Erasmus University Rotterdam
Erasmus School of Economics
Econometric Institute

Portfolio risk management: a new model for asymptotic (in)dependence

Author:
Kevin R. R. Kroondijk

Supervised by:
A. A. Kiriliouk & C. Zhou
Co-reader:
P. Wan

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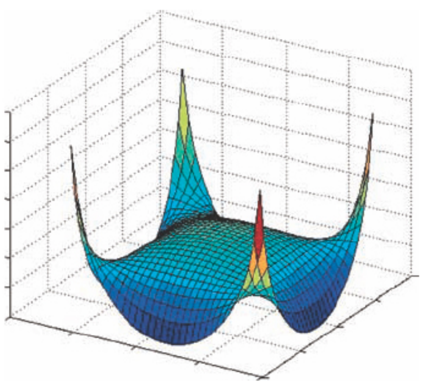
Abstract

This thesis analyses the dependence structure of 11 global stock markets to estimate portfolio risk measures. A parametric copula model from Huser and Wadsworth (forthcoming 2018) is applied to model the co-exceedances over a threshold. The model allows a smooth transition between asymptotic dependence and asymptotic independence. We find that the strongest spillover effect exists for countries within the European Union as opposed to more geographically diverse countries. The risk estimates based on the Huser and Wadsworth (forthcoming 2018) model outperform the benchmark estimates based on conventional copula models.

Keywords : asymptotic (in)dependence, copula, Value-at-Risk



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1 Introduction

Through the integration of capital markets, the global financial system becomes increasingly entangled. A negative shock in one market causes more volatility spillover on related markets as they become more strongly connected. A good example of this is the crash of financial markets in 2008, which suggests a large dependence between assets worldwide during a crisis. The overall losses were so severe, that the impact was disproportionate to any empirical evidence, leading to portfolio performance worse than the modeled worst-case scenarios. An accurate estimate of the dependence structure of global markets could have resulted in more informed decision-making when constructing portfolio of geographically diversified assets. Estimating risk measures incorporating the dependence of extreme events potentially provides better information regarding portfolio losses during extreme events.

This thesis analyzes the dependence structure of crashes across financial markets and estimates portfolio risk measures in accordance with that dependence structure. There are two potential cases that may occur when analyzing the behaviour of joint extreme events of a pair of markets: asymptotic dependence or asymptotic independence. Understanding the difference is crucial for both model development and data application as standard models only allow either asymptotic dependence or asymptotic independence. When chosen the incorrect model, extrapolation will yield inaccurate results. The difference between the two depends on whether the relation between risk factors remains constant when considering more extreme events. The dependence of extreme events is usually measured by the conditional probability that one is above high threshold, given the other is above a high threshold. In this thesis, the thresholds refer to quantiles with the same probability of the corresponding marginal loss distributions. As the tail probability defining the marginal threshold tends to zero, we are considering more and more extreme events. If the conditional probability remains at a constant level when considering more and more extreme events, the relation between markets is called asymptotic dependent. Then the conditional probability that one market crashes, given that another market crashes, remains constant when crashes get more extreme. If the conditional probability diminished to zero when considering more and more extreme events, the relation between markets is called asymptotic independent. Here, the conditional probability that one market crashes, given that another market crashes, decreases to zero when crashes become more extreme. However, stating whether asymptotic (in)dependence holds is problematic in general. The traditional method to distinguish between asymptotic dependence and asymptotic independence is through a tail dependence coefficient. The coefficient performs well in classifying the dependence structure, but neglects informing about the remaining level of dependence in asymptotically independent distributions. Therefore in our study the tail coefficient is an undesirable candidate for understanding the asymptotic behaviour.

Instead, we estimate the asymptotic (in)dependence of stock markets through the parameter space in a model without distinguishing the two cases ex ante. In addition, when joint extremes follow an asymptotic independent structure, this thesis differentiates between the rate at which the dependence structure reaches asymptotic independence. This is referred to as the sub-asymptotic dependence. To complement existing research this research focuses on two main research questions. The first question is the following:

1) What is the asymptotic relation between pairs of large global stock markets?

Capturing the asymptotic relation within stock markets enables extrapolation towards the most extreme events, those which affect the financial stability of stock markets the strongest. Due to the impact of these crashes, regulators and policy makers require that financial institutions, i.e. banks, hold risk capital that is enough to cover portfolio losses in unlikely events. The amount of capital a financial institution must hold, is derived from common standard measurements for portfolio risk. Having a large cash reserve diminishes potential profits, while too small reserves causes an increase in the default probability. Industry standard risk measures exist to summarize the risk in portfolios, i.e. those conform Basel I & II regulations. These measures largely depend on the extreme quantiles and therefore on assumptions made about the loss distribution. Since a portfolio loss is a linear combination of losses of each asset, the dependence across the assets in part determines the distribution of portfolio losses, particularly the dependence structure for extreme losses. Modelling the dependence of the extreme losses across assets could contribute to more accurate portfolio risk estimates than applying standard risk models. The second question therefore focuses on these widely used risk models:

2) Can standard risk models be improved by incorporating a new model for the asymptotic (in)dependence between indices?

The co-movements of financial markets are usually summarized by the Pearson correlation coefficient using all observations (Bekaert and Harvey (2003)). The advantage is that it is one characteristic to compare and allows for fast decision making. However, it does not distinguish between extreme and moderate observations. A different approach would be to use a copula model to study the co-movements. Copula refers to a multivariate distribution function that links the univariate marginal distributions (margins) of individual random vectors to a cumulative distribution function (c.d.f). Since it allows to model the dependence structure with non-Gaussian margins, it is often used to model the asymptotic dependence structure in financial markets.

A common restriction of copulas is, depending on the choice of copula, one either assumes asymptotic dependence or asymptotic independence. When selecting a dependence model the (incorrect) assumption of asymptotic (in)dependence could severely bias the risk measures. The parametric copula model in Huser and Wadsworth (forthcoming 2018) (for future reference noted as H&W) allows for the possibility of a smooth transition between asymptotic independence and asymptotic dependence through two components. The parameter space specifies the asymptotic (in)dependence and therefore no assumptions on the asymptotic dependence structure are made. The asymptotic independent component is flexible and contains different parametric forms. In this thesis, simulations are done to compare the estimation of parameters between the different specifications. The parametric fit of the copula model and non-parametric estimates of the tail dependence together determine the goodness of fit.

Based on modelling the asymptotic (in)dependence between indices, this thesis estimates the risk measure Value-at-Risk (VaR) of a portfolio. We consider three approaches to estimating the VaR, including a method where the dependence structure of two indices is modeled by the H&W model.

The methodology in this thesis consists of two parts. Firstly, an in-sample analysis fitting all data to the proposed model. In this part, we model the asymptotic (in)dependence by the model in H&W and use a heteroskedastic volatility model for the marginals. Testing the asymptotic relation within pairs of financial markets is based on the assumption of asymptotic normality of the estimators for the parameters. Secondly, to test whether our new method can improve the VaR estimation, we conduct an out-of-sample analysis that incorporates temporal changes in the asymptotic (in)dependence. As the aim is to accurately measure portfolio risk, parametric and non-parametric estimates for the VaR of a portfolio are backtested, including VaR estimated using the copula model in H&W.

The structure of this thesis is as follows. Firstly, Section 2 summarizes the relevant literature of the asymptotic (in)dependence in financial markets through a copula model. Next, an introduction to a copula model and in particular an extension of the copula model in H&W is in Section 3. The modelling of the return series and calculation of the VaR estimates are in Section 4. A simulation study concerning the parameter estimation of the copula model is given in Section 5. Characteristics of the empirical data are in 6. Section 7 focuses on the implementation of the model and the respective risk measurements. Last, a conclusion of this thesis is provided in Section 8.

2 Literature

There are many studies analyzing stock market integration over time. Bekaert and Harvey (2003) and Longin and Solnik (2001) report that more market integration has had a positive effect on the dependence structure between markets. Furthermore, Poon et al. (2003) finds evidence of increasing dependence between extremes by investigating global stock markets. Dividing the estimation sample into arbitrarily chosen sub-periods, Poon et al. (2003) calculates standard dependence measures in each sub-period via a regression model. They find an increase in asymptotic dependence, but whether the increase is statistically significant remains uncertain. The markets analyzed in Poon et al. (2003) are included in this thesis for comparison, however a different sample period is chosen. Castro-Camilo et al. (2018) build upon the findings of Poon et al. (2003) and propose a rolling window estimation in combination with a regression model. In this thesis, we analyze the asymptotic dependence by the model in H&W, which is less complex than that in Castro-Camilo et al. (2018) and more suited towards our research questions. To account for the non-stationary dependence structure in markets, we adopt a similar rolling window approach.

Longin and Solnik (2001) report that heteroskedastic volatility affects asymptotic dependence. A period with high volatility is more likely to be followed by a period with high volatility, than a period with low volatility. To filter the heteroskedastic volatility, we implement a flexibile GARCH-model before estimating the asymptotic (in)dependence. Lastly, Forbes and Rigobon (2002) provide empirical evidence for asymmetry in most of the pairs of extremes within 21 stock markets. The asymmetry in pairs refers to stronger dependence when markets go down, as opposed to going up and incorporating this characteristic could improve the fit of a model.

Assuming asymptotic (in)dependence between indices when selecting a dependence model, could severely over- or underestimate portfolio risk. Since the relation is often unclear, H&W introduce a flexible copula model that enables the range between the asymptotic dependence and asymptotic independence in the parameter space. The model builds upon Huser et al. (2017), who construct a Gaussian scale mixtures model to enable asymptotic dependence. The model is shown to outperform non-parametric dependence statistics when using a censored likelihood estimation. Other models estimating the asymptotic relation include, but are not limited to, a limiting Poisson process (Engelke et al. (2015)), a generalized Pareto process (Ferreira et al. (2014)) or a factor copula model (Krupskii et al. (2018))). However, these models either lack the tail flexibility of the model in H&W or only allow asymptotic independence at boundary points. The only possible advantage of the aforementioned models compared to the model in H&W, is that they do not suffer from large computational burden in higher dimensions. Also, the numerical estimation is less prone to error than a model incorporating max-stable distributions. However, as this study investigates pairwise dependence only, the computational burden remains moderate. Due to the tail flexibility and the possibility to test for asymptotic (in)dependence, we expect that the H&W model is more suitable for our research. We consider four different variants of the model: a Gaussian, Dirichlet, Logistic and Asymmetric-Logistic variant. The expectation is that a variant which inhabits an asymmetric structure(i.e. Dirichlet, Asymmetric-Logistic) would perform better than with a symmetric structure(i.e. Gaussian, Logistic), as Forbes and Rigobon (2002) indicates asymmetry in financial stock markets.

3 Modelling bivariate tail dependence structure

This section covers an introduction to copula and the theory regarding the H&W model. The model allows for asymptotic (in)dependence between two random variables, without assuming asymptotic (in)dependence upfront, possibly contributing to more accurate risk analysis. The general methodology for modelling the return series, using the dependence model from this section, is given in Section 4.

Note that throughout Section 3 & 4, like in general literature, the extremes are defined as positive exceedances above a threshold. However, the extremes of interest in our study lay below a pre-specified threshold (crashes). Therefore we consider the upper tail of the loss distribution of the returns in our application (Section 7) which aligns with the theory and methodology in Section 3 & 4.

3.1 Copula

Suppose a bivariate random vector (X_1, X_2) follows the corresponding c.d.f. of the pair be $F(x_1, x_2)$, with marginal c.d.f. $F_j(x_j)$ for $j = 1, 2$. Sklar (1959) shows that if the margins are continuous and strictly increasing, a copula function C_F exists and is unique for c.d.f. F

$$\begin{aligned} F(x_1, x_2) &= \Pr\{X_1 < x_1, X_2 < x_2\} \\ &= \Pr\{U_1 < F_1(x_1), U_2 < F_2(x_2)\} \\ &= C_F(F_1(x_1), F_2(x_2)). \end{aligned} \tag{1}$$

Here $(U_1, U_2) = (F_1(X_1), F_2(X_2))$ is a bivariate random vector and the copula $C_F \in [0, 1]^2$ is a bivariate distribution function that links the two margins. Due to the property of copula invariance for monotonously increasing margins, the information between the copula and the margins are mutually exclusive, which allows investigating the two separately.

In this thesis, the dependence of values above a extreme marginal threshold (u_1^*, u_2^*) are of importance. Define the extreme pairs (Y_1, Y_2) as the subset where at least one r.v. in the pair (X_1, X_2) exceeds its extreme marginal threshold. In the next section we shall introduce a copula model by H&W, C_H, which models the dependence in the pairs of extremes (Y_1, Y_2). The asymptotic dependence of the extremes can then be evaluated through the copula C_H. To measure the asymptotic relation of extremes in C_H, we use the tail dependence measure for copulas, χ, which is the probability that one variable exceeds a high threshold, conditional that the other variable exceeds a high threshold,

$$\chi = \lim_{u \to 1} \Pr\{F_1(X_1) > u | F_2(X_2) > u\}. \tag{2}$$

The dependence measure $\chi \in [0, 1]$ measures the top-right asymptotic (in)dependence level of the copula and separates the relation into two classes: asymptotic dependent or asymptotic independent. When $\chi > 0$ there exists asymptotic dependence, since the conditional probability is strictly positive. At the boundary $\chi = 0$, there is asymptotic independence between a pair of vectors and the probability that both vectors exceed their respective marginal thresholds simultaneously, converges to zero.

3.2 The Huser and Wadsworth(2018) model

Due to the limitations of specifying the asymptotic distribution family (either asymptotic dependence or asymptotic independence) prior to fitting the model, H&W introduces a new model for modelling the copula of a bivariate random vector. Consider

$$(H_1, H_2) = (P^\delta W_1^{1-\delta}, P^\delta W_2^{1-\delta}), \quad \delta \in [0, 1], \tag{3}$$

where the first component, P, follows a standard Pareto random variable and the second component (W_1, W_2) is asymptotic independent with standard Pareto margins and copula

$$\Pr\{W_1 > x, W_2 > x\} = L(x) x^{-1/\eta_W} \tag{4}$$

for $x \geq 1$. Here $L(x)$ denotes a slowly varying function, where $\lim_{x \to \infty} \frac{L(ax)}{L(x)} \to 1 \ \forall \ a > 0$. The η_W is called the coefficient of tail independence between (W_1, W_2), which ranges from $0 < \eta_W < 1$. The coefficient η_W measures the speed of convergence of (W_1, W_2) towards asymptotic independence as it is invariant to the margins of (W_1, W_2) (H&W). The copula of (H_1, H_2) can be used for modelling dependence without assuming asymptotic (in)dependence in advance.

In this model, the parameter δ determines the level of asymptotic dependence. When $\delta \to 1$, there exists full dependence. If $\delta \to 0$, (H_1, H_2) has exactly the same dependence structure as (W_1, W_2) which is asymptotic independence. The parameter space of δ allows for a continuous transition between asymptotic dependence and asymptotic independence. When $0.5 < \delta < 1$, the H&W model is asymptotic dependent and the dependence measure

$$\chi_H = \frac{2\delta - 1}{\delta} \mathbb{E}\big[\min(W_1, W_2)^{(1-\delta)/\delta}\big] > 0. \tag{5}$$

When, $0 < \delta < 0.5$, $\chi_H = 0$ and the model is asymptotic independent with a coefficient of tail independence η_H given as (H&W)

$$\eta_H = \begin{cases} 1 & \text{if} \quad \delta \geq 1/2, \\ \frac{\delta}{1-\delta} & \text{if} \quad \frac{\eta_W}{1+\eta_W} < \delta < 1/2, \\ \eta_W & \text{if} \quad \delta \leq \frac{\eta_W}{1+\eta_W}. \end{cases} \tag{6}$$

There are two types of models that satisfy the characterization of (W_1, W_2) in this thesis. The first case is a Gaussian model and the second case an inverted max-stable model. First, the properties with a Gaussian model are in Section 3.2.1. For the inverted max-stable model, an introduction to extreme-value theory is given in Section 3.2.2. The properties of different max-stable distributions are given in Section 3.2.3.

3.2.1 A Gaussian approach

Let us assume that a pair (X_1, X_2) follows a Gaussian distribution with correlation function ρ and standard Gaussian margins Φ. Then the distribution of

$$(W_1, W_2) = \Big(\frac{1}{1 - \Phi(X_1)}, \frac{1}{1 - \Phi(X_2)}\Big) \tag{7}$$

possesses Pareto margins and a Gaussian copula with $\eta_W = \frac{(1+\rho)}{2}$ (H&W). A Gaussian copula is a symmetric copula. Based on (W_1, W_2) following a Gaussian copula, we can establish (H_1, H_2) as in equation (3). The dependence measure χ_H for the model (H_1, H_2) can be calculated through numerical integration. If $\delta < 0.5$, the coefficient of tail independence for (H_1, H_2),

$$\eta_H = \begin{cases} 1 & \text{if} \quad \delta \geq 1/2, \\ \frac{\delta}{1-\delta} & \text{if} \quad \frac{1+\rho}{3+\rho} < \delta < 1/2, \\ \frac{1+\rho}{2} & \text{if} \quad \delta \leq \frac{1+\rho}{3+\rho}. \end{cases} \tag{8}$$

3.2.2 Extreme value theory

Before reviewing the inverted max-stable model, a brief overview regarding extreme value theory (EVT), the fundamentals for max-stable models. Fisher and Tippett (1928) and Gnedenko (1943) were among the first to introduce the early principals for EVT by proving that the distribution of extreme values of an i.i.d. sample from a c.d.f. F may converge in the limit towards only three kinds of distributions.

Let us consider the univariate approach, where the quantity of interest is the maximum in a block of the sample. Let i.i.d. random variables $\{X_1, ..., X_n\}$ follow a common d.f. F. Define the maximum as $M_n = \max\{X_1, ..., X_n\}$. Theoretically, the d.f. of M_n,

$$\begin{aligned} \Pr\{M_n \leq x\} &= \Pr\{X_1 \leq x, ..., X_n \leq x\} \\ &= \Pr\{X_1 \leq x\} \times ... \times \Pr\{X_n \leq x\} \\ &= \{F(x)\}^n, \end{aligned}$$

where we use the assumption that $\{X_1, ..., X_n\}$ are i.i.d.. Note that for any value of x smaller than the endpoint of the right tail of F, $\{F(x)\}^n \to 0$ as $n \to \infty$. By renormalizing the maximum M_n by sequences of constants that depend on the sample, the limit may turn to be non-degenerated. Therefore the focus shifts to the probability $P\{\tilde{M}_n \leq x\}$ with

$$\tilde{M}_n = \frac{M_n - b_n}{a_n}, \tag{9}$$

where $a_n > 0$ and b_n are sequences of constants. von Mises (1936) shows that when this probability converges to a non-degenerate d.f. $G(x)$, then $G(x)$ is an univariate Generalized-Extreme-Value (GEV) distribution

$$\lim_{n\to\infty} \Pr\{\tilde{M}_n < x\} \to^d G(x) = \exp\{-[1 + \xi(\frac{x-\mu}{\sigma})]^{-1/\xi}\}, \tag{10}$$

where $\mu \in (-\infty, \infty)$, $\sigma \in (0, \infty)$ and $\xi \in (-\infty, \infty)$. The shape parameter ξ differentiates between the type of the GEV-distribution, which encompasses the 3 types: Gumbel ($\xi = 0$), Fréchet ($\xi > 0$) and the Weibull ($\xi < 0$) distribution. The GEV-distribution is a max-stable distribution (Segers (2012)).

Next, we switch from univariate EVT to bivariate EVT to study pairs of extremes. Consider $\{(X_{1,1}, X_{1,2}), ..., (X_{n,1}, X_{n,2})\}$ as i.i.d. bivariate random vectors. Let $M_{n,j}$, $j = \{1, 2\}$, be the maximum of the univariate random variables $\{X_{1,j}, ..., X_{n,j}\}$. Then the bivariate vector of renormalized maxima $\tilde{\boldsymbol{M}}_{\boldsymbol{n}}$ is as follows

$$\tilde{\boldsymbol{M}}_{\boldsymbol{n}} = (\tilde{M}_{n,1}, \tilde{M}_{n,2}) = (\frac{M_{n,1} - b_{n,1}}{a_{n,1}}, \frac{M_{n,2} - b_{n,2}}{a_{n,2}}), \tag{11}$$

where $a_{n,1}, a_{n,2}, b_{n,1}, b_{n,2}$ are sequences of constants. If the renormalized maxima converge to a distribution, as $n \to \infty$, that distribution has to be a bivariate GEV-distribution (Segers (2012))

$$\lim_{n\to\infty} \Pr\{\tilde{\boldsymbol{M}}_{\boldsymbol{n}} < \boldsymbol{x}\} \to^d G(\boldsymbol{x}). \tag{12}$$

Segers (2012) show that a bivariate GEV-distribution $G(\boldsymbol{x})$ can be written into two components, the univariate GEV-margins $G_j(x_j)$ and a max-stable copula C^*. The max-stable copula C^* can be written as a tail dependence function $V(x_1, x_2)$ as follows

$$C^*(\exp\{-1/x_1\}, \exp\{-1/x_2\}) = \exp\{-V(x_1, x_2)\}. \tag{13}$$

The function $V(x_1, x_2)$ is the exponent function and is homogeneous of order (-1), i.e. $V(ax_1, ax_2) = a^{-1}V(x_1, x_2)$ for $a \neq 0$. Throughout this thesis we denote the max-stable copula in the form of an exponent function and compare four different parametric variants for the exponent function.

3.2.3 An inverted max-stable approach

In this subsection we introduce an inverted max-stable model as the (W_1, W_2) component in the model of H&W. Assume that (X_1, X_2) follows a bivariate max-stable distribution $G(x_1, x_2)$ as in (12). Define

$$(W_1, W_2) = (\frac{1}{G_1(X_1)}, \frac{1}{G_2(X_2)}). \tag{14}$$

Then, (W_1, W_2) possesses Pareto margins and an inverted max-stable copula. Note that the copula of (W_1, W_2) is asymptotic independent. We derive the relation between the exponent function of $G(x_1, x_2)$ and the coefficient of tail independence η_W as follows. Write $W_j = \exp\{1/X_j\}$ where $(X_1, X_2) \sim G(X_1, X_2)$. Then

$$\begin{aligned}
\Pr\{W_1 > x, W_2 > x\} &= \Pr\{\exp\{1/X_1\} > x, \exp\{1/X_2\} > x\} \\
&= \Pr\{X_1 < \frac{1}{\log(x)}\}, X_2 < \frac{1}{\log(x)}\} \\
&= G(\frac{1}{\log(x)}, \frac{1}{\log(x)}) \\
&= \exp\{-V(\frac{1}{\log(x)}, \frac{1}{\log(x)})\} \\
&= \exp\{-V(1,1)\log(x)\} \\
&= x^{-V(1,1)},
\end{aligned}$$

where we use the homogeneous property of $V(x_1, x_2)$. Therefore we get that $\eta_W = 1/V(1,1)$.

Further if $0.5 < \delta < 1$, χ_H in equation (5) becomes (H&W)

$$\chi_H = \frac{2\delta - 1}{1 - (1-\delta)(1+\eta_W)}, \tag{15}$$

where η_W is the tail independence coefficient of (W_1, W_2).

We choose symmetric and asymmetric forms for the exponent function $V(x_1, x_2)$. The first inclusion is the Dirichlet distribution in Coles and Tawn (1991). The distribution allows an asymmetric dependence structure through the parameters $(\alpha, \beta) > (0, 0)$ of a

Beta-distribution (Be)

$$V(x_1, x_2;\ \theta) := \frac{1}{x_1}\{1 - Be(\frac{\alpha x_2}{\alpha x_2 + \beta x_1};\ \alpha+1, \beta)\} + \frac{1}{x_2}\{Be(\frac{\alpha x_2}{\alpha x_2 + \beta x_1};\ \alpha, \beta+1)\}. \quad (16)$$

The case of asymptotic dependence occurs when $\alpha = \beta$ both go to infinity. The case of complete independence occurs in two situations: either when $\alpha = \beta = 0$, or when $\alpha(\beta)$ is stable, while $\beta(\alpha)$ converges to zero. The corresponding coefficient of tail independence is

$$\eta_W = \{1 - \text{Be}(\frac{\alpha}{\alpha+\beta}, \alpha+1, \beta) + \text{Be}(\frac{\alpha}{\alpha+\beta}, \alpha+1, \beta)\}^{-1} \in [1/2, 1). \quad (17)$$

Tawn (1988) provides a second exponent function that allows for asymmetric dependence: the Asymmetric-Logistic distribution. The respective exponent function

$$V(x_1, x_2;\ \theta) := (1-\tau_1)\frac{1}{x_1} + (1-\tau_2)\frac{1}{x_2} + [(\frac{\tau_1}{x_1})^{\frac{1}{\tau_3}} + (\frac{\tau_2}{x_2})^{\frac{1}{\tau_3}}]^{\tau_3}, \quad (18)$$

where the asymmetry parameters $(\tau_1, \tau_2) \in [0, 1]^2$ and $\tau_3 \in (0, 1]$. If (τ_1, τ_2) are significantly different from $(1, 1)$, a symmetric structure is less likely, since there is evidence for an asymmetric dependence structure. The coefficient of tail independence is

$$\eta_W = \frac{1}{2 - \tau_1 - \tau_2 + (\tau_1^{1/\tau_3} + \tau_2^{1/\tau_3})^{\tau_3}} \in [1/2, 1). \quad (19)$$

The models is asymptotic independent when either $\tau_3 = 1$, $\tau_1 = 0$ or $\tau_2 = 0$. There is complete dependence when, at the boundary point $\tau_1 = \tau_2 = 1$, τ_3 converges to zero. When the restriction $\tau_1 = \tau_2 = 1$ holds and τ_3 does not converge to zero, the dependence structure is identical to that of a standard Logistic copula. The case with a symmetric Logistic exponent function we consider separately from the Asymmetric-logistic function, where

$$V(x_1, x_2;\ \theta) := (x_1^{-\frac{1}{\tau_4}} + x_2^{-\frac{1}{\tau_4}})^{\tau_4} \quad (20)$$

with parameter $\tau_4 \in (0, 1]$ and $\eta_W = \frac{1}{2^{\tau_4}} \in [1/2, 1)$.

4 Methodology

This section covers the procedure for risk analysis on portfolio returns with a copula model. We first filter the univariate returns $\{R_{t,j}\}$ with a heteroskedastic volatility model. Secondly, we estimate the dependence structure of the two stock markets $\{R_{t,1}, R_{t,2}\}$ and derive a test for the asymptotic (in)dependence. Here, the dependence structure is modelled by the model in Section 3. Lastly, the calculation of three VaR models for the portfolio returns $R_{t,p}$ are shown together with back-tests to evaluate their accurateness, where we define portfolio returns in period t as follows

$$R_{t,p} = \sum_{j=1}^{2} w_j R_{t,j}, \tag{21}$$

where w_j is the weight and $R_{t,j}$ is the return of financial market j in period t.

4.1 Heteroskedastic volatility filter

Assume the return serie possesses heteroskedastic volatility. We first model $\{R_{t,j}\}_{t=1}^{T}$ for each j by an Autoregressive(AR)-Glosten-Jagannathan-Runkle(GJR)-Generalized Autoregressive Conditional Heteroskedastic(GARCH) model following the model in Glosten et al. (1993). The AR-GJR-GARCH model is the simplest GARCH model that incorporates the stylized fact of asymmetric volatility in stock markets as a negative shock increases volatility to stock prices more than a positive shock does, when the sizes of the shocks are equal (leverage effect). GARCH models that allow for asymmetry in the conditional variance have shown to systematically outperform symmetric volatility GARCH models in forecasting stock movements at short horizons (Brownlees et al. (2011)). The specification of the model is as follows

$$\begin{aligned} R_{t,j} &= \phi_{1,j} + \phi_{2,j} R_{t-1,j} + \nu_{t,j} Z_{t,j}, \\ \nu_{t,j}^2 &= \phi_{3,j} + (\phi_{4,j} + \gamma_j I_{t-1}) Z_{t-1,j} + \phi_{5,j} \nu_{t-1,j}^2, \end{aligned} \tag{22}$$

$$I_{t-1} := \begin{cases} 0 \quad \text{if} \quad R_{t-1,j} \geq \phi_{1,j}, \\ 1 \quad \text{if} \quad R_{t-1,j} < \phi_{1,j}, \end{cases} \tag{23}$$

where $(\phi_{1,j}, \phi_{2,j}, \phi_{3,j}, \phi_{4,j}, \phi_{5,j}, \gamma_j)$ are its parameters and γ_j models the leverage effect. The heteroskedastic volatilities are $\{\nu_{t,j}\}_{t=1}^T$ and the residuals $\{Z_{t,j}\}_{t=1}^T$ are i.i.d. with mean zero and variance one. By performing quasi-maximum likelihood estimation(QMLE), one can estimate the quasi-maximum likelihood parameters $(\hat{\phi}_{1,j}, \hat{\phi}_{2,j}, \hat{\phi}_{3,j}, \hat{\phi}_{4,j}, \hat{\phi}_{5,j}, \hat{\gamma}_j)$, the residuals $\{\hat{Z}_{t,j}\}_{t=2}^T$ and the estimated time varying volatilities $\{\hat{\nu}_{t,j}\}_{t=2}^T$, while assuming the distribution of $\{Z_{t,j}\}_{t=1}^T$ to be incorrectly specified to a certain extent as opposed to maximum likelihood estimation(MLE) which assumes the correct distribution.

4.2 Copula estimation

Next, we model the cross-sectional dependence between the series. For that purpose, we consider using copula to model the dependence in $(Z_{t,1}, Z_{t,2})$. Since we do not model the marginal distribution of $Z_{t,j}$ parametrically, we use a non-parametric marginal transformation to transform $(Z_{t,1}, Z_{t,2})$ to the uniform marginals. This is achieved by using the empirical c.d.f.,

$$\begin{aligned} \hat{U}_{t,j} &= \hat{F}_j(\hat{z}_{t,j}), \\ &= \frac{1}{T+1} \sum_{t=1}^{T} \mathbb{1}_{\hat{Z}_{t,j} \leq z_j}. \end{aligned} \tag{24}$$

Then we use the copula of (H_1, H_2) in equation (3) to model the dependence. We fit the H&W model on $\{(\hat{U}_{t,1}, \hat{U}_{t,2})\}$. First, to avoid computation complexity, we look at a logarithmic transformation of the model. Note that maximizing a likelihood function yields the same optimal parameters as maximizing the logarithm of that likelihood, since it does not affect the location of the maxima. Secondly, the estimation depends on the value of the marginal thresholds (u_1^*, u_2^*). By using a censored likelihood to estimate the parameters for the copula model, values below the threshold (u_1^*, u_2^*) do not contribute to the maximum likelihood optimalization.

The maximization problem becomes

$$\ell(\psi) = \sum_{t=1}^{T} \log(L_t(\psi)), \tag{25}$$

where $L_t(\psi)$ depends on whether a pair of observation exceeds a marginal thresholds and $\psi = (\delta, \theta)$, where θ are the parameters of the exponent function for (W_1, W_2). Let $y_t = \{j : \hat{U}_{t,j} > u_j^*\} \subseteq \{1, 2\}$ describe the set of individual markets that exceed their marginal thresholds at time $t \in \{1, ..., T\}$. Then, the contribution to the censored likelihood for observation t,

$$L_t(\psi) = \begin{cases} C_H(u_1^*, u_2^*;\ \psi) & y_t = \emptyset, \\ c_H(\hat{U}_{t,1}, \hat{U}_{t,2};\ \psi) & y_t = \{1, 2\}, \\ C_H^y(\max(\hat{U}_{t,1}, u_1^*), \max(\hat{U}_{t,2}, u_2^*);\ \psi) & y_t = \{1\} \cup y_t = \{2\}. \end{cases} \tag{26}$$

The first case in equation (26) corresponds to no exceedances, the second case when both observations are exceeding the threshold and the last case occurs if only one market exceeds a marginal threshold. The definitions for the the copula distribution function C_H, density function c_H and partial derivative of the copula distribution function C_H^y are in H&W. The parameters from the maximum likelihood estimation are $\hat{\psi} = (\hat{\delta}, \hat{\theta})$, where $\hat{\delta}$ is the asymptotic dependence parameter and $\hat{\theta}$ are the parameters of (W_1, W_2).

In H&W, it is argued that the estimator $\hat{\psi}$ converges to the true value ψ with asymptotic normality given as follows: as the number of exceedances $n \to \infty$,

$$\sqrt{n}(\hat{\psi} - \psi) \xrightarrow{d} N(0, \Sigma_\psi). \tag{27}$$

The criteria to which we compare the different models for (W_1, W_2) is by comparing AIC scores. A lower AIC score indicates a better fit of a model, relative to other models.

To classify the dependence class as a characteristic of each pair in asymptotic dependence and asymptotic independence, we perform a test on $\hat{\delta}$. Depending on the value of

$\hat{\delta}$ one tests the hypothesis

$$H_0 : \delta > 0.5 \text{ (Asympt. Dep.)} \quad \text{vs.} \quad H_a : \delta \leq 0.5 \text{ (Asympt. Indep.)}$$
$$H_0 : \delta < 0.5 \text{ (Asympt. Indep.)} \quad \text{vs.} \quad H_a : \delta \geq 0.5 \text{ (Asympt. Dep.)}$$

where we assume asymptotic normality of $\hat{\delta}$ which should hold true for some conditions (H&W).

4.3 Value-at-Risk

4.3.1 Definition and estimation procedure

The VaR states, for a given probability, the risk to lose at least the specified amount over a certain time period. Mathematically speaking, the VaR is the minimum amount k that would mitigate a negative outcome at a fixed probability level $(1-q)$,

$$VaR_q := \inf\{k \in \mathbb{R} : F^{-1}(k) \geq q\}, \tag{28}$$

where F^{-1} is the quantile function of the distribution of portfolio returns and $q \in (0, 1)$. Note that this study restricts itself to only study the risk measure for a portfolio with long positions who risk a decline in value. We compare three different approaches to estimate the VaR: the variance-covariance method, the historical method and a method based on Monte Carlo simulation.

The variance-covariance method assumes the portfolio returns $R_{p,t}$ are i.i.d. and follow a normal distribution. The variance of the portfolio returns σ_p^2 is equal to

$$\sigma_p^2 = W^T \Sigma W, \tag{29}$$

where W is the weight vector and Σ a covariance matrix. Then the risk measure is

$$VaR_q = \mu_p + \sigma_p \Phi_q^{-1} \tag{30}$$

with the unconditional mean $\mu_p = \mathbb{E}[R_p]$ and Φ_q^{-1} the quantile function of the Normal distribution.

The historical method is a non-parametric estimation of the unconditional VaR. In this method the portfolio returns from a sample are sorted dependent on the size and the q quantile of the sample is directly the VaR for the next period.

For our H&W copula approach, we use Monte Carlo simulation to obtain estimates for the VaR. Here, the procedure to estimate the VaR of a portfolio consists out of four steps. First, we simulate l values from a copula C_H with parameters $\hat{\psi}$ to obtain a sufficient amount of pairs $(\hat{U}^{(l)}_{T+1,1}, \hat{U}^{(l)}_{T+1,2})$, $l = 1, 2, ..., m$. Secondly, through the inverse of the empirical cumulative distribution function in equation (24) we transform the values of the simulation into simulated residuals

$$(\hat{Z}^{(l)}_1, \hat{Z}^{(l)}_2) = (\hat{F}^{-1}_1(\hat{U}^{(l)}_{T+1,1}), \hat{F}^{-1}_2(\hat{U}^{(l)}_{T+1,2})). \tag{31}$$

Thirdly, to coincise the pair $(\hat{Z}^{(l)}_1, \hat{Z}^{(l)}_2)$ with the heteroskedastic volatility, we use the parameters from the AR-GJR-GARCH(1,1) model in equation (22). Finally, to estimate $(\hat{R}^{(l)}_{T+1,1}, \hat{R}^{(l)}_{T+1,2})$, we predict the 1-step ahead volatility

$$\hat{\nu}^2_{T+1,j} = \hat{\phi}_{3,j} + (\hat{\phi}_{4,j} + \frac{\hat{\gamma}}{2} + \hat{\phi}_{5,j})\nu^2_{T,j} \tag{32}$$

and forecast the returns as follows

$$\hat{R}^{(l)}_{T+1,j} = \hat{\phi}_{1,j} + \hat{\phi}_{2,j} R_{T,j} + \hat{\nu}_{T+1,j} \hat{Z}^{(l)}_j. \tag{33}$$

Then the VaR_q is the q quantile of the l simulated portfolio returns $\hat{R}^{(l)}_{T+1,p} = \sum_{j=1}^{2} w_j \hat{R}^{(l)}_{T+1,j}$. If the H&W copula model is a good fit for the data generating process, we expect that the Monte Carlo method based on the copula model should overperform the benchmarks that are the variance-covariance and the historical method. Through backtesting the risk estimates are compared.

4.3.2 Backtesting

To study which VaR model is most adequate, we test the violations of the VaR estimates in two parts, where a violation refers to a portfolio return exceeding the VaR in that period.

First we test the unconditional coverage which examines whether the number of violations are in proportion to the sample size. The second test concerns the independence of the violations, where if the clustering of violations is too great we deem the model inaccurate.

The proportions of failures (PoF) test in Kupiec (1995) evaluates the unconditional coverage of the violations. Under the null hypothesis that VaR is properly forecasted, the number of violations n in a sample of $\tilde{T}$ observations follow a Bernoulli distribution with the probability of a violation p. When the observed rate $\hat{p}$ differs significantly of the expected rate p, the null hypothesis will be rejected and the VaR model found inaccurate. The test statistic used is as follows (Kupiec (1995))

$$LR_{PoF} = -2\ln\Big(\frac{(1-p)^{\tilde{T}-n}p^n}{[1-(\hat{p})]^{\tilde{T}-n}(\hat{p})^n}\Big) \sim \chi^2(1). \tag{34}$$

We reject the null hypothesis if the LR_{PoF}-statistic exceeds the 95% critical value of the $\chi^2(1)$ distribution.

Christoffersen (1998) introduces a test on the serial dependence of the violations. When a VaR model is adequate, a violation in a period should not depend on whether there was a violation the period before. Therefore, the probability that a violation occurs in the current period should be equal, independent of previous period. Define a indicator variable that depends on the occurrence of a violation

$$I_t = \begin{cases} 1 & \text{if} \quad R_t < VaR_t \\ 0 & \text{if} \quad R_t \geq VaR_t \end{cases}$$

Furthermore, let $n_{uv} = \sum_{t=T+1}^{T+\tilde{T}}(I_t = v|I_{t-1} = u)$, where $u, v \in \{0, 1\}$. Then, we split the whole period $t = T+1, .., T+\tilde{T}$ into four scenarios. Table 1 provides an overview of the seperate scenarios for n_{uv}.

	$I_{t-1} = 0$	$I_{t-1} = 1$	
$I_t = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$I_t = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	$\tilde{T}$

Table 1: Contingency table with $n_{uv} = \sum_{t=T+1}^{T+\tilde{T}}(I_t = v|I_{t-1} = u)$.

In addition, let $\pi_i = \Pr(I_t = 1|I_{t-1} = i)$ and $\pi = \Pr(I_t = 1)$. Then they can be estimated by

$$\hat{\pi}_0 = \frac{n_{01}}{n_{00} + n_{01}}, \quad \hat{\pi}_1 = \frac{n_{11}}{n_{10} + n_{11}} \quad \text{and} \quad \hat{\pi} = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}. \tag{35}$$

The null hypothesis states that last period should not affect the probability of a violation in the current period, i.e. $H_0 : \pi_0 = \pi_1$. The test statistic that is used to test the null hypothesis is

$$LR_{IND} = -2\ln\Big(\frac{(1-\hat{\pi})^{n_{00}+n_{10}}\hat{\pi}^{n_{01}+n_{11}}}{(1-\hat{\pi}_0)^{n_{00}}\hat{\pi}_0^{n_{01}}(1-\hat{\pi}_1)^{n_{10}}\hat{\pi}_1^{n_{11}}}\Big) \sim \chi^2(1). \tag{36}$$

We reject the null hypothesis if the LR_{IND}-statistic exceeds the 95% critical value of the $\chi^2(1)$ distribution.

5 Simulation

In this part we evaluate the maximum likelihood estimation of the parameters $\psi = \{\delta, \theta\}$ of the H&W model in a simulation study. The study is split into two parts. The first part investigates the estimation of the asymptotic dependence parameter δ and the second part investigates the estimation of the remaining parameter(s) θ.

In order to evaluate the estimation performance of each parameter in ψ, we simulate 1000 independent pairs (H_1, H_2) and estimate the parameter of interest by the censored maximum likelihood method. We treat all remaining parameters in ψ as known. The entire procedure is repeated 100 times for each specification of (W_1, W_2) at each scenario $\delta \in [0.1, ..., 0.9]$ (from asymptotic independence to asymptotic dependence). Throughout this section the choice of θ for a Gaussian/Dirichlet/Logistic/Asymmetric-Logistic specification corresponds to $\rho = 0.4/(\alpha, \beta) = (0.25, 0.6)/\tau_4 = 0.4/(\tau_1, \tau_2, \tau_3) = (0.25, 0.6, 0.4)$ respectively.

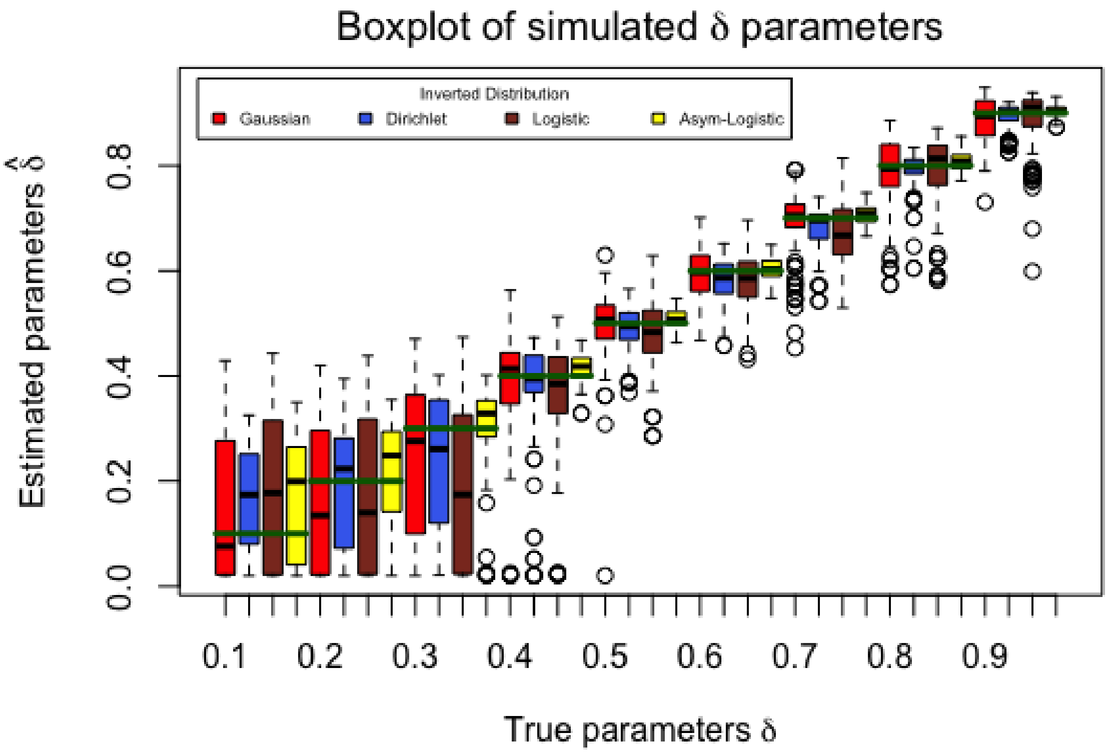

Figure 1: Box plot comparing the true δ as opposed to the estimated $\hat{\delta}$. The green line represents the true value δ. The dotted lines represent the 95% confidence interval. The copula C_H is fitted 100 times for each of the four specifications of (W_1, W_2) at each scenario $\delta \in [0.1, ..., 0.9]$.

Figure 1 displays the variability and bias of the estimates $(\hat{\delta})$ of δ. As $\delta \in [0.1, 0.2, 0.3]$ the copula structure of the copula model (H_1, H_2) closely follows the copula structure of (W_1, W_2). According to the definition of η_H in equation (6), η_H is constant for $\delta \in [0.1, 0.2, 0.3]$ since $\eta_W > 0.5$ in all four models. Therefore low values of δ yield a similar dependence structure for each model. In addition, there exists large variability in the estimation for the scenario's $\delta < 0.5$. However, when the underlying data generating process is asymptotically dependent, the estimation of δ becomes more accurate for higher levels of asymptotic dependence. The interpretation is that δ is identified in χ_H for any δ, but not necessarily in η_H when the data generating process is asymptotic independent. Furthermore, the assumption of normality of $\hat{\delta}$ appears unreasonable when the true parameter δ lies close to the boundaries 0 and 1. The four specifications show a slight downward bias, with the exception of the Asymmetric-Logistic specification that exhibits an upward bias.

The second part investigates the estimation of θ for each specification of (W_1, W_2), which assumes δ as known. If θ is a vector of parameters, only the parameter of interest is estimated, while the remaining parameter(s) are also known.

Figure 2 shows the estimation of ρ is for the copula model with a Gaussian specification. The estimator $\hat{\rho}$ is reliable in estimating the true value ρ. Variability increases when the data generating process exhibits more asymptotic dependence. The estimator $\hat{\rho}$ performs well for $\delta \leq 0.5$, where the model closely follows the structure of (W_1, W_2), indicating that the assumption of normality for $\hat{\rho}$ may only hold for $\delta \leq 0.5$.

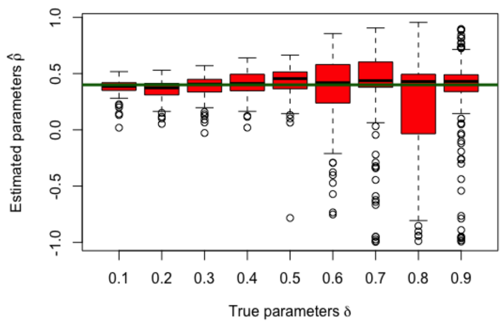

Figure 2: Box plot evaluating the estimation of ρ. The green line represents the true value ρ. The dotted lines represent the 95% confidence interval. The copula C_H is fitted 100 times on a simulated sample of 1000 observations at each scenario $\delta \in [0.1, ..., 0.9]$.

Figure 3 display the results of estimating the copula model with a Dirichlet specification for (W_1, W_2). The estimated parameters $\hat{\alpha}$ and $\hat{\beta}$ are both around the respective true values $\alpha = 0.25$ and $\beta = 0.6$. However, we observe a large number of outliers for both parameters at any $\delta \in [0.1, ..., 0.9]$. One possibility is that the identification problem stems from the increase in the number of parameters.

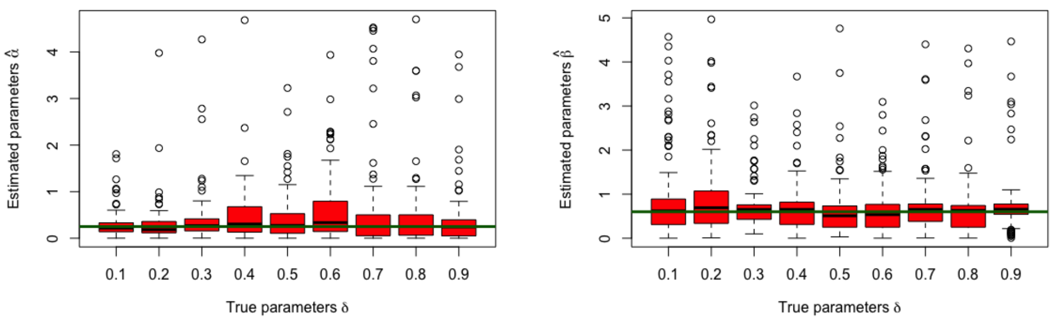

Figure 3: Box plot evaluating the estimation of α/β with fixed parameter $\beta = 0.6/\alpha = 0.25$. The green line represents the true value α/β. The dotted lines represent the 95% confidence interval. The copula C_H is fitted 100 times on a simulated sample of 1000 observations at each scenario $\delta \in [0.1, ..., 0.9]$.

This simulation study concludes with Figure 4, which shows the results of the simulation for a Logistic and Asymmetric-Logistic specification. The estimated Asymmetric-Logistic parameters $\hat{\tau}_1, \hat{\tau}_2$ & $\hat{\tau}_3$ contain bias and large variability. Although the estimation of the parameter δ is accurate, the estimation of remaining parameters τ_1, τ_2 & τ_3 is inaccurate. The fourth panel presents the results for a simulation with a Logistic specification (τ_4) and in general, the estimator $\hat{\tau}_4$ is accurate for $\delta \leq 0.5$, however there appears a strong increase in the variability for $\delta > 0.5$. A possibility is that the assumption of normality may only hold for $\hat{\tau}_4 < 0.5$. This evidence is in agreement with our findings with a Gaussian specification.

In summary, the simulation study indicates that the H&W model contains large variability estimating δ when $\delta < 0.5$ or when δ is close to the boundaries 0 and 1. The estimation of θ shows larger variability in the case for Gaussian and Logistic variants at $\delta \geq 0.5$. The Asymmetric-Logistic variant is inaccurate when estimating θ for any $\delta \in [0.1, ..., 0.9]$ and the estimator $\hat{\theta}$ in the Dirichlet variant is unbiased, but provides large outliers.

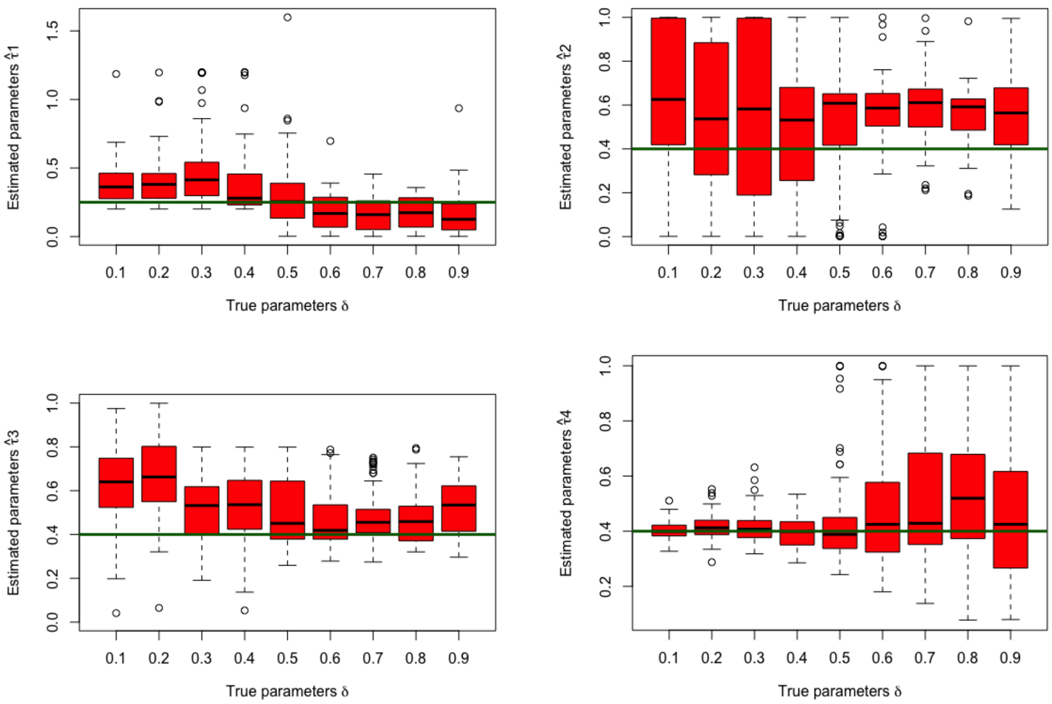

Figure 4: Box plot evaluating the estimation of $\tau_1/\tau_2/\tau_3/\tau_4$. The green line represents the true value. The dotted lines represent the 95% confidence interval. The copula C_H is fitted 100 times on a simulated sample of 1000 observations at each scenario $\delta \in [0.1, ..., 0.9]$.

6 Data and preliminary analysis

A strong indicator of the performance in financial markets is through indices returns. In this thesis daily index data of the largest index of a country from the Wharton Research Data Services (WRDS) database are used. These 11 countries are the United States of America, the Netherlands, Spain, South Africa, France, Germany, Japan, Australia, United Kingdom, Norway and Denmark. The data ranges from January 1992 - May 2018. A period with a missing observation in a country is deleted from the sample. Due to this elimination there remain 5721 observations for each index. Table 2 contains the summary statistics of the standard returns in percentages.

	Mean	Std. Dev.	Minimum	Skewness	Kurtosis	LB-stat	P-val
USA	0.02	1.13	-9.47	-0.49	11.18	42.9	0.00
Netherlands	-0.01	1.33	-9.59	-0.27	9.04	69.9	0.00
Spain	0.01	1.43	-13.39	-0.15	8.87	42.0	0.00
South Africa	0.03	1.20	-13.66	-0.56	9.42	48.7	0.00
France	-0.00	1.32	-9.42	-0.17	7.55	58.4	0.00
Germany	0.00	1.37	-8.20	-0.25	7.06	30.7	0.00
Japan	-0.01	1.37	-10.3	-0.18	8.71	16.5	0.08
Australia	0.02	0.97	-8.76	-0.44	8.53	22.2	0.01
United Kingdom	-0.00	1.01	-9.21	-0.25	8.80	47.9	0.00
Norway	0.02	1.43	-11.04	-0.48	10.55	33.3	0.00
Denmark	0.02	1.18	-11.61	-0.41	8.25	38.4	0.00

Table 2: Summary statistics of daily return percentages from 11 countries in the range of January 1988 - May 2018. P-values are estimated with 1 degree of freedom.

The data exhibits negative skewness and large kurtosis, typical for financial assets. A Ljung-Box test measures the amount of autocorrelation in a sample, where we reject the sample to be i.i.d. when P-val < 0.05. In our sample, there appears a large amount of autocorrelation in the data (only for Japan we do not reject the daily returns to be i.i.d.), signalling that the returns might not be i.i.d. due to the presence of heteroskedastic volatility. As a result the asymptotic dependence could be overestimated.

To filter the heteroskedastic volatility, we implement an AR-GJR-GARCH(1,1) model on the daily losses defined as the negative of the daily returns. The results of the Ljung-Box tests and a graphical representation are in Appendix A. For all countries, the residuals of an AR-GJR-GARCH(1,1) model show no significant autocorrelation in the first ten lags.

We transform the loss residuals to the Uniform scale of which we show three examples in Figure 5. The red and blue dots represent the top 10% extreme losses of the individual stock markets, whereas purple dots represent the scenario where both markets co-exceed their top 10% threshold. The France - United States of America panel shows a higher concentration of observations in the upper right corner. Therefore, the probability that an extreme loss occurs in the United States of America, given an extreme loss in France, is relatively high in comparison to the other two panels. Consequently the relation appears more strongly asymptotic dependent than the other two. The probability decreases in the case of Norway - France and in the case of Australia - United States of America. Especially the latter appears strongly asymptotic independent as a large portion of the extremes are scattered outside the upper right corner. In addition, we observe that the extremes are not necessarily spread symmetrically, suggesting an asymmetric structure in our model could approximate the dependence structure of the extremes more accurately.

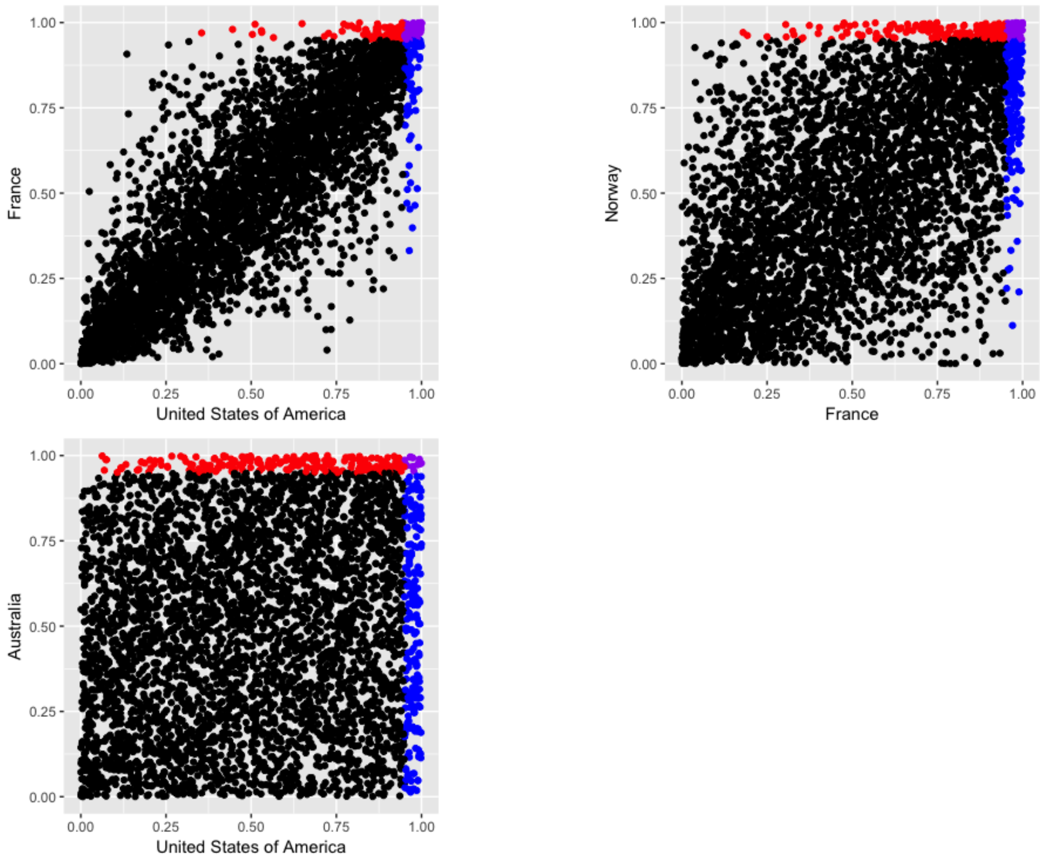

Figure 5: The AR-GJR-GARCH(1,1) loss residuals on the Uniform scale for pairs France-United States of America, Norway-France and United States of America-Australia. Red and blue dots represent the top 10% observations where an individual country exceeds its marginal threshold $u^* = 0.9$, purple dots where both countries exceed their marginal threshold at the same day.

7 Application

In this part we fit the model of H&W to pairs of residuals from the AR-GJR-GARCH(1,1) model in Section 6. Then, we select two specifications for the H&W models out of four possible candidates. We base our selection on which model achieves the lowest AIC-scores and through comparing the parametric and non-parametric estimate of the dependence measure χ. Furthermore, the asymptotic dependence structure of the residual pairs are summarized in a network analysis, differentiating between asymptotic dependent and asymptotic independent pairs. We construct portfolios of two indices that classify as either an asymptotic dependent or an asymptotic independent portfolio based on this network analysis. Thereafter, we determine the risk estimates of the portfolios from two benchmark VaR models and one that incorporates the H&W model. Lastly, the risk estimates are compared via backtesting.

7.1 Model selection

The censored likelihood estimation of the H&W model is performed for all 55 unique pairs of residuals, after filtering the loss returns by a AR-GJR-GARCH(1,1) model. The marginal thresholds for the censored likelihood were set at $(u_1^*, u_2^*) = (0.95, 0.95)$ in all cases for consistency, which also agrees with H&W. A summary of the model selection is in Table 3, while the individual AIC-scores are in the Appendix.

The first observation is that the model with an Asymmetric-Logistic specification for (W_1, W_2) has a relatively high AIC-score in all cases. In addition, the numerical optimization often can not evaluate the integrals in the censored likelihood estimation to derive the standard errors. The absence of standard errors means there is no method to test the asymptotic dependence. The model with a Gaussian specification achieves the lowest AIC-score in the majority of time and appears to be a good fit for our research. The model with a symmetric Logistic specification and asymmetric Dirichlet specification both perform modestly, each achieving the lowest AIC-score in a fifth of the cases.

Specification (W_1, W_2)	Gaussian	Dirichlet	Logistic	Asymmetric-Logistic	Total
Optimal AIC	33	10	12	0	55

Table 3: Summary table of the number of times a model achieves the lowest AIC-score. The marginal thresholds chosen for the censored likelihood estimation are $(u_1^*, u_2^*) = (0.95, 0.95)$.

7.2 Goodness of fit

Here, we complement Section 7.1 by performing a goodness-of-fit analysis on the parametric and the empirical estimate of the dependence measure χ for the different specifications.

Figure 6 shows the the empirical and parametric estimates of χ for three examples. The dashed lines represent a 95% bootstrap confidence interval that measures the uncertainty. The interval is based on 200 bootstrap samples and the individual dots are the empirical estimates of χ at varying thresholds u. The colored lines represent the parametric estimates of χ at varying thresholds u.

Each panel hints at asymptotic independence as $\chi \to 0$ for $u \to 1$. At the boundary $u \to 1$ the uncertainty increases, shown by the increasing size of the 95% confidence interval. In addition, although the copula model only directly models the dependence structure of the top 5% extreme losses, it still appears as a good fit for the top 10% extreme losses.

In the France - Norway panel, the parametric χ of the model with an Asymmetric-Logistic specification lies outside the 95% bootstrap interval. In general, such a model tends to overestimate δ and is inferior to other models in this study. One reason could be that the uncertainty in the additional parameters cause the maximum likelihood estimation to converge to a sub-optimal solution. Therefore we conclude, also evident by sub-optimal AIC-scores and varying parameter estimates in our simulation study, that the model with an Asymmetric-Logistic specification is not suitable for our data. The Logistic and Dirichlet specifications closely resemble each other, however the Logistic variant estimates δ lower relative to the other models. The advantage of the Dirichlet specification is its flexibility to model asymmetry in our data. Therefore, we select the copula model with a symmetric Gaussian or an asymmetric Dirichlet specification and omit the model with a Logistic and Asymmetric-Logistic specification in further analysis. Through this selection, we reduce the computational time and have a robust representation of the model with and without asymmetry.

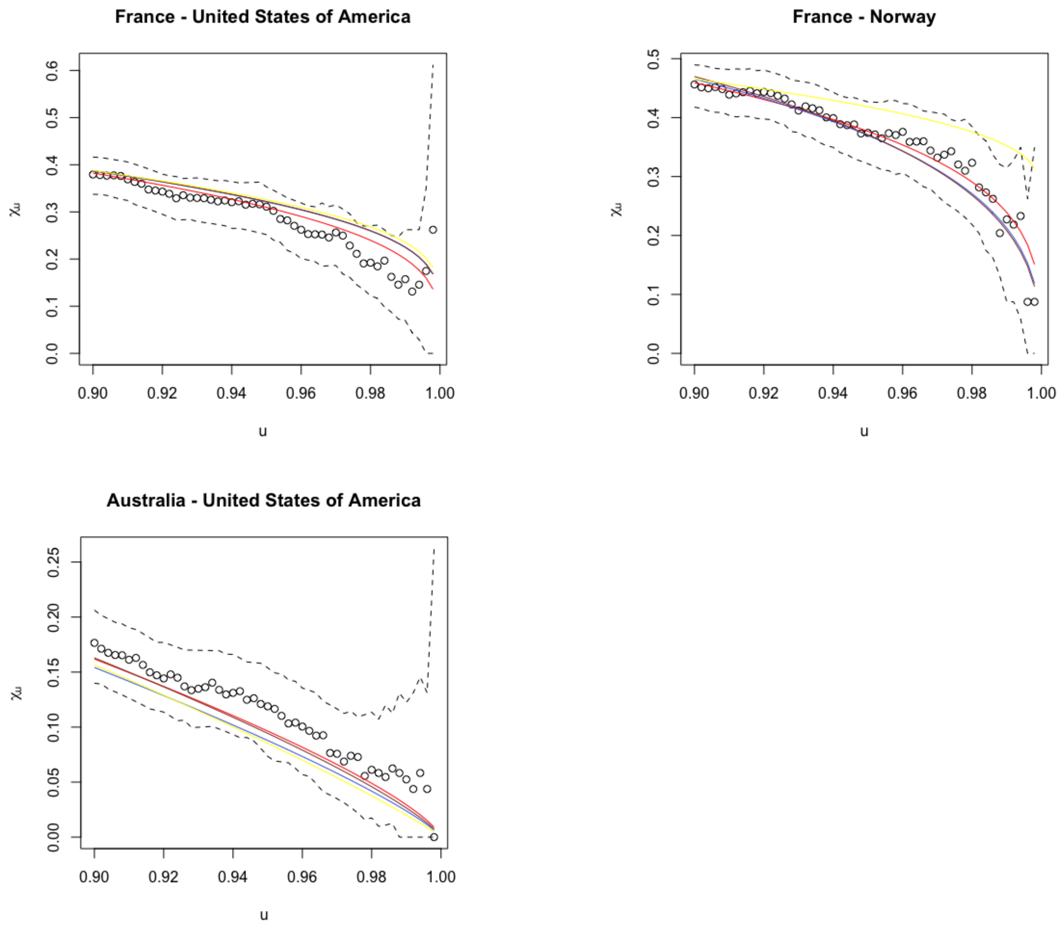

Figure 6: Three examples of the parametric and empirical estimate of χ. The dots are the empirical estimates of χ. The different colored lines represent estimates of χ according to the four different models. The yellow/brown/blue/red line represents the model using the Asymmetric-Logistic/Logistic/Dirichlet/Gaussian specification.

7.3 Asymptotic (in)dependence analysis

Here, the asymptotic relation in global stock markets is established by estimating the dependence parameter δ. The estimated asymptotic dependence parameters $\hat{\delta}$ of the copula model with an asymmetric(symmetric) specification are in Table 9(10) of the Appendix. The classification of asymptotic dependent and asymptotic independent pairs is on the basis that $\hat{\delta}$ approximately follows a Normal distribution. As the differences between the copula model with a Gaussian and Dirichlet specification are small, we only present the figures for the Gaussian variant here and refer to the Dirichlet wherever necessary.

Figure 7 shows the asymptotic strength of dependent stock markets by comparing the dependence measure χ_H. The European countries Spain, Netherlands, France and Germany contain asymptotic dependent structures for the losses. The strongest relations exist between the northern Europe countries France, Germany and the Netherlands, which is shown by the thickness of the line. The asymptotic dependence relation with the northern European countries and Spain is slightly weaker and in the case of a Dirichlet specification the relation is significantly asymptotic independent. For the cases that our test can not reject asymptotic dependence ($\delta > 0.5$) the lines are dotted. The United Kingdom has a weak and insignificant asymptotic dependent relation with the rest of the northern European countries, whereas in the Dirichlet variant it is asymptotic dependent with Netherlands and France. Other cases where we do not reject the relation to be asymptotic dependent are Norway - Germany and Australia - Japan.

In conclusion, the class of asymptotic dependent pairs of countries is primarily reserved for stock markets within the European Union (with the exception of Australia - Japan). Due to the integration of stock markets in the European Union, they produce large spillover effects when extreme losses occur, relative to more geographically distant stock markets.

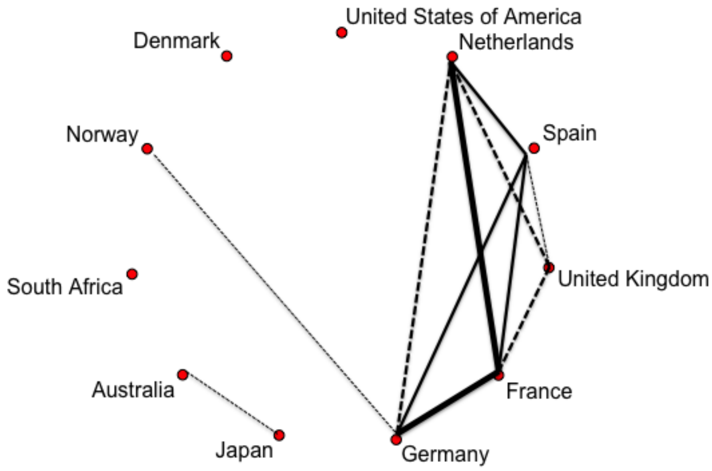

Figure 7: The countries that exhibit asymptotic dependence according to the model with a Gaussian specification. The solid lines represent connections with a $\delta > 0.5$ whereas the dotted lines represent the cases where we cannot reject $\delta > 0.5$. The thickness indicates the value of the dependence measure χ_H, where a thicker line equates to a higher value.

Figure 8 captures the asymptotic independent relations when δ is significantly lower than 0.5. Here, the color of the line indicates the strength of the asymptotic independence measure η_H. Even though Norway and Denmark are not asymptotically dependent with the rest of the European countries, the asymptotic independence measure is relatively high. The asymptotic independence measure is low for pairs that include the United States of America and Australia. A possible reason could be due to the measurement error of different time zones of the daily observations. Surprisingly, the asymptotic independence measures between South Africa and nearly all other countries are high, whereas for Japan it is only strong for combinations with countries in the European Union.

Based on these findings, we divide pairs into two portfolio classes for future analysis using equal weights. The pairs in Figure 7 represent the portfolios with asymptotically dependent countries and the pairs in Figure 8 represent the portfolios with asymptotically independent countries.

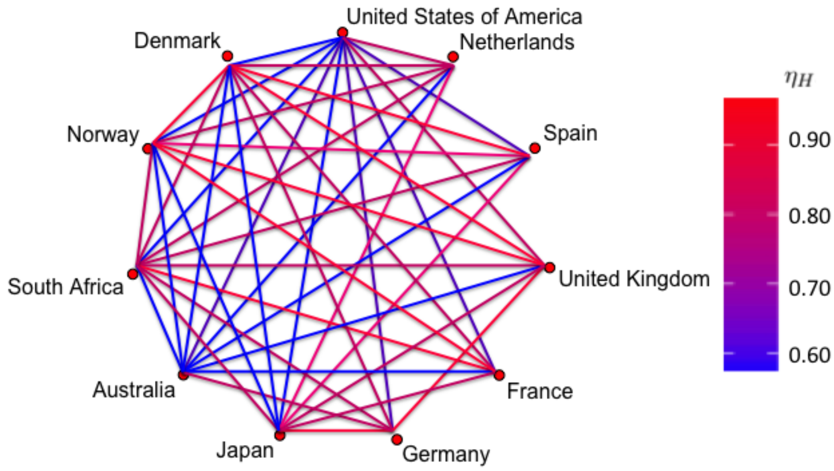

Figure 8: The countries that exhibit asymptotic dependence according to the model with a Gaussian specification. The lines represent pairs with a $\delta < 0.5$. The color indicates the value of the independence measure η_H.

7.4 Risk analysis

7.4.1 Estimation

In this section we compare the general differences between four risk models through an example by visually inspecting 1-day ahead VaR estimates.

Here, the two benchmarks to beat are a Historical and a Variance-Covariance model. The VaR estimates of the Monte Carlo model, which incorporates the H&W copula model, contains two variants: either with a Gaussian or with a Dirichlet specification for (W_1, W_2). The rolling window contains 3000 observation to enable convergence of the censored likelihood. We re-estimate the copula models yearly w.r.t. $\Psi = \{\theta, \delta\}$ and fix δ throughout the year, while re-estimating the remaining parameters θ monthly. When the integral in the copula density function can not be evaluated through numerical optimization it is a sign of no convergence in the parameter estimates. In that case, the old estimates in last month were set to be the new estimates. Throughout our analysis, the longest period for which this happened is 6 months. The out-of-sample estimation starts at the year 2007, just before the financial crisis, containing 2571 days. The number of simulations $m = 10000$. For the marginals the AR-GJR-GARCH(1,1) model is re-estimated monthly to forecast the volatilities, while keeping the computational time to a reasonable degree. Four different levels of α for the VaR are considered, namely $\alpha \in [0.10, 0.05, 0.025, 0.01]$.

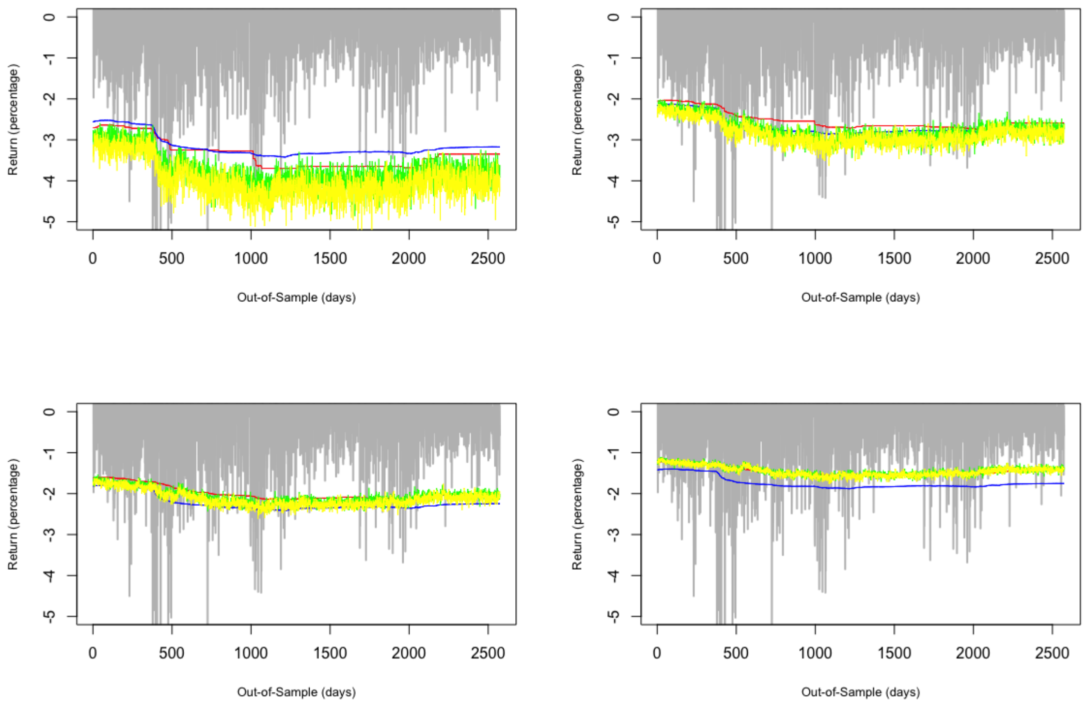

Figure 9: An example of the estimation of different levels of the VaR series for the models shown simultaneously. The example portrays a portfolio of France and the USA with weights $(w_1, w_2) = (0.5, 0.5)$. The yellow/green/blue/red line represents the value at risk measure using the MC-Dirichlet/MC-Gaussian/Var-Cov/Historical method.

In Figure 9 we present an example to illustrate the VaR estimates for an equally-weighted portfolio consisting of the largest stock market in France and the USA. The most recent financial crisis is visible at the start, where irrespective of the risk model a cluster of violations occur. Both the benchmark models closely follow each other, where the Historical model is more conservative at $\alpha = 0.01$ and the Variance-Covariance model at $\alpha = 0.10$. The difference in the two models is due to the data not perfectly following a Normal distribution. Relative difference between the risk measures from the MC-Gaussian and MC-Dirichlet is not as clear cut. However, the risk estimates of MC-

Dirichlet varies more over time.

Furthermore, analyzing the differences between the benchmark and the copula risk models we observe that the benchmark models are less erratic, whereas the Monte Carlo method varies more as it is based on simulations and involves the forecast for volatility. When $\alpha = 0.01$ the MC-Gaussian and MC-Dirichlet are in general more conservative relative to the benchmark models. Note that for the MC-Dirichlet and MC-Gaussian the copula model only models the extremes above the marginal thresholds $(u_1^*, u_2^*) = (0.95, 0.95)$ and therefore partly models the dependence structure of the extremes in case of $\alpha = 0.10$. However, H&W notes that, even though the marginal threshold is set at $(u_1^*, u_2^*) = (0.95, 0.95)$, the copula model still reasonably fits the data. This agrees with our findings in Section 7.2. In case of $\alpha \in [0.025, 0.05]$ the four VaR models appear similar in the example, especially when $\alpha = 0.05$, with small deviations. For $\alpha = 0.10$ the Variance-Covariance model produces significantly lower VaR estimates, relative to the other three models.

Whether the relative relations of the risk models here hold for all portfolios is not apparent by the visual example of one VaR series alone. To evaluate the performance in depth and summarize the performance of our risk models, we move towards backtesting the VaR estimates of the entire set of portfolios.

7.4.2 Backtesting

In this section two tests determine the performance of the risk models: the unconditional coverage test in Kupiec (1995) (PoF) and the independence test of Christoffersen (1998) (IND). Backtesting the risk models is on the entire out-of-sample dataset, consisting out of 2571 days. Table 4 summarizes the backtest results for 55 portfolios. Here, PoF indicates the number of portfolios where we do not reject the number of violations to be proportionate to the sample size. In addition, IND indicates the number of portfolios where we do not reject that the violations are independently distributed.

The risk estimates from the Variance-Covariance model is the best representation for VaR at $\alpha = 0.025$. The assumption of normality for the distribution of portfolio returns is inaccurate at $\alpha = 0.10$, as the frequency of violations deviates too large from what is to be expected. A non-parametric Historical VaR performs excellent as both tests indicate it performs the best, relative to the other models, for $\alpha \in [0.025, 0.05, 0.10]$.

In the case of a parametric risk models that incorporates the H&W copula model, the Monte Carlo with a Gaussian specification performs slightly better for both tests. This is in line with the result that this model approximates the majority of the portfolios best. For $\alpha = 0.01$ it performs slightly better than the Historical VaR model. The case with a Dirichlet specification scores slightly lower, indicating the addition of asymmetry in general does not increase the accuracy as much for a VaR model in our data relative to a symmetric Gaussian specification.

$\alpha = 0.01$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.008	0.002	42	32
MC-Dirichlet	0.007	0.002	35	26
Var-Covariance	0.014	0.014	26	21
Historical	0.012	0.003	39	31

$\alpha = 0.025$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.019	0.005	27	13
MC-Dirichlet	0.019	0.005	22	13
Var-Covariance	0.024	0.004	50	24
Historical	0.028	0.004	39	19

$\alpha = 0.05$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.041	0.009	21	8
MC-Dirichlet	0.041	0.010	17	7
Var-Covariance	0.040	0.005	22	4
Historical	0.054	0.006	40	13

$\alpha = 0.10$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.087	0.012	16	4
MC-Dirichlet	0.087	0.013	16	6
Var-Covariance	0.071	0.007	0	0
Historical	0.102	0.007	50	21

Table 4: Summary table of the results for backtesting VaR estimates on 55 portfolios. f represents the frequency of violations and s_f the respective standard deviation across the selection of 55 portfolios.

Whether the separate risk models perform differently depending on asymptotic dependent or asymptotic independent portfolios, we examine the risk models in the portfolios of countries based on the differentiation we made in Section 7.3. A summary of the test results for the asymptotic dependent portfolios is in Table 5. Results show that the MC-Dirichlet method displays similair results as MC-Gaussian for asymptotic dependent portfolios. However, the Historical method is superior in comparison with both models. The Variance-Covariance method performs relatively good at $\alpha \in [0.01, 0.025]$ and under performs at $\alpha \in [0.05, 0.10]$.

$\alpha = 0.01$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.007	0.002	6	5
MC-Dirichlet	0.007	0.002	5	6
Var-Covariance	0.013	0.013	7	8
Historical	0.010	0.003	9	10

$\alpha = 0.025$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.018	0.004	4	2
MC-Dirichlet	0.019	0.005	4	5
Var-Covariance	0.024	0.003	10	8
Historical	0.026	0.004	9	8

$\alpha = 0.05$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.040	0.006	4	3
MC-Dirichlet	0.042	0.006	4	4
Var-Covariance	0.038	0.005	3	1
Historical	0.051	0.005	10	6

$\alpha = 0.10$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.084	0.006	3	1
MC-Dirichlet	0.089	0.007	5	4
Var-Covariance	0.070	0.008	0	0
Historical	0.099	0.006	11	9

Table 5: Summary table of the results for backtesting VaR estimates on 11 asymptotic dependent portfolios in line with Section 7.3. f represents the frequency of violations and s_f the respective standard deviation across the selection of 11 portfolios.

Finally, the results of asymptotic independent portfolios are in Table 6. The risk estimates from the MC-Gaussian method performs better than all other risk models in this study at $\alpha = 0.01$. Therefore, the inclusion of the H&W model for asymptotic independent portfolios leads to more accurate risk estimates at the most extreme quantile. In all other cases the Historical benchmark method performs the best.

In summary, we find that a Monte Carlo simulation in combination with the copula model in H&W with either a Gaussian or Dirichlet specification under performs in comparison with a Historical method in almost every case, with the exception of asymptotic independent portfolios at $\alpha = 0.01$. As regulators and investors are interested in the most extreme cases, our study provides an additional tool to model risk accurately for the most extreme case. At $\alpha \in [0.05, 0.10]$ the copula based risk models perform badly, which indicates that even though the 10% extremes appear to follow the model when estimated for the top 5% extremes, it does not help for estimating the VaR. The Variance-Covariance method gets outperformed with our model at $\alpha \in [0.01, 0.10]$ and outperforms our model at $\alpha = [0.025]$.

$\alpha = 0.01$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.008	0.002	36	27
MC-Dirichlet	0.008	0.002	30	19
Var-Covariance	0.014	0.014	18	12
Historical	0.012	0.002	29	20

$\alpha = 0.025$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.020	0.006	23	11
MC-Dirichlet	0.019	0.006	18	8
Var-Covariance	0.024	0.004	39	16
Historical	0.029	0.004	29	11

$\alpha = 0.05$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.042	0.010	17	5
MC-Dirichlet	0.041	0.011	13	3
Var-Covariance	0.040	0.005	19	3
Historical	0.055	0.006	29	6

$\alpha = 0.10$				
Method	$\overline{f}$	s_f	PoF	IND
MC-Gaussian	0.087	0.014	13	2
MC-Dirichlet	0.087	0.015	11	2
Var-Covariance	0.072	0.007	0	0
Historical	0.103	0.007	38	11

Table 6: Summary table of the results for backtesting VaR estimates on 44 asymptotic independent portfolios in line with Section 7.3. f represents the frequency of violations and s_f the respective standard deviation across the selection of 44 portfolios.

8 Conclusion and discussion

Being conscious about the risk level could improve the performance of allocating wealth into investment portfolios. This study examines the asymptotic dependence of international stock markets and considers three different VaR models for corresponding portfolios. The focus is on a VaR model that incorporates a new copula model by H&W using Monte Carlo simulation. The copula model estimates the asymptotic (in)dependence in the data by letting the data 'speak' for itself through the dependence parameter δ. The asymptotic dependence in markets were tested and through simulation from the copula model the estimates from a VaR model were tested.

The extreme losses in international stock markets exhibit asymptotic dependence only for pairs of countries inside the European Union, whereas countries outside the EU show asymptotic independence with countries inside and outside the EU. A notable exceptions is Japan - Australia, where we can not reject asymptotic dependence. The strongest asymptotic dependent pairs include the countries Germany, France and the Netherlands, where the cause for the level of stock market integration is often linked to their participation in the European Monetary Union. In general, stock markets of countries appear asymptotic independent in our sample period when corrected for heteroskedastic volatility.

The risk analysis in this study shows that for asymptotic independent portfolios at the most extreme level $\alpha = 0.01$, a Monte Carlo method with a Gaussian specification of the H&W model outperforms all other risk models in most cases. The result is based on a portfolio analysis of 11 international stock markets of which the 1-day ahead VaR is calculated. In case of asymptotic dependent portfolios the Historical benchmark model performs best in most cases.

The simulation of our copula model shows that in estimating the parameters of different specifications contains significant bias and high variability. In addition, the model with an inverted max-stable specification is prone for numerical error in the estimation of the censored likelihood. There are a plethora of other inverted max-stable distributions for independence modelling. However, an in-depth analysis on the performance of the model is necessary before selecting a model for real-world applications as the estimation of δ possibly differs between specifications.

The results in this thesis are best used for linear portfolios without shorting and other financial instrument such as options. An extension would be to investigate longer horizons and calculate n-day ahead risk estimates. As risk managers are interested in longer periods than 1 days, one could look at the standard 10 days VaR estimates.

The performance of the risk model with our copula model could increase by certain factors. Firstly, empirical data with a higher frequency to filter between different time zones. In this study the closing prices of international indices are at different time periods. In addition, during our risk analysis we include the financial recession which greatly affects the entire risk estimation as the estimation window used was large. Note that estimating the copula model required a large estimation window. Separating a period with financial crisis and more tranquil period could provide new insights. However, a risk model should not only be able to perform for standard data as financial crises are unpredictable. Finally, only considering VaR estimates creates tunnel vision. Different forms of risk could cause a larger effect on the losses such as political risk, liquidity risk and regulatory risk which should be a part of the risk analysis before investing.

Appendix A Appendix

A.1 - Autocorrelation of AR-GJR-GARCH(1,1) residuals.

	USA	NL	SP	SA	FR	DE	JP	AUS	UK	NW	DK
LB	11.2	7.4	6.5	5.5	10.8	8.4	6.9	16.3	10.7	18.3	10.4
P-val	0.34	0.69	0.77	0.85	0.37	0.59	0.74	0.09	0.38	0.05	0.40

Table 7: The result of Ljung-Box tests on the univariate timeseries using only the first 10 lag coefficients.

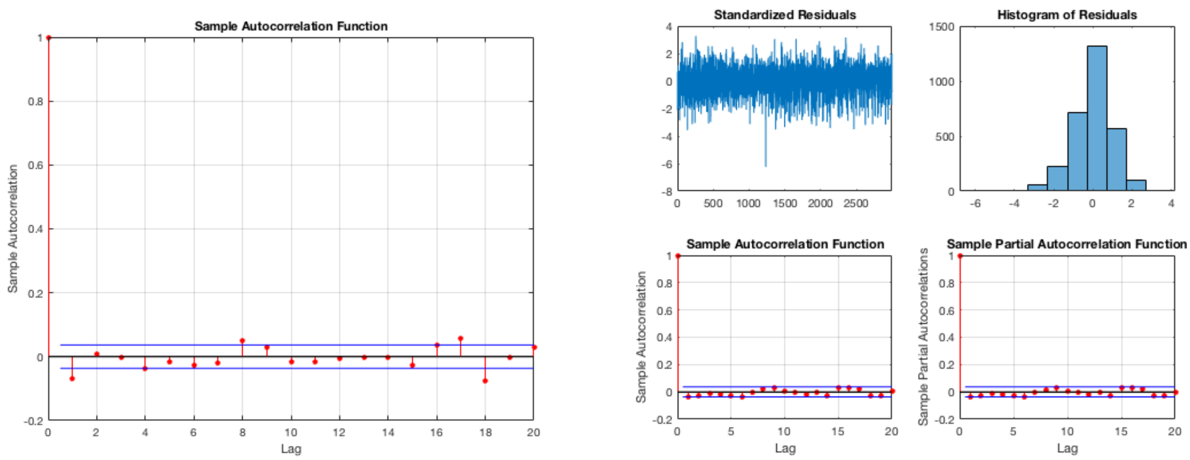

Figure 10: Autocorrelation before(left) and after(right) correcting for the heteroskedastic volatility by an AR-GJR-GARCH(1,1) model. Here, the data shown is of the USA.

A.2 - Akaike-Information-Criteria scores.

	Gaussian	Dirichlet	Logistic	Asym-Logistic
USA - Netherlands	546.5	547.2	**545.5**	554.0
USA - Spain	**637.8**	641.2	640.1	646.2
USA - South Africa	899.4	897.3	**896.7**	901.8
USA - France	548.1	546.5	**545.8**	549.9
USA - Germany	**571.8**	573.0	573.3	578.4
USA - Japan	**996.7**	998.9	997.9	1003
USA - Australia	1001	**989.5**	1001	1004
USA - United Kingdom	**507.3**	519.3	509.1	526.4
USA - Norway	**799.3**	802.7	800.9	806.5
USA - Denmark	854.7	**852.8**	855.8	865.1
Netherlands - Spain	-14.31	**-20.02**	-13.86	6.21
Netherlands - South Africa	**614.0**	620.3	619.0	634.3
Netherlands - France	-586.6	-561.9	**-590.5**	
Netherlands - Germany	**-350.6**	-319.7	-347.7	
Netherlands - Japan	835. 5	832.5	**830.8**	836.3
Netherlands - Australia	**824.2**	826.2	824.9	830.9
Netherlands - United Kingdom	**-294.6**	-265.9	-293.9	
Netherlands - Norway	**209.1**	214.9	211.8	
Netherlands - Denmark	341.3	**332.5**	337.9	374.9
Spain - South Africa	**746.1**	755.1	749.9	764.0
Spain - France	-179.1	**-179.4**	-178.1	
Spain - Germany	**-20.89**	-18.7	-17.20	-1.328
Spain - Japan	938.8	935.4	**933.2**	937.3
Spain - Australia	933.7	935.1	**933.6**	
Spain - United Kingdom	131.6	**128.1**	134.7	152.9
Spain - Norway	**496.5**	497.6	499.1	511.3
Spain - Denmark	517.9	518.7	**515.7**	541.7
South Africa - France	**659.7**	664.72	666.4	672.4
South Africa - Germany	645.0	643.4	**642.3**	645.8
South Africa - Japan	**866.3**	869.1	867.6	875.4
South Africa - Australia	**783.3**	784.5	783.5	787.6
South Africa - United Kingdom	**586.0**	590.5	588.8	601.1
South Africa - Norway	**525.49**	530.7	529.1	535.9
South Africa - Denmark	**703.7**	707.7	706.0	718.3
France - Germany	**-467.3**			
France - Japan	859.2	860.0	**858.7**	863.3
France - Australia	**886.4**	888.5	886.8	891.0
France - United Kingdom	-349.1	-314.9	**-354.3**	-312.6
France - Norway	237.9	**237.6**	237.7	273.3
France - Denmark	357.8	**345.9**	354.1	379.9
Germany - Japan	**861.7**	864.3	862.4	868.4
Germany - Australia	**863.3**	869.6	868.6	874.9
Germany - United Kingdom	**-127.8**	-120.1	-124.2	-79.27
Germany - Norway	**305.0**	310.3	307.7	
Germany - Denmark	323.4	**315.6**	321.1	344.9
Japan - Australia	**506.5**	510.9	509.9	516.0
Japan - United Kingdom	**819.5**	823.4	820.8	835.1
Japan - Norway	**840.6**	843.4	862.4	868.4
Japan - Denmark	**880.8**	880.8	881.2	886.1
Australia - United Kingdom	858.6	859.9	**858.4**	862.5
Australia - Norway	789.2	**789.0**	789.7	793.8
Australia - Denmark	**821.1**	823.0	821.2	829.4
United Kingdom - Norway	**299.5**	301.1	300.9	
United Kingdom - Denmark	**426.6**	427.0	427.3	443.6
Norway - Denmark	**452.3**	457.1	454.1	470.3

Table 8: The AIC-scores according to $AIC = 2k - 2L$.

Dirichlet	δ	$\Pr(\delta) > 0.5$ or $\Pr(\delta) < 0.5$
Asymptotic Dependent		$\Pr(\delta) > 0.5$
Netherlands - France	0.65	0.00
Netherlands - Germany	0.61	0.00
Netherlands - United Kingdom	0.61	0.00
France - United Kingdom	0.62	0.00
Asymptotic Independent		$\Pr(\delta) < 0.5$
USA - Netherlands	0.45	0.01
USA - Spain	0.44	0.00
USA - South Africa	0.36	0.00
USA - France	0.48	0.05
USA - Germany	0.46	0.03
USA - Japan	0.16	0.00
USA - Australia	0.03	0.00
USA - United Kingdom	0.47	0.15
USA - Norway	0.40	0.00
USA - Denmark	0.34	0.00
Netherlands - Spain	0.48	0.18
Netherlands - South Africa	0.38	0.00
Netherlands - Japan	0.38	0.00
Netherlands - Australia	0.37	0.00
Netherlands - Norway	0.45	0.01
Netherlands - Denmark	0.39	0.00
Spain - South Africa	0.40	0.00
Spain - France	0.49	0.28
Spain - Germany	0.48	0.02
Spain - Japan	0.38	0.00
Spain - Australia	0.36	0.00
Spain - United Kingdom	0.47	0.13
Spain - Norway	0.44	0.00
Spain - Denmark	0.39	0.00
South Africa - France	0.42	0.00
South-Africa - Germany	0.46	0.01
South-Africa - Japan	0.36	0.00
South-Africa - Australia	0.43	0.00
South-Africa - United Kingdom	0.42	0.00
South-Africa - Norway	0.46	0.05
South-Africa - Denmark	0.38	0.00
France - Japan	0.39	0.00
France - Australia	0.39	0.00
France - Norway	0.40	0.02
France - Denmark	0.41	0.00
Germany - Japan	0.37	0.00
Germany - Australia	0.37	0.00
Germany - United Kingdom	0.43	0.00
Germany - Norway	0.44	0.00
Germany - Denmark	0.42	0.00
Japan - Australia	0.48	0.12
Japan - United Kingdom	0.30	0.00
Japan - Norway	0.39	0.00
Japan - Denmark	0.38	0.00
Australia - United Kingdom	0.40	0.00
Australia - Norway	0.41	0.00
Australia - Denmark	0.37	0.00
United Kingdom - Norway	0.45	0.01
United Kingdom - Denmark	0.44	0.00
Norway - Denmark	0.43	0.00
Numerically unstable		
France - Germany		

Asymmetric Logistic	δ	$\Pr(\delta) > 0.5$ or $\Pr(\delta) < 0.5$
Asymptotic Dependent		$\Pr(\delta) > 0.5$
Netherlands - Spain	0.58	0.00
Netherlands - Denmark	0.51	0.06
Spain - Germany	0.58	0.00
Spain - United Kingdom	0.55	0.00
France - United Kingdom	0.62	0.00
Germany - United Kingdom	0.59	0.00
Germany - Denmark	0.52	0.00
Japan - Australia	0.50	0.38
United Kingdom - Denmark	0.51	0.34
Asymptotic Independent		$\Pr(\delta) < 0.5$
USA - Netherlands	0.49	0.11
USA - Spain	0.47	0.00
USA - South Africa	0.39	0.00
USA - France	0.49	0.12
USA - Germany	0.48	0.03
USA - Japan	0.32	0.00
USA - Australia	0.27	0.00
USA - United Kingdom	0.49	0.16
USA - Norway	0.42	0.00
USA - Denmark	0.39	0.00
Netherlands - South Africa	0.47	0.00
Netherlands - Japan	0.41	0.00
Netherlands - Australia	0.41	0.00
Spain - South Africa	0.43	0.00
Spain - Japan	0.38	0.00
Spain - Norway	0.49	0.14
Spain - Denmark	0.48	0.03
South Africa - France	0.46	0.00
South-Africa - Germany	0.47	0.00
South-Africa - Japan	0.40	0.00
South-Africa - Australia	0.43	0.00
South-Africa - United Kingdom	0.47	0.00
South-Africa - Norway	0.49	0.20
South-Africa - Denmark	0.47	0.00
France - Japan	0.41	0.00
France - Australia	0.40	0.00
France - Norway	0.44	0.00
France - Denmark	0.42	0.00
Germany - Japan	0.40	0.00
Germany - Australia	0.40	0.00
Germany - Norway	0.48	0.12
Japan - United Kingdom	0.41	0.12
Japan - Norway	0.41	0.00
Japan - Denmark	0.39	0.00
Australia - United Kingdom	0.41	0.00
Australia - Norway	0.43	0.00
Australia - Denmark	0.41	0.00
United Kingdom - Norway	0.44	0.00
Norway - Denmark	0.46	0.01
Numerically unstable		
Netherlands - France		
Netherlands - Germany		
Netherlands-United Kingdom		
Netherlands Norway		
Spain - France		
Spain - Australia		
France - Germany		

Table 9: Censored likelihood estimation of the copula model on the whole data period. Asymptotic normal P-values are based on the Hessian. Numerically unstable implies the integral in the censored likelihood function could not be evaluated. Includes the asymmetric Dirichlet(left) and Asymmetric-Logistic(right) specification.

Gaussian	δ	$\Pr(\delta) > 0.5$ or $\Pr(\delta) < 0.5$	Logistic	δ	$\Pr(\delta) > 0.5$ or $\Pr(\delta) < 0.5$
Asymptotic Dependent		$\Pr(\delta) > 0.5$	Asymptotic Independent		$\Pr(\delta) < 0.5$
Netherlands - Spain	0.51	0.40	USA - Netherlands	0.44	0.01
Netherlands - France	0.53	0.05	USA - Spain	0.44	0.01
Spain - France	0.52	0.36	USA - South Africa	0.37	0.00
Spain - Germany	0.51	0.39	USA - France	0.48	0.26
France - Germany	0.60	0.00	USA - Germany	0.42	0.00
Asymptotic Independent		$\Pr(\delta) < 0.5$	USA - Japan	0.17	0.01
USA - Netherlands	0.44	0.00	USA - Australia	0.03	0.00
USA - Spain	0.43	0.00	USA - United Kingdom	0.41	0.00
USA - South Africa	0.26	0.00	USA - Norway	0.39	0.00
USA - France	0.46	0.03	USA - Denmark	0.34	0.00
USA - Germany	0.46	0.01	Netherlands - Spain	0.46	0.06
USA - Japan	0.17	0.04	Netherlands - South Africa	0.37	0.00
USA - Australia	0.09		Netherlands - France	0.43	0.01
USA - United Kingdom	0.42	0.00	Netherlands - Germany	0.41	0.00
USA - Norway	0.39	0.00	Netherlands - Japan	0.39	0.00
USA - Denmark	0.34	0.00	Netherlands - Australia	0.38	0.00
Netherlands - South Africa	0.37	0.00	Netherlands - United Kingdom	0.42	0.00
Netherlands - Germany	0.49	0.41	Netherlands- Norway	0.43	0.00
Netherlands - Japan	0.25	0.00	Netherlands- Denmark	0.38	0.00
Netherlands - Australia	0.38	0.00	Spain - South Africa	0.34	0.00
Netherlands - United Kingdom	0.49	0.35	Spain - France	0.46	0.08
Netherlands - Denmark	0.42	0.00	Spain - Germany	0.46	0.09
Netherlands - Norway	0.47	0.05	Spain - Japan	0.38	0.00
Spain - South Africa	0.28	0.00	Spain - Australia	0.37	0.02
Spain - Japan	0.07	0.00	Spain - United Kingdom	0.47	0.03
Spain - Australia	0.36	0.00	Spain - Norway	0.44	0.00
Spain - United Kingdom	0.48	0.21	Spain - Denmark	0.37	0.01
Spain - Norway	0.45	0.01	South Africa - France	0.36	0.01
Spain - Denmark	0.41	0.00	South Africa - Germany	0.46	0.02
South Africa - France	0.41	0.00	South Africa - Japan	0.35	0.00
South-Africa - Germany	0.41	0.00	South-Africa - Australia	0.43	0.02
South-Africa - Japan	0.33	0.00	South-Africa - United Kingdom	0.42	0.00
South-Africa - Australia	0.42	0.00	South-Africa - Norway	0.46	0.08
South-Africa - United Kingdom	0.41	0.00	South-Africa - Denmark	0.38	0.00
South-Africa - Norway	0.46	0.02	France - Japan	0.39	0.00
South-Africa - Denmark	0.38	0.00	France - Australia	0.39	0.00
France - Japan	0.37	0.00	France - United Kingdom	0.40	0.00
France - Australia	0.38	0.00	France - Norway	0.37	0.00
France - United Kingdom	0.48	0.28	France - Denmark	0.39	0.00
France - Norway	0.42	0.01	Germany - Japan	0.37	0.00
France - Denmark	0.44	0.00	Germany - Australia	0.18	0.00
Germany - Japan	0.31	0.04	Germany - United Kingdom	0.38	0.00
Germany - Australia	0.21	0.00	Germany - Norway	0.43	0.00
Germany - United Kingdom	0.44	0.02	Germany - Denmark	0.41	0.00
Germany - Norway	0.45	0.15	Japan - Australia	0.47	0.11
Germany - Denmark	0.45	0.01	Japan - United Kingdom	0.26	0.00
Japan - Australia	0.47	0.05	Japan - Norway	0.38	0.00
Japan - United Kingdom	0.24	0.00	Japan - Denmark	0.38	0.00
Japan - Norway	0.37	0.01	Australia - United Kingdom	0.41	0.00
Japan - Denmark	0.37	0.00	Australia - Norway	0.41	0.00
Australia - United Kingdom	0.40	0.00	Australia - Denmark	0.37	0.00
Australia - Norway	0.41	0.00	United Kingdom - Norway	0.43	0.00
Australia - Denmark	0.37	0.00	United Kingdom - Denmark	0.42	0.02
United Kingdom - Norway	0.45	0.02	Norway - Denmark	0.42	0.00
United Kingdom - Denmark	0.45	0.01	Numerically unstable		
Norway - Denmark	0.44	0.00	France - Germany		

Table 10: Censored likelihood estimation of the copula model on the whole data period. Asymptotic normal P-values are based on the Hessian. Numerically unstable implies the integral in the censored likelihood function could not be evaluated. Includes the symmetric Gaussian(left) and Logistic(right) specification.

References

G. Bekaert and C. R. Harvey. Market integration and contagion. Working Paper 9510, National Bureau of Economic Research, February 2003. URL http://www.nber.org/papers/w9510.

C. T. Brownlees, R. F. Engle, and B. T. Kelly. A practical guide to volatility forecasting through calm and storm. *Available at SSRN 1502915*, 2011.

D. Castro-Camilo, M. de Carvalho, J. Wadsworth, et al. Time-varying extreme value dependence with application to leading european stock markets. *The Annals of Applied Statistics*, 12(1):283–309, 2018.

P. F. Christoffersen. Evaluating interval forecasts. *International Economic Review*, 39 (4):841–862, 1998.

S. G. Coles and J. A. Tawn. Modelling extreme multivariate events. *Journal of the Royal Statistical Society. Series B (Methodological)*, 53(2):377–392, 1991. ISSN 00359246. URL http://www.jstor.org/stable/2345748.

S. Engelke, A. Malinowski, Z. Kabluchko, and M. Schlather. Estimation of husler–reiss distributions and brown–resnick processes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 77(1):239–265, 2015.

A. Ferreira, L. De Haan, et al. The generalized pareto process; with a view towards application and simulation. *Bernoulli*, 20(4):1717–1737, 2014.

R. A. Fisher and L. H. C. Tippett. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Mathematical Proceedings of the Cambridge Philosophical Society*, 24(2):180–190, 1928. doi: 10.1017/S0305004100015681.

K. J. Forbes and R. Rigobon. No contagion, only interdependence: measuring stock market comovements. *The Journal of Finance*, 57(5):2223–2261, 2002.

L. R. Glosten, R. JAGANNATHAN, and D. E. RUNKLE. On the relation between the expected value and the volatility of the nominal excess return on stocks. *The Journal of Finance*, 48(5):1779–1801, 1993. doi: 10.1111/j.1540-6261.1993.tb05128.x. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1540-6261.1993.tb05128.x.

B. Gnedenko. Sur la distribution limite du terme maximum d'une serie aleatoire. *Annals of Mathematics*, 44(3):423–453, 1943. ISSN 0003486X. URL http://www.jstor.org/stable/1968974.

R. Huser, T. Opitz, and E. Thibaud. Bridging asymptotic independence and dependence in spatial extremes using gaussian scale mixtures. *Spatial Statistics*, 21:166 – 186, 2017. ISSN 2211-6753. doi: https://doi.org/10.1016/j.spasta.2017.06.004. URL http://www.sciencedirect.com/science/article/pii/S221167531730088X.

R. G. Huser and J. L. Wadsworth. Modeling spatial processes with unknown extremal dependence class. *Journal of the American Statistical Association*, forthcoming 2018. doi: 10.1080/01621459.2017.1411813. URL https://doi.org/10.1080/01621459.2017.1411813.

P. Krupskii, R. Huser, and M. G. Genton. Factor copula models for replicated spatial data. *Journal of the American Statistical Association*, 113(521):467–479, 2018.

P. Kupiec. Techniques for verifying the accuracy of risk measurement models. *Journal of Derivatives*, 3:73–84, 1995.

F. Longin and B. Solnik. Extreme correlation of international equity markets. *The Journal of Finance*, 56(2):649–676, 2001.

S.-H. Poon, M. Rockinger, and J. Tawn. Modelling extreme-value dependence in international stock markets. *Statistica Sinica*, 13(4):929–953, 2003.

J. Segers. Max-stable models for multivariate extremes. *arXiv preprint arXiv:1204.0332*, 2012.

M. Sklar. Fonctions de repartition a n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959. URL https://ci.nii.ac.jp/naid/10011938360/en/.

J. A. Tawn. Bivariate extreme value theory: Models and estimation. *Biometrika*, 75(3): 397–415, 1988. doi: 10.1093/biomet/75.3.397. URL http://dx.doi.org/10.1093/biomet/75.3.397.

R. von Mises. La distribution de la plus grande de n valeurs. *Revue Math. de l'Union Interbalkanique*, 1:141–160, 1936.