ERASMUS UNIVERSITY ROTTERDAM

Erasmus School of Economics

Thesis [Masters in Quantitative Finance]

Title of Thesis: Analyzing Single-State and Multi-state Momentum Factors with VAR and MSVAR Models

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Abstract

The momentum factor has always been attracting great interest from investors, and recent studies has helped its implementation into portfolio constructions. We take a closer look at one of these studies, the paper by Moskowitz et al. (2012), and intend to further refine it as well as applying an extension in order to provide more insight regarding the momentum factor. We propose new estimation methods, namely the Generalized Least Squares (GLS) and the Maximum Likelihood Estimation (MLE), in addition to the Ordinary Least Squares (OLS) performed by Moskowitz et al. (2012). The new estimation methods are introduced as we find one of his assumptions regarding uncorrelated errors to be not as applicable in a real-world scenario, which is a requirement for the OLS they used. The model is then extended by introducing a multi-state approach, where we utilize Markov-switching processes for the momentum factor states. Our data spanned from 1982 to 2018 across 9 indices that are selected by Moskowitz et al. (2012), and our findings show different results once we implement our proposed GLS and MLE. As we introduce the multi-state model approach as an extension, we find links between the low-momentum and high-momentum states with the recession periods in the market. Based on these results, the momentum factor can still be utilized in portfolio construction although it may not be as straightforward as previously believed.

Contents

1 Introduction

It has been widely known that the performance of stocks in one period may have an impact on its performance in the next period. This effect is often called the momentum factor, and it has been convincing investors to trade against the traditional philosophy of buying low and selling high; momentum investors has been trying to buy high and sell low, with hopes that they can sell even higher or buy even lower. The momentum factor has been studied for years and a lot of papers have proven its effectiveness, including Moskowitz et al. (2012), which studies the momentum factur based on data from 1965 to 2009. The recent studies on momentum factors have been attracting more and more investors into applying it in their own investing strategy. Since the momentum factor is mostly seen via technical analysis instead of fundamental analysis, it allows us to try and explain, capture, and predict its behaviour by using various econometric models.

A way to model the momentum factor, as proposed by Moskowitz et al. (2012), is the restricted multivariate Auto-regressive (AR) model. Simply put, this model does a regression of a return on one of its past returns. The model works in a similar way as a regular AR model, with the condition that each index return series has its own regression constant, but shares the same momentum coefficient across all series. The multivariate AR model can be written as a Vector Auto-regressive (VAR) model, as it allows cross-sectional interactions between the variables. The VAR model allows us to analyze multiple variables and their lags at the same time, and to see how they interact with one another. Considering how various stock indices in the real market often move together in the same direction, including the lagged variable of another stock index may improve our explanation and forecasting power for a stock index.

In this paper, we aim to correct and further refine the research of Moskowitz et al. (2012), by proposing the use of new estimation methods as well as a multi-state model extension. Moskowitz et al. (2012) includes multiple index return series in their estimation, as it increases the statistical power of the momentum factor in their regression by looking at the momentum of the economy in general instead of in only one particular series. However, by using the Ordinary Least Squared (OLS) in their approach, they assumed that all return series are independent with one another, which often does not hold true in the real world. Based on the perspective of investors, it is extremely important for these momentum models to replicate the real world economy as much as possible as they look to implement the momentum factors in their portfolios. The estimation methods we propose corrects this problem by taking the correlation and covariance of each series into account. As further extension to this model, we also introduce a multi-state approach to this model in order to see how the model expands when it is no longer restricted to a single state or regime. We analyze the momentum factor on more recent data, spanning from 1982 to 2018.

The VAR model itself can be estimated in multiple ways, with one of them being the multivariate OLS estimation used by Moskowitz et al. (2012). Although the multivariate

OLS gives a consistent and unbiased estimation under the correct circumstances, we believe that the required assumptions for the regression may be too restrictive and unrealistic to be applied in the financial sector. One of the assumptions for the OLS includes constant variance and uncorrelated errors, which we find to be unfit for the financial data in the real world, as stock returns across different markets and periods are very often heavily correlated with one another. In their analysis, Moskowitz et al. (2012) ignored the covariance matrix of the index returns as well as in the errors, which we seek to correct in our analysis. In order to address this problem, we extend the estimation method of the VAR model by introducing the Generalized Least Squared (GLS) estimation, as well as the Maximum Likelihood Estimation (MLE). Although both the GLS and MLE estimations still require the normality assumption, both methods do not require the uncorrelated errors or constant variance assumptions. This property allows us to further refine the momentum factor estimation results of Moskowitz et al. (2012), now taking the errors into account. The GLS method allows the use of the covariance matrix resulting from our OLS estimation to be used as a proxy of the real covariance matrix, which is then used to scale our variables and perform the regression again, this time correcting for the error covariance matrix. The MLE provides further tools to estimate these parameters, as it allows us to estimate the regression coefficients together with the covariance matrix of the returns.

As we try to further refine our VAR model, we realize that a downside of the regular VAR model is that it ignores the presence of different regimes, such as the bear or bull regimes. This may cause the estimation results of the regular VAR model to be less accurate, especially when these regimes have a big impact on the financial markets (e.g. the 2008 financial crisis). As the findings of Cooper et al. (1995) suggests, momentum portfolios in the stock index markets are often most profitable following a bull market. The paper also finds that during other states of the economy, the momentum portfolios may show slightly less profitability but it is still not eliminated. As mentioned by Cooper et al. (1995), the high momentum returns following a bull regime is in line with the overreaction model findings of Daniel et al. (1998).

The Markov-Switching VAR (MSVAR) model gives an additional tool to further improve upon the VAR model, as it allows our VAR model to fit different parameters for different regimes. If the estimated parameters of each regime is significantly different from one another, it indicates that the momentum factor is indeed heavily influenced by the regimes and thus the MSVAR model should be a significant improvement upon the VAR model. The MSVAR model is estimated by introducing several unobserved state variables, which represents the current regime at that time period. We define two possible unobserved states of the market, a low-momentum state and a high-momentum state. We can then estimate a different coefficient in the VAR model for every state or regime, which will give a more accurate representation of the real-life financial stocks behaviour in different regimes. The MSVAR also introduces a transition matrix that defines how the realized state at a time period may affect the probability of a regime switch at the next time period. It is thus of our interest to see how well the MSVAR model operate on a financial market.

Both the VAR and the MSVAR models are applied on our dataset that includes 9 different stock indices across the world. Our data selection is based on the stock indices used by Moskowitz et al. (2012), such that we can try to replicate the OLS results as much as possible. In our 9 stock indices, we include more recent data that dates up to September 2018, with different start dates for each indices that varies depending on the availability of the data. Our longest index is S&P 500 with 27 years of daily returns, with FTSE/MIB being the shortest index with only 14 years of daily returns. The daily returns of each index is compounded over every month in order to create a monthly return series, as we are interested in monthly lags based on Moskowitz et al. (2012).

Our aim in this paper is to see how the VAR and MSVAR model can fit the data in a financial setting, which allows us to further refine and improve upon the momentum factor proposed by Moskowitz et al. (2012) much more extensively. By now taking the covariance matrix of returns into account, we can expect considerable differences in our VAR estimation with that of Moskowitz et al. (2012). The multi-state perspective of the MSVAR model then allows us to check how much of an impact does the lagged returns represent based on the regime in that time period, further extending the scope of the original VAR model. The results of this paper can be used in a lot of cases, mostly in portfolio construction as an accurate estimation of the VAR and MSVAR model between stock indices may help investors adjust their portfolio based on the current performance of stock indices, and adjusting it depending on what regime the economy is currently in.

Based on our results, we find that there are indeed significant momentum factors across all lags, but we find some differences with the results of Moskowitz et al. (2012) once we take into account the error covariances with GLS and MLE. As we apply the GLS and MLE on the VAR model, we find that although there are still significant momentum effects across all lags, they no longer show uniformly positive signs, which may make future returns more difficult to predict due to the alternating momentum directions between different lags. Our contrasting findings from Moskowitz et al. (2012) indicates that the removal of independent series assumption has a significant impact on the momentum factor results. This implicates that the mainly positive results from Moskowitz et al. (2012) may be invalid in a real world setting to a certain extent, and its implementation in portfolio construction may be limited.

Our findings in the MSVAR model further complicates this issue, the momentum factor also alternates between low-momentum and high-momentum states, sometimes even with opposite directions in each state. In our MSVAR model, we observe several significant negative momentum factors on certain lags, which suggests that there are some apparent mean-reversal effects on some lags. Our test statistic shows significant difference in momentum factor between the two states, and the state probabilities are affected by the economic regime-switching up to a certain extent. We find that during the specific recession periods such as the 2002 and 2008 crises, we can expect significant shifts in the momentum states between a positive momentum state and a low or even a mean-reversal state.

Our different proposed estimation methods are followed by different results from that of Moskowitz et al. (2012) which is to be expected. The significant difference in our results indicates that the underlying assumption of uncorrelated index return series severely impacts results of Moskowitz et al. (2012), where we find this assumption hard to justify due to the apparent cross-correlation between indices in the real-world market. As we try to correct this problem ourselves, we find that the momentum factor is no longer consistently positive across all lags, which means that investors should be more careful in constructing their portfolios with hopes of riding the momentum of past returns. Momentum factor can still be used in portfolio construction, with the most reliable strategy being to utilize the positive momentum and the mean-reversal effects during and after a recession. We do know however, that accurately predicting these recessions and their tipping points has always proven to be relatively difficult.

2 Literature review

As previously mentioned, Moskowitz et al. (2012) proposed the use of cross-sectional AR model in order to model the momentum factor, which can be transformed into a VAR model. The paper found that there is significant proof of the momentum factor, which is caused mainly by the positive covariance between the excess returns of next month and the 1-year lagged excess returns. The findings of the paper shows partial reversal of the momentum effect, which means that the same momentum effect will not be constantly present across time. This behavior of the momentum factor explains the initial under-reaction and delayed over-reaction in the returns, as mentioned by Moskowitz et al. (2012). The paper also finds that the correlations across time series are bigger than the correlation within the time series itself, which suggests that there is a common underlying factor that carries more impact to the momentum factor than the single time series itself. This underlying factor can be interpreted as an underlying state in our case, which will be taken into account in our MSVAR model. Moskowitz et al. (2012) also finds how different investors utilize the same momentum factor, as speculators will ride along the positive trends in the early months and reduce their position once the trends start to reverse, while hedging investors will use this reversing trend to their advantage.

The more recent findings of Giampietro et al. (2018) suggests that regime-switching autoregressive models do outperform their single-state counterparts, giving more accurate estimations based on their information criterions. The results are obtained from a regression of a mix of stocks, bonds, and commodities portfolios on 3 factors, namely the average, carry, and momentum factors. The paper also finds that there are significant correlations within commodities itself, as well as between commodities and the other assets. Although both the findings of Moskowitz et al. (2012) and Giampietro et al. (2018)

are mainly focused on commodities data, we expect to find similar patterns on stock indices. The paper by Tang and Xiong (2012) suggests that the recent financialization of commodities have caused its prices not to be driven only by the supply and demand of its underlying assets anymore, but by other financial factors and investor behaviors as well.

The paper by Cooper et al. (1995) provides further basis for our regime-switching approach on the momentum factor. Cooper et al. (1995) found that the profits of a momentum factor portfolio heavily relies on the lagged returns, and is barely affected by macroeconomic factors. These lagged returns also provides better out-of-sample forecasting accuracy, compared to the macroeconomic factors. The paper defines two states of the economy, an UP (bull) regime and a DOWN (bear) regime. Their findings show that the momentum-based portfolio shows significantly higher profits following a bull regime, which thus supports the existence of regime-switching behaviour in the momentum factor itself. The low and high momentum regimes may be closely related to bear and bull regimes of the market, but there may also be other external factors that affect the momentum regimes.

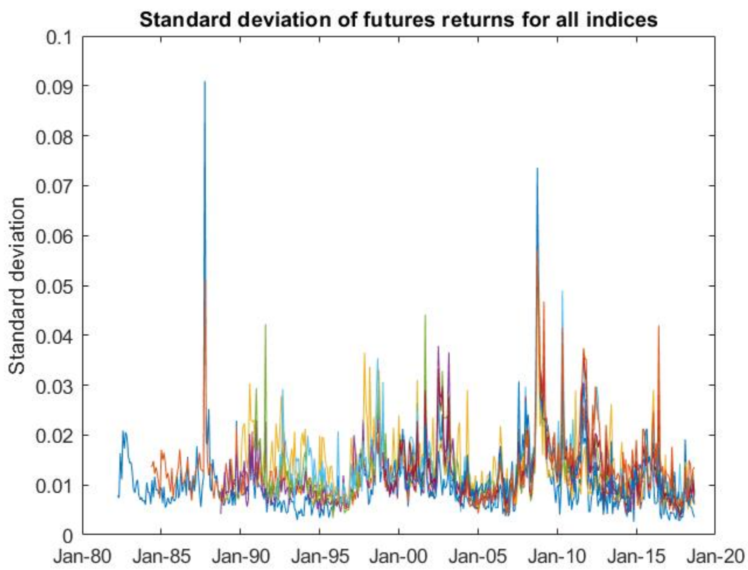
3 Data

For this paper, we analyze 9 different stock index futures returns. These 9 indices are also used by Moskowitz et al. (2012), as we hope to refine the results based on our findings. We prefer to exclude the commodities, bonds, and currencies from their original dataset of 24 assets, and focus solely on the financial indices. The 9 indices included in our research are ASX SPI 200 (Australia), DAX (Germany), IBEX 35 (Spain), CAC 40 (France), FTSE/MIB (Italy), TOPIX (Japan), AEX (Netherlands), FTSE 100 (UK) and S&P 500 (USA). As much as we would like to replicate the data used by Moskowitz et al. (2012), we are unable to obtain the full dataset length that he used. We make up for our more recent starting date by using more recent data up till September 2018. When compared side-to-side, our dataset contains almost as many observations, and our 10 years difference in time period span may prove to be useful in getting insights and relevance in the current state of the market.

All 9 of our indices have different starting periods, as not all futures contracts of these indices start at the same time. We opt to include these various starting dates in order to accommodate as many observations as we can get, which results in an unbalanced panel as a consequence. Performing our estimation techniques on these unbalanced panels is a point of interest in this paper, as we would be required to make several adjustments on our models in order to properly analyze these unbalanced panel. Similar models may be performed in a much simpler manner on a balanced panel, but in our case it would mean sacrificing up to 22 years of observations, which contributes up to more than half the length of our dataset. The index with the earliest starting date in our dataset is the S&P 500, as it is active starting from May 1982, consisting of 36 years of data and making it our longest asset. Our shortest asset is the Italian index FTSE/MIB, which starts from April 2004 and consists of only 14 years of observations.

All futures prices are taken from the closing prices of the continuous futures series, obtained from Datastream up to September 2018. The continuous futures series, as defined by Datastream, indicates a perpetual price series of the futures contracts of a particular asset. The prices start at the beginning of the futures contract, which is then transferred to the price of the next contract once the existing contract is expired. This way, the continuous series does not expire as long as there are still futures contracts of that asset available. The descriptive statistics of each futures returns are given in Table 1. The annualized mean and volatility included in the table are calculated from the monthly returns of each index over their respective horizons. The annualized mean is calculated by taking the mean of the monthly returns, which is then annualized by compounding it over 12 months. The annualized standard deviation is calculated by multiplying the monthly standard deviation by $\sqrt{12}$.

Figure 1: Standard deviation of futures returns over time



Although we selected the time period from 1982 to 2018 due to the availability of the data, the specified time period consists of several interesting occurrences in the financial sector. The selected time period includes two major recessions not only in the United States but also the global financial market, namely the 2001 dot-com bubble as well as the 2007 sub-prime mortgage crisis. As mentioned by Schwert (2011), the volatility of stock returns during recessions increases significantly, which is mainly due to the increased level of uncertainty during those periods. These recessions at 2001 and 2007 will serve as a good point of interest for our MSVAR model, as it may have significant impact on our defined low and high momentum regimes. These recession periods can also be seen from Figure 1, where some time periods clearly show higher volatility across all indices compared to other time periods. Figure 1 shows the monthly volatility of every index, calculated by taking the standard deviations of the daily index returns over every month.

Table 1: Descriptive statistics

	Futures returns		Start date
	Annualized mean	Annualized volatility	
ASX	4.63%	21.70%	May-00
IBEX 35	3.25%	23.83%	May-92
CAC 40	0.80%	20.96%	Jan-99
FTSE/MIB	-2.53%	25.16%	Apr-04
TOPIX	-0.20%	20.24%	Sep-88
AEX	5.31%	20.45%	Nov-88
DAX	6.97%	22.10%	Dec-90
FTSE 100	5.17%	17.99%	May-84
S&P 500	8.92%	14.97%	May-82

[1] The annualized mean is calculated by compounding the mean monthly returns of each index over 12 months. The annualized volatility is calculated by multiplying the standard deviation of the monthly returns by $\sqrt{12}$.
[2] Despite the different starting dates, all indices have the same end date of September 2018.

As seen from Table 1, the American index S&P 500 shows the highest annualized mean returns yet the lowest annualized volatility, which explains the high interest from investors in this index. The British index FTSE 100 is also heavily desired by investors, as it shows relatively low volatility with considerably positive mean returns. Our shortest index, FTSE/MIB, spans only 14 years and it shows the lowest mean returns at -2.53% yet the highest volatility at 25.16%. We believe that this may be partially caused by its relatively short window compared to the other indices, which allows negative returns and higher volatilities of recession periods to be more heavily weighted. Other than FTSE/MIB, our European indices show relatively similar volatilities, with the German index DAX being the best performing index with an annualized mean of 6.97%, and the French index CAC 40 being the lowest with 0.80%.

We scale our future returns with its first lagged standard deviation for each month as was done by Moskowitz et al. (2012). This scaling procedure is performed in order to allow us to compare the performance different indices from a better perspective, since the different volatilities of each indices has been taken into account. We use the first lagged annualized volatility in the scaling in order to reduce any look-ahead bias. The scaling procedure can be written as:

$$r_{i,t}^* = r_{i,t}/\sigma_{i,t-1} \tag{1}$$

where $r_{i,t}$ indicates the monthly returns of index i at time t, and $\sigma_{i,t-1}$ indicates its lagged monthly annualized volatility. Although we use the scaling based on Moskowitz et al. (2012), we do not use their ex-ante volatility measure as we believe that the regular lagged

Table 2: Descriptive statistics of scaled returns

	Mean	Volatility
ASX	0.1453	1.4159
IBEX 35	0.0895	1.5135
CAC 40	0.0660	1.2936
FTSE/MIB	0.0052	1.5380
TOPIX	-0.0101	1.3226
AEX	0.1502	1.4817
DAX	0.1860	1.4067
FTSE 100	0.1566	1.3754
S&P 500	0.3043	1.3230

[1] The scaled returns are calculated by scaling the monthly returns of each index by its first lagged monthly volatility.

annualized volatility is good enough as a measure of volatility in our scaling system. The descriptive statistics of the scaled returns for each index is shown in Table 2. As seen from the table, S&P 500 still shows the highest mean scaled returns at 0.3043. FTSE/MIB shows a higher scaled return mean compared to TOPIX, although it has lower annualized mean returns. This can be explained by the larger weight of recession periods in FTSE/MIB, as the negative returns of those periods are accompanied by higher volatilities as well, causing its impact to be partially diminished once the returns are scaled by their volatilities.

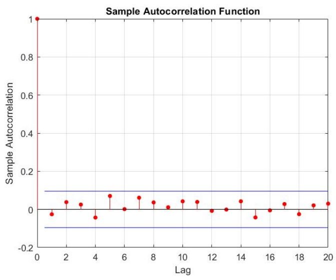
Despite the presence of momentum factor described by Moskowitz et al. (2012) in the selected indices, each indices individually does not present significant autocorrelation, as seen from Figure 2. As mentioned by Lewellen (2002), the lack of apparent positive autocorrelation in the index returns does not necessarily indicate the lack of momentum factor. The momentum factor is analyzed jointly for all indices in a cross-sectional manner, while the autocorrelation only shows the correlation between time-series observations. As seen from Table 3, the scaled returns show relatively high correlations with one another, with an average cross-correlation of 0.696.

Table 3: Cross-correlations between scaled return series

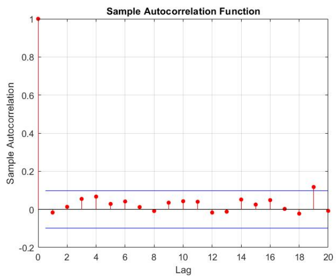
i	1	2	3	4	5	6	7	8	9
1	1.000	0.667	0.518	0.713	0.706	0.569	0.681	0.620	0.583
2	0.667	1.000	0.521	0.839	0.794	0.735	0.819	0.736	0.741
3	0.518	0.521	1.000	0.518	0.537	0.473	0.499	0.537	0.486
4	0.713	0.839	0.518	1.000	0.873	0.800	0.914	0.654	0.827
5	0.706	0.794	0.537	0.873	1.000	0.819	0.916	0.637	0.834
6	0.569	0.735	0.473	0.800	0.819	1.000	0.863	0.625	0.875
7	0.681	0.819	0.499	0.914	0.916	0.863	1.000	0.640	0.884
8	0.620	0.736	0.537	0.654	0.637	0.625	0.640	1.000	0.599
9	0.583	0.741	0.486	0.827	0.834	0.875	0.884	0.599	1.000

[1] The correlations are calculated pair-wise, due to the uneven lengths of the scaled return series. Each pair-wise correlation is then calculated for the number of elements in the shorter series between the two indices.

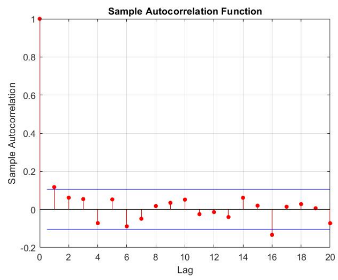
Figure 2: Autocorrelation of scaled returns for each index



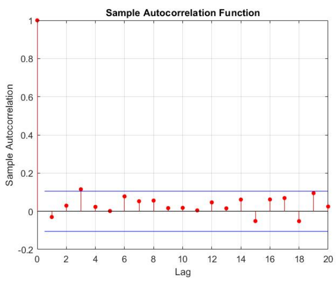
S&P 500



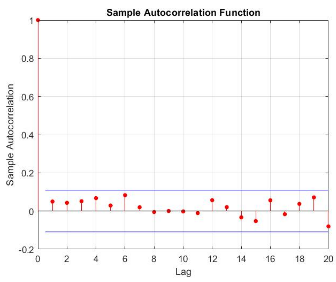
FTSE 100



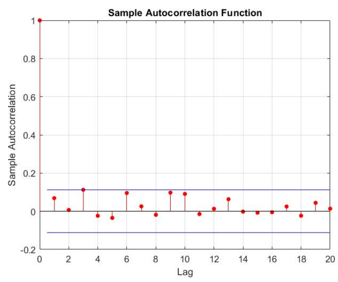
TOPIX



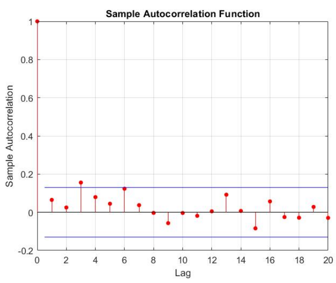
AEX



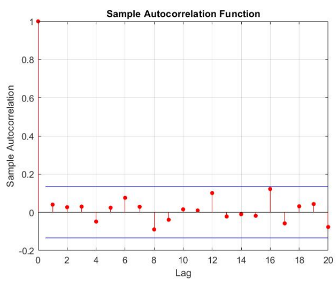
DAX



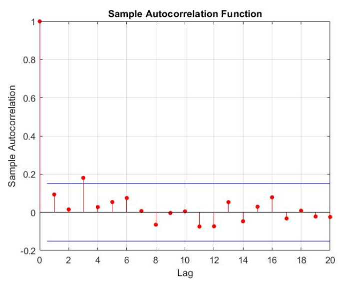
IBEX 35



CAC 40



ASX 200



MIB

4 Methodology

As mentioned in the previous section, this paper will utilize the MSVAR model to analyze the 9 stock indices monthly excess returns. We take a step-by-step approach in this section by first specifying the VAR model, as well as the adjusted maximum likelihood estimation method that is used in to analyze the VAR model. We start the methodology by first replicating what Moskowitz et al. (2012) did via the OLS. The OLS gives unbiased estimators, although it requires a certain set of assumptions and it is not always suitable for testing due to its assumption of uncorrelated errors. For this reason, we refine the estimation method of the VAR model by including the covariance matrix via Feasible GLS (GLS) and MLE. Based on the MLE, we further extend the VAR model by assuming the two underlying states, which we define as a low-momentum state and a high-momentum state. The inclusion of these two underlying states transforms the VAR model into a MSVAR model, which is analyzed by using the Expectation-Maximization (EM) algorithm with the aim of properly capturing the parameters that goes along with each of the underlying states.

4.1 VAR model specification

The auto-regressive (AR) models are often used to describe time-series observations, which in our case are represented by index returns. The degree of how far shall we look into the past observations are called the lags. The AR model allows us to analyze time-series observations with any amount of desired lags, no matter how far into the past it maybe, given that we have enough observations. In our model, we will be looking at 12 different monthly lags, starting from a 1-month up to 12-months lags.

Although the AR model serves a good purpose when analyzing previous observations, it is characterized as a univariate model that can only estimate parameters on a single time-series. As we often see in the financial setting, analyzing a single stock index may not give the best insight on how the whole market performs. For this reason, we are interested in analyzing multiple stock indices at the same time, which forces us to use multivariate estimation methods to accommodate our multivariate time-series data. In order to perform the estimation on a multivariate setting, we have to expand the AR model into the VAR model. The VAR model uses similar techniques as the AR model, which means that it can still be estimated by performing the OLS or MLE. The VAR model also allows us to put various conditions on how the different time-series would be able to interact with one another, which is certainly a benefit when used in a financial study setting.

The momentum factor model proposed by Moskowitz et al. (2012) can be represented by a restricted VAR(H) model, with H indicating the number of lags in the model. The model is estimated with the monthly excess returns that is scaled by its first lagged volatility, in order to correct for the different volatility across all our assets over time. The first lag of the annualized volatility is used in order to correct for the look-ahead bias in the volatility estimate. The general VAR(H) model for the scaled returns used by Moskowitz et al. (2012) is written as:

$$r_{i,t}^* = \alpha_{i,h} + \beta_{i,h} r_{i,t-h}^* + \varepsilon_{i,t} \tag{2}$$

where $r_{i,t}^*$ denotes the monthly scaled returns of asset i at time t previously mentioned in Equation (1), with $\alpha_{i,h}$ being the constant and $\beta_{i,h}$ being the regression coefficient. The lag indicator $h = 1, 2, ..., H$ denotes the lagged variable order, and we set $H = 12$ for a 12-months lag horizon. As we consider 9 stock indices, we set $i = 1, 2, ..., I$ with $I = 9$. As seen from this model, there is no cross-sectional term in any of the parameters. This indicates that the model can be estimated separately for each index i when we assume that errors are uncorrelated, which is one of the assumptions of the OLS soon to be discussed.

By following the momentum factor model of Moskowitz et al. (2012), we now restrict this general VAR(H) model with $\beta_{1,h} = \beta_{2,h} = ... = \beta_{9,h} = \beta_h$. The restriction indicates that the lag-specific coefficient β_h have the same cross-sectional effect on all 9 of our stock indices. We put this cross-sectional restriction on β_h in order to analyze the aggregate momentum effect across the whole market in general. As seen previously in Table 3, the high correlations between each series further encourages us to analyze the momentum factor as a whole, instead of for each specific index. The introduction of this restriction forces us to estimate the whole model for all indices at once, as each index are no longer independent of each other due to β_h and thus can no longer be estimated separately. The constants $\alpha_{i,h}$ are still kept separate for each index, in order to correct for the different means of each index scaled returns. This restricted VAR model can now be rewritten as:

$$r_{i,t}^* = \alpha_{i,h} + \beta_h r_{i,t-h}^* + \varepsilon_{i,t} \tag{3}$$

The restricted model can be rewritten again in matrix notation, which is given as:

$$\boldsymbol{y_t} = \boldsymbol{a_h} + \beta_h \boldsymbol{y_{t-h}} + \boldsymbol{\varepsilon_t} \tag{4}$$

We can also merge the constants matrix along with the β_h coefficients into a single matrix form, such that the new equation and the new parameters matrix $\boldsymbol{B_h}$ is defined as:

$$\boldsymbol{y_t} = \boldsymbol{X_{t,h}} \boldsymbol{B_h} + \boldsymbol{\varepsilon_t} \tag{5}$$

with

$$\boldsymbol{y_t} = \begin{bmatrix} r_{1,t}^* \\ r_{2,t}^* \\ \vdots \\ r_{I,t}^* \end{bmatrix}, \quad \boldsymbol{X_{t,h}} = \begin{bmatrix} 1 & \dots & 0 & r_{1,t-h}^* \\ \vdots & \ddots & 0 & r_{2,t-h}^* \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & r_{I,t-h}^* \end{bmatrix}, \quad \boldsymbol{B_h} = \begin{bmatrix} \alpha_{1,h} \\ \alpha_{2,h} \\ \vdots \\ \alpha_{I,h} \\ \beta_h \end{bmatrix}$$

As not all of our indices contain the same number of observations, the vector $\boldsymbol{y_t}$ as well as the matrix $\boldsymbol{X_{t,h}}$ have uneven lengths for every time t. The changing of dimensions can be done in several ways, and we utilize 2 different methods that we will discuss further in the estimation sections. In our OLS and GLS approaches, we will stack all our indices into one vector with all scaled returns. By doing this method, the regressions can be performed over all indices as if they were a single regression. On the other hand, on our MLE approach we perform the estimation over time across the indices. As we perform the MLE method, we introduce a new transformation matrix $\boldsymbol{H_t}$ that allows us to change the dimensions of the matrices, such that the likelihood function for each time period t can include the correct amount of active indices at that time. Once we define $\boldsymbol{H_t}$, the Equation (5) can now be written as:

$$\boldsymbol{H_t y'_t} = (\boldsymbol{H_t X'_{t,h}})\boldsymbol{B^*_{h,t}} + \boldsymbol{H_t \varepsilon'_t} \tag{6}$$

where $\boldsymbol{B^*_{h,t}}$ is an adjusted parameter vector containing $\alpha_{i,h}$ with varying dimension and β_h. This model is further elaborated in the MLE estimation section.

4.2 OLS and GLS estimation for VAR model with unbalanced panel

OLS assumptions

As in line with the methods performed by Moskowitz et al. (2012), we perform the OLS in a multivariate setting for the VAR model. The OLS is a regression method that allows us to estimate the parameters of a model by minimizing the squared errors of the said model. In estimating the OLS, we apply the standard set of assumptions regarding linearity in the model. We assume that the errors of the OLS model is homoskedastic with no autocorrelation, as in line with Cai and Hayes (2008), as heteroskedastic errors may cause Type 1 errors. We also need to use the additional assumption regarding normality, where the errors terms are multivariate normally distributed with mean 0 and covariance matrix $\sigma^2 I$. Lastly, we also require the assumption of linearly independent explanatory variables.

OLS estimation

In a multivariate setting, the OLS can be done in a similar way as a regular OLS by replacing the relevant dependent and explanatory variables with their multivariate counterpart. Based on our monthly collected data, T is set at 438, with $T = 1$ denoting May 1982 and $T = 438$ denoting September 2018. As not all of our indices start on the same date, only the S&P 500 index contains the full 438 observations while other indices contains less observations, with MIB being the shortest index with only 175 observations. In MIB's case as an example, the index will 'activate' at $t = 264$, with the same end date at $t = 438$ on September 2018. As previously mentioned, this results in uneven vectors of $\boldsymbol{y_t}$ and $\boldsymbol{X_{t,h}}$ from Equation (5), which requires a slightly adjusted estimation method when compared to a common balanced panel. In order to accommodate our different data lengths, we set the matrices in our equation to have time-varying dimensions, which will depend on any specific time period t. For example, $\boldsymbol{y_t}$ has a dimension of $I_t \times 1$, which indicates that it will be a 1×1 matrix at time $t = 1$ (as in this period, only the S&P 500 index is active), while it will transform into a 9×1 matrix at time $t = 438$ once every indices are active.

We rearrange our time-series panel dataset into a seemingly undated non-panel data in order to deal with this unbalanced panel problem. This sorting is performed by stacking all index returns in $\boldsymbol{y_t}$ for $t = 1, ..., T_i$ in a single vector, where the length of this vector can be written as $\sum_{i=1}^{I} T_i$. We sort these returns based on its indices, then on its time period. The sorting of the indices is started with the longest index, defined as the index with the highest T_i, up to the shortest index. This process can also be referred to as pooling, which helps us analyze these types of cross-sectional and time-series data.

Once our dependent variables vector has been established, we are able to take its corresponding lag variables along with a modified identity matrix as the independent variable. These new stacked and sorted matrices can be seen as the following:

$$\boldsymbol{r^*_{i,T_i}} = \begin{bmatrix} r^*_{i,1} \\ r^*_{i,2} \\ \vdots \\ r^*_{i,T_i-1} \\ r^*_{i,T_i} \end{bmatrix}, \quad \boldsymbol{\tilde{y}} = \begin{bmatrix} \boldsymbol{r^*_{1,T_1}} \\ \boldsymbol{r^*_{2,T_2}} \\ \vdots \\ \boldsymbol{r^*_{I,T_I}} \end{bmatrix}, \quad \boldsymbol{\tilde{X}_h} = \begin{bmatrix} 1 & \dots & 0 & \boldsymbol{r^*_{1,T_1-h}} \\ \vdots & \ddots & 0 & \boldsymbol{r^*_{2,T_2-h}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & \boldsymbol{r^*_{I,T_I-h}} \end{bmatrix}$$

The modified identity matrix in $\boldsymbol{\tilde{X}_h}$ is constructed as a regular $I \times I$ identity matrix, with each row being replicated T_i times, resulting in a $(I * T) \times I$ matrix of ones and zeroes. Instead of $\boldsymbol{y_t}$ and $\boldsymbol{X_{t,h}}$, the OLS estimation can now be done directly on $\boldsymbol{\tilde{y}}$ and $\boldsymbol{\tilde{X}_h}$

As an extension of the methods proposed by Moskowitz et al. (2012), we will expand on the OLS method by applying multivariate GLS in order to estimate the momentum effects while taking correlations between the residuals into account. We believe that residuals in a financial dataset may often be correlated with one another, and we can check how much does the GLS improve upon the OLS by alleviating the assumption on uncorrelated residuals.

FGLS assumptions

In estimating the GLS model, we also apply the same set of assumptions that we used in the OLS model with a single alteration. We now no longer assume that the errors are homoskedastic with no cross-correlation, but instead it is normally distributed with mean 0 and its covariance matrix Σ. The assumption of no autocorrelation still holds as in the case for the OLS. Although the model is supposed to be estimated by using the true covariance matrix Σ, we are not able to obtain this true covariance matrix as it is still unknown. We can, however, use an estimate of this covariance matrix, denoted as $\hat{\Sigma}$ and assume that this estimator is close enough to the true covariance matrix. By using the estimated covariance instead of the true covariance matrix, the estimation method is now called the feasible GLS (FGLS). As the main difference between FGLS and OLS will be the the covariance matrix of the errors, we do expect some differences in the estimated parameters due to how correlated each return series are, even after being scaled by its volatility. As both our FGLS and OLS models are using lagged variables as the independent variable, we can also expect that the FGLS may show relatively less significant parameter estimates compared to the parameters in the OLS, due to the additional correction for correlation of the scaled returns.

FGLS estimation

The FGLS estimation for unbalanced panel can be done similarly as the OLS, with some slight adjustments. The FGLS is structured exactly in the same way as the OLS, but we scale down both $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{X}}_{\boldsymbol{h}}$ by the Cholesky decomposition of the estimated covariance matrix that is obtained from the residuals our original OLS estimation. The covariance matrix based on the residuals of our OLS regression can be written as:

$$\boldsymbol{\varepsilon}^*_{\boldsymbol{i},\boldsymbol{T_i},\boldsymbol{h}} = \begin{bmatrix} \varepsilon^*_{i,1,h} \\ \varepsilon^*_{i,2,h} \\ \vdots \\ \varepsilon^*_{i,T_i-1,h} \\ \varepsilon^*_{i,T_i,h} \end{bmatrix}, \quad \tilde{\boldsymbol{\varepsilon}}_{\boldsymbol{h}} = \begin{bmatrix} \boldsymbol{\varepsilon}^*_{\boldsymbol{1},\boldsymbol{T_1},\boldsymbol{h}} \\ \boldsymbol{\varepsilon}^*_{\boldsymbol{2},\boldsymbol{T_2},\boldsymbol{h}} \\ \vdots \\ \boldsymbol{\varepsilon}^*_{\boldsymbol{I},\boldsymbol{T_I},\boldsymbol{h}} \end{bmatrix}, \quad \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{h}} = \begin{bmatrix} \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{11}} & \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{12}} & \cdots & \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{1I}} \\ \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{21}} & \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{22}} & \cdots & \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{2I}} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{I1}} & \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{I2}} & \cdots & \tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{II}} \end{bmatrix},$$

$$\tilde{\boldsymbol{\Sigma}}_h = \Lambda \, \Lambda'$$

where $\tilde{\boldsymbol{\Sigma}}^{\boldsymbol{h}}_{\boldsymbol{ij}}$ indicates the pair-wise covariance matrix between $\boldsymbol{\varepsilon}^*_{\boldsymbol{i},\boldsymbol{T_i},\boldsymbol{h}}$ and $\boldsymbol{\varepsilon}^*_{\boldsymbol{j},\boldsymbol{T_j},\boldsymbol{h}}$. It is important to note that due to our rearranged $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{X}}_{\boldsymbol{h}}$, the covariance matrix Σ_{OLS} also need to be adjusted based on the sorting of these variables. As we sort $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{X}}_{\boldsymbol{h}}$ from the longest index to the shortest index, the resulting covariance matrix will have the variance of the longest index on the top-leftmost corner, which moves diagonally for each indices up till the shortest index on the bottom-rightmost corner. We choose to perform the pair-wise covariance calculation as we need the same number of observations in both $\boldsymbol{\varepsilon}^*_{\boldsymbol{i},\boldsymbol{T_i},\boldsymbol{h}}$ and $\boldsymbol{\varepsilon}^*_{\boldsymbol{j},\boldsymbol{T_j},\boldsymbol{h}}$ in order to calculate the covariance. By using the pair-wise calculation, we do not need to limit all our indices strictly to the number of observations in the smallest index. Although we still need to leave out some observations when calculating the covariance between two index residuals with uneven lengths, the information that we leave out is relatively minimal.

As the covariance matrix $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{h}}$ is a positive semi-definite square matrix with a full rank, we can show that it is an invertible matrix. The Cholesky decomposition of the covariance matrix, which is denoted as Λ, is a invertible lower-triangular matrix that serves as our scaling component for $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{X}}_{\boldsymbol{h}}$. Our adjusted FGLS can then be performed by using the same multivariate OLS regression techniques on $\tilde{\boldsymbol{y}} \, \Lambda^{-1}$ and $\tilde{\boldsymbol{X}}_{\boldsymbol{h}} \, \Lambda^{-1}$.

4.3 MLE for VAR model with unbalanced panel

Although the OLS and FGLS can be performed on the unbalanced VAR model, it is difficult to expand it for later use in the MSVAR model. For that reason, we will look into the MLE method, that is able to be implemented well with the EM algorithm that is used in order to estimate the MSVAR model. Conventional multivariate MLE methods may not work on our VAR model with unbalanced panel without sacrificing some of the observations. For that reason, we construct an adjusted MLE method that allows us estimate the model parameters while using the whole observation sample.

MLE assumptions

As with the conventional comparisons between OLS and MLE, the MLE is able to provide the same unbiased estimates when uncorrelated residuals are assumed. Once we alleviate that assumption, the FGLS however may provide slightly different results compared to the MLE due to the different way of how the covariance matrix is taken into account. The FGLS uses the estimated covariance matrix obtained from our OLS as a proxy of the true covariance matrix, while MLE optimizes not only over the coefficients but also over the covariance matrix itself. The estimated covariance matrix obtained via MLE can be considered consistent, while in the FGLS it only holds asymptotically. In order to do the MLE, we assume that the log-likelihood function is continuously differentiable and the expected likelihood is bounded, with identifiable parameters. Lastly, we also assume normality in our estimation.

MLE estimation

The adjusted MLE can first be written as a regular MLE for a multivariate problem, which can be written as:

$$\theta_h^{MLE} = \arg\max_{\theta_h} \sum_{t=1}^{T} \mathcal{L}(\boldsymbol{y_t}; \boldsymbol{X_{t,h}}, \theta_h) \tag{7}$$

where we define θ_h as:

$$\theta_h = \left\{\boldsymbol{B_h},\ \Lambda\right\}$$

with Λ being the Cholesky decomposition of the covariance matrix, Σ, which is written as:

$$\Sigma = \Lambda\ \Lambda'$$

The aim of the maximum likelihood method will be to select the optimal $\boldsymbol{B_H}$ and Λ such that the log likelihood will be maximized. Note that we choose the Cholesky decomposition Λ as the maximizing parameter instead of the covariance matrix Σ, in order to ensure a positive semi-definite covariance matrix. In order to calculate the likelihood function, we assume that the errors are multivariate normally distributed with mean 0 and covariance matrix Σ. We also assume that the log-likelihood function $\mathcal{L}(\boldsymbol{y_t}; \boldsymbol{X_{t,h}}, \theta_h)$ is continuously differentiable and bounded around the true parameter. By assuming that the errors are normally distributed, we know that the marginal distributions of our multivariate distribution are also normal distributions.

Due to the unbalanced nature of our dataset, the log likelihood of each time period t must be calculated separately before being finally summed together. At each time period t, we have to check how many indices are active, which will change the dimensions of $\boldsymbol{y_t}$, $\boldsymbol{X_{t,h}}$, $\boldsymbol{B_h}$, and Σ. The dimension changes are done by introducing a transformation matrix $\boldsymbol{H_t}$ which has a dimension of $i_t \times i_T$. The transformation matrix $\boldsymbol{H_t}$ can be written as an identity matrix with dimension i_t for how many indices are active at time t, along with a $i_t \times (i_T - i_t)$ matrix of zeroes. As previously seen in Equation (6), for every time period t the equation can then be written as:

$$\boldsymbol{H_t y_t'} = (\boldsymbol{H_t X_{t,h}'})\boldsymbol{B_{h,t}^*} + \boldsymbol{H_t \varepsilon_t'}$$

This allows $\boldsymbol{H_t y_t'}$ to have a dimension of $i_t \times 1$. The lagged return matrix $\boldsymbol{H_t X_{t,h}'}$ has a dimension of $i_t \times (i_t + 1)$ as it includes a time-varying $i_t \times i_t$ identity matrix and the $i_t \times 1$ lagged returns vector. The parameter vector $\boldsymbol{B_{h,t}^*}$ for time t is an adjusted form of $\boldsymbol{B_h}$, which includes only the active constants $\alpha_{i,h}$ at time t but always includes β_h as the last element. The adjusted vector $\boldsymbol{B_{h,t}^*}$ has a dimension of $(i_t + 1) \times 1$, as it contains the i_t amount of constants along with β_h.

Once we have the proper $\boldsymbol{H_t y_t'}$, $\boldsymbol{H_t X_{t,h}'}$, and $\boldsymbol{B_{h,t}^*}$ for time t, we can calculate the error terms $\boldsymbol{H_t \varepsilon_t'}$ which is a $i_t \times 1$ vector. The likelihood ratio can then be obtained by using the multivariate normal density function $\boldsymbol{\varepsilon_t} \sim MVN(0, \boldsymbol{H_t}\Sigma\boldsymbol{H_t'})$. Note that although Σ is constant over time, we need to use only the top-left $i_t \times i_t$ submatrix of Σ in the multivariate normal distribution, which can be denoted as $\boldsymbol{H_t}\Sigma\boldsymbol{H_t'}$. This process requires $\boldsymbol{y_t}$ and $\boldsymbol{X_{t,h}}$ to be ordered from the longest index in the first column, up to the shortest index in the last column. This process can then be done for all time t, and the log likelihood of each time period is then summed as a total. The likelihood maximization itself is solved by utilizing numerical optimization, since obtaining an analytical solution for this problem will prove to be too complicated.

4.4 MSVAR model specification

As an extension from the VAR model, the Markov-Switching VAR model is applied by introducing several unobserved state variables in the data. The inclusion of state variables in our model allows us to analyze whether any of the momentum effects found in our VAR model is consistently present regardless of the state of the market, or do they vary along with each underlying state. By assuming that there are more than one states of the economy, the MSVAR model allows us to assign parameters of the VAR model to each underlying state. For the sake of simplicity, we will assume that there are only 2 states in the economy, which is defined as a low-momentum state ($S_t = 1$) and a high-momentum state ($S_t = 2$). The MSVAR(H) model can be written in an explicit formula, similar to Equation (3) for the VAR(H) model, as:

$$r^*_{i,t} = \begin{cases} \alpha_{i,h,1} + \beta_{h,1}\, r^*_{i,t-h} + \varepsilon_{i,t}, & \text{if } S_t = 1 \\ \alpha_{i,h,2} + \beta_{h,2}\, r^*_{i,t-h} + \varepsilon_{i,t}, & \text{if } S_t = 2 \end{cases} \tag{8}$$

where S_t is the unobserved state of the market. When rewritten into a matrix form, Equation (8) becomes:

$$\boldsymbol{y_t} = \boldsymbol{X_{t,h} B_{h,S_t}} + \boldsymbol{\varepsilon_t} \tag{9}$$

with $S_t = \{1, 2\}$. The introduction of state variables essentially multiplies the number of parameters we need to estimate by the number of states, which in this case doubles our number of parameters. We can see how this may be a problem when we introduce even more state variables into the model, as the computation time and the complexity of the problem may grow exponentially.

4.5 EM algorithm for MSVAR model

As previously mentioned, in order to estimate the MSVAR model it is not possible to simply perform the FGLS, and our adjusted MLE method will not be able to perform well due to the presence of unobserved underlying states. For this reason, we need to use the Expectation-Maximization (EM) algorithm introduced by Dempster et al. (1977). The EM algorithm is a popular way of estimating a time-series model that relies on an underlying state variable. The EM algorithm can be split into two steps, one being the 'expecation' step, while the other being the 'maximization' step. Both steps have to be done repeatedly for multiple iterations in order to reach convergence.

Expectation step

The EM algorithm is performed by first taking the joint likelihood of $\boldsymbol{y_t}$ and S_t, conditional on past information. The true log likelihood can be written as:

$$\mathcal{L}(\boldsymbol{y_t}, S_t; \theta_{h,S_t}) = \sum_t^T \log \Big(\sum_{s=1}^S f(\boldsymbol{y_t}; \theta_{h,S_t} | S_t = s) * P(S_t = s | \boldsymbol{y_{t-1}}) \Big) \tag{10}$$

Note that optimizing over the true likelihood is hard to do, due to the fact that the probability $P(S_t = s|\boldsymbol{y_{t-1}})$ has a recursive function which forces us to perform a state augmentation and thus further increases the dimensionality by a significant amount. In order to avoid this problem, we will instead optimize over the expected likelihood instead of the true likelihood function. The expected likelihood uses the smoothed probabilities $P(S_t = s|\boldsymbol{y_T})$ instead of $P(S_t = s|\boldsymbol{y_{t-1}})$ in order to eliminate our recursion problem. The expected likelihood can be written as:

$$\hat{\mathcal{L}}(\boldsymbol{y_t}, S_t; \theta_{h,S_t}) = \sum_{s=1}^{S} \sum_{t}^{T} \log\left(f(\boldsymbol{y_t}; \theta_{h,S_t} | S_t = s)\right) * P(S_t = s | \boldsymbol{y_T}) \tag{11}$$

The smoothed probability $P(S_t = s|\boldsymbol{y_T})$ denotes the occurrence probability of a state given all available information up till time T, which is obtained by performing the probability smoother proposed by Hamilton (1994).

The Hamilton filter allows us to calculate the predicted transition probabilities between the states, as well as the estimated state variables $\boldsymbol{\hat{\xi}_{t|t}}$. If we assume that the state variables at time t is given, then we can get the predicted states $\boldsymbol{\hat{\xi}_{t+1|t}}$ by using:

$$\boldsymbol{\hat{\xi}_{t+1|t}} = \begin{pmatrix} p_{11} & 1 - p_{22} \\ 1 - p_{11} & p_{22} \end{pmatrix} \boldsymbol{\hat{\xi}_{t|t}} = \boldsymbol{P}\boldsymbol{\hat{\xi}_{t|t}}$$

where $\boldsymbol{P}$ indicates the transition probabilities between states. By using the fact that $P(S_t = s_t|\boldsymbol{y_t}) = f(\boldsymbol{y_t}, S_t = s_t; \theta_{h,S_t}|\boldsymbol{y_{t-1}})/f(\boldsymbol{y_t}; \theta_{h,S_t}|\boldsymbol{y_{t-1}})$, we can write the Hamilton updating step as:

$$\boldsymbol{\hat{\xi}_{t|t}} = \frac{\begin{pmatrix} f(\boldsymbol{y_t}|S_t = 1) \\ f(\boldsymbol{y_t}|S_t = 2) \end{pmatrix} \odot \boldsymbol{\hat{\xi}_{t|t-1}}}{\begin{pmatrix} 1 & 1 \end{pmatrix} \left[\begin{pmatrix} f(\boldsymbol{y_t}|S_t = 1) \\ f(\boldsymbol{y_t}|S_t = 2) \end{pmatrix} \odot \boldsymbol{\hat{\xi}_{t|t-1}} \right]}$$

The Hamilton filter is done iteratively for every t from $t = 1$, until we reach T. Once we have obtained the filtered and predicted states, we can calculate the smoothed probabilities, given as:

$$\boldsymbol{\hat{\xi}_{t|T}} = \boldsymbol{\hat{\xi}_{t|t}} \odot \boldsymbol{P'}(\boldsymbol{\hat{\xi}_{t+1|T}} \oslash \boldsymbol{\hat{\xi}_{t+1|t}})$$

$$\boldsymbol{P}^*(t) = \boldsymbol{P} \odot (\boldsymbol{\hat{\xi}_{t|T}}\boldsymbol{\hat{\xi}'_{t-1|t-1}}) \oslash (\boldsymbol{\hat{\xi}_{t|t-1}} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix})$$

where $\odot$ and $\oslash$ indicates element-wise multiplication and division respectively. Unlike the Hamilton filter, the Hamilton smoother is performed iteratively from the last observation, T, until the first observation at $t = 1$.

In order to perform the EM algorithm, we first take our initialized $\theta_{h,S_t}^{(0)}$ as a given, which we then use in the Hamilton filter and smoother written above to obtain the initialized expected probabilities $P(S_t = s|\boldsymbol{y_T})^{(0)}$. Once we have the expected probabilities, we proceed to calculate the newly calculated parameters $\theta_{h,S_t}^{(1)}$. These newly obtained parameters can then be used again to calculate the new expected probabilities of the first iteration, $P(S_t = s|\boldsymbol{y_T})^{(1)}$. This process is repeated iteratively until the true likelihood $\mathcal{L}(\boldsymbol{y_t}, S_t; \theta_{h,S_t}^{(k)})$ reach convergence. We decide whether the algorithm has reached convergence at the K^{th} iteration by setting the condition that $\mathcal{L}(\boldsymbol{y_t}, S_t; \theta_{h,S_t}^{(K)}) - \mathcal{L}(\boldsymbol{y_t}, S_t; \theta_{h,S_t}^{(K-1)}) < \epsilon$, where we set

$\epsilon = 0.01$. The condition is applied with the restriction that the true likelihood is strictly increasing over each iteration.

Maximization step

In order to obtain $\theta_{h,S_t}^{(k)}$ from $P(S_t = s|\boldsymbol{y_T})^{(k-1)}$, we calculate:

$$\theta_{h,S_t}^{(k)} = \underset{\theta_{h,S_t}}{\arg\max}\ E\Big(\mathcal{L}(\boldsymbol{y_t}, S_t; \theta_{h,S_t})|\boldsymbol{y_t}; \theta_{h,S_t}^{(k-1)}\Big) \tag{12}$$

As we estimate the parameters also via numerical optimization, the process is done in a similar way as our adjusted MLE for unbalanced panel, with some slight differences that allows us to optimize over the parameters of each underlying state. The optimization algorithms are done separately for parameters of each state variable, in order to improve efficiency due to the limitations of performing a joint optimization with multiple parameters in MATLAB. Each state parameters are then used to calculate the joint true likelihood that decides whether we should do another iteration or not, where the algorithm will stop once the true likelihood of the current iteration increases by less than 0.01.

The MLE procedure for the low momentum state ($S_t = 1$) in our MSVAR model is initialized as the MLE parameters from our VAR model, but only with $\beta_{h,1}^{(0)} = 0.2\ \beta_h^{VAR(MLE)}$, where (0) indicates the 0^{th} iteration or the initialization. As for our high momentum state ($S_t = 2$), the parameters are initialized as $\beta_{h,2}^{(0)} = 5.0\ \beta_h^{VAR(MLE)}$. The likelihood optimization is also slightly adjusted in both states, as the likelihood function for each time period t has been weighted by the smoothed probability of the previous iteration, denoted as $P(S_t = s|\boldsymbol{y_T})^{(k-1)}$. As previously mentioned, the smoothed probabilities is used to calculate the expected likelihood only in our optimization function, as the convergence condition as well as information criterion calculation utilize the true likelihood and not the expected likelihood.

4.6 Sign regression

In addition to the specified VAR and MSVAR models on the scaled returns, we expand the sign regression model proposed by Moskowitz et al. (2012). The sign regression is an alternative, simpler way to analyze the momentum factor, and it provides a different insight from the regression on scaled returns. The sign regression treats all explanatory variables as either 1 or -1 depending on whether the return at time t is positive or negative, which allows us to ignore the magnitude of the lagged returns at any given time period. We perform the same VAR model with OLS, GLS, and MLE on these sign variables in the same way, which we then extend into the MSVAR model as well. As in line with Moskowitz et al. (2012), the sign VAR model can be written as:

$$r_{i,t}^{*} = \alpha_{i,h,sign} + \beta_{h,sign} sign(r_{i,t-h}) + \varepsilon_{i,t} \tag{13}$$

where $r^*_{i,t}$ indicates the scaled returns of index i at time t, otherwise written as $r_{i,t}/\sigma_{i,t-1}$. The same formulation is also used in our sign MSVAR model, which is denoted as follows:

$$r^*_{i,t} = \begin{cases} \alpha_{i,h,1,sign} + \beta_{h,1,sign}\ sign(r_{i,t-h}) + \varepsilon_{i,t}, & \text{if } S_t = 1 \\ \alpha_{i,h,2,sign} + \beta_{h,2,sign}\ sign(r_{i,t-h}) + \varepsilon_{i,t}, & \text{if } S_t = 2 \end{cases} \tag{14}$$

The sign VAR model and the sign MSVAR model serve as extensions to our VAR and MSVAR models, providing an analysis of the momentum factor from a different perspective. The coefficient results of the sign and non-sign models are not directly comparable due to the different explanatory variables, although we can still check the significance levels of the t-statistics for both cases.

4.7 Testing

Wald Test

We perform a Wald test on the MSVAR model coefficients in order to test whether the parameters of $S_t = 1$ and $S_t = 2$ are statistically different. The Wald test is performed by taking the difference in the coefficients of both states, which is then scaled by the standard errors of the two coefficients. The Wald test, as in line with Clogg et al. (1995), can be written as:

$$W_h = \frac{\beta_{h,1} - \beta_{h,2}}{\sqrt{(SE^2_{h,1} + SE^2_{h,2} - 2cov(\beta_{h,1}, \beta_{h,2}))}} \quad \sim Z \tag{15}$$

for each lags h, where W_h indicates the Wald test statistic for lag h. $\beta_{h,1}$ and $\beta_{h,2}$ denotes the estimated momentum coefficients of state 1 and 2 , while $SE_{h,1}$, $SE_{h,1}$ and $cov(\beta_{h,1}, \beta_{h,2})$ denotes their standard errors and covariance respectively for every lag h. As the Wald test statistics follows the standard normal statistic Z, we can test for 5% significance level by checking if W_h is larger than 1.96.

Akaike Information Criterion (AIC) comparison

We calculate the AIC for both the VAR model, estimated by the MLE, as well as the MSVAR model, estimated by the EM algorithm. The formula used for calculating the AIC is as follows:

$$AIC = 2k - 2LogL \tag{16}$$

where k indicates the number of parameters, while $LogL$ indicates the maximized log likelihood of the estimated model. In calculating the AIC of the EM algorithm, we use the true likelihood calculated from our estimated parameters, instead of the expected likelihood.

5 Results

5.1 VAR model estimation

As we first construct our model as a single-state VAR model, the following tables report the different estimation results from our three estimation methods of the VAR model. Table 4 reports the regression coefficients and the t-statistics of our VAR model, respectively, while Table 5 reports the coefficients and t-statistics of the same VAR model but on lagged sign variables instead of lagged scaled returns. The t-statistics of the VAR model and the sign VAR model are also plotted in Figures 3 and 4.

Table 4: β_h of VAR model

h	β_h^{OLS}	β_h^{GLS}	β_h^{MLE}	$t\text{-}stats_h^{OLS}$	$t\text{-}stats_h^{GLS}$	$t\text{-}stats_h^{MLE}$
1	3.249 (1.873)	-3.052 (1.874)	-2.883 (1.870)	1.73	-1.63	-1.54
2	2.924 (1.874)	3.524 (1.871)	-0.620 (1.869)	1.56	1.88	-0.33
3	8.121 (1.867)	2.949 (1.869)	2.338 (1.864)	4.35	1.58	1.25
4	0.737 (1.874)	0.817 (1.871)	-3.123 (1.855)	0.39	0.44	-1.68
5	2.804 (1.882)	1.263 (1.873)	3.435 (1.862)	1.49	0.67	1.84
6	5.022 (1.883)	2.688 (1.874)	4.785 (1.861)	2.67	1.43	2.57
7	2.338 (1.885)	8.720 (1.869)	1.909 (1.860)	1.24	4.66	1.03
8	0.154 (1.898)	0.621 (1.184)	-1.026 (1.868)	0.08	0.52	-0.55
9	1.877 (1.912)	8.015 (1.882)	0.843 (1.874)	0.98	4.26	0.45
10	3.458 (1.912)	9.459 (1.881)	-0.318 (1.879)	1.81	5.03	-0.17
11	-0.038 (1.913)	1.768 (1.886)	1.841 (1.879)	-0.02	0.94	0.98
12	1.364 (1.914)	5.305 (1.869)	3.889 (1.881)	0.71	2.84	2.07

[1] The numbers in parentheses indicates the standard errors of the estimated β_h. Both the values of β_h and the standard errors are multiplied by $\times 10^2$.

Table 5: β_h of sign VAR model

h	β_h^{OLS}	β_h^{GLS}	β_h^{MLE}	$t\text{-}stats_h^{OLS}$	$t\text{-}stats_h^{GLS}$	$t\text{-}stats_h^{MLE}$
1	1.993 (2.665)	-2.443 (2.335)	-3.197 (2.087)	0.75	-1.05	-1.53
2	6.901 (2.663)	4.628 (2.334)	1.751 (2.082)	2.59	1.98	0.84
3	10.863 (2.659)	4.210 (2.330)	1.395 (2.076)	4.09	1.81	0.67
4	-0.010 (2.672)	-1.997 (2.349)	-2.623 (2.070)	0.00	-0.85	-1.27
5	5.126 (2.675)	0.336 (2.342)	5.151 (2.068)	1.92	0.14	2.49
6	6.230 (2.678)	7.669 (2.348)	3.012 (2.068)	2.33	3.27	1.46
7	3.831 (2.685)	9.283 (2.346)	2.544 (2.064)	1.43	3.96	1.23
8	-3.018 (2.691)	-1.415 (1.642)	-0.746 (2.068)	-1.12	-0.86	-0.36
9	1.810 (2.695)	9.563 (2.328)	2.375 (2.066)	0.67	4.11	1.15
10	2.258 (2.698)	4.095 (2.323)	-0.713 (2.067)	0.84	1.76	-0.34
11	4.506 (2.698)	1.525 (2.321)	0.548 (2.076)	1.67	0.66	0.26
12	-0.543 (2.701)	2.059 (2.325)	-2.159 (2.081)	-0.20	0.89	-1.04

[1] The numbers in parentheses indicates the standard errors of the estimated β_h. Both the values of β_h and the standard errors are multiplied by $\times 10^2$.

Table 6: Log likelihood of VAR and sign VAR models estimation results

h	$LogL^{OLS}$	$LogL^{GLS}$	$LogL^{MLE}$	$LogL^{OLS}_{sign}$	$LogL^{GLS}_{sign}$	$LogL^{MLE}_{sign}$
1	-5413.4	-3855.3	-3776.7	-5415.8	-3862.9	-3776.8
2	-5387.9	-3837.9	-3757.9	-5383.7	-3839.5	-3757.6
3	-5348.5	-3808.1	-3737.0	-5350.6	-3816.8	-3737.5
4	-5350.9	-3811.8	-3712.5	-5351.1	-3820.7	-3713.1
5	-5332.2	-3794.2	-3699.0	-5330.7	-3801.1	-3697.6
6	-5312.0	-3779.4	-3682.8	-5313.7	-3793.6	-3685.0
7	-5300.4	-3776.1	-3669.1	-5299.9	-3786.0	-3668.9
8	-5284.9	-3778.5	-3660.2	-5283.9	-3788.2	-3660.3
9	-5267.0	-3782.3	-3648.1	-5267.5	-3792.6	-3647.6
10	-5245.2	-3766.9	-3635.5	-5247.7	-3770.7	-3635.4
11	-5224.2	-3746.9	-3618.7	-5221.5	-3753.4	-3619.1
12	-5204.3	-3692.8	-3606.2	-5204.8	-3704.8	-3607.7

The regression constants $\alpha_{i,h}$ for the OLS, GLS, and MLE methods can be found in the Appendix. The regressions used here utilizes different constants for each lag and each index. As seen from Table 4, most of the regression coefficients of the OLS, GLS and MLE methods show positive momentum factor, which is in line with the findings of Moskowitz et al. (2012). Based on the t-statistics from Table 4, we can see that only a few of these results are sigificant under the 5% confidence level. The estimated results of the sign VAR models as well as the t-statistics are shown in Table 5. The resulting t-statistics of the non-sign regressions are shown in Figure 3, and t-statistics of the sign regressions inFigure 4. As our OLS method tried to replicate the methodology from Moskowitz et al. (2012), we do obtain similar results as the first 12 months of their results. However, as we try to correct for correlations between equity indices, we start to see significant differences in our results.

As we start with the GLS, performing the linear regression by scaling the variables with the OLS-estimated covariance matrix does not change the results very much other than on the first lag. While the momentum factor for lags 2 to 12 still show positive results, we obtain negative effect on the first lag. Similar to our findings in the regressions on scaled returns, the sign regressions show mainly positive the momentum effect under the OLS. When we perform the GLS, some lags also turn from positive to negative values, although none of the negative values prove to be significant under the 5% level. As we perform the MLE on the VAR models on scaled returns and sign variables, we observe more significant changes in our estimated results from the OLS, compared to the GLS. For both the VAR model and the sign VAR model, we can see that even more non-positive momentum effects on the different lags, such that we may not see the whole 12 lags as to be jointly positive anymore. The log likelihood of each estimation methods for both models are shown in Table 6, and we can see that GLS and MLE offers significantly better fit than the OLS, mainly due to the inclusion of covariance matrix in the estimation methods. The MLE offers slightly better fit than the GLS as well, as the MLE optimizes over the covariance matrix as well instead of only using a proxy as previously mentioned.

Table 7: Covariance-correlation matrix of VAR model for $h = 1$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7490	0.6541	0.3817	0.6781	0.6541	0.5496	0.6601	0.6230	0.5717
2	0.6541	1.8849	0.4329	0.7635	0.7234	0.6925	0.7845	0.7448	0.7113
3	0.3817	0.4329	1.7879	0.4161	0.3810	0.3764	0.4074	0.5259	0.3828
4	0.6781	0.7635	0.4161	2.2986	0.8413	0.7185	0.8965	0.6469	0.8052
5	0.6541	0.7234	0.3810	0.8413	2.1212	0.7317	0.9080	0.5970	0.8110
6	0.5496	0.6925	0.3764	0.7185	0.7317	2.4219	0.8174	0.6290	0.8644
7	0.6601	0.7845	0.4074	0.8965	0.9080	0.8174	1.7372	0.6467	0.8828
8	0.6230	0.7448	0.5259	0.6469	0.5970	0.6290	0.6467	2.2346	0.6093
9	0.5717	0.7113	0.3828	0.8052	0.8110	0.8644	0.8828	0.6093	2.4694

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

As the MLE provides the estimated covariance matrix for each lag, we can observe high correlations in the scaled returns between each equity indices for every lag, with an average correlation across all indices for each lags being around 0.66. The covariance-correlations table estimated from the MLE for the first lag can be seen in Table 7, while the covariance-correlations tables for the rest of the lags as well as for the sign VAR models can be seen in the Appendix. This gives an indication that the OLS results that show mostly positive momentum factors may be flawed to a certain degree, as it ignores significant amount of information regarding the covariance matrix and the correlations between indices.

Figure 3: t-Statistics of VAR model estimation with OLS, GLS and MLE

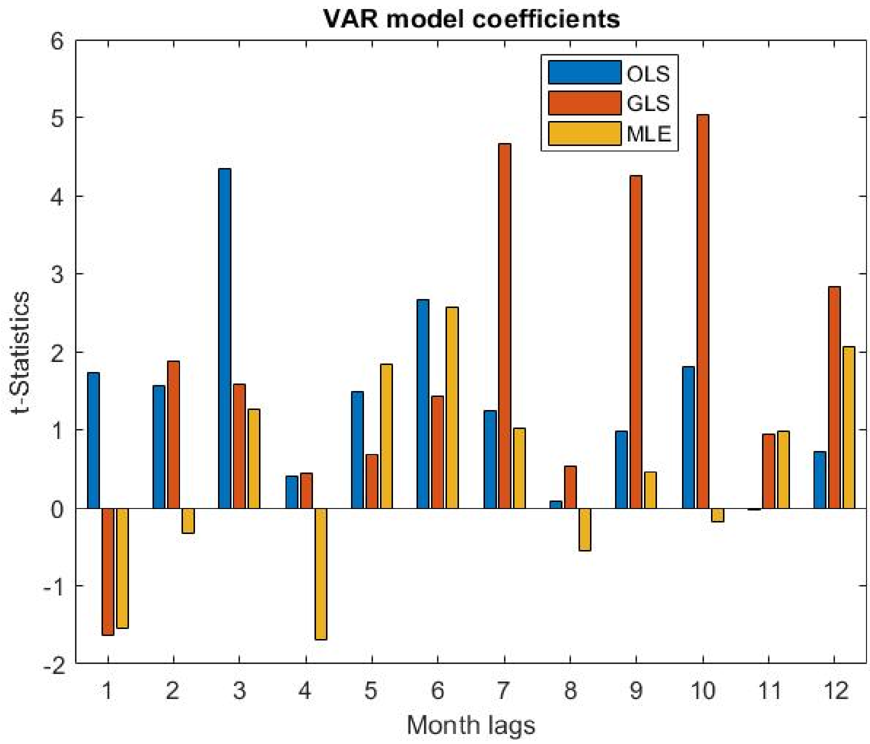
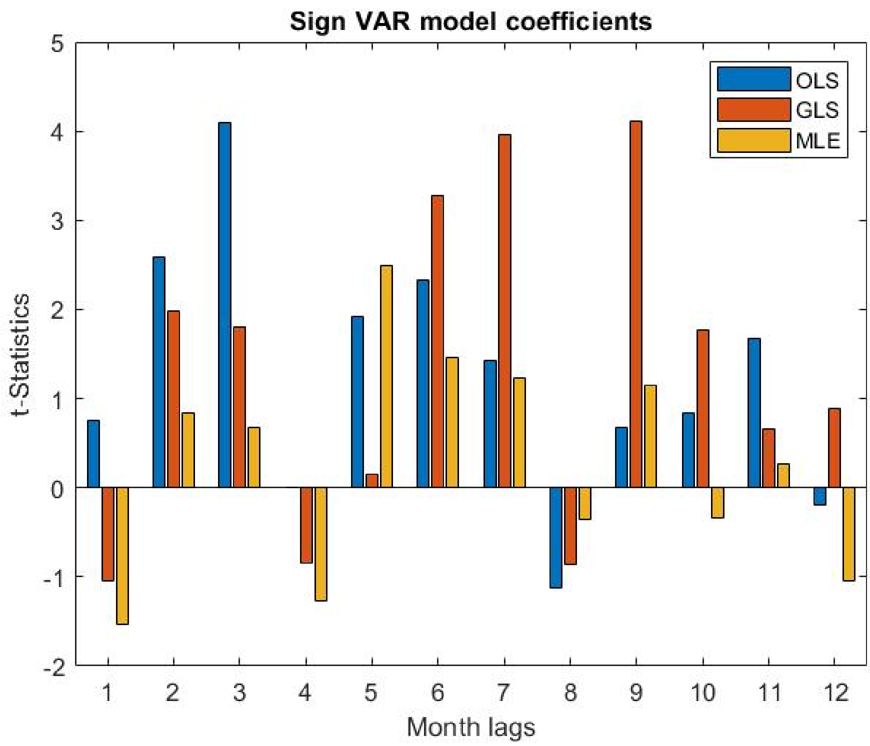


Figure 4: t-Statistics of sign VAR model estimation with OLS, GLS and MLE



5.2 EM algorithm for MSVAR model

As we apply the EM algorithm in the MSVAR model, we obtained two beta values for each lags in both states. As our EM algorithm utilizes our adjusted MLE method in the optimization process in every iteration, we initialize our parameters based on the MLE results of the VAR model. As a result, our estimated parameters for the MSVAR model shows a closer resemblance to our MLE results of the VAR model compared to the OLS and GLS, displaying a more diverged coefficients centered around zero rather than the jointly positive coefficients across all lags. The estimated results of the MSVAR model is shown in Table 8, and the t-statistics are plotted in Figure 5.

Table 8: $\beta_h^{S_t}$ results obtained from EM algorithm for MSVAR model

h	$\beta_h^{S_t=1}$	$\beta_h^{S_t=2}$	$t\text{-}stats_h^{S_t=1}$	$t\text{-}stats_h^{S_t=2}$
1	-11.350 (2.925)	2.868 (2.262)	-3.88	1.27
2	-0.869 (2.590)	0.138 (2.664)	-0.34	0.05
3	-2.566 (2.493)	7.891 (2.891)	-1.03	2.73
4	-6.978 (1.962)	-0.327 (2.703)	-3.56	-0.12
5	-0.301 (2.307)	6.965 (2.819)	-0.13	2.47
6	0.175 (2.136)	7.506 (2.755)	0.08	2.72
7	-9.579 (2.571)	4.841 (2.125)	-3.73	2.28
8	-10.414 (2.690)	5.229 (2.176)	-3.87	2.40
9	-0.804 (2.809)	4.521 (2.095)	-0.29	2.16
10	-3.424 (1.780)	4.990 (2.252)	-1.92	2.22
11	-2.031 (2.191)	3.797 (2.853)	-0.93	1.33
12	1.785 (2.514)	6.451 (2.123)	0.71	3.04

[1] The numbers in parentheses indicates the standard errors of the estimated β_h. Both the values of β_h and the standard errors are multiplied by $\times 10^2$.

Table 9: $\beta_h^{S_t}$ results obtained from EM algorithm for sign MSVAR model

h	$\beta_{h,sign}^{S_t=1}$	$\beta_{h,sign}^{S_t=2}$	$t\text{-}stats_{h,sign}^{S_t=1}$	$t\text{-}stats_{h,sign}^{S_t=2}$
1	-28.911 (3.035)	1.615 (1.797)	-9.52	0.90
2	0.319 (3.905)	2.832 (2.981)	0.08	0.95
3	-5.216 (4.308)	9.157 (2.989)	-1.21	3.06
4	-4.224 (3.047)	0.105 (2.356)	-1.39	0.04
5	-0.186 (2.275)	10.525 (3.576)	-0.08	2.94
6	-1.038 (2.637)	7.250 (3.407)	-0.39	2.13
7	-7.651 (3.009)	7.825 (2.031)	-2.54	3.85
8	-4.691 (4.160)	0.230 (3.165)	-1.13	0.07
9	-0.775 (2.258)	7.932 (3.203)	-0.34	2.48
10	-5.467 (2.978)	3.936 (2.101)	-1.84	1.87
11	-3.652 (5.095)	2.782 (1.997)	-0.72	1.39
12	-6.849 (3.202)	3.259 (2.262)	-2.14	1.44

[1] The numbers in parentheses indicates the standard errors of the estimated β_h. Both the values of β_h and the standard errors are multiplied by $\times 10^2$.

Figure 5: t-Statistics of lagged returns in MSVAR model

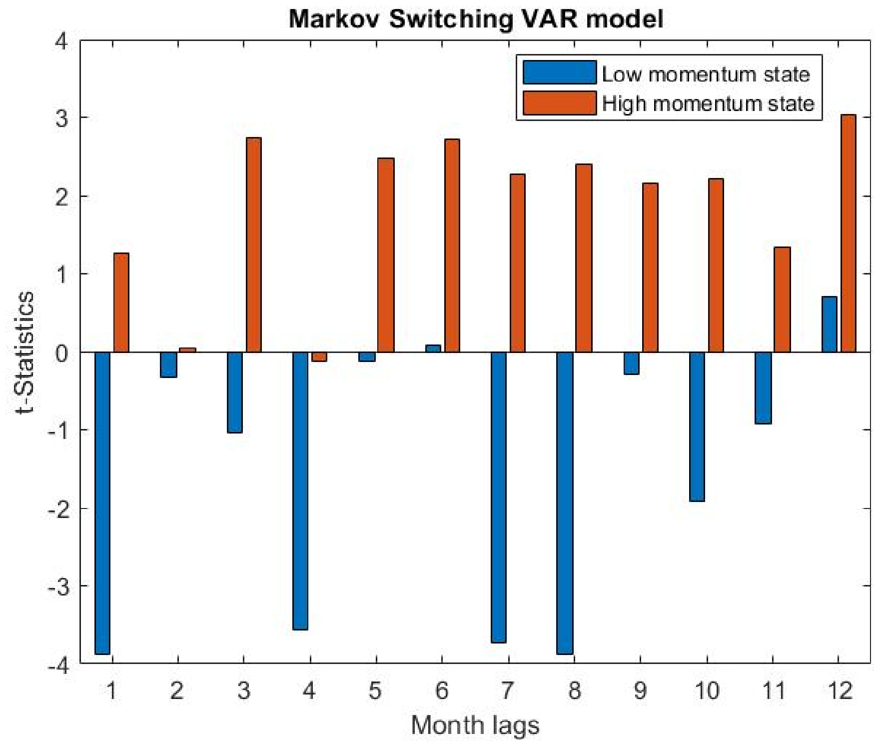


Figure 6: t-Statistics of lagged returns in sign MSVAR model

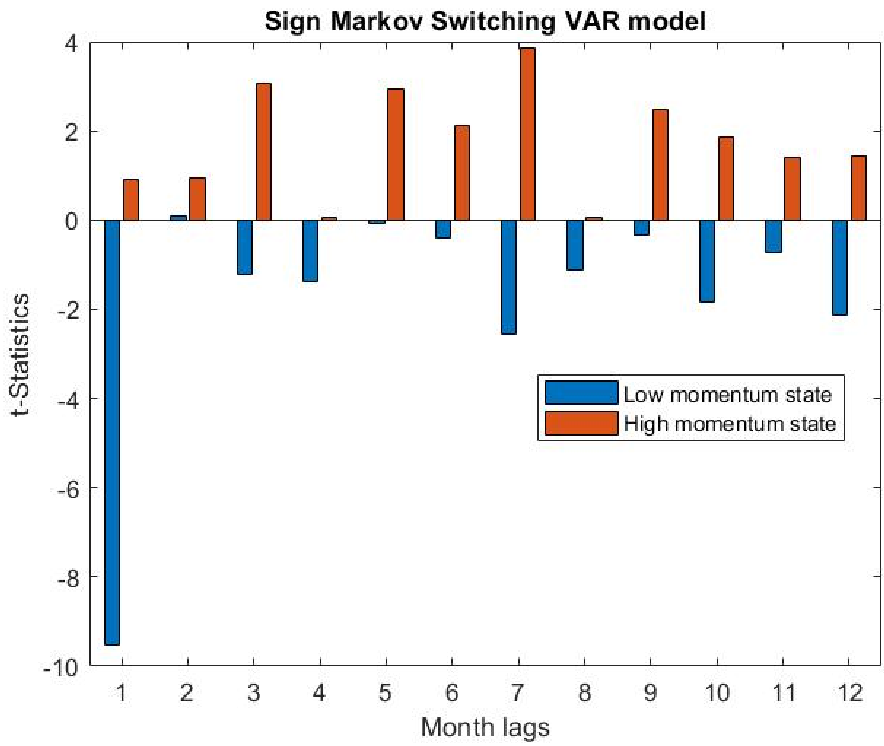
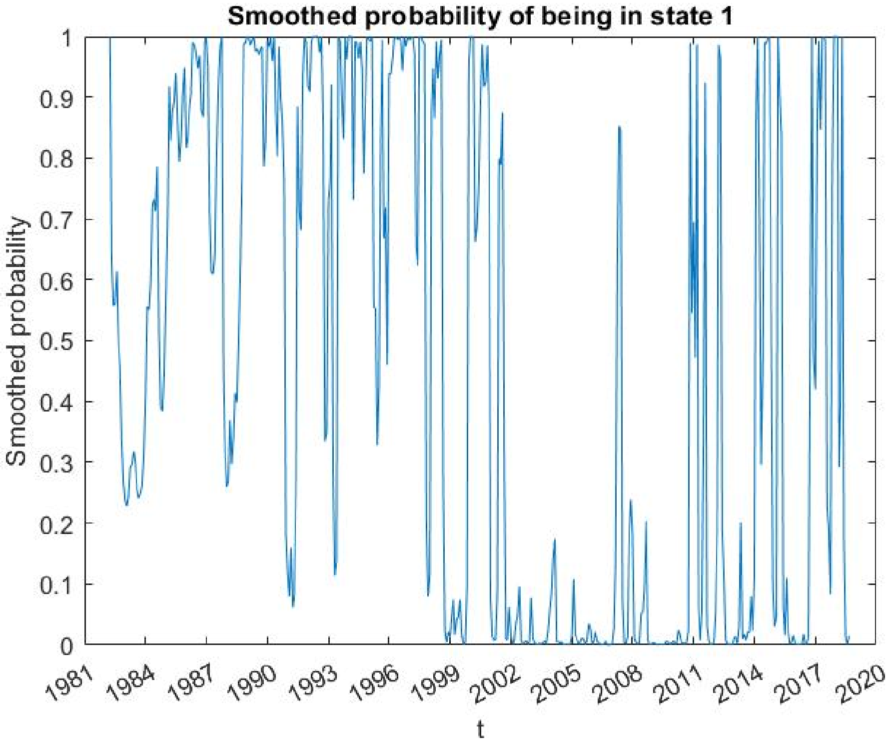


Figure 7: Smoothed probability of $S_t = 1$ for $h = 1$



As seen from Table 8, the momentum coefficients in the low-momentum state ($S_t = 1$) are often very close to zero, or significantly negative which indicates a mean-reversal effect. The high-momentum state do show significant momentum factor across almost all lags, with the exception being on $h = 2$ and $h = 4$. We observe that in 8 out of 12 lags, the momentum factor show different signs across the low-momentum state and the high-momentum state. On lags $h = 7, 8, 10$, we observe two significant coefficients with opposing signs across both states. For these lags with opposite signs for each state, we can argue that the states may be defined as a mean-reversal state and a high-momentum state, although it does not apply for all lags. As with our MLE results on the VAR model, lags $h = 1, 4, 8$ shows significant mean-reversal effects on $S_t = 1$. In Figure 6, we can see that the t-statistic of the first lag in the sign MSVAR model has a very significant negative value in the low-momentum state, indicating an extreme mean-reversal effect in $S_t = 1$. Although a similar significant mean-reversal effect is also present in the non-sign MSVAR model, it is not nearly as drastic as the one in the sign MSVAR model. In $h = 4$, both the non-sign and the sign MSVAR model shows very small momentum factor even in the high-momentum state, which may indicate that the regime switches between a mean-reversal state and a random walk state where there is no significant momentum effect in either direction for this particular lag.

Based on the smoothed probabilities plot in Figure 7, there are noticeable regime changes during recession periods such as the 2001 and 2007 crises. We can also see a significant regime change from before and after the 2001 dot-com crisis. The remaining smoothed probabilities plots for $h = 2, ..., 12$, as well as the smoothed probabilities for the sign

MSVAR model can be seen in the Appendix. As in line with our MLE results for the VAR model, we cannot observe a jointly positive momentum effect as with the OLS results and the results of Moskowitz et al. (2012) across both states, which is also the case for the sign MSVAR model results. The contrast between mean-reversal and positive momentum factors between the two states may bring some difficulties in constructing a specific strategy that only utilizes the momentum effect, mainly due to the unpredictability of the momentum states outside of recession periods and the varying state probabilities for each monthly lags.

5.3 Wald test results

Table 10: Test statistic and p-value of the Wald test

h	Wald statistic	Wald statistic (sign)	p-value	p-value (sign)
1	-3.845	-2.956	0.000	0.005
2	-0.270	-0.442	0.385	0.362
3	-2.818	-2.357	0.008	0.025
4	-2.188	-1.121	0.036	0.213
5	-2.107	-2.499	0.043	0.018
6	-2.066	-1.962	0.047	0.058
7	-4.247	-4.158	0.000	0.000
8	-4.563	-0.829	0.000	0.283
9	-1.591	-2.347	0.113	0.025
10	-2.657	-2.752	0.012	0.009
11	-1.649	-1.275	0.102	0.177
12	-1.488	-2.685	0.132	0.011

As seen from Table 10, the Wald test shows that there is significance difference in the momentum factors between the low-momentum state and the high-momentum state across most lags, other than for $h = 2, 9, 11, 12$ under the 5% significance level. On the second lag, we observe that the beta coefficients between $S_t = 1$ and $S_t = 2$ does not significantly differ, but as previously seen from Figure 5, the momentum effect on both states in $h = 2$ are very close to zero. The highest Wald statistics are observed in lags where the coefficient in the two states show opposite signs, such as for $h = 1, 3, 7, 8$. In general, the results of the Wald test supports our hypothesis that there are indeed two different states of the momentum factor in the economy, as the momentum coefficients in the two states are statistically different from one another. The Wald test statistics on the sign MSVAR model also indicates that the coefficients at the low-momentum state is significantly different from that of the high-momentum state, except for lags $h = 2, 4, 8, 11$ under the 5% significance level.

5.4 AIC calculation

Table 11: AIC values for the VAR and MSVAR models, for regular and sign variables

h	AIC_h^{VAR}	AIC_h^{MSVAR}	$AIC_{h,sign}^{VAR}$	$AIC_{h,sign}^{MSVAR}$
1	7573.48	7131.89	7573.53	7136.18
2	7535.70	6958.52	7535.13	6954.42
3	7493.91	7118.67	7494.98	6917.27
4	7445.02	6983.65	7446.21	6883.24
5	7417.98	7037.13	7415.28	6861.43
6	7385.64	6815.21	7390.00	7007.25
7	7358.16	6978.18	7357.75	6849.72
8	7340.49	7015.85	7340.65	6825.29
9	7316.24	6755.26	7315.17	6911.57
10	7290.98	6786.16	7290.89	7175.29
11	7257.34	6844.52	7258.20	6727.89
12	7232.31	6722.02	7235.41	6817.64

[1] AIC values for VAR models uses the MLE method in estimating the VAR models.

As the AIC compares the likelihood of each model relative to the number of parameters, it can serve as a method to compare our VAR models and the MSVAR models. As seen from Table 11, the AIC indicates that our two-state MSVAR model fits the data better than the single-state model, for both the scaled return regressions as well as the sign regressions. The better fit of the MSVAR model can be explained by the significant increase in likelihood that outweighs the increase in number of parameters of the MSVAR model. When comparing between the non-sign models and the sign VAR models, we find that the AIC values are relatively close to one another, which indicates that neither of the non-sign and the sign models provide significantly better fit than the other. In the MSVAR model, non-sign and sign explanatory variables do show some differences in AIC, with the sign MSVAR model showing lower AIC than the non-sign MSVAR model in 5 out of 12 lags.

6 Conclusion

It has long been known that the momentum factor has a significant impact in stock returns, and our results supports this claim. Although our findings slightly differ from the results of Moskowitz et al. (2012), we find that the momentum factor still has significant effect for the first 12-month lags across all 9 indices. Once we correct for error covariance and correlations, starting with the GLS and then with the MLE, we find less and less positive momentum effects across the lags. The similar case can be found when we perform the estimations on the sign VAR model. Upon introducing a two-state MSVAR model, we find significant differences between the momentum factor of the low-momentum and high-momentum states, which is also the case in the sign MSVAR model. The transition between low-momentum and high-momentum states is affected by the economic regimes to a certain extent, as apparent in the 2008 crisis.

It can be concluded that although the momentum effect in stock indices are still present, implementing it into portfolio construction may not be as straightforward as it to be. Not only do the momentum factors fail in showing consistent positive coefficients across all lags in the single-state model, there is also an apparent regime-switching between low-momentum or mean-reversal states and high-momentum states that is linked to the recession or expansion regimes in the economy. We believe that investors should put more caution in utilizing past returns into their portfolio construction, which is often done by simply assuming positive momentum across any time period. The momentum factor based on our findings can still be reliably implemented in portfolios mostly only during recession periods, where the momentum factor is often significantly positive. However, the positive momentum is soon followed by a mean-reversal state right after the recession which may be relatively risky, considering the fact that recession periods and its recovery are often unpredictable.

As for future research, it may be beneficial to implement more in-depth studies regarding recession periods and its forecasting methods to combine with our research. By implementing forecasting studies on recession periods, we may obtain more insight on the forecasting ability of the momentum factor while taking recessions into account, which may be of great interest to investors. As we currently assume only two states in the economy, introducing more state variables may also provide better insight on the momentum factor. As our results find mostly significantly positive or negative momentum, the introduction of a third state may show perhaps a neutral momentum state, and the transition between these three states may provide investors with better strategy in their portfolio construction.

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7 Appendix

7.1 VAR model results

Table 12: Regression constants $\alpha_{i,h}$ of VAR model estimated with OLS

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	0.2951	0.1519	-0.0097	0.1457	0.1805	0.0869	0.0642	0.1412	0.0052
2	0.3016	0.1531	-0.0170	0.1499	0.1824	0.0761	0.0700	0.1294	0.0031
3	0.2866	0.1481	-0.0275	0.1381	0.1726	0.0791	0.0663	0.1310	-0.0009
4	0.3108	0.1562	-0.0301	0.1525	0.1853	0.0955	0.0685	0.1419	-0.0042
5	0.2993	0.1547	-0.0297	0.1486	0.1877	0.0944	0.0714	0.1479	0.0013
6	0.2932	0.1516	-0.0307	0.1440	0.1822	0.1023	0.0654	0.1515	0.0043
7	0.2973	0.1559	-0.0262	0.1412	0.1859	0.1070	0.0683	0.1538	-0.0080
8	0.3035	0.1591	-0.0273	0.1506	0.1968	0.1065	0.0698	0.1525	-0.0210
9	0.2984	0.1554	-0.0221	0.1457	0.1911	0.1047	0.0636	0.1485	-0.0431
10	0.2929	0.1559	-0.0179	0.1382	0.1880	0.0984	0.0550	0.1531	-0.0616
11	0.3032	0.1548	-0.0251	0.1501	0.1949	0.1023	0.0515	0.1740	-0.0559
12	0.2991	0.1523	-0.0208	0.1452	0.1951	0.0981	0.0367	0.1607	-0.0659

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.

Table 13: Regression constants $\alpha_{i,h}$ of VAR model estimated with GLS

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	0.2018	0.1685	-0.0116	0.2312	0.2888	0.0831	0.0741	0.1296	0.0035
2	0.1831	0.1743	-0.0077	0.2539	0.2808	0.0663	0.0822	0.1125	0.0019
3	0.1867	0.1650	-0.0390	0.2237	0.2706	0.0758	0.0793	0.1211	0.0000
4	0.1710	0.1949	-0.0010	0.2902	0.3012	0.0961	0.0774	0.1245	-0.0027
5	0.1548	0.1960	-0.0051	0.2827	0.3113	0.0953	0.0862	0.1334	0.0013
6	0.1482	0.1882	-0.0105	0.2663	0.3125	0.1143	0.0800	0.1378	0.0033
7	0.1222	0.2014	0.0105	0.2870	0.2896	0.1086	0.0756	0.1277	-0.0069
8	0.1268	0.2312	0.0267	0.3215	0.3283	0.1055	0.0773	0.1323	-0.0137
9	0.1180	0.2629	0.0305	0.3330	0.3045	0.0882	0.0563	0.1182	-0.0293
10	0.1189	0.2642	0.0209	0.3069	0.2908	0.0778	0.0435	0.1228	-0.0421
11	0.1299	0.2566	-0.0024	0.3346	0.3091	0.0824	0.0395	0.1480	-0.0372
12	0.1695	0.1469	-0.0222	0.1882	0.2559	0.0756	0.0122	0.1348	-0.0443

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.

Table 14: Regression constants $\alpha_{i,h}$ of VAR model estimated with MLE

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	0.3137	0.1526	-0.0010	0.1781	0.2041	0.1096	0.1676	0.2228	0.0801
2	0.3123	0.1555	-0.0037	0.1841	0.2088	0.1079	0.1747	0.2082	0.0806
3	0.3038	0.1530	-0.0144	0.1755	0.2031	0.1118	0.1687	0.2078	0.0796
4	0.3223	0.1649	-0.0148	0.2031	0.2270	0.1310	0.1908	0.2267	0.0968
5	0.2974	0.1529	-0.0101	0.1849	0.2121	0.1223	0.1764	0.2135	0.0875
6	0.2939	0.1507	-0.0130	0.1821	0.2069	0.1299	0.1731	0.2146	0.0927
7	0.2985	0.1506	-0.0098	0.1777	0.2044	0.1229	0.1693	0.2084	0.0793
8	0.3071	0.1563	-0.0096	0.1852	0.2131	0.1247	0.1733	0.2117	0.0803
9	0.3014	0.1552	-0.0062	0.1804	0.2125	0.1241	0.1686	0.2099	0.0725
10	0.3040	0.1594	-0.0018	0.1864	0.2173	0.1238	0.1692	0.2120	0.0678
11	0.2978	0.1492	-0.0072	0.1827	0.2117	0.1197	0.1615	0.2086	0.0634
12	0.2920	0.1445	-0.0038	0.1752	0.2072	0.1182	0.1573	0.1966	0.0646

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.

Table 15: Covariance-correlation matrix of VAR model for $h = 2$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7419	0.6539	0.3830	0.6745	0.6514	0.5510	0.6588	0.6269	0.5719
2	0.6539	1.8852	0.4300	0.7633	0.7224	0.6894	0.7866	0.7458	0.7114
3	0.3830	0.4300	1.7611	0.4234	0.3873	0.3769	0.4110	0.5190	0.3924
4	0.6745	0.7633	0.4234	2.2948	0.8402	0.7162	0.8967	0.6410	0.8052
5	0.6514	0.7224	0.3873	0.8402	2.1118	0.7279	0.9066	0.5987	0.8084
6	0.5510	0.6894	0.3769	0.7162	0.7279	2.3796	0.8165	0.6294	0.8633
7	0.6588	0.7866	0.4110	0.8967	0.9066	0.8165	1.7214	0.6461	0.8830
8	0.6269	0.7458	0.5190	0.6410	0.5987	0.6294	0.6461	2.1977	0.6120
9	0.5719	0.7114	0.3924	0.8052	0.8084	0.8633	0.8830	0.6120	2.4456

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 16: Covariance-correlation matrix of VAR model for $h = 3$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7439	0.6527	0.3937	0.6743	0.6534	0.5499	0.6556	0.6308	0.5687
2	0.6527	1.8815	0.4329	0.7633	0.7211	0.6866	0.7822	0.7475	0.7065
3	0.3937	0.4329	1.7160	0.4264	0.3889	0.3751	0.4026	0.5123	0.3875
4	0.6743	0.7633	0.4264	2.2806	0.8395	0.7186	0.8967	0.6406	0.8058
5	0.6534	0.7211	0.3889	0.8395	2.1101	0.7317	0.9069	0.6015	0.8094
6	0.5499	0.6866	0.3751	0.7186	0.7317	2.3503	0.8189	0.6254	0.8616
7	0.6556	0.7822	0.4026	0.8967	0.9069	0.8189	1.7136	0.6430	0.8831
8	0.6308	0.7475	0.5123	0.6406	0.6015	0.6254	0.6430	2.1821	0.6079
9	0.5687	0.7065	0.3875	0.8058	0.8094	0.8616	0.8831	0.6079	2.4306

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 17: Covariance-correlation matrix of VAR model for $h = 4$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7438	0.6544	0.3933	0.6886	0.6608	0.5609	0.6667	0.6325	0.5821
2	0.6544	1.9053	0.4342	0.7901	0.7393	0.6994	0.7994	0.7498	0.7280
3	0.3933	0.4342	1.7176	0.4232	0.3881	0.3736	0.4026	0.5170	0.3876
4	0.6886	0.7901	0.4232	2.3525	0.8456	0.7277	0.9017	0.6509	0.8132
5	0.6608	0.7393	0.3881	0.8456	2.1715	0.7353	0.9099	0.6015	0.8137
6	0.5609	0.6994	0.3736	0.7277	0.7353	2.3813	0.8214	0.6322	0.8650
7	0.6667	0.7994	0.4026	0.9017	0.9099	0.8214	1.7737	0.6458	0.8859
8	0.6325	0.7498	0.5170	0.6509	0.6015	0.6322	0.6458	2.2027	0.6157
9	0.5821	0.7280	0.3876	0.8132	0.8137	0.8650	0.8859	0.6157	2.4897

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 18: Covariance-correlation matrix of VAR model for $h = 5$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7318	0.6544	0.3961	0.6872	0.6643	0.5626	0.6680	0.6286	0.5822
2	0.6544	1.8913	0.4448	0.7905	0.7370	0.6968	0.7989	0.7509	0.7251
3	0.3961	0.4448	1.7342	0.4295	0.3916	0.3766	0.4052	0.5266	0.3889
4	0.6872	0.7905	0.4295	2.3575	0.8450	0.7282	0.9020	0.6544	0.8120
5	0.6643	0.7370	0.3916	0.8450	2.1541	0.7347	0.9099	0.6062	0.8119
6	0.5626	0.6968	0.3766	0.7282	0.7347	2.3931	0.8209	0.6326	0.8648
7	0.6680	0.7989	0.4052	0.9020	0.9099	0.8209	1.7661	0.6491	0.8842
8	0.6286	0.7509	0.5266	0.6544	0.6062	0.6326	0.6491	2.2079	0.6165
9	0.5822	0.7251	0.3889	0.8120	0.8119	0.8648	0.8842	0.6165	2.4878

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 19: Covariance-correlation matrix of VAR model for $h = 6$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7459	0.6524	0.4065	0.6897	0.6639	0.5679	0.6675	0.6370	0.5838
2	0.6524	1.8950	0.4485	0.7917	0.7366	0.7030	0.7997	0.7477	0.7258
3	0.4065	0.4485	1.7657	0.4352	0.3961	0.3844	0.4085	0.5309	0.3971
4	0.6897	0.7917	0.4352	2.3539	0.8459	0.7378	0.9045	0.6558	0.8144
5	0.6639	0.7366	0.3961	0.8459	2.1521	0.7353	0.9088	0.6038	0.8087
6	0.5679	0.7030	0.3844	0.7378	0.7353	2.3485	0.8236	0.6345	0.8663
7	0.6675	0.7997	0.4085	0.9045	0.9088	0.8236	1.7593	0.6455	0.8843
8	0.6370	0.7477	0.5309	0.6558	0.6038	0.6345	0.6455	2.2121	0.6150
9	0.5838	0.7258	0.3971	0.8144	0.8087	0.8663	0.8843	0.6150	2.4747

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 20: Covariance-correlation matrix of VAR model for $h = 7$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7360	0.6547	0.4067	0.6898	0.6626	0.5627	0.6655	0.6370	0.5813
2	0.6547	1.9001	0.4507	0.7919	0.7371	0.7043	0.8007	0.7601	0.7261
3	0.4067	0.4507	1.7446	0.4347	0.3986	0.3848	0.4090	0.5264	0.4022
4	0.6898	0.7919	0.4347	2.3403	0.8440	0.7398	0.9032	0.6539	0.8164
5	0.6626	0.7371	0.3986	0.8440	2.1510	0.7344	0.9068	0.6072	0.8067
6	0.5627	0.7043	0.3848	0.7398	0.7344	2.3717	0.8254	0.6377	0.8663
7	0.6655	0.8007	0.4090	0.9032	0.9068	0.8254	1.7609	0.6492	0.8847
8	0.6370	0.7601	0.5264	0.6539	0.6072	0.6377	0.6492	2.2391	0.6159
9	0.5813	0.7261	0.4022	0.8164	0.8067	0.8663	0.8847	0.6159	2.4666

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 21: Covariance-correlation matrix of VAR model for $h = 8$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7451	0.6552	0.4074	0.6920	0.6633	0.5629	0.6683	0.6374	0.5828
2	0.6552	1.9055	0.4502	0.7911	0.7365	0.7051	0.8015	0.7569	0.7249
3	0.4074	0.4502	1.7459	0.4339	0.3964	0.3818	0.4048	0.5295	0.3992
4	0.6920	0.7911	0.4339	2.3460	0.8437	0.7408	0.9027	0.6520	0.8163
5	0.6633	0.7365	0.3964	0.8437	2.1495	0.7355	0.9070	0.6046	0.8061
6	0.5629	0.7051	0.3818	0.7408	0.7355	2.3722	0.8265	0.6368	0.8656
7	0.6683	0.8015	0.4048	0.9027	0.9070	0.8265	1.7606	0.6474	0.8839
8	0.6374	0.7569	0.5295	0.6520	0.6046	0.6368	0.6474	2.2397	0.6131
9	0.5828	0.7249	0.3992	0.8163	0.8061	0.8656	0.8839	0.6131	2.4619

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 22: Covariance-correlation matrix of VAR model for $h = 9$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7474	0.6560	0.4110	0.6927	0.6650	0.5654	0.6706	0.6401	0.5865
2	0.6560	1.9122	0.4451	0.7910	0.7403	0.7060	0.8014	0.7561	0.7232
3	0.4110	0.4451	1.7413	0.4347	0.3960	0.3806	0.4001	0.5278	0.3950
4	0.6927	0.7910	0.4347	2.3513	0.8458	0.7420	0.9043	0.6530	0.8189
5	0.6650	0.7403	0.3960	0.8458	2.1709	0.7377	0.9083	0.6070	0.8091
6	0.5654	0.7060	0.3806	0.7420	0.7377	2.3839	0.8285	0.6364	0.8629
7	0.6706	0.8014	0.4001	0.9043	0.9083	0.8285	1.7775	0.6475	0.8866
8	0.6401	0.7561	0.5278	0.6530	0.6070	0.6364	0.6475	2.2529	0.6088
9	0.5865	0.7232	0.3950	0.8189	0.8091	0.8629	0.8866	0.6088	2.4480

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 23: Covariance-correlation matrix of VAR model for $h = 10$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7518	0.6567	0.4109	0.6939	0.6658	0.5673	0.6715	0.6410	0.5870
2	0.6567	1.9146	0.4459	0.7921	0.7413	0.7105	0.8039	0.7587	0.7275
3	0.4109	0.4459	1.7413	0.4365	0.3979	0.3830	0.4008	0.5291	0.3951
4	0.6939	0.7921	0.4365	2.3670	0.8462	0.7423	0.9050	0.6553	0.8179
5	0.6658	0.7413	0.3979	0.8462	2.1811	0.7386	0.9081	0.6102	0.8091
6	0.5673	0.7105	0.3830	0.7423	0.7386	2.3942	0.8301	0.6391	0.8624
7	0.6715	0.8039	0.4008	0.9050	0.9081	0.8301	1.7772	0.6513	0.8879
8	0.6410	0.7587	0.5291	0.6553	0.6102	0.6391	0.6513	2.2705	0.6117
9	0.5870	0.7275	0.3951	0.8179	0.8091	0.8624	0.8879	0.6117	2.4270

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 24: Covariance-correlation matrix of VAR model for $h = 11$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7534	0.6607	0.4107	0.6965	0.6668	0.5701	0.6751	0.6424	0.5894
2	0.6607	1.8989	0.4418	0.7913	0.7400	0.7085	0.8029	0.7552	0.7260
3	0.4107	0.4418	1.7337	0.4302	0.3929	0.3803	0.3924	0.5259	0.3880
4	0.6965	0.7913	0.4302	2.3507	0.8453	0.7418	0.9061	0.6487	0.8174
5	0.6668	0.7400	0.3929	0.8453	2.1767	0.7374	0.9089	0.6047	0.8083
6	0.5701	0.7085	0.3803	0.7418	0.7374	2.3885	0.8274	0.6413	0.8604
7	0.6751	0.8029	0.3924	0.9061	0.9089	0.8274	1.7638	0.6472	0.8864
8	0.6424	0.7552	0.5259	0.6487	0.6047	0.6413	0.6472	2.2294	0.6096
9	0.5894	0.7260	0.3880	0.8174	0.8083	0.8604	0.8864	0.6096	2.4159

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 25: Covariance-correlation matrix of VAR model for $h = 12$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7632	0.6622	0.4078	0.6993	0.6689	0.5699	0.6752	0.6387	0.5869
2	0.6622	1.9105	0.4438	0.7943	0.7398	0.7102	0.8025	0.7587	0.7281
3	0.4078	0.4438	1.7351	0.4269	0.3929	0.3851	0.3910	0.5263	0.3888
4	0.6993	0.7943	0.4269	2.3479	0.8454	0.7424	0.9060	0.6492	0.8175
5	0.6689	0.7398	0.3929	0.8454	2.1695	0.7372	0.9099	0.6024	0.8072
6	0.5699	0.7102	0.3851	0.7424	0.7372	2.4022	0.8265	0.6405	0.8608
7	0.6752	0.8025	0.3910	0.9060	0.9099	0.8265	1.7705	0.6438	0.8864
8	0.6387	0.7587	0.5263	0.6492	0.6024	0.6405	0.6438	2.2048	0.6060
9	0.5869	0.7281	0.3888	0.8175	0.8072	0.8608	0.8864	0.6060	2.4392

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

7.2 Sign VAR model results

Table 26: Regression constants $\alpha_{i,h,sign}$ of sign VAR model estimated with OLS

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	0.2992	0.1550	-0.0109	0.1474	0.1834	0.0878	0.0645	0.1419	0.0033
2	0.2906	0.1508	-0.0201	0.1430	0.1768	0.0717	0.0656	0.1193	-0.0035
3	0.2798	0.1503	-0.0330	0.1329	0.1706	0.0757	0.0618	0.1208	-0.0103
4	0.3130	0.1574	-0.0302	0.1537	0.1868	0.0962	0.0690	0.1430	-0.0041
5	0.2932	0.1539	-0.0324	0.1444	0.1846	0.0921	0.0684	0.1419	-0.0031
6	0.2907	0.1532	-0.0339	0.1414	0.1818	0.1012	0.0629	0.1466	-0.0004
7	0.2934	0.1556	-0.0282	0.1383	0.1840	0.1054	0.0661	0.1497	-0.0114
8	0.3126	0.1626	-0.0258	0.1561	0.2023	0.1099	0.0730	0.1590	-0.0173
9	0.2988	0.1566	-0.0232	0.1456	0.1919	0.1048	0.0631	0.1478	-0.0446
10	0.2967	0.1591	-0.0193	0.1397	0.1912	0.0996	0.0550	0.1537	-0.0631
11	0.2905	0.1502	-0.0270	0.1426	0.1872	0.0973	0.0465	0.1652	-0.0611
12	0.3045	0.1551	-0.0208	0.1482	0.1988	0.1002	0.0382	0.1638	-0.0648

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.

Table 27: Regression constants $\alpha_{i,h,sign}$ of sign VAR model estimated with GLS

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	0.2018	0.1685	-0.0116	0.2312	0.2888	0.0831	0.0741	0.1296	0.0035
2	0.1831	0.1743	-0.0077	0.2539	0.2808	0.0663	0.0822	0.1125	0.0019
3	0.1867	0.1650	-0.0390	0.2237	0.2706	0.0758	0.0793	0.1211	0.0000
4	0.1710	0.1949	-0.0010	0.2902	0.3012	0.0961	0.0774	0.1245	-0.0027
5	0.1548	0.1960	-0.0051	0.2827	0.3113	0.0953	0.0862	0.1334	0.0013
6	0.1482	0.1882	-0.0105	0.2663	0.3125	0.1143	0.0800	0.1378	0.0033
7	0.1222	0.2014	0.0105	0.2870	0.2896	0.1086	0.0756	0.1277	-0.0069
8	0.1268	0.2312	0.0267	0.3215	0.3283	0.1055	0.0773	0.1323	-0.0137
9	0.1180	0.2629	0.0305	0.3330	0.3045	0.0882	0.0563	0.1182	-0.0293
10	0.1189	0.2642	0.0209	0.3069	0.2908	0.0778	0.0435	0.1228	-0.0421
11	0.1299	0.2566	-0.0024	0.3346	0.3091	0.0824	0.0395	0.1480	-0.0372
12	0.1695	0.1469	-0.0222	0.1882	0.2559	0.0756	0.0122	0.1348	-0.0443

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.

Table 28: Regression constants $\alpha_{i,h,sign}$ of sign VAR model estimated with MLE

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	0.3142	0.1516	0.0006	0.1786	0.2039	0.1100	0.1674	0.2233	0.0823
2	0.3054	0.1526	-0.0044	0.1799	0.2046	0.1052	0.1714	0.2027	0.0785
3	0.3068	0.1552	-0.0152	0.1772	0.2055	0.1130	0.1706	0.2094	0.0801
4	0.3204	0.1628	-0.0135	0.2022	0.2252	0.1306	0.1895	0.2255	0.0976
5	0.2931	0.1526	-0.0130	0.1817	0.2100	0.1206	0.1744	0.2100	0.0845
6	0.2998	0.1548	-0.0146	0.1850	0.2115	0.1324	0.1768	0.2192	0.0944
7	0.2971	0.1508	-0.0111	0.1765	0.2039	0.1224	0.1689	0.2070	0.0779
8	0.3061	0.1554	-0.0093	0.1846	0.2123	0.1243	0.1728	0.2110	0.0805
9	0.2972	0.1539	-0.0074	0.1776	0.2101	0.1224	0.1662	0.2067	0.0697
10	0.3051	0.1596	-0.0014	0.1871	0.2179	0.1242	0.1699	0.2128	0.0686
11	0.3016	0.1516	-0.0076	0.1849	0.2146	0.1213	0.1637	0.2114	0.0643
12	0.3090	0.1529	-0.0035	0.1860	0.2194	0.1256	0.1674	0.2104	0.0719

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.

Table 29: Covariance-correlation matrix of sign VAR model for $h = 1$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7477	0.6547	0.3834	0.6784	0.6547	0.5492	0.6591	0.6232	0.5722
2	0.6547	1.8837	0.4355	0.7625	0.7236	0.6915	0.7838	0.7427	0.7128
3	0.3834	0.4355	1.7827	0.4177	0.3828	0.3781	0.4090	0.5261	0.3860
4	0.6784	0.7625	0.4177	2.2991	0.8403	0.7167	0.8956	0.6452	0.8052
5	0.6547	0.7236	0.3828	0.8403	2.1178	0.7301	0.9078	0.5959	0.8096
6	0.5492	0.6915	0.3781	0.7167	0.7301	2.4149	0.8161	0.6277	0.8645
7	0.6591	0.7838	0.4090	0.8956	0.9078	0.8161	1.7358	0.6442	0.8834
8	0.6232	0.7427	0.5261	0.6452	0.5959	0.6277	0.6442	2.2279	0.6079
9	0.5722	0.7128	0.3860	0.8052	0.8096	0.8645	0.8834	0.6079	2.4646

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 30: Covariance-correlation matrix of sign VAR model for $h = 2$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7390	0.6532	0.3818	0.6736	0.6521	0.5515	0.6591	0.6260	0.5727
2	0.6532	1.8845	0.4312	0.7626	0.7217	0.6894	0.7865	0.7453	0.7115
3	0.3818	0.4312	1.7573	0.4230	0.3860	0.3774	0.4091	0.5212	0.3899
4	0.6736	0.7626	0.4230	2.2912	0.8391	0.7157	0.8965	0.6407	0.8044
5	0.6521	0.7217	0.3860	0.8391	2.1105	0.7275	0.9063	0.5980	0.8076
6	0.5515	0.6894	0.3774	0.7157	0.7275	2.3775	0.8165	0.6278	0.8636
7	0.6591	0.7865	0.4091	0.8965	0.9063	0.8165	1.7198	0.6454	0.8830
8	0.6260	0.7453	0.5212	0.6407	0.5980	0.6278	0.6454	2.1962	0.6121
9	0.5727	0.7115	0.3899	0.8044	0.8076	0.8636	0.8830	0.6121	2.4393

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 31: Covariance-correlation matrix of sign VAR model for $h = 3$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7432	0.6531	0.3941	0.6739	0.6525	0.5490	0.6559	0.6297	0.5687
2	0.6531	1.8826	0.4325	0.7636	0.7220	0.6870	0.7824	0.7473	0.7071
3	0.3941	0.4325	1.7172	0.4260	0.3893	0.3737	0.4021	0.5140	0.3869
4	0.6739	0.7636	0.4260	2.2880	0.8398	0.7187	0.8967	0.6410	0.8051
5	0.6525	0.7220	0.3893	0.8398	2.1137	0.7327	0.9074	0.6018	0.8098
6	0.5490	0.6870	0.3737	0.7187	0.7327	2.3567	0.8191	0.6254	0.8633
7	0.6559	0.7824	0.4021	0.8967	0.9074	0.8191	1.7181	0.6427	0.8828
8	0.6297	0.7473	0.5140	0.6410	0.6018	0.6254	0.6427	2.1787	0.6082
9	0.5687	0.7071	0.3869	0.8051	0.8098	0.8633	0.8828	0.6082	2.4417

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 32: Covariance-correlation matrix of sign VAR model for $h = 4$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7464	0.6564	0.3953	0.6883	0.6613	0.5607	0.6664	0.6330	0.5829
2	0.6564	1.9004	0.4352	0.7898	0.7385	0.6971	0.7991	0.7502	0.7265
3	0.3953	0.4352	1.7168	0.4242	0.3881	0.3723	0.4022	0.5172	0.3849
4	0.6883	0.7898	0.4242	2.3470	0.8449	0.7275	0.9008	0.6515	0.8119
5	0.6613	0.7385	0.3881	0.8449	2.1660	0.7344	0.9094	0.6038	0.8116
6	0.5607	0.6971	0.3723	0.7275	0.7344	2.3794	0.8210	0.6307	0.8655
7	0.6664	0.7991	0.4022	0.9008	0.9094	0.8210	1.7687	0.6462	0.8850
8	0.6330	0.7502	0.5172	0.6515	0.6038	0.6307	0.6462	2.2039	0.6146
9	0.5829	0.7265	0.3849	0.8119	0.8116	0.8655	0.8850	0.6146	2.4887

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 33: Covariance-correlation matrix of sign VAR model for $h = 5$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7326	0.6557	0.3991	0.6895	0.6648	0.5629	0.6682	0.6315	0.5822
2	0.6557	1.8899	0.4427	0.7901	0.7365	0.6963	0.7987	0.7516	0.7237
3	0.3991	0.4427	1.7276	0.4292	0.3909	0.3757	0.4022	0.5280	0.3894
4	0.6895	0.7901	0.4292	2.3619	0.8452	0.7282	0.9014	0.6543	0.8118
5	0.6648	0.7365	0.3909	0.8452	2.1583	0.7339	0.9103	0.6051	0.8103
6	0.5629	0.6963	0.3757	0.7282	0.7339	2.3932	0.8202	0.6315	0.8656
7	0.6682	0.7987	0.4022	0.9014	0.9103	0.8202	1.7657	0.6495	0.8831
8	0.6315	0.7516	0.5280	0.6543	0.6051	0.6315	0.6495	2.2063	0.6172
9	0.5822	0.7237	0.3894	0.8118	0.8103	0.8656	0.8831	0.6172	2.4756

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 34: Covariance-correlation matrix of sign VAR model for $h = 6$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7445	0.6519	0.4063	0.6892	0.6639	0.5673	0.6675	0.6339	0.5830
2	0.6519	1.8950	0.4474	0.7899	0.7366	0.7041	0.8001	0.7496	0.7267
3	0.4063	0.4474	1.7532	0.4330	0.3965	0.3825	0.4067	0.5267	0.3938
4	0.6892	0.7899	0.4330	2.3533	0.8450	0.7361	0.9035	0.6553	0.8139
5	0.6639	0.7366	0.3965	0.8450	2.1540	0.7347	0.9081	0.6044	0.8074
6	0.5673	0.7041	0.3825	0.7361	0.7347	2.3502	0.8238	0.6352	0.8662
7	0.6675	0.8001	0.4067	0.9035	0.9081	0.8238	1.7601	0.6463	0.8834
8	0.6339	0.7496	0.5267	0.6553	0.6044	0.6352	0.6463	2.2117	0.6151
9	0.5830	0.7267	0.3938	0.8139	0.8074	0.8662	0.8834	0.6151	2.4766

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 35: Covariance-correlation matrix of sign VAR model for $h = 7$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7347	0.6537	0.4073	0.6887	0.6615	0.5628	0.6656	0.6360	0.5807
2	0.6537	1.8998	0.4497	0.7915	0.7369	0.7052	0.8002	0.7610	0.7265
3	0.4073	0.4497	1.7409	0.4337	0.3974	0.3839	0.4074	0.5246	0.4015
4	0.6887	0.7915	0.4337	2.3420	0.8438	0.7396	0.9038	0.6529	0.8171
5	0.6615	0.7369	0.3974	0.8438	2.1516	0.7344	0.9066	0.6078	0.8076
6	0.5628	0.7052	0.3839	0.7396	0.7344	2.3789	0.8267	0.6392	0.8652
7	0.6656	0.8002	0.4074	0.9038	0.9066	0.8267	1.7647	0.6499	0.8857
8	0.6360	0.7610	0.5246	0.6529	0.6078	0.6392	0.6499	2.2380	0.6164
9	0.5807	0.7265	0.4015	0.8171	0.8076	0.8652	0.8857	0.6164	2.4579

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 36: Covariance-correlation matrix of sign VAR model for $h = 8$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7432	0.6550	0.4077	0.6915	0.6631	0.5635	0.6678	0.6377	0.5832
2	0.6550	1.9052	0.4499	0.7906	0.7363	0.7049	0.8009	0.7568	0.7253
3	0.4077	0.4499	1.7459	0.4344	0.3963	0.3820	0.4047	0.5293	0.3987
4	0.6915	0.7906	0.4344	2.3441	0.8438	0.7414	0.9029	0.6520	0.8167
5	0.6631	0.7363	0.3963	0.8438	2.1491	0.7360	0.9068	0.6048	0.8065
6	0.5635	0.7049	0.3820	0.7414	0.7360	2.3734	0.8267	0.6373	0.8659
7	0.6678	0.8009	0.4047	0.9029	0.9068	0.8267	1.7613	0.6472	0.8841
8	0.6377	0.7568	0.5293	0.6520	0.6048	0.6373	0.6472	2.2424	0.6142
9	0.5832	0.7253	0.3987	0.8167	0.8065	0.8659	0.8841	0.6142	2.4645

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 37: Covariance-correlation matrix of sign VAR model for $h = 9$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7443	0.6554	0.4111	0.6928	0.6651	0.5662	0.6716	0.6419	0.5876
2	0.6554	1.9125	0.4459	0.7908	0.7402	0.7062	0.8014	0.7562	0.7221
3	0.4111	0.4459	1.7411	0.4351	0.3964	0.3809	0.4008	0.5273	0.3965
4	0.6928	0.7908	0.4351	2.3546	0.8460	0.7424	0.9039	0.6543	0.8181
5	0.6651	0.7402	0.3964	0.8460	2.1703	0.7381	0.9079	0.6085	0.8094
6	0.5662	0.7062	0.3809	0.7424	0.7381	2.3807	0.8293	0.6370	0.8630
7	0.6716	0.8014	0.4008	0.9039	0.9079	0.8293	1.7770	0.6489	0.8867
8	0.6419	0.7562	0.5273	0.6543	0.6085	0.6370	0.6489	2.2464	0.6101
9	0.5876	0.7221	0.3965	0.8181	0.8094	0.8630	0.8867	0.6101	2.4478

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 38: Covariance-correlation matrix of sign VAR model for $h = 10$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7521	0.6566	0.4110	0.6940	0.6662	0.5675	0.6719	0.6413	0.5871
2	0.6566	1.9146	0.4459	0.7921	0.7415	0.7106	0.8044	0.7585	0.7278
3	0.4110	0.4459	1.7417	0.4365	0.3977	0.3832	0.4002	0.5294	0.3948
4	0.6940	0.7921	0.4365	2.3660	0.8461	0.7423	0.9049	0.6552	0.8178
5	0.6662	0.7415	0.3977	0.8461	2.1808	0.7385	0.9080	0.6105	0.8087
6	0.5675	0.7106	0.3832	0.7423	0.7385	2.3939	0.8302	0.6393	0.8625
7	0.6719	0.8044	0.4002	0.9049	0.9080	0.8302	1.7772	0.6515	0.8878
8	0.6413	0.7585	0.5294	0.6552	0.6105	0.6393	0.6515	2.2710	0.6119
9	0.5871	0.7278	0.3948	0.8178	0.8087	0.8625	0.8878	0.6119	2.4286

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 39: Covariance-correlation matrix of sign VAR model for $h = 11$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7547	0.6597	0.4079	0.6973	0.6675	0.5687	0.6754	0.6408	0.5891
2	0.6597	1.9005	0.4412	0.7911	0.7399	0.7078	0.8028	0.7552	0.7253
3	0.4079	0.4412	1.7311	0.4300	0.3921	0.3786	0.3916	0.5249	0.3881
4	0.6973	0.7911	0.4300	2.3473	0.8452	0.7411	0.9057	0.6481	0.8171
5	0.6675	0.7399	0.3921	0.8452	2.1743	0.7370	0.9088	0.6041	0.8081
6	0.5687	0.7078	0.3786	0.7411	0.7370	2.3841	0.8268	0.6395	0.8603
7	0.6754	0.8028	0.3916	0.9057	0.9088	0.8268	1.7614	0.6464	0.8861
8	0.6408	0.7552	0.5249	0.6481	0.6041	0.6395	0.6464	2.2272	0.6082
9	0.5891	0.7253	0.3881	0.8171	0.8081	0.8603	0.8861	0.6082	2.4104

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 40: Covariance-correlation matrix of sign VAR model for $h = 12$ estimated with MLE

i	1	2	3	4	5	6	7	8	9
1	1.7604	0.6596	0.4107	0.6980	0.6697	0.5685	0.6756	0.6410	0.5906
2	0.6596	1.9080	0.4394	0.7918	0.7396	0.7091	0.8029	0.7542	0.7297
3	0.4107	0.4394	1.7268	0.4273	0.3925	0.3838	0.3908	0.5219	0.3909
4	0.6980	0.7918	0.4273	2.3573	0.8467	0.7430	0.9061	0.6501	0.8201
5	0.6697	0.7396	0.3925	0.8467	2.1740	0.7370	0.9089	0.6057	0.8071
6	0.5685	0.7091	0.3838	0.7430	0.7370	2.3980	0.8273	0.6431	0.8616
7	0.6756	0.8029	0.3908	0.9061	0.9089	0.8273	1.7683	0.6470	0.8868
8	0.6410	0.7542	0.5219	0.6501	0.6057	0.6431	0.6470	2.2042	0.6132
9	0.5906	0.7297	0.3909	0.8201	0.8071	0.8616	0.8868	0.6132	2.4276

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

7.3 MSVAR model results

Table 41: Regression constants $\alpha_{i,h,1}$ of MSVAR model for $S_t = 1$

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	41.793	20.050	-28.186	33.070	21.517	-3.074	22.131	14.213	10.265
2	22.500	4.579	-14.543	8.374	6.785	-12.510	7.425	-9.334	-10.345
3	48.478	31.035	0.384	45.866	34.551	15.520	33.004	25.834	22.721
4	32.619	9.883	3.785	10.359	19.657	14.988	10.494	19.145	3.043
5	18.700	5.561	7.088	0.189	12.444	8.531	7.826	15.949	-4.690
6	39.520	31.471	17.577	30.378	38.532	35.238	32.108	46.942	26.992
7	47.281	33.637	-12.932	45.373	26.551	7.739	50.268	-62.701	20.179
8	44.588	37.443	-18.790	32.423	36.762	23.167	30.505	24.372	31.894
9	16.731	0.601	-22.095	2.756	0.845	-16.277	-1.749	-14.783	0.536
10	35.553	17.982	6.000	21.260	31.638	21.090	20.549	24.471	10.906
11	11.589	-6.409	2.373	-10.872	2.927	-0.954	-4.986	7.845	-9.085
12	36.902	18.164	-7.104	27.809	25.824	12.917	21.280	5.183	14.841

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.
[2] The numbers indicates the estimated constants $\alpha_{i,h,1}$ multiplied by $\times 10^2$.

Table 42: Regression constants $\alpha_{i,h,2}$ of MSVAR model for $S_t = 2$

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	25.933	13.479	17.774	9.380	19.643	17.782	14.496	24.812	2.953
2	43.155	30.625	17.846	32.185	38.270	37.414	31.577	46.961	26.391
3	12.319	-0.255	-3.881	-10.774	4.630	4.137	0.461	9.649	-8.919
4	30.264	23.179	-10.597	32.202	22.641	3.339	34.393	35.134	15.399
5	42.045	25.919	-14.256	39.956	29.739	12.391	27.427	27.330	19.309
6	21.106	2.775	-15.701	8.823	6.458	-5.879	5.093	-5.999	-7.279
7	22.180	5.990	2.995	5.129	15.690	10.204	8.802	20.543	0.209
8	20.355	-1.681	10.693	6.093	8.157	3.400	6.978	16.977	-4.813
9	44.379	31.649	20.863	34.769	41.439	39.086	35.068	44.472	21.868
10	20.428	11.249	-14.312	12.984	-1.453	-8.781	10.964	7.986	-7.082
11	48.514	36.589	-7.277	49.344	39.877	21.922	39.717	37.793	5.536
12	16.691	8.311	8.215	1.186	12.265	8.099	6.241	24.416	-6.611

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.
[2] The numbers indicates the estimated constants $\alpha_{i,h,2}$ multiplied by $\times 10^2$.

Table 43: Covariance-correlation matrix of MSVAR model for $h = 1$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	2.4067	0.6363	0.3461	0.6285	0.5318	0.4666	0.5938	0.5877	0.5159
2	0.6363	2.6794	0.3400	0.6856	0.5896	0.5631	0.6609	0.6124	0.4893
3	0.3461	0.3400	2.5424	0.3373	0.2482	0.2885	0.2122	0.3711	0.3675
4	0.6285	0.6856	0.3373	2.6655	0.7900	0.6635	0.8725	0.5571	0.7867
5	0.5318	0.5896	0.2482	0.7900	2.5108	0.6363	0.8750	0.3931	0.6566
6	0.4666	0.5631	0.2885	0.6635	0.6363	3.3650	0.8034	0.3834	0.8116
7	0.5938	0.6609	0.2122	0.8725	0.8750	0.8034	2.1162	0.4344	0.8446
8	0.5877	0.6124	0.3711	0.5571	0.3931	0.3834	0.4344	2.7828	0.3982
9	0.5159	0.4893	0.3675	0.7867	0.6566	0.8116	0.8446	0.3982	2.9948

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	1.1866	0.6797	0.4386	0.7321	0.7625	0.6439	0.7266	0.6202	0.6288
2	0.6797	1.2223	0.5343	0.8511	0.8279	0.8023	0.8657	0.7709	0.8401
3	0.4386	0.5343	1.1475	0.5094	0.4854	0.4328	0.5132	0.5906	0.4275
4	0.7321	0.8511	0.5094	1.8570	0.8732	0.7827	0.9016	0.7043	0.8246
5	0.7625	0.8279	0.4854	0.8732	1.7230	0.8125	0.9200	0.6881	0.8706
6	0.6439	0.8023	0.4328	0.7827	0.8125	1.6893	0.8420	0.7106	0.8685
7	0.7266	0.8657	0.5132	0.9016	0.9200	0.8420	1.4279	0.7182	0.8821
8	0.6202	0.7709	0.5906	0.7043	0.6881	0.7106	0.7182	1.7417	0.6804
9	0.6288	0.8401	0.4275	0.8246	0.8706	0.8685	0.8821	0.6804	2.0023

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 44: Covariance-correlation matrix of MSVAR model for $h = 2$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.3272	0.6689	0.3575	0.6639	0.6157	0.5176	0.6370	0.6160	0.5612
2	0.6689	2.5981	0.3983	0.7432	0.6652	0.6124	0.7370	0.6687	0.6058
3	0.3575	0.3983	2.4386	0.3871	0.3326	0.3256	0.3540	0.5096	0.3727
4	0.6639	0.7432	0.3871	3.2856	0.8295	0.6747	0.8846	0.5684	0.7851
5	0.6157	0.6652	0.3326	0.8295	2.7221	0.6681	0.8958	0.4575	0.7321
6	0.5176	0.6124	0.3256	0.6747	0.6681	2.9644	0.7984	0.5123	0.8251
7	0.6370	0.7370	0.3540	0.8846	0.8958	0.7984	2.3040	0.5377	0.8455
8	0.6160	0.6687	0.5096	0.5684	0.4575	0.5123	0.5377	2.9991	0.4997
9	0.5612	0.6058	0.3727	0.7851	0.7321	0.8251	0.8455	0.4997	3.1243
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	0.9331	0.5932	0.4274	0.7019	0.7073	0.5816	0.6796	0.5732	0.5474
2	0.5932	0.8765	0.4879	0.8243	0.8296	0.8208	0.8762	0.7938	0.8617
3	0.4274	0.4879	0.8226	0.5161	0.4849	0.4385	0.4987	0.5063	0.4290
4	0.7019	0.8243	0.5161	0.9639	0.8662	0.8198	0.9082	0.7328	0.8311
5	0.7073	0.8296	0.4849	0.8662	1.2085	0.8273	0.9082	0.7616	0.8826
6	0.5816	0.8208	0.4385	0.8198	0.8273	1.4012	0.8518	0.7195	0.8870
7	0.6796	0.8762	0.4987	0.9082	0.9082	0.8518	0.9467	0.7461	0.9078
8	0.5732	0.7938	0.5063	0.7328	0.7616	0.7195	0.7461	1.1339	0.6985
9	0.5474	0.8617	0.4290	0.8311	0.8826	0.8870	0.9078	0.6985	1.5361

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 45: Covariance-correlation matrix of MSVAR model for $h = 3$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	1.7908	0.5572	0.2446	0.5716	0.4629	0.3747	0.5131	0.4277	0.4233
2	0.5572	1.8815	0.3194	0.6932	0.6143	0.5545	0.6642	0.5156	0.5007
3	0.2446	0.3194	2.4768	0.2972	0.2263	0.2295	0.2023	0.3923	0.2682
4	0.5716	0.6932	0.2972	1.9471	0.7785	0.6130	0.8510	0.4998	0.7522
5	0.4629	0.6143	0.2263	0.7785	1.9660	0.5779	0.8735	0.2928	0.6289
6	0.3747	0.5545	0.2295	0.6130	0.5779	2.4916	0.7294	0.4226	0.7821
7	0.5131	0.6642	0.2023	0.8510	0.8735	0.7294	1.5657	0.4070	0.8215
8	0.4277	0.5156	0.3923	0.4998	0.2928	0.4226	0.4070	1.5584	0.3806
9	0.4233	0.5007	0.2682	0.7522	0.6289	0.7821	0.8215	0.3806	2.4296

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	1.6465	0.7576	0.6732	0.7712	0.8285	0.7375	0.7870	0.7745	0.7266
2	0.7576	1.8134	0.6498	0.8344	0.8187	0.8089	0.8630	0.8446	0.8393
3	0.6732	0.6498	0.9718	0.6694	0.6729	0.6288	0.6878	0.6890	0.6464
4	0.7712	0.8344	0.6694	2.5234	0.8916	0.8272	0.9274	0.7382	0.8660
5	0.8285	0.8187	0.6729	0.8916	2.3151	0.8762	0.9348	0.7785	0.9142
6	0.7375	0.8089	0.6288	0.8272	0.8762	2.3105	0.8980	0.7429	0.9077
7	0.7870	0.8630	0.6878	0.9274	0.9348	0.8980	1.8768	0.7669	0.9171
8	0.7745	0.8446	0.6890	0.7382	0.7785	0.7429	0.7669	2.5129	0.7392
9	0.7266	0.8393	0.6464	0.8660	0.9142	0.9077	0.9171	0.7392	2.5228

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 46: Covariance-correlation matrix of MSVAR model for $h = 4$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	1.1938	0.6733	0.5629	0.7126	0.7196	0.5695	0.6639	0.6149	0.5605
2	0.6733	1.2808	0.5358	0.8241	0.8273	0.7658	0.8340	0.7055	0.7749
3	0.5629	0.5358	0.9573	0.5414	0.5190	0.5075	0.5029	0.5756	0.4543
4	0.7126	0.8241	0.5414	1.5717	0.8751	0.8051	0.9129	0.6055	0.8188
5	0.7196	0.8273	0.5190	0.8751	1.6333	0.8182	0.9182	0.6110	0.8348
6	0.5695	0.7658	0.5075	0.8051	0.8182	1.7748	0.8468	0.6274	0.8750
7	0.6639	0.8340	0.5029	0.9129	0.9182	0.8468	1.4333	0.6137	0.8578
8	0.6149	0.7055	0.5756	0.6055	0.6110	0.6274	0.6137	1.9382	0.5894
9	0.5605	0.7749	0.4543	0.8188	0.8348	0.8750	0.8578	0.5894	2.0866

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	2.4488	0.6445	0.2811	0.6718	0.5929	0.5391	0.7082	0.6344	0.6433
2	0.6445	2.6986	0.3670	0.7597	0.6297	0.6101	0.7651	0.7900	0.5985
3	0.2811	0.3670	2.8130	0.3505	0.2847	0.2621	0.3386	0.4040	0.4187
4	0.6718	0.7597	0.3505	3.3886	0.8098	0.6397	0.8750	0.8441	0.8183
5	0.5929	0.6297	0.2847	0.8098	2.8256	0.6219	0.8738	0.6220	0.7620
6	0.5391	0.6101	0.2621	0.6397	0.6219	3.1944	0.8310	0.5365	0.8071
7	0.7082	0.7651	0.3386	0.8750	0.8738	0.8310	2.1128	0.6780	0.9363
8	0.6344	0.7900	0.4040	0.8441	0.6220	0.5365	0.6780	1.6165	0.5593
9	0.6433	0.5985	0.4187	0.8183	0.7620	0.8071	0.9363	0.5593	2.6473

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 47: Covariance-correlation matrix of MSVAR model for $h = 5$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	1.3263	0.7011	0.5005	0.7355	0.7956	0.6787	0.7498	0.6684	0.6696
2	0.7011	1.3519	0.5764	0.8467	0.8322	0.8034	0.8659	0.7930	0.8378
3	0.5005	0.5764	1.1462	0.5574	0.5480	0.4994	0.5652	0.6242	0.5042
4	0.7355	0.8467	0.5574	2.0057	0.8803	0.8012	0.9143	0.6829	0.8434
5	0.7956	0.8322	0.5480	0.8803	1.8599	0.8357	0.9254	0.7037	0.8884
6	0.6787	0.8034	0.4994	0.8012	0.8357	1.9612	0.8720	0.6787	0.8894
7	0.7498	0.8659	0.5652	0.9143	0.9254	0.8720	1.5617	0.6948	0.9011
8	0.6684	0.7930	0.6242	0.6829	0.7037	0.6787	0.6948	2.1410	0.6600
9	0.6696	0.8378	0.5042	0.8434	0.8884	0.8894	0.9011	0.6600	2.1764

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	2.1773	0.6245	0.3217	0.6277	0.5105	0.4307	0.5719	0.5339	0.4804
2	0.6245	2.4990	0.3459	0.7294	0.6181	0.5513	0.6850	0.5569	0.5142
3	0.3217	0.3459	2.4721	0.3340	0.2424	0.2464	0.1819	0.1959	0.2872
4	0.6277	0.7294	0.3340	2.6035	0.7931	0.6279	0.8699	0.5785	0.7659
5	0.5105	0.6181	0.2424	0.7931	2.3974	0.5882	0.8727	0.3673	0.6318
6	0.4307	0.5513	0.2464	0.6279	0.5882	2.8670	0.7257	0.4503	0.7745
7	0.5719	0.6850	0.1819	0.8699	0.8727	0.7257	1.8877	0.4860	0.8174
8	0.5339	0.5569	0.1959	0.5785	0.3673	0.4503	0.4860	1.2142	0.4194
9	0.4804	0.5142	0.2872	0.7659	0.6318	0.7745	0.8174	0.4194	2.7409

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 48: Covariance-correlation matrix of MSVAR model for $h = 6$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	0.8669	0.6124	0.4066	0.7107	0.7522	0.6004	0.6893	0.5874	0.5830
2	0.6124	0.8169	0.4773	0.8155	0.8110	0.7966	0.8793	0.7637	0.8519
3	0.4066	0.4773	0.8215	0.4944	0.4559	0.4185	0.4909	0.4833	0.4177
4	0.7107	0.8155	0.4944	0.8877	0.8591	0.7961	0.9058	0.7145	0.8142
5	0.7522	0.8110	0.4559	0.8591	1.1143	0.8065	0.8993	0.7547	0.8704
6	0.6004	0.7966	0.4185	0.7961	0.8065	1.2484	0.8406	0.6840	0.8723
7	0.6893	0.8793	0.4909	0.9058	0.8993	0.8406	0.9394	0.7394	0.9183
8	0.5874	0.7637	0.4833	0.7145	0.7547	0.6840	0.7394	1.0154	0.6758
9	0.5830	0.8519	0.4177	0.8142	0.8704	0.8723	0.9183	0.6758	1.2608
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	2.3527	0.6599	0.3935	0.6821	0.6265	0.5450	0.6517	0.6298	0.5649
2	0.6599	2.6251	0.4228	0.7845	0.7015	0.6519	0.7528	0.6885	0.6507
3	0.3935	0.4228	2.4154	0.4076	0.3529	0.3478	0.3424	0.5365	0.3727
4	0.6821	0.7845	0.4076	3.4059	0.8411	0.7156	0.9000	0.5905	0.7946
5	0.6265	0.7015	0.3529	0.8411	2.8149	0.6948	0.9030	0.4779	0.7478
6	0.5450	0.6519	0.3478	0.7156	0.6948	3.0514	0.8073	0.5652	0.8509
7	0.6517	0.7528	0.3424	0.9000	0.9030	0.8073	2.2876	0.5367	0.8440
8	0.6298	0.6885	0.5365	0.5905	0.4779	0.5652	0.5367	3.2180	0.5361
9	0.5649	0.6507	0.3727	0.7946	0.7478	0.8509	0.8440	0.5361	3.4132

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 49: Covariance-correlation matrix of MSVAR model for $h = 7$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.3933	0.5988	0.2789	0.5979	0.4294	0.4573	0.5638	0.5641	0.6028
2	0.5988	2.6187	0.3493	0.6959	0.5943	0.5946	0.7298	0.5729	0.5421
3	0.2789	0.3493	3.1886	0.3448	0.2152	0.2739	0.3160	0.2759	0.2271
4	0.5979	0.6959	0.3448	2.4588	0.8207	0.6202	0.9537	0.1135	0.8840
5	0.4294	0.5943	0.2152	0.8207	2.1340	0.5769	0.8855	0.2022	0.6478
6	0.4573	0.5946	0.2739	0.6202	0.5769	3.0252	0.7043	0.5896	0.7030
7	0.5638	0.7298	0.3160	0.9537	0.8855	0.7043	2.3791	0.2219	0.7816
8	0.5641	0.5729	0.2759	0.1135	0.2022	0.5896	0.2219	7.7222	0.1915
9	0.6028	0.5421	0.2271	0.8840	0.6478	0.7030	0.7816	0.1915	2.5876
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.3755	0.7057	0.5553	0.7294	0.7673	0.6400	0.7198	0.6758	0.6195
2	0.7057	1.4536	0.5481	0.8447	0.7970	0.7532	0.8349	0.7609	0.7598
3	0.5553	0.5481	1.0778	0.5373	0.5286	0.4642	0.5193	0.5521	0.4826
4	0.7294	0.8447	0.5373	2.1443	0.8498	0.7922	0.9001	0.7392	0.8200
5	0.7673	0.7970	0.5286	0.8498	2.0204	0.7931	0.9125	0.6991	0.8285
6	0.6400	0.7532	0.4642	0.7922	0.7931	2.0320	0.8555	0.6781	0.8736
7	0.7198	0.8349	0.5193	0.9001	0.9125	0.8555	1.5873	0.7221	0.8935
8	0.6758	0.7609	0.5521	0.7392	0.6991	0.6781	0.7221	1.7177	0.6643
9	0.6195	0.7598	0.4826	0.8200	0.8285	0.8736	0.8935	0.6643	2.3135

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 50: Covariance-correlation matrix of MSVAR model for $h = 8$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	1.9102	0.6334	0.5160	0.6774	0.7304	0.5495	0.6537	0.6245	0.4855
2	0.6334	2.5095	0.4033	0.7593	0.7488	0.6822	0.8114	0.7343	0.6708
3	0.5160	0.4033	2.0705	0.3672	0.4185	0.3508	0.3844	0.4993	0.3395
4	0.6774	0.7593	0.3672	1.6398	0.8538	0.8108	0.9240	0.5738	0.7697
5	0.7304	0.7488	0.4185	0.8538	1.7629	0.8231	0.8924	0.6073	0.8349
6	0.5495	0.6822	0.3508	0.8108	0.8231	3.1206	0.8907	0.5622	0.9398
7	0.6537	0.8114	0.3844	0.9240	0.8924	0.8907	1.6534	0.6041	0.8909
8	0.6245	0.7343	0.4993	0.5738	0.6073	0.5622	0.6041	3.0952	0.5474
9	0.4855	0.6708	0.3395	0.7697	0.8349	0.9398	0.8909	0.5474	3.5898
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.5773	0.6900	0.3443	0.7050	0.6299	0.5679	0.6751	0.6776	0.6596
2	0.6900	1.3077	0.5402	0.8463	0.7445	0.6921	0.7960	0.7959	0.7758
3	0.3443	0.5402	1.4768	0.5019	0.4117	0.4224	0.4287	0.5860	0.4942
4	0.7050	0.8463	0.5019	2.6060	0.8323	0.7174	0.8914	0.7231	0.8927
5	0.6299	0.7445	0.4117	0.8323	2.2770	0.6823	0.9137	0.6087	0.8116
6	0.5679	0.6921	0.4224	0.7174	0.6823	1.8788	0.7738	0.6940	0.7702
7	0.6751	0.7960	0.4287	0.8914	0.9137	0.7738	1.7248	0.6771	0.9014
8	0.6776	0.7959	0.5860	0.7231	0.6087	0.6940	0.6771	1.7130	0.6858
9	0.6596	0.7758	0.4942	0.8927	0.8116	0.7702	0.9014	0.6858	1.8478

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 51: Covariance-correlation matrix of MSVAR model for $h = 9$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.4119	0.6754	0.3803	0.6987	0.6356	0.5432	0.6854	0.6514	0.5586
2	0.6754	2.7391	0.4072	0.7753	0.6803	0.6121	0.7468	0.6555	0.6106
3	0.3803	0.4072	2.5159	0.3971	0.3238	0.3184	0.3078	0.5357	0.3488
4	0.6987	0.7753	0.3971	3.6124	0.8345	0.6923	0.8963	0.5569	0.8251
5	0.6356	0.6803	0.3238	0.8345	2.8734	0.6515	0.8923	0.4223	0.7348
6	0.5432	0.6121	0.3184	0.6923	0.6515	2.9185	0.7974	0.4841	0.8486
7	0.6854	0.7468	0.3078	0.8963	0.8923	0.7974	2.3534	0.5028	0.8866
8	0.6514	0.6555	0.5357	0.5569	0.4223	0.4841	0.5028	3.4452	0.4584
9	0.5586	0.6106	0.3488	0.8251	0.7348	0.8486	0.8866	0.4584	3.0695
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	0.9795	0.5876	0.4260	0.6624	0.6909	0.5529	0.6214	0.5576	0.5529
2	0.5876	0.9554	0.4771	0.8234	0.8303	0.8270	0.8638	0.7993	0.8601
3	0.4260	0.4771	0.8420	0.4815	0.4820	0.4096	0.4706	0.4891	0.4256
4	0.6624	0.8234	0.4815	0.9938	0.8683	0.8401	0.9067	0.7277	0.8038
5	0.6909	0.8303	0.4820	0.8683	1.2744	0.8449	0.9098	0.7561	0.8646
6	0.5529	0.8270	0.4096	0.8401	0.8449	1.5619	0.8660	0.7187	0.8656
7	0.6214	0.8638	0.4706	0.9067	0.9098	0.8660	1.0751	0.7203	0.8585
8	0.5576	0.7993	0.4891	0.7277	0.7561	0.7187	0.7203	1.1859	0.7197
9	0.5529	0.8601	0.4256	0.8038	0.8646	0.8656	0.8585	0.7197	1.7084

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 52: Covariance-correlation matrix of MSVAR model for $h = 10$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	1.1615	0.6565	0.5126	0.7036	0.7245	0.5796	0.6565	0.6013	0.5446
2	0.6565	1.2893	0.5234	0.8240	0.8334	0.7588	0.8416	0.7273	0.7642
3	0.5126	0.5234	1.0856	0.5248	0.4952	0.4776	0.4829	0.5258	0.4283
4	0.7036	0.8240	0.5248	1.3490	0.8716	0.7850	0.9060	0.6496	0.7862
5	0.7245	0.8334	0.4952	0.8716	1.4744	0.7921	0.9069	0.6531	0.8241
6	0.5796	0.7588	0.4776	0.7850	0.7921	1.7001	0.8302	0.6285	0.8573
7	0.6565	0.8416	0.4829	0.9060	0.9069	0.8302	1.2777	0.6406	0.8706
8	0.6013	0.7273	0.5258	0.6496	0.6531	0.6285	0.6406	1.9625	0.5964
9	0.5446	0.7642	0.4283	0.7862	0.8241	0.8573	0.8706	0.5964	1.8896
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	2.8274	0.6559	0.3080	0.6828	0.5818	0.5212	0.7104	0.6901	0.6512
2	0.6559	3.0488	0.3641	0.7657	0.6068	0.6149	0.7322	0.7783	0.6009
3	0.3080	0.3641	2.9977	0.3573	0.2680	0.2546	0.2566	0.6153	0.4282
4	0.6828	0.7657	0.3573	4.4208	0.8186	0.6875	0.8803	0.7125	0.8582
5	0.5818	0.6068	0.2680	0.8186	3.4244	0.6377	0.8794	0.4839	0.6938
6	0.5212	0.6149	0.2546	0.6875	0.6377	3.6061	0.8527	0.6054	0.8217
7	0.7104	0.7322	0.2566	0.8803	0.8794	0.8527	2.8039	0.6223	0.8503
8	0.6901	0.7783	0.6153	0.7125	0.4839	0.6054	0.6223	2.1919	0.5783
9	0.6512	0.6009	0.4282	0.8582	0.6938	0.8217	0.8503	0.5783	3.0435

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 53: Covariance-correlation matrix of MSVAR model for $h = 11$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	1.2873	0.7408	0.6074	0.7805	0.8103	0.6903	0.7613	0.6793	0.7034
2	0.7408	1.4881	0.6034	0.8680	0.7802	0.7992	0.8497	0.6747	0.8018
3	0.6074	0.6034	0.9414	0.6224	0.6062	0.5851	0.5978	0.6199	0.5688
4	0.7805	0.8680	0.6224	2.3948	0.8668	0.8285	0.9120	0.6764	0.8843
5	0.8103	0.7802	0.6062	0.8668	2.0942	0.8013	0.9180	0.6212	0.8299
6	0.6903	0.7992	0.5851	0.8285	0.8013	1.6699	0.8640	0.6818	0.8750
7	0.7613	0.8497	0.5978	0.9120	0.9180	0.8640	1.6542	0.6603	0.8995
8	0.6793	0.6747	0.6199	0.6764	0.6212	0.6818	0.6603	1.7295	0.6584
9	0.7034	0.8018	0.5688	0.8843	0.8299	0.8750	0.8995	0.6584	1.7604
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	2.1792	0.5992	0.3189	0.6130	0.5243	0.4825	0.6063	0.6059	0.5528
2	0.5992	2.2297	0.3691	0.6923	0.6842	0.6317	0.7330	0.8953	0.6743
3	0.3189	0.3691	2.6929	0.3330	0.2647	0.2739	0.2876	0.4344	0.2365
4	0.6130	0.6923	0.3330	1.8989	0.8045	0.6792	0.9128	0.6455	0.8023
5	0.5243	0.6842	0.2647	0.8045	2.1387	0.6817	0.8800	0.6105	0.8532
6	0.4825	0.6317	0.2739	0.6792	0.6817	3.3397	0.7996	0.5669	0.8087
7	0.6063	0.7330	0.2876	0.9128	0.8800	0.7996	1.7692	0.6373	0.9179
8	0.6059	0.8953	0.4344	0.6455	0.6105	0.5669	0.6373	3.3424	0.5376
9	0.5528	0.6743	0.2365	0.8023	0.8532	0.8087	0.9179	0.5376	4.1479

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 54: Covariance-correlation matrix of MSVAR model for $h = 12$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.2030	0.6321	0.3716	0.6595	0.5920	0.5066	0.6168	0.5702	0.4971
2	0.6321	2.3854	0.3850	0.7596	0.6943	0.6585	0.7406	0.6969	0.6349
3	0.3716	0.3850	2.3159	0.3574	0.3093	0.3111	0.2902	0.4737	0.2920
4	0.6595	0.7596	0.3574	2.4413	0.8100	0.7174	0.8906	0.4832	0.7598
5	0.5920	0.6943	0.3093	0.8100	2.4223	0.6836	0.8943	0.4122	0.7274
6	0.5066	0.6585	0.3111	0.7174	0.6836	3.0877	0.7976	0.5734	0.8496
7	0.6168	0.7406	0.2902	0.8906	0.8943	0.7976	1.9728	0.4777	0.8312
8	0.5702	0.6969	0.4737	0.4832	0.4122	0.5734	0.4777	3.2248	0.4857
9	0.4971	0.6349	0.2920	0.7598	0.7274	0.8496	0.8312	0.4857	3.3060
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	0.9975	0.7621	0.5442	0.7942	0.8477	0.7367	0.8059	0.7583	0.7879
2	0.7621	1.1011	0.6184	0.8770	0.8338	0.8106	0.8945	0.8128	0.8965
3	0.5442	0.6184	0.8584	0.6275	0.6164	0.6039	0.6270	0.6161	0.6534
4	0.7942	0.8770	0.6275	2.0261	0.8972	0.8039	0.9236	0.8159	0.9041
5	0.8477	0.8338	0.6164	0.8972	1.6863	0.8425	0.9303	0.8211	0.9170
6	0.7367	0.8106	0.6039	0.8039	0.8425	1.4397	0.8691	0.7552	0.8760
7	0.8059	0.8945	0.6270	0.9236	0.9303	0.8691	1.3981	0.8115	0.9552
8	0.7583	0.8128	0.6161	0.8159	0.8211	0.7552	0.8115	1.2468	0.7818
9	0.7879	0.8965	0.6534	0.9041	0.9170	0.8760	0.9552	0.7818	1.5435

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Figure 8: Smoothed probability of $S_t = 1$ for $h = 2$

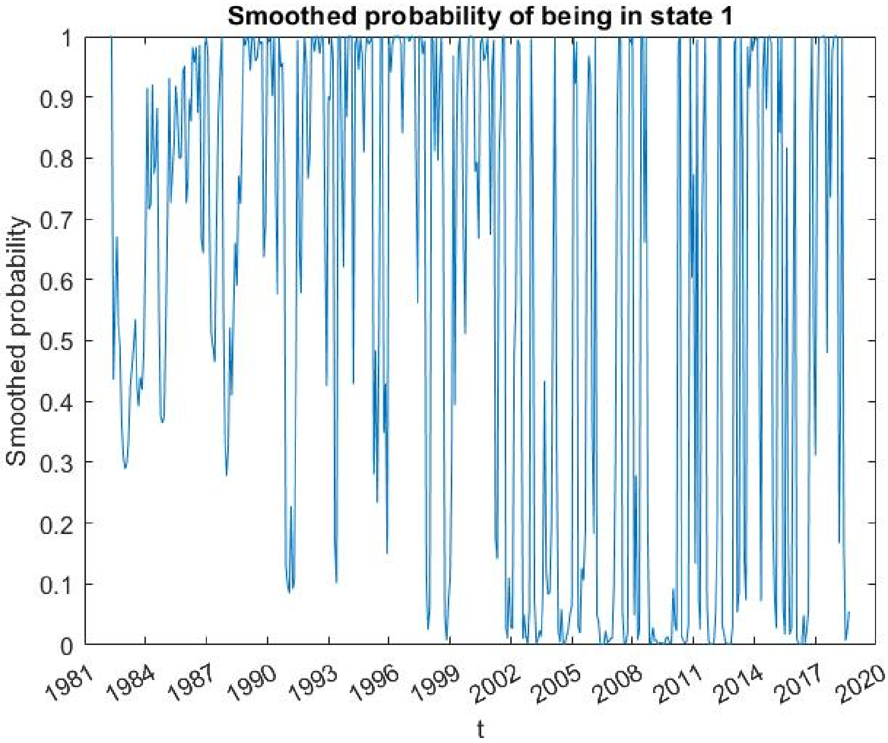


Figure 9: Smoothed probability of $S_t = 1$ for $h = 3$

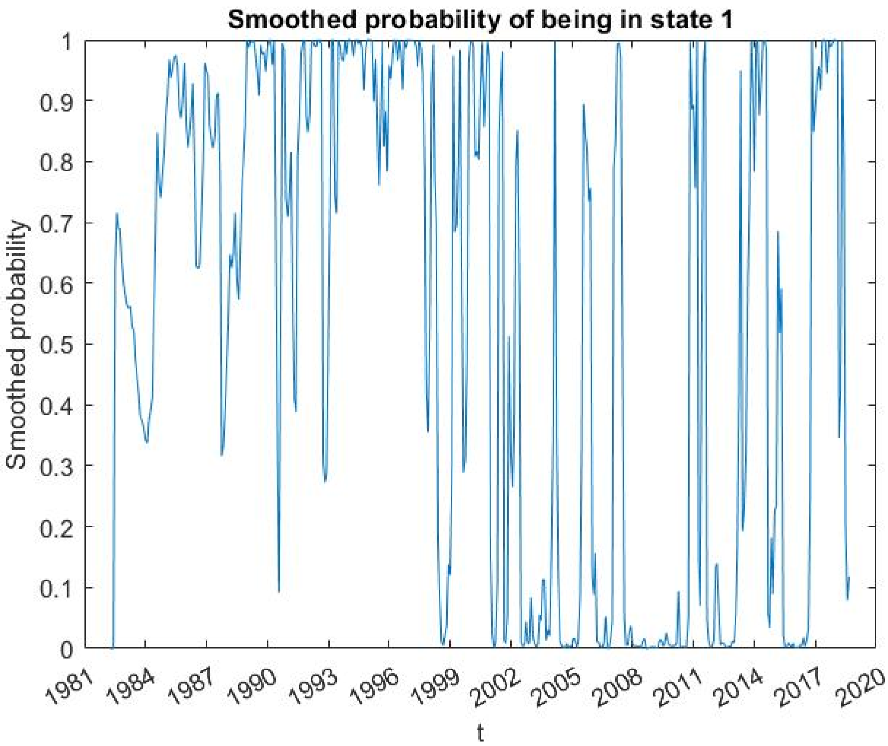


Figure 10: Smoothed probability of $S_t = 1$ for $h = 4$

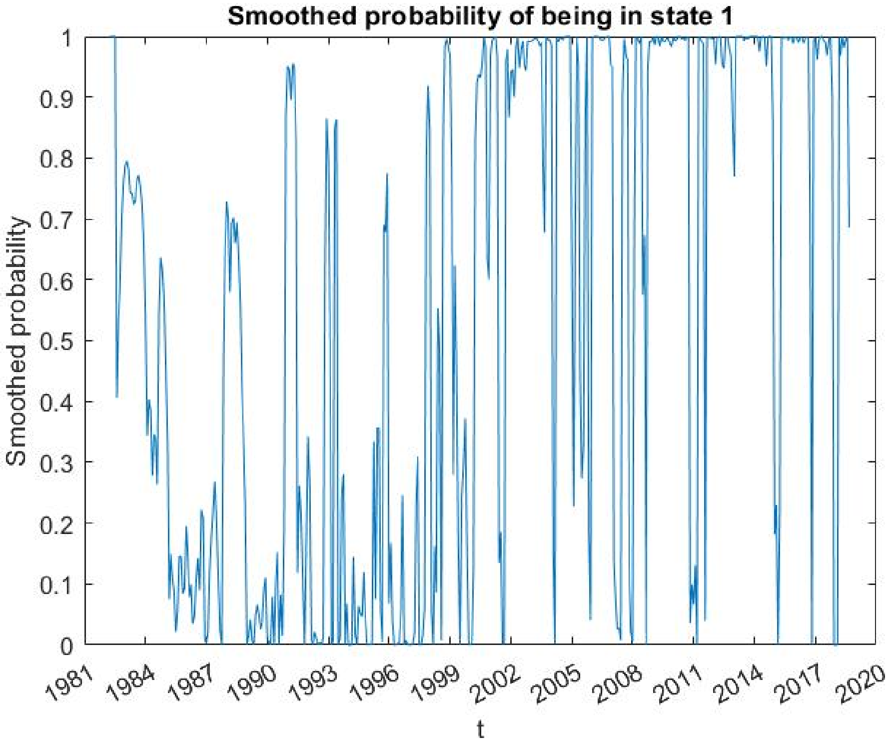


Figure 11: Smoothed probability of $S_t = 1$ for $h = 5$

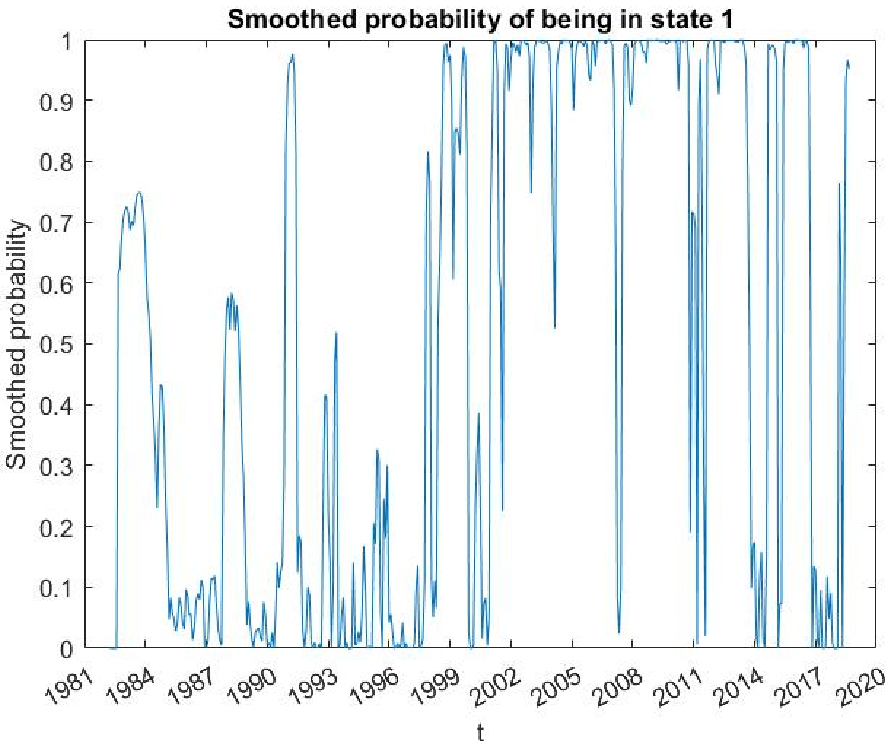


Figure 12: Smoothed probability of $S_t = 1$ for $h = 6$

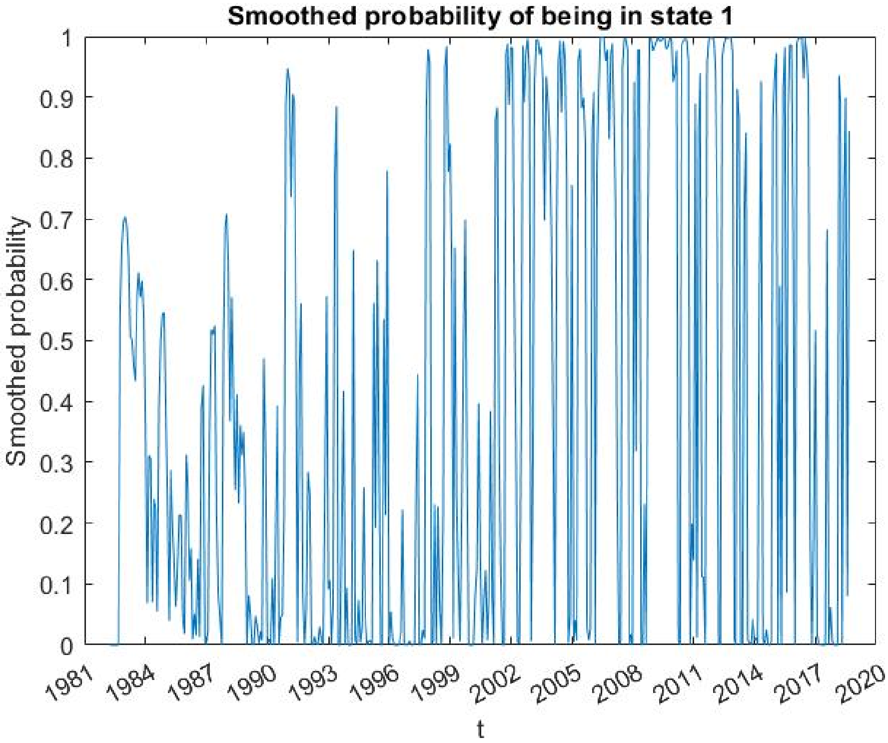


Figure 13: Smoothed probability of $S_t = 1$ for $h = 7$

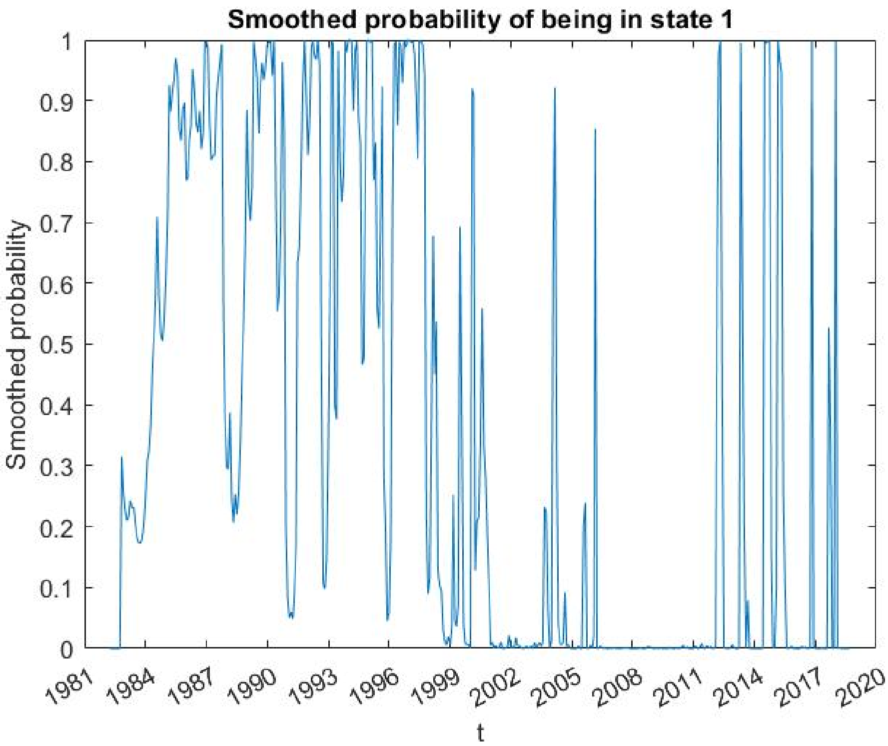


Figure 14: Smoothed probability of $S_t = 1$ for $h = 8$

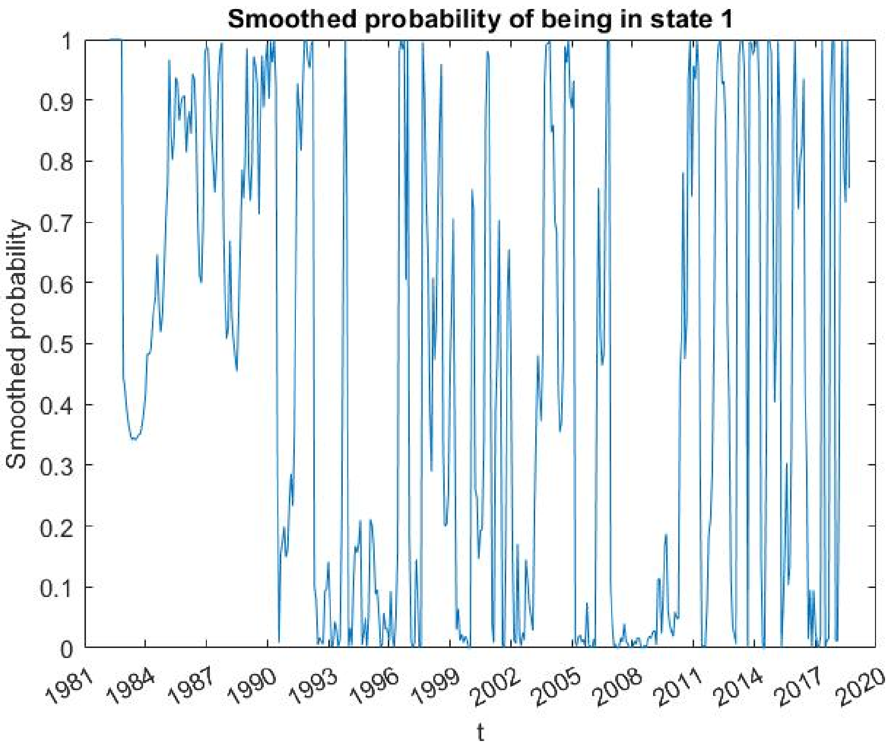


Figure 15: Smoothed probability of $S_t = 1$ for $h = 9$

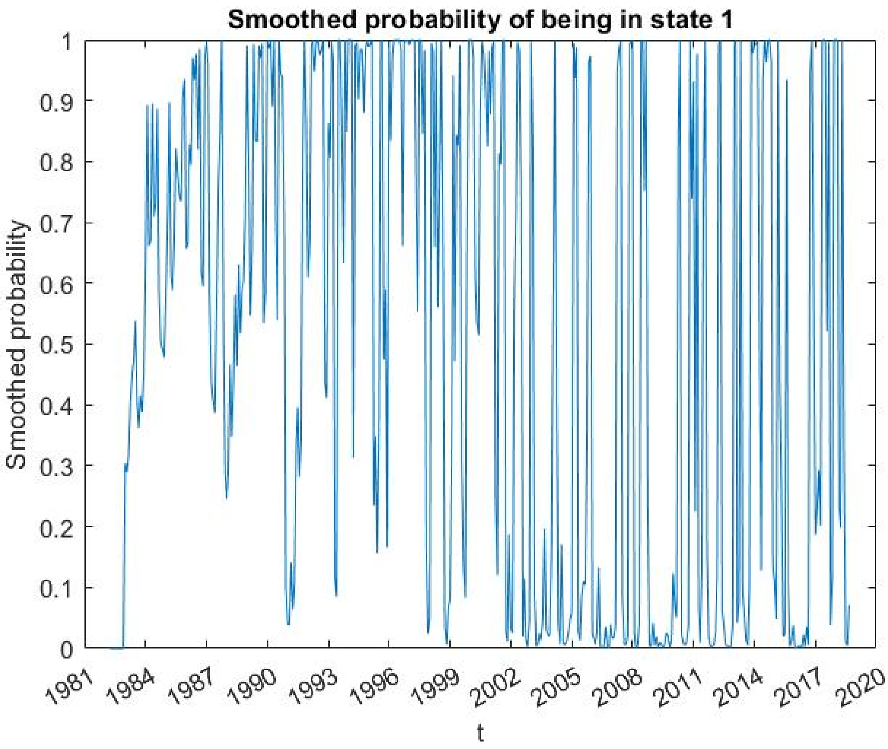


Figure 16: Smoothed probability of $S_t = 1$ for $h = 10$

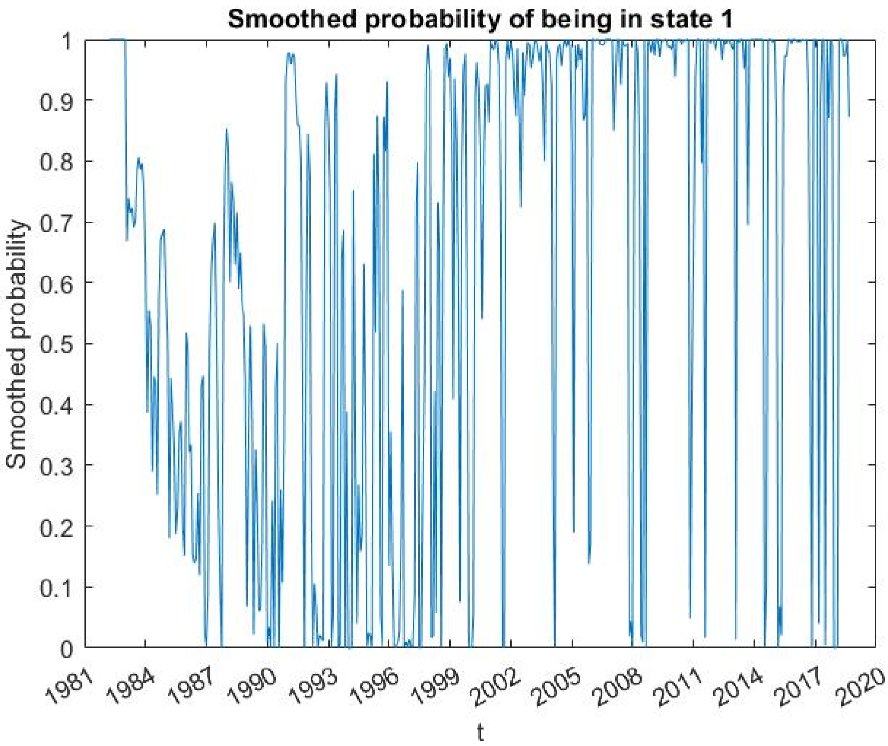


Figure 17: Smoothed probability of $S_t = 1$ for $h = 11$

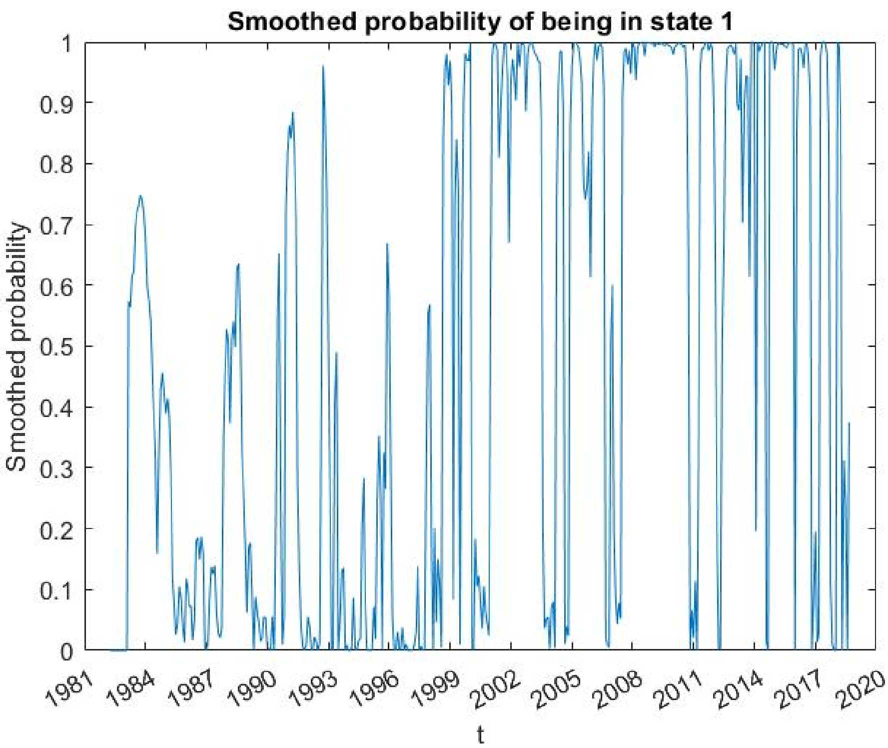
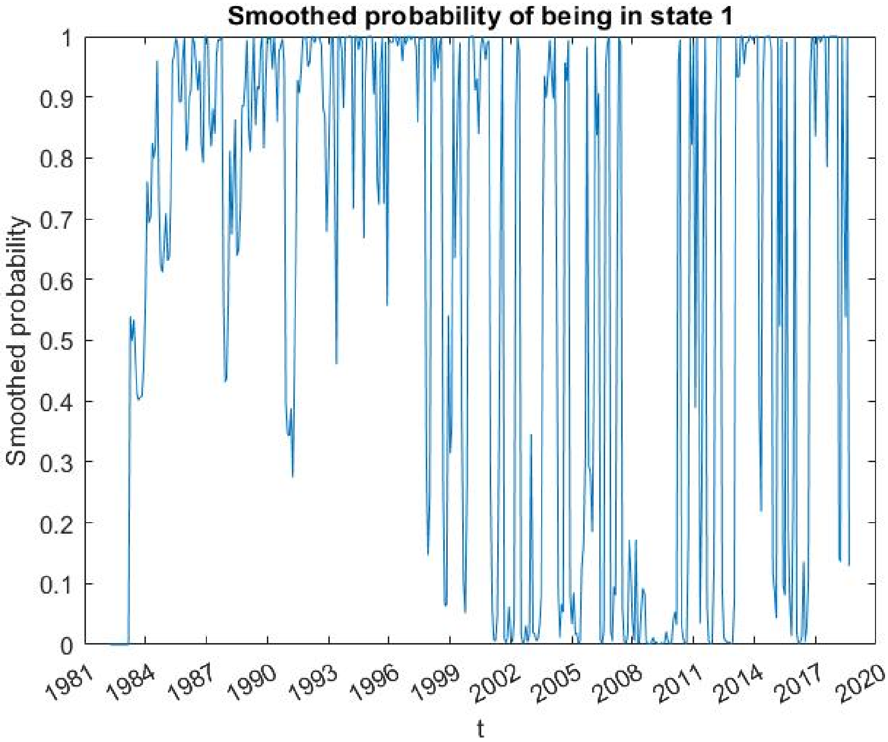


Figure 18: Smoothed probability of $S_t = 1$ for $h = 12$



7.4 Sign MSVAR model results

Table 55: Regression constants $\alpha_{i,h,sign,1}$ of sign MSVAR model for $S_t = 1$

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	46.494	35.682	-17.943	45.779	31.953	6.833	32.377	-13.998	24.986
2	31.331	13.539	-10.490	13.739	11.296	-4.091	11.092	33.167	0.266
3	19.474	8.383	-18.269	8.377	6.808	-7.754	9.111	-3.260	7.843
4	23.747	5.548	-16.121	10.468	7.802	-8.986	8.600	-8.688	-1.416
5	34.213	22.411	9.522	26.296	35.937	32.469	27.237	24.293	26.621
6	20.944	9.436	-0.589	6.628	18.278	12.094	11.591	29.290	-0.574
7	26.408	17.079	-10.098	19.736	9.951	-1.382	17.384	34.295	0.695
8	42.729	25.128	14.355	27.408	33.807	25.934	25.595	44.978	13.073
9	2.064	-4.041	6.505	-13.681	2.457	-3.928	-2.183	14.707	-14.607
10	38.970	16.317	7.830	25.550	22.678	10.068	18.541	18.357	11.247
11	20.755	5.806	-28.936	11.740	-6.448	-17.605	3.906	49.902	-48.940
12	49.867	32.992	-8.461	42.256	32.743	18.315	34.257	20.871	10.695

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.
[2] The numbers indicates the estimated constants $\alpha_{i,h,1}$ multiplied by $\times 10^2$.

Table 56: Regression constants $\alpha_{i,h,sign,2}$ of sign MSVAR model for $S_t = 2$

h	S&P 500	FTSE 100	TOPIX	AEX	DAX	IBEX 35	CAC 40	ASX 200	MIB
1	27.371	6.442	7.816	7.159	15.427	11.100	10.867	19.576	2.764
2	30.168	17.609	9.873	22.907	29.610	24.418	23.223	15.525	15.306
3	44.541	25.616	16.452	29.597	35.911	28.618	27.571	38.822	13.266
4	40.988	29.153	15.565	31.691	38.541	35.500	31.567	45.087	22.205
5	23.407	6.586	-14.799	8.261	1.338	-16.229	5.358	20.646	-23.580
6	39.795	21.630	-4.293	32.309	22.262	11.325	26.494	-13.277	21.590
7	33.721	13.883	4.196	17.670	26.769	19.990	17.532	16.410	8.750
8	22.011	8.658	-13.098	12.270	11.270	-0.166	12.258	-13.695	8.622
9	54.539	32.794	-11.074	46.851	37.208	27.106	31.734	13.752	24.914
10	22.569	15.803	-7.407	12.425	21.081	14.805	15.671	23.595	3.098
11	35.034	20.124	11.615	22.079	33.166	24.130	22.091	21.830	17.849
12	11.867	-2.003	4.452	-4.606	9.869	3.101	0.822	12.097	-5.878

[1] The indices are sorted from left to right, with S&P 500 being denoted as $i = 1$ and MIB as $i = 9$.
[2] The numbers indicates the estimated constants $\alpha_{i,h,2}$ multiplied by $\times 10^2$.

Table 57: Covariance-correlation matrix of sign MSVAR model for $h = 1$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.3798	0.6162	0.2240	0.6003	0.5199	0.4742	0.6485	0.5278	0.5982
2	0.6162	2.6692	0.3492	0.6320	0.6491	0.6880	0.6841	0.7046	0.8155
3	0.2240	0.3492	3.3808	0.2891	0.2129	0.2884	0.3291	0.2538	0.2891
4	0.6003	0.6320	0.2891	2.6111	0.7929	0.6322	0.9505	0.2154	0.7250
5	0.5199	0.6491	0.2129	0.7929	2.3247	0.6635	0.7942	0.4312	0.6982
6	0.4742	0.6880	0.2884	0.6322	0.6635	4.4901	0.7991	0.6826	0.9526
7	0.6485	0.6841	0.3291	0.9505	0.7942	0.7991	2.0908	0.3719	0.8455
8	0.5278	0.7046	0.2538	0.2154	0.4312	0.6826	0.3719	7.9317	0.6338
9	0.5982	0.8155	0.2891	0.7250	0.6982	0.9526	0.8455	0.6338	3.2963
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.3563	0.6898	0.5574	0.7395	0.7345	0.6360	0.7155	0.6467	0.6185
2	0.6898	1.4207	0.5291	0.8332	0.7656	0.7375	0.8195	0.7133	0.7230
3	0.5574	0.5291	1.0149	0.5451	0.5150	0.4910	0.5290	0.5838	0.4858
4	0.7395	0.8332	0.5451	2.0043	0.8472	0.7809	0.8939	0.7042	0.8137
5	0.7345	0.7656	0.5150	0.8472	1.9112	0.7793	0.9094	0.6348	0.8087
6	0.6360	0.7375	0.4910	0.7809	0.7793	1.6965	0.8356	0.6410	0.8500
7	0.7155	0.8195	0.5290	0.8939	0.9094	0.8356	1.5169	0.6753	0.8705
8	0.6467	0.7133	0.5838	0.7042	0.6348	0.6410	0.6753	1.6268	0.6154
9	0.6185	0.7230	0.4858	0.8137	0.8087	0.8500	0.8705	0.6154	2.0996

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 58: Covariance-correlation matrix of sign MSVAR model for $h = 2$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.4014	0.6301	0.3191	0.6381	0.5876	0.5118	0.6357	0.6464	0.5395
2	0.6301	2.6880	0.3629	0.7373	0.6729	0.6476	0.7439	0.7564	0.6640
3	0.3191	0.3629	2.4128	0.3584	0.3076	0.2904	0.3420	0.3441	0.3536
4	0.6381	0.7373	0.3584	3.4298	0.8223	0.6701	0.9005	0.6846	0.7688
5	0.5876	0.6729	0.3076	0.8223	2.9951	0.6868	0.9107	0.6180	0.7558
6	0.5118	0.6476	0.2904	0.6701	0.6868	3.4545	0.7935	0.6467	0.8523
7	0.6357	0.7439	0.3420	0.9005	0.9107	0.7935	2.3286	0.6548	0.8364
8	0.6464	0.7564	0.3441	0.6846	0.6180	0.6467	0.6548	2.0882	0.6639
9	0.5395	0.6640	0.3536	0.7688	0.7558	0.8523	0.8364	0.6639	3.9054
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.0114	0.7261	0.5226	0.7788	0.7928	0.6468	0.7138	0.6153	0.6489
2	0.7261	0.9897	0.5787	0.8299	0.8132	0.7708	0.8435	0.7443	0.7935
3	0.5226	0.5787	1.0426	0.5771	0.5383	0.5471	0.5338	0.6428	0.5216
4	0.7788	0.8299	0.5771	1.0771	0.8731	0.8126	0.8902	0.6646	0.8425
5	0.7928	0.8132	0.5383	0.8731	1.1498	0.7926	0.8877	0.6364	0.8484
6	0.6468	0.7708	0.5471	0.8126	0.7926	1.2332	0.8419	0.6446	0.8683
7	0.7138	0.8435	0.5338	0.8902	0.8877	0.8419	1.0312	0.6575	0.9305
8	0.6153	0.7443	0.6428	0.6646	0.6364	0.6446	0.6575	1.9352	0.5942
9	0.6489	0.7935	0.5216	0.8425	0.8484	0.8683	0.9305	0.5942	1.1871

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 59: Covariance-correlation matrix of sign MSVAR model for $h = 3$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	2.3925	0.6708	0.3402	0.6714	0.6122	0.5252	0.6843	0.6282	0.6039
2	0.6708	2.7415	0.4001	0.7405	0.6511	0.5975	0.7409	0.6665	0.6164
3	0.3402	0.4001	2.4943	0.3848	0.3158	0.3223	0.3534	0.4966	0.4513
4	0.6714	0.7405	0.3848	3.4660	0.8291	0.6646	0.9052	0.5587	0.8841
5	0.6122	0.6511	0.3158	0.8291	2.8078	0.6610	0.9074	0.4486	0.7802
6	0.5252	0.5975	0.3223	0.6646	0.6610	2.7069	0.7758	0.5662	0.7327
7	0.6843	0.7409	0.3534	0.9052	0.9074	0.7758	2.2364	0.5669	0.8888
8	0.6282	0.6665	0.4966	0.5587	0.4486	0.5662	0.5669	3.3199	0.5409
9	0.6039	0.6164	0.4513	0.8841	0.7802	0.7327	0.8888	0.5409	2.3942

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	0.9719	0.6020	0.4701	0.6733	0.6923	0.5348	0.5959	0.5631	0.5105
2	0.6020	0.8921	0.4688	0.8353	0.8463	0.8172	0.8539	0.7953	0.8225
3	0.4701	0.4688	0.8375	0.5077	0.5007	0.4079	0.4703	0.4907	0.4207
4	0.6733	0.8353	0.5077	1.0078	0.8585	0.8342	0.8887	0.7141	0.8055
5	0.6923	0.8463	0.5007	0.8585	1.2503	0.8207	0.8931	0.7488	0.8446
6	0.5348	0.8172	0.4079	0.8342	0.8207	1.7547	0.8661	0.6516	0.8980
7	0.5959	0.8539	0.4703	0.8887	0.8931	0.8661	1.0992	0.6749	0.8725
8	0.5631	0.7953	0.4907	0.7141	0.7488	0.6516	0.6749	1.1658	0.6489
9	0.5105	0.8225	0.4207	0.8055	0.8446	0.8980	0.8725	0.6489	2.0303

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 60: Covariance-correlation matrix of sign MSVAR model for $h = 4$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.3826	0.6690	0.3711	0.6802	0.6245	0.5228	0.6411	0.6155	0.5588
2	0.6690	2.6549	0.4055	0.7750	0.6796	0.6107	0.7489	0.6585	0.6123
3	0.3711	0.4055	2.3985	0.3900	0.3270	0.3105	0.3366	0.5155	0.3647
4	0.6802	0.7750	0.3900	3.4546	0.8362	0.6842	0.8849	0.5665	0.8190
5	0.6245	0.6796	0.3270	0.8362	2.7842	0.6656	0.8957	0.4318	0.7124
6	0.5228	0.6107	0.3105	0.6842	0.6656	2.9608	0.8071	0.4914	0.8361
7	0.6411	0.7489	0.3366	0.8849	0.8957	0.8071	2.4005	0.5095	0.8545
8	0.6155	0.6585	0.5155	0.5665	0.4318	0.4914	0.5095	3.0561	0.4732
9	0.5588	0.6123	0.3647	0.8190	0.7124	0.8361	0.8545	0.4732	2.8401
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	0.9305	0.6072	0.4291	0.7029	0.7171	0.5987	0.6892	0.5877	0.5699
2	0.6072	0.9359	0.4805	0.8290	0.8379	0.8295	0.8819	0.8055	0.8709
3	0.4291	0.4805	0.8504	0.4981	0.4868	0.4494	0.4893	0.5055	0.4397
4	0.7029	0.8290	0.4981	0.9921	0.8626	0.8155	0.9092	0.7379	0.8120
5	0.7171	0.8379	0.4868	0.8626	1.2912	0.8268	0.9114	0.7690	0.8837
6	0.5987	0.8295	0.4494	0.8155	0.8268	1.4730	0.8442	0.7334	0.8743
7	0.6892	0.8819	0.4893	0.9092	0.9114	0.8442	0.9898	0.7574	0.8954
8	0.5877	0.8055	0.5055	0.7379	0.7690	0.7334	0.7574	1.1947	0.7301
9	0.5699	0.8709	0.4397	0.8120	0.8837	0.8743	0.8954	0.7301	1.8085

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 61: Covariance-correlation matrix of sign MSVAR model for $h = 5$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	1.1005	0.7148	0.5104	0.7744	0.7802	0.6509	0.7207	0.6252	0.6343
2	0.7148	1.1522	0.5420	0.8422	0.8387	0.7949	0.8654	0.8031	0.8271
3	0.5104	0.5420	1.0046	0.5640	0.5200	0.5133	0.5178	0.6148	0.4339
4	0.7744	0.8422	0.5640	1.2191	0.8806	0.8074	0.9020	0.7012	0.8263
5	0.7802	0.8387	0.5200	0.8806	1.3341	0.8079	0.9049	0.6898	0.8571
6	0.6509	0.7949	0.5133	0.8074	0.8079	1.5190	0.8293	0.6966	0.8784
7	0.7207	0.8654	0.5178	0.9020	0.9049	0.8293	1.1418	0.7046	0.9050
8	0.6252	0.8031	0.6148	0.7012	0.6898	0.6966	0.7046	2.0652	0.6565
9	0.6343	0.8271	0.4339	0.8263	0.8571	0.8784	0.9050	0.6565	1.6338

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	2.4356	0.6305	0.3209	0.6521	0.5914	0.5068	0.6417	0.6276	0.5306
2	0.6305	2.6950	0.3711	0.7643	0.6579	0.6046	0.7373	0.6533	0.5989
3	0.3209	0.3711	2.5044	0.3527	0.2891	0.2609	0.3017	0.3654	0.4189
4	0.6521	0.7643	0.3527	3.7496	0.8231	0.6750	0.8951	0.6862	0.7818
5	0.5914	0.6579	0.2891	0.8231	3.0438	0.6641	0.9034	0.5493	0.7077
6	0.5068	0.6046	0.2609	0.6750	0.6641	3.2671	0.8135	0.5294	0.8095
7	0.6417	0.7373	0.3017	0.8951	0.9034	0.8135	2.4702	0.5961	0.8178
8	0.6276	0.6533	0.3654	0.6862	0.5493	0.5294	0.5961	1.6922	0.5665
9	0.5306	0.5989	0.4189	0.7818	0.7077	0.8095	0.8178	0.5665	3.3215

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 62: Covariance-correlation matrix of sign MSVAR model for $h = 6$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	1.3031	0.7098	0.5051	0.7404	0.8027	0.6763	0.7856	0.7032	0.6854
2	0.7098	1.2910	0.5472	0.8428	0.8266	0.7802	0.8761	0.8216	0.8649
3	0.5051	0.5472	1.1697	0.5234	0.5156	0.4468	0.5293	0.5373	0.4752
4	0.7404	0.8428	0.5234	2.1313	0.8584	0.7166	0.9113	0.7746	0.8236
5	0.8027	0.8266	0.5156	0.8584	2.1695	0.7524	0.9218	0.7703	0.8660
6	0.6763	0.7802	0.4468	0.7166	0.7524	2.0526	0.8233	0.7230	0.8547
7	0.7856	0.8761	0.5293	0.9113	0.9218	0.8233	1.5532	0.7859	0.9129
8	0.7032	0.8216	0.5373	0.7746	0.7703	0.7230	0.7859	1.5602	0.7506
9	0.6854	0.8649	0.4752	0.8236	0.8660	0.8547	0.9129	0.7506	2.3003

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	2.2665	0.6126	0.3256	0.6304	0.4874	0.4301	0.5126	0.5889	0.4626
2	0.6126	2.6342	0.3648	0.7469	0.6281	0.5762	0.6985	0.6193	0.5134
3	0.3256	0.3648	2.5668	0.3474	0.2527	0.2988	0.2378	0.5718	0.2497
4	0.6304	0.7469	0.3474	2.4375	0.8154	0.7583	0.8774	0.4475	0.7847
5	0.4874	0.6281	0.2527	0.8154	1.8608	0.6879	0.8871	0.2764	0.6501
6	0.4301	0.5762	0.2988	0.7583	0.6879	2.5986	0.8002	0.4363	0.8648
7	0.5126	0.6985	0.2378	0.8774	0.8871	0.8002	1.9917	0.3687	0.8040
8	0.5889	0.6193	0.5718	0.4475	0.2764	0.4363	0.3687	3.6805	0.3634
9	0.4626	0.5134	0.2497	0.7847	0.6501	0.8648	0.8040	0.3634	2.6852

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 63: Covariance-correlation matrix of sign MSVAR model for $h = 7$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.5569	0.6407	0.3103	0.6691	0.5851	0.5541	0.6759	0.7002	0.6230
2	0.6407	2.7377	0.3908	0.7724	0.6362	0.6254	0.7332	0.7756	0.6072
3	0.3103	0.3908	2.7454	0.3678	0.2811	0.2756	0.3154	0.4840	0.4780
4	0.6691	0.7724	0.3678	3.8005	0.8219	0.6933	0.9113	0.7485	0.8911
5	0.5851	0.6362	0.2811	0.8219	2.9923	0.6529	0.9115	0.5795	0.7537
6	0.5541	0.6254	0.2756	0.6933	0.6529	2.8557	0.7742	0.7167	0.7018
7	0.6759	0.7332	0.3154	0.9113	0.9115	0.7742	2.2264	0.6915	0.8666
8	0.7002	0.7756	0.4840	0.7485	0.5795	0.7167	0.6915	1.9779	0.6304
9	0.6230	0.6072	0.4780	0.8911	0.7537	0.7018	0.8666	0.6304	2.3543
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.0842	0.6785	0.5707	0.7212	0.7493	0.5674	0.6706	0.6115	0.5525
2	0.6785	1.2155	0.5252	0.8313	0.8416	0.7648	0.8482	0.7454	0.7791
3	0.5707	0.5252	0.9858	0.5654	0.5334	0.4876	0.5155	0.5526	0.4336
4	0.7212	0.8313	0.5654	1.2537	0.8752	0.8065	0.9079	0.6366	0.7970
5	0.7493	0.8416	0.5334	0.8752	1.4212	0.8008	0.9011	0.6554	0.8320
6	0.5674	0.7648	0.4876	0.8065	0.8008	1.8407	0.8542	0.6067	0.8921
7	0.6706	0.8482	0.5155	0.9079	0.9011	0.8542	1.2848	0.6319	0.8729
8	0.6115	0.7454	0.5526	0.6366	0.6554	0.6067	0.6319	1.9921	0.5883
9	0.5525	0.7791	0.4336	0.7970	0.8320	0.8921	0.8729	0.5883	2.0463

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 64: Covariance-correlation matrix of sign MSVAR model for $h = 8$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	0.9249	0.6016	0.4420	0.6898	0.7263	0.5506	0.6668	0.5610	0.5232
2	0.6016	0.8965	0.4964	0.8159	0.8248	0.8001	0.8630	0.7744	0.8281
3	0.4420	0.4964	0.9429	0.4544	0.4614	0.3751	0.4585	0.4490	0.4201
4	0.6898	0.8159	0.4544	0.9545	0.8604	0.8292	0.9086	0.7054	0.8087
5	0.7263	0.8248	0.4614	0.8604	1.2051	0.8169	0.8970	0.7419	0.8544
6	0.5506	0.8001	0.3751	0.8292	0.8169	1.7001	0.8794	0.6595	0.9022
7	0.6668	0.8630	0.4585	0.9086	0.8970	0.8794	1.0049	0.6907	0.9050
8	0.5610	0.7744	0.4490	0.7054	0.7419	0.6595	0.6907	1.1232	0.6449
9	0.5232	0.8281	0.4201	0.8087	0.8544	0.9022	0.9050	0.6449	2.0534

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	2.3341	0.6690	0.3827	0.6906	0.6283	0.5536	0.6557	0.6340	0.6094
2	0.6690	2.6307	0.4231	0.7808	0.6913	0.6453	0.7593	0.6857	0.6502
3	0.3827	0.4231	2.3352	0.4184	0.3520	0.3604	0.3396	0.6021	0.3393
4	0.6906	0.7808	0.4184	3.4211	0.8366	0.7067	0.8949	0.5669	0.8700
5	0.6283	0.6913	0.3520	0.8366	2.8121	0.6814	0.9017	0.4565	0.7682
6	0.5536	0.6453	0.3604	0.7067	0.6814	2.7731	0.7873	0.5808	0.7927
7	0.6557	0.7593	0.3396	0.8949	0.9017	0.7873	2.3277	0.5433	0.8743
8	0.6340	0.6857	0.6021	0.5669	0.4565	0.5808	0.5433	3.4097	0.5341
9	0.6094	0.6502	0.3393	0.8700	0.7682	0.7927	0.8743	0.5341	2.5144

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 65: Covariance-correlation matrix of sign MSVAR model for $h = 9$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	1.4820	0.7588	0.5183	0.7905	0.8482	0.7697	0.8233	0.7846	0.7656
2	0.7588	1.6127	0.5787	0.8755	0.7923	0.7964	0.8718	0.8130	0.8366
3	0.5183	0.5787	1.1901	0.5887	0.5584	0.5481	0.5852	0.6454	0.5857
4	0.7905	0.8755	0.5887	2.5247	0.8812	0.8186	0.9253	0.8159	0.9005
5	0.8482	0.7923	0.5584	0.8812	2.2174	0.8346	0.9276	0.7939	0.8938
6	0.7697	0.7964	0.5481	0.8186	0.8346	1.8500	0.8770	0.7736	0.8656
7	0.8233	0.8718	0.5852	0.9253	0.9276	0.8770	1.7601	0.8173	0.9510
8	0.7846	0.8130	0.6454	0.8159	0.7939	0.7736	0.8173	1.8743	0.7625
9	0.7656	0.8366	0.5857	0.9005	0.8938	0.8656	0.9510	0.7625	1.8316

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	1.8341	0.5600	0.3855	0.5740	0.4711	0.3934	0.5152	0.5383	0.4490
2	0.5600	2.1148	0.3733	0.7016	0.6867	0.6201	0.6954	0.6868	0.6181
3	0.3855	0.3733	2.3168	0.3624	0.2962	0.2842	0.2446	0.3685	0.2810
4	0.5740	0.7016	0.3624	1.9439	0.8022	0.6781	0.8781	0.3904	0.7196
5	0.4711	0.6867	0.2962	0.8022	2.0814	0.6521	0.8821	0.3343	0.7115
6	0.3934	0.6201	0.2842	0.6781	0.6521	2.9207	0.7697	0.5291	0.8664
7	0.5152	0.6954	0.2446	0.8781	0.8821	0.7697	1.7592	0.3858	0.8011
8	0.5383	0.6868	0.3685	0.3904	0.3343	0.5291	0.3858	3.3326	0.4710
9	0.4490	0.6181	0.2810	0.7196	0.7115	0.8664	0.8011	0.4710	3.4239

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 66: Covariance-correlation matrix of sign MSVAR model for $h = 10$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	1.7305	0.6257	0.3699	0.6620	0.6121	0.5390	0.6246	0.5961	0.5288
2	0.6257	1.8812	0.4224	0.7863	0.7151	0.6678	0.7779	0.6924	0.6480
3	0.3699	0.4224	1.7022	0.4116	0.3599	0.3747	0.3553	0.4965	0.3756
4	0.6620	0.7863	0.4116	2.0353	0.8232	0.7485	0.8994	0.5625	0.7830
5	0.6121	0.7151	0.3599	0.8232	1.9740	0.7200	0.8910	0.5020	0.7557
6	0.5390	0.6678	0.3747	0.7485	0.7200	2.2090	0.8133	0.5666	0.8535
7	0.6246	0.7779	0.3553	0.8994	0.8910	0.8133	1.6436	0.5522	0.8570
8	0.5961	0.6924	0.4965	0.5625	0.5020	0.5666	0.5522	2.2554	0.5136
9	0.5288	0.6480	0.3756	0.7830	0.7557	0.8535	0.8570	0.5136	2.4490
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.7660	0.6899	0.4448	0.7224	0.7198	0.5970	0.7190	0.6894	0.6424
2	0.6899	1.9441	0.4708	0.8011	0.7664	0.7544	0.8314	0.8219	0.8227
3	0.4448	0.4708	1.7661	0.4559	0.4303	0.3969	0.4387	0.5582	0.3999
4	0.7224	0.8011	0.4559	2.6826	0.8653	0.7400	0.9108	0.7344	0.8498
5	0.7198	0.7664	0.4303	0.8653	2.3872	0.7536	0.9218	0.7039	0.8587
6	0.5970	0.7544	0.3969	0.7400	0.7536	2.5895	0.8450	0.7079	0.8721
7	0.7190	0.8314	0.4387	0.9108	0.9218	0.8450	1.9138	0.7400	0.9186
8	0.6894	0.8219	0.5582	0.7344	0.7039	0.7079	0.7400	2.3311	0.7121
9	0.6424	0.8227	0.3999	0.8498	0.8587	0.8721	0.9186	0.7121	2.4279

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 67: Covariance-correlation matrix of sign MSVAR model for $h = 11$

$S_t = 1$									
i	1	2	3	4	5	6	7	8	9
1	2.9558	0.6634	0.2920	0.6910	0.5981	0.5142	0.7037	0.6485	0.6422
2	0.6634	3.1593	0.3527	0.7577	0.6219	0.6004	0.7310	0.7933	0.5595
3	0.2920	0.3527	3.1807	0.3215	0.2527	0.2212	0.2414	0.5018	0.4815
4	0.6910	0.7577	0.3215	4.1894	0.8174	0.6771	0.8850	0.8249	0.6774
5	0.5981	0.6219	0.2527	0.8174	3.5117	0.6399	0.8908	0.6212	0.6187
6	0.5142	0.6004	0.2212	0.6771	0.6399	3.8982	0.8467	0.5665	0.8736
7	0.7037	0.7310	0.2414	0.8850	0.8908	0.8467	2.7014	0.6499	0.8108
8	0.6485	0.7933	0.5018	0.8249	0.6212	0.5665	0.6499	2.1865	0.6578
9	0.6422	0.5595	0.4815	0.6774	0.6187	0.8736	0.8108	0.6578	5.0308
$S_t = 2$									
i	1	2	3	4	5	6	7	8	9
1	1.1651	0.6538	0.5143	0.7003	0.7214	0.5976	0.6692	0.6173	0.5767
2	0.6538	1.2779	0.5186	0.8237	0.8239	0.7704	0.8441	0.7364	0.7971
3	0.5143	0.5186	1.0275	0.5454	0.4998	0.4978	0.4971	0.5651	0.4272
4	0.7003	0.8237	0.5454	1.4920	0.8685	0.7844	0.9054	0.6215	0.8420
5	0.7214	0.8239	0.4998	0.8685	1.4646	0.7916	0.9073	0.6471	0.8322
6	0.5976	0.7704	0.4978	0.7844	0.7916	1.6389	0.8302	0.6469	0.8369
7	0.6692	0.8441	0.4971	0.9054	0.9073	0.8302	1.3085	0.6445	0.8821
8	0.6173	0.7364	0.5651	0.6215	0.6471	0.6469	0.6445	1.9355	0.5994
9	0.5767	0.7971	0.4272	0.8420	0.8322	0.8369	0.8821	0.5994	1.6839

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Table 68: Covariance-correlation matrix of sign MSVAR model for $h = 12$

$S_t = 1$

i	1	2	3	4	5	6	7	8	9
1	2.2683	0.5994	0.3453	0.6215	0.5489	0.4853	0.5991	0.5682	0.5387
2	0.5994	2.3018	0.3692	0.7046	0.6935	0.6596	0.7233	0.8047	0.6906
3	0.3453	0.3692	2.6767	0.3308	0.2714	0.2915	0.3074	0.4123	0.2471
4	0.6215	0.7046	0.3308	2.2757	0.8264	0.7076	0.9292	0.4481	0.7270
5	0.5489	0.6935	0.2714	0.8264	2.3400	0.6970	0.9037	0.4353	0.8014
6	0.4853	0.6596	0.2915	0.7076	0.6970	3.5911	0.8040	0.5764	0.9001
7	0.5991	0.7233	0.3074	0.9292	0.9037	0.8040	2.0140	0.4622	0.8502
8	0.5682	0.8047	0.4123	0.4481	0.4353	0.5764	0.4622	3.8367	0.5109
9	0.5387	0.6906	0.2471	0.7270	0.8014	0.9001	0.8502	0.5109	4.3006

$S_t = 2$

i	1	2	3	4	5	6	7	8	9
1	1.1938	0.7439	0.5653	0.7772	0.8112	0.7048	0.7667	0.7112	0.6980
2	0.7439	1.4454	0.5798	0.8688	0.7808	0.7810	0.8529	0.7166	0.7965
3	0.5653	0.5798	0.9271	0.6103	0.5938	0.5766	0.5791	0.5995	0.5549
4	0.7772	0.8688	0.6103	2.1935	0.8594	0.8098	0.9043	0.7397	0.8756
5	0.8112	0.7808	0.5938	0.8594	1.9343	0.7957	0.9107	0.6898	0.8203
6	0.7048	0.7810	0.5766	0.8098	0.7957	1.5572	0.8510	0.7035	0.8329
7	0.7667	0.8529	0.5791	0.9043	0.9107	0.8510	1.5395	0.7295	0.9036
8	0.7112	0.7166	0.5995	0.7397	0.6898	0.7035	0.7295	1.5965	0.6993
9	0.6980	0.7965	0.5549	0.8756	0.8203	0.8329	0.9036	0.6993	1.6441

[1] The diagonal elements indicate the variance of the scaled return of each index, while the off-diagonal elements indicate the correlation coefficient between the two indices.

Figure 19: Smoothed probability of $S_t = 1$ for $h = 1$ of sign MSVAR model

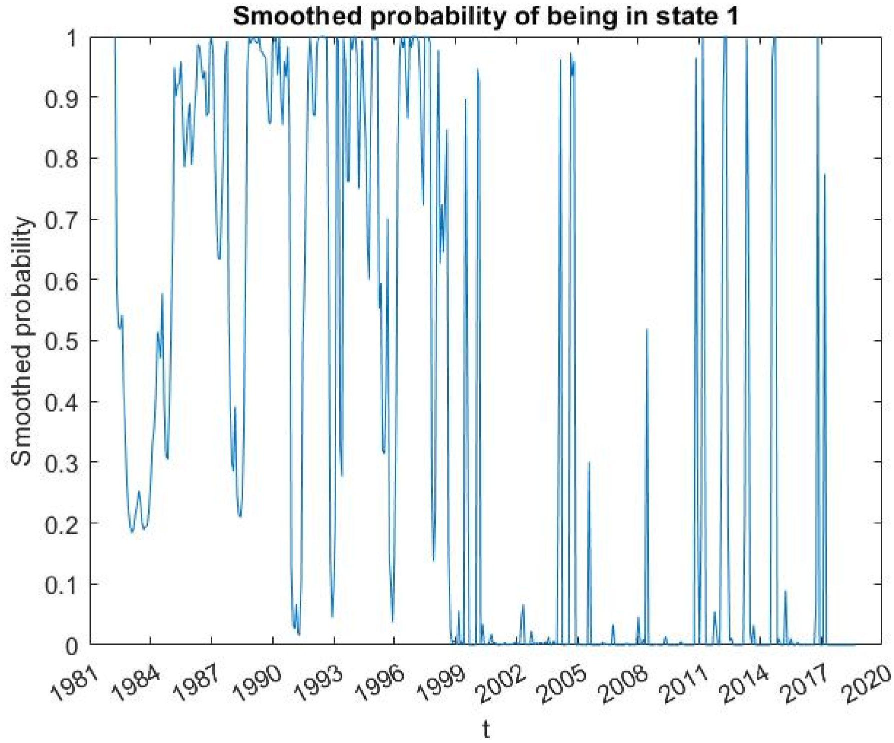


Figure 20: Smoothed probability of $S_t = 1$ for $h = 2$ of sign MSVAR model

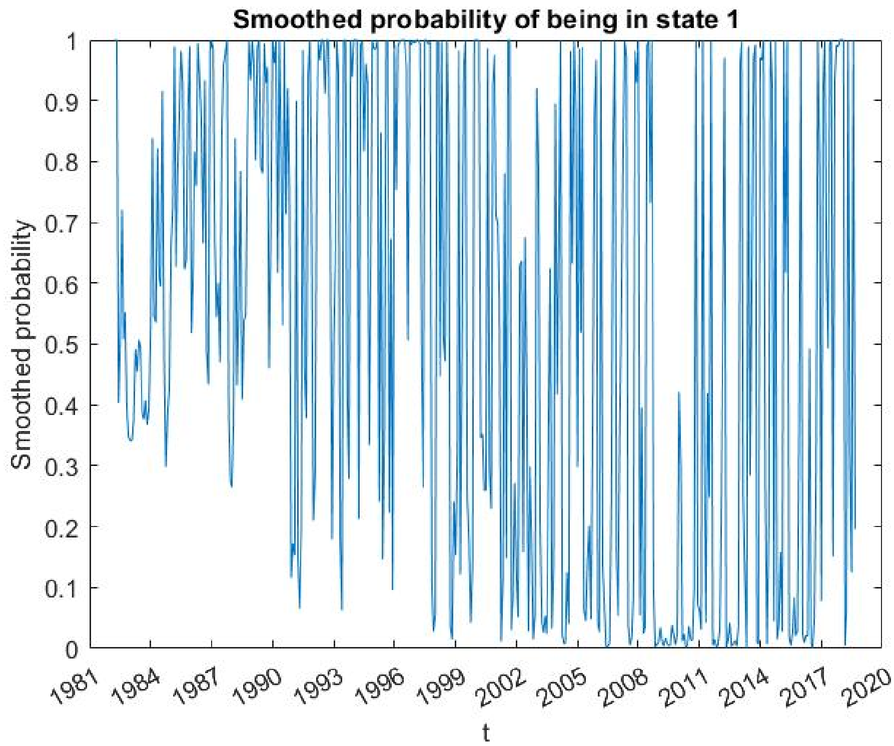


Figure 21: Smoothed probability of $S_t = 1$ for $h = 3$ of sign MSVAR model

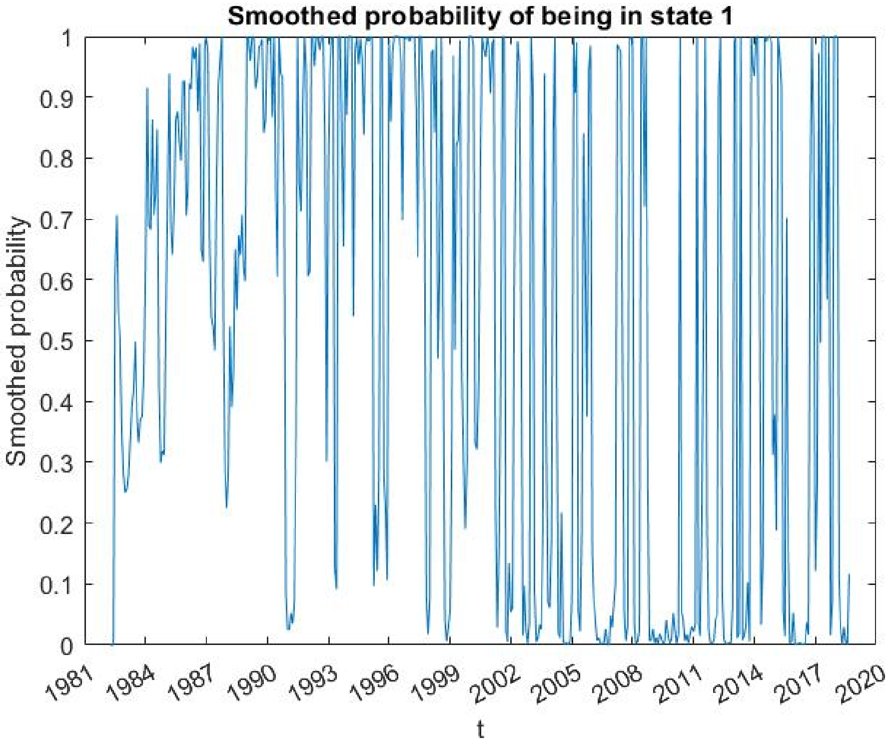


Figure 22: Smoothed probability of $S_t = 1$ for $h = 4$ of sign MSVAR model

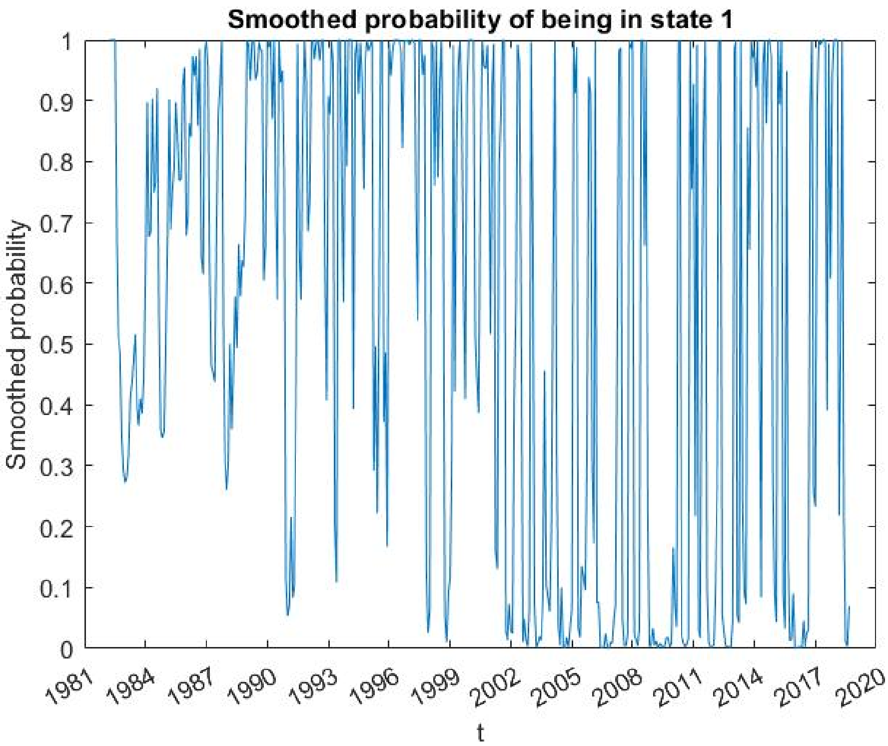


Figure 23: Smoothed probability of $S_t = 1$ for $h = 5$ of sign MSVAR model

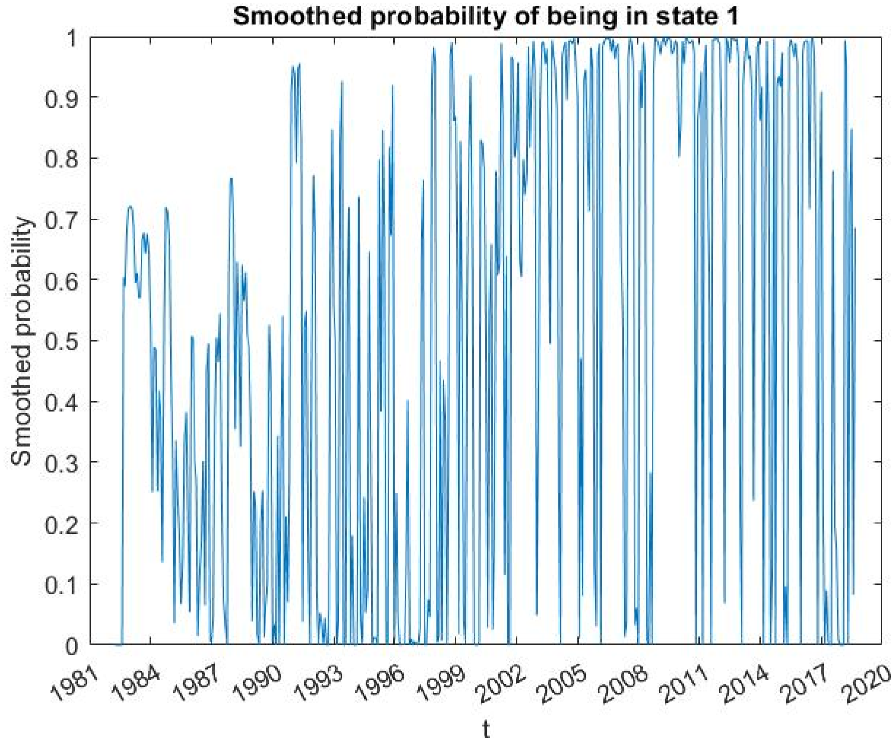


Figure 24: Smoothed probability of $S_t = 1$ for $h = 6$ of sign MSVAR model

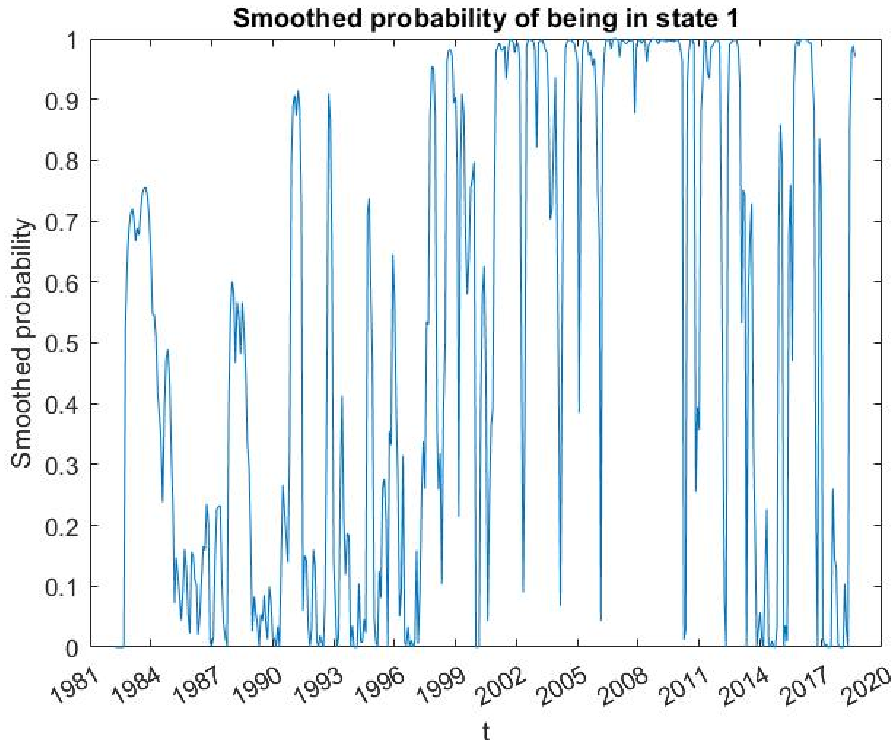


Figure 25: Smoothed probability of $S_t = 1$ for $h = 7$ of sign MSVAR model

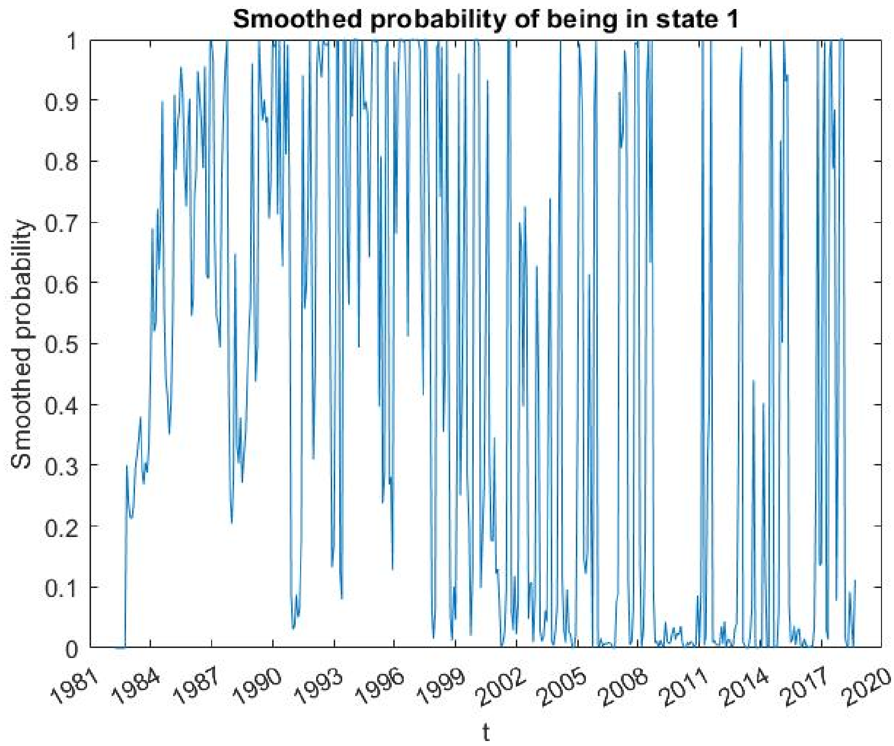


Figure 26: Smoothed probability of $S_t = 1$ for $h = 8$ of sign MSVAR model

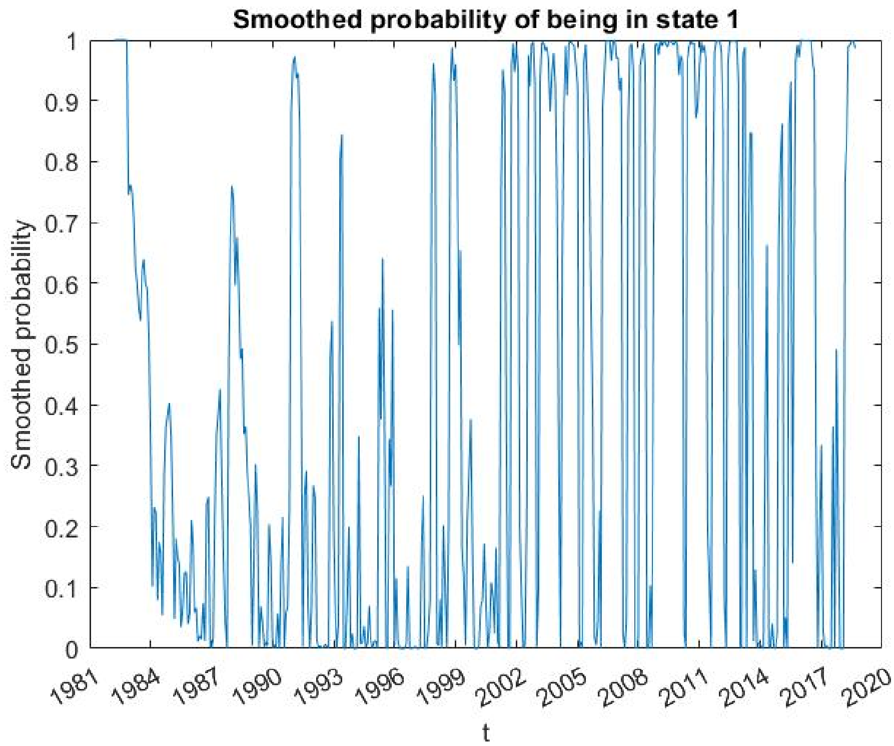


Figure 27: Smoothed probability of $S_t = 1$ for $h = 9$ of sign MSVAR model

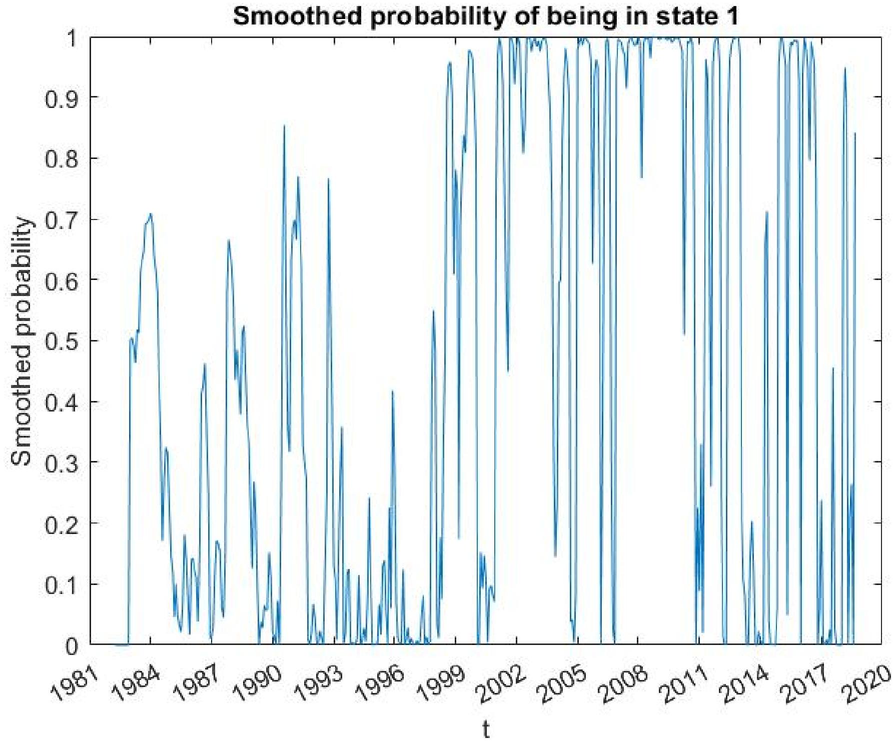


Figure 28: Smoothed probability of $S_t = 1$ for $h = 10$ of sign MSVAR model

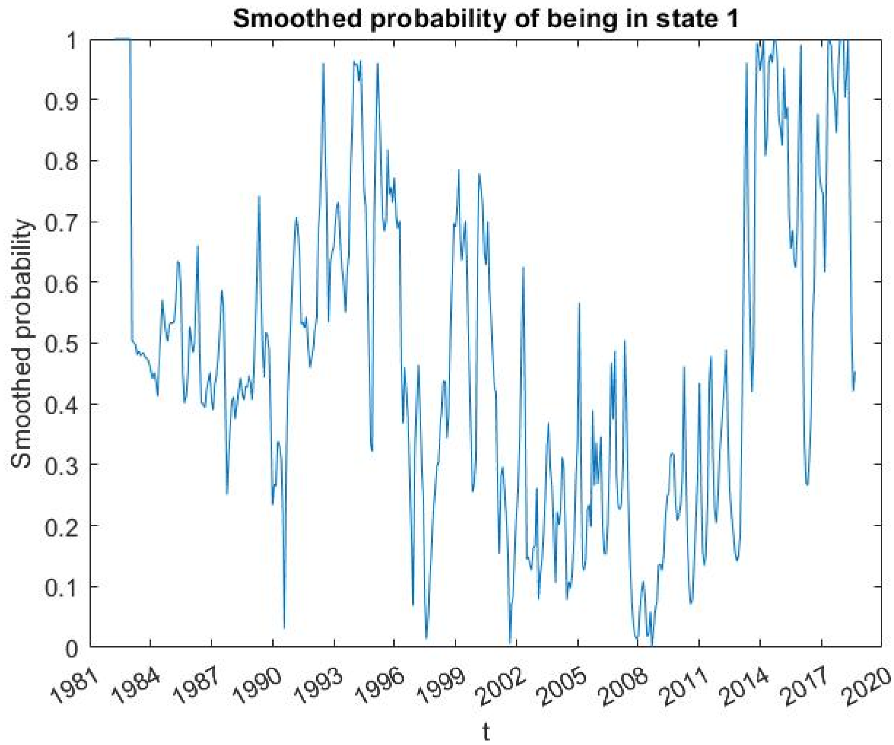


Figure 29: Smoothed probability of $S_t = 1$ for $h = 11$ of sign MSVAR model

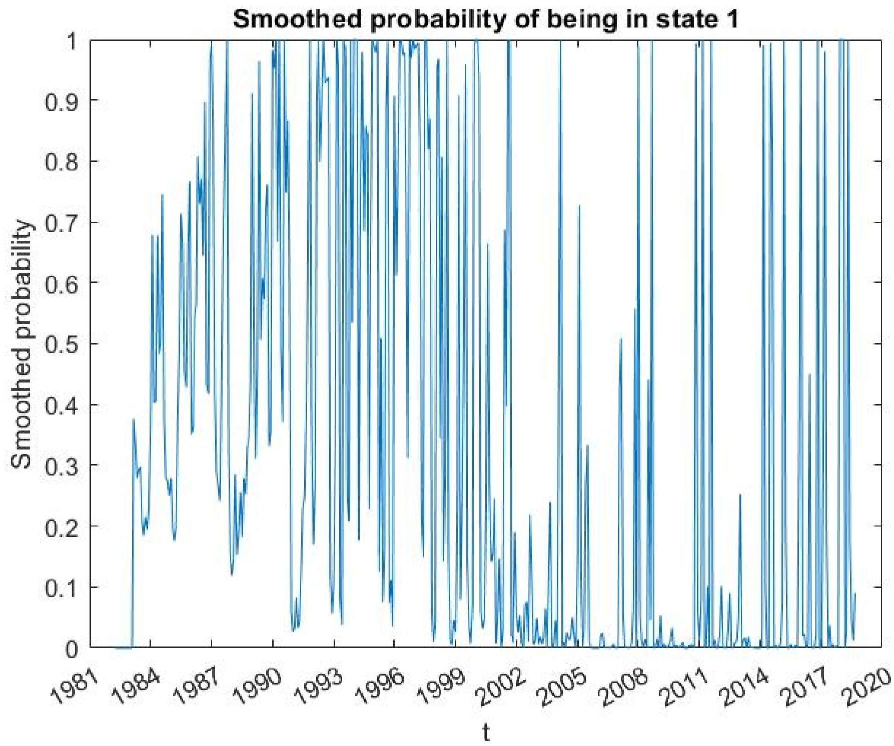


Figure 30: Smoothed probability of $S_t = 1$ for $h = 12$ of sign MSVAR model