ERASMUS UNIVERSITY ROTTERDAM

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# A General Bundling Theory; an Application to Complementary Demand and Separate Sellers

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ABSTRACT - The goal of this thesis is to investigate the effect of unilateral bundle price change on profit, when considering a mixed bundling strategy, complementary goods and separate sellers. Specifically, the research question *'How does the profit depend on unilateral bundle price change?'* is answered by establishing a theoretical proposition on profitability, deriving results for one specific case and analyzing the effect of varying levels of complementarity. Under some reasonable assumptions, we obtain the monotonicity result that introducing a small unilateral bundle premium can positively affect the firm's profit. When adding the assumption of symmetric prices and bundle premium values and considering a specific case example, we find that offering a higher bundle price does not necessarily give a higher industry profit. Lastly, we find that the strength of synergy between the two complementary products positively affects the industry profit. For all possible degrees of complementarity, however, the highest industry profit is obtained under linear pricing.

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# 1 Introduction

Providing a package of products is becoming increasingly popular amongst online shops such as Coolblue and MediaMarkt. They sell, for instance, a laptop individually as well as a bundle in which the laptop is packaged with a mouse and a laptop case. Offering such bundles is not only becoming more attractive for consumer goods but also for services such as energy suppliers and insurance companies (Cooper et al., 2006).

Specific bundling practices have been studied since the moment of the old-school strategy of offering a number of movies in one package, which proved to be profitable (Stigler, 1963). Stigler (1963) was the first in economic literature who introduced the concept of *bundling* as a tool of price discrimination – the practice of offering an array of products or services in a single package at one price. Stigler's paper additionally introduces the concept of reservation price, which illustrates the customer's willingness-to-pay for a certain product. The aforementioned profitability conclusion can only be drawn under a few rather strong assumptions; reservation prices are negatively correlated, additive and their aggregations are similar for all consumers.

Adams and Yellen (1976) continued with Stigler's model (1963), yet introduced three different strategies; separate pricing, pure bundling and mixed bundling. Moreover, they developed the bundling model of a monopolist with two products in a two-dimensional graphical framework. The assumptions related to the distribution of the consumer reservation prices and the cost structure are crucial to determine which one of the three strategies is most profitable. As a variety of assumptions can be applied to Adams and Yellen's model, their contributions lie at the heart of the bundling research and below mentioned frameworks are all extensions of this model.

Schmalensee (1982) enhances the Adams and Yellen model (1976) by allowing situations in which a product from a single-product monopolist is paired with a second good from a competitive industry. This study shows that mixed bundling will boost profits dependent on the distribution of reservation prices. Similar to Stigler's finding (1963), Schmalensee (1982) demonstrates that mixed bundling can increase the monopoly's profits when reservation prices are negatively correlated. In 1984, Schmalensee et al. published new research results, which made them the first to use numerical criteria to solve the bundling model. They specified the profitability condition of a bundling strategy as a function of production costs and reservation price characteristics (e.g. mean and variance). Furthermore, they used a bivariate normal distribution as the distribution for reservation prices. Schmalensee et al.'s main finding (1984) is that a reduction in diversity of consumers allows higher extraction of *consumer surplus* (the value difference between the consumer's reservation price and the actual price paid), and in turn, higher profits for both mixed and pure bundling. Their study also concluded that bundling may increase profits when the valuations of two goods are either negatively, positively (but not perfectly) correlated or uncorrelated. This finding is contradictory to Schmalensee's (1982) and Stigler's (1963), nevertheless the resemblance among Schmalensee (1982), Schmalensee et al. (1984) and Stigler (1963) is that they all draw similar conclusions for mixed and pure bundling. McAfee et al. (1989) adds to the literature by introducing a number of conditions in which mixed bundling is preferred over pure bundling in a two goods model. Moreover, McAfee et al. (1989) concludes that the addition of a bundle discount can be profitable when reservation prices are independent and additive when sellers are integrated.

***All the aforementioned studies assume that reservation prices are 'strictly additive'.*** This means that consumers' valuation of a bundle ($v_b$) is assumed to be equal to the sum of valuations of the separate components ($v_b = v_1 + v_2 + \cdots + v_z$ for arbitrary natural number $z$). Strict additivity can only be a realistic assumption in case all products in the market are independent, meaning that the individual valuation of one good is not affected when one also consumes another good. However, in reality commodity bundling is widely used for *interrelated products*, products for which $v_1$ is affected by $v_2$, and vice versa. Interrelated products can be divided in two groups of goods; substitutes ($v_b \leq v_1 + v_2 + \cdots + v_z$) and complements ($v_b \geq v_1 + v_2 + \cdots + v_z$). An example of two products that can be substituted are Coca Cola and Pepsi Cola, which contain overlapping benefits and compete for similar resources. The consumer valuation of the bundle of such goods is not as highly valued as the sum of the components. An example of three products that are complements is the example given in the introduction; a laptop, a mouse and a laptop case. Contrarily to substitutable products, the reservation price of a bundle is greater than the sum of the individual products as the multiple products are used together.

***Another classical assumption made in all the above-mentioned bundling models is that supply is integrated.*** There is integrated supply when all the bundle components are produced by one single firm. This is a realistic assumption when the model only applies to similar products. For instance, it is acceptable to assume that a matching fork and knife originate from the same producer. Still, there are also numerous examples in which the paired products may very well stem from separate sellers, e.g. when bundling coffee beans and a coffee machine or a laptop and a laptop case. In addition, the number and size of online shops that bring separate sellers together on one online platform has significantly increased over the past years. Consequently, this development has enlarged the set of possibilities to purchase a bundle with goods from different firms (Cooper et al., 2006).

These two rather strong assumptions restrict the number of situations the studies' findings apply to. By allowing for non-additive consumer valuations and separate sellers, the model in this thesis relaxes these assumptions. Hence, this research adds to the literature by considering a model that accounts for a set of situations that is not broadly considered in the economic literature.

This thesis is organized as follows. Section 2 provides an overview of the relevant related literature and introduces the research question. In this section it will also become clear which theoretical model is used as foundation for this thesis and how it is extended. Section 3 covers the assumptions and main properties of the demand side of the model and Section 4 presents explanations on the supply side. In Section 5 three complementary analyses are presented. A proposition on the profitability of premium pricing under the discussed assumptions is presented. Furthermore, for one specific case results are derived and compared to Armstrong (2012). Finally, effects of varying synergies between the two complementary products are treated. Section 6 contains a discussion on the limitations of the study and discusses possible further research. Finally, Section 7 gives a conclusion and concise summary of the thesis.

## 2 Review of Related Literature

In this section we discuss bundling literature that is particularly relevant to this thesis. First, findings on non-additive consumer valuations are examined, with special focus on complementary demand, followed by bundling research that allows for separate sellers. The third paragraph covers the two main papers that account for both contingent reservation prices and separate sellers. Here it will become clear which model is used as a foundation for the research in this thesis and how it will be extended. Moreover, the research question addressed in this thesis is stated at the end of this section.

The model in this thesis allows for non-additive consumer valuations by specifically looking at complementary goods. According to Oxenfeldt (1966), complementary goods have relationships that can be divided in three categories: products that are complements because of economies in time and effort from purchasing them together, products that enhance customers' levels of satisfaction with other products and products that enhance the overall image of the seller so that all products are valued more highly.[1] Another property of complementary demand is studied by Guiltinan (1987), who states that when such services are bundled, the consumer surplus of one service can be transferred to the other service. Guiltinan specifically looks at *cross-selling*, when consumers shift from buying one good to buying the bundle, and *customer acquisitions*, when consumers shift from buying nothing to buying the bundle. Moreover, this paper establishes conditions on which of these two types is most successful. Another bundling model that allows for non-additive reservation prices is studied by Bakos and Brynjolfsson (1999). They treat non-additive consumer valuations and come up with an equation that specifies the profits of pure bundling with such valuations. Their study is an extension of the well-known Adams and Yellen model (1976), discussed in the introduction. Bakos and Brynjolfsson are the first to successfully introduce a statistical model that shows profitability of bundling a very large number of goods. The law of large numbers is applied to consumer valuations and shows that bundles with a larger number of products have a lower variance, and therefore the reservation prices of large bundles can be more precisely predicted. Note that this model only applies to information goods with no or significantly small marginal costs. Another study related to contingent reservation prices is the one from Venkatesh

[1] Consumers that fall in category 3 'products that enhance the overall image of the seller so that all products are valued more highly' are not accounted for in this thesis, because only bundles from which the products originate from different sellers are considered.

(2003), studying which type of bundling is preferred for contingent valuations in a dual product monopoly framework. This research provides a comparison between the pricing strategies of complements and substitutes to additive valuations. In this study it is concluded that the degree of synergy between the two products as well as the level of marginal costs determine which bundling strategy is optimal according to firm's profit maximization. For instance, under a relatively low value of marginal costs, mixed bundling is found to be optimal for moderate-to-weak substitutes and weak complements. With moderate-to-strong complements, everyone will stop buying the individual goods and therefore pure bundling is most profitable. These findings are in line with the ones from Lewbel (1984), who studies exploiting demand complementarity as a reason for bundling. According to these results, complementary demand is not necessary or sufficient for any type of bundling to be the most profitable selling strategy. Cready (1991) adds a new type of bundling to the list by extending the mixed bundling strategy defined by Adams and Yellen (1976) to allow for a bundle price greater than the sum of the individual bundle components. For this new strategy, named 'premium bundling', Cready showed profitability when consumer reservation prices are correlated in various ways. For instance, premium bundling maximizes profit for certain contingency ranges of complementary goods as well as substitutable goods. Long (1984) is unique in the contingent valuations literature as this study is the first to use standard demand theory, which makes the model tractable for non-additive as well as additive consumer valuations. This paper reflected on Schmalensee et al.'s research (1984), discussed in the introduction, and aimed to introduce a more general framework based on integrated firms. Under weaker assumptions as compared to Schmalensee et al., Long develops certain conditions in which bundling increases profitability. From all the above mentioned contingent reservation prices related literature, we will use the definitions for complementary demand introduced by Lewbel (1984) and allow for premium pricing like Cready (1991).

This thesis considers a model in which the bundle components originate from different sellers. Gans and King (2006) also account for non-integrated supply by studying the effect of a bundling discount in an oligopoly setting and discussing the bundle discount's welfare effects. More specifically, Gans and King's model incorporates four separate producers that all produce two products and all unique products are produced by two different firms. However, their model is based on additive consumer valuations and the products studied are unrelated and independent. Gans and King find that introducing a bundle discount for two unrelated products can cause these products to become strategically interdependent, which increases the profits of the two producers involved. Similar to Gans and King, Matutes and Regibeau (1992) show that some combinations of assumptions will not necessarily benefit profit or welfare. However, Matutes and Regibeau's study focusses on a duopoly setting and a larger number of components that could be sold in a package and can be assembled by the customer himself. They find that it is profitable for firms to produce various compatible components but also that a bundle discount is only beneficial when all available components are purchased together. However, note that the model discussed in Matutes and Regibeau (1992) allows for bundle discounts only when components are purchased from the same firm, which is contradictory to the discount possibilities in Gans and King paper (2006) and this thesis. In our model we allow a single-product firm to provide a discount or premium to a customer who also buys a product from another firm.

This thesis is not the first to study these two assumptions together. There are two main papers that conduct very similar research by studying a model that accounts for non-additive consumer valuations as well as non-integrated supply. First, Calzolari and Denicolo (2013) study exclusive contracts; each of the two firms in their model can set two distinct prices, one for customers that want to buy from one firm only and one for customers that want to buy products from both firms. Therefore firms have the opportunity of introducing a unilateral discount when its own product is bundled with the other firm's product. Calzolari and Denicolo do not specifically look at non-additive consumer valuations, but it is implicit in the model as a strict concave utility function is assumed. This function represents an increased utility for purchasing products from both companies. In this setting consumers who buy only from one firm perceive the product as a perfect substitute. Calzolari and Denicolo's study concludes that the availability of exclusive contracts lowers prices and that both firms benefit if no unilateral discounts is offered. The second main paper, that serves as a starting point for this thesis, is from Armstrong (2012) who analyzes a mixed bundling model in which all goods are perceived as substitutes. Furthermore, this study compares a model in which supply is integrated to a model in which supply is separated. Armstrong (2012) differs from Calzolari and Delicolo (2013) as consumers can purchase at most one product of each firm (unit demand) and the level of substitutability cannot differ for each consumer. Similar to Calzolari and Delicolo, Armstrong's separate sellers model allows firms to offer a discount to consumers that purchase the other firm's product without any coordination between the two firms. Moreover, they both consider a mixed bundling strategy, motivated by the fact that a significant amount of the economic literature shows that such a strategy is most profitable. For instance, Schmalensee et al. (1984) explain that mixed bundling captures the best option when '*selling the bundle to a group of buyers with reduced effective heterogeneity, while charging high markups to those on the fringes of the taste distribution who are mainly interested in only one of the two goods'* (Schmalensee, 1984, p.227). Armstrong's main finding is that irrespective of whether supply is integrated or not, introducing a unilateral bundle discount will positively affect the firms' profit in a market with substitutable goods.

In this thesis we extend the separate sellers' model of Armstrong (2012) to account for complementary goods.[2] Therefore, we assume that the valuation for the bundle is always greater or equal to the sum of the individual components' valuations. Examples of such packages are coffee beans and a coffee

[2] Note that one of Armstrong's proposed extensions to his analysis is '*How do the results change if the products are complements rather than substitutes?*' (Armstrong 2012, p.26).

machine, a laptop and a laptop case, or a flight ticket and hotel accommodation. These types of goods are widely available in the market and rarely studied in the economic bundling literature.

Given this model, in which a mixed bundling strategy, complementary goods and separate sellers are considered, as a further contribution to this research field we study the following specific research question:

**How does the profit depend on unilateral bundle price change?**

Note that a 'bundle price change' accounts for selling the bundle at a premium or discount. This question will be answered by establishing a theoretical proposition on profitability, deriving results for one specific case and analyzing the effect of varying levels of complementarity.

## 3 Model A: Characterizing Demand

As mentioned in Section 2, this thesis adopts the separated sellers model by Armstrong (2012) and extends it to allow for complementary demand as defined in Lewbel (1984). In this section the demand side of the model is explained. First we will discuss the basic economic assumptions of the model after which the mathematical model is specified. Given this model, three lemmas are derived that are related to demand properties, two of which are dependent on complementary demand.

### *3.1 Economic Assumptions*

We consider a market with two products ($i$ and $j$) and a seller that uses a mixed bundling strategy, implying that product $i$ and $j$ can be purchased both individually as well as together in one package. The consumer can make a choice based on three prices; $p_i$, $p_j$, and $p_b$, which correspond to product $i$, product $j$ and the bundle, respectively. The bundle price depends negatively on δ as can be seen in (1), which amounts to a discount (when $\delta > 0$) or a premium (when $\delta < 0$). Note that $\delta = 0$ corresponds to linear prices. All consumers satisfy unit-demand, implying that customers can choose to buy either one unit or none of each product. This assumption is introduced to assure that variations in demand are due to changes in the number of consumers.

The consumer's willingness-to-pay for products $i$ and $j$ is reflected in the reservation prices $v_i$ and $v_j$, respectively. As consumer valuations are assumed to satisfy complementary demand, the valuation for the bundle ($v_b$) is greater than the sum of the individual valuations[3], cf. (2). Moreover, there is no

[3] Note that there are various other methods of modelling sub-additive consumer valuations. For instance, Venkatesh (2003) considers the relative measure θ = (Reservation price for bundle – sum of stand-alone reservation prices for product 1 and 2)/(sum of stand-alone reservation prices for products 1 and 2), which judges differences in proportional terms rather than absolute terms and Bakos and Brynjolfsson (1999) capture the contingency of the goods in a power term denoted by α which is positive for complements and negative for substitutes. The higher the absolute value of θ and α, the higher the contingency.

negative valuation from receiving an extra good as there is free disposal (cf. (3)). The consumer reservation prices ($v_i$, $v_j$, $v_b$) fluctuate across people based on a known distribution[4].

The value difference between the consumer's reservation price and the actual price paid is called the consumer surplus. Suppose the consumer maximizes utility by purchasing one of the following four options, which yields the highest surplus: solely product $i$, solely product $j$, the bundle or nothing. The only option to buy both goods is by purchasing the bundle, which is done if and only if $v_b - p_b \geq \max\{v_i - p_i, v_j - p_j, 0\}$ holds. The consumer will purchase solely product $i$ if and only if $v_i - p_i \geq \max\{v_b - p_b, v_j - p_j, 0\}$ holds. A similar statement holds for purchasing solely product $j$. When all options deliver a non-positive surplus, the consumer will choose to buy nothing. These discrete choices are combined in (4) to account for all situations in which product $i$ is purchased. Note that these four discrete options exclude the possibility of buying both stand-alone products together, which is no severe restriction when the bundle is sold at a discount[5]. However, when selling the bundle at a premium, discarding this option is often not very realistic and restricts the number of situations in which this model applies[6]. In many situations, when both products are available together at a lower price than the bundle, people will switch from buying the bundle to purchasing the stand-alone products together. Hence, mixed bundling then collapses to a pure component strategy (Venkatesh and Kamakura, 2003).

In this model, demand depends on $p_i, p_j$ and δ and is denoted by $Q_i = Q_i(p_i, p_j, \delta)$, $Q_j = Q_j(p_i, p_j, \delta)$, and $Q_b = Q_b(p_i, p_j, \delta)$ for product $i$, product $j$ and the bundle, respectively. The demand is measured as a proportion of all consumers and $Q_i$ (like $Q_j$) accounts for demand of stand-alone product $i$ (product $j$). As the analysis requires frequent comparison to linear prices ($\delta = 0$), the simplified notation (5) is introduced.

The above discussed economic assumptions lead to the mathematical model A below. Note that the same relationships hold when $i$ and $j$ are interchanged.

[4] In Section 5, we consider a uniform distribution for $(v_i, v_j)$ to simulate the model. Note that all kinds of distributions can be applied to this model, such as the normal and beta distribution. The only requirement is that the demand functions are differentiable.

[5] Because no consumer would want to purchase the two stand-alone products together when the bundle is available at a lower price. Therefore, in this situation there would be no demand for buying both stand-alone products together.

[6] When the bundle is sold at a premium, the model only applies to situations in which consumer demand is known, as consumers are not willing to voluntary expose their preferences when this results in higher prices. A perfect example is an online shop (such as AliExpress and Ebay) in which different sellers can promote their products and obtain information on consumer demand from their online purchasing history. In such a situations, a seller can see a consumer's preferences and offer a price depending on the previously purchased products.

| | | |
|---|---|---|
| **MODEL A** | $p_b = p_i + p_j - \delta$ | ( 1 ) |
| | $v_b \geq v_i + v_j$ | ( 2 ) |
| | $v_b \geq max\{v_i, v_j\}$ | ( 3 ) |
| | $max\{v_b - (p_i + p_j - \delta), v_i - p_i\} \geq max\{v_j - p_j, 0\}$ | ( 4 ) |
| | $q_i(p_i, p_j) \equiv Q_i(p_i, p_j, 0);\ q_b(p_i, p_j) \equiv Q_b(p_i, p_j, 0)$ | ( 5 ) |

## *3.2 Demand Properties*

In this subsection several lemmas are presented that describe properties of demand of model A. The lemmas will be used in the theoretical analysis given in Section 5. Only in Lemma 1 and 2, the complementary condition (2) is used.

**Lemma 1**. *Consider model A with* $\delta = 0$. *The following holds*

$$\frac{\partial(q_b + q_i)}{\partial p_j} \leq 0.$$

*This means that when linear prices are used, the demand for product* $i$ *weakly decreases with the price of the other good.*

This statement can be proven using both calculus and graphical arguments. The idea of the proof of both methods is based on

$$max\{v_b - p_i - p_j, v_i - p_i\} \geq max\{v_j - p_j, 0\} \quad (6)$$

which follows from taking $\delta = 0$ in (4). The consumers that satisfy (6) will purchase product $i$ either in the bundle or as a stand-alone product. Therefore, (6) characterizes demand as the proportion of consumers with vector $(v_i, v_j, v_b)$ that satisfy this inequality is equal to $q_i + q_b$ (total demand for product $i$).

To prove Lemma 1 using calculus we distinguish the four possible situations that follow from (6) and discuss the influence of $p_j$ on every individual situation. For the graphical proof (6) is rewritten as follows

$$max\{v_b - p_i - p_j, v_i - p_i\} + min\{p_j - v_j, 0\} \geq 0 \quad (7)$$

which is illustrated in Figure 1. The weakly decreasing behavior of the green line in Figure 1 provides the proof for Lemma 1 (all omitted proofs can be found in Appendix A). Note that the proof based on graphical arguments also provides additional information on the behavior of $q_i + q_b$ as a function of $p_j$ values. As can be seen in Figure 1, when $A > B$, total demand of product $i$ is constant for small and large values of $p_j$, and decreasing for moderate values of $p_j$. When $A = B$, total demand of product $i$ is independent of all possible values of $p_j$.

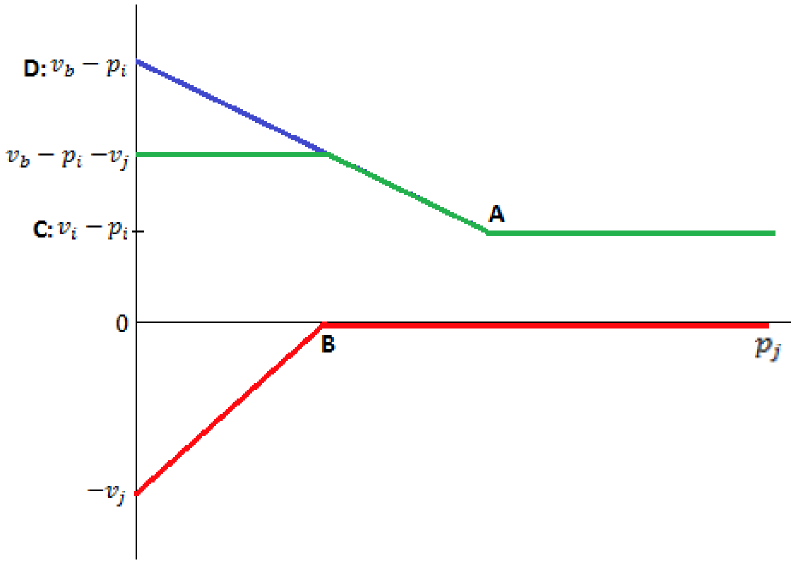


Figure 1: The green graph depicts the relationship between the total demand for product $i$ and the price level of good $j$. Moreover, the green line represents the summation of the blue and red line (for $p_j \geq p_j^B$ the green line lies on top of the blue line). The red line corresponds to the minimization part and the blue line corresponds to the maximization part of (7). Note that this figure is an example of a situation when $p_j^A > p_j^B$. However, when $p_j^A = p_j^B$, the green line is constant as the slope of the blue and the red line would offset each other until the exact same point. Lastly, note that the green line can also be shifted up and down depending on $v_b - p_i - v_j$, as long as the $y$ intercept is between point $C$ and $D$ (note: point $C$ and $D$ can also move up and down, even below 0), the trend however stays the same.

The economic interpretation of Lemma 1 is easily explained as it relies on complementary goods. When the price of good $j$ goes up ceteris paribus, the bundle will become more expensive under linear prices as $p_b = p_i + p_j$ (cf. (1)), which directly decreases bundle demand. When the price of the bundle increases, there are three different groups of consumers with a bundle valuation that fall just below the increased bundle price. The first group will now purchase stand-alone product $j$, the second group stand-alone product $i$ and the last group will stop buying at all. The second group of people leave the total demand for product $i$ unaffected. On the other hand, group 1 and 3 negatively affect the total demand for product $i$. In addition, this effect is strengthened due to complementary demand; the stronger the synergy between the two products, the smaller the number of people who are willing to buy individual products at all (Venkatesh, 2003). This implies that when the synergy is very strong, the above mentioned group 2 will become very small. The result of Lemma 1 can also be logically interpreted when looking at an example. Consider the complementary products gasoline and gas cars. When the price of gasoline significantly increases, consumers will look for alternative options such as electronic cars. So due to this price increase for gasoline, consumers can decide to substitute gas cars by electronic cars which thus decreases the demand for gas cars.

**Lemma 2.** *Consider model A with $q_i > 0$. The following holds for the consumers corresponding to $q_i$*

$$v_j \leq p_j.$$

*This means that for those consumers who buy stand-alone product $i$ under linear prices, the decision to buy product j on its own is not dependent on a price changes in product i*[7].

When the inequalities $v_b \geq v_i + v_j$ (complementarity) and $v_i - p_i \geq v_b - p_i - p_j$ (decision to buy good $i$ on its own) are combined, one easily obtains inequality $v_j \leq p_j$ (details are given in Appendix B).

How to interpret Lemma 2 is most easily explained by sketching two possible situations. First, consider a small price increase in $p_i$, which causes some consumers from the initial group $q_i$ to decide to buy nothing at all. In this situation, Lemma 2 shows that no consumers from the initial group $q_i$ will instead purchase product *j* on its own. Secondly, consider a small decrease in $p_i$ ceteris paribus, which will in turn lower the bundle price. In this situation some consumers that would not buy anything in the initial situation will now want to buy the bundle or stand-alone product $i$. Here Lemma 2 implies that due to this price decrease no consumer that would buy product $i$ in the initial situation would now want to buy product *j* on its own. The result of Lemma 2 is plausible for both situations when considering that the price of product *j* is not affected by a change in $p_i$. Therefore, the group of consumers with a reservation price above $p_j$ remains unchanged in both situations.

**Lemma 3[8].** *Consider model A except the complementary demand inequality (2). The following holds*

$$\left.\frac{\partial(Q_i+Q_b)}{\partial\delta}\right|_{\delta=0} = -\frac{\partial q_b}{\partial p_i}. \tag{8}$$

*This means that the effect of a small bundle discount on product $i$'s total demand is equal to the effect of a price change in product $i$ on the linear bundle demand (same relationship holds for product j).*

This lemma is proven by applying the Slutsky Symmetry of cross-price effects to the consumers' discrete options in (4). In short, the Slutsky Symmetry states that the derivative of demand of good $i$ to the price of good *j* is equal to the derivative of demand of good *j* to the price of good $i$ ($\frac{\delta D_i}{\delta p_j} = \frac{\delta D_j}{\delta p_i}$). To apply this theory, we first consider the widely known economic rule that the negative integral of the demand function is equal to the consumer surplus. Accordingly, by differentiating the consumer

[7] The conclusion in Lemma 2 does not hold when all consumers in the population have a reservation price of $v_i = p_i$.

[8] The result of Lemma 3 is completely identical as the demand property discussed by Armstrong (Armstrong, p.7, 2012). This is due to the fact that this property applies similarly to substitutes as to complements. Therefore, the proof and economic interpretation are just shortly discussed. This Lemma is however included as it is crucial for the following analysis.

surpluses in (4), we obtain the demand functions. Having taken the first derivative, we can now consider the symmetry of second derivatives[9]. Therefore by using the equality of the second derivatives and (5) we obtain (8) (more details of the proof are given in Appendix C).

This lemma does not contain important economic interpretations at this stage. It is merely a mathematical property that is used for further analysis.

The three lemmas in this section explain the demand properties in this model and are used in the analysis of Section 5. Specifically,  Lemma 1 and 2 are used to draw the graphs in Figures 3 and 4, and in turn, to define the demand functions for the analysis. Lemma 3 is used when proving Proposition 1, 2 and 3.

## 4 Model B: Characterizing Supply

In this section, the supply side of the model is explained. The additional economic assumptions related to separate supply will be discussed and are translated into mathematical expressions in model B.

### *4.1 Economic Assumptions*

Suppose that each of the two products is supplied by a different firm. Firm $i$ can offer a unilateral discount or premium to consumers that buy product $j$, thus transparency of consumer demand is assumed. This assumption is not very restrictive in case a bundle discount is considered, because consumers are willing to expose their demand when they are rewarded with a discount. Contrarily, when considering a bundle premium this assumption restricts the number of situations the model applies to as consumers will try to hide their demand decisions. Therefore, under premium bundling only products that can be monitored without customers' contribution are reflected in this model[10]. Both firms choose their prices simultaneously without negotiation. Accordingly, it is possible that both firms decide to offer a discount, in which case the bundle price is $p_i + p_j - \delta_i - \delta_j$. Firm $i$ chooses its optimal linear price $p_i^*$ by optimizing the linear profit function (cf. (9)), with $c_i$ being the marginal costs of firm $i$. The firm can also maximize a profit function that includes both the optimal price and discounting (cf. (10)). We assume that at linear prices there is at least some demand for the individual products as well as the bundle (cf. (11)).

The above discussed economic assumptions lead to the supply part of the model, given in mathematical model B below. Note that the same relationships hold when $i$ and $j$ are interchanged.

[9] This theory states that all second derivatives from the same function must be symmetric.

[10] An example of a situation for which this assumption holds is an online shop (such as AliExpress and Ebay) in which different sellers can promote their products and obtain information on consumer demand from their online purchasing history. Note that this assumption restrict for similar situations as the assumption in model A on the four discrete choice, which leaves out the option to buy the stand-alone products together.

| | | |
|---|---|---|
| **MODEL B** | $\pi_L = (p_i - c_i)(q_i + q_b)$ | ( 9 ) |
| | $\pi_D = (p_i^* - c_i)(Q_i + Q_b) - \delta Q_b$ | ( 10 ) |
| | $q_i > 0;\ q_b > 0$ | ( 11 ) |

# 5 Analysis of Model A and B

In this section various analyses are conducted to answer the research question *'How does the profit depend on unilateral bundle price change?'*. This question is answered by deriving a theoretical proposition on profitability in Subsection 5.1, deriving results for one specific case and comparing these to Armstrong's (2012) in Subsection 5.2 and analyzing the effect of varying synergies between the two products under a uniform distribution in Subsection 5.3. A crucial difference between these subsections is that in the first we investigate the profitability of bundle premium pricing for firm $i$ taking the prices of firm $j$ as fixed, while in the last two all prices of both firms are taken variable.

## *5.1 Propositions on the Profitability of a Bundle Price Change*

The purpose of this subsection is to establish a proposition on the profitability of a bundle price change under complementary demand based on model A and B. Specifically, we observe the equilibrium of firm $i$ when introducing a small unilateral bundle premium given the prices of firm $j$ remain unchanged. To obtain such a result and use a tractable structure, we divided the analysis in two parts. First, two similar propositions are presented on the profitability of introducing bundle discounts and premiums based on an elasticity assumption that applies to additive consumer valuations. Secondly, complementary demand is considered and an assumption is made on the hazard rate of $v_i$. The result of the second part of the analysis implies the elasticity assumption used in Proposition 2. This then yields the concluding statement, which is formulated in Proposition 3.

**Proposition 1.** *Consider model A and B except (2) and suppose* $v_b = v_i + v_j$ *and* $\frac{-\partial q_i/\partial p_i}{q_i} < \frac{-\partial q_b/\partial p_i}{q_b}$ [11]. *The following holds*

$$\left.\frac{\partial \pi_{Discounting}}{\partial \delta}\right|_{\delta=0} > 0 \qquad (12)$$

*This means that when the elasticity of demand is lower for buying firm i's product only than for buying the bundle, introducing a small unilateral bundle discount is profitable for firm i, when consumer valuations are additive.*

[11] Note that in these elasticity definitions (with a standard format of $\frac{p}{q}\frac{dq}{dp}$), the $p$ cancels.

When reversing the elasticity assumption ($\frac{-\partial q_i/\partial p_i}{q_i} < \frac{-\partial q_b/\partial p_i}{q_b}$), the relationship between $\delta$ and profit changes as presented in Proposition 2 below.

**Proposition 2.** *Consider model A and B except (2) and suppose $v_b = v_i + v_j$ and $\frac{-\partial q_i/\partial p_i}{q_i} > \frac{-\partial q_b/\partial p_i}{q_b}$. The following holds*

$$\left.\frac{\partial \pi_{Discounting}}{\partial \delta}\right|_{\delta=0} < 0 \tag{13}$$

*This means that when the elasticity of demand is higher for buying firm i's product only than for buying the bundle, introducing a small unilateral bundle premium is profitable for firm $i$, when consumer valuations are additive.*

We sketch the idea of the proof of these propositions and show the main difference. As the unilateral bundle premium of firm $j$ is assumed to be fixed, δ stands for the unilateral bundle premium introduced by firm $i$ only. For both cases one can derive the following identical identities

$$\frac{\partial \pi_{Linear\ pricing}}{\partial p_i} = q_i\left[1 - \frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^* - c_i)\right] + q_b\left[1 - \frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^* - c_i)\right] = 0 \tag{14}$$

and

$$\left.\frac{\partial \pi_{Discounting}}{\partial \delta}\right|_{\delta=0} = -(p_i^* - c_i)\frac{\partial q_b}{\partial p_i} - q_b. \tag{15}$$

(14) Is obtained by maximizing linear profits of firm $i$, defined in (9), with respect to the price of product $i$ . To get (15), the profit function defined in (10), is maximized with respect to δ. Moreover, the derivative in (15) is evaluated at $\delta = 0$. The difference between Proposition 1 and 2 results from the following analysis of the sign of the whole quantity in (15).

Considering (11), we observe that the two expressions in brackets in (14) do not have the same sign. Because when, for example, both [·] values were to be positive, we have a sum of two positive values which can never be 0. Therefore, one of these [·] must be positive and the other negative. The sign of these individual brackets can be determined by using the two elasticity assumptions used in Propositions 1 and 2. As (15) is exactly the same as the right part of (14), but multiplied by $-1$, the sign of the quantity on the right hand side of (15) can be determined (depending on which of the two assumptions is fulfilled). The complete proof can be found in Appendix D and E.

To understand the economic interpretation of Proposition 1 and 2, we first clarify the meaning of the elasticity assumptions. The price elasticity of demand describes the response of the quantity demanded to a change in price, ceteris paribus. Therefore, when the elasticity of demand is lower for buying stand-alone product $i$ than for buying the bundle (assumed in Proposition 1), changes in the price have a relatively larger impact on the bundle demand as compared to demand for stand-alone product $i$. In this case, introducing a discount to the bundle will result in the largest positive impact on demand. Here Proposition 1 implies that the amount of gained demand has a more dominant effect on profit than the small amount of lost revenue per consumer due to the discount. Similarly the economic interpretation of Proposition 2 is explained. When the elasticity of demand is higher for buying stand-alone product $i$ than for buying the bundle, changes in the price have a relatively lower impact on the bundle demand as compared to demand for stand-alone product $i$. In this case, introducing a premium to the bundle amounts to the smallest decrease in demand. Here Proposition 2 yields that the impact of the decrease in demand is not as powerful as the impact of the gain in revenue due to charging a higher price. Lastly it is important to understand to which situations the results (12) and (13) apply. The expressions show the slope of the profit function when $\delta = 0$. Four potential trends of the profit function $\pi_D$ are depicted in Figure 2. In these graphs it can be seen that a positive slope at $\delta = 0$ does nog guarantee profitability of larger discount values. Therefore results of both propositions only account for *small* bundle discounts or premiums.

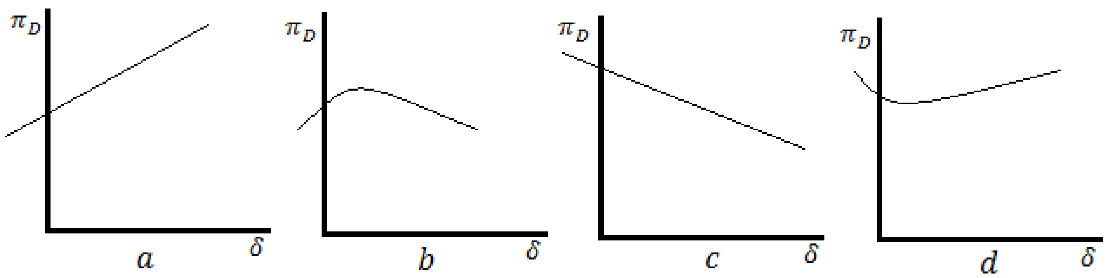


Figure 2: Four examples of curvatures of the profit function $\pi_D$. This illustration is included to clarify the interpretation for (12) and (13). Proposition 1 is in line with graphs a and b; Proposition 2 is in line with graphs c and d. Note that in this figure only the uncertainty of trends for values of $\delta$ greater than 0 is depicted. However, for values $\delta < 0$, the curvatures may fluctuate in a similar manner.

Having derived Propositions 1 and 2, we can now analyze profitability under complementary demand.

**Proposition 3.** *Consider model A and B and suppose that $v_i$ and $v_j$ are independently distributed and the hazard rate $\frac{f_i(\cdot)}{[1-F_i(\cdot)]}$ is strictly increasing, with $F_i(\cdot)$ and $f_i(\cdot)$ being the cumulative distribution function and probability density function of $v_i$, respectively. The following holds*

$$\left.\frac{\partial \pi_{Discounting}}{\partial \delta}\right|_{\delta=0} < 0.$$

*This means that when the stand-alone reservation prices are independently distributed and the hazard rate is strictly increasing, introducing a small unilateral bundle premium is profitable for firm $i$, when complementary demand is considered.*

The idea of the proof of this proposition is shortly summarized below. Consumers that purchase complementary goods have a constant increased utility of joint consumption. As complementary demand is defined as $v_b \geq v_i + v_j$ in (2), the following must hold

$$v_b \equiv v_i + v_j + z^{12} \text{ with } z \geq 0.$$

In this notation, $z$ denotes the additional willingness-to-pay for buying the goods together as compared to buying them separately. In Figure 3 the linear demand pattern can be observed when this parameter $z$ is included. As can be seen, in the model there exist consumers that purchase the bundle while the valuations of both goods are below the individual selling price.

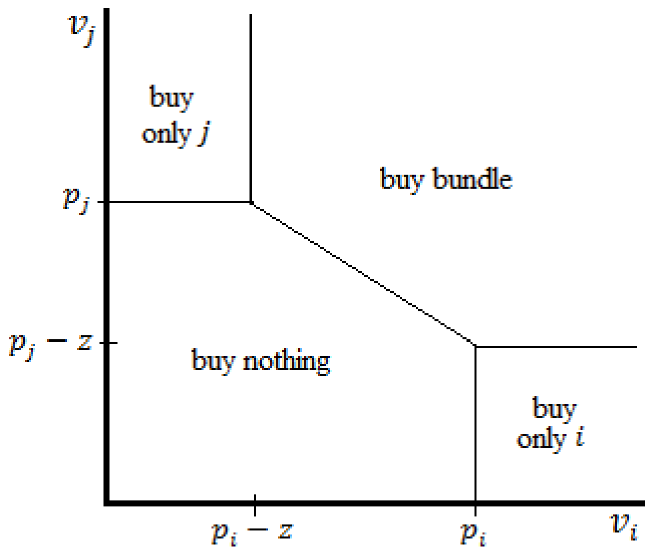


Figure 3: The pattern of complementary demand when $v_b \equiv v_i + v_j + z$ and $\delta = 0$. Lemma 1 and 2 are used to draw this graph.

By using Figure 3, the linear demand functions for the bundle and good $i$ can be defined as

$$q_b = \int_{p_j - z}^{p_j} \int_{p_j + p_i - z - v_j}^{\infty} f_j(v_j) f_i(v_i)\, dv_i dv_j + \int_{p_j}^{\infty} \int_{p_i - z}^{\infty} f_j(v_j)\, f_i(v_i)\, dv_i dv_j$$

and

$$q_i = \int_{p_i}^{\infty} \int_{0}^{p_j - z} f_j(v_j) f_i(v_i) dv_j dv_i.$$

The derivatives of both demand functions are taken with respect to $p_i$ and are rewritten until the hazard rate appears. By using the assumed trend of the hazard rate, a similar inequality can be defined for $q_b$

[12] This notation for non-additive reservation prices was first introduced by Lewbel (1985), who used $r_b = r_1 + r_2 + X$. The consumer has complementary demand when $X > 0$ and perceives the products as substitutes when $X < 0$.

and $q_i$. Combining these equations gives $-\frac{1}{q_b}\frac{\partial q_b}{\partial p_i} \leq -\frac{1}{q_i}\frac{\partial q_i}{\partial p_i}$, which is the same inequality as the elasticity assumption used for Proposition 2. Now we can apply Proposition 2 which then yields the statement of Proposition 3 (details of the proof can be found in Appendix F).

As the finding of Proposition 3 is the same as the one in Proposition 2, the economic interpretation of the result is the same. Only the new assumptions made for this proposition need to be clarified. The 'Hazard Rate' is referred to as the failure rate as it quantifies a statement for the likelihood that something will survive to a next point in time. Assumptions on the hazard rate are frequently used in the economic literature as they are generally not very restrictive. Also in this model, considering a strictly increasing hazard rate in $v_i$ is a justifiable assumption because of the following. The denominator $[1 - F_i(\cdot)]$ will always be a decreasing function as the probability that some random value is smaller than $v_i$ increases with $v_i$ regardless of the distribution of $(v_i, v_j)$. This means that for distributions in which $f_i(\cdot)$ is constant or increasing (e.g. the uniform distribution), the hazard rate assumption must hold[13]. The independence assumption, although used in many economic studies, is more stringent as in many cases it is not very realistic for complementary goods that the product valuation of one good is independent of the valuation of the other good.

From this subsection, the aforementioned research question can be answered. Under some assumptions, introducing a small unilateral bundle premium can positively affect the firm's profit when complementary demand is considered. Note that this conclusion is only valid when all prices of other firms in the market remain unchanged.

## *5.2 Case Study: An Application of the Model*

This subsection provides results for a specific example of the economic bundling model explained in Section 3 and 4. Furthermore, it presents a comparison between our model and the separate sellers model of Armstrong (2012) based on substitutes. The structure of this subsection is as follows. The specifics of the case study are clarified after which the results are presented. Finally, we discuss our findings and make a comparison to Armstrong (2012).

To apply the model and make a comparison to Armstrong (2012), we consider a uniform distribution of reservation prices. This distribution is selected due to its analytical tractability and its correspondence to the assumptions made in the model[14]. $(v_i, v_j)$ is uniformly distributed on the unit square $[1,2]^2$ with $c = 1$. To allow an easy comparison to Armstrong (2012) we consider $z = \frac{1}{4}$, as the separate sellers

[13] More specifically, the hazard rate assumption is fulfilled as long as the distribution satisfies $f_i{}'(\cdot)\big(1 - F_i(\cdot)\big) + f_i(\cdot)^2 > 0$. Note that certain distributions that are not only constant and increasing do satisfy this inequality and can also be used.

[14] Under a uniform distribution, $v_i$ and $v_j$ are independently distributed and the hazard rate $\frac{f_i(\cdot)}{[1-F_i(\cdot)]}$ is strictly increasing (the specific assumptions introduced for Proposition 3).

model of Armstrong is simulated for the same level of synergy for substitutable goods[15]. In this case study, we make two additional restrictions and there is a slight change in notation compared to the model explained in Section 3 and 4. The first restriction is that, based on the results derived in Proposition 3, only premium prices are considered. Secondly, prices as well as bundle premium values of firm $i$ and $j$ are assumed to be symmetric, which is an essential condition to solve the model. This is a critical new assumption, but reasonable from an economic perspective. When both firms face the same marginal costs and level of complementarity and choose their optimal prices by maximizing their own profits (assumed in Model A and B), it is a legitimate expectation that they will end up choosing the same prices and premium values as they face the same situation. Concerning the notation, to simplify the interpretation of the results of this part of the analysis, we slightly changed the mathematical notation of the profit function defined in (10) in which positive values of δ quantify discounts. Since we only look at bundle premium prices, it is a natural notation to allow positive values of δ to account for bundle premiums[16].

In the analysis we will calculate the following indicators; equilibrium price, industry profit, the share of consumers buying the bundle, the share of consumers buying something, the welfare level and the premium. These indicators are calculated for the linear situation ($\delta = 0$), for the situation when $\delta > z$ and $\delta < z$. The latter two situations are depicted in Figure 4 below.

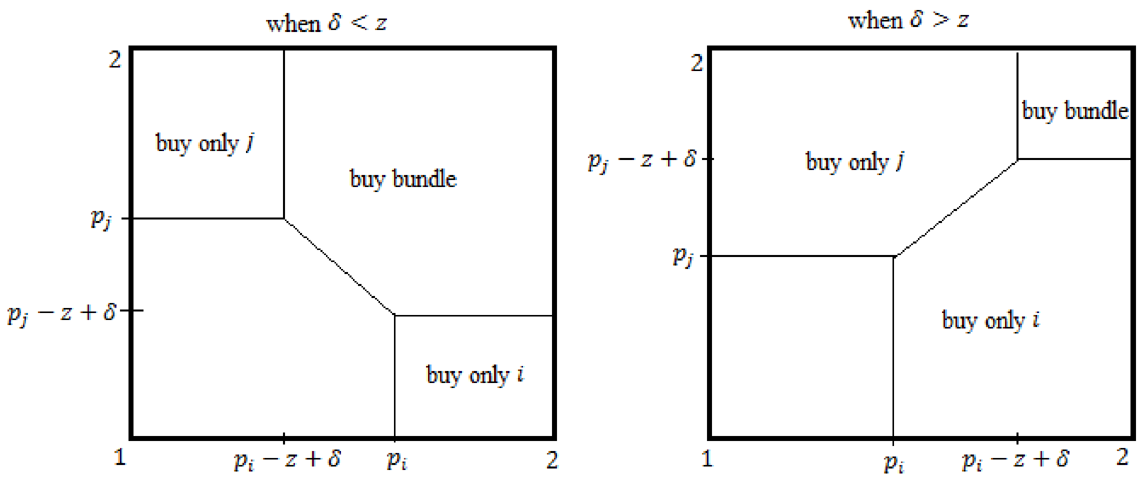


Figure 4: Graphical illustration of demand for the situations $\delta > z$ and $\delta < z$. Only premium bundle prices are considered ($\delta > 0$). $v_i$ is on the x-axis and $v_j$ on the y-axis. For simplicity we used the notation $\delta = \delta_i + \delta_j$. Lemma 1 and 2 are used to draw the left graph ($\delta < z$). The demand dynamics used for drawing the right graph can be found on p.7 and p.8 of Armstrong (2012).

[15] In our notation, the synergy value for the comparable substitutable good would be $z = -\frac{1}{4}$, as there is a constant disutility of joint consumption for these kind of goods.

[16] Thus $\pi_i = (p_i - c)(Q_i + Q_b) - \delta_i Q_b$ (cf.(10)) in which $\delta > 0$ accounts to discounts is changed to $\pi_i = (p_i - c)(Q_i + Q_b) + \delta_i Q_b$ in which $\delta > 0$ stands for premiums. Moreover, (1) becomes $p_b = p_i + p_j + \delta$.

The results of our case study results based on Figure 4 can be found in Table 1 in the third column. In addition, the case findings of Armstrong's separate sellers model (2012) are given in the second column to provide a comprehensive overview for the comparison.

| | **Substitutes** | | | **Complements** | | |
|---|---|---|---|---|---|---|
| | *Linear* | *allowing for discounts* | | *Linear* | *allowing for premiums* | |
| | | For small bundle discounts | For large bundle discounts | | For small bundle premiums ($\delta < z$) | For large bundle premiums ($\delta > z$) |
| **Equilibrium price** | 1.446 | 1.476 | | 1.569 | 1.529 | |
| **Industry profit** | 0.399 | 0.421 | | 0.649 | 0.626 | |
| **% of consumers buying the bundle** | 9.2% | 14.1% | | 43.2% | 38.3% | |
| **% of consumers buying something** | 80.1% | 77.3% | | 70.7% | 73.3% | |
| **Total welfare** | 0.658 | 0.665 | | 0.91 | 0.895 | |
| **δ ($\boldsymbol{\delta_i + \delta_j}$)** | 0 | 0.101 | NO EQUILIBRIUM | 0 | 0.092 | NO EQUILIBRIUM |

Table 1: Results of the uniform distribution in which $(v_i, v_j)$ is distributed over the unit square $[1,2]^2$ with $c = 1$. For substitutes the synergy in this simulation is $z = -\frac{1}{4}$, and for complements $z = \frac{1}{4}$ (according to the notation used in this thesis). The results from the substitute case can be found in Armstrong (2012), and the derivations for the complements case in Appendix G. Note that δ accounts to a bundle discount for substitutes and a bundle premium for complements. For both a large bundle discount and a large bundle premium there is no equilibrium found in the allowed range of δ.

We now discuss the case study results based on complementary demand. While considering these findings it is important to understand the two counteracting effects of δ on the industry profit. First, due to the positive sign of δ in the profit function, an increase in discount premium has a direct positive effect on industry profit. The seller's gain consists of the additional amount paid by the bundle consumers which is a direct source of revenue. Secondly, knowing that price is inversely related to demand, an increase in premium value negatively effects profits as it decreases bundle demand (indirect effect). Whether the seller makes a net gain or loss depends on which effect is most influential. When $\delta > z$, the finding that there is no equilibrium in the possible range of δ indicates that the second counteracting effect has a stronger influence on profits as compared to the first. Clearly the seller does not maximize profit when the bundle price is above bundle reservation prices as no consumers will find the bundle attractive. Therefore, in this situation the influence of lost bundle demand is more substantial than the won revenue from δ. When $\delta < z$, a premium equilibrium value of 0.046 is found for each of the two firms. Since the additional willingness-to-pay the bundle is greater than the additional bundle price ($\frac{1}{4} > 0.092$), bundle demand is not lost and the existence of an equilibrium point is not surprising. Justifiable as well as unexpected findings stem from comparing the linear results with the small bundle premium results. Allowing premium pricing yields the appearance of two distinctly priced sub-markets; consumers buying the stand-alone products and consumers buying the bundle. Due to complementary demand, the stand-alone products are relatively more elastic as compared to the bundle. In this situation, price discrimination is expected to cause the stand-alone prices to go down and the bundle price to go up. This theory matches our results; the bundle price increases from 1.569 to 1.621 and the stand-alone price decreases from 1.569 to 1.529 (taking linear pricing as a starting point). When considering these price developments and applying the basic economic law of supply and demand, the proportion of consumers buying the bundle and the proportion buying stand-alone product $i$ change in a plausible manner; as the bundle price has gone up, the bundle demand has gone down from 43.2% to 38.3% (similar negative relationship holds for the stand-alone products). Interestingly, the percentage of consumers buying *something* has increased from 70.7% to 73.3%, showing that the effect of the stand-alone price change on overall demand is relatively stronger than the effect of the bundle price change. So far, all these discussed results are in line with our expectations of the findings. However, a crucial difficulty in this case study concerns the decline in industry profit. Both Proposition 3 and the fact that profit maximization is applied suggest that introducing a small discount would increase industry profit, relative to the linear pricing case. The paragraph below substantiates this decline in industry profit.

The mathematical reason for the decrease in industry profit is that a rather special definition of stationarity is used, which relies on the symmetry assumption of equal prices and equal premium values. Specifically, the definition for 'stationary point' used in this thesis is as follows.[17]

$$\left(p_i^*, p_j^*, \delta_i^*, \delta_j^*\right) \textit{ is a stationary point}^{18} \textit{ when } \left.\frac{\partial \pi_i(p_i, p_j, \delta)}{\partial p_i}\right|_{\left(p_i^*, p_j^*, \delta^*\right)} = 0, \left.\frac{\partial \pi_j(p_j, p_i, \delta)}{\partial p_j}\right|_{\left(p_i^*, p_j^*, \delta^*\right)} = 0,$$

$$\left.\frac{\partial \pi_i(\delta_i, \delta_j, p)}{\partial \delta_i}\right|_{\left(\delta_i^*, \delta_j^*, p^*\right)} = 0 \textit{ and } \left.\frac{\partial \pi_j(\delta_j, \delta_i, p)}{\partial \delta_j}\right|_{\left(\delta_i^*, \delta_j^*, p^*\right)} = 0.^{19}$$

Here, the $p_j$ and $\delta_j$ are assumed to be fixed when calculating the maximum of $\pi_i$, and similarly for $\pi_j$. This is an economically plausible assumption as both firms are only capable of varying their own prices and premium values. Nonetheless, by using this definition there exist combinations of prices with higher profits. This is because when firm $i$ varies $p_i$ and $\delta_i$, $p_j$ and $\delta_j$ react to this change and are thus *indirectly dependent*. When the objective is to calculate the real maximum value of profit, math requires that the derivative of $\pi_i\left(p_i, p_j, \delta_i, \delta_j\right)$ should be taken and equalized to 0 for *all* variables $p_i, p_j, \delta_i$ and $\delta_j$. This, however, is economically not correct as one firm cannot vary another firm's prices. This is why the industry profit in our analysis can decrease from 0.649 to 0.626. The point 0.626 is merely a stationary point in the sense as defined above rather than a maximum level of profit. A simple example, showing what exactly is calculated in the bundling model when symmetry is assumed and why this is not

[17] Note that many economic studies, like Armstrong (2012) and Venkatesh and Kamakura (2003), have used this definition of stationary point in their bundling models when considering a uniform distribution.

[18] Acknowledge that a stationary point does not necessarily give a maximum. Whether this point corresponds to a saddle point, minimum or maximum depends on the sign of the second derivative of the profit function.

[19] Note that we induced symmetry of prices ($p_i = p_j = p$) before taking the derivatives to $\delta$ and symmetry of premium values ($\delta_i = \delta_j = \frac{1}{2}\delta$) before taking the derivatives to $p$. However, leaving both the prices and premium values asymmetric until after the derivatives would make no difference.

necessarily an optimal solution, can be found in Appendix H. Figure 5, depicting the profit function of this example, shows that the equilibrium found when using the defined 'stationary point' (lying on the red line) deviates from the maximum profit value (the pink dot).

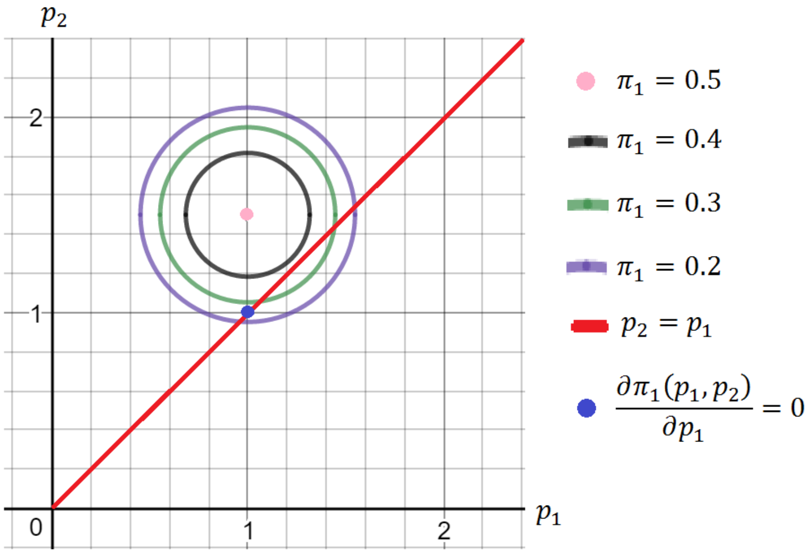


Figure 5: Graph that depicts the profit function of firm 1 $\pi_1(p_1, p_2) = 0.5 - (p_1 - 1)^2 - (p_2 - 1.5)^2$ for four different values of $\pi_1$. The corresponding example can be found in Appendix H. Note that the pink dot represents the maximum level op profit.

Applying Proposition 3 to this case study we find that there is a prisoner's dilemma. When substituting $\delta = 0$ and using that $p > 0$ in (27), we obtain a positive partial derivate of the profit to $\delta_i$ in point $\delta = 0$ as proven in Proposition 3. This means that it is profitable for one firm to introduce a small bundle premium when the other firm does not change its prices. On the contrary, when *both* firms introduce a bundle premium the firms' profits are negatively affected (cf. Table 1). This difference is caused by the unequal vs. equal distribution of demand; when only one firm introduces premium bundle pricing it can draw away demand from the other firm, but when both firms introduce premium bundle pricing less demand is spread equally over both firms (relative to linear pricing). The prisoner's dilemma in this specific case is represented by the payoff matrix in Table 2. Though it would be in both firms' best interest to cooperate, the nash equilibrium is for both firms to introduce premium bundle pricing.

| | **Firm *i*: Preserve Linear Pricing** | **Firm *i*: Introduce (small) Premium Bundle Pricing** |
|---|---|---|
| **Firm *j*: Preserve Linear Pricing** | Profit of both firms is 0.3245 | Firm $i$'s profit: *more* than 0.3245<br>Firm $j$'s profit: *less* than 0.3245 |
| **Firm *j*: Introduce (small) Premium Bundle Pricing** | Firm $i$'s profit: *less* than 0.3245<br>Firm $j$'s profit: *more* than 0.3245 | Profit of both firms is 0.313<br>***= nash equilibrium*** |

Table 2: The prisoner's dilemma payoff matrix for this specific case study. Both firms have two options: to introduce small premium bundle pricing or to preserve linear pricing. Note that the two situations in which the two firms do not choose the same option cannot occur in this case study as symmetry is assumed. The profit values come from the industry profit in Table 1. We make the assumption that when the premium price is unilaterally introduced by solely one firm, the price change is 'small' such that Proposition 3 can be applied.

Having discussed all the case results established in this thesis, we can now make a comparison to Armstrong (2012). As Armstrong's case analysis is carried out for the same specifics, we can study the difference between complementary goods and substitutable goods under the assumptions of the model.

The first crucial difference is that all indicators (except the % of consumers buying something) of the substitutes case have a much lower *linear* value as compared to the complements case. This results from the huge difference in bundle demand; the percentage of consumers buying the bundle is 9.2 for substitutes and 43.2 for complements (cf. Table 1). This low percentage for the substitute case is explained by the characteristic that two substitutable goods compete for the same resources. A low equilibrium price, industry profit and welfare is a consequence of low demand for the bundle. For both kinds of products there does not exist an equilibrium price change value when the discount or premium is larger than the reduced or additional willingness-to-pay. Logically, the seller does not maximize profit by asking a bundle price above the consumer's willingness-to-pay. When again taking linear prices as a starting point, it is interesting that substitutes show an effect of similar size in the opposing direction as compared to complements when introducing a unilateral bundle price change. Lastly, when a price change is introduced, profit maximizing results in a higher profit level for substitutes, but not complements.

Considering the results in this subsection, the research question of this thesis can be answered. Under certain assumptions, specifically symmetry, introducing a bundle premium does not necessarily give a higher industry profit in case of complementary demand. In the discussed case study, in which the strength of complementarity is ¼, industry profit decreases. Moreover, we found that the model has the structure of a prisoner's dilemma and that the payoffs of both firms would be higher if they would collaborate and preserve linear prices.

### *5.3 Effects of Varying Strengths of Complementarity*

In this subsection we investigate the effect of all possible complementarity levels on industry profit. To study the effects of different strengths of complementarity ($z$) we study graphs that show the relations between certain variables of the model. We consider $z$ to be variable, which makes a closed-form solution of the model impossible. The derivations of this subsection are according to the same distribution and assumptions as explained in Subsection 5.2 and can be found in Appendix G.2. Based on the findings of the previous subsection, we decided to only look at situations in which $\delta < z$. The three dependencies that we will discuss are: $\delta(z)$, $p(z, \delta)$ and $\pi_{industry}(z, \delta)$. To understand the effect on equilibrium price and industry profit, the relation $\delta(z)$ should first be explained.

Figure 6 characterizes the relation between the bundle premium value ($\delta$) and the degree of complementarity ($z$) in the symmetric equilibrium. This graph depicts all possible combinations of $\delta$ and $z$ that can appear in equilibrium for the allowed range. When $z = \frac{1}{4}$ and calculating the stationary point as defined in Subsection 5.2, we obtain an optimal premium value of 0.092 as calculated in the previous analysis (cf. point $B$ in Figure 6). The line in this graph is particularly important as it restricts the set of occurring possibilities of the other variables in the model. Using the optimization method as explained, combinations of $\delta$ and $z$ not appearing on the blue part of this line cannot be obtained when introducing a premium value (nor the corresponding price and profit levels). An interesting property of this line is its non-monotonic behavior and the absence of equilibria for high degrees of complementarity. An expected trend is that when the consumers' willingness-to-pay for the bundle increases, the seller benefits from asking a higher bundle price. This explains the monotonically increasing relation between $\delta$ and $z$ until point $C$. From this point on, the line makes a sharp drop and there are no equilibria for synergies above 0.740. Considering a mixed bundling strategy only causes this unexpected behavior. The corresponding assumption for this strategy is that there must be demand for the bundle, stand-alone product $i$ *and* stand-alone product $j$ (cf. (11)). However, for high degrees of complementarity the demand is lost for at least one of these groups[20]. Economically this is also plausible. When the willingness to consume both products together increases, the demand for the stand-alone products decreases. This will result in less, or eventually no, demand for stand-alone product $i$ and/or $j$. Zero demand for either of these stand-alone products, however, cannot happen as it contradicts the assumptions of the model. In view of this, considering a mixed bundling strategy only is the reason for the absence of equilibria for high synergy levels[21]. A final interesting property of Figure 6 is the existence of two stationary points for z values between 0.732 and 0.740. The equilibria corresponding to the green line in Figure 6, however, give a lower profit value. These solutions are inferior and will not occur as a solution because the objective of the firms in our model is to maximize profit.

[20] Mathematical argument; a rough measure of the slope of Figure 6 (up to point C) is 0.2773. This means that the bundle premium only offsets a small part of an increase in complementarity. This is an attractive strategy since you will gain total demand. However, in our model, this strategy cannot continue to work for higher values of z. The reason for this can be explained by looking at Figure 4. Given the slope of 0.2773 and $p < 2$, it can be concluded that $p - z + \delta$ will go to 1. When $p - z + \delta$ becomes 1 (look at Figure 4), there is no demand left for the stand-alone products. This is why the model also requires the strict relation $1 < p - z + \delta$. Hence, for high values of z, when $p - z + \delta$ is close to 1, the relationship between δ and z must change in a way that the aforementioned relationship holds. Moreover, this dynamic of the model aligns with the findings of Venkatesh (2003), who concludes that pure bundling is most profitable for moderate-to-strong complements.

[21] Further research is needed to explain why this nonappearance of equilibria in this model happens specifically from the point $z = 0.74$ onwards.

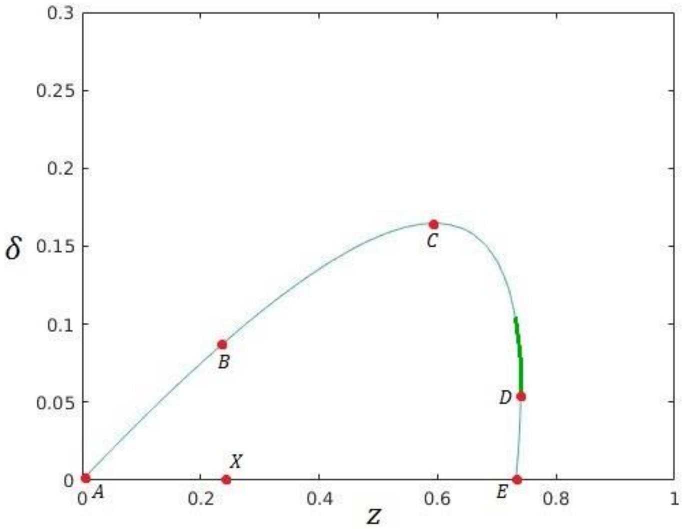


Figure 6: δ as a function of $z$ in symmetric equilibrium. Using MATLAB, this graph is the plot of (39). $A(0,0)$; $B(0.250,0.092)$; $C(0.595,0.165)$; $D(0.740,0.057)$; $E(0.732,0)$; $X(0.250,0)$. Note that these equilibrium points are 'stationary points' as defined in Subsection 5.2. Points $B$ and $X$ are discussed in the example of Subsection 5.2. $\delta$ is endogenous and $z$ is exogenous. Point $E$ has a lower z-value than point $D$. Hence, there are two stationary points for $z$ values between $0.732$ and $0.740$. However, the equilibrium points with the higher premium values ($\delta$), which are market in green, give a lower profit value. Therefore, the set of solutions corresponding to this line give a less optimal solution, and will not be selected. Note: D is not the highest occurring z value, but just an arbitrary example which happens to be close to this point.

Having discussed the relationship between $\delta$ and $z$, we now consider the effect of varying strengths of complementarity on the equilibrium price and industry profit. For this analysis we distinguish between the degrees of complementarity up to 0.732 (cf. Figure 7 and 8) and between 0.732 and 0.740 (cf. Figure 9 and 10). The reason for this is the different behavior of the model in the specific ranges.

For all degrees of complementarity up to 0.732 there exists precisely one equilibrium premium value. The strict increasing relationship to this point makes it possible to depict both price and profit as a function of z only (up to 0.732). Figure 7 shows the dependence among $p$ and $z$ under linear pricing as well as premium pricing. From the graph one can see that the stand-alone equilibrium price is positively related to the strength of complementarity under premium pricing as well as linear pricing. Considering the supply and demand theory, this result is not surprising when looking at total demand. Given the relationship in Figure 6 and 4, when products become more complementary, total consumer demand increases and so do prices. As the stand-alone price is a direct source of revenue, an increase in this variable positively affects industry profit. Consistently, Figure 8 shows that the industry profit increases when the synergy between the two products becomes stronger. Moreover, it can be seen that the profit values under linear pricing ($\delta = 0$) give the highest industry profit. This maybe unexpected effect can occur due to the symmetry assumption and the fact that in essence all variables $\delta_i, \delta_j, p_i, p_j$ vary, but firm $i$ takes $\delta_j$ and $p_j$ as given (and visa versa). This issue is extensively discussed in Subsection 5.2.

But to be able to draw conclusions for all possible degrees of complementarity, the multiple equilibria range must also be considered. The set of solutions belonging to the line drawn by the crosses (cf. Figure 9 and 10) give the lowest profit values and are therefore not a solution since a firm wishes to maximize profit. This means that both the price and profit jump up at 0.732. The economic interpretation of the findings for this range are not as clear as for the synergy range up to 0.732. The reason for this is the aforementioned complication of the model that it only considers a mixed bundling strategy combined with the value limits imposed by Figure 4. These assumptions especially limit the model for high degrees of complementarity, which makes the solutions of the model in the multiple equilibria range a mathematical solution rather than a representation of the economy. All solutions in the multiple equilibria range somewhat resemble the sketch of the demand model in Figure 11. One way or the other, with a higher premium value combined with a lower price, or a lower premium value combined with a higher price, the model moves towards pure bundling, which is excluded to happen by assumption. A critical finding, however, is that the linear pricing strategy gives higher industry profit in the multiple equilibria range.

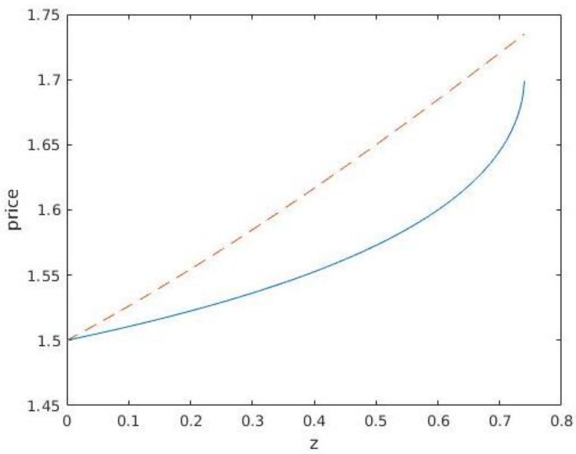


Figure 7: Symmetric equilibrium price as a function of $z$ ($z$ max = 0.732). The blue line represents the model under premium pricing ($\delta > 0$). The dotted orange line shows the linear price as a function of $z$. To obtain this graph, we plotted equation (36) with MATLAB, while simultaneously considering the relationship between $\delta$ and $z$ (39). $\delta$ and $p$ are endogenous and $z$ is exogenous. A graph that shows the symmetric equilibrium price as a function of $z$ and $\delta$ can be found in Appendix I.

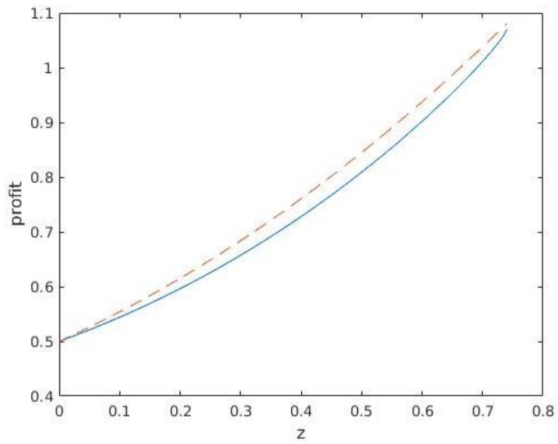


Figure 8[22]: Industry Profit as a function of $z$ ($z$ max = 0.732) in the symmetric equilibrium. Equations (40) and (39) were solved in MATLAB to obtain this graph. The blue line shows the relation under premium pricing and the dotted orange line under linear pricing. $\delta$ and $\pi$ are endogenous and $z$ is exogenous. A graph that shows the Industry profit as a function of $z$ and $\delta$ can be found in Appendix I.

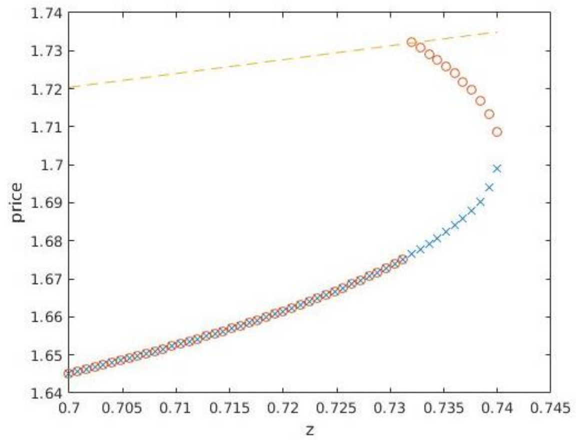


Figure 9: Symmetric equilibrium price as a function of $z$ ($0.700 < z < 0.740$). The orange circles and the blue crosses represent the model under premium pricing ($\delta > 0$). Specifically, the circles correspond to the stationary points with the lower premium value (cf. Figure 6) and the crosses to the higher premium value (cf. green market line in Figure 6). The dotted orange line shows the linear price as a function of z. To obtain this graph, we plotted equation (36) with MATLAB, while simultaneously considering the relationship between $\delta$ and z (39). $\delta$ and $p$ are endogenous and $z$ is exogenous. Note that the premium pricing equilibria lie on the line drawn by the crosses and circles. The individual points are merely used to make a clear distinction between the higher and lower group of stationary points.

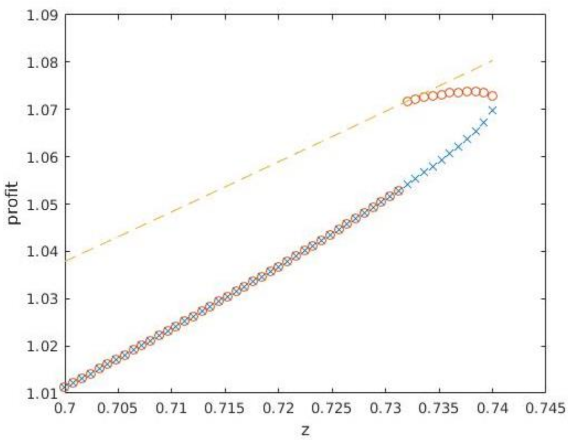


Figure 10: Industry Profit as a function of $z$ ($0.700 < z < 0.740$) in the symmetric equilibrium. The orange circles and the blue crosses represent the model under premium pricing ($\delta > 0$). Specifically, the circles correspond to the stationary points with the lower premium value (cf. Figure 6) and the crosses to the higher premium value (cf. green market line in Figure 6). The dotted orange line shows the relation under linear pricing. Equations (40) and (39) were solved in MATLAB to obtain this graph. $\delta$ and $\pi$ are endogenous and $z$ is exogenous. Note that the premium pricing equilibria lie on the line drawn by the crosses and circles. The individual points are merely used to make a clear distinction between the higher and lower group of stationary points.

[22] When constructing a similar graph but taken $\delta_j$ and $p_j$ as fixed, we see that profit increases for small values of delta. This result is in line with the findings in Subsection 5.1.

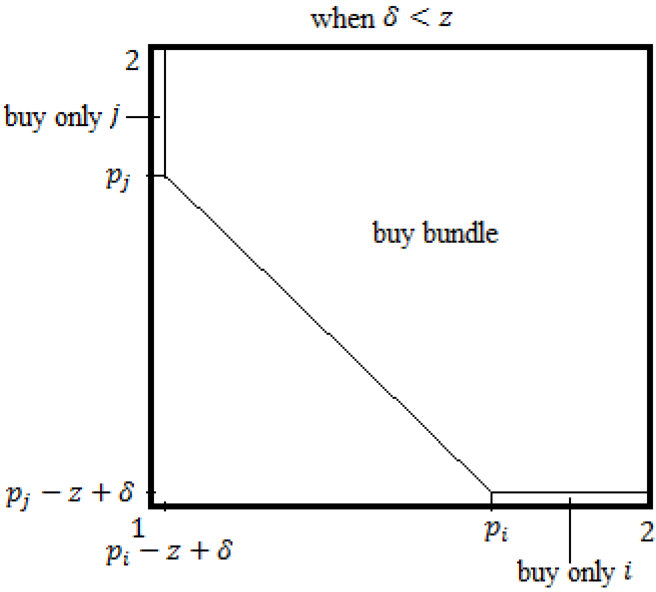


Figure 11: Graphical illustration of demand for the situations $\delta < z$ in the multiple equilibria range. Only premium bundle prices are considered ($\delta > 0$). $v_i$ is on the x-axis and $v_j$ on the y-axis. For simplicity we used the notation $\delta = \delta_i + \delta_j$.

Considering all discussed results in this subsection, the aforementioned research question of this thesis can be answered. Under the same assumptions as discussed in Subsection 5.2 we find that the stronger the synergy between the two products, the higher the industry profit. It can be concluded, for all possible strengths of complementarity, that introducing a bundle premium does not benefit industry profit; the highest industry profit is obtained under linear pricing.

# 6 Discussion

The findings of this thesis rely on some critical assumptions. The most restrictive assumptions are shortly discussed in this section, along with other main limitations and some suggestions for further research.

A property of the model, which may be a limitation in certain cases, is that the model only includes a mixed bundling strategy. In some situations component selling or pure bundling gives higher profits or reflects the preferred consumer choice. For instance, when the package is priced at a premium under mixed bundling, a number of consumers would prefer to buy both stand-alone goods together promoting component selling. Further research is needed to investigate whether adding a fifth discrete choice ‘buying the stand-alone products together’ to (4) and dropping (11) can extend our model to account for all three bundling strategies. Furthermore, most products are perceived differently per consumer, and thus assuming the same degree of complementarity for all consumers in the market is not a very realistic assumption in most cases. It is even possible that some consumers regard a product as a substitute while others identify the product as a complement. Another restrictive assumption is the independent distribution of reservation prices considered in Proposition 3. Although used in many economic studies for simplicity, the number of situations in which the product valuations of two complementary goods are independent is rather restricted. Therefore forthcoming studies may want to investigate whether the profitability conclusion still holds when independence is not assumed. Another useful addition to the bundling literature would be to study the specific case and synergy analysis when considering other distributions than the uniform one. A uniform distribution, applied to the model in this thesis, does not always reflect the distribution of consumer valuations in real life. Finally, further research is required for this model to study the effect of different values of marginal costs.

# 7 Conclusion

The goal of this thesis was to investigate the effect of unilateral bundle price change on profit, when considering a mixed bundling strategy, complementary goods and separate sellers. This monotonicity result holds when *only* the bundle premium of the maximizing firm changes, and all other variables in the model remain unchanged.

Furthermore, in a specific case study we obtain a lower industry profit under premium pricing as compared to linear pricing, and thus we get the result that offering a higher bundle price does not necessarily give a higher industry profit. Additionally, when combining this finding with the monotonicity result we find that the model has the structure of a prisoner's dilemma and that the payoffs of both firms would be higher if they would collaborate and preserve linear prices. This case study example, which is based on a uniform distribution of reservation prices, makes use of the additional assumption of symmetric prices and bundle premium values. We present an explanation of why profit maximization under this assumption does not naturally give the highest level of industry profit.

Lastly, we find that the strength of synergy between the two complementary products positively affects the industry profit. For all possible degrees of complementarity, however, introducing a bundle premium does not show to be beneficial to industry profit; the highest industry profit is always obtained under linear pricing.

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# 9 Appendix

## A Proof of Lemma 1

When assuming linear pricing ($\delta = 0$) and using (4), a consumer with the vector $(v_i, v_j, v_b)$ purchases product $i$ if and only if

$$max\{v_b - p_i - p_j, v_i - p_i\} \geq max\{v_j - p_j, 0\}. \quad (16)$$

(16) Characterizes total demand for product $i$ as $q_i + q_b$ is the proportion of people having a vector $(v_i, v_j, v_b)$ that satisfies this inequality.

The statement that (16) weakly decreases in $p_j$, for all vectors, can be proven both using calculus and graphical arguments. Below we present two different proofs using these types of arguments.

***Calculus**:* Four possible situations follow from (16) which are distinguished below.

I. Assume $v_b - p_i - p_j < v_i - p_i$ and $v_j - p_j > 0$. This case will *not occur* because of the following:
   Situation $v_b - p_i - p_j < v_i - p_i$ can be rewritten as $v_b < v_i + p_j$
   and $v_j - p_j > 0$ as $p_j < v_j$.
   Combining these two we get $v_b < v_i + p_j < v_i + v_j$.
   Implying $v_b < v_i + v_j$, which is contradictory to the complementary goods assumption (2).
II. Assume $v_b - p_i - p_j \geq v_i - p_i$ and $v_j - p_j \geq 0$. Then the following holds
   $v_b - p_i - p_j \geq v_j - p_j$,
   which can be rewritten as
   $v_b - p_i - v_j \geq 0$.
   In this situation $p_j$ cancels, and therefore the inequality is independent of $p_j$. Hence, $q_i + q_b$ is a *constant* function of $p_j$.
III. Assume $v_b - p_i - p_j \leq v_i - p_i$ and $v_j - p_j \leq 0$. Then the following holds
   $v_i - p_i \geq 0$.
   The inequality is *independent* of $p_j$. Hence, $q_i + q_b$ is a *constant* function of $p_j$.
IV. Assume $v_b - p_i - p_j \geq v_i - p_i$ and $v_j - p_j \leq 0$. Then the following holds
   $v_b - p_i - p_j \geq 0$,
   which can be rewritten as
   $v_b - p_i \geq p_j$.
   In the set of all $(v_i, v_j, v_b)$ values, there are less that satisfy this inequality when $p_j$ increases. Hence, $q_i + q_b$ in a *decreasing* function of $p_j$.

The above shows that all occurring cases (II, III and IV) are either constant or decreasing in $p_j$, thus $q_i + q_b$ weakly decreases with $p_j$. ■

***Graphical proof***: Equation (16) can be rewritten as

$$max\{v_b - p_i - p_j, v_i - p_i\} + min\{p_j - v_j, 0\} \geq 0. \quad (17)$$

For (17) we introduce the following notation

$$g(p_j) = max\{v_b - p_i - p_j, v_i - p_i\} + min\{p_j - v_j, 0\}. \quad (18)$$

This function can be graphically depicted in three steps.

First, the maximization term in (18) which denotes the maximum surplus when buying product $i$, is depicted in Figure 12 below. Note that point $A$ is equal to $v_b - p_i - p_j = v_i - p_i$ which simplifies to $v_b = v_i + p_j$.

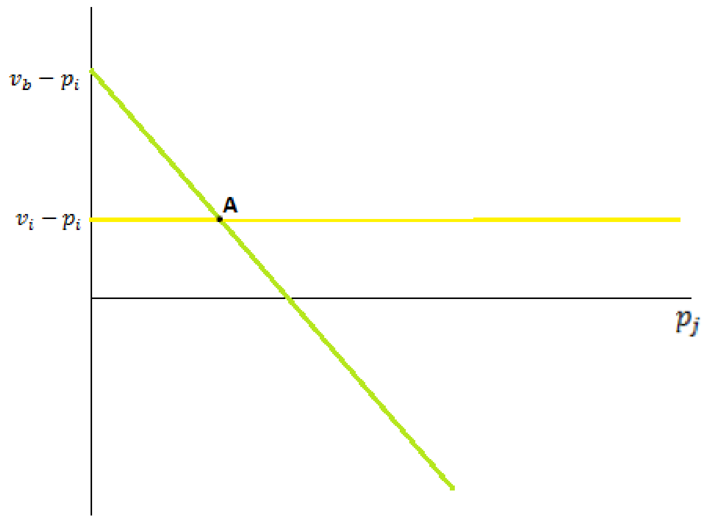


Figure 12: Graph of Maximization part of (18). $v_b - p_i - p_j$ is depicted in the green line and $v_i - p_i$ in the yellow line.

Secondly, the minimization term in (18) which denotes the minimum surplus when not buying product $i$, is characterized in Figure 13. Note that point $B$ is equal to $p_j - v_j = 0$ which can be reduced to $p_j = v_j$.

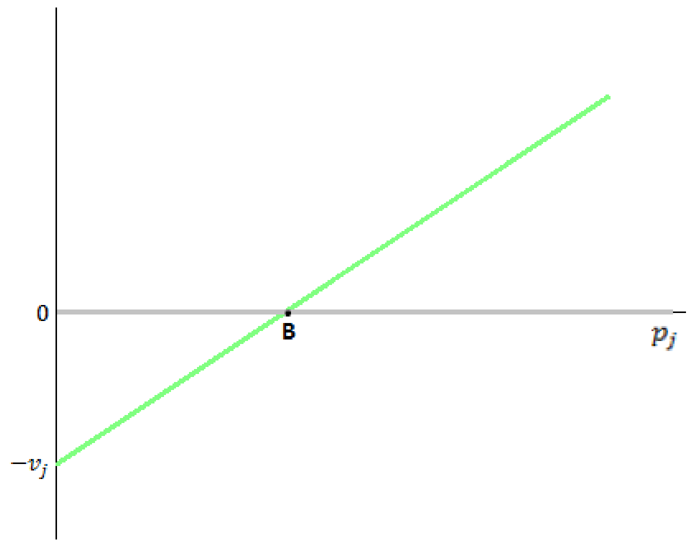


Figure 13: Graph of Minimization part of (18). Note that the grey line lies on the x-axis and $p_j - v_j$ is depicted in the green line.

Thirdly, the sum must be taken of the maximum of Figure 12 and the minimum of Figure 13. To do so, two properties of the model are used to draw the graph in Figure 14:

- At point $A$ in Figure 12 we have a $p_j$ where $p_j^A = v_b - v_i$ and at point $B$ in Figure 13 we have $p_j^B = v_j$. Using $v_b \geq v_i + v_j$ we get $p_j^A = v_b - v_i \geq v_i + v_j - v_i = v_j = p_j^B$. Hence point $A$ has a $p_j$ value similar or greater than point $B$.
- The intersection with the y-axis should be above or equal to point $C$ as $v_b \geq v_i + v_j$ can be rewritten as $v_b - v_j - p_i \geq v_i - p_i$. Moreover, the y-intercept must be below or equal to point $D$ because $v_j \geq 0$.

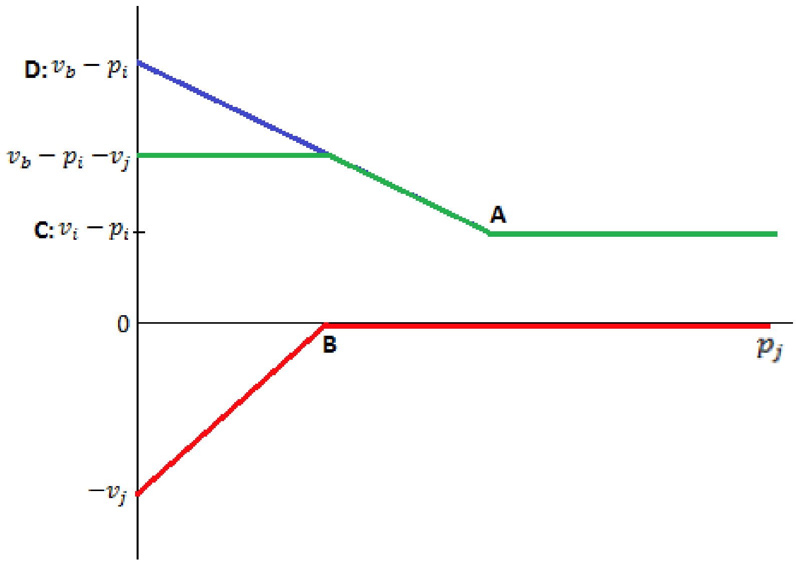


Figure 14: Green graph depicts the relationship between the demand for product $i$ and the price level of good $j$. Moreover, the green line represents the summation of the blue and red line (for $p_j \geq p_j^B$ the green line lies on top of the blue line). The red line corresponds to the minimization part and the blue line corresponds to the maximization part of (18). Note that this figure is an example of a situation when $p_j^A > p_j^B$. However, when $p_j^A = p_j^B$, the green line is constant as the slope of the blue and the red line would offset each other until the exact same point. Lastly, note that the green line can also be shifted up and down depending on $v_b - p_i - v_j$, as long as the $y$ intercept is between point $C$ and $D$ (note: point $C$ and $D$ can also move up and down, even below 0), the trend however stays the same.

Combining (17) and (18) we get $g(p_j) \geq 0$, which is now considered in combination with Figure 14. As can be seen from the $p_j$ values of the green line in Figure 14, there are three possible situations:

- Consider $p_j \leq p_j^B$. The following holds

  $v_b - p_i - v_j \geq 0,$

  which is independent of $p_j$. Hence, $q_i + q_b$ is a *constant* function of $p_j$.
- Consider $p_j^B \leq p_j \leq p_j^A$. The following holds

  $v_b - p_i - p_j \geq 0,$

  which can be rewritten as

  $v_b - p_i \geq p_j.$

  In the set of all $(v_i, v_j, v_b)$ values, there are less consumers that satisfy this inequality when $p_j$ increases. Hence, $q_i + q_b$ in a *decreasing* function of $p_j$.
- Consider $p_j^A \leq p_j$. The following holds

  $v_i - p_i \geq 0,$

  which is independent of $p_j$. Hence, $q_i + q_b$ is a *constant* function of $p_j$.

The above shows that the course of the green line in Figure 14 proofs that $q_i + q_b$ weakly decreases with $p_j$. Note that this proof also provides additional information on the behavior of $q_i + q_b$ as a function of $p_j$ values. When $A > B$, total demand of product $i$ is independent of relative small and large values of $p_j$, and decreasing in moderate values of $p_j$. When $A = B$, total demand of product $i$ is independent of all possible values of $p_j$. ■

### B Proof of Lemma 2

When combining (2) with the inequality that follows from the decision to buy good $i$ on its own at linear prices ($v_i - p_i \geq v_b - p_i - p_j$), the following holds for those consumers who buy stand-alone product $i$

$$v_i - p_i \geq v_b - p_1 - p_2 \geq v_1 + v_2 - p_1 - p_2.$$

This can be divided into the inequalities

$v_i - p_i \geq v_b - p_i - p_j$ and $v_b - p_i - p_j \geq v_i + v_j - p_i - p_j$

which can be deducted to

$v_i \geq v_b - p_j$ and $v_b \geq v_i + v_j$.

When inserting $v_b$ of the right inequality into the left inequality we obtain

$v_i \geq v_i + v_j - p_j$,

which leads to

$v_j \leq p_j$.

Note that this inequality is independent of $p_i$, and therefore the conclusion holds. ■

### C Proof of Lemma 3

To apply the Slutsky Symmetry, we first define $F(p_i, p_j, \delta)$ as the expectation of the four discrete choices in (4), which gives

$$F(p_i, p_j, \delta) = \mathbb{E}\left[max\{v_b - (p_i + p_j - \delta), v_i - p_i, v_j - p_j, 0\}\right].$$

Use the economic relationship between the consumer surplus and the demand function; the negative integral of the demand function is equal to the consumer surplus. Therefore, taking the derivative of the surpluses we obtain

$$\frac{\partial F}{\partial p_i} \equiv -(Q_i + Q_b);\ \frac{\partial F}{\partial p_j} \equiv -(Q_j + Q_b);\ \frac{\partial F}{\partial \delta} \equiv Q_b.$$

The symmetry of second derivatives states that the second derivative of the same function (in this case $F(p_i, p_j, \delta)$) must always be equal. Using this rule, we obtain

$$\frac{\partial(Q_i + Q_b)}{\partial\delta} = -\frac{\partial Q_b}{\partial p_i};\ \frac{\partial(Q_j + Q_b)}{\partial\delta} = -\frac{\partial Q_b}{\partial p_j}.$$

It can be seen that $i$ and $j$ hold the exact same relationship with respect to the bundle demand. Therefore we define the following relationship that holds for both $i$ an $j$. When using (5) we get to the proving statement

$$\left.\frac{\partial(Q_i + Q_b)}{\partial\delta}\right|_{\delta=0} = -\frac{\partial q_b}{\partial p_i}.$$

**D Proof of Proposition 1**

The idea of the proof is as follows. The linear profits are maximized to find the optimal price $p_i^*$, then a profit function that includes a discount for customers that also buy product $j$ is maximized with respect to δ to untangle the effect of δ.

Profits under linear pricing ($\delta = 0$) are defined as (cf.(9))

$$\pi_L = (p_i - c_i)(q_i + q_b).$$

By profit maximization with respect to $p_i$ we obtain

$$\frac{\partial\pi_L}{\partial p_i} = (q_i + q_b) + (p_i^* - c_i)\frac{\partial(q_i+q_b)}{\partial p_i} = 0,$$

which can be rewritten as

$$q_i + \frac{\partial q_i}{\partial p_i}(p_i^* - c_i) + q_b + \frac{\partial q_b}{\partial p_i}(p_i^* - c_i) = 0$$

$$q_i\left[1 - \frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^* - c_i)\right] + q_b\left[1 - \frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^* - c_i)\right] = 0. \quad (19)$$

To investigate the effect of this discount we again maximize profits but now a bundle discount δ (which is a discount over $p_i^*$) is included in the profit function.

Profits including a discount are defined as (cf. (10)).

$$\pi_D = (p_i^* - c_i)(Q_i + Q_b) - \delta Q_b.$$

By profit maximization with respect to $\delta$ we obtain

$$\frac{\partial\pi_D}{\partial\delta} = (p_i^* - c_i)\frac{\partial(Q_i+Q_b)}{\partial\delta} - Q_b - \delta\frac{\partial Q_b}{\partial\delta}.$$

To show that small discounting is profitable it is sufficient to show that the derivate at $\delta = 0$ is strictly positive. Hence we consider

$$\left.\frac{\partial \pi_D}{\partial \delta}\right|_{\delta=0} = -(p_i^* - c_i)\frac{\partial q_b}{\partial p_i} - q_b. \tag{20}$$

From equation (20) it cannot yet be seen what effect δ has on $\pi_D$. Therefore the below comparisons are made to find out the sign of the quantity. Note that if equation (20) would be positive, this implies that introducing a discount will positively affects firm $i$'s profit.

Combining the assumption that $q_i > 0$ and $q_b > 0$ (cf. (11)) with (19), it must be true that the two expressions in brackets do not contain the same sign. Because in case both [·] values were to be positive, we have a sum of two positive values which can never be 0. Same reasoning when both [·] values would be negative. Therefore, one of these [·] must be positive and the other negative to have a sum of zero.

Now we turn to use the elasticity assumption mentioned in the proposition:

$$\frac{-\partial q_i/\partial p_i}{q_i} < \frac{-\partial q_b/\partial p_i}{q_b}.$$

First we multiply both sides by $(p_i^* - c_i)$ which gives[23]

$$\frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^* - c_i) < \frac{-\partial q_b/\partial p_i}{q_b}(p_i^* - c_i),$$

then we multiply both sides by $-1$ which gives

$$-\frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^* - c_i) > -\frac{-\partial q_b/\partial p_i}{q_b}(p_i^* - c_i),$$

and lastly we add 1 to both sides and we end up with

$$1 - \frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^* - c_i) > 1 - \frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^* - c_i).$$

When comparing this inequality with (19) it can be seen that the right and the left hand side are exactly the same as the [·] parts of (19). This taken together with the fact that one of these [·] must be positive and the other negative to have a sum of zero, the following can be concluded:

$$q_i\left[1 - \frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^* - c_i)\right] > 0;\ q_b\left[1 - \frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^* - c_i)\right] < 0$$

[23] Note that this step can be taken without change in the inequality sign as $(p_i^* - c_i)$ must be strictly positive.

And this now proofs that (20) must be positive as it is exactly the same as $q_b\left[1-\frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^*-c_i)\right]$, but multiplied by $-1$. Now the proof is completed that firm $i$ can raise its profits by offering a unilateral bundle discount. ■

### E Proof of Proposition 2

The idea of the proof is similar to the one from Proposition 1. Therefore, only the parts that differ are presented below.

First of all, when profit is maximized, the same equations are obtained as (19) and (20). The change is in defining the sign of the whole quantity in (20). Note that if this equation would be negative, this implies that introducing a premium ($\delta < 0$) will positively affects firm $i$'s profit.

Using the elasticity assumption mentioned in Proposition 2 we have

$$\frac{-\partial q_i/\partial p_i}{q_i} > \frac{-\partial q_b/\partial p_i}{q_b}.$$

First we multiply both sides by $(p_i^*-c_i)$ which gives

$$\frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^*-c_i) > \frac{-\partial q_b/\partial p_i}{q_b}(p_i^*-c_i),$$

then we multiply both sides by -1 which leads to

$$-\frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^*-c_i) < -\frac{-\partial q_b/\partial p_i}{q_b}(p_i^*-c_i),$$

and lastly we add 1 to both sides and we end up with

$$1-\frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^*-c_i) < 1-\frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^*-c_i).$$

When comparing this inequality with (19) it can be seen that the right and the left hand side are again exactly the same as the $[\cdot]$ parts of (19). This taken together with the fact that one of these $[\cdot]$ must be positive and the other negative to have a sum of zero, the following can be concluded:

$$q_i\left[1-\frac{-\frac{\partial q_i}{\partial p_i}}{q_i}(p_i^*-c_i)\right] < 0;\ q_b\left[1-\frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^*-c_i)\right] > 0$$

And this now proofs that (20) must be negative as it is exactly the same as $q_b\left[1-\frac{-\frac{\partial q_b}{\partial p_i}}{q_b}(p_i^*-c_i)\right]$, but multiplied by $-1$. Now the proof is completed that firm $i$ can raise its profits by offering a unilateral bundle premium. ■

### F Proof of Proposition 3

The idea of the proof is as follows. The demand functions for the bundle and good $i$ are constructed by using Figure 3. The derivatives of both demand functions are taken with respect to $p_i$ and are rewritten until the hazard rate appears. By using the hazard rate assumption stated in the proposition, a similar inequality can be defined for $q_b$ and $q_i$. Combining these equations leads to the elasticity assumption used for Proposition 2.

Using Figure 3, the demand for the bundle can be defined as

$$q_b=\int_{p_j-z}^{p_j}\int_{p_j+p_i-z-v_j}^{\infty}f_j(v_j)f_i(v_i)\,dv_i dv_j+\int_{p_j}^{\infty}\int_{p_i-z}^{\infty}f_j(v_j)\,f_i(v_i)\,dv_i dv_j.$$

When rewriting and using cumulative distribution functions we obtain

$$q_b=\int_{p_j-z}^{p_j}[1-F_i(p_j+p_i-z-v_j)]\,f_j(v_j)dv_j+[1-F_i(p_i-z)][1-F_j(p_j)].$$

Taking the derivative with respect to $p_i$ gives

$$-\frac{\partial q_b}{\partial p_i}=\int_{p_j-z}^{p_j}f_i(p_j+p_i-z-v_j)\,f_j(v_j)dv_j+f_i(p_i-z)[1-F_j(p_j)],$$

and by rewriting we obtain

$$-\frac{\partial q_b}{\partial p_i}=\int_{p_j-z}^{p_j}\frac{f_i(p_j+p_i-z-v_j)}{[1-F_i(p_j+p_i-z-v_j)]}[1-F_i(p_j+p_i-z-v_j)]f_j(v_j)dv_j$$
$$+\frac{f_i(p_i-z)}{[1-F_i(p_i-z)]}[1-F_i(p_i-z)][1-F_j(p_j)].$$

By using the hazard rate related assumption, defined in Proposition 3, the first hazard rate can be taken out of the integral. As $v_j$ is subtracted in the nominator and denominator of the hazard rate, plugging in the lowest value of the integral (and knowing that the hazard rate is strictly increasing) gives the highest value of the hazard rate possible which leads to the inequality

$$-\frac{\partial q_b}{\partial p_i}\leq\frac{f_i(p_i)}{[1-F_i(p_i)]}\int_{p_j-z}^{p_j}[1-F_i(p_j+p_i-z-v_j)]\,f_j(v_j)dv_j$$
$$+\frac{f_i(p_i-z)}{[1-F_i(p_i-z)]}[1-F_i(p_i-z)][1-F_j(p_j)].$$

Knowing that $z \geq 0$, the presence of z in the second hazard rate will weakly decrease the value of this term, therefore taking $z = 0$ yields the maximum value

$$-\frac{\partial q_b}{\partial p_i} \leq \frac{f_i(p_i)}{[1-F_i(p_i)]}\int_{p_j-z}^{p_j}\left[1-F_i\left(p_j+p_i-z-v_j\right)\right]f_j\left(v_j\right)dv_j$$
$$+\frac{f_i(p_i)}{[1-F_i(p_i)]}[1-F_i(p_i-z)][1-F_j\left(p_j\right)].$$

Factoring out the hazard rate simplifies the equation as only $q_b$ is left within the brackets, this leads to

$$-\frac{\partial q_b}{\partial p_i} \leq \frac{f_i(p_i)}{[1-F_i(p_i)]}q_b$$

So the concluding statement from the calculation of the bundle demand is

$$\boxed{-\frac{1}{q_b}\frac{\partial q_b}{\partial p_i} \leq \frac{f_i(p_i)}{[1-F_i(p_i)]}}$$

We now conduct the same analysis for $q_i$, all steps are the same and therefore are not repeatedly mentioned. Using Figure 3, the demand for the product $i$ can be defined as

$$\begin{aligned}
q_i &= \int_{p_i}^{\infty}\int_0^{p_j-z} f_j(v_j)f_i(v_i)dv_j dv_i \\
&= [1-F_i(p_i)]\int_0^{p_j-z} f_j(v_j)dv_j \\
-\frac{\partial q_i}{\partial p_i} &= f_i(p_i)\int_0^{p_j-z} f_j(v_j)dv_j \\
&= \frac{f_i(p_i)}{[1-F_i(p_i)]}[1-F_i(p_i)]\int_0^{p_j-z} f_j(v_j)dv_j \\
&= \frac{f_i(p_i)}{[1-F_i(p_i)]}q_i
\end{aligned}$$

So the concluding statement from the calculation of product $i$'s demand is

$$\boxed{-\frac{1}{q_i}\frac{\partial q_i}{\partial p_i} = \frac{f_i(p_i)}{[1-F_i(p_i)]}}$$

When combing the concluding statements based on $q_b$ and $q_b$, we obtain

$$-\frac{1}{q_b}\frac{\partial q_b}{\partial p_i} \leq -\frac{1}{q_i}\frac{\partial q_i}{\partial p_i}$$

which is the elasticity assumption used for Proposition 2. Now we can apply Proposition 2 which then yields the statement. Now the proof is completed that firm $i$ can raise its profits by offering a unilateral bundle premium when there is complementary demand. ■

## G Output of Uniform Distribution

$(v_i, v_j)$ is uniformly distributed on the unit square $[1,2]^2$ with $c = 1$. G.1 contains the calculations in case $z = \frac{1}{4}$ and G.2 presents similar calculations when taking z as variable. Note that for all derivations in this section we slightly changed the notation of the profit function defined in (10). As we only consider premium pricing, which is denoted by negative values of δ in the original notation, double negative values cause cumbersome interpretations. In our new notation we simply use positive values of δ and multiply the double minus sign, which leads to

$$\pi_i = (p_i - c)(Q_i + Q_b) + \delta_i Q_b$$

being the profit of firm $i$.

*G.1 When $z = \frac{1}{4}$*

First we will calculate the optimal price, the industry profit, the fraction of consumers that buys the bundle, the fraction of consumers that buys something and total welfare under linear pricing ($\delta = 0$). Secondly, we will allow $\delta$ in the profit function, for calculating the same indicators as for the linear situation we need to distinguish between the cases when $\delta > z$ and $\delta < z$.

*G.1.1 Linear situation*

The linear pricing situation is depicted in the figure below which is used for the calculations.

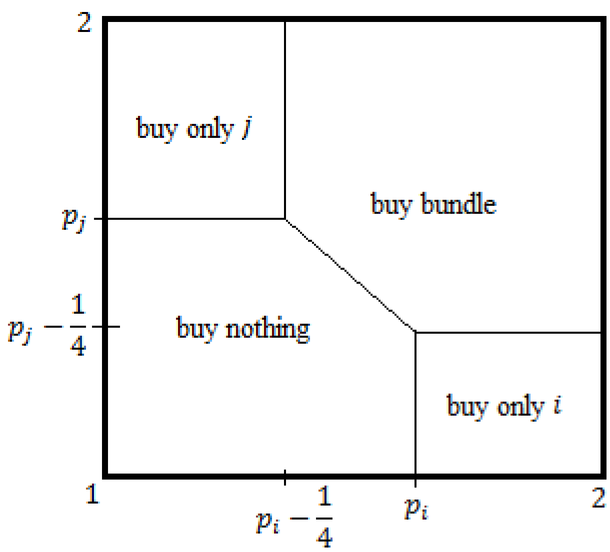


Figure 15: The pattern of complementary demand when $v_b \equiv v_i + v_j + z$ under linear pricing. $v_i$ is on the x-axis and $v_j$ on the y-axis. $(v_i, v_j)$ is uniformly distributed on the unit square $[1,2]^2$ with $c = 1$ and $z = \frac{1}{4}$. Lemma 2 is used to draw this graph.

To find the equilibrium linear price $p^*$, we optimize the profits of firm $i$; $\pi_L = (p_i - c_i)(q_i + q_b)$. To do so, we will first calculate $q_i$ and $q_b$[24].

$$\begin{aligned}
q_b &= \int_{p_j-z}^{p_j}\int_{p_j+p_i-z-v_j}^{\infty} f_j(v_j)f_i(v_i)\,dv_i dv_j + \int_{p_j}^{\infty}\int_{p_i-z}^{\infty} f_j(v_j)\,f_i(v_i)\,dv_i dv_j \\
&= \int_{p_j-\frac{1}{4}}^{p_j}\int_{p_j+p_i-\frac{1}{4}-v_j}^{2} 1\ dv_i dv_j + \int_{p_j}^{2}\int_{p_i-\frac{1}{4}}^{2} 1\ dv_i dv_j \\
&= \int_{p_j-\frac{1}{4}}^{p_j} (2 - p_j - p_i + \frac{1}{4} + v_j)\,dv_j + \int_{p_j}^{2}\left(2 - p_i + \frac{1}{4}\right) dv_j \\
&= \left(2 - p_j - p_i + \frac{1}{4}\right)\left(p_j - p_j + \frac{1}{4}\right) + \left[\frac{1}{2}v_j^2\right]_{p_j-\frac{1}{4}}^{p_j} + \left(2 - p_i + \frac{1}{4}\right)(2 - p_j) \\
&= \frac{9}{16} - \frac{1}{4}p_j - \frac{1}{4}p_i + \left(\frac{1}{2}p_j^2 - \frac{1}{2}\left(p_j - \frac{1}{4}\right)^2\right) + \left(\frac{9}{4} - p_i\right)(2 - p_j) \\
&= \frac{9}{16} - \frac{1}{4}p_j - \frac{1}{4}p_i - \frac{1}{32} + \frac{1}{4}p_j + \frac{18}{4} - 2p_i - \frac{9}{4}p_j + p_i p_j \\
&= \frac{161}{32} - \frac{9}{4}p_j - \frac{9}{4}p_i + p_i p_j
\end{aligned}$$

$$\begin{aligned}
q_i &= \int_{p_i}^{\infty}\int_{0}^{p_j-z} f_j(v_j) f_i(v_i) dv_j dv_i \\
&= \int_{p_i}^{2}\int_{1}^{p_j-\frac{1}{4}} 1\,dv_j dv_i \\
&= (p_j - \frac{1}{4} - 1)(2 - p_i) \\
&= (p_j - \frac{5}{4})(2 - p_i) \\
&= 2p_j - p_j p_i - \frac{5}{2} + \frac{5}{4}p_i
\end{aligned}$$

When substituting the equations found for $q_i$ and $q_b$ into the linear profit function $\pi_L = (p_i - c_i)(q_i + q_b)$, with $c = 1$, we obtain

$$\begin{aligned}
\pi_L &= (p_i - 1)(2p_j - p_j p_i - \frac{5}{2} + \frac{5}{4}p_i + \frac{161}{32} - \frac{9}{4}p_j - \frac{9}{4}p_i + p_i p_j) \\
&= (p_i - 1)\left(\frac{81}{32} - \frac{1}{4}p_j - p_i\right).
\end{aligned}$$

When optimizing with respect to $p_i$ we get

[24] Note that for $q_i$ and $q_b$ we used the equality constructed in the proof of Proposition 3. As $(v_i, v_j)$ is uniformly distributed, it is also possible to simply define the area in Figure 3 without using integrals.

$$\frac{\partial \pi_L}{\partial p_i} = \left(\frac{81}{32} - \frac{1}{4}p_j - p_i\right) - (p_i - 1) = 0.$$

Now we impose symmetry such that $p_j = p_i = p^*$, which leads to

$$\left(\frac{81}{32} - \frac{1}{4}p^* - p^*\right) - (p^* - 1) = 0$$

$$\frac{81}{32} - \frac{5}{4}p^* - p^* + 1 = 0$$

$$\frac{113}{32} = \frac{9}{4}p^*$$

| |
|---|
| the symmetric linear equilibrium price $p^* = \frac{113}{72} \approx 1.569.$ |

To find the linear industry profit we substitute $q_i$, $q_b$ and $p^*$ in the linear profit function $\pi_L = (p_i - c_i)(q_i + q_b)$ and multiply by two. We multiply the profit function by two as there are two sellers in the market. By doing so we get

$$\pi_{industry} = 2\,\pi_L^* = 2\,(\frac{113}{72} - 1)(\frac{81}{32} - \frac{1}{4}\frac{113}{72} - \frac{113}{72})$$

| |
|---|
| the linear industry equilibrium profit $\pi_{industry} = \frac{1681}{2592} \approx 0.649.$ |

Knowing $p^*$and $\pi^*_{industry}$, the percentage of consumers buying the bundle and buying something (the sum of 'buy $i$ only', 'buy $j$ only' and 'buy bundle' in Figure 15) is easily obtained by calculating the areas in Figure 15[25]. By doing so we obtain

$$q_b^* = \left(2 - p^* + \frac{1}{4}\right)^2 - \frac{1}{2}\left(p^* - p^* + \frac{1}{4}\right)^2 = \left(2 - \frac{113}{72} + \frac{1}{4}\right)^2 - \frac{1}{2}\left(\frac{1}{4}\right)^2$$

$$q_i^* = q_j^* = (2 - p^*)\left(p^* - \frac{1}{4} - 1\right) = \left(2 - \frac{113}{72}\right)\left(\frac{113}{72} - \frac{5}{4}\right) = \frac{713}{5184}$$

| |
|---|
| the percentage of consumers buying the bundle $q_b^* = \frac{2239}{5184} \approx 43.2\%$ and the percentage of consumers buying something $q_i^* + q_j^* + q_b^* = \frac{3665}{5184} \approx 70.7\%.$ |

The last indicator to be calculated is the linear total welfare. Welfare stands for the surplus minus the costs, and is defined for every distinct choice as

Welfare of buying only product $i$: $\iint (v_i - 1)$

Welfare of buying only product $j$: $\iint \left(v_j - 1\right)$

[25] Using the integrals for $q_i$ and $q_b$, used on page 42, yields the same result.

Welfare of buying the bundle: $\iint (v_i - v_j + z - 2)$.

Note: the welfare of product *i* is identical to the welfare of product *j* as the products are symmetric.

Below we define formulas for the total welfare which include $\delta$, as these formulas can then also be used for the next sections.

Two situations need to be considered separately; when $\delta > z$ and $\delta < z$.

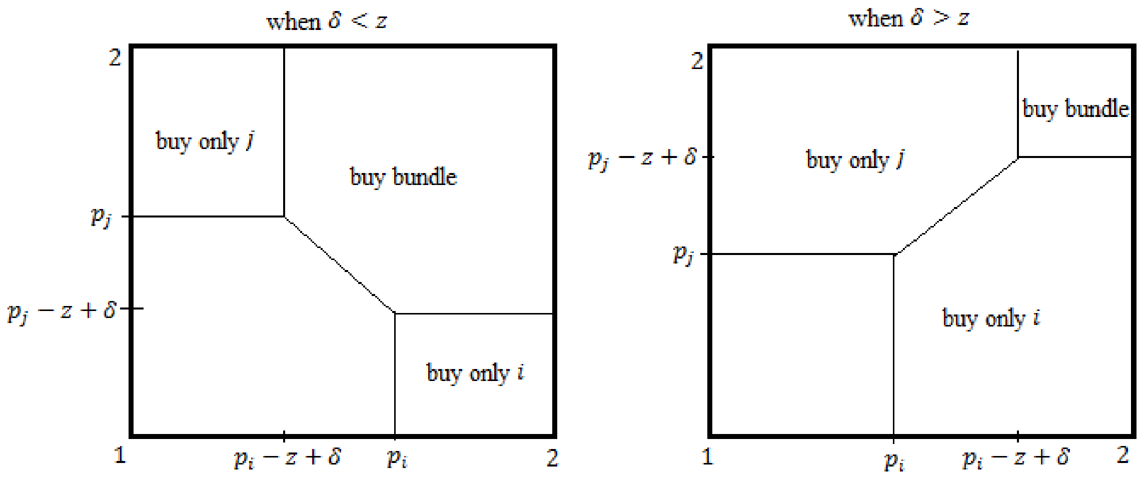


Figure 16: Demand situation when $\delta > z$ and $\delta < z$. Only premium bundle prices are considered ($\delta > 0$). $v_i$ is on the x-axis and $v_j$ on the y-axis. For simplicity we used the notation $\delta = \delta_i + \delta_j$. Note that we only allow for the situations in which $\delta > 0$, this only includes premium pricing (as we found premium pricing to be profitable for complementary demand in proposition 1). This means that bundling discounts are not evaluated in the calculations based on these graphs.

The level of welfare depends on these two distinct situations and therefore different formulas are constructed.

The total welfare when $\delta > z$ is equal to

$$W_{\delta>z} = 2\left[\int_{p}^{p-z+\delta} (v_i - 1)^2 dv_i + \int_{p-z+\delta}^{2} (v_i - 1)(p - z + \delta - 1) dv_i\right] + \int_{p-z+\delta}^{2}\int_{p-z+\delta}^{2} (v_i + v_j + z - 2) dv_j dv_i$$

Note that the part between brackets is the welfare for the consumers that buy only product $i$, and is exactly the same size for the consumers that buy only product $j$, and therefore the whole part is multiplied by two. These integrals can be solved using calculus, of which the steps are shown below.

$$W_{\delta>z} = 2\left[\int_{p}^{p-z+\delta} (v_i - 1)^2 dv_i + \int_{p-z+\delta}^{2} (v_i - 1)(p - z + \delta - 1) dv_i\right] + 2\int_{p-z+\delta}^{2}\int_{p-z+\delta}^{2} (v_i) dv_j dv_i + \int_{p-z+\delta}^{2}\int_{p-z+\delta}^{2} (z - 2) dv_j dv_i$$

$$= 2\left[\left[\frac{1}{3}(v_i-1)^3\right]_p^{p-z+\delta} + \left[\frac{v_i}{2}(v_i-2)(p-z+\delta-1)\right]_{p-z+\delta}^2\right] + (z-2)(2-p+z-\delta)^2$$

$$+ 2\int_{p-z+\delta}^{2}\left[\left(\frac{1}{2}{v_j}^2\right)\right]_{p-z+\delta}^{2} dv_i$$

$$= 2\left[\frac{1}{3}(p-z+\delta-1)^3 - \frac{1}{3}(p-1)^3 + \frac{2}{2}(2-2)(p-z+\delta-1)\right.$$

$$\left. - \frac{p-z+\delta}{2}(p-z+\delta-2)(p-z+\delta-1)\right] + (z-2)(2-p+z-\delta)^2$$

$$+ \int_{p-z+\delta}^{2}(2^2-(p-z+\delta)^2)\, dv_i$$

$$= \frac{2}{3}(p-z+\delta-1)^3 - \frac{2}{3}(p-1)^3 - (p-z+\delta)(p-z+\delta-2)(p-z+\delta-1)$$

$$+ (z-2)(2-p+z-\delta)^2 + (2-(p-z+\delta))(2+(p-z+\delta))\,(2-(p-z+\delta))$$

$$= \frac{2}{3}(p-z+\delta-1)^3 - \frac{2}{3}(p-1)^3 - (p-z+\delta)(p-z+\delta-2)(p-z+\delta-1)$$

$$+ (z-2)(2-p+z-\delta)^2 + (2-p+z-\delta)^2(2+(p-z+\delta))$$

$$= \frac{2}{3}(p-z+\delta-1)^3 - \frac{2}{3}(p-1)^3 - (p-z+\delta)(p-z+\delta-2)(p-z+\delta-1)$$

$$+ (z-2+2+p-z+\delta)(2-p+z-\delta)^2$$

$$W_{\delta>z} = \tfrac{2}{3}(p-z+\delta-1)^3 - \tfrac{2}{3}(p-1)^3 - (p-z+\delta)(p-z+\delta-2)(p-z+\delta-1) + (p+\delta)(2-p+z-\delta)^2 \qquad (21)$$

Similarly, the total welfare when $\delta < z$ is calculated. Therefore, the detailed derivations for total welfare when $\delta < z$ is omitted.

$$W_{\delta<z} = 2\left[\int_p^2 (v_i-1)(p-z+\delta-1)dv_i\right] + \int_{p-z+\delta}^{p}\int_{2p-z+\delta-v_i}^{2}\big(v_i+v_j+z-2\big)dv_j dv_i$$

$$+ \int_p^2\int_{p-z+\delta}^{2}\big(v_i+v_j+z-2\big)dv_j dv_i$$

$$W_{\delta<z} = -p(p-2)(\delta+p-z-1) + \tfrac{1}{6}(\delta-z)(2\delta^2+\delta(9p-z-12)+9p^2-3p(z+8)-z^2+12) + \tfrac{1}{2}(p-2)(\delta^2+\delta(3p-2)+2p^2-p(z+4)-z(z+2)) \qquad (22)$$

Now we can calculate the linear total welfare by substituting $p^* = \frac{113}{72}, z = \frac{1}{4}$ and $\delta = 0$ in (19), which gives

$$\begin{aligned}\mathrm{W}_{0<\frac{1}{4}} = &-\frac{113}{72}\left(\frac{113}{72}-2\right)\left(0+\frac{113}{72}-\frac{1}{4}-1\right)\\ &+\frac{1}{6}\left(0-\frac{1}{4}\right)\left(2\times 0^2+0\left(9\times\frac{113}{72}-\frac{1}{4}-12\right)+9\times\left(\frac{113}{72}\right)^2-3\times\frac{113}{72}\left(\frac{1}{4}+8\right)\right.\\ &\left.-\left(\frac{1}{4}\right)^2+12\right)\\ &+\frac{1}{2}\left(\frac{113}{72}-2\right)\left(0^2+0\left(3\times\frac{113}{72}-2\right)+2\times(\frac{113}{72})^2-\frac{113}{72}\left(\frac{1}{4}+4\right)-\frac{1}{4}\left(\frac{1}{4}+2\right)\right)\\ = &-\frac{113}{72}\times\left(-\frac{31}{72}\right)\times\left(\frac{23}{72}\right)+\frac{1}{6}\times\left(-\frac{1}{4}\right)\times\left(9\times\left(\frac{113}{72}\right)^2-3\times\frac{113}{72}\times(8.25)-\left(\frac{1}{4}\right)^2+12\right)\\ &+\frac{1}{2}\times\left(-\frac{31}{72}\right)\times\left(2\times(\frac{113}{72})^2-\frac{113}{72}\times(4.25)-\frac{1}{4}\times(2.25)\right)\approx 0.910\end{aligned}$$

| the linear total welfare level of $\mathrm{W}_{0<\frac{1}{4}}$ ≈0.910 |
|---|

*G.1.2 Non-linear situation when $\delta < z$*

Knowing these linear indicators, we now allow for $\delta$ to be included in the calculations. The effect of $\delta$ depends on the size of $z$, as can be seen in Figure 16. Therefore a distinction is made between small values of δ (so that $\delta < z$) and big values of δ (so that $\delta > z$). Note that in all below derivations we assume $\delta > 0$, which means that only premium bundle prices are included.

We first consider $\delta < z$ (depicted in the left graph of Figure 16). To find the equilibrium price $p^*$ and $\delta$, we look at the profits of firm $i$[26]; $\pi_D = (p_i - c_i)(Q_i + Q_b) + \delta_i Q_b$. In this profit function $p_i$ is the stand-alone price of product $i$ and δ is a premium that is added to price $p_i$ when the customer also buys the other firm's product. Note that in this situation $p_i$ takes on values in the range $1 + z - \delta \le p_i \le 2$. To maximize the profit function, we will first formulate equations for $Q_i$ and $Q_b$, based on Figure 16 and using $\delta = \delta_i + \delta_j$.

$$Q_i = (2 - p_i)\left(p_j - z + \delta_i + \delta_j - 1\right) \qquad (23)$$

$$Q_b = \left(2 - p_j + z - \delta_i - \delta_j\right)\left(2 - p_i + z - \delta_i - \delta_j\right) - \tfrac{1}{2}(z - \delta_i - \delta_j)^2 \qquad (24)$$

Now we substitute $c = 1, z = \frac{1}{4}$, $Q_i$ and $Q_b$ in the profit function which gives

[26] As mentioned at the beginning of this Appendix, we slightly changed the notation of the profit function defined in (10). As we only consider premium pricing, which is denoted by negative values of δ in the original notation, double negative values cause cumbersome interpretations. In the new notation used for the uniform distribution application we simply use positive values of δ and multiply the double minus sign, which leads to $\pi_i = (p_i - c)(Q_i + Q_b) + \delta_i Q_b$

$$\pi_D = (p_i - 1)\left((2-p_i)\left(p_j - \tfrac{1}{4} + \delta_i + \delta_j - 1\right) + \left(2 - p_j + \tfrac{1}{4} - \delta_i - \delta_j\right)\left(2 - p_i + \tfrac{1}{4} - \delta_i - \delta_j\right) - \tfrac{1}{2}\left(\tfrac{1}{4} - \delta_i - \delta_j\right)^2\right) + \delta_i\left(\left(2 - p_j + \tfrac{1}{4} - \delta_i - \delta_j\right)\left(2 - p_i + \tfrac{1}{4} - \delta_i - \delta_j\right) - \tfrac{1}{2}\left(\tfrac{1}{4} - \delta_i - \delta_j\right)^2\right) \quad (25)$$

$$= (p_i - 1)\left((2-p_i)\left(p_j - \frac{5}{4} + \delta_i + \delta_j\right) + \left(\frac{9}{4} - p_j - \delta_i - \delta_j\right)\left(\frac{9}{4} - p_i - \delta_i - \delta_j\right) - \frac{1}{2}\left(\frac{1}{4} - \delta_i - \delta_j\right)^2\right) + \delta_i\left(\left(\frac{9}{4} - p_j - \delta_i - \delta_j\right)\left(\frac{9}{4} - p_i - \delta_i - \delta_j\right) - \frac{1}{2}\left(\frac{1}{4} - \delta_i - \delta_j\right)^2\right).$$

When considering symmetric discounts with $\delta_i = \delta_j = \frac{1}{2}\delta$, and optimizing the profit function with respect to $p_i$, we get

$$\pi_D = (p_i - 1)\left((2-p_i)\left(p_j - \frac{5}{4} + \delta\right) + \left(\frac{9}{4} - p_j - \delta\right)\left(\frac{9}{4} - p_i - \delta\right) - \frac{1}{2}\left(\frac{1}{4} - \delta\right)^2\right) + \frac{1}{2}\delta\left(\left(\frac{9}{4} - p_j - \delta\right)\left(\frac{9}{4} - p_i - \delta\right) - \frac{1}{2}\left(\frac{1}{4} - \delta\right)^2\right)$$

$$\frac{\partial \pi_D}{\partial p_i} = (2-p_i)\left(p_j - \frac{5}{4} + \delta\right) + \left(\frac{9}{4} - p_j - \delta\right)\left(\frac{9}{4} - p_i - \delta\right) - \frac{1}{2}\left(\frac{1}{4} - \delta\right)^2 + \left(-\left(p_j - \frac{5}{4} + \delta\right) - \left(\frac{9}{4} - p_j - \delta\right)\right)(p_i - 1) - \frac{1}{2}\delta\left(\frac{9}{4} - p_j - \delta\right).$$

When using symmetry for prices such that $p_i = p_j = p$, the optimum price can be defined as a function of δ

$$(2-p)\left(p - \frac{5}{4} + \delta\right) + \left(\frac{9}{4} - p - \delta\right)^2 - \frac{1}{2}\left(\frac{1}{4} - \delta\right)^2 - (p-1) - \frac{1}{2}\delta\left(\frac{9}{4} - p - \delta\right) = 0$$

$$2p - \frac{10}{4} + 2\delta - p^2 + \frac{5}{4}p - \delta p + \left(\frac{9}{4}\right)^2 - \frac{18}{4}p - \frac{18}{4}\delta + p^2 + 2p\delta + \delta^2 - \frac{1}{32} + \frac{1}{4}\delta - \frac{1}{2}\delta^2 - p + 1 - \frac{9}{8}\delta + \frac{1}{2}\delta p + \frac{1}{2}\delta^2 = 0$$

$$2p + \frac{5}{4}p - \frac{18}{4}p - p + 2p\delta + \frac{1}{2}\delta p - \delta p - \frac{10}{4} + \left(\frac{9}{4}\right)^2 + 1 - \frac{1}{32} + 2\delta - \frac{18}{4}\delta + \frac{1}{4}\delta - \frac{9}{8}\delta + \delta^2 - \frac{1}{2}\delta^2 + \frac{1}{2}\delta^2 = 0$$

$$-\frac{9}{4}p + \frac{3}{2}\delta p + \frac{113}{32} - \frac{27}{8}\delta + \delta^2 = 0$$

$$\frac{9}{4}p - \frac{3}{2}\delta p = \frac{113}{32} - \frac{27}{8}\delta + \delta^2$$

$$p = \frac{\frac{113}{32} - \frac{27}{8}\delta + \delta^2}{\frac{9}{4} - \frac{3}{2}\delta}$$

$$p = \frac{113 - 108\delta + 32\delta^2}{72 - 48\delta}. \tag{26}$$

When considering symmetric prices with $p_{\mathrm{i}} = p_{\mathrm{j}} = p$, and optimizing the profit function with respect to $\delta_i$, we get

$$\pi_D = (p-1)\left((2-p)\left(p - \frac{5}{4} + \delta_i + \delta_j\right) + \left(\frac{9}{4} - p - \delta_i - \delta_j\right)^2 - \frac{1}{2}\left(\frac{1}{4} - \delta_i - \delta_j\right)^2\right) + \delta_i\left(\left(\frac{9}{4} - p - \delta_i - \delta_j\right)^2 - \frac{1}{2}\left(\frac{1}{4} - \delta_i - \delta_j\right)^2\right)$$

$$\frac{\partial \pi_D}{\partial \delta_i} = (p-1)(2-p) - 2\left(\frac{9}{4} - p - \delta_i - \delta_j\right)(p-1) + \left(\frac{1}{4} - \delta_i - \delta_j\right)(p-1) + \left(\frac{9}{4} - p - \delta_i - \delta_j\right)^2 - \frac{1}{2}\left(\frac{1}{4} - \delta_i - \delta_j\right)^2 - 2\delta_i\left(\frac{9}{4} - p - \delta_i - \delta_j\right) + \delta_i\left(\frac{1}{4} - \delta_i - \delta_j\right).$$

When using symmetry for $\delta$ such that $\delta_i = \delta_j = \frac{1}{2}\delta$, we get

$$(p-1)(2-p) - 2\left(\frac{9}{4} - p - \delta\right)(p-1) + \left(\frac{1}{4} - \delta\right)(p-1) + \left(\frac{9}{4} - p - \delta\right)^2 - \frac{1}{2}\left(\frac{1}{4} - \delta\right)^2 - \delta\left(\frac{9}{4} - p - \delta\right) + \frac{1}{2}\delta\left(\frac{1}{4} - \delta\right) = 0$$

$$2p - p^2 - 2 + p - \frac{18}{4}p + 2p^2 + 2\delta p + \frac{18}{4} - 2p - 2\delta + \frac{1}{4}p - \frac{1}{4} - \delta p + \delta + \frac{81}{16} - \frac{18}{4}p - \frac{18}{4}\delta + 2\delta p + p^2 + \delta^2 - \frac{1}{32} + \frac{1}{4}\delta - \frac{1}{2}\delta^2 - \frac{9}{4}\delta + p\delta + \delta^2 + \frac{1}{8}\delta - \frac{1}{2}\delta^2 = 0$$

$$-2 + \frac{18}{4} - \frac{1}{4} + \frac{81}{16} - \frac{1}{32} + 2p + p - \frac{18}{4}p - 2p + \frac{1}{4}p - \frac{18}{4}p - p^2 + 2p^2 + p^2 + 2\delta p + 2\delta p + p\delta - \delta p - 2\delta + \delta - \frac{18}{4}\delta + \frac{1}{4}\delta - \frac{9}{4}\delta + \frac{1}{8}\delta + \delta^2 - \frac{1}{2}\delta^2 + \delta^2 - \frac{1}{2}\delta^2 = 0$$

$$\frac{233}{32} - \frac{31}{4}p + 2p^2 + 4\delta p - \frac{59}{8}\delta + \delta^2 = 0 \tag{27}$$

We know $p$ as a function of $\delta$ from (26). As (27) is a quadratic function in $p$ and $\delta$, substituting $p$ in (27) will not simply give one equilibrium value of δ. However, we know that $\delta < z$, and $\delta > 0$.

Therefore, with Mathematica we can find the equilibrium solution of (27) in the specific interval. So plotting the following equation in Mathematica for $\delta \in \left[0,\frac{1}{4}\right]$ gives

$$\frac{233}{32}-\frac{31}{4}\left(\frac{113-108\delta+32\delta^2}{72-48\delta}\right)+2\left(\frac{113-108\delta+32\delta^2}{72-48\delta}\right)^2+4\delta\left(\frac{113-108\delta+32\delta^2}{72-48\delta}\right)-\frac{59}{8}\delta+\delta^2=0$$

| the equilibrium premium value when $\delta < z$ is $\delta^* \approx 0.092$. |
|---|

Knowing δ, we can now substitute this value in (26) to obtain

$$p=\frac{113-108(0.092)+32(0.092)^2}{72-48(0.092)}$$

| the equilibrium price when $\delta < z$ is $p^* \approx 1.529$. |
|---|

Knowing δ and $p$ we can now find the industry profits by substituting and multiplying (25) by two, so that

$$\pi_{industry}=2\left[(1.529-1)\left((2-1.529)\left(1.529-\frac{5}{4}+0.092\right)+\left(\frac{9}{4}-1.529-0.092\right)^2-\frac{1}{2}\left(\frac{1}{4}-0.092\right)^2\right)+\frac{1}{2}(0.092)\left(\left(\frac{9}{4}-1.529-0.092\right)^2-\frac{1}{2}\left(\frac{1}{4}-0.092\right)^2\right)\right]$$

| the equilibrium industry profit when $\delta < z$ is $\pi^*_{industry} \approx 0.626$. |
|---|

Knowing $p^*$and $\pi^*_L$, the percentage of consumers buying the bundle and buying something is obtained by using (23) and (24). By doing so we obtain

$$Q_i=(2-1.529)\left(1.529-\frac{1}{4}+0.092-1\right)\approx 0.175$$

$$Q_b=\left(2-1.529+\frac{1}{4}-0.092\right)\left(2-1.529+\frac{1}{4}-0.092\right)-\frac{1}{2}\left(\frac{1}{4}-0.092\right)^2\approx 0.383$$

| the percentage of consumers buying the bundle $Q^*_b \approx 38.3\%$ and the percentage of consumers buying something $Q^*_i+Q^*_j+Q^*_b \approx 73.3\%$. |
|---|

Lastly we calculate the total welfare when $\delta < z$ by substituting the obtained equilibrium values in (22), which gives

$$W_{\delta<z} = -1.529(1.529-2)\left(0.092+1.529-\frac{1}{4}-1\right)$$
$$+\frac{1}{6}\left(0.092-\frac{1}{4}\right)\left(2\times 0.092^2+0.092\left(9\times 1.529-\frac{1}{4}-12\right)+9(1.529)^2-3\right.$$
$$\left.\times 1.529\left(\frac{1}{4}+8\right)-\left(\frac{1}{4}\right)^2+12\right)+\frac{1}{2}(1.529-2)(0.092^2+0.092(3\times 1.529-2)$$
$$+2(1.529)^2-1.529\left(\frac{1}{4}+4\right)-\frac{1}{4}\left(\frac{1}{4}+2\right))$$

| the linear total welfare level when $\delta < z$ of $W^*_{\delta<z}$ ≈0.895 |
|---|

*G.1.3 Non-linear situation when $\delta > z$*

Now we turn to the situation when $\delta > z$ (depicted in the right graph of Figure 16). To find the equilibrium price $p^*$ and $\delta^*$, we again look at the profit of firm $i$. Using $\delta = \delta_i + \delta_j$, we will first define $Q_i$ and $Q_b$, based on Figure 16 ($Q_i$ and $Q_b$ are the main difference between the two situations).

$$Q_i = (2-p_i)\left(p_j - z + \delta_i + \delta_j - 1\right) - \tfrac{1}{2}(\delta_i + \delta_j - z)^2 \quad (28)$$

$$Q_b = \left(2 - p_i + z - \delta_i - \delta_j\right)\left(2 - p_j + z - \delta_i - \delta_j\right) \quad (29)$$

Now we substitute $c = 1, z = \frac{1}{4}$, $Q_i$ and $Q_b$ in the profit function $\pi_D = (p_i - c_i)(Q_i + Q_b) + \delta_i Q_b$ which gives

$$\pi_D = (p_i-1)\left((2-p_i)\left(p_j-\tfrac{1}{4}+\delta_i+\delta_j-1\right)-\tfrac{1}{2}\left(\delta_i+\delta_j-\tfrac{1}{4}\right)^2+\left(2-p_i+\tfrac{1}{4}-\delta_i-\delta_j\right)\left(2-p_j+\tfrac{1}{4}-\delta_i-\delta_j\right)\right)+\delta_i\left(\left(2-p_i+\tfrac{1}{4}-\delta_i-\delta_j\right)\left(2-p_j+\tfrac{1}{4}-\delta_i-\delta_j\right)\right) \quad (30)$$
$$= (p_i-1)\left((2-p_i)\left(p_j-\frac{5}{4}+\delta_i+\delta_j\right)-\frac{1}{2}\left(\delta_i+\delta_j-\frac{1}{4}\right)^2\right.$$
$$\left.+\left(\frac{9}{4}-p_i-\delta_i-\delta_j\right)\left(\frac{9}{4}-p_j-\delta_i-\delta_j\right)\right)$$
$$+\delta_i\left(\left(\frac{9}{4}-p_i-\delta_i-\delta_j\right)\left(\frac{9}{4}-p_j-\delta_i-\delta_j\right)\right)$$

We conduct the same analysis as for the $\delta < z$ case (therefore, most of the derivations are omitted). First, we consider symmetric premium values with $\delta_i = \delta_j = \frac{1}{2}\delta$, and optimize the profit function with respect to $p_i$. After taking the first order condition we impose symmetric prices such that $p_i = p_j = p$. The equilibrium price[27] can be defined as a function of $\delta$, and takes on

[27] The equilibrium price calculated here is exactly the same as (26), in which $\delta < z$.

$$p = \frac{113 - 108\delta + 32\delta^2}{72 - 48\delta}.$$

Next, we consider symmetric prices with $p_i = p_j = p$, and optimize the profit function with respect to $\delta_i$. After taking the first order condition we impose symmetric premium values such that $\delta_i = \delta_j = \frac{1}{2}\delta$, which gives

$$(p-1)\left(\delta + p - \frac{9}{4}\right) + \left(\frac{9}{4} - p - \delta\right)^2 - \delta\left(\frac{9}{4} - p - \delta\right) = 0.$$

As this equality is a quadratic function, substituting $p$ will not simply give one optimal value of $\delta$. In Figure 16 the allowed ranges of $\delta$ and $p$ can be observed. By plotting the graph of this function for the allowed range of $\delta$ with Mathematica, we find no equilibrium bundle premium value.

*G.2 When $z$ taken as variable*

We now conduct a similar analysis as G.1.2, but take $z$ as variable. By doing so we will be able to construct equations of variables that can be plotted in MATLAB. For this analysis we only look at the situation when $\delta < z$. For the analysis we introduce the following notation

$$\alpha = z - \delta_i - \delta_j \tag{31}$$

$$\beta = p_j - 1 - \alpha. \tag{32}$$

Using this notation, the demand for stand-alone product $i$ and the bundle can be defined as

$$Q_i = (2 - p_i)\beta \tag{33}$$

$$Q_b = (1-\beta)(2 - p_i + \alpha) - \frac{1}{2}\alpha^2 \tag{34}$$

As explained at the beginning of this appendix we use the following profit function[28]

$$\pi_i = (p_i - 1)(Q_i + Q_b) + \delta_i Q_b.$$

We will first take the derivative with respect to $p_i$ which gives

$$\frac{\partial \pi_i}{\partial p_i} = Q_i + Q_b + (p_i - 1)\left(\frac{\partial Q_i}{\partial p_i} + \frac{\partial Q_b}{\partial p_i}\right) + \delta_i \frac{\partial Q_b}{\partial p_i} = 0. \tag{35}$$

Using the notation (31) and (32), the derivatives of $Q_i$ and $Q_b$ are

$$\frac{\partial Q_i}{\partial p_i} = -\beta$$

[28] Note that we still use $c_i = c_j = 1$.

$$\frac{\partial Q_b}{\partial p_i} = -(1-\beta) = \beta - 1.$$

Substituting these derivatives in (35) and simplifying gives

$$\frac{\partial \pi_i}{\partial p_i} = Q_i + Q_b + (p_i - 1)(-1) + \delta_i(\beta - 1) = 0.$$

Now can substitute the demand functions (33) and (34) and simplify to

$$(2-p_i)\beta + (1-\beta)(2 - p_i + \alpha) - \frac{1}{2}\alpha^2 + 1 - p_i + \delta_i(\beta - 1) = 0$$

$$p_i(-\beta - 1 + \beta - 1) + 2\beta + (1-\beta)(2+\alpha) - \frac{1}{2}\alpha^2 + 1 + \beta\delta_i - \delta_i = 0$$

$$p_i = \frac{1}{2}(2\beta + 2 - 2\beta + \alpha - \beta\alpha - \frac{1}{2}\alpha^2 + 1 + \beta\delta_i - \delta_i)$$

$$p_i = \frac{1}{2}\left(3 + \alpha - \beta\alpha - \frac{1}{2}\alpha^2 + \beta\delta_i - \delta_i\right).$$

Now we impose symmetry such that $\delta_i = \delta_j = \frac{1}{2}\delta$ and $p_i = p_j = p$ and substitute $\beta$ defined in (32) which gives

$$\begin{aligned} p &= \frac{1}{2}\left(3 + \alpha - (p - 1 - \alpha)\alpha - \frac{1}{2}\alpha^2 + (p - 1 - \alpha)\frac{1}{2}\delta - \frac{1}{2}\delta\right) \\ &= \frac{1}{2}\left(3 + \alpha - p\alpha + \alpha + \alpha^2 - \frac{1}{2}\alpha^2 + \frac{1}{2}\delta p - \frac{1}{2}\delta - \frac{1}{2}\delta\alpha - \frac{1}{2}\delta\right) \\ &= \frac{1}{2}\left(3 + 2\alpha + \frac{1}{2}\alpha^2 - \delta - \frac{1}{2}\delta\alpha\right) + p\left(-\frac{1}{2}\alpha + \frac{1}{4}\delta\right). \end{aligned}$$

When we substitute α as defined in (31), we obtain

$$p = \frac{1}{2}\left(3 + 2z - 2\delta + \frac{1}{2}(z^2 - 2z\delta + \delta^2) - \delta - \frac{1}{2}\delta(z - \delta)\right) + p\left(-\frac{1}{2}z + \frac{1}{2}\delta + \frac{1}{4}\delta\right)$$

$$p\left(1 + \frac{1}{2}z - \frac{3}{4}\delta\right) = \frac{1}{2}\left(3 + 2z - 2\delta + \frac{1}{2}z^2 - z\delta + \frac{1}{2}\delta^2 - \delta - \frac{1}{2}\delta z + \frac{1}{2}\delta^2\right)$$

$$p = \frac{\frac{1}{2}\left(3 + 2z - 3\delta + \frac{1}{2}z^2 - \frac{3}{2}z\delta + \delta^2\right)}{\left(1 + \frac{1}{2}z - \frac{3}{4}\delta\right)}. \qquad (36)^{29}$$

Which is the first important equation as it reflects the symmetric equilibrium price as a function of $\delta$ and $z$. Plotting (36) with MATLAB provides a figure in which the relationship between $p$, $\delta$ and $z$ can be observed. The graph is depicted in Figure 18.

[29] Check: when substituting $z = \frac{1}{4}$ and rewriting, we obtain (26).

Now we will take the derivative of the profit function with respect to $\delta_i$ which leads to

$$\frac{\partial \pi_i}{\partial \delta_i} = (p_i - 1)\left(\frac{\partial Q_i}{\partial \delta_i} + \frac{\partial Q_b}{\partial \delta_i}\right) + Q_b + \delta_i \frac{\partial Q_b}{\partial \delta_i} = 0. \qquad (37)$$

Using the notation (31) and (32), the derivatives of $Q_i$ and $Q_b$ are

$$\frac{\partial Q_i}{\partial \delta_i} = 2 - p_i$$

$$\frac{\partial Q_b}{\partial \delta_i} = -(2 - p_i + \alpha) - (1 - \beta) + \alpha = -2 + p_i + \beta - 1.$$

Substituting (34) and these derivatives in (37) gives

$$\begin{aligned}(p_i - 1)(2 - p_i - 2 + p_i + \beta - 1) + (1 - \beta)(2 - p_i + \alpha) - \frac{1}{2}\alpha^2 + \delta_i(-2 + p_i + \beta - 1) &= 0\\ (p_i - 1)(\beta - 1) + (1 - \beta)(2 - p_i + \alpha) - \frac{1}{2}\alpha^2 - 2\delta_i + p_i\delta_i + \delta_i(\beta - 1) &= 0\\ (\beta - 1)(p_i - 1 - 2 + p_i - \alpha + \delta_i) - \frac{1}{2}\alpha^2 - 2\delta_i + p_i\delta_i &= 0\\ (\beta - 1)(2p_i - 3 - \alpha + \delta_i) - \frac{1}{2}\alpha^2 - 2\delta_i + p_i\delta_i &= 0.\end{aligned}$$

Now we impose symmetry such that $\delta_i = \delta_j = \frac{1}{2}\delta$ and $p_i = p_j = p$ and substitute $\beta$, which gives

$$(p - 1 - \alpha - 1)\left(2p - 3 - \alpha + \frac{1}{2}\delta\right) - \frac{1}{2}\alpha^2 - \delta + \frac{1}{2}p\delta = 0.$$

Next, when also substituting α, we obtain

$$(p - 2 - z + \delta)\left(2p - 3 - z + \delta + \frac{1}{2}\delta\right) - \frac{1}{2}z^2 + z\delta - \frac{1}{2}\delta^2 - \delta + \frac{1}{2}p\delta = 0$$

$$(p - 2 - z + \delta)\left(2p - 3 - z + \frac{3}{2}\delta\right) - \frac{1}{2}z^2 + z\delta - \frac{1}{2}\delta^2 - \delta + \frac{1}{2}p\delta = 0. \qquad (38)$$

When we substitute (36) in (38) we get to the second important equation which characterizes the relationship between $\delta$ and $z$ in the symmetric equilibrium. (39) can be plotted with MATLAB and the ranges of $\delta$ and $z$ depend on $\delta < z$. The graph is depicted in Figure 6.

$$\boxed{\left(\frac{\frac{1}{2}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)} - 2 - z + \delta\right)\left(\frac{\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)} - 3 - z + \frac{3}{2}\delta\right) - \frac{1}{2}z^2 + z\delta - \frac{1}{2}\delta^2 - \delta + \frac{\frac{1}{4}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)}\delta = 0 \qquad (39)}$$

We are also interested in the effect of varying product synergies on the industry profit. The formula for the industry profit is $\pi_{inudstry} = 2\left[(p_i - 1)(Q_i + Q_b) + \frac{1}{2}\delta Q_b\right]$ with $p$ being the symmetric equilibrium price and $\delta_i = \delta_j = \frac{1}{2}\delta$. Substituting (31), (32), (33), (34) gives

$$\pi_{inudstry} = 2\Bigg[(p-1)\Big((2-p)(p-1-z+\delta)+(1-p+1+z-\delta)(2-p+z-\delta) - \frac{1}{2}(z-\delta)^2\Big) + \frac{1}{2}\delta((1-p+1+z-\delta)(2-p+z-\delta)-\frac{1}{2}(z-\delta)^2)\Bigg]$$

$$= 2\Bigg[(p-1)\Big((2-p)(p-1-z+\delta)+(2-p+z-\delta)^2-\frac{1}{2}(z-\delta)^2\Big) + \frac{1}{2}\delta((2-p+z-\delta)^2-\frac{1}{2}(z-\delta)^2)\Bigg].$$

When also substituting the symmetric equilibrium price (36) we obtain the industry profit as a function of $\delta$ and $z$. Depicting (40) graphically with MATLAB provides a figure in which the relationship among industry profit, $\delta$ and $z$ can be observed. This graph can be found in Figure 19.

$$\pi_{inudstry} = 2\left[\left(\frac{\frac{1}{2}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)}-1\right)\left((2-\frac{\frac{1}{2}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)})(\frac{\frac{1}{2}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)}-1-z+\delta)+\left(2-\frac{\frac{1}{2}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)}+z-\delta\right)^2-\frac{1}{2}(z-\delta)^2\right)+\frac{1}{2}\delta(\left(2-\frac{\frac{1}{2}\left(3+2z-3\delta+\frac{1}{2}z^2-\frac{3}{2}z\delta+\delta^2\right)}{\left(1+\frac{1}{2}z-\frac{3}{4}\delta\right)}+z-\delta\right)^2-\frac{1}{2}(z-\delta)^2)\right]. \tag{40}$$

## H Example

Take into account all the discussed assumptions of the model and suppose that $\pi_1(p_1, p_2)$ and $\pi_2(p_2, p_1)$ correspond to the profits of firm 1 and 2, with $p$ being the price of the subscript matching firm[30]. Moreover, consider the specific profit function[31] $\pi_1(p_1, p_2) = 0.5 - (p_1 - 1)^2 - (p_2 - 1.5)^2$ and we calculate the optimal values of $p_1$ and $p_2$ by applying the same method as in the conducted analysis. Using that $x^2 + y^2 = \alpha^2$ is a circle with midpoint 0 and radius $\alpha$, we sketch the profit function for various constant values of $\pi_1$ in Figure 17. To calculate the equilibrium price level, the derivative of the profit function with respect to $p_1$ is made equal to 0. This leads to $\frac{\partial \pi_1(p_1,p_2)}{\partial p_1} = -2(p_1 - 1) = 0$, such that $p_1 = 1$. If we now introduce symmetry, such that $p_2 = p_1$ holds, the optimal price value of both firms must lie on the red and the optimal value of $p_1 = p_2$ is indicated by the blue point in Figure 17. By substituting these values in the profit function we get an 'equilibrium' profit of 0.25. This, however, is no maximum. By for instance taking $p_1 = p_2 = 1.25$ we obtain a profit of 0.375 which thus gives a higher level of profit.

[30] Note that in our analysis the profit function of the two firms depend on four variables, as $\pi_i\left(p_i, p_j, \delta_i, \delta_j\right)$ and $\pi_j\left(p_j, p_i, \delta_j, \delta_i\right)$. However, for simplicity we use only two variables in this example. The take away of the example also holds for profit functions that depend on four variables.

[31] This profit function is a mathematical example with less economic relevance. It is selected to resemble, but mostly simplify, the profit functions used in the analysis of this thesis. In Appendix G.1 it can be seen that the profit functions are at least quadratic in $p_i, p_j, \delta_i, \delta_j$, which is why this function is also quadratic in $p_1$ and $p_2$.

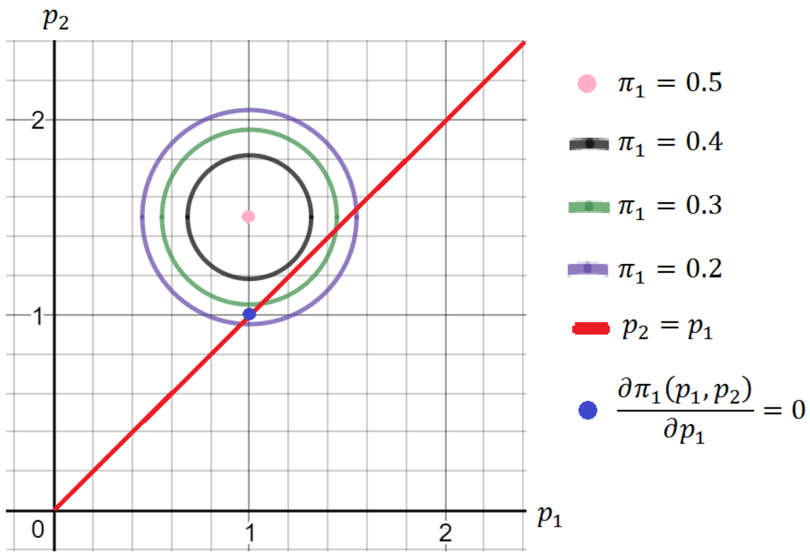


Figure 17: Graph that depicts the profit function of firm 1 $\pi_1(p_1, p_2) = 0.5 - (p_1 - 1)^2 - (p_2 - 1.5)^2$ for four different values of $\pi_1$. Note that the pink dot represents the maximum level op profit.

## I Additional Graphs

For the Analysis in Subsection 5.3 we plotted the model in MATLAB. Figures 18 and 19 were used to understand the outcome of the model.

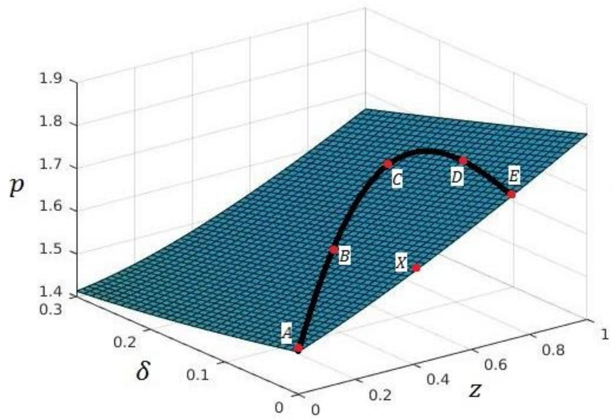


Figure 18: Symmetric equilibrium price as a function of $z$ and $\delta$. To obtain the area of the graph, we plotted equation (36) with MATLAB. The black line is a rough representation of the line in Figure 6. We obtain the following stand-alone symmetric prices: $p_A = 1.5$; $p_B = 1.529$; $p_C = 1.598$; $p_D = 1.704$; $p_E = 1.732$; $p_X = 1.569$. $\delta$ and $p$ are endogenous and $z$ is exogenous.

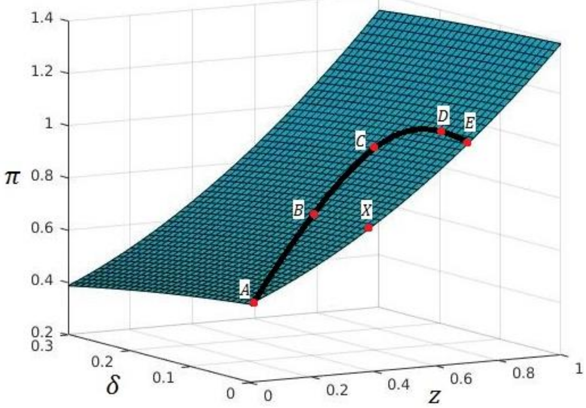


Figure 19: Industry Profit as a function of $z$ and $\delta$ in the symmetric equilibrium. Equation (40) was plotted with MATLAB to obtain the area in this graph. The black line is a rough representation of the line in Figure 6. We obtain the following industry profit values: $\pi_A = 0.5$; $\pi_B = 0.626$; $\pi_C = 0.897$; $\pi_D = 1.071$; $\pi_E = 1.071$; $\pi_X = 0.649$. $\delta$ and $\pi$ are endogenous and $z$ is exogenous.