction of the pill and the pyramid is quite close to a pillar indicating a more stable population than Germany. Germany is more diamond shaped with the large cohorts more than twice the size of the smallest cohort. The fertility rate in Germany drops from 2.5 to 1.4 from 1960 to 1973. This leads to shrinking cohorts because the fertility rate has become structurally lower than the reproduction rate. The only exception is the echo effect 20-25 years later of the large cohort. The diamond shaped population distribution in Germany differs substantially from a population distribution in a balanced growth path. In France the deviation from a balanced population distribution is less obvious than in Germany.

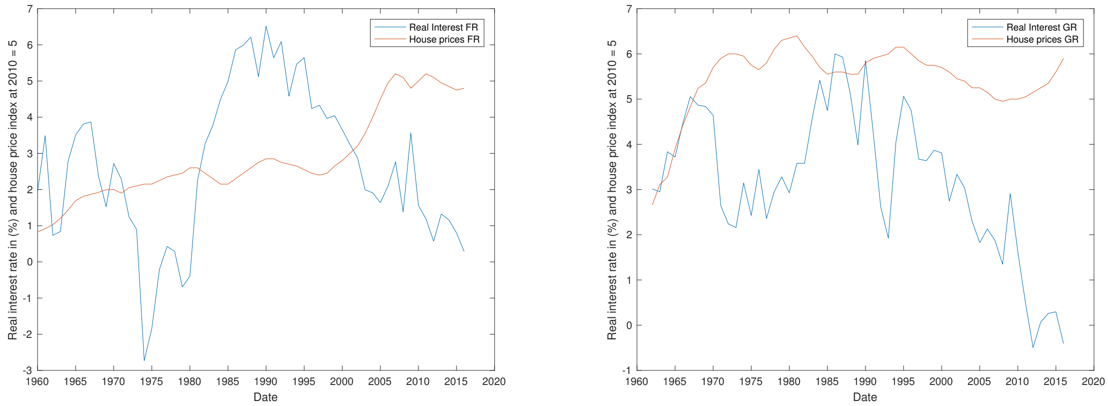


(a) 2016 France



(b) 2016 Germany

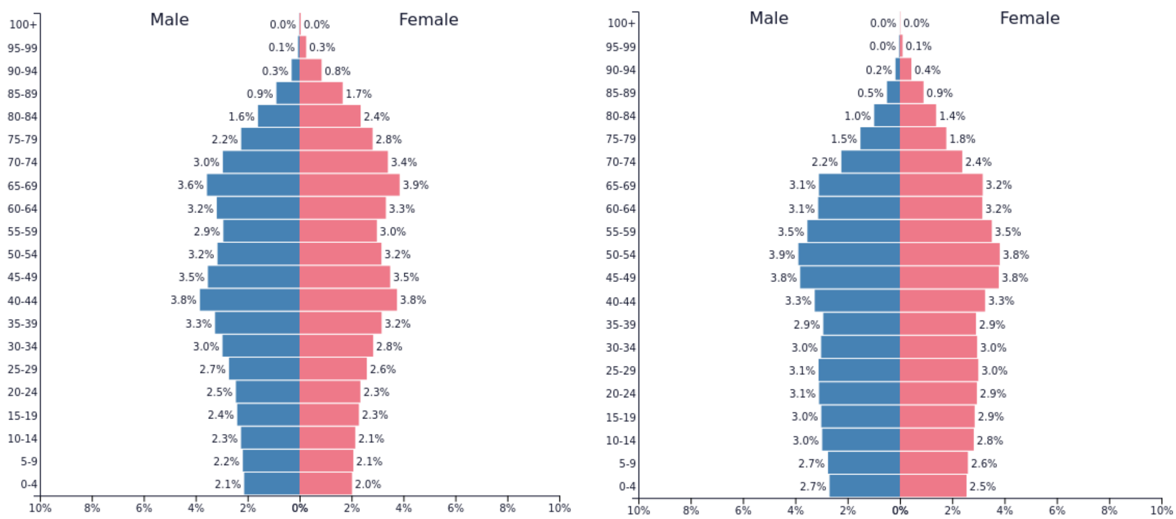
Figure 1: The population pyramids of France and Germany in 2016.



(a) (b)

Figure 2: The real interest rate plotted against the real house prices indexed at 2010 = 5 of France in 2a and of Germany in 2b

Figure 2a shows a decreasing real interest rate in France since 1990. This is followed by an increase in house prices in the late 1990s. The contrast with Figure 2b is interesting. The real interest rate in Germany declines earlier, from 1985, and more severely than in France. This can be caused by the larger demographic shock in Germany. However, there is no direct reaction in the house prices. The house prices in Germany do not start to rise until 2010. This might indicate that houses are not (or to a lesser extent) used in Germany as a bubbly asset during demographic transition to solve the asset shortage problem.

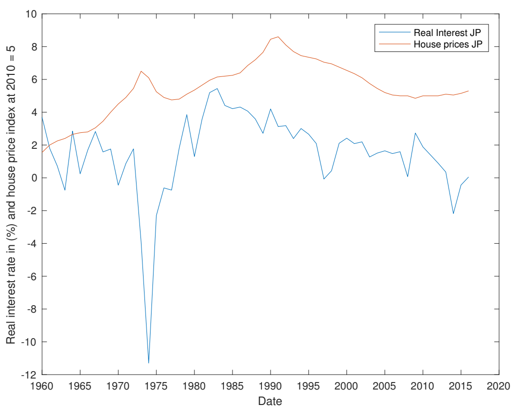


(a) 2016 Japan (b) 2016 the Netherlands

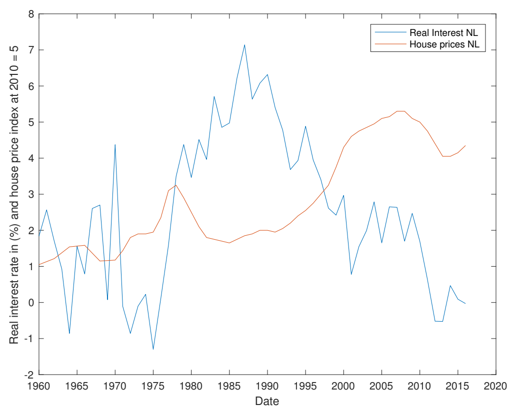
Figure 3: The population pyramids of Japan and the Netherlands in 2016.

The large cohort of Japan in Figure 3a is born after the Second World War and is aged 60-69. Their offspring (the 'echo effect') is aged 40-49. Again there is a decrease in cohort size after the baby boom and a drop of about 30% in fertility rate. The demographic distribution is further evolved than in most other well-developed countries in the sense that the large cohort is about 15 years older and starting to retire. This makes Japan a perfect test case because of the different timing. Figure 3b shows the current population distribution of the Netherlands. It reflects the same diamond shape as Germany. The fertility rate of about 3 in the 1960s drops to 2 in 1973 and decreases even further in the following decades.

The real interest rates and house prices of Japan and the Netherlands are shown in Figure 4. The real interest rates in Figure 4a show a lesser decline in Japan than in the other countries. In addition, the house prices are increasing already from the start of my sample and start to fall from 1990 and onwards. The Netherlands is again like Germany except for the house prices. In the Netherlands, the house prices start increasing from 1990.

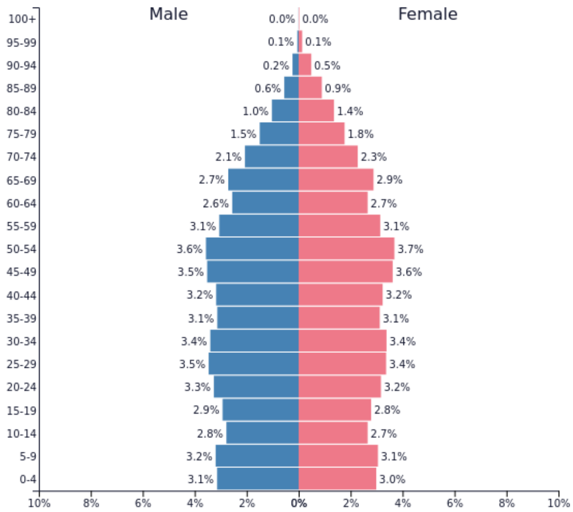


(a)

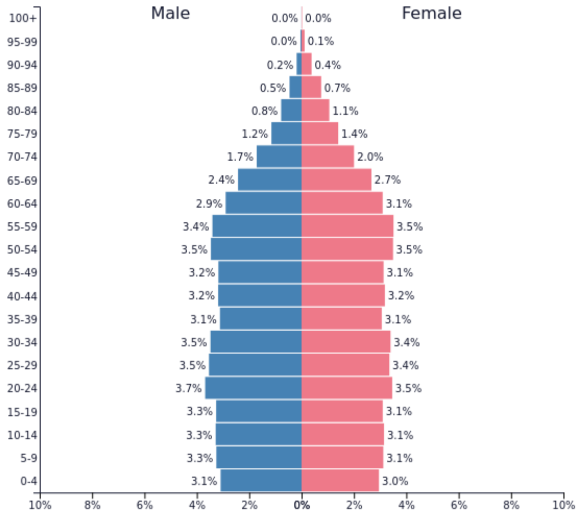


(b)

Figure 4: The real interest rate plotted against the real house prices indexed at 2010 = 5 of Japan in 4a and of the Netherlands in 4b



(a) 2016 the United Kingdom

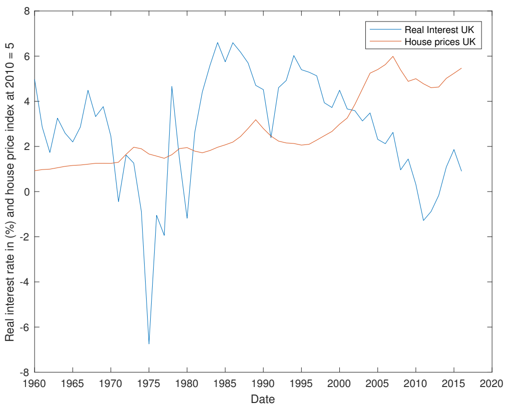


(b) 2016 the United States

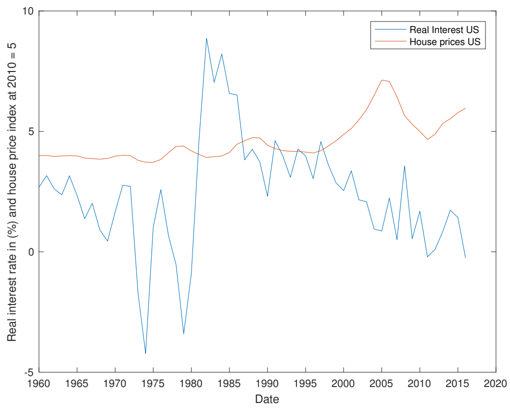
Figure 5: The population pyramids of the United Kingdom and the United States in 2016.

The population distribution of the United Kingdom and the United States are more pillar-shaped than that of Germany, Japan and the Netherlands, as can be seen in Figures 5a and 5b. The large cohorts are again aged $45-54$. The fertility rates drop by approximately 25% from 2.5 to 1.9. These new stable fertility rates are approximately the same as in France, whereas the new stable level of Germany, Japan and the Netherlands is much lower leading to a decrease in relative cohort sizes.

Figure 6a shows that the real interest rates of the United Kingdom and the United states have been declining for some decades followed by an increase in house prices.



(a)



(b)

Figure 6: The real interest rate plotted against the real house prices indexed at 2010 $=$ 5 of the United Kingdom in 6a and of the United States in 6b

It should be noted that the real interest rates and house prices follow the pattern described above in the big picture over the longer term. There are other influences that cause short-term fluctuations in the series. While all countries have well-developed economies and the Euro-zone is a closed capital market, the extent to which unexpected events, for example the last financial crisis, affect a country differs. Furthermore, local circumstances and regulations vary. For instance, in Germany houses are to a large extent funded without bank debt, while in the Netherlands primarily bank debt (secured with mortgages) is used to finance houses because the interest on such loans is tax deductible in the Netherlands. These varying local circumstances may explain differences in the short-term movements of house prices in the various countries.

3 Methods

To model the various interest rates paths and the size and duration of the bubbles during the demographic transition, I use to a large extend the methods of Lu & Teulings (2016). We first model the demographic change. Then a stylized model is introduced that is capable of illustrating the effect of the demographic transition on the real interest rate and on the growth of a bubble. Finally, we propose an OLG model to simulate the real interest rate paths for each country individually.

3.1 Demographic Transition

Households live for J periods and the size of a cohort born in period t is N_t. The total population alive at period t is now:

$$P_t = \sum_{i=0}^{J-1} N_{t-i}. \tag{1}$$

Women are fertile from age $\underline{F}$ to $\overline{F}$. The size of the newborn cohort in period t is

$$N_t = b_t \sum_{i=\underline{F}}^{\overline{F}} N_{t-i}, \tag{2}$$

where b_t is the birth-rate at time t. I assume women between the age of $\underline{F} = 18$ and $\overline{F} = 30$ are fertile. Lu & Teulings (2016) derive the constant rate of population growth g for a fixed birth-rate. This relation is given by rewriting equation 2 to

$$b = \frac{1}{\sum_{i=\underline{F}}^{\overline{F}} (1+g)^{-i}}. \tag{3}$$

The relationship between the birth-rate b and the constant rate of population growth g makes it possible to model the effect of the pill. First there is a high constant population growth which is equal to a high birth-rate. A sudden drop in b results in a fertility shock, which copies the effect of the pill. After the transition there is a lower birth-rate which gives a lower population growth according to equation 3. The change in birth-rate can be summarized as follows:

$$b_t = \begin{cases} b_H & \text{if} \quad t < t^*, \\ b_L & \text{if} \quad t \geq t^*, \end{cases} \tag{4}$$

where $b_H > b_L$. The high and low birth-rates are calibrated for each country specifically. The high birth-rate is derived from the average fertility rate from 1960 to 1969 whereas the low birth rate is derived from the average fertility rate from 1970 to 2016.

3.2 Stylized model

To illustrate the effect of a demographic shock a simpler model is introduced. In this model a household lives for two periods. In the first period, income is earned and saved for retirement. In the second period, a household is in retirement and spending its savings. In the two-generation economy the fertility shock is modelled as a one-period positive shock to cohort size. A household born in period τ has a constant elasticity of substitution (CES) utility function

$$U_\tau = \begin{cases} \frac{c_{\tau,0}^{1-\theta}}{1-\theta} + \beta \frac{c_{\tau,1}^{1-\theta}}{1-\theta} & \text{for} \quad \theta \neq 1, \\ \log(c_{\tau,0}) + \beta \log(c_{\tau,1}) & \text{for} \quad \theta = 1, \end{cases} \tag{5}$$

where $c_{\tau,i}$ denotes the consumption of the cohort born in period τ at age i, β is the discount or impatience factor. The lower β, the more impatient households are and prefer to consume immediately. The choice by Lu & Teulings (2016) to use the CES utility function seems logical. The CES utility function is a specific case of homothetic preferences. Homothetic preferences and quasilinear preferences are the two well-known types of preferences in ordinal utility because of their clear graphical interpretation and analytical traceability [De Jaeger (2008)]. Quasilinear utility functions do not have a wealth effect whereas in the stylized model and in the OLG model changes in wealth should influence consumption choice. This important characteristic is supported by the CES utility function chosen by Lu & Teulings (2016). Maximising the utility function subject to the budget constraint leads to the following Euler equation[5]:

$$c_{\tau,1} = \beta^{\frac{1}{\theta}}(1 + r_{\tau+1})^{\frac{1}{\theta}} c_{\tau,0}, \tag{6}$$

where $\frac{1}{\theta}$ is the intertemporal rate (elasticity) of substitution. It is a measure of concavity of the utility function and it measures the degree to which households are willing to substitute consumption across time (in response to intertemporal price changes). $r_{\tau+1}$ is the real interest rate between periods τ to $\tau + 1$. If the utility of a household is maximised, a household should be indifferent between consuming in this period or in the next period. This is stated in the Euler equation.

The supply side of the economy is a standard Cobb-Douglas production function with respect to capital and labour. We model no production growth from technological progress, such that

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \tag{7}$$

where α is the share of income spent on capital and $1 - \alpha$ the share spent on labour. The parameter α measures the degree to which capital can absorb savings. Capital fully depreciates between each period, thus the depreciation rate $\delta = 1$ and $K_{t+1} = I_t$, where I_t is the level of capital investment in period t. Furthermore, there is perfect competition in the labour and

[5] In short: a household has a utility it wants to maximise

$$\max_{c_t, c_{t+1}} U = u(c_t) + \beta u(c_{t+1}),$$
$$\text{s.t.} \quad c_t + \frac{c_{t+1}}{1 + r_{t+1}} = w_t.$$

Rewrite the budget constraint to $c_{t+1} = (1 + r_{t+1})(w_t - c_t)$ and rewrite the utility function to

$$\max_{c_t} u(c_t) + \beta u((1 + r_{t+1})(w_t - c_t)).$$

Then, take the derivative with respect to c_t and set it equal to zero

$$u'(c_t) + \beta u'(c_{t+1})(1 + r_{t+1})(-1) = 0.$$

Rewrite the FOC to

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1}).$$

Finally, taking the derivatives with respect to c_t and c_{t+1} in equation 5 leads to the Euler equation

$$c_{\tau,1} = \beta^{\frac{1}{\theta}}(1 + r_{\tau+1})^{\frac{1}{\theta}} c_{\tau,0}.$$

capital market leading to[6]

$$k_t = \left(\frac{\alpha}{r_t+1}\right)^{\frac{1}{1-\alpha}} \Leftrightarrow f'(k) = r + \delta, \tag{8}$$

$$w_t = (1-\alpha)k_t^\alpha \Leftrightarrow \frac{\partial Y}{\partial L} = w_t, \tag{9}$$

where $k_t \equiv \frac{K_t}{N_t}$ is the level of capital per worker and w_t the wage as a function of capital per worker, and implicit a function of the real interest rate by rewriting equation 8. The wage earned at t is allocated between current and future consumption.

We are interested in the case of an asset shortage with $r < g$, thus where households oversave and trade in bubbles to increase welfare across all future cohorts. Barro and Sala-i-Martin (2004) derive the condition for when there is sufficient store for value. This condition can be used to check if a bubble is efficient.

Proposition 1a: The condition for the existence of a bubble.
The economy finds its itself in a dynamically inefficient region if the steady stage value k^* exceeds the optimal value of capital k^{opt}, i.e. $k^* > k^{opt}$, where k^* in case of the Cobb-Douglas production function is determined by the nonlinear equation

$$(1+g)\left(1+\left(\frac{1}{\beta}\right)^{\frac{1}{\theta}}\left(1+\alpha\left(k^*\right)^{\alpha-1}-\delta\right)^{\frac{-(1-\theta)}{\theta}}\right) = (1-\alpha)\left(k^*\right)^{\alpha-1}, \tag{10}$$

and

$$k^{opt} = \left(\frac{\alpha}{g+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{11}$$

Derivations in this section are presented in Appendix A.3, unless stated otherwise.

Remark 1a: *To make it as simple as possible, set the constant growth of population $g = 0$. In a balanced growth path the efficient real interest rate is also equal to zero. Furthermore if $\delta = 1$ and $\theta = 1$, the condition for the existence of an efficient bubble is*

$$\frac{1-\alpha}{\alpha}\frac{\beta}{1+\beta} > 1. \tag{12}$$

[6] $Y = K^\alpha L^{1-\alpha}$. The constant output-labour ratio y is obtained by dividing the output Y by the labour force $L = N$

$$\frac{Y}{N} = \frac{K^\alpha N^{1-\alpha}}{N},$$
$$y = \left(\frac{K}{N}\right)^\alpha = k^\alpha = f(k),$$
$$Y = Lf(k).$$

$Y = Lf(k)$, the marginal products of the factor inputs are $\frac{\partial Y}{\partial K} = f'(k) = r + \delta$ and $\frac{\partial Y}{\partial L} = f(k) - kf'(k) = w$.

In equation 12, I recover the same no-bubble condition as Lu & Teulings (2016). The condition that $k^* > k^{opt}$ for $\theta \neq 1$ (such that the utility function is not equal to a logarithmic utility function) can be studied numerically. Oversaving is more likely to occur if the impatience factor, β, and the depreciation rate, δ, are large and if the capital share, α, and population growth, g, are small. Intuitively, oversaving cannot occur if α is close to zero because wages are then close to 0 followed by little capacity for the young generation to save. Specifically for $\alpha = 0.3$ and $\beta = 1$, the baseline parameters, equation 12 does not hold and therefore a bubble is efficient.
To illustrate the growth of the bubble, we model the size of the bubble. The size is simply equal to the price of the bubble since the supply (available supply of land) can be fixed at one.

Proposition 1b: The size of a bubble.
The size of the bubble is equal to the difference between savings and capital:

$$p = \frac{f(k^*) - k^* f'(k^*)}{\left(1 + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} (1 + r_t)^{\frac{-(1-\theta)}{\theta}}\right)} - k^*. \tag{13}$$

Remark 1b: *For $g = 0, \delta = 1$ and $\theta = 1$, I recover equation (8) in Lu & Teulings (2016):*

$$p = \left(\frac{\beta}{1+\beta} \frac{1-\alpha}{\alpha} - 1\right) k^*, \tag{14}$$

where $k^* = \alpha^{\frac{1}{1-\alpha}}$ from equation 8.

During the transition path, a rational bubble appreciates at the rate of the real interest rate [Tirole (1985)] as follows:

$$p_{t+1} = (1 + r_t)p_t, \tag{15}$$

where p_{t+1} is the price the elderly generation receives for the asset at time t. This money can be used for consumption in period t. In case of the population growth $g = 0$, we can trade in the bubbly asset to create the efficient $r = g$ outcome [Samuelson (1958)]. The size of the bubble is such that we have the dynamically efficient real interest rate $r = 0$.

The simple model is based on households living for two periods. A drop in birthrate b leads immediately to the new cohort size growth at the lower level with $N_t = b_L N_{t-1}$. This simple transition is unable to replicate the existence of a disproportional large cohort caused by the past WWII baby boom before the introduction of the pill and therefore unable to resemble the demographic distribution that countries face at present. The existence of this large cohort drives the real interest rates to undershoot the new BGP as a response. Without the large cohort the transition would be a gradual convergence process of the real interest rate to the new growth rate. To model the complete demographic transition a demographic shock is added. At time t^* there is a one-period increase in cohort size, which returns to normal after t^*, i.e.

$$N_t = \begin{cases} N_H & t \quad = t^*, \\ N_L & t \quad \neq t^*, \end{cases} \tag{16}$$

where N_L is normalised to 1 and N_H is calibrated to 1.5 to represent the size difference between the larger cohort and the smaller cohort of Germany. Germany is used for illustration because it has the biggest fall in cohort size.

There are two ways in which the demographic transition can enter the described economy. Either the arrival of a demographic transition is not known beforehand and arrives as a shock at t^*, or the arrival of a demographic transition is known beforehand at $t < t^*$. This way the effect of prior knowledge can be separated from the effect of the shock. The steady-state size of the bubble, given by equation 14, is equal across models but the behaviour of the interest rate during the transition period differs. Both cases are partially dealt with by Lu & Teulings (2016).

In the first case, at the time of the demographic shock t^*, the size of the bubble jumps. Hereafter the transition follows equation 15. It remains to find the size of the bubble at t^* that enables the real interest rate to return at its efficient level $r = 0$. For an initial bubble size at t^* equation 13 with $\theta = \delta = 1$ is solved for r_{t^*}. The bubble evolves according to equation 15 after which equation 13 is solved for r_{t^*+1} and so on until r_t converges. The bisection method is used to find a new initial bubble size at t^* and the algorithm is iterated until an initial bubble size is found for which r_t converges to 0.
In the second case, Lu & Teulings (2016) assume that the news of demographic shock arrives before the consumption decision at $t = -2$. The demographic shock only influences the interest rates through its effect on future interest rates. The transition path can be described by the following propositions.

Proposition 2a: Analytic solution for the simple model for $t > 1$.
The focus is still on the simpler case with steady-state growth $g = 0$ and $\theta = \delta = 1$. For $t > 1$ the real interest rate can be described by

$$(1+r_t)^{\frac{\alpha}{\alpha-1}} = \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_t)(1+r_{t-1})^{\frac{\alpha}{\alpha-1}}\frac{N_{t-1}}{N_t} + \frac{\alpha}{\gamma}(1+r_{t+1})^{\frac{1}{\alpha-1}}\frac{N_{t+1}}{N_t}, \quad \textit{for} \quad t > 1, \qquad (17)$$

where $\gamma = 1 - \frac{1-\alpha}{1+\beta}$.

Equation 17 is a second order difference equation and has an analytical solution

$$1 + r_t = C^{\alpha^t}, \quad \text{for} \quad t > 1, \qquad (18)$$

where C is a constant of integration. For any α in the unit interval, $1 + r_t$ converges to one for $t \to \infty$ and $r_\infty = 0$, which is our steady state rate. The derivation of equation 18 is also included in Appendix A.3.
To describe the real interest rate transition path from $t = -1$ to $t = 1$ directly after the arrival of the news a system of equations is formed using equations 14, 15, 17 and 18.

Proposition 2b: Analytic solution for the simple model.
Again the focus is on the simpler case with steady state growth $g = 0$ and $\theta = \delta = 1$.

Then an equilibrium path for $r_t\ \forall\ t > -1$ satisfies the following system of equations:

$$1 = \frac{\alpha}{\gamma}\left(1 + \frac{p_{-1}}{k}C^A\left(\prod_{t=-1}^{1}(1+r_t)\right)\right) + \frac{\alpha}{\gamma}(1+r_{-1})^{\frac{1}{\alpha-1}}, \tag{19}$$

$$\begin{aligned}(1+r_{-1})^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_{-1}) + \frac{\alpha}{\gamma}(1+r_0)^{\frac{1}{\alpha-1}}N_H,\\ (1+r_0)^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_0)(1+r_{-1})^{\frac{\alpha}{\alpha-1}}N_H^{-1} + \frac{\alpha}{\gamma}(1+r_1)^{\frac{1}{\alpha-1}}N_H^{-1},\\ (1+r_1)^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_1)(1+r_0)^{\frac{\alpha}{\alpha-1}}N_H + \frac{\alpha}{\gamma}(C^{\alpha^2})^{\frac{1}{\alpha-1}},\end{aligned}$$

$$\frac{p_{-1}}{k} = \left(\frac{\beta}{\alpha}\frac{1-\alpha}{1+\beta} - 1\right)C^{-A}\left(\prod_{t=-1}^{1}(1+r_t)\right)^{-1}, \tag{20}$$

where $A = \frac{\alpha^2}{1-\alpha}$, k is the steady state level of capital-labour ratio and p_{-1} is the resale value of the bubble at $t = -2$.

The system consists of five equations with five unknowns: C, p_{-1}, r_{-1}, r_0 and r_1. The solution gives an illustration of the reaction of the real interest rate and the asset bubble to the demographic shock. The lines of equation 19 all have the same structure[7]; on the left the output minus the consumption of the current workforce, the first term on the right is the consumption of the retirees and the second term on the right is the investment in next period's capital. The capital-labour ratio k_{-2} in the first line is at its steady state value k since it was decided before the arrival of the information on the demographic shock, hence $r_{-2} = 0$. The consumption of the retirees is equal to the return on the capital stock, $(1 + r_{-2})k_{-2} = k$. The shock N_H raises the investment demand in the second line. In the third line, the demographic shock reduces consumption of retirees and investments relative to the current size of the workforce. In the fourth line, the shock raises consumption of the retirees. Equation 20 is obtained by substitution of equation 14 in a rewritten equation 15.

3.3 Overlapping generations model

To simulate the market-clearing interest rates during a more realistic transition, the lifetime of a household is expanded from two generations to a complete lifetime. A market-clearing interest rate is an interest rate that causes the quantity supplied and quantity demanded to be equal. A general OLG model is proposed to perform the

[7]This structure is clearly visible from the derivation in Appendix A.3.

simulation of the real interest rate paths. This model makes it possible to study aggregate implications of life-cycle savings of individuals. The supply side of the economy is subject to a constant elasticity of substitution (CES) production function. I use the same neoclassical production function as Lu & Teulings (2016) as it provides an easy interpretable relation between capital and labour, and keeps my results comparable to theirs. See section 5.1 for a discussion on the production function. Production takes two inputs, capital and labour, and exhibits constant returns to scale (CRTS):

$$Y_t = \begin{cases} (\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) L_t^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} & \text{for} \quad \sigma \neq 1, \\ K_t^{\alpha} L_t^{1-\alpha} & \text{for} \quad \sigma = 1. \end{cases} \tag{21}$$

If the elasticity of substitution between capital and labour $\sigma = 1$, we get the Cobb-Douglas production function from Section 3.2. Capital does not fully depreciate each period but depends on the depreciation rate $\delta \in (0, 1]$. Capital accumulates according to

$$K_{t+1} = I_t + (1-\delta) K_t, \tag{22}$$

where I_t is the capital investment in period t. The first order condition (FOC) of equation 21 with respect to capital and labour, under perfect competition, gives

$$r_t + \delta = \begin{cases} \alpha k_t^{-\frac{1}{\sigma}} [\alpha k_t^{\frac{\sigma-1}{\sigma}} + 1 - \alpha]^{\frac{1}{\sigma-1}} & \text{for} \quad \sigma \neq 1, \\ \alpha k_t^{\alpha-1} & \text{for} \quad \sigma = 1, \end{cases} \tag{23}$$

and

$$w_t = \begin{cases} (1-\alpha) [\alpha k_t^{\frac{\sigma-1}{\sigma}} + 1 - \alpha]^{\frac{1}{\sigma-1}} & \text{for} \quad \sigma \neq 1, \\ (1-\alpha) k_t^{\alpha} & \text{for} \quad \sigma = 1. \end{cases} \tag{24}$$

A household born in period τ now lives for a total of J periods instead of two periods. Its utility function changes, now age can be greater than 1, slightly to

$$U_\tau = \begin{cases} \sum_{i=0}^{J-1} \beta^i \frac{c_{\tau,i}^{1-\theta}}{1-\theta} & \text{for} \quad \theta \neq 1 \\ \sum_{i=0}^{J-1} \beta^i \log(c_{\tau,i}) & \text{for} \quad \theta = 1 \end{cases} \tag{25}$$

The household problem is given by the corresponding Euler condition of equation 25 and is similar to the Euler condition in the two-generations economy. The Euler condition is as follows:

$$c_{\tau,i+1} = \beta^{\frac{1}{\theta}} (1 + r_{\tau+i+1})^{\frac{1}{\theta}} c_{\tau,i}. \tag{26}$$

A household born in period τ earns a total income:

$$W_\tau = \sum_{i=\chi}^{\psi-1} P_{\tau,i} w_{\tau+i}, \tag{27}$$

where $P_{s,i}$ is the intertemporal price of period $s + i$ consumption in terms of period s income, χ is the age of entering the labour market and ψ is the retirement age. The wage is given by equation 24 and the intertemporal price is

$$P_{s,i} = \prod_{j=0}^{i-1} (1 + r_{s+j+1})^{-1}. \tag{28}$$

The total labour force at any t is given by

$$L_t = \sum_{i=\chi}^{\psi-1} N_{t-i}. \tag{29}$$

The life-time budget constraint of a household born at period τ is given by

$$\sum_{i=0}^{J-1} c_{\tau,i} P_{\tau,i} \leqslant W_\tau. \tag{30}$$

The whole transition depends on the existence of a balanced growth path (BGP) equilibrium, where the population grows at a constant rate and where all per capital variables are constant and therefore $r = g$. Without the existence of a BGP equilibrium, there is no convergence of the real interest rates to a steady-state after the reaction to the demographic shock and the real interest rates will be unpredictable. Lu & Teulings (2016) prove the existence of a $r = g$ equilibrium by approximately following the argument of Samuelson (1958). They verify market-clearing by proving that aggregate capital income equals aggregate investments and therefore that $r = g$ generates a BGP equilibrium.

The development of the real interest rates during the demographic transition in the production economy has no analytical solution. Lu & Teulings (2016) propose an iterative algorithm to numerically compute the transition equilibrium in a production economy. First they initiate the transition window. The transition window begins at t^*, the arrival of the demographic shock, and consists of tJ observations with $t = 10$ an arbitrary integer large enough to let the transition play out in full but small enough to keep the computational time workable and J the lifespan of a household. After tJ periods the economy has converged to the terminal BGG with the real interest rate equal to the lower population growth. There are J observations after convergence. Before the transition window we also have J observations equal to the initial BGP where $r = g_H$. For the first iteration, we make the assumption that the interest linearly falls from the initial level to the terminal level. Each subsequent iteration is calculated in the following way. The initial linear interest rate is denoted as r_t^0. For the $k+1st$ iteration we start at r_t^k and derive the implied capital, investments, output and wages. Using the implied path for wages and r_t^k we solve the household problem in every period for every cohort by deriving the cohort's consumption using the Euler equation in equation 26 and the life-time budget constraint in equation 30. The cohorts' consumption allows us to calculate aggregate consumption demand in each period t as

$$C_t^{k+1} = \sum_{i=0}^{J-1} N_{t-i} c_{t-i,i}^{k+1}, \tag{31}$$

where $c_{t-i,i}^{k+1}$ is the demanded consumption in period t by the cohort born in period $t-i$, in iteration $k+1$. Now aggregate saving supply is given by aggregate income minus aggregate consumption demand

$$S_t^{k+1} = w_t^{k+1} L_t - C_t^{k+1}. \tag{32}$$

Savings is equal to investments minus return on capital. That is, savings is the additional quantity devoted to asset accumulation after capital returns are reinvested. The next periods desired asset position is

$$A_{t+1}^{k+1} = A_t^{k+1}(1 + r_{t+1}^k) + S_t^{k+1}, \tag{33}$$

where r_{t+1}^k denotes the real interest rate in period t, iteration k, and the initial asset position is given by $A_{t^*-1}^{k+1} = K_{t^*}$. This result gives the desired asset position through time. The implied capital stock, consistent with the desired asset position, can be derived by

$$K_{t+1}^{k+1} = A_t^{k+1}. \tag{34}$$

Note that the implied capital stock is derived from the desired asset position. The actual asset position may be different from the desired asset position. In this algorithm we are interested in the desired asset position hence we assume that assets grow with the real interest rate of the previous iteration $(1 + r_t^k)$. The capital stock per worker is easily obtained by dividing aggregate capital by the labour force. Using equation 23 we derive the implied real interest rate path r_{t+1}^I from k_{t+1}^{k+1}. The real interest path for the next iteration is given by

$$r_t^{k+1} = (1 - \phi) r_t^k + \phi r_t^I, \tag{35}$$

where $\phi = 0.05$, a parameter that controls the degree to which we adjust per iteration. To check for convergence the euclidean distance between r_t^k and r_t^I is computed. If the distance is lower than 10^{-5}, the distance is sufficiently small, the algorithm is terminated and the equilibrium real interest rate path is found. Convergence of the algorithm is verified with the market-clearing condition, i.e. aggregate consumption demand must equal aggregate income minus aggregate investment.

3.4 Parameter optimisation

Some of the parameter values in section 3.3 are based on the optimal values found in literature. Chirinko, Fazzari and Meyer (2004) report a concensus value of the average elasticity of substitution between capital and labour σ of approximately 0.4. Nadiri and Prucha (1996) find that the mean depreciation rate of plant and equipment capital δ in the US is equal to 0.059 and Havránek (2015) finds that the mean elasticity of intertemporal substitution in consumption $\frac{1}{\theta}$ for asset holders is around 0.3 to 0.4 with a 95% confidence interval of $[-0.2, 0.8]$. These parameters are supposed to be the optimal parameters in general. The baseline specification of the parameters mentioned above in Lu & Teulings (2016) are $\delta = 0.075$, $\sigma = 0.4$ and $\frac{1}{\theta} = 0.5$ (i.e. $\theta = 2$). These values are approximately the same and thus deemed appropriate. Other parameters are life expectancy J, retirement age ψ and age of entering the working population χ. The baseline specification of these parameters is $J = 75$, $\psi = 65$ and $\chi = 20$. The only argument here might be about the life expectancy. According to the world health organization, a life expectancy of developed countries of around 80 is more fitting. Therefore, J is changed to 80 in the baseline specification.

The remaining two parameters are the elasticity of capital α and the discount factor or impatience factor β. Lu & Teulings (2016) set $\alpha = 0.3$ and $\beta = 1$ in their baseline specification. An increase in α increases the capital share of income, and hence increases the productivity of capital. This makes capital a better store for value. Lu & Teulings (2016) show that the size of the bubble is smaller both in the steady states and in the transition periods for $\alpha = 0.4$ instead of 0.3. The baseline specification with $\alpha = 0.3$ makes sure that no-bubble condition in equation 12 is violated which means that there is not sufficient store for value and consequently a bubble is needed for the economy to be efficient. However, the values used by Lu & Teulings (2016) may not be the optimal values to replicate the path of the real interest rates. Different parameter values may be more suitable and in addition these parameters may also differ per country. Consequently, the resulting values from optimising the parameters may satisfy the no-bubble condition instead of violating it. Frederick, Loewenstein, and O'Donoghue (2002) have conducted empirical research on intertemporal choice and summarize the implicit discount rates from all the studies they reviewed. Most find a discount rate varying between 0.8 and 1. To keep the computational burden workable, I set $\beta = 0.99$ just as Song, Storesletten, and Zilibotti (2011). Cho & Schreyer (2017) find for 28 OECD countries, under which the six countries that are subject in this paper, a labour share from a production perspective varying on average between 50% to 70% from 1995 to 2014. The capital share in developed countries therefore varies on average between 30% to 50%. With interval steps of 0.01, I will vary the capital share of the economy between 0.3 and 0.5. To optimise α three different loss functions are used for robustness. These functions are:

$$\min_{\alpha} S_{abs}(\alpha) = \sum_{t=t^*}^{T} |r_t - \hat{r}_t(\alpha)| \,, \tag{36}$$

$$\min_{\alpha} S(\alpha) = \sum_{t=t^*}^{T} (r_t - \hat{r}_t(\alpha))^2 \,, \tag{37}$$

$$\arg\min_{\alpha} \max |r_t - \hat{r}_t(\alpha)| \quad for \quad t \in [t^*, T], \tag{38}$$

subject to $0.3 \leq \alpha \leq 0.5$, where r_t is the observed real interest rate and $\hat{r}_t$ the predicted real interest rate from the OLG model. t^* is equal to 1970, the start of the demographic transition (except for Japan) and T is equal to 2016, the last observation from the sample.

4 Results

4.1 Stylized model

In Figure 7 the news of the existence of an extraordinary demographic distribution at $t = 0$ is not known beforehand. The real interest rate falls as a result of oversaving by the large cohort during the demographic transition at $t = 0$. In response there is a spike in the bubble size to accommodate for the shortage in assets. The bubble in

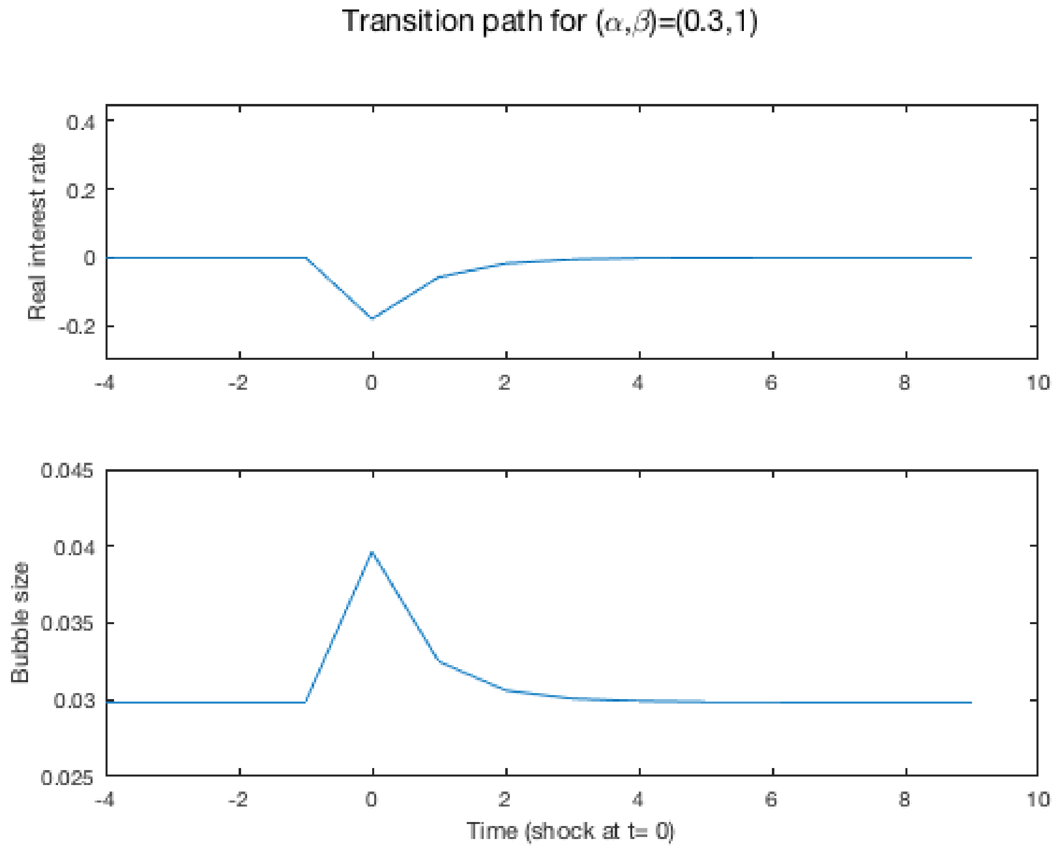


Figure 7: Development of the real interest rate and a bubble during the transition when the news arrives as a shock at t^*.

turn increases the value of the available assets and consequently decreases the shortage in assets, which, at the end the day, limits the fall in interest rate. As the loss in net utility by a fall in interest rate dominates higher wages from working, reducing the drop in interest rate improves welfare across all cohorts. Therefore the spike in the bubble improves the net utility across all cohorts. After several periods the shock dies out and the interest rate returns to its equilibrium level. The bubble size follows the development of the real interest rate described by equation 15.

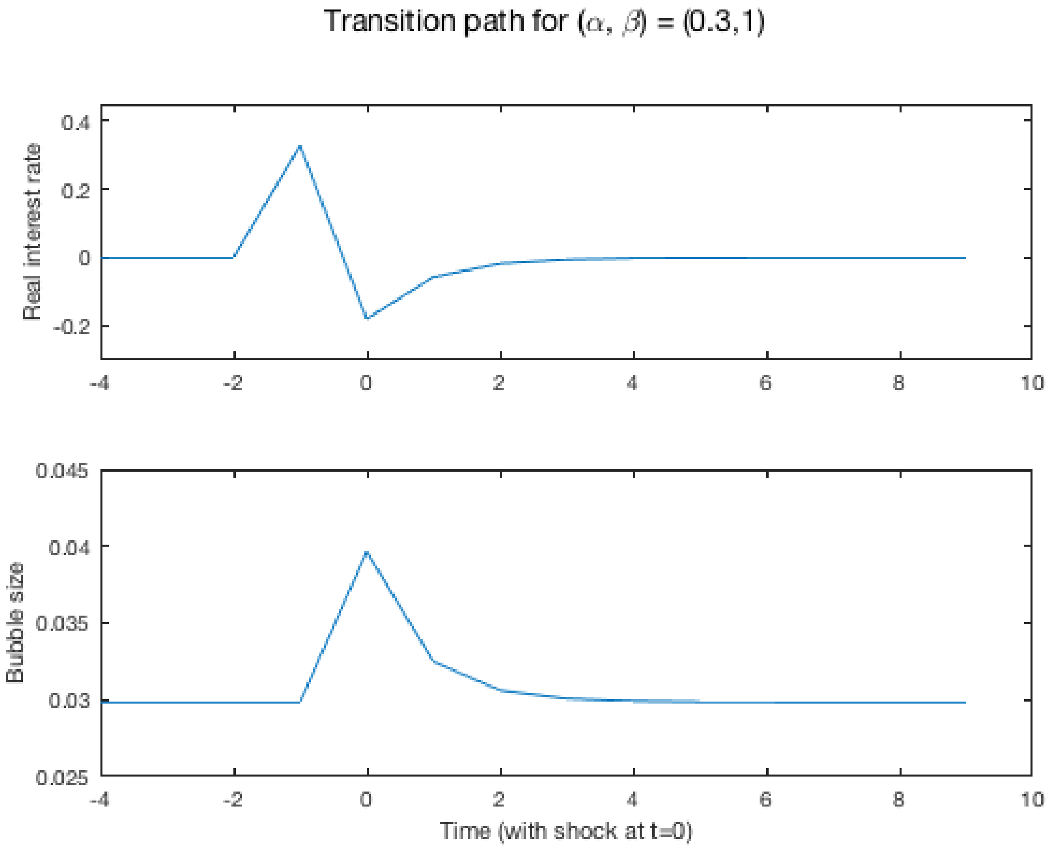


Figure 8: Development of the real interest rate and a bubble during the transition when the information of the demographic shock arrives at $t = -2$ before t^*.

In Figure 8 the information on a demographic shock arrives at $t = -2$. Real interest rates rise as a reaction to the news. At $t = -1$, people start borrowing to buy extra assets. Thus the demand for savings increases. The extraordinary large cohort starts to oversave at $t = 0$ causing the interest rate to drop and a spike in bubble size which again leads to increase welfare across cohorts. The bubble size behaves exactly the same as if the news is unknown beforehand. The bubble size follows the real interest rate according to equation 15.

We have to interpret the real interest rates observations as quantities per generation. So the negative interest rate corresponds to the years the large cohort is active on the labour market. The large cohort is born in around 1970. Therefore t^* implicates a period from around 1985 to 2035. The start of the fall in real interest rates around 1985 is exactly what we observe if we look at developed economies.

Regarding sensitivity with respect to the baseline parameters, I refer to Lu & Teulings (2016) who have done a more extensive sensitivity analysis on the parameters. An increase in α increases the capital share of income and makes capital a better store of value. As a result an increase of α is paired with a decrease in steady-state bubble size and spike in bubble size. For $\beta = 1$, if $\alpha \geq \frac{1}{3}$ then equation 12 is violated and there is sufficient store for value. The existence of a bubble is not efficient and therefore does not occur at the steady-state. Furthermore, for a higher α the decrease in interest rates is smaller.

4.2 OLG model

The results in this section for all countries are all based on the baseline parameters specified in section 3.4: $(J, \sigma, \frac{1}{\theta}, \alpha, \beta, \underline{F}, \overline{F}, \chi, \psi) = (80, 0.4, 0.5, 0.3, 0.99, 18, 30, 20, 65)$. α is equal to 0.3 which is the optimal value as we will see later on in section 4.3.

Figures 9 and 10 show the development of cohort size and the effect of the fertility shock. Cohort size starts declining immediately after the fertility shock. The decline is followed by a period of increase of cohort size because the cohort size of mothers is still increasing. This shock echoes every 24 years which corresponds to the average age of a mother at child birth. When the smaller cohorts enter the fertile sub-population, cohort size starts to fall again. The shock diminishes as time goes by until the population distribution converges to its stable steady-state growth.

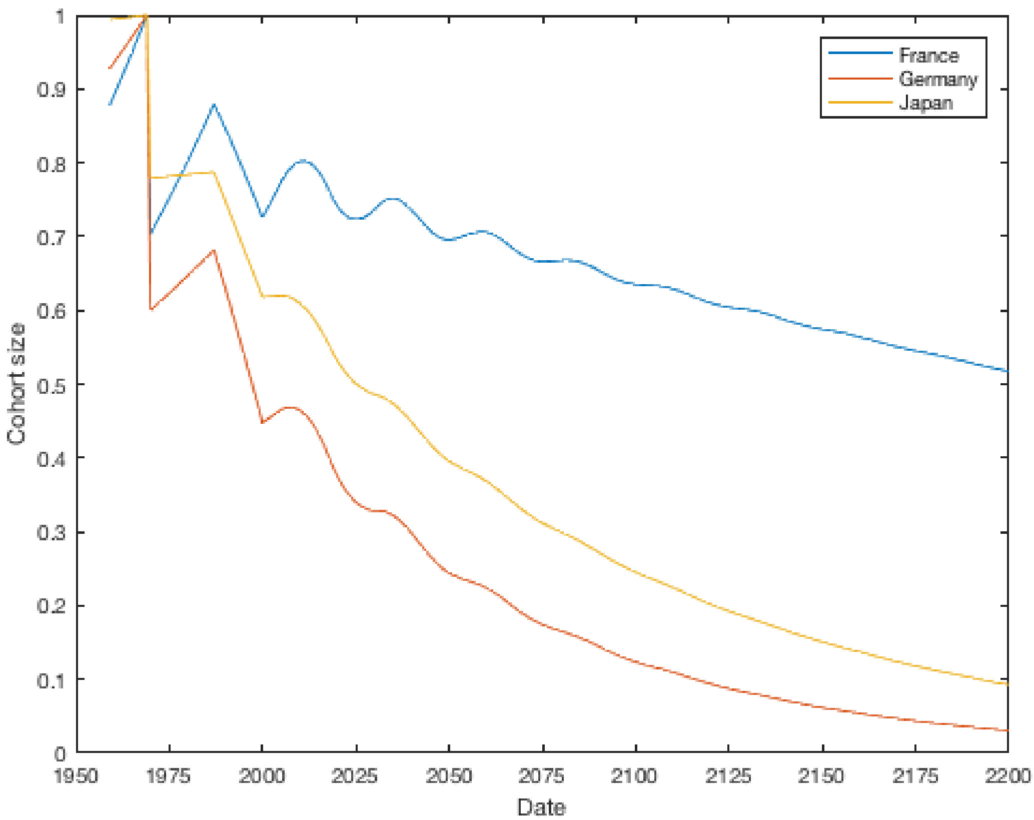


Figure 9: The evolution of cohort size of France, Germany and Japan.

The Netherlands and Germany are most influenced by the change in demographic distribution resulting in the largest difference between high and low birthrate. The UK is influenced to a lesser extent whereas France and the US experience the least influence from the fertility shock. This is in line with the demographic structures from Section 2. Japan is different from the other countries in the sense that the population was already slightly declining for 15 years before the fertility shock. In this case, the fertility shock leads to a further decline of the population.

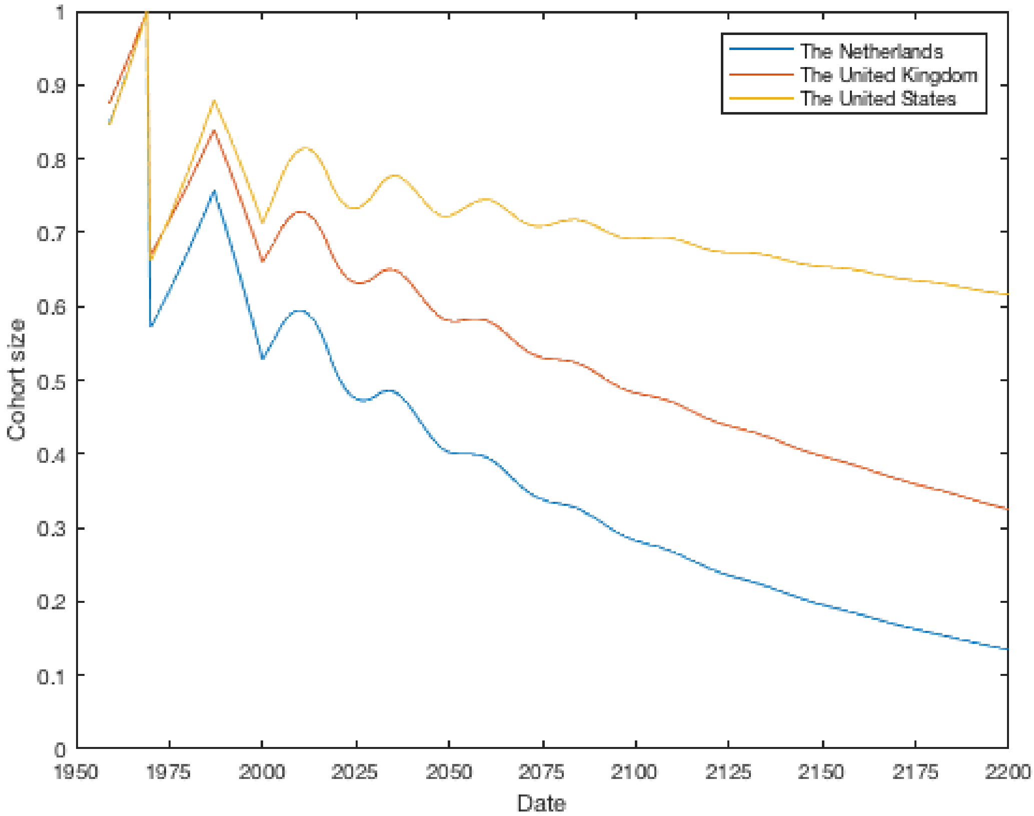


Figure 10: The evolution of cohort size of the Netherlands, the United Kingdom and the United States.

The development of the real interest rates paths of the six countries are shown in Figures 11 and 12. The first result that strikes the eye is the initial increase in real interest rates. In the beginning, the demographic structure is hardly affected by the change in demographics. The older cohorts realise that future cohort size shrinks which will decrease demand on the goods market. Consequently, they feel richer because their future income will be more valuable. In return, the older cohorts increase consumption leading to an increase in real interest rates. From 1985, when the extraordinary large cohorts enters employment and starts saving for retirement, the real interest rates start declining. After 2030, when the larger cohorts retire, the pressure on the real interest rates loosens and starts to rise as the new retirees deplete their savings for consumption. In accordance with cohort size evolution, the real interest rate paths show the same cyclical behaviour. Generations of the first post-shock cohorts are relatively smaller than surrounding cohorts. This again leads to a drop in real interest rate followed by an upwards adjustment. This echo effect slowly fades away. The real interest rates converge to the lower growth rates juster after 2200.

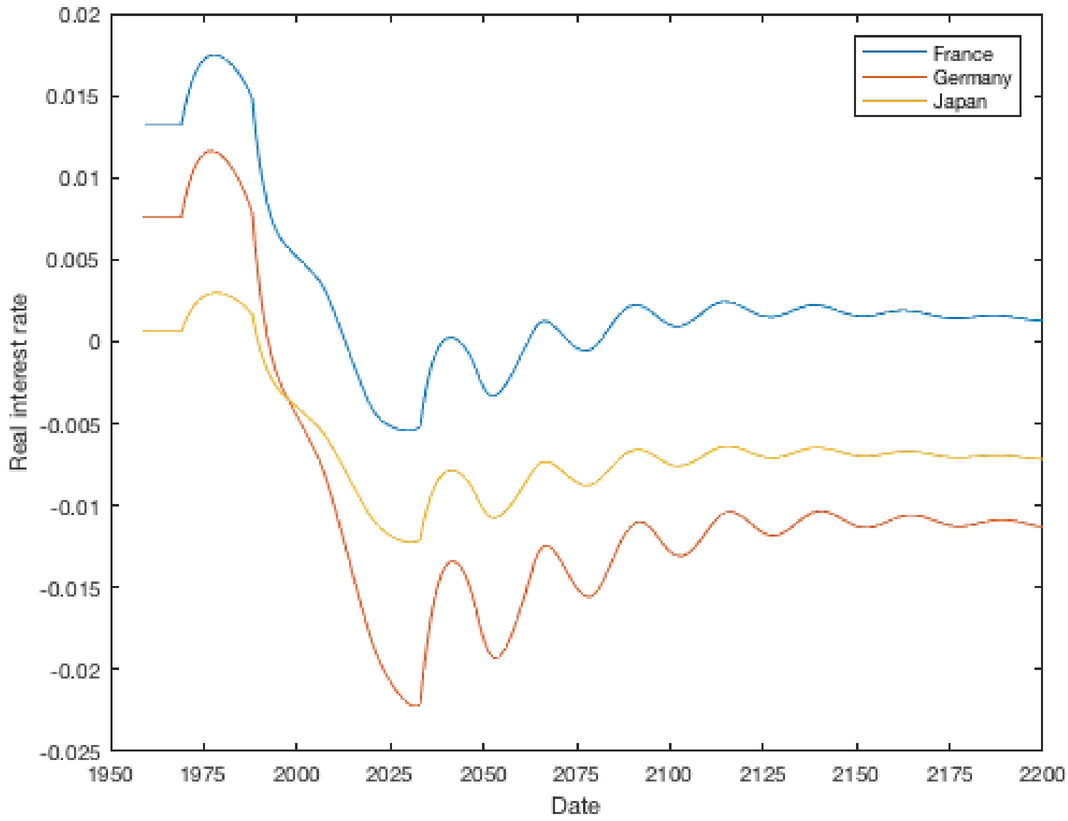


Figure 11: The real interest rates paths of France, Germany and Japan.

Germany and the Netherlands experience the biggest fall in real interest rate. Germany's real interest rate settles a little under -1%. The real interest rate of the Netherlands settles slightly higher at -0.5%. France and the US maintain a positive real interest rate as their low growth rates are 0.1% and 0.2% respectively. The real interest rate path of the UK converges to approximately 0%. At last, Japan already starts at a low growth rate and eventually it decreases even more. Their demographic transition and thus decline in real interest rate started 15 years before the other countries and serves as a laboratory for the duration of the low real interest rates. The continuing presence of a low level in Japan indicates that the real interest rates will remain low for at least upcoming 15 years in the other five countries and supports the findings from the OLG model that the duration of the first drop in real interest rates will last until at least 2030. At present, the large cohort in Japan is retiring. Interesting would be to examine the resulting effect on the real interest rates.

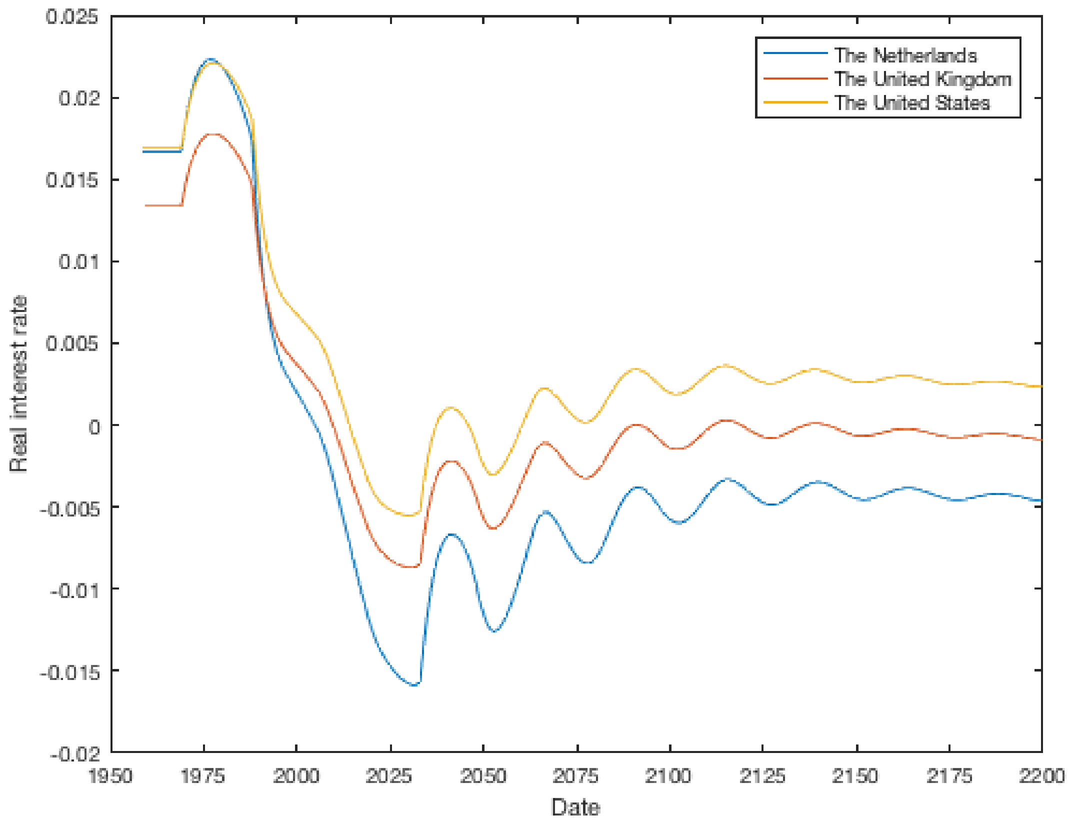


Figure 12: The real interest rates paths of the Netherlands, the United Kingdom and the United States.

Lu & Teulings (2016) show some sensitivity analyses for a number of the aforementioned parameters. A higher life expectancy or lower retirement age, keeping the other parameters constant, leads to a deeper fall in the real interest rates because the number of years in retirement for which a household needs to save rises. A lower elasticity of capital-labour substitution inflates the cyclicality of the real interest path caused by the existence of life cycles as physical capital is less able to smooth the variations in savings. Finally, a lower elasticity of intertemporal substitution, so a higher θ, implies that households are less tolerable to variations consumptions resulting in an increase of their saving behaviour, which in turn results consumption, resulting in a deeper fall of real interest rates.

4.3 The existence of a bubble

The part of the predicted real interest rate paths that is compared to the actual observations is the first small increase initiated in 1970 followed by the steep fall until 2016. This is far from perfect as demographics are unable to predict short-term fluctuations. The increase in real interest rates is the same for all capital share values so we are searching for the capital share where the fall in real interest rates is most similar to the real situation. The drop is slightly less steep for a higher α than if capital is less absorbative. For all three loss functions, the optimal α between 0.3 and 0.5 for all countries to replicate the real interest rate paths is 0.3. This means that a slightly steeper fall in real interest rate better represents the observed real interest rates. The capital share for which the general bubble condition in equation 12 holds, using the new growth rates per country and that $\theta = 2$, $\delta = 0.075, \sigma = 0.4$ and $\beta = 0.99$, is $\alpha \leq 0.31$ for Germany, Japan and the Netherlands and $\alpha \leq 0.32$ for France, the UK and the US. Therefore we can conclude that the growth of a bubble is a rational phenomenon for all countries considered. If we look back at house prices in Figures 2, 4 and 6 we observe an increase in prices matching the time period of the demographic transition. The only exception is Germany, where the prices started to rise in 2010. These results indicate that a response to the current demographic transition and the decline in real interest rates is the growth of a bubble which expresses itself in rising house prices.[8]

5 Conclusion

This paper shows that the demographic transition most developed countries are momentarily in is able to explain the decline in real interest rates in recent decades. The demographic transition is caused by the existence of a large cohort, baby boomers now aged 45-54, followed by a fertility shock mainly caused by the introduction of the pill. All six investigated countries are affected by the demographic transition but to a varying extent. Only when the large cohorts retire in 2035 and increase consumption, the real interest rates will start to rise. This prediction is supported by Japan. Its demographic structure leads the other countries by 15 years and still Japan encounters low real interest rates. This especially has consequences for pension funds that have to

[8] Remember that land is a favourable asset for a bubble since the supply of land is largely fixed.

discount their liabilities using a low discount rate while the population is ageing. The real interest rates will eventually settle around the new population growth rates.

In addition to the effect on real interest rates, this paper shows that the growth of a bubble is a natural and efficient reaction enforced by the market to the demographic transition and subsequent fall in real interest rates. The growth of a bubble decreases the surplus of savings and reduces the asset shortage and thereby limits the fall in real interest rates and consequently improves utility across all cohorts. This is exactly what we observe looking at real house prices. Hence the growth of a bubble manifests itself in growing house prices.

5.1 Discussion

First, it is unclear whether the real interest rate paths generated by the OLG model are unique. There may be different transitions which will also converge to the new growth rates. Furthermore, the OLG model currently does not generate a path for the bubble size. A rational bubble grows with the real interest rate. Since this rate turns negative while the assumption is that the bubble grows, further research is needed to explain the development of the bubble size and to generate a path for the bubble size. Second, throughout this paper, several assumptions are made regarding the models. By determining the birth- and growth rates from fertility rates, I assumed that each cohort consists of 50% men and 50% women. The individual cohorts could be further tailored to reflect the actual population distribution within the cohorts. Regarding the assumed production function, the neoclassical production function with diminishing return to capital has some unsatisfactory features. Burmeister (2000) provides a relevant overview. For instance, capital is not a well-defined concept. There are many types of capital equipment, which makes capital equipment 'impossible' to aggregate. In addition, there are also many conceptual types of capital such as human capital, institutional capital, physical capital etc. Another example is the assumption of only one wage rate in the model whereas in the real world there are many different wages. Despite these shortcomings, the neoclassical production function is still widely used for lack of a better one. Individuals in the OLG model are assumed to have a finite horizon and do not care about the welfare of their descendants. Altruism may be added for a better reflection of reality [Barro (1974) and Barro and Sala-i-Martin (2004)]. As real-world interest rates, money supply and the behaviour of individuals are highly influenced by Central Banks and governments, further extensions to the OLG model could be the addition of Central Banks, which are able to influence interest rates through lending and borrowing money [De Santis (2016)], and governments that are able to collect taxes and make government purchases. Finally, one could consider adding open economy by allowing international borrowing and lending or adding time-varying parameters.

References

Ayres, R. U., & Warr, B. (2009). Energy efficiency and economic growth: the 'rebound effect' as a driver. *Palgrave Macmillan UK*.

Barro, R., & Sala-i Martin, X. (2004). *Economic Growth* (2nd ed.). MIT Press.

Barro, R. J. (1974). Are government bonds net wealth? *Journal of Political Economy*, *82*(6), 1095–1117.

Burmeister, E. (2000). The capital theory controversy. In *Critical essays on piero sraffa's legacy in economics* (p. 305–314). Cambridge University Press.

Carvalho, C., Ferrero, A., & Nechio, F. (2016). Demographics and real interest rates: Inspecting the mechanism. *European Economic Review*, *88*(Supplement C), 208 - 226.

Chirinko, R. S., Fazzari, S. M., & Meyer, A. P. (2004). That elusive elasticity: A long-panel approach to estimating the capital-labor substitution elasticity. *CESifo Working Paper Series*(1240).

Cho, T., Hwang, S., & Schreyer, P. (2017). Has the labour share declined?: It depends. *OECD Statistics Working Paper*.

De Santis, R. A. (2016). Impact of the asset purchase programme on Euro area government bond yields using market news. *ECB Working Paper*(1939).

Diamond, P. A. (1965). National debt in a neoclassical growth model. *The American Economic Review*, *55*(5), 1126–1150.

Eichholtz, P. (1997). A long run house price index: The Herengracht index, 1628–1973. *Real Estate Economics*, *25*(2), 175–192.

Favero, C. A., Gozluklu, A. E., & Tamoni, A. (2011). Demographic trends, the dividend-price ratio, and the predictability of long-run stock market returns. *Journal of Financial and Quantitative Analysis*, *46*(5), 1493–1520.

Favero, C. A., Gozluklu, A. E., & Yang, H. (2016). Demographics and the behavior of interest rates. *IMF Economic Review*, *64*(4), 732–776.

Favero, C. A., & Tamoni, A. (2010). Demographics and the term structure of stock market risk. *available at http:\www. igier.unibocconi.it\favero*.

Frederick, S., Loewenstein, G., & O'Donoghue, T. (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature*, *40*(2), 351-401.

Geanakoplos, J., Magill, M., & Quinzii, M. (2004). Demography and the long-run predictability of the stock market. *Brookings Papers on Economic Activity*(1380R).

Gordon, R. J. (2012). Is U.S. economic growth over? Faltering innovation confronts the six headwinds. *NBER Working Papers*.

Havránek, T. (2015). Measuring intertemporal substitution: The importance of method choices and selective reporting. *Journal of the European Economic Association*, *13*(6), 1180–1204.

Knoll, K., Schularick, M., & Steger, T. (2014). No price like home: Global house prices, 1870-2012. *CEPR Discussion Paper 10166*(5006).

Kris, D. J. (2008). Benchmark two-good utility functions*. *The Manchester School*, *76*(1), 44-65.

Lu, J., & Teulings, C. (2016). Falling real interest rates, house prices, and the introduction of the pill. *Cambridge Working Papers in Economics*.

Nadiri, M. I., & Prucha, I. R. (1996). Estimation of the depreciation rate of physical and R&D capital in the U.S. total manufacturing sector. *Economic Inquiry*, *34*(1), 43–56.

Rachel, L., & Smith, T. (2015). Secular drivers of the global real interest rate. *Bank of England Working Paper No. 571*.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, *66*(6), 467–482.

Song, Z., Storesletten, K., & Zilibotti, F. (2011). Growing like China. *American Economic Review*, *101*(1), 196-233.

Teulings, C., & Baldwin, R. (2014). Secular stagnation: facts, causes, and cures. *London: Centre for Economic Policy Research-CEPR.*, *15*.

Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica: Journal of the Econometric Society*, 1071–1100.

A Appendix

A.1 Data Sources

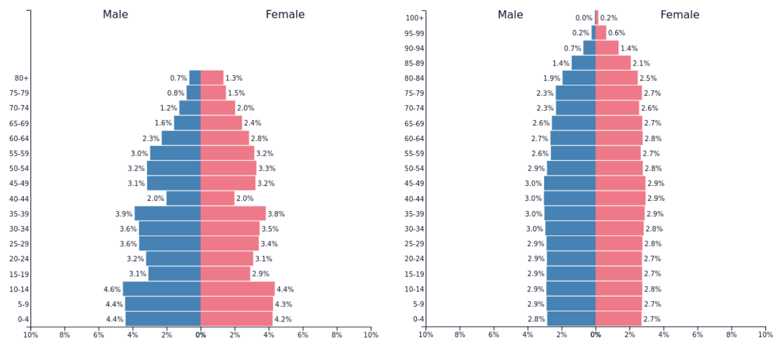
Demographics: The United Nations population division of the Department of Economic and Social Affairs provides quinquennial population data by five-year age groups (cohorts) in their World Population Prospects of 2017.
CPI: All the CPI data was collected from the database of Federal Reserve Bank of St.Louis.
Interest rates: The interest rates are Government bond yields and were collected from the International Financial Statistics database of the International Monetary Fund (IMF).
House prices: The Organisation for Economic Co-operation and Development (OECD) has a comprehensive database on house prices from 1956 - 2016. The quality of this database was verified with the research of Knoll, Schularick & Steger (2014). They did extensive research on national house prices and collected annual house prices from 1870-2012 for 14 advanced economies. Their sources were used to collect missing data or to adjust incorrect data from the OECD. The following steps were taken to complete the OECD data on house prices:

- France: The missing house prices from 1960 - 1970 were collected from the Conseil General de l'Environnement et du Developpement Durable (CGPC). The series can be downloaded from *http://www.cgedd.developpement-durable.gouv.fr/house-prices-in-france-property-a1117.html*. The CGPC series is nominal and was therefore corrected for inflation to obtain real house prices.
- Germany: Here too, the prices from 1960-1970 were missing. I used a report from the Federal Statistical office of Germany: Kaufwerte fur Bauland, Fachserie 17 Reihe 5, 2015 to obtain data back to 1962. I corrected the missing house prices for inflation to derive real house prices. I was not able to find reliable sources for the missing two years.
- Japan: The OECD data was complete and correct.
- Netherlands: Again, data from 1960-1970 was missing. The only source that covers the time before 1970 is the 'Herengracht-index' by Eichholtz (1997). This is a biannual index, which I interpolated to obtain annual house prices.

- United Kingdom: The data of the OECD for the UK was incomplete and unreliable. I used data from the Nationwide Building Society to complete the series. This data can be downloaded from *http://www.nationwide.co.uk/about/house-price-index /download-data#xtab:uk-series*. The data contains quarterly nominal house prices from 1952-2016. The data was first averaged to obtain annual observations and then corrected for inflation.
- United States: The OECD data was incomplete and unreliable. The Shiller index is the most complete database for house prices in the US. This dataset contains real annual observations prior to 1953. Thereafter, the data contains monthly observations. The monthly data was averaged to obtain real annual house prices from 1960-2016.

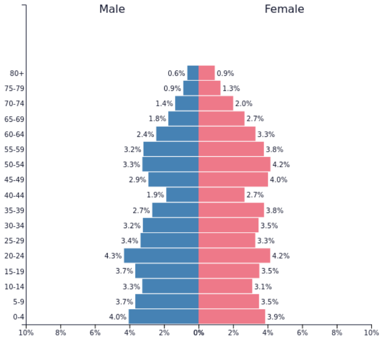
Fertility rates: The fertility rates for the six countries were collected from the OECD for 1960-2015. For 2016, I assumed the same rate as 2015.

A.2 Population Pyramids

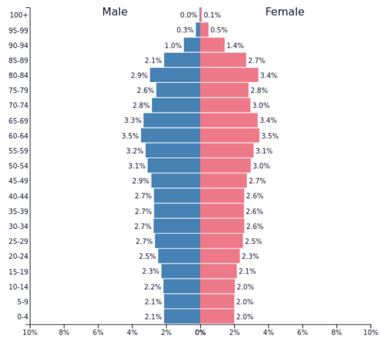


(a) 1960 (b) 2050

Figure 13: The population pyramid of France in 1960 and 2050.

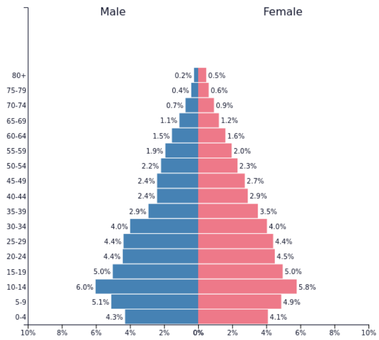


(a) 1960

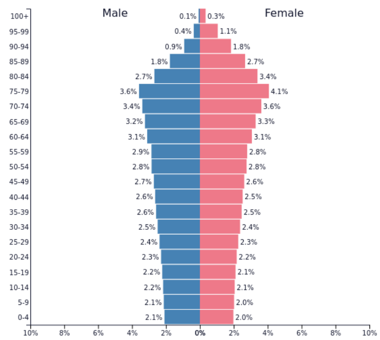


(b) 2050

Figure 14: The population pyramid of Germany in 1960 and 2050.

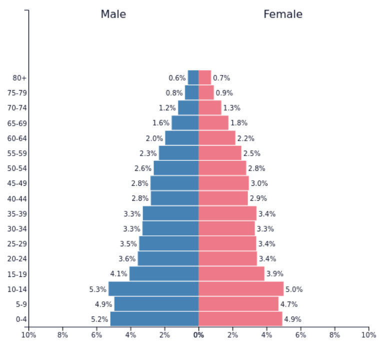


(a) 1960

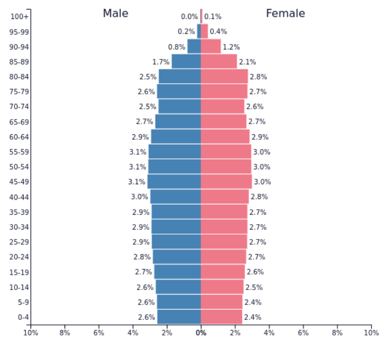


(b) 2050

Figure 15: The population pyramid of Japan in 1960 and 2050.

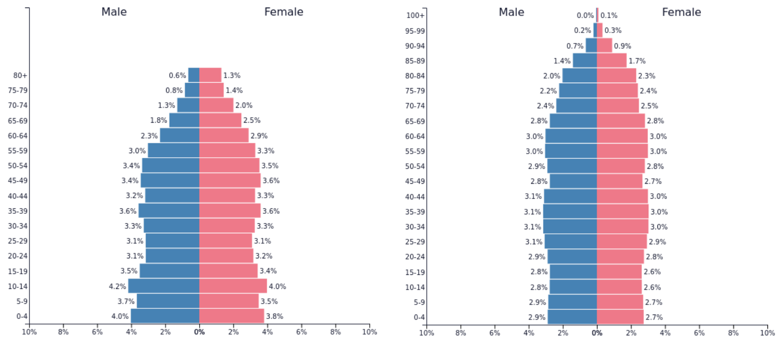


(a) 1960



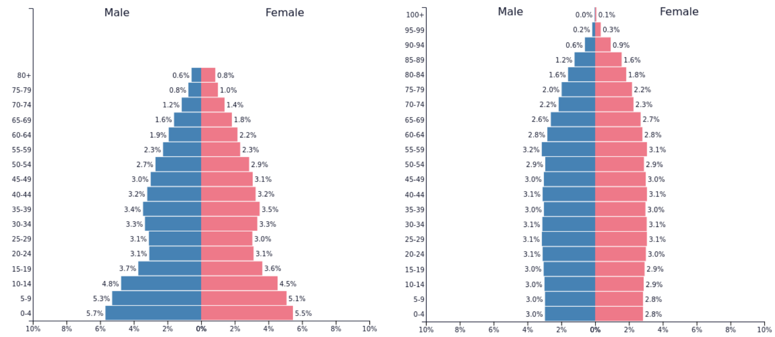
(b) 2050

Figure 16: The population pyramid of the Netherlands in 1960 and 2050.



(a) 1960 (b) 2050

Figure 17: The population pyramid of the United Kingdom in 1960 and 2050.



(a) 1960 (b) 2050

Figure 18: The population pyramid of the United States in 1960 and 2050.

A.3 Derivations simple model

A.3.1 Proposition 1a

To derive the no-bubble condition we start with market-clearing, the condition in which total income equals aggregate consumption plus aggregate investments:

$$\begin{aligned} Y_t &= C_t + I_t, \\ Y_t &= C_t + K_{t+1} - (1-\delta)K_t, \\ w_t L_t + (r_t + \delta)K_t &= C_t + K_{t+1} - (1-\delta)K_t, \\ w_t L_t + r_t K_t &= C_t + (K_{t+1} - K_t), \\ K_{t+1} - K_t &= w_t L_t + r_t K_t - c_{t,0} N_t - c_{t-1,1} N_{t-1}, \\ K_{t+1} &= K_t + w_t L_t + r_t K_t - c_{t,0} N_t - c_{t-1,1} N_{t-1}. \end{aligned} \tag{39}$$

Consumption is now replaced by the budget constraint. The budget constraint is

$$c_{t,0} + s_t = w_t, \tag{40}$$

where $s_t = \frac{c_{t,1}}{1+r_{t+1}}$, the amount saved in period t. If we substitute for consumption we find

$$\begin{aligned}
K_{t+1} &= w_t L_t + (1+r_t)K_t - (w_t - s_t)N_t - (1+r_t)s_{t-1}N_{t-1}, \\
&= s_t L_t + (1+r_t)(K_t - s_{t-1}L_{t-1}) \quad \text{using} \quad L_t = N_t, \\
&= s_t L_t + (1+r_t)(K_t - K_t), \\
&= s_t L_t.
\end{aligned}$$

To derive the third line from the second line, we use the fact that the old generation L_t in period t has saved in period $t-1$ for consumption in t by investing I_{t-1}, which is equal to K_t on an aggregate level. This result holds under the assumption that the old generation has no benefit from ending up with assets (no altruism). Consequently, they sell all their assets to the next generation in order to use the proceeds for consumption. The next periods aggregate capital is equal to all the capital owned by the old cohort plus the increase in capital and must be purchased by the young generation with their savings. Next, we try to describe the future path of capital. The capital per capita can be expressed as $k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{s_t}{1+g}$. Now $k_{t+1}(1+g) = s_t$, where $s_t = \frac{w_t}{\psi_{t+1}}$ and $\psi_{t+1} = \left(1 + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} (1+r_{t+1})^{\frac{-(1-\theta)}{\theta}}\right)$.[9]
Substitution of s_t leads to

$$\begin{aligned}
k_{t+1}(1+g)\psi_{t+1} &= w_t, \\
k_{t+1}(1+g)\left(1 + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} (1+r_{t+1})^{\frac{-(1-\theta)}{\theta}}\right) &= w_t, \\
k_{t+1}(1+g)\left(1 + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} (1+r_{t+1})^{\frac{-(1-\theta)}{\theta}}\right) &= f(k_t) - k_t f'(k_t).
\end{aligned} \tag{41}$$

This equation describes the future path of capital for a given initial value of k_t. To compute the steady-state capital intensity of k, we use that $k_{t+1} = k_t = k^*$. Divide the last line of equation 41 by k^* to obtain

$$(1+g)\left(1 + \left(\frac{1}{\beta}\right)^{\frac{1}{\theta}} (1 + f'(k^*) - \delta)^{\frac{-(1-\theta)}{\theta}}\right) = \frac{f(k^*)}{k^*} - f'(k^*), \tag{42}$$

[9] Compute the first-order condition with respect to s, $\frac{\partial U}{\partial s_t} = 0$ to obtain this relation.

where $f(k_t)$ is the Cobb-Douglas production function $f(k_t) = k_t^\alpha$. Remember that $f'(k) = r + \delta$. This simplifies equation 42 to

$$(1+g)\left(1+\left(\frac{1}{\beta}\right)^{\frac{1}{\theta}}\left(1+\alpha\,(k^*)^{\alpha-1}-\delta\right)^{\frac{-(1-\theta)}{\theta}}\right) = (1-\alpha)\,(k^*)^{\alpha-1}.$$

For log utility, so $\theta = 1$, this function can be solved analytically for k^* resulting in

$$k^* = \left(\frac{1-\alpha}{(1+g)(1+\frac{1}{\beta})}\right)^{\frac{1}{1-\alpha}}. \tag{43}$$

The question is now, which level of k^* allows for oversaving? Oversaving occurs when k^* is higher than the level of capital that maximises consumption. The maximisation of consumption of a household occurs at the k^{opt} that satisfies $f'(k^{opt}) = g + \delta$, because $r = g$ is the optimal value [Samuelson (1958)]. This means that $k^{opt} = \left(\frac{\alpha}{g+\delta}\right)^{\frac{1}{1-\alpha}}$. The condition for the steady-state capital intensity to exceed the golden-rule/optimal value of capital and hence for the economy to be in the dynamically inefficient region is therefore

$$\begin{aligned}\frac{1-\alpha}{(1+g)(1+\frac{1}{\beta})} &> \frac{\alpha}{g+\delta},\\ \frac{1-\alpha}{(1+g)(\frac{\beta+1}{\beta})} &> \frac{\alpha}{g+\delta}.\end{aligned} \tag{44}$$

For $g = 0$ and $\delta = \theta = 1$, we recover the no-bubble condition in equation 12. The condition that $k^* > k^{opt}$ for $\theta \neq 1$ such that the production function is not equal to the Cobb-Douglas production function can be studied numerically.

A.3.2 Proposition 1b

The supply of the bubbly asset is fixed at one. Therefore the size of the bubble is equal to the price and is given by the difference between savings and capital:

$$\begin{aligned}p &= s(k^*) - k^*,\\ &= \frac{w(k^*)}{\psi(k^*)} - k^*,\\ &= \frac{f(k^*) - k^* f'(k^*)}{\left(1+\left(\frac{1}{\beta}\right)^{\frac{1}{\theta}}(1+f'(k^*)-\delta)^{\frac{-(1-\theta)}{\theta}}\right)} - k^*,\end{aligned} \tag{45}$$

where $k^* = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$. This proves Proposition 1b: The size of a bubble. If we assume a Cobb-Douglas production function, $g = 0$ and $\delta = \theta = 1$ then

$$\begin{aligned} p &= \frac{(k^*)^\alpha - k^*\alpha\,(k^*)^{\alpha-1}}{\frac{\beta+1}{\beta}} - k^*, \\ &= \frac{\beta}{1+\beta}(1-\alpha)(k^*)^\alpha - k^*, \\ &= \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha}k^* - k^*, \\ &= \left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} - 1\right)k^*, \end{aligned} \tag{46}$$

where $k^* = \alpha^{\frac{1}{1-\alpha}}$. We arrive at Remark 1b (equation 15).

A.3.3 Proposition 2a

To derive the real interest rate path we look at the simple case where the intertemporal rate of substitution $\theta = 1$, capital fully depreciates between each period so $\delta = 1$ and population growth $g = 0$ such that the steady-state capital per worker $k = \alpha^{\frac{1}{1-\alpha}} = \alpha^{\frac{-1}{\alpha-1}} = \frac{1}{\alpha}^{\frac{1}{\alpha-1}}$. The constant output-labour ratio is once again obtained by dividing the output Y by the labour force $L = N$ resulting in $y = k^\alpha = \alpha^{\frac{\alpha}{1-\alpha}}$.
Optimal consumption $c_{t,0}$ and $c_{t,1}$ of a household born in period t is found by substituting the budget constraint $c_{t,0} + \frac{c_{t,1}}{1+r_{t+1}} = w_t$ in the Euler equation given in equation 6. The optimal consumption of a household is

$$\begin{aligned} c_{t,0}^{opt} &= \frac{1}{1+\beta}w_t, \\ c_{t,1}^{opt} &= \frac{\beta}{1+\beta}(1+r_{t+1})w_t. \end{aligned} \tag{47}$$

Lu & Teulings (2016) derive three equations which in combination with the derivations above lead to the transition path for the real interest rates. First, if we rewrite equation 8 to $k_t = \frac{1+r_t}{\alpha}^{\frac{1}{\alpha-1}}$ and substitute $\frac{1}{\alpha}^{\frac{1}{\alpha-1}}$ for k then the level of capital per worker can be written as

$$k_t = k(1+r_t)^{\frac{1}{\alpha-1}}. \tag{48}$$

Second, wage can be rewritten using the same trick. Substitute k_t in $w_t = (1-\alpha)k_t^\alpha$ and use the results above to rewrite the equation as follows:

$$
\begin{aligned}
w_t &= (1-\alpha)k_t^{\alpha}, \\
&= (1-\alpha)\left(\left(\frac{1+r_t}{\alpha}\right)^{\frac{1}{\alpha-1}}\right)^{\alpha}, \\
&= (1-\alpha)\left(\frac{1+r_t}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}, \\
&= (1-\alpha)\left(\frac{1+r_t}{\alpha}\right)^{\frac{1}{\alpha-1}+1}, \\
&= (1-\alpha)\left(\frac{1+r_t}{\alpha}\right)\left(\frac{1+r_t}{\alpha}\right)^{\frac{1}{\alpha-1}}, \\
&= \boldsymbol{\frac{1-\alpha}{\alpha}(1+r_t)k_t}, \\
&= \frac{1-\alpha}{\alpha}(1+r_t)k(1+r_t)^{\frac{1}{\alpha-1}}, \\
&= \frac{1-\alpha}{\alpha}\alpha^{\frac{1}{1-\alpha}}(1+r_t)(1+r_t)^{\frac{1}{\alpha-1}}, \\
&= (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(1+r_t)^{\frac{\alpha}{\alpha-1}}, \\
&= \boldsymbol{(1-\alpha)y(1+r_t)^{\frac{\alpha}{\alpha-1}}}.
\end{aligned}
\tag{49}
$$

Third, we look at the capital-labour ratio from a market-clearing perspective. We start at the market-clearing condition

$$
\begin{aligned}
Y_t &= C_t + I_t, \\
&= C_t + K_{t+1} - (1-\delta)K_t, \quad \text{where} \quad \delta = 1, \\
&= C_t + K_{t+1}, \\
Y_t &= c_{t,0}N_t + c_{t-1,1}N_{t-1} + K_{t+1}
\end{aligned}
\tag{50}
$$

Divide by N_t to obtain the output per capita

$$
\begin{aligned}
y_t &= c_{t,0} + c_{t-1,1}\frac{N_{t-1}}{N_t} + \frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}}, \\
y_t &= c_{t,0} + c_{t-1,1}\frac{N_{t-1}}{N_t} + k_{t+1}\frac{N_{t+1}}{N_t}, \\
k_t^{\alpha} &= c_{t,0} + c_{t-1,1}\frac{N_{t-1}}{N_t} + k_{t+1}\frac{N_{t+1}}{N_t}.
\end{aligned}
\tag{51}
$$

The last line holds because $y_t = k_t^{\alpha}$. To obtain the transition path for the real interest rates, the consumption of the current workforce is subtracted from the output per worker. Then consumption is substituted by their optimal values given by equation 47, and k_t and w_t are substituted by equations 48 and 49.

$$
\begin{aligned}
k_t^\alpha - c_{t,0} &= c_{t-1,1}\frac{N_{t-1}}{N_t} + k_{t+1}\frac{N_{t+1}}{N_t},\\
k^\alpha(1+r_t)^{\frac{\alpha}{\alpha-1}} - \frac{1-\alpha}{1+\beta}y(1+r_t)^{\frac{\alpha}{\alpha-1}} &= \beta\frac{1-\alpha}{1+\beta}(1+r_t)(1+r_{t-1})^{\frac{\alpha}{\alpha-1}}y\frac{N_{t-1}}{N_t} + k(1+r_{t+1})^{\frac{1}{\alpha-1}}\frac{N_{t+1}}{N_t},\\
y(1+r_t)^{\frac{\alpha}{\alpha-1}} - \frac{1-\alpha}{1+\beta}y(1+r_t)^{\frac{\alpha}{\alpha-1}} &= \beta\frac{1-\alpha}{1+\beta}(1+r_t)(1+r_{t-1})^{\frac{\alpha}{\alpha-1}}y\frac{N_{t-1}}{N_t} + k(1+r_{t+1})^{\frac{1}{\alpha-1}}\frac{N_{t+1}}{N_t},\\
y\left(1-\frac{1-\alpha}{1+\beta}\right)(1+r_t)^{\frac{\alpha}{\alpha-1}} &= \beta\frac{1-\alpha}{1+\beta}(1+r_t)(1+r_{t-1})^{\frac{\alpha}{\alpha-1}}y\frac{N_{t-1}}{N_t} + k(1+r_{t+1})^{\frac{1}{\alpha-1}}\frac{N_{t+1}}{N_t},\\
(1+r_t)^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_t)(1+r_{t-1})^{\frac{\alpha}{\alpha-1}}\frac{N_{t-1}}{N_t} + \frac{\alpha}{\gamma}(1+r_{t+1})^{\frac{1}{\alpha-1}}\frac{N_{t+1}}{N_t},
\end{aligned} \tag{52}
$$

where $\gamma = 1 - \frac{1-\alpha}{1+\beta}$ and $\frac{k}{y} = \alpha$. The last line is equal to Proposition 2a and describes the development of the real interest rate.

To derive equation 18, lets assume that the last line of equation 52 can be written as:

$$v_{t+1}^{1+a} = bv_{t+1}v_t^{1+a} + cv_{t+2}^a. \tag{53}$$

Now lets assume that $v_t = Ah^{m^t}$. Now rewrite 53 to

$$A^{(1+a)}h^{m(1+a)m^t} = bAh^{mm^t}A^{(1+a)}h^{(1+a)m^t} + cA^a h^{m^2am^t}. \tag{54}$$

Superscript $\frac{\alpha}{\alpha-1}$ in equation 52 is equal to $\frac{1}{\alpha-1}+1$. This means that $a = \frac{1}{\alpha-1}$. Consequently $\alpha = \frac{1}{a} + 1 = \frac{1+a}{a} = m$. It remains to find the value for A for which equation 52 holds. We have

$$A = bA^2 + c, \tag{55}$$

where $b = \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}$ and $c = \frac{\alpha}{\gamma}$. The only possible solution to this equation is $A = 1$. Hence C^{α^t} is a solution to the second order difference equation.

A.3.4 Proposition 2b

The system of five equations is given by

$$\begin{aligned}
1 &= \frac{\alpha}{\gamma}\left(1+\frac{p_{-1}}{k}C^{A}\left(\prod_{t=-1}^{1}(1+r_t)\right)\right)+\frac{\alpha}{\gamma}(1+r_{-1})^{\frac{1}{\alpha-1}},\\
(1+r_{-1})^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_{-1})+\frac{\alpha}{\gamma}(1+r_0)^{\frac{1}{\alpha-1}}N_H,\\
(1+r_0)^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_0)(1+r_{-1})^{\frac{\alpha}{\alpha-1}}N_H^{-1}+\frac{\alpha}{\gamma}(1+r_1)^{\frac{1}{\alpha-1}}N_H^{-1},\\
(1+r_1)^{\frac{\alpha}{\alpha-1}} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_1)(1+r_0)^{\frac{\alpha}{\alpha-1}}N_H+\frac{\alpha}{\gamma}(C^{\alpha^2})^{\frac{1}{\alpha-1}},\\
\frac{p_{-1}}{k} &= \left(\frac{\beta}{\alpha}\frac{1-\alpha}{1+\beta}-1\right)C^{-A}\left(\prod_{t=-1}^{1}(1+r_t)\right)^{-1},
\end{aligned}$$

where $A=\frac{\alpha^2}{1-\alpha}$, k is the steady state level of the capital-labour ratio and p_{-1} is the resale value of the bubble at $t=-2$. The starting point of the system of five equations is the second order difference equation given in the last line of equation 52:

$$(1+r_t)^{\frac{\alpha}{\alpha-1}} = \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}(1+r_t)(1+r_{t-1})^{\frac{\alpha}{\alpha-1}}\frac{N_{t-1}}{N_t}+\frac{\alpha}{\gamma}(1+r_{t+1})^{\frac{1}{\alpha-1}}\frac{N_{t+1}}{N_t}.$$

The second, third and fourth line of the system of equations follow immediately by using that $r_{-2}=0$ and $1+r_2=C^{\alpha^2}$. Because k_{-2} is decided on before the information on the demographic shock arrives, it is on its steady- state level k and thus r is at its steady-state level as well. The last line follows from rewriting equation 15, the growth of a rational bubble, by substituting $1+r_t$ by equation 18 leading to

$$\frac{p}{p_{-1}}=\prod_{t=2}^{\infty}C^{\alpha^t}\prod_{t=-1}^{1}(1+r_t)=C^{A}\prod_{t=-1}^{1}(1+r_t)\,, \tag{56}$$

where $A=\frac{\alpha^2}{1-\alpha}$. A is the sum of the geometric series α^t from $t=2$ to infinity. Substitution of equation 14 by equation 56 results in

$$\frac{p_{-1}}{k}=\left(\frac{\beta}{\alpha}\frac{1-\alpha}{1+\beta}-1\right)C^{-A}\left(\prod_{t=-1}^{1}(1+r_t)\right)^{-1}, \tag{57}$$

which is equal to the last line in Proposition 2b. At last, the first line follows by first using that $(1+r_{-2})$ and $(1+r_{-3})=1$. The remainder of the first term on the right hand side is $\frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}$. Equation 52, using that $\frac{p}{k}=\frac{\beta}{\alpha}\frac{1-\alpha}{1+\beta}$ from equation 14, now results in

$$\frac{\alpha}{\gamma}(\frac{p}{k}+1)=\frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}. \tag{58}$$

Substituting p by equation 56 gives the first line of the system of equations.