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Pricing Derivatives in Periods of Low or Negative Interest Rates

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Abstract

This study investigates which pricing model is more appropriate in a low or negative interest rate setting in terms of pricing accuracy. We compare six different option pricing models where each model assume different dynamics of the interest rate and the volatility. The models incorporate either a constant, Vasicek or CIR type dynamic for the interest rate and the models incorporate either a constant or Heston type dynamic for the volatilty. We test which pricing model has the better performance in either interest rate setting. Our simulation study shows that the pricing models that rely on a CIR type short-rate process exhibit abnormal parameter values and drastically underperform in terms of pricing accuracy in comparison with models based on a Vasicek type short-rate process. The models are calibrated on real world S&P 500 option data from the time period 2010 to 2015 and we find that the Heston-Vasicek model has the best pricing performance. In contrast, the Heston-CIR models shows a relatively poor out-of-sample pricing performance.

Keywords: Option Pricing; Negative Interest Rates; Stochastic Interest Rates; Stochastic Volatility

Contents

1 Introduction

In recent events we observe a major shift in nominal interest rates across the world. Since the financial crisis in 2008, interest rates have reached all-time lows were we even observe negative interest rates on long-term maturities in Europe, the US and other parts of the world. Several banks introduced a negative interest rate policy since the crisis of 2008 in order to bolster the local economy. Central banks such as the Swiss National Bank, the European Central Bank and the Bank of Japan employed negative rates. This has major economic consequences and also has considerable technical implications. Modeling in derivative pricing usually require (high) positive risk-free rates as certain models do not allow for negative values. For example, the Cox-Ingersoll-Ross (CIR) short-rate model only allows positive inputs and can therefore not be used to model negative interest rates. Furthermore, the diffusion term of the CIR short-rate model is dependent on the current level of the interest rate. When the interest rate goes to zero, the diffusion term will go to zero as well which does not occur in practice. The question then remains what derivative pricing model is appropriate in this low-interest-rate environment and more importantly, more accurate in terms of pricing accuracy.

Several recent studies have investigated the matter of negative interest rates on derivative pricing. Giribone et al. (2017) study the effects of negative rates on the pricing of interest rate options with regards to the option sensitivities (the Greeks). They examine a normal framework as opposed to the traditionally used log-normal framework and assess the Greek discrepancies between the two. Cafferata et al. (2017) study the effect of negative interest rates on the pricing of American calls and highlights the differences between estimation using quasi-closed formulas and an approximation method represented by a stochastic trinomial tree. Recchioni et al. (2017) investigate if models that allow for negative interest rates improve option pricing and implied volatility forecasting. They use an adjusted Heston model where they allow the interest rate to follow a stochastic Vasicek process. The latter study forms an important focus for this thesis as we use the same model to price S&P 500 index call and put options. This model incorporates both stochastic volatility and stochastic interest rates.

In this study, we relax the assumption of a constant interest rate and we consider stochastic short rate models as they are able to capture the dynamics of interest rate dynamics in order to price financial derivatives. Additionally, some of those models are highly tractable that can lead to closed-form approximations of financial derivatives which depend on the interest rate. Numerous short-rate models have been developed to model the stochastic interest rates. A few examples are the Vasicek (1977) model, the Hull and White (1993) model, the Cox-Ingersoll-Ross (1985) model and the Heath et al. (1992) model. We focus in this study on the models of Cox-Ingersoll-Ross (CIR) and Vasicek as closed-form solutions for call and put options are available. The Vasicek model assumes a normal distribution of the interest rate and thus they assign positive probabilities to interest rate levels considerably lower than zero. Traditionally, this was considered a major flaw of the model as interest rates were believed never to reach negative values (Brigo & Mercurio, 2001). For the same reasoning, the CIR model was a more realistic approach in the sense that only positive inputs were allowed for the short rate. However, in recent events we do observe negative interest rates and it is a point of interest how the Vasicek model is able to facilitate the pricing of derivatives compared to the CIR model. Incorporating stochastic interest rates has practical use as Abudy and Izhakian (2011) finds that derivative pricing models where the interest rate dynamic is modeled by a Vasicek process has better pricing performance in terms of a lower mean squared errors.

In addition to stochastic interest rates, we also consider stochastic volatility of the underlying asset. There are two kinds of volatility models, which in comparison with the Black-Scholes model, relaxes the assumption of a constant volatility. These are the local volatility models and the stochastic volatility models. The local volatility model, first introduced by Dupire et al. (1994), is a model where the volatility is a deterministic function of the stock price. This model was able to accurately account for the volatility smile observed in current real markets. However, local volatility models have considerable drawbacks. Hagan et al. (2002) finds that local volatility models are not able to accurately predict future smiles and skews. Specifically, the local volatility models actually predict the exact opposite behavior observed in real markets and suggests the use of the stochastic volatility models instead. The Heston model is an example of a stochastic volatility model. Although it is not able to perfectly match the volatility smile of current market prices, it is able to provide realistic volatility surfaces at future moments. Stochastic volatility models have considerable positive impact on modeling derivative prices and is documented by several papers, such as Bakshi et al. (1997), Nandi (1998) and Jones (2003). The aforementioned studies consider single-factor stochastic volatility. However, Christoffersen et al. (2009) argue that single-factor stochastic volatility models are still overly restrictive in generating the proper smiles and smirks of the volatility surface. Christoffersen et al. (2009) find that two-factor stochastic volatility models are better able to model the stylized facts of volatility smiles and smirks and result in an improved statistical fit.

Numerous studies have incorporated multi-factor stochastic volatility models where one of the factors is the stochastic interest rate. Zhu (2000) establishes a multi-factor model that is able to generate skew patterns for equity. We note that the stochastic interest rate is uncorrelated with the equity. Andreasen (2006) uses a multi-factor Heston stochastic volatility model where the model specifies an indirect correlation between the stochastic interest rate and the equity. Ahlip (2008) presents a derivative model for foreign exchange options which incorporates both stochastic volatility and stochastic interest rates. The volatility process is mean-reverting and is directly correlated with the exchange rate. This study derives an analytical expression for the price of a call option on the foreign exchange rate. Haastrecht et al. (2010) introduces a Schobel-Zhu-Hull-White model where the short rate and volatility dynamics follow an Ornstein-Uhlenbeck process. Initially, the model is not of an affine form but can be made affine by adding a fourth state variable. Grzelak and Oosterlee (2011) extends the Heston stochastic volatility model with stochastic interest rates and specifies two variants: one variant where the short rate is specified by the Hull-White process and the other variant where the short rate is specified by the CIR process. More recently, the work of Recchioni and Sun (2016) and Recchioni et al. (2017) modifies the multi-factor Heston model proposed by Grzelak and Oosterlee (2011). The model still preserves the main features of the model, but in addition is now analytically tractable allowing for explicit formula's for the moments of the asset price variable and allows for simple one-dimensional approximations for European call and put options.

The goal of this thesis is to gain insight how option pricing models are able to price call and put options on the S&P 500 index in periods of low or negative interest rates. We consider six pricing models: the standard Black-Scholes (BS) model, the Black-Scholes-Vasicek (BS-VS) model, the Black-Scholes-CIR (BS-CIR) model, the standard Heston (H) model, the Heston-Vasicek (H-VS) model and the Heston-CIR (H-CIR) model. In order to gain an understanding of the pricing performance of the six models, we perform both a simulation study and an empirical study.

In the empirical study we apply the models to real market data where the sample data spans the period from 2010 to 2015 where interest rates reached record lows. We obtain data for a total of 86,020 weekly observations. In-sample results show that the standard BS model has the worst performance. Extensions of the BS model with the Vasicek and CIR stochastic interest rates show considerable pricing improvement. The BS-VS model performs better than the BS-CIR model and we attribute this to the CIR model not being able to properly model the dynamics of low and negative interest rates. We find the same results for the Heston models. Including either of the stochastic rates increases pricing performance compared to the standard Heston model. The H-VS model outperforms the H-CIR model in terms of pricing accuracy when we consider the in-sample results. When we consider the out-of-sample results, we still observe that the BS-VS and BS-CIR models outperform the standard BS model. However, the H-CIR model is considerably under performing compared to the H-VS and standard Heston models. Apparently, the H-CIR is not well specified to deal with low or negative interest rates.

We contribute to existing literature in three ways. First, we test multiple models in a setting of low and negative interest rates. We know that negative interest rates have a huge impact on valuing derivatives and this makes a lot of models unsuitable for use. However, we do not know exactly to what extent this hurts option pricing. This study will measure the performance of the pricing models using in and out-of-sample goodness of fit tests where the sample period is selected in a period of low and negative interest rates. Second, we perform a simulation study under a setting of low interest rates and under a setting of negative interest rates. Through this study, we will be able to identify misspecification in the model parameters and examine the consequences to the pricing performance. And third, we use option data from a relatively large sample period of time containing a wide variety of in and out-of-the-money options across different maturities compared to our benchmark study of Recchioni et al. (2017). We use a 6-year time period, while our benchmark study only selects a time period of 2 months with a very narrow range of option quotes.

Section 2 considers the methodology used in this study and introduces the option pricing models and the calibration method. Section 3 presents the results of the simulation study. In Section 4 we apply our models to empirical data. Results regarding the implied parameters and the real pricing performance will be presented here. Section 5 concludes.

2 Methodology

This section describes the methodology where we start with briefly discussing the option pricing models used in this study. Section 2.1 describes the Black-Scholes model. We then relax the assumption of a constant interest rate by allowing the interest rate to be stochastic. Sections 2.2 and 2.3 describe the Black-Scholes-Vasicek and the Black-Scholes-CIR models, respectively. Section 2.4 describes the Heston stochastic volatility model. Sections 2.5 and 2.6 discuss the Heston-Vasicek and Heston-CIR models, respectively. Afterwards, we describe the calibration procedure in section 2.7.

2.1 Black-Scholes model

The Black-Scholes model assumes a geometric Brownian motion for the underlying stock price S_t such that

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1}$$

under the risk-neutral measure, where r is the interest rate, σ is the volatility of the stock price and W_t is a standard Brownian Motion. In order to price a call option V the Black-scholes formula is used and is given by

$$V = S_0\Phi(d1) - Ke^{-rT}\Phi(d2), \tag{2}$$

$$d1 = \frac{\log(\frac{S_0}{K}) + (r_0 + \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{3}$$

$$d2 = d1 - \sigma\sqrt{T}, \tag{4}$$

where $\Phi(.)$ is the cumulative probability function of a standard normal distribution. This model is only a simple representation of real-world circumstances and makes several assumptions. For example, the Black-Scholes model assumes a log-normal distribution of the stock returns and assumes that during the lifetime of an option no dividends are paid out. Additionally, the model assumes that the volatility of the underlying and the interest rate are constant over time. The Black-Scholes model can be extended such that it incorporates stochastic interest rates. We discuss this in the next section.

2.2 Black-Scholes Vasicek model

We consider a Black-Scholes economy where we relax the assumption of a constant risk-free rate. Instead, we assume a Vasicek (1977) stochastic short rate process such that S_t and r_t are given by

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{5}$$

$$dr_t = \lambda(\theta - rt)dt + \eta dZ_t, \tag{6}$$

under the risk-neutral measure, where λ represents the speed of mean-reversion of the interest rate, θ the long-run mean, η is the volatility of the short rate and dZ_t and dW_t are standard Wiener process correlated with the instantaneous correlation parameter ρ. We follow the deriviation of Abudy and Izhakian (2011) which presents the value of a call options V as

$$\begin{aligned} V =& S\Phi\Big[\frac{\ln\frac{S}{K} + A - \frac{1}{2}\upsilon^2\tau}{\sigma\sqrt{\tau}} + \upsilon\sqrt{\tau}\Big] \\ & - Ke^{-A+\frac{1}{2}B^2\tau}\Phi\Big[\frac{\ln\frac{S}{K} + A - \frac{1}{2}\upsilon^2\tau}{\sigma\sqrt{\tau}} - B\sqrt{\tau}\Big], \end{aligned} \tag{7}$$

$$\tag{8}$$

where

$$\upsilon^2 = \sigma^2 + B^2 - 2\rho\sigma B, \tag{9}$$

$$A = \frac{\alpha}{\beta}\tau + (r_t - \frac{\alpha}{\beta})\lambda, \tag{10}$$

$$B = \frac{\epsilon}{\beta}\sqrt{\tau - \lambda - \frac{\beta}{2}\lambda^2}, \tag{11}$$

$$\lambda = \frac{1}{\beta}(1 - e^{-\beta\tau}). \tag{12}$$

The intuition behind expression (7) is that the price of a call option is dependent whether the unexpected excess returns is higher than a given threshold (Abudy & Izhakian, 2011).

2.3 Black-Scholes-CIR model

We consider the same Black-Scholes economy but instead we relax the assumption that the interest rate is constant. We assume that the stochastic interest rate follows a process introduced by Cox et al. (1985) such that S_t and r_t are given by

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{13}$$

$$dr_t = \lambda(\theta - rt)dt + \eta\sqrt{r_t} dZ_t, \tag{14}$$

under the risk-neutral measure, where λ represents the speed of mean-reversion of the interest rate, θ is the long-run mean of the interest rate, η is the volatility of the interest rate, W_t and Z_t are standard Wiener process correlated with the instantaneous correlation parameter ρ. The feller condition $2\lambda\theta \geq \eta^2$ ensures that the interest rate process remains positive. Kim (2002) presents the value of the of call option V under a CIR type stochastic interest rate process as

$$\begin{aligned} V =& \Big[S_0\Phi(d_1) - Ke^{-\int_0^T r_t^* dt}\Phi(d_2)\Big] \\ &+ \eta C_0\Big[S_0\phi(d1) - Ke^{-\int_0^T r_t^* dt}(\phi(d2) - \delta\sqrt{T}\Phi(d_2))\Big] \\ &+ \eta C_1\Big[d_2 S_0\phi(d1) - d_1 Ke^{-\int_0^T r_t^* dt}\phi(d_2)\Big], \end{aligned} \tag{15}$$

where the quantities C_0, C_1, C_{11} and ψ are given in (A1), (A2), (A3) and (A4)

2.4 Heston model

The model of Heston (1993) is a well-known stochastic volatility model, which in contrast to the Black-Scholes model, relaxes the assumption that the volatility of the underlying is constant. The Heston model assumes that the variance follows a CIR type square root process that is correlated with the underlying asset. The stock price S_t and the variance v_t are given by the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{1,t}, \tag{16}$$

$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t} dW_{2,t}, \tag{17}$$

where $E^{\mathbb{P}}[dW_{1,t}, dW_{2,t}] = \rho dt$, μ is the drift of the process of the stock, χ is the speed of mean-reversion for the variance, v^* is the long-run mean of the variance and γ is the volatility of the volatility (vol-of-vol).[1] The feller condition $2\chi v^* \geq \gamma^2$ ensures that the variance process remains strictly positive. The processes in equation (16) and (17) are under the historical measure $\mathbb{P}$. However, for option pricing we require the processes to be under the risk-neutral measure $\mathbb{Q}$. Applying Girsanov's theorem we obtain the following process for the stock price and variance:

$$dS_t = r S_t dt + \sqrt{v_t} S_t d\tilde{W}_{1,t}, \tag{18}$$

$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t} d\tilde{W}_{2,t}, \tag{19}$$

where $E^{\mathbb{Q}}[d\tilde{W}_{1,t}, d\tilde{W}_{2,t}] = \rho dt$ under the risk-neutral measure $\mathbb{Q}$ and r is the risk-free rate. Heston (1993) provides a closed-form solution for the value of a call options V. The complete derivation of the Heston formula is well documented and we omit the details here. We simply state that the value of a call option V is given by

$$V = S_t P_1 - K e^{r\tau} P_2 \tag{20}$$

[1] We note that traditionally, the parameters of the speed of mean-reversion, the long-run mean and the vol-of-vol for the Heston model are given by κ, θ and σ, respectively. We rename the parameter names here to avoid confusion when we introduce a multi-factor Heston model later on.

where P_1 and P_2 are

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Big[\frac{e^{\iota\psi \ln K f_i(\phi;x;v)}}{\iota\phi}\Big] d\phi, \tag{21}$$

$$f_j(\phi; x_j, v_t) = exp(C_j(\tau,\phi) + D_j(\tau,\phi)v_t + \iota\phi x_t), \tag{22}$$

$$D_j(\tau;\phi) = \frac{b_j - \rho\sigma\iota\phi + d_j}{\sigma^2}\Big(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_t\tau}}\Big), \tag{23}$$

$$C_j = r\iota\phi\tau + \frac{a}{\sigma^2}\Big[(b_j - \rho\sigma\iota\phi + d_j)\tau - \ln\Big(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\Big)\Big], \tag{24}$$

$$g = \frac{b_j - \rho\sigma\phi + d}{\sigma^2}\Big(\frac{1 - e^{dr}}{1 - ge^{dr}}\Big), \tag{25}$$

$$d = \sqrt{(\rho\sigma\phi\iota - b_j)^2 - \sigma^2(2u_j\phi\iota - \phi^2)}, \tag{26}$$

where $\alpha = \chi v^*$. The integral in expression (21) cannot be evaluated analytically but has to be evaluated numerically.

2.5 Heston-Vasicek model

Recchioni et al. (2017) provides a framework for a multi-factor Heston model where the model is able to incorporate negative interest rates. This framework is an adjusted model of the Heston-CIR in Recchioni and Sun (2016) where now the Vasicek model used to model stochastic interest rates, instead of the CIR model. In this study, we call this model the Heston-Vasicek model. The process for the stock price S_t, the variance v_t and the interest rate r_t is given by

$$dS_t = S_t r_t dt + S_t\sqrt{v_t}dW_t^v + S_t\Delta\sqrt{v_t}dZ_t^v + S_t\Omega dW_t^r, \tag{27}$$

$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dZ_t^v, \tag{28}$$

$$dr_t = \lambda(\theta - r_t)dt + \eta dZ_t^r, \tag{29}$$

where Ω and Δ are non-negative constants. χ is the speed of mean reversion for the variance, v^* is the long-run mean of the variance and γ is the vol-of-vol. χ, v^* and γ are positive constants. The variance v_t stays positive if the feller condition $\frac{2\chi v^*}{\gamma^2} > 1$ holds.

λ is the speed of mean-reversion for the interest rate, θ is the long-run mean and η is the volatility of the interest rate. λ and η are positive numbers, while θ can be both positive or negative as the Vasicek allows for negative interest rates. W_t^v, W_t^u, Z_t^v and Z_t^r are Wiener process with the following correlations:

$$E(dW_t^v dZ_t^v) = \rho_v dt, \tag{30}$$

$$E(dW_t^r dZ_t^r) = \rho_r dt. \tag{31}$$

The formulation of a multi-factor Heston model with stochastic interest rates is not new and has been done before. Grzelak and Oosterlee (2011) provides a similar framework of a multi-factor Heston model where the main difference lies in the parameter Ω. Grzelak and Oosterlee (2011) assumes Ω to be variable and dependent on v_t while Recchioni and Sun (2016) assumes Ω to be constant. By assuming a constant Ω the model becomes analytically tractable such that explicit moments of the price variables are available. As a consequence, it is possible to approximate European call and put options with only one-dimensional integrals which can be efficiently computed using numerical integration. We provide a brief explanation of the relevant expressions used to determine European call and put options. For futher details we refer to the work of Recchioni et al. (2017). To deduce the analytical formula's for the value of call and put options, we first apply ito's lemma and derive the following SDEs with the log-price process $x_t = \ln(S_t/S_0)$:

$$dx_t = \left[r_t - \frac{1}{2}(\tilde{\psi} v_t + \Omega^2)\right]dt + \sqrt{v_t}dW_t^v + \Delta\sqrt{v_t}dZ_t^v + \Omega dW_t^r, \tag{32}$$

$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dZ_t^v, \tag{33}$$

$$dr_t = \lambda(\theta - r_t)dt + \eta\, dZ_t^r, \tag{34}$$

where $\tilde{\psi}$ is given by

$$\tilde{\psi} = 1 + \Delta^2 + 2\rho_v\Delta. \tag{35}$$

The price of a vanilla call and put options is given by

$$C(S_0, T, K, r_0, v_0) = E^Q\Big(e^{-\int_0^T r_t dt}(S_0 e^{xT} - K)_+\Big), \tag{36}$$

$$P(S_0, T, K, r_0, v_0) = E^Q\Big(e^{-\int_0^T r_t dt}(K - S_0 e^{xT})_+\Big), \tag{37}$$

where $(.)_+ = \max\{., 0\}$. The expressions for vanilla call and put options in (36) and (37) require the evaluation of three-dimensional integrals. Numerical integration would not be efficient and would take a long time to compute. Recchioni et al. (2017) approximates term $e^{\int_0^T r(t)dt}$ using

$$e^{\int_0^T r(t)dt} \approx e^{-r_0(1-\omega)T} e^{-rT}, \tag{38}$$

where ω is

$$\omega = \frac{1}{T}\frac{(T - \Psi_{1,\lambda}(T))}{\lambda \Psi_{1,\lambda}(T)}. \tag{39}$$

The stochastic integral in (36) and (37) can then be approximated by using a quadrature rule. The weights are chosen such that it incorporates the features of this particular stochastic interest rate process. Using the results of the expressions of the various moments of the pricing variable, expressions (36), (37) and (38) we get an approximation of the value of a European vanilla call and put option, V_a, which is

$$\begin{aligned} V_a(S_0, T, K, r_0, v_0) = & e^{-(\theta T + (r_0 - \theta)\Psi_{1,\lambda}(T))} e^{(\omega T)^2 \frac{\eta^2}{2} \Psi_{2,\lambda}(T)} \\ & \times \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} \frac{\left(\frac{S_0}{K}\right)^{q-1-\iota k} e^{Q_{v,q}(T, v_0, k; \Theta_v)} e^{Q_{r,q}(T, r_0, k; \Theta)}}{-k^2 - (q1 - 1)\iota k + q(q-1)} \\ & \times e^{(\iota k - q)\omega T\left(\Omega \rho_r \eta \Psi_{1,\lambda}(T) + \frac{\eta^2}{\lambda}(\Psi_{1,\lambda} - \Psi_{2,\lambda})\right)}. \end{aligned} \tag{40}$$

The quantities (A5) to (A14) are required and are presented in the appendix. Expression (40) approximates the value of a call option for $q > 1$, and approximates the value of a put option for $q < -1$. The integral that appears in this expression is a smooth function, therefore numerical integration can be done efficiently using a quadrature rule. It is possible to take the limit of $\Omega \to 0^+$, $\lambda \to 0^+$ and $\eta \to 0^+$ which eliminates the stochastic interest rate and reverts the Heston-Vasicek model into the standard Heston model such that expression (40) turns into

$$V_H(S_0, T, K, r_0, v_0) = e^{-r_0 T} \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} \frac{\left(\frac{S_0}{K}\right)^{q-1-\iota k} e^{Q_{v,q}(T, v_0, k; \Theta)} e^{-(\iota k - q) r_0 T}}{-k^2 - (q1-1)\iota k + q(q-1)} dk, \tag{41}$$

This formula approximates the value of a call option for $q > 1$ and a put option for $q < -1$ under constant interest rates. We verify that (41) approximates the closed-form Heston expression (20).

2.6 Heston-CIR model

For the Heston model where the interest rate follows a CIR type process, we again follow the procedures as mentioned in Recchioni and Sun (2016). We start with the process dynamics of the stock price S_t, the variance v_t and the interest rate r_t

$$dS_t = S_t r_t dt + S_t \sqrt{v_t} dW_t^v + S_t \Delta \sqrt{v_t} dZ_t^v + S_t \Omega \sqrt{r_t} dW_t^r, \tag{42}$$

$$dv_t = \chi(v^* - v_t)dt + \gamma \sqrt{v_t} dZ_t^v, \tag{43}$$

$$dr_t = \lambda(\theta - r_t)dt + \eta \sqrt{r_t} dZ_t^r, \tag{44}$$

where Ω and Δ are non-negative constants. χ is the speed of mean reversion for the variance, v^* is the long-run mean of the variance and γ the vol-of-vol. χ, v^* and γ are positive constants. The variance v stays positive if the feller condition $\frac{2\chi v^*}{\gamma^2} > 1$ holds. λ is the speed of mean-reversion for the interest rate, θ is the long-run mean and η is the volatility of the interest rate. λ, θ and η are positive numbers. W_t^v, W_t^u, Z_t^v and Z_t^r are Wiener process with the following correlations:

$$E(dW_t^v dZ_t^v) = \rho_v dt, \tag{45}$$

$$E(dW_t^r dZ_t^r) = \rho_r dt, \tag{46}$$

To derive the value of a call option, we follow the steps similar to the Heston-Vasicek model. We rewrite with ito's lemma with $x_t = \ln(S_t/S_0)$:

$$dx_t = [r_t - \tfrac{1}{2}(\tilde{\psi} v_t + \Omega^2 r_t)]dt + \sqrt{v_t}dW_t^{p,v} + \Delta\sqrt{v_t}dW_t^v + \Omega\sqrt{r_t}dW_t^{p,r}, \tag{47}$$

$$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \tag{48}$$

$$dr_t = \lambda(\theta - r_t)dt + \eta\sqrt{r_t}dW_t^r, \tag{49}$$

where $\tilde{\psi}$ is

$$\tilde{\psi} = 1 + \Delta^2 + 2\rho_v\Delta. \tag{50}$$

Similar to the Heston-VS model in section 2.5, we use the expressions of the various moments of the pricing variable and expressions (36)-(39) to derive an approximation of the value V of an option for the Heston-CIR model and is given by

$$\begin{aligned} V_a(S_0, T, K, r_0, v_0) =& e^{-(\theta T + (r_0-\theta)\Psi_{1,\lambda}(T))} e^{(\omega T)^2 \frac{\eta^2}{2}\Psi_{2,\lambda}(T)} \\ &\times \frac{S_0}{2\pi}\int_{-\infty}^{+\infty} \frac{\left(\frac{S_0}{K}\right)^{q-1-\iota k} e^{Q_{v,q}(T,v_0,k;\Theta_v)} e^{Q_{r,q}(T,r_0,k;\Theta)}}{-k^2 - (q1-1)\iota k + q(q-1)} \\ &\times e^{(\iota k - q)\omega T\left(\Omega\rho_r\eta\Psi_{1,\lambda}(T) + \frac{\eta^2}{\lambda}(\Psi_{1,\lambda} - \Psi_{2,\lambda})\right)}, \end{aligned} \tag{51}$$

where the quantities (A15) to (A25) are required. This formula approximates the value of a call option for $q > 1$ and a put option for $q < -1$.

2.7 Calibration Method

In this study, the loss function approach is chosen where we minimize the error between the quoted (simulated) price and the theoretical model price. We minimize the dollar-

Table 1: Upper and Lower bounds of the parameters

	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
Upper bound	5	1	1	1	5	1	1	1	1	4	3
Lower bound	0	-1/0	0	-1	0	0	0	0	-1	0	0
Starting value	3	0.05	0.01	-0.8	0.5	0.05	0.1	0.1	-0.9	0.3	1.5

measure of the mean squared error ($MSE) loss function which is given by

$$\begin{aligned} \min_{\Theta_m} \frac{1}{N+M}\Big[&\sum_{i=1}^{N}(C_o(S_t,K_i,\tau_i) - C_\Pi(\Theta_\Pi;S_t,K_i,\tau_i)^2 \\ &+ \sum_{j=1}^{M}(P_o(S_t,K_i,\tau_i) - P_\Pi(\Theta_\Pi;S_t,K_i,\tau_i)^2\Big] \\ &+ \alpha(\Theta_\Pi - \Theta_0)'(\Theta_\Pi - \Theta_0), \end{aligned} \tag{52}$$

where $C_o(S_t,K_i,\tau_i)$ is the observed call option price, $P_o(S_t,K_i,\tau_i)$ the observed put option price, $C_\Pi(\Theta_\Pi;S_t,K_i,\tau_i)$ is the theoretical model call price, $P_\Pi(\Theta_\Pi;S_t,K_i,\tau_i)$ is the theoretical model put price, N is the amount of call options and M the amount of put options. We use matlab's in-built function *lsqnonlin* to minimize our objective function. The solver requires additional inputs, namely the starting values of the calibration process. Because the solver is not a global optimizer, the starting values can provide bias in the estimation of the parameters. For this reason we set multiple starting points using Matlab's *multistart* function. This function uses uniformly distributed starting points within the upper and lower limit of the bounds specified by the user. Table 1 displays the upper and lower bounds of all the variables. We note that the *multistart* function still prompts us for an initial starting value. This starting value is also displayed in the same table.

The last term in expression (52) is the regularization term where α is the regularization parameter, Θ_Π is the parameter set to be calibrated from the theoretical model and Θ_0 is a initial set of parameter values. The regularization term is essential for the parameters resulting from the calibration procedure to remain stable. Regarding regularization, there is a trade-off between accuracy and regularity and the choice of α is therefore crucial (Zeng et al., 2014). If α is too small, the parameter estimates will vary too greatly and will lead to instability. If α is too large, the results will be oversmoothed and might cause bias which is introduced in the initial set Θ_0. In general, there is no widely accepted method to determine the regularization parameter α and there are two approaches to determine α. The first approach is to use a priori methods where the value of α is determined based on the noise of the data. The second approach is to use a posteriori methods and is the most frequently used approach in financial literature (Zeng et al., 2014). An example of an a posteriori method is the use of discrepancy principles where the regularization parameter α is set to the level so that the data fidelity is not greater than the noise of the observations. Several recent contributions have been made on determining the implied volatility's of financial derivatives by Wang and Yang (2014) and Liu and Liu (2018) who implement a total variation regularization strategy. This regularization technique is able to better characterize the properties of the implied volatility's compared to other regularization techniques. Determining the optimal value of α is outside the scope of this study as the expressions required for the regularization algorithm are not readily available for the pricing models used in this study. We leave it up to future research to explore this topic further.

3 Simulation Study

In this section we perform a simulation study regarding the performance of the six pricing models from section 2 when we apply them in periods of low or negative interest rates. Our goal is twofold. First, we want to verify if the calibration procedure is adequate such that our pricing models are able to closely replicate simulated option prices. And second, we want to examine how the pricing models behave in terms of their parameter values in periods of low and negative interest rates. Section 3.1 explains the process of simulating the option prices and calibrating the pricing models to these options. Section 3.2 evaluates the models in terms of their pricing errors and parameter values.

3.1 Simulation Procedure

A set of call option contracts are made with varying maturities and strike prices. Four time-to-maturities τ_i are considered where $\tau_i \in \{\frac{1}{12}, \frac{1}{4}, \frac{1}{2}, 1\}$ for i = 1, ..., 4. This represents a time-to-maturity of 1-month, 3-month, 6-month and 1-year, respectively. Six different strikes K_j are selected with $K_j = \frac{S_0}{1.08-0.03(j-1)}$ for j = 1, 2, ..., 6 where S_0 is the current spot price. This represents six option contracts with different strike prices. We use the moneyness ratio (S/K) to obtain a balanced set of in-the-money and out-of-the-money call options. With this procedure, we obtain a set of 24 (4×6) options with different time-to-maturities and strikes. We then construct the simulated option market prices $C_{MP}(S_0, K_j, \tau_i,)$ using the closed-form approximations of the pricing models explained in section 2. We use the notation $C_\Pi(\Theta_\Pi; S_0, K_j, \tau_i,)$ for the closed-form approximation where C_Π with $\Pi \in$ {BS, BS-VS, BS-CIR, H, H-VS, H-CIR}[2] represents one of the six pricing models and Θ_Π represents the respective parameter set for model Π.

In this study we do not model the prices directly by simulating the risk-neutral

[2]BS and H stand for the Black-Scholes and Heston models, respectively. VS and CIR stand for the Vasicek and CIR stochastic interest rate process, respectively.

processes of the respective models. We only consider vanilla call and put options where the closed-form expressions in section 2 are able to correctly approximate the values of the options. If we were to simulate the paths directly, given enough simulations, the average value of the options resulting from each path would converge to the value of the closed-form approximation. Therefore, directly simulating the paths is not necessary.

In order to construct the simulated option price $C_{MP}(S_0, K_j, \tau_i)$, we first need to determine the model parameters which represents a setting of low or negative interest rates. We create a set of parameters for the pricing models and we use the parameter values from Recchioni and Sun (2016) and Recchioni et al. (2017) as benchmark. In the aforementioned studies the H-CIR and H-VS models are calibrated on empirical data in a period where the interest rate was negative. We assume that the parameter values from the studies capture the interest rate dynamics under these extreme circumstances. Another point of interest is how option models behave when the interest rate is very low, instead of negative. Therefore, we create another set where we slightly adjust the parameters of the first set to a context more befitting of a period with very low interest rates. Table 2 displays the chosen parameter sets in order to determine the simulated option prices. Panel A of table 2 shows the parameter values under period of low interest rates where we set $r_0 = 0.005$. Panel B shows the parameter values under negative interest rates where we set $r_0 = -0.005$. We note that we only let the parameters vary regarding the stochastic interest rate process as this is our point of interest. The parameters regarding the stochastic volatility remain constant for all models.

After determining the parameter values, we are able to obtain simulated prices for a set of call option contracts using the six pricing models as the data generating process (DGP). We can now begin to calibrate the models using our calibration procedure explained in section 2.7. The amount of starting points is set to 50, such that 50 calibrations are performed for each model. We set the regularization parameter α for the simulation study equal to zero. Our reasoning is that it is unclear what the proper value for the initial parameter set Θ_0 is. Instead, we let Matlab's *multistart* function

Table 2: Initial Parameters for the Simulated Option Prices

Panel A: Low interest rates ($r_0 = 0.005$)	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
BS								0.05			
BS-VS	3.62	0.003	0.02	−0.81				0.05			
BS-CIR	3.62	0.003	0.0098	−0.81				0.05			
H					0.65	0.0345	0.08	0.05	−0.97		
H-VS	3.62	0.003	0.02	−0.81	0.65	0.0345	0.08	0.05	−0.97	0.26	1.23
H-CIR	3.62	0.003	0.0098	−0.81	0.65	0.0345	0.08	0.05	−0.97	2.51	1.98
Panel B: Negative interest rates (r_0 = -0.005)	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
BS								0.05			
BS-VS	3.62	0.002	0.02	−0.81				0.05			
H					0.65	0.0345	0.08	0.05	−0.97		
H-VS	3.62	0.002	0.02	−0.81	0.65	0.0345	0.08	0.05	−0.97	0.26	1.23

This table presents the initial parameters sets required to simulate the option prices for each model. Panel A shows the parameter sets under a low interest rate setting. Panel B shows the parameters sets under a negative interest rate setting. The BS-CIR and H-CIR models are not used under a negative interest rate setting.

search for a global optimum.

3.2 Simulation Results

We now discuss the results produced in our simulation study. As a result of the calibration process, we obtain for each model a set of parameters and the pricing performance. A proper model should be able to accurately price the simulated option prices. Table 3 displays the average price of the models after calibration where the percentage error is shown in parenthesis. The simulated options are created using the models displayed on the top of each column (DGP) in Table 3 for each panel. The models on the left-hand side of each panel are the models calibrated on the aforementioned simulated options. No options are created for the BS-CIR and H-CIR models in panel B, as they do not allow for negative inputs.

Table 3: Average Price and Pricing errors

Panel A: Low interest rates ($r_0 = 0.005$)	DGP					
	BS	BS-VS	BS-CIR	H	H-VS	H-CIR
BS	122.855	124.905	122.811	118.527	143.140	152.158
	(0.000)	(0.923)	(-0.015)	(-0.993)	(-0.644)	(-0.812)
BS-VS	122.855	124.568	122.814	118.681	143.376	152.510
	<u>(0.000)</u>	(0.000)	(-0.005)	(-0.727)	(-0.094)	(-0.112)
BS-CIR	122.855	124.563	122.816	119.272	143.401	152.541
	(0.000)	(-0.052)	(0.000)	(0.969)	(-0.081)	(-0.087)
H	122.853	124.556	122.812	119.236	143.409	152.563
	(-0.007)	(0.008)	(-0.009)	(0.000)	(-0.021)	(-0.005)
H-VS	122.855	124.568	122.816	119.234	143.416	152.563
	(0.000)	<u>(-0.002)</u>	<u>(0.000)</u>	<u>(-0.030)</u>	(0.000)	<u>(-0.001)</u>
H-CIR	122.855	124.568	122.816	119.234	143.418	152.564
	(0.000)	(0.004)	(0.009)	(-0.033)	<u>(0.007)</u>	(0.000)
Panel B: Negative interest rates (r_0 = -0.005)	**DGP**					
	BS	BS-VS	BS-CIR	H	H-VS	H-CIR
BS	118.611	123.650		114.011	141.880	
	(0.000)	(2.464)		(-0.820)	(0.435)	
BS-VS	118.611	122.859		114.198	141.662	
	<u>(0.000)</u>	(0.000)		(-0.957)	(-0.109)	
BS-CIR	117.275	119.341		115.722	139.052	
	(-1.483)	(-2.778)		(1.463)	(-1.838)	
H	118.609	122.858		114.749	141.705	
	(-0.007)	(-0.113)		(0.000)	<u>(0.081)</u>	
H-VS	118.611	122.859		114.754	141.704	
	(0.000)	<u>(0.000)</u>		<u>(0.036)</u>	(0.000)	
H-CIR	116.621	120.384		113.082	139.351	
	(-1.453)	(-1.670)		(-1.302)	(-1.632)	

This table presents the average price of the options. The percentage error is shown in parenthesis. The simulated options are created using the models displayed on the top of each column (DGP). The models on the left-hand side of each panel are the models calibrated on the aforementioned simulated options. No options are created for the BS-CIR and H-CIR models in panel B, as they do not allow for negative inputs. <u>Underlined</u> values are the best performing models for each DGP (values on the diagonal are excluded).

In panel A were we consider a setting of low interest rates (r_0 = 0.005), we observe that on the diagonal that each calibrated model is able to accurately price the options where the simulated prices are derived from the same model. The percentage error is 0.000% for these models. The BS model has overall the worst performance of all six models as the percentage error percentage error are generally higher compared to the other models. This is expected as the BS model is not able to model either the stochastic interest rate or the stochastic volatility. For example, the standard BS model has a percentage error of -0.644% and -0.812% when it tries to match the data generated by the H-VS and H-cir models, respectively. The BS-V model and BS-CIR model have more or less the same performance. The H-VS and H-CIR have the most accurate performance with percentage errors close to 0% across all the different DGP's. These models are able to both incorporate stochastic volatility and interest rates. We note that these models contain more parameters (11 total) which greatly aids in fitting the data (compared to the 5 parameters for the BS-VS and BS-CIR models).

Panel B considers the setting of negative interest rates (r_0 = -0.005) and we find stark differences in percentage error of the models which rely on the CIR stochastic interest rate process. The BS-CIR has percentage errors of -1.483% and -2.778% where the DGP are the BS model and BS-VS model, respectively. These percentage errors were only 0.000% and -0.052% under periods of low interest rates in panel A. With the H model and H-VS models as DGP, the BS-CIR has percentage errors of 1.463% and -1.838%, respectively. Which is quite higher than the percentages of 0.969% and -0.081% under periods of low negative interest rate. Apparently, the CIR interest rate is not able to properly model the option prices under negative rates. We observe the same pattern for the H-CIR model, which has higher percentage errors across the board under the setting of negative interest rates in panel B, compared to the setting in panel A. Next we will look at the parameter estimates of the model. Based on this finding, we expect that the parameter estimates for the models extended with the CIR framework to greatly differ under a negative interest rate setting leading to unusual high (or low) parameter settings.

Tables 4 and 5 and show the calibrated parameters where the DGP are BS-VS and H-VS models, respectively. Table 4 shows the calibrated parameters where the DGP is the BS-VS model. In panel A, we observe that the the BS-CIR model takes on extraordinary parameters. The long-run mean Θ is 0.002 and the volatility of the short rate η is 0.770 which is remarkably high. The correlation parameter ρ_r is also significantly different then the starting value of -0.800 with 0.382. Yet, the pricing of this model was accurate according to table 3 in Panel A with the average price only being -0.052% off. The same applies to the H-CIR model where the volatility of the interest rate η and the correlation parameter ρ_r are considerably different with a value of 0.679 and 0.171 respectively. Yet the pricing error to the simulated price is small with only 0.004%. It seems that the calibration procedure assigns high volatility and correlation parameter values in order for the models to accurately approximate the simulated prices even though the parameters do not accurately describe the underlying dynamics of the DGP. Next we look at Panel B where we consider a setting of negative interest rates where we set the initial interest rate r_0 equal to -0.005. We expect that the models which assume a CIR type interest rate will have difficulty in establishing proper parameters as they do not allow for negative inputs. The BS-CIR model assumes a value of 1.000 for η which is equal to the value of the earlier established upper bounds. The correlation parameter ρ_r is also equal to the upper bound of 1. The pricing error of this model as mentioned earlier is -2.778% and coupled with the inaccurate parameter estimates indicate that the BS-CIR model is not correctly specified to handle this negative interest rate setting. The H-CIR model, which assumes the same CIR type short rate process, does not exhibit the same extreme values as the BS-CIR model. However, The initial volatility v_0 and the long run mean v^* are considerably higher than the initial variance of the BS-CIR model. It is likely that the parameters of the stochastic process of the volatility are trying to (unsuccessfully) capture the variation in the data. This is unsuccessful as we established earlier from table 3 that average pricing error is considerable for the H-CIR model under negative interest rates.

Table 4: Calibration results with the Black-Scholes-Vasicek model as DGP

DGP : $Black-Scholes-Vasicek$ $dS_t = S_t r_t dt + S_t \sigma dW_t^r$
$dr_t = \lambda(\theta - r_t)dt + \eta dZ_t^r$

Panel A: Low interest rates ($r_0 = 0.005$)	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value	3.620	0.003	0.020	−0.800				0.050			
BS								0.052			
								(4.036)			
BS-VS	3.620	0.003	0.020	−0.800				0.050			
	(0.000)	(0.000)	(0.000)	(0.000)				(-0.001)			
BS-CIR	0.525	0.002	0.770	0.382				0.050			
	(-85.508)	(-18.469)	(3752.217)	(-147.763)				(-0.698)			
H					0.616	0.061	0.013	0.050	0.435		
								(-0.316)			
H-VS	2.133	0.002	0.161	−0.188	2.235	0.131	0.001	0.075	−0.931	0.199	1.003
	(-41.073)	(-16.731)	(703.241)	(-76.532)				(49.016)			
H-CIR	0.205	0.001	0.679	0.171	1.057	0.182	0.033	0.160	−0.834	0.000	0.916
	(-94.336)	(-39.989)	(3293.711)	(-121.355)				(219.27)			

Panel B: Negative interest rates (r_0 = -0.005)	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value	3.620	0.002	0.020	−0.810				0.050			
BS								0.055			
								10.008			
BS-VS	3.620	0.002	0.020	−0.810				0.050			
	(0.000)	(0.000)	(0.000)	(0.000)				(-0.)			
BS-CIR	1.790	0.001	1.000	0.999				0.049			
	(-50.548)	(-41.379)	(4900.00)	(-223.333)				(-1.308)			
H					0.393	0.096	0.125	0.050	−0.166		
								(-0.326)			
H-VS	3.800	0.002	0.097	−0.410	1.889	0.091	0.001	0.021	−0.920	0.215	1.043
	(4.963)	(-9.722)	(383.011)	(-49.37)				(-57.585)			
H-CIR	0.986	0.010	0.182	0.428	5.000	0.216	0.261	0.214	−0.882	0.835	0.959
	(-72.77)	(379.586)	(810.484)	(-152.842)				(327.262)			

This table presents the calibration parameters of the six pricing models on simulated option prices where the Black-Scholes-Vasicek is the DGP. The percentage error between the calibrated parameter and the true value is shown in parenthesis (if applicable). Panel A shows the results where the true parameters of the DGP are chosen in a low interest rate setting. Panel B shows the results where the true parameters of the DGP are chosen in a negative interest rate setting.

Table 5: Calibration results with the Heston-Vasicek model as DGP

DGP : $Heston - Vasicek$

$$dS_t = S_t r_t dt + S_t \sqrt{v_t} dW_t^v + S_t \Delta \sqrt{v_t} dZ_t^v + S_t \Omega dW_t^r$$
$$dv_t = \chi(v^* - v_t)dt + \gamma \sqrt{v_t} dZ_t^v$$
$$dr_t = \lambda(\theta - r_t)dt + \eta dZ_t^r$$

Panel A: Low interest rates ($r_0 = 0.005$)											
	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value	3.620	0.003	0.020	−0.800	0.650	0.035	0.080	0.050	−0.970	0.260	1.230
BS								0.071			
								(42.99)			
BS-VS	3.082	0.001	0.092	0.999				0.073			
	(-14.87)	(-97.74)	(360.06)	(-224.87)				(46.33)			
BS-CIR	2.525	0.002	0.546	−0.437				0.074			
	(-30.26)	(-35.79)	(2632.43)	(-45.41)				(47.86)			
H					1.897	0.068	0.021	0.074	0.418		
					(191.92)	(97.86)	(-73.58)	(47.78)	(-143.13)		
H-VS	3.620	0.003	0.020	−0.800	0.650	0.035	0.080	0.050	−0.970	0.260	1.230
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
H-CIR	0.513	0.024	0.001	−0.913	0.686	0.073	0.065	0.101	−0.592	0.180	0.881
	(-85.82)	(687.48)	(-92.54)	(14.17)	(5.49)	(110.2)	(-18.47)	(101.3)	(-39.)	(-30.95)	(-28.39)
Panel B: Negative interest rates r_0 = -0.005)											
	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value	3.620	0.002	0.020	−0.810	0.650	0.035	0.080	0.050	−0.970	0.260	1.230
BS								0.075			
								(49.71)			
BS-VS	4.937	0.000	0.143	−0.999				0.073			
	(36.37)	(-105.71)	(616.22)	(23.33)				(46.15)			
BS-CIR	4.264	0.001	1.000	−0.999				0.072			
	(17.79)	(-98.11)	(4900.)	(23.33)				(44.76)			
H					0.080	0.173	0.148	0.073	−0.113		
					(-87.69)	(400.)	(84.52)	(46.94)	(-88.33)		
H-VS	3.620	0.002	0.020	−0.810	0.650	0.035	0.080	0.050	−0.970	0.260	1.230
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
H-CIR	0.222	0.029	0.136	0.555	0.760	0.280	0.060	0.334	−0.918	0.295	1.166
	(-93.87)	(1342.13)	(577.51)	(-168.47)	(16.98)	(711.71)	(-24.93)	(567.86)	(-5.33)	(13.54)	(-5.21)

This table presents the calibration parameters of the six pricing models on simulated option prices where the Heston-Vasicek is the DGP. The percentage error between the calibrated parameter and the true value is shown in parenthesis (if applicable). Panel A shows the results where the true parameters of the DGP are chosen in a low interest rate setting. Panel B shows the results where the true parameters of the DGP are chosen in a negative interest rate setting.

4 Empirical Results

This section presents the results regarding the performance of the pricing models using the empirical data on S&P 500 options in the period from 2010 to 2015. Section 4.1 provides the discrition of the data used in this study. Section 4.2 explains the estimation procedure. Section 4.3 presents the results of the estimated parameters as a result of the calibration procedure. Section 4.4 presents the MSE results of the pricing models.

4.1 Data Description

This study uses the prices of S&P 500 call and put options that are obtained from OptionMetrics. We choose the time period from January 1st, 2010 to December 31st, 2015 as around this time period interest rates reached record lows. We use the mid-point of the current day's lowest ask price and highest bid price as a proxy for the market price. The US three-month government bond yields are used as values of the risk-free interest rates and are obtained from Bloomberg[3]. The closing value of the S&P 500 index is used as the current day's spot price. The S&P 500 index includes dividend pay-outs and needs to be excluded for this study and this requires that the index needs to be adjusted. OptionMetrics provides the annualized S&P 500 dividend yield and we use this to discount the spot price to the remaining lifespan of the option. Therefore we adjust the spot price using

$$\bar{S}_{t,i} = S_{t,i} \times e^{-div_t \times \tau_i} \tag{53}$$

where $S_{t,i}$ is the unadjusted spot price, div_t is the annualized dividend yield for the respective day and τ_i is the time to maturity in days for the i-th contract.

Furthermore, we apply several common option filters which are used throughout other option pricing literature such as Bakshi et al. (1997). We remove options where the time-to-maturity is smaller than 10 days, and we remove options where the time-

[3] The yield series is called USGG3M in Bloomberg.

to-maturity is larger than 365 days. We also remove options where the trading volume on a giving day is smaller than 50. Options where the mid-point of the bid-ask quotes is below 0.5 are removed. Additionally, options where the implied volatility is higher than 70% are also excluded. Finally, we only select the option prices on Wednesdays to significantly reduce computation time for our calibration problem. Wednesday excludes end-of-the-week effects and is the day of the week that is the least likely day to be a holiday. If Wednesday is a holiday, the next following trading day is selected. After applying the option filters, we have a total of 86,020 weekly observations (32,782 calls; 53,238 puts) which spans a six year period (01/01/2010 - 31/12/15).

Table 6 provides the descriptive statistics of the S&P 500 call and put options. The data is separated in several moneyness and maturity bins where we define moneyness as S/K where S is the spot price and K is the strike price. We consider call options in-the-money (ITM) if $S/K >$1.03 and out-of-the-money (OTM) if $S/K <$0.97. Values between 0.97 and 1.03 are considered at-the-money (ATM). The opposite holds true for put options such that $S/K >$1.03 is OTM and $S/K <$0.97 are in-the-money options. We also categorize options based on the days to maturity (DTM) of the option contract. We consider short-term (DTM <60), medium-term (60 $\leq$ DTM <120), long-term (120 $\leq$ DTM <180) and very long-term maturities (DTM $\geq$ 180).

We observe that 48 % of the call options are traded ATM and are the most actively traded option contracts. About 6% percent of the call options are traded ITM while 46% of the call options are traded OTM. The call options also differ in terms of the time to maturity where options with short, medium, long and very long time to maturities account for 61%, 21%, 7% and 11% of all call options, respectively. The average call option price is $30.26 however we note that the average call options can greatly vary when we consider the different maturity and moneyness bins. For example, the average price for a deep ITM call options with a very long maturity is $235.03. High values of the call price can have considerable impact on our results as our loss function for the calibration procedure is based on the $MSE. The calibration procedure will then place larger weights on the high value option contracts as they contribute a large amount to

Table 6: Summary Statistics of S&P 500 Call and Put Options

Panel A: Call Options

	D <60		60 ≤ D <120		120 ≤ D <180		D ≥ 180		Total Sample	
Number of Observations										
S/K <0.94	2169		1944		889		1690		6692	
0.94 ≤ S/K <0.97	5809		1709		426		593		8537	
0.97 ≤ S/K <1.00	8028		1863		460		875		11226	
1.00 ≤ S/K <1.03	3042		869		197		303		4411	
1.03 ≤ S/K <1.06	659		173		66		96		994	
S/K ≥ 1.06	413		234		100		175		922	
Total	20120		6792		2138		3732		32782	
Average Price (Standard Deviation)										
S/K <0.94	2.19	(2.41)	5.04	(5.11)	9.92	(8.22)	22.69	(16.77)	9.22	(12.51)
0.94 ≤ S/K <0.97	4.09	(4.3)	14.80	(9.1)	29.97	(11.04)	59.00	(17.26)	11.34	(16.41)
0.97 ≤ S/K <1.00	14.08	(9.62)	36.68	(12.55)	55.83	(13.02)	84.99	(18.16)	25.07	(23.5)
1.00 ≤ S/K <1.03	37.98	(13.19)	59.46	(14.34)	79.15	(14.82)	106.43	(18.54)	48.76	(23.79)
1.03 ≤ S/K <1.06	74.96	(17.02)	91.91	(18.88)	107.77	(17.52)	137.09	(19.46)	86.09	(26.11)
S/K ≥ 1.06	124.59	(44.02)	156.34	(57.81)	187.62	(59.36)	235.03	(64.72)	172.60	(51.3)
Total	18.82	(22.54)	35.38	(25.38)	48.52	(31.19)	72.14	(43.08)	30.26	(33.43)

Panel B: Put Options

	D <60		60 ≤ D <120		120 ≤ D <180		D ≥ 180		Total Sample	
Number of Observations										
S/K <0.94	179		112		60		91		442	
0.94 ≤ S/K <0.97	402		159		80		140		781	
0.97 ≤ S/K <1.00	3028		1140		343		737		5248	
1.00 ≤ S/K <1.03	6790		1655		432		633		9510	
1.03 ≤ S/K <1.06	6138		1332		372		489		8331	
S/K ≥ 1.06	16767		6391		2260		3508		28926	
Total	33304		10789		3547		5598		53238	
Average Price (Standard Deviation)										
S/K <0.94	110.39	(33.46)	141.70	(45.92)	184.10	(49.04)	218.55	(56.78)	151.30	(48.70)
0.94 ≤ S/K <0.97	75.52	(19.68)	87.09	(17.98)	103.33	(21.17)	123.31	(19.94)	89.29	(26.62)
0.97 ≤ S/K <1.00	36.79	(12.76)	56.86	(13.27)	73.96	(14.16)	101.85	(18.03)	52.72	(26.64)
1.00 ≤ S/K <1.03	18.88	(9.55)	40.80	(11.6)	58.81	(12.69)	83.48	(16.60)	28.81	(21.2)
1.03 ≤ S/K <1.06	9.71	(6.94)	27.36	(9.51)	44.07	(10.49)	68.57	(16.31)	17.52	(17.76)
S/K ≥ 1.06	3.81	(4.01)	9.56	(8.51)	16.60	(12.66)	28.98	(20.71)	9.13	(12.59)
Total	12.67	(19.21)	25.09	(35.83)	36.13	(47.44)	53.63	(46.95)	21.06	(32.35)

This table shows the descriptive statistics of the call and put options on the S&P 500 index for the period from 01/01/2010 to 12/31/2015. The upper section of each panel shows the number of observations. The lower part of each panel shows the average price where the standard deviation is displayed in parenthesis. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins.

the squared errors in the loss function. Overall, the average call option price rises if either the moneyness and/or the time-to-maturity increases.

The proportion of the amount of options contracts across the moneyness bins are considerably different for put options. Around 70% of the put options are traded OTM and are the most actively traded contracts. Followed by the ATM put options with 28% and ITM contracts with 2%. The average put price is 21.06 and, similarly to the call option contracts, we observe high fluctuations of the average prices across the different moneyness and time-to-maturity bins. The highest prices are when contracts are ITM (S/K <0.97) and will lead in turn for the calibration procedure to lay greater emphasis on these contracts. This partially offsets the low average prices of the call values in the same moneyness bin. The same applies for the higher moneyness bins (S/K >1.03) where the low average price of the put options and the high average price of the call options counteract each other which results for the calibration procedure to place equal weights on options across all the moneyness bins. The proportion of the amount of the amount of put options across maturities is roughly the same as the call options. For short, medium, long and very long maturities we observe the respective proportions of 63%, 19%, 7% and 11%

Our sample data is considerably different then Recchioni and Sun (2016) and Recchioni et al. (2017). Their sample uses data from the period 2 April 2012 to 2 July 2012 where the expiry date from these options is on 16 March 2013. The data sample only selects a 2-month period where the call and put options have long time-to-maturities. Our reasoning for choosing a di fferent and larger time period is that the period of low and negative spans the entire period from 2010 to 2015. We also include options with shorter expiry dates in order to make a better comparison how the option models are able to price options across the different time-to-maturity bins.

4.2 Estimation Procedure

As mentioned in section 4.1, we use quoted prices from the time period 1 January 2010 to 31 December 2015 where we use the daily observations on every Wednesday. This leaves us with 312 windows of daily quoted prices of call and put options. On each single window we perform the calibration process where we minimize the dollar-measure mean squared errors using the objective function (52) in section 2.7.

We perform this for every pricing model. For every calibration run on each daily window, we set multiple starting points using Matlab's *multistart*. We set the amount of starting points to 3. Ideally, we would like this amount to be higher but the calibration time over the whole time period is already quite cumbersome. The last term in expression (52) is the regularization term where α is the regularization parameter, Θ_{Π} is the parameter set to be calibrated from the theoretical model and Θ_0 is a initial set of parameter values. We use a pragmatic approach and set α equal to 1 where the stability of our calibrated parameters seemed acceptable.[4] To determine Θ_0, we first perform a calibration run on the total set without regularization. We set Θ_0 equal to the long-run average of the calibrated parameters.

4.3 Implied Parameters

In this section we will analyze the implied parameters which are obtained after performing the calibration procedure as explained in section 4.2. A point of interest is how the models manage to model the different parameters under periods of low and negative interest rates. Figure 1 shows the implied parameters over the complete time period of 01/01/2010 to 31/12/2015. The graphs show the moving average of the last 10 daily calibrated parameters. Table 7 shows the mean values of the implied parameters. The calibration process is performed on the full sample from 01/01/2010 to 31/12/2015. The standard deviation of the implied parameters is shown in parenthesis. We note that

[4] We perform additional calibration runs with different values of α. Calibration runs with $\alpha < 1$ generate very unstable parameter estimates while calibration runs with $\alpha > 5$ result in very biased parameter estimates which are caused by the initial set of parameters θ_0.

Table 7: Implied Parameters of the pricing models (01/01/2010 - 31/12/2015)

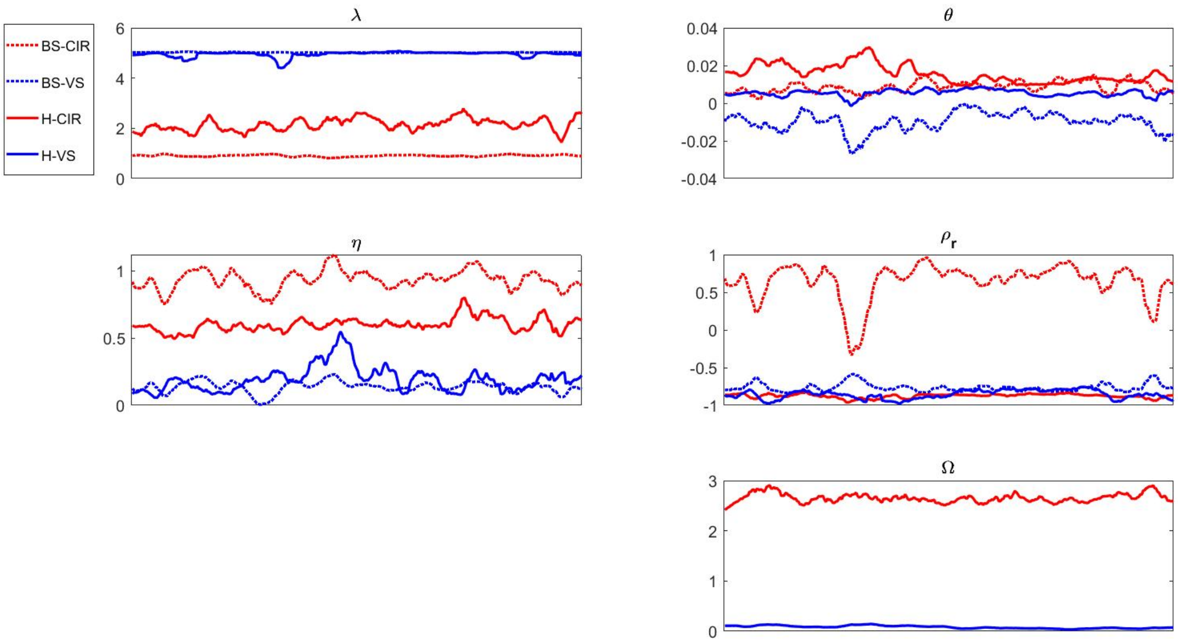
Model	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
BS								0.033			
								(0.018)			
BS-VS	5.018	-0.010	0.137	-0.765				0.027			
	(0.024)	(0.008)	(0.061)	(0.095)				(0.018)			
BS-CIR	0.908	0.009	0.761	0.645				0.031			
	(0.063)	(0.006)	(0.114)	(0.332)				(0.019)			
H					4.376	0.063	1.110	0.028	-0.735		
					(1.647)	(0.026)	(0.342)	(0.026)	(0.056)		
H-VS	4.950	0.005	0.193	-0.856	4.022	0.059	1.359	0.033	-0.947	0.082	0.225
	(0.201)	(0.003)	(0.145)	(0.096)	(0.223)	(0.030)	(0.220)	(0.031)	(0.046)	(0.03)	(0.097)
H-CIR	2.150	0.016	0.599	-0.876	3.913	0.030	0.032	0.065	-0.929	2.653	1.437
	(0.524)	(0.006)	(0.107)	(0.051)	(0.313)	(0.013)	(0.109)	(0.045)	(0.022)	(0.202)	(0.211)

This table presents the average value of the implied parameters derived from the calibration process on S&P 500 index call and put options for the period from 01/01/2010 to 31/12/2015. The standard deviation is shown in parenthesis. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek (1977) and Cox-Ingersroll-Rox (1985), respectively.

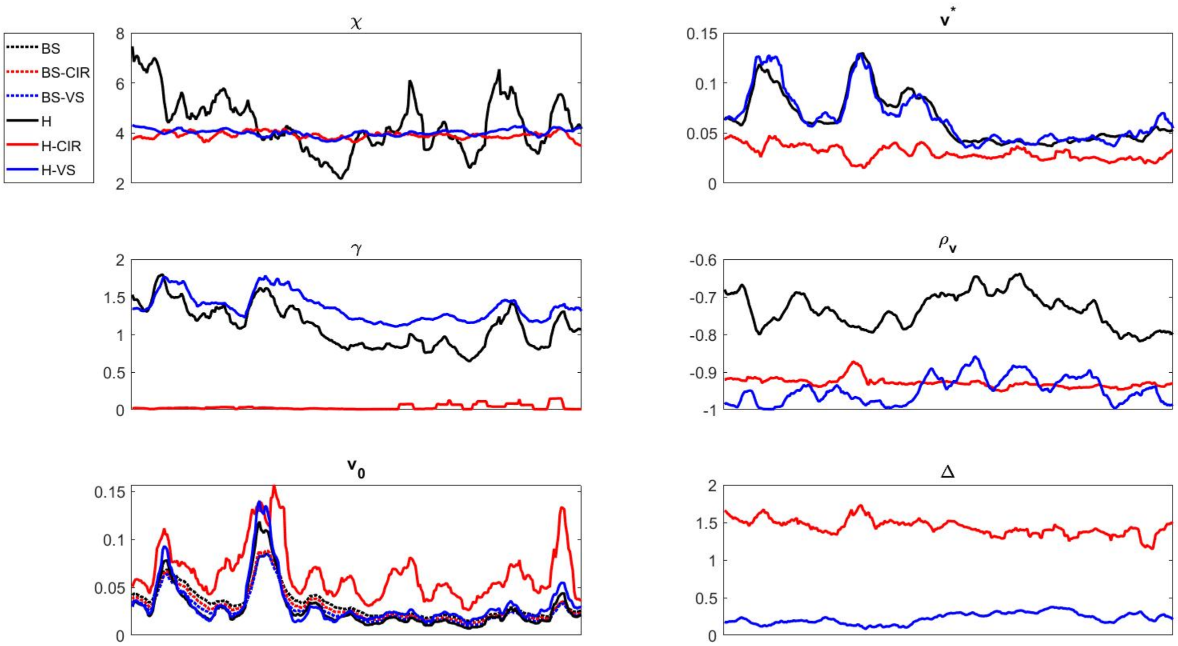
the Black-Scholes (BS) model uses the parameter σ for the volatility of the underlying while the Heston model uses v_t for the variance. In order to make results comparable, we present the results from both models as the variance v_t. We therefore compute the variance from the Black-Scholes models by squaring the volatility's derived from the calibration procedure.

We will first discuss the calibrated parameters of the BS model and their extensions. The average value of the variance parameter v_0 is 0.033 for the normal BS model. This is marginally higher than the variance of the BS-VS and BS-CIR models of 0.027 and 0.031 respectively. This makes sense as the normal BS model only consists of one parameter, while the BS-VS and BS-CIR extensions contains five parameters each. It seems that part of the variation in the data is explained by the parameters of the stochastic interest rate process which in turn leads to a decrease of the variance parameter. When we look closely at the differences between the BS-VS and BS-CIR models, we observe that the

Figure 1: Implied Parameters of the pricing models (01/01/2010 - 31/12/2015)



(a) Moving average of the last 10 daily calibrated parameters of the stochastic interest rate process where λ is the speed of mean reversion, θ is the long-run mean, η is the volatility of the short rate, ρ_r is the correlation between the short rate and the underlying. Ω is a nonnegative constant required for the H-VS and H-CIR models.



(b) Moving average of the last 10 daily calibrated parameters of the stochastic volatility process where is the speed of mean reversion, v^* is the long-run mean of the volatility, γ is the volatility of volatility, ρ_v is the correlation between volatility and underlying, v_0 is the initial stochastic volatility. Δ is a nonnegative contstant required for the H-VS and H-CIR models.

long-run mean of the short rate θ is negative with -0.010 for the BS-VS model and just slightly positive 0.009 for the BS-CIR model. The BS-VS model has the ability to model negative interest rates while the BS-CIR model cannot. We notice that the volatility of the short rate η is 0.137 for the BS-VS model, while this value is significantly larger for the BS-CIR model with a mean of 0.761. Additionally, the implied parameter for the correlation ρ_r is positive and high with 0.645. It seems that due to the inability for the BS-CIR model to incorporate low and negative interest rates, a large part of the variation of the data is explained by a high volatility of the short rate and high possitive corrleation value. The speed of mean reversion λ is also relatively low for the BS-CIR with a value of 0.91. This leads to a very low drift value, where changes in the short rate are primarily determined by the high shocks in the short rate.

Next, we will discuss the Heston models and their stochastic interest rate extensions. The volatility of the H model is lower than H-VS, this is in line with the BS models in the sense that the stochastic interest rate parameters manage to explain some part of the data which in turn leads to the H-VS model attributing less weight to the stochastic volatility process. The stochastic volatility process of the H-CIR model is different than the other two models. Whereas the standard Heston and H-VS models have a low initial variance v_0 of around 0.030, and high long term mean v^* around 0.06. The H-CIR model has a high initial variance of 0.065 and a lower long-run mean of only 0.030. We note that the H-CIR model has a considerable lower vol-of-vol γ value of 0.032 (vs. 1.110 for the Heston model; 1.359 for H-VS). Additionally, the volatility of the interest rate η is much higher for H-CIR (0.599) than H-VS (0.145). This is in line with the notion that the CIR process is not able to properly model the low or negative interest rates, and requires a high volatility parameter η to accurately match the model with the data.

4.4 Goodness of Fit tests

This section shows the performance of the pricing models in terms of the dollar-measure mean-squared errors ($MSE) when applied to empirical data. The sample data contains

Table 8: Average $MSE on Call and Put Options on the S&P 500 index(2010-2015)

Panel A: Call Options	In-Sample	1-step ahead Out-of-Sample	5-step ahead Out-of-Sample
BS	41.55	54.18	82.89
BS-VS	24.98	38.93	67.30
BS-CIR	33.65	48.39	79.59
H	1.35	17.04	45.90
H-VS	0.90	17.39	44.79
H-CIR	0.92	22.89	48.13

Panel B: Put Options	In-Sample	1-step ahead Out-of-Sample	5-step ahead Out-of-Sample
BS	40.77	50.73	70.67
BS-VS	34.98	47.54	68.20
BS-CIR	35.93	46.68	67.56
H	1.91	16.77	43.73
H-VS	0.66	16.07	38.74
H-CIR	0.71	18.01	46.36

This table presents the in-sample, 1-step ahead out-of-sample and 5-step ahead out-of-sample $MSEs of call and put options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively.

the entire period from January 1, 2010 to December 31, 2015.

Table 8 presents the total average $MSE of the call and put options on the S&P 500 index. The table shows the in-sample, 1 step-ahead OOS and 5 step-ahead OOS results. We first discuss the in-sample results. Panel A of Table 8 shows $MSE of the call options. The BS model performs by far the worst with a MSE of 41.55 for call options. This is roughly 25% more worse than the BS-CIR model which reports an MSE of 33.65, and 73% more worse than the BS-VS model which reports 24.98. Apparently, modeling the risk-free interest rate as a stochastic process improves pricing accuracy compared to assuming a constant interest rate. We reach the same conclusion when we look at the pricing performance of the put options in panel B. The BS model has the worst performance with a $MSE of 40.77 for put options. This is roughly 16% higher than the $MSEs of the BS-VS and BS-CIR models of 34.98 and 35.93, respectively. The bad performance of the standard BS model is expected, as it is quite an naive model that doesn't assume a complex process for either the interest rate or the volatility.

According to our expectations, the BS-VS model has a better performance than the BS-CIR model for both call and put options. As we know that the CIR process is unable to properly model negative or low interest rates, we appoint this the cause of the bad performance of the BS-CIR model. Table 9 shows the average in-sample $MSE results for call options where the data is categorized in moneyness and maturity bins. We observe at the bottom of Table 9 that the differences between the $MSE values of the BS-VS and BS-CIR models grow larger as the time-to-maturity increases.

Next, we look at the Heston model which relaxes the assumption of a constant volatility of the underlying. We observe that the pricing performance of the Heston models and their extensions is considerably better than the BS models. As shown in Table 8, the standard Heston model reports an in-sample $MSE of 1.35 for call options, while the H-VS and H-CIR models report 0.90 and 0.92, respectively. The performance of the standard Heston model is roughly 50% more worse. When we consider only puts, the standard Heston model performs roughly 190% more worse with a $MSE of 1.91 compared to the $MSE of the H-VS and H-CIR models of 0.66 and 0.71, respectively. Our findings are similar to the results reported in Recchioni et al. (2017). Their results, spanning a time window over 2 months in 2012, show that the standard Heston model has considerable higher percentage errors compared to the H-VS model and attributes this to the ability of the H-VS model to incorporate stochastic interest rates. Additionally, they find that the H-VS outperforms the H-CIR and contributes this to the fact that the H-VS allows for negative interest rates. In our sample, which covers the period from 2010 to 2015, we are able to report similar findings that the H-VS model outperforms the H-CIR model.

We note that the number of parameters of a model plays a crucial role in how the model is able to explain the data. A model with more parameters is capable of more accurately fitting the data compared to a model with less parameters. This does come at a cost, as models with more parameters are more likely to overfit the data which makes them unsuitable for out-of-sample analysis. A point of interest is whether the extended Heston models with their high number of parameters are able to correctly

Table 9: In-Sample Average $MSE on Call Options on the S&P 500 index(2010-2015)

		D <60	60 ≤ D <120	120 ≤ D <180	D ≥ 180	Total Sample
S/K < 0.94	BS	23.88	59.01	82.53	92.33	59.16
	BS-VS	15.05	39.37	63.47	102.55	50.64
	BS-CIR	21.03	51.61	80.05	137.34	67.13
	H	0.28	0.39	0.60	2.20	0.84
	H-VS	0.49	0.82	0.92	2.44	1.14
	H-CIR	0.53	0.74	0.85	2.42	1.16
0.94 <S/K <0.97	BS	34.35	71.13	50.56	27.42	42.04
	BS-VS	15.43	37.02	35.58	28.34	21.65
	BS-CIR	25.49	54.53	53.49	47.48	34.23
	H	0.69	0.78	1.02	1.30	0.77
	H-VS	0.52	0.74	0.77	1.33	0.63
	H-CIR	0.51	0.80	0.81	1.32	0.64
0.97 <S/K <1.00	BS	34.78	27.36	8.72	65.66	34.89
	BS-VS	9.13	7.44	6.18	18.69	9.48
	BS-CIR	20.32	17.09	8.46	17.49	19.07
	H	0.75	1.29	2.16	3.98	1.15
	H-VS	0.81	0.78	0.64	0.83	0.80
	H-CIR	0.76	0.78	0.63	0.84	0.82
1.00 <S/K <1.03	BS	11.63	6.77	21.11	140.21	19.93
	BS-VS	5.24	12.69	30.78	56.37	11.36
	BS-CIR	6.41	7.00	18.85	42.41	9.55
	H	1.80	1.29	2.97	6.98	2.10
	H-VS	1.40	0.99	0.53	0.44	1.21
	H-CIR	1.41	0.91	0.49	0.43	1.22
1.03 <S/K <1.06	BS	10.27	26.55	71.64	239.58	39.33
	BS-VS	21.37	63.67	98.42	160.92	47.32
	BS-CIR	13.95	39.25	66.89	127.51	32.83
	H	1.80	1.85	4.23	10.17	2.78
	H-VS	1.58	0.69	0.49	0.52	1.25
	H-CIR	1.44	0.68	0.52	0.56	1.30
S/K ≥ 1.06	BS	12.84	46.15	122.27	345.12	96.23
	BS-VS	19.97	72.96	169.63	281.29	99.25
	BS-CIR	14.58	53.41	123.02	239.57	78.90
	H	3.30	12.34	7.93	11.93	7.73
	H-VS	1.63	0.86	0.56	0.97	1.19
	H-CIR	1.70	0.80	0.54	0.90	1.24
Total Sample	BS	28.73	45.42	56.14	95.29	41.55
	BS-VS	11.62	28.38	48.62	77.23	24.98
	BS-CIR	19.46	36.91	55.32	91.79	33.65
	H	0.93	1.30	1.69	3.53	1.35
	H-VS	0.82	0.81	0.77	1.61	0.90
	H-CIR	0.87	0.81	0.83	1.66	0.92

This table presents the in-sample $MSE of call options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins. Underlined values are the smallest $MSE values within their respective bins.

price out-of-sample option contracts.

Table 8 also presents the 1 step-ahead OOS MSE for call options. The $MSE for the Black-Scholes model is 54.18 for the whole sample, whereas this is 38.93 and 48.39 for the BS-VS and BS-CIR, respectively. Even in a out-of-sample setting, the BS-VS model is the better performer. The same holds when we view the 1-day out-of-sample MSE for put options. The MSE for the BS-VS model is 47.54 (50.73 for BS; 46.68 for BS-CIR). It seems that the Vasicek interest rate specification is able to more accurately price options than the CIR interest type specification. Furthermore, we observe that the standard Heston model marginally outperforms the H-VS and H-CIR models with a MSE of 17.04 (17.39 for H-VS; 18.89 for H-CIR) for 1-step OOS call options. We expect that the H-VS and H-CIR models would outperform the standard Heston model because the specifications of the H-VS and H-CIR models allows them to more intricately model the dynamic processes regarding the stochastic interest rates. It seems that the high amount of parameters for the H-VS and H-CIR models allowed the models to overfit the data, leading to relatively bad out-of-sample results. In panel B of Table 8 we observe that the $MSE of the H-VS model is 16.07 is marginally better (vs. 16.77 for the standard Heston model; 18.01 for H-CIR). The H-CIR model has relatively bad performance given the case that the in-sample fit was fairly decent. Apparently, the calibrated parameters for the H-CIR model are not adequate for out-of-sample use and we attribute this to the fact that the CIR process is not able to correctly model low and negative interest rates.

Table 8 shows the 5-day out-of-sample results for call and put options, respectively. We first examine the Black-Scholes models and we observe that the BS-VS model ($MSE of 67.30) outperforms the standard BS model (82.89) and the BS-CIR model (79.59) by a significant margin when we consider call options for the whole sample. This is in line with the notion that the Vasicek interest rate type is a better specification of the current interest rate dynamics compared to the constant and CIR type rates. Similarly, for the Heston models we observe a better performance for the H-VS models compared to the standard Heston and H-CIR models. For the 5-day out-of-sample put options we observe a MSE of 38.74 for the H-VS model, while this is 43.73 and 46.36 for the Heston and H-CIR model, respectively.

Table 10 presents the Diebold-Mariano (DM) statistics which compares the pricing performance between the six models. The test is performed using the Harvey-Leybourne-Newbold bias correction. The differential $\bar{d}$ is calculated using the pricing errors of the models in the left column minus the pricing errors of the models in the top row. The values above the diagonal are the pricing errors derived from the 1 step-ahead OOS forecasts, while the values below the diagonal are the pricing errors derived from the 5 step-ahead OOS forecasts. We observe in Table 10 that all pricing differences between the models are significant based on a confidence level of 5%. We note that the DM-test infers several assumptions about the models which may not be applicable in our case. The DM-test assumes that the models are non-nested and linear. However, our pricing models do not satisfy these assumptions. We still present our findings but one must take care in further interpreting the results.

Based on our empirical study, we find that the models that incorporate a Vasicek type interest rate process seems to have better a pricing performance compared to a constant or CIR type interest rate process. Our results indicate that this happens across all maturity and moneyness bins. We note that the H-CIR model performed well regarding the in-sample test in comparison with the standard Heston model. However, the out-of-sample results for the H-CIR model are slightly worse than the standard Heston model. We further note that the performance of the standard Heston model is considerably better than the standard Black-Scholes model across all maturity and moneyness bins. Adding a stochastic interest rate process on top of the standard models only yields a marginally small increase in performance for the Heston model, while it has a considerable impact on the increase in performance for the Black-Scholes model.

Table 10: Diebold-Mariano test for comparing predictive accuracy between models

	BS	*BS-VS*	*BS-CIR*	*H*	*H-VS*	*H-CIR*
BS		8.90	14.92	40.09	39.68	36.60
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
BS-VS	−5.37		0.90	29.19	28.74	24.85
	(0.000)		(0.371)	(0.000)	(0.000)	(0.000)
BS-CIR	−4.30	6.57		47.05	46.48	46.72
	(0.000)	(0.000)		(0.000)	(0.000)	(0.000)
H-std	−11.81	−28.13	−27.84		1.79	−3.67
	(0.000)	(0.000)	(0.000)		(0.039)	(0.000)
H-VS	−13.74	−30.30	−33.82	−4.38		−3.38
	(0.000)	(0.000)	(0.000)	(0.000)		(0.000)
H-CIR	−10.27	−21.44	−23.19	3.47	6.71	
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	

This table presents the Diebold-Mariano (DM) test statistics with the Harvey-Leybourne-Newbold bias correction. The DM-test compares predictive accuracy between models by testing if the differential between models has expectation zero. The differential $\bar{d} = f(model_i) - f(model_j)$, where $f(.)$ is the predictive error of $model_i$, and i and j are one of the six pricing models. The model at the left of each row is $model_i$. The model at the top of each column is $model_j$. The values above the diagonal are the pricing error derived from the 1 step-ahead OOS forecasts. The values below the diagonal are the pricing errors derived from the 5 step-ahead OOS forecasts.

5 Conclusion

In this study, we consider a setting of low or negative interest rates as these scenario's have considerable technical implications on option pricing. We examine the Vasicek and Cox-Ingersoll-Ross (CIR) type short rate processes and expect that the CIR type short rate model is inadequate in modelling option prices due to the inherent limitation of not accepting negative inputs. Six pricing models are considered namely the standard Black-Scholes (BS) model, the BS-VS model, the BS-CIR model, the standard Heston (H) model, the H-VS model and the H-CIR model. Our goal is to gain insight how the models are able to capture the interest rate dynamics under either low or negative interest rates and how accurate they are in terms of option pricing. A simulation study is performed which considers different levels of interest rates. Additionally, an empirical study is implemented which applies the six option model to historical data.

The simulation study has shown that the models which assume a CIR type interest rate process are not able to adequately model the simulated option prices under negative rates. The average percentage errors between simulated prices and theoretical prices are considerably higher compared to the other models that do no rely on a CIR type interest rate. Additionally, the calibrated parameter values under both low and negative rates show considerably different values compared to the initial values specified by the DGP. Notably, the volatility of the stochastic interest rate process and the correlation parameter show abnormal values. The question remains whether the out-of-sample performance for the models that rely on a CIR type interest rate is adequate as the calibrated parameters show that the models assume different dynamics of the underlying dynamics.

For the empirical study, we calibrate the pricing models on historical data of S&P 500 index options from the period 01/01/2010 to 31/12/2015 where interest interests are extremely low. The standard BS model has the worst pricing performance which is to be expected as the BS model is not able to intricately model either the volatility or the interest rate. Between the BS-VS and BS-CIR models, the BS-VS has the better pricing performance for both in-sample and out-of-sample. We attribute this to the fact that the CIR type interest rate is not able to properly model the dynamics of low interest rates in our sample period. The standard Heston model has a considerable better pricing performance compared to the BS models and their extensions. Apparently, modeling the stochastic volatility has a high impact on pricing performance. The H-VS model is able to outperform the standard Heston model in terms of both the in-sample and out-of-sample fit. The H-CIR model is also able to beat the standard Heston model in terms of the in-sample fit. However, the out-of-sample results show that the H-CIR models performs worse than the standard Heston model. It seems that the the H-CIR model is misspecified as the CIR stochastic interest rate is not able to properly model the interest rate dynamics.

We have a few remarks about our study. First, as we mentioned earlier, the regularization method of the calibration procedure plays an important role in determining the estimated parameters. While we do not expect that the conclusions from this study will drastically change. We do expect that the relative pricing performance between models can vary to a certain degree as a result of a different regularization method. Second, we only consider relatively simple financial products as closed-form expressions of the more complex products are not available. While computationally cumbersome, additional analysis can be done on more complex financial instruments using simulations. We suggest one topic for further research which is the use of a shifted CIR model in combination with the Heston stochastic volatility. A shifted CIR model shifts the short rate upwards away from the zero lower bound such that the model is still applicable in periods of low or negative interest rates. However, this has technical implications as we then introduce another parameter to be estimated, namely the shift parameter.

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A Option Pricing Models

In this section, we display the remaining required quantities which are required to compute the value of an option for the respective model.

A.1 Black-Scholes-CIR

$$C_0 = \frac{1}{\eta\sqrt{T}}\Big[\frac{\zeta(r_0-\theta)}{\lambda}\Big(\frac{1-e^{-\lambda T}}{\lambda} - Te^{-\lambda T}\Big) + \frac{\zeta\theta T}{\lambda}\Big(1 - \frac{1-e^{-\lambda T}}{\lambda}\Big)\Big] \tag{A1}$$

$$C_1 = -\frac{\rho_r}{\eta T}C_11 \tag{A2}$$

$$C_{11} = \frac{2\sqrt{\theta}\Big((1+2e^{\lambda T}\sqrt{r_0} - 3e^{\frac{\lambda T}{2}}\sqrt{r_0-\theta(1-e^{\lambda T})}\Big) + \Big(\theta(1+2e^{\lambda T}) - r_0\Big)\psi}{2e^{\lambda T}\lambda^2\sqrt{\theta}} \tag{A3}$$

$$\psi = \log\Big(\frac{\theta(2e^{\lambda T} - 1 + r_0 + 2e^{\frac{\lambda T}{2}}\sqrt{\theta^2(e+\theta r_0)}}{(\sqrt{r_0}+\sqrt{\theta})^2}\Big) \tag{A4}$$

A.2 Heston-Vasicek

$$
\begin{aligned}
\Psi_{j,\lambda}(\tau) =& \frac{1 - e^{-j\lambda\tau}}{j\lambda}, \tau > 0 && \text{(A5)} \\
\phi_q(k) =& \frac{k^2}{2} + \iota\frac{k}{2}(sq - 1) - \frac{1}{2}(q^2 - q), && \text{(A6)} \\
\mu_{q,v} =& -\frac{1}{2}(\chi + (\iota k - q)\gamma(\rho_v + \Delta)) && \text{(A7)} \\
\zeta_{q,v} =& \frac{1}{2}[4\mu_v^2 + 2\gamma^2\phi_q(k)\tilde{\psi}]^{1/2} && \text{(A8)} \\
s_{q,v,g} =& 1 - e^{-2\zeta_{q,v}\tau} && \text{(A9)} \\
s_{q,v,b} =& (\zeta_{q,v} + \mu_{q,v})e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} - \mu_{q,v}) && \text{(A10)} \\
Q_0(\tau, k) =& \psi_q(k)\Omega^2\tau - a_{q,r}(k)\frac{(\iota k - q)}{\lambda}(\tau - \Psi_{1,\lambda}(\tau)) && \text{(A11)} \\
&+ \frac{\eta^2}{2}\frac{(\iota k - q)^2}{\lambda^2}(\tau - 2\Psi_{1,\lambda}(\tau) + \Psi_{2,\lambda}(\tau)) \\
a_{q,r} =& \lambda\theta - (\iota k - q)\Omega\rho_r\eta && \text{(A12)} \\
Q_{v,q}(\tau, v, ; \Theta_v) =& \frac{-2v^*}{\gamma^2}\ln(s_{q,v,b}/(2\zeta_{q,v})) && \text{(A13)} \\
&- \frac{2\chi v^*}{\gamma^2}(\zeta_{q,v} + \mu_{q,v})\tau \\
&- \frac{2v}{\gamma^2}(\zeta_{q,v}^2 - \mu_{q,v}^2)\frac{s_{q,v,g}}{s_{q,v,b}} \\
Q_{r,q}(\tau, r, k; \Theta_r) =& -r(\iota k - q)\Psi_{1,\lambda}(\tau) + Q_0(\tau, k) && \text{(A14)}
\end{aligned}
$$

A.3 Heston-CIR

$$\phi_q(k) = \frac{k^2}{2} + \iota\frac{k}{2}(sq-1) - \frac{1}{2}(q^2-q), \tag{A15}$$

$$\mu_{q,v} = -\frac{1}{2}(\chi + (\iota k - q)\gamma(\rho_v + \Delta)) \tag{A16}$$

$$\zeta_{q,v} = \frac{1}{2}[4\mu_v^2 + 2\gamma^2\phi_q(k)\tilde{\psi}]^{1/2} \tag{A17}$$

$$s_{q,v,g} = 1 - e^{-2\zeta_{q,v}\tau} \tag{A18}$$

$$s_{q,v,b} = (\zeta_{q,v} + \mu_{q,v})e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} - \mu_{q,v}) \tag{A19}$$

$$W_{v,q} = \mathrm{e}^{-\frac{2\,T\,\chi\,v_s\,(\mu_{\mathrm{qv}}+\zeta_{\mathrm{qv}})}{\gamma^2}}\,\mathrm{e}^{-\frac{2\,\chi\,v_s\,\ln\left(\frac{s_{\mathrm{qvb}}}{2\,\zeta_{\mathrm{qv}}}\right)}{\gamma^2}}\,\mathrm{e}^{\frac{2\,s_{\mathrm{qvg}}\,v_0\,\left(\mu_{\mathrm{qv}}{}^2-\zeta_{\mathrm{qv}}{}^2\right)}{\gamma^2\,s_{\mathrm{qvb}}}} \tag{A20}$$

$$\mu_{q,r} = -\frac{1}{2}(\chi + (\iota k - q)\gamma\Omega\rho_{p,r}) \tag{A21}$$

$$\zeta_{q,r} = \frac{1}{2}[4\mu_r^2 + 2\eta^2\phi_q(k)\Omega^2 - q + \iota k)]^{1/2} \tag{A22}$$

$$s_{q,r,g} = 1 - e^{-2\zeta_{q,r}\tau} \tag{A23}$$

$$s_{q,r,b} = (\zeta_{q,r} + \mu_{q,r})e^{-2\zeta_{q,r}\tau} + (\zeta_{q,r} - \mu_{q,r}) \tag{A24}$$

$$W_{r,q} = \mathrm{e}^{-\frac{2\,T\,\lambda\,\mathrm{theta}\,(\mu_{\mathrm{qr}}+\zeta_{\mathrm{qr}})}{\eta^2}}\,\mathrm{e}^{-\frac{2\,\lambda\,\mathrm{theta}\,\ln\left(\frac{s_{\mathrm{qrb}}}{2\,\zeta_{\mathrm{qr}}}\right)}{\eta^2}}\,\mathrm{e}^{\frac{2\,r_0\,s_{\mathrm{qrg}}\,\left(\mu_{\mathrm{qr}}{}^2-\zeta_{\mathrm{qr}}{}^2\right)}{\eta^2\,s_{\mathrm{qrb}}}} \tag{A25}$$

B Calibrated Parameters on Simulated Data

[Results on next page]

Table 11: Calibration results with the Black-Scholes model as DGP

DGP : $Black-Scholes-Vasicek \quad dS_t = S_t r_t dt + S_t \sigma dW_t^r$											
Panel A: Low interest rates (r_0 = 0.005)											
	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value								0.050			
BS								0.050 (0.0)			
BS-VS	1.135	0.005	0.000	0.892				0.050 (-0.)			
BS-CIR	1.619	0.005	0.272	0.000				0.050 (0.)			
H					4.998	0.050	0.075	0.050 (0.089)	0.001		
H-VS	0.594	0.005	0.030	−0.707	0.752	0.112	0.003	0.023 (-53.25)	−0.937	0.216	1.059
H-CIR	0.081	0.005	0.008	−0.879	0.000	0.288	0.001	0.086 (71.155)	−0.698	0.585	0.923
Panel B: Negative interest rates (r_0 = -0.005)											
	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value								0.050			
BS								0.050 (0)			
BS-VS	0.000	0.000	0.000	0.998				0.050 (-0.)			
BS-CIR	0.007	0.013	0.991	−0.981				0.049 (-2.62)			
H					4.967	0.050	0.082	0.050 (0.101)	0.000		
H-VS	0.007	0.001	0.001	−0.748	0.208	0.126	0.001	0.113 (126.116)	−0.949	0.185	1.148
H-CIR	2.667	0.192	0.236	0.040	1.662	0.670	0.717	0.953 (1806.531)	0.863	2.807	2.989

This table presents the calibration parameters of the six pricing models on simulated option prices where the Black-Scholes model is the DGP. The percentage error between the calibrated parameter and the true value is shown in parenthesis (if applicable). Panel A shows the results where the true parameters of the DGP are chosen in a low interest rate setting. Panel B shows the results where the true parameters of the DGP are chosen in a negative interest rate setting.

Table 12: Calibration results with the Heston model as DGP

DGP : *Heston* $dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^v$
$dv_t = \chi(v^* - v_t)dt + \gamma\sqrt{v_t} dW_t^v$

Panel A: Low interest rates ($r_0 = 0.005$)											
	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value					0.650	0.034	0.800	0.050	−0.970		
BS								0.046 (-8.289)			
BS-VS	4.599	0.026	1.000	0.999				0.044 (-11.545)			
BS-CIR	2.215	0.038	1.000	−0.999				0.051 (1.979)			
H					0.669 (2.966)	0.036 (4.592)	0.122 (-84.783)	0.050 (0.045)	−0.721 (-25.686)		
H-VS	2.099	0.001	0.031	−0.775	0.255 (-60.822)	0.042 (22.187)	0.144 (-81.967)	0.046 (-8.23)	−0.954 (-1.679)	0.115	0.111
H-CIR	3.681	0.002	0.791	−0.999	0.434 (-33.257)	0.189 (456.134)	0.294 (-63.279)	0.252 (403.687)	−0.947 (-2.346)	0.180	0.639
Panel B: Negative interest rates r_0 = -0.005)											
	λ	θ	η	ρ_r	χ	v^*	γ	v_0	ρ_v	Ω	Δ
True Value					0.650	0.034	0.800	0.050	−0.970		
BS								0.046 (-8.773)			
BS-VS	4.565	0.016	1.000	0.999				0.044 (-11.535)			
BS-CIR	1.612	0.030	1.000	−0.999				0.048 (-3.971)			
H					0.659 (1.337)	0.035 (4.096)	0.122 (-84.804)	0.050 (0.041)	−0.722 (-25.58)		
H-VS	0.001	0.000	0.063	−0.783	0.556 (-14.448)	0.085 (149.602)	0.247 (-69.066)	0.066 (32.299)	−0.876 (-9.671)	0.130	0.359
H-CIR	3.198	0.020	0.442	−0.999	1.785 (174.685)	0.042 (23.899)	0.008 (-98.94)	0.126 (152.673)	−0.893 (-7.973)	0.591	1.329

This table presents the calibration parameters of the six pricing models on simulated option prices where the Heston model is the DGP. The percentage error between the calibrated parameter and the true value is shown in parenthesis (if applicable). Panel A shows the results where the true parameters of the DGP are chosen in a low interest rate setting. Panel B shows the results where the true parameters of the DGP are chosen in a negative interest rate setting.

C Additional $MSE results

[Results on next page]

Table 13: In-Sample Average $MSE on Put Options on the S&P 500 index (2010-2015)

		D <60	60 ≤ D <120	120 ≤ D <180	D ≥ 180	Total Sample
S/K<0.94	*BS*	13.67	65.84	89.04	160.09	67.26
	BS-VS	16.52	104.54	154.64	377.24	131.84
	BS-CIR	13.65	59.83	87.33	228.02	79.49
	H	2.24	<u>2.37</u>	3.87	29.50	8.11
	H-VS	<u>1.99</u>	2.74	<u>0.71</u>	<u>2.06</u>	<u>2.02</u>
	H-CIR	2.10	2.80	0.73	2.18	2.20
0.94≤ S/K <0.97	*BS*	30.16	72.01	68.91	39.94	44.40
	BS-VS	24.07	72.55	123.87	195.75	74.94
	BS-CIR	25.90	53.92	72.49	67.90	43.91
	H	<u>1.80</u>	2.14	3.66	13.43	4.14
	H-VS	2.24	1.39	<u>1.57</u>	4.97	2.49
	H-CIR	2.31	<u>1.28</u>	1.68	<u>4.59</u>	<u>2.70</u>
0.97≤ S/K <1.00	*BS*	28.27	25.66	11.99	19.27	25.37
	BS-VS	9.29	16.70	30.84	103.65	25.56
	BS-CIR	15.21	15.04	11.75	26.21	16.49
	H	<u>1.03</u>	1.00	1.94	8.99	2.20
	H-VS	1.56	0.82	0.93	<u>2.00</u>	<u>1.42</u>
	H-CIR	1.44	<u>0.79</u>	<u>0.93</u>	2.09	1.45
1.00≤ S/K <1.03	*BS*	10.47	7.54	12.73	68.29	13.91
	BS-VS	6.54	5.10	6.99	29.33	7.82
	BS-CIR	6.69	7.00	10.70	18.47	7.71
	H	1.63	<u>0.75</u>	1.10	4.08	1.62
	H-VS	<u>0.74</u>	0.82	<u>0.96</u>	<u>1.76</u>	<u>0.83</u>
	H-CIR	0.75	0.78	0.98	1.83	0.84
S/K>1.06	*BS*	12.30	25.52	65.17	164.33	25.70
	BS-VS	19.62	34.44	39.89	25.13	23.22
	BS-CIR	15.42	35.79	61.22	72.16	24.05
	H	1.03	<u>0.44</u>	0.68	1.65	0.96
	H-VS	0.49	0.53	0.61	1.33	<u>0.55</u>
	H-CIR	<u>0.46</u>	0.51	<u>0.61</u>	<u>1.31</u>	0.56
S/K >1.06	*BS*	12.01	42.53	99.93	264.31	56.22
	BS-VS	14.00	47.19	95.06	168.91	46.46
	BS-CIR	12.79	46.72	101.29	211.26	51.27
	H	2.63	1.42	1.12	1.26	2.08
	H-VS	<u>0.37</u>	0.46	0.47	<u>0.62</u>	<u>0.43</u>
	H-CIR	0.37	<u>0.44</u>	<u>0.45</u>	0.63	0.45
Total Sample	*BS*	13.46	33.95	76.28	193.84	40.77
	BS-VS	13.22	36.90	73.99	136.03	34.98
	BS-CIR	12.42	36.17	76.51	149.63	35.93
	H	1.98	1.17	1.26	3.39	1.91
	H-VS	<u>0.61</u>	<u>0.60</u>	0.62	<u>1.12</u>	<u>0.66</u>
	H-CIR	0.62	0.61	<u>0.62</u>	1.16	0.71

This table presents the in-sample average $MSE of put options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins. <u>Underlined</u> values are the smallest $MSE values within their respective bins.

Table 14: 1-Day Out-Of-Sample average $MSE on Call options on the S&P 500 index (2010-2015)

		D <60	60 ≤ D <120	120 ≤ D <180	D ≥ 180	Total Sample
S/K<0.94	*BS*	24.47	63.31	95.09	115.22	68.07
	BS-VS	16.36	43.80	76.25	126.66	60.17
	BS-CIR	22.95	56.58	96.64	166.36	78.76
	H	3.10	6.90	11.79	19.96	9.62
	H-VS	4.18	8.15	14.01	21.58	11.04
	H-CIR	4.67	9.83	16.83	20.77	13.54
0.94≤ S/K <0.97	*BS*	39.75	84.74	80.03	66.40	52.61
	BS-VS	22.06	51.26	61.44	54.73	32.13
	BS-CIR	33.35	69.89	89.86	82.04	46.86
	H	6.94	18.29	23.75	36.86	12.13
	H-VS	7.93	19.13	24.45	36.81	13.00
	H-CIR	7.43	18.71	24.10	49.12	12.56
0.97≤ S/K <1.00	*BS*	45.53	50.98	40.73	99.15	50.41
	BS-VS	21.90	31.98	33.84	45.00	25.86
	BS-CIR	34.53	40.91	39.31	46.45	36.71
	H	16.67	30.77	32.24	35.05	21.08
	H-VS	16.78	30.98	31.29	32.19	20.93
	H-CIR	22.29	41.13	31.76	33.62	28.03
1.00≤ S/K <1.03	*BS*	25.47	37.33	46.82	159.75	37.94
	BS-VS	20.63	43.10	55.46	80.67	30.71
	BS-CIR	23.02	37.14	42.33	68.38	29.76
	H	22.45	36.46	31.12	34.07	26.39
	H-VS	22.73	38.13	31.60	32.50	26.82
	H-CIR	22.59	37.30	31.36	33.28	35.40
S/K>1.06	*BS*	17.77	44.94	61.83	244.32	47.37
	BS-VS	29.36	78.71	96.15	183.18	57.28
	BS-CIR	22.50	56.32	65.91	123.86	41.08
	H	16.00	35.39	22.22	35.42	21.66
	H-VS	16.97	38.42	26.03	31.60	22.72
	H-CIR	16.49	48.70	31.53	45.32	22.19
S/K >1.06	*BS*	16.24	51.09	116.13	348.03	98.98
	BS-VS	24.65	81.75	182.09	395.66	126.67
	BS-CIR	18.23	58.70	124.60	237.71	81.70
	H	9.79	26.82	15.76	23.38	17.32
	H-VS	8.26	10.39	12.17	13.81	10.28
	H-CIR	12.29	27.54	19.22	26.39	19.57
Total Sample	*BS*	37.05	61.11	75.89	121.57	54.18
	BS-VS	21.46	44.53	67.78	106.48	38.93
	BS-CIR	30.48	53.21	78.30	119.20	48.39
	H	13.11	21.50	20.86	27.88	17.04
	H-VS	13.60	21.86	21.72	27.27	17.39
	H-CIR	17.73	24.85	26.29	36.87	18.89

This table presents the 1-day out-of-sample $MSE of call options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins. Underlined values are the smallest $MSE values within their respective bins.

Table 15: 1-Day Out-Of-Sample average $MSE on Put options on the S&P 500 index (2010-2015)

		D <60	60 ≤ D <120	120 ≤ D <180	D ≥ 180	Total Sample
S/K<0.94	*BS*	15.04	52.12	121.52	202.24	77.49
	BS-VS	18.89	92.31	212.01	604.13	184.41
	BS-CIR	18.08	46.44	122.45	282.73	94.03
	H	12.49	14.89	25.27	71.72	27.06
	H-VS	13.33	19.48	30.26	49.41	24.63
	H-CIR	17.08	17.19	27.76	84.47	25.84
0.94≤ S/K <0.97	*BS*	37.10	72.81	83.02	76.99	56.18
	BS-VS	33.94	72.34	147.76	247.70	91.56
	BS-CIR	40.39	61.52	106.39	108.83	63.67
	H	30.03	43.67	62.31	64.99	42.35
	H-VS	32.75	48.03	71.68	70.67	46.61
	H-CIR	31.39	45.85	87.76	67.83	44.48
0.97≤ S/K <1.00	*BS*	40.92	46.36	38.95	54.12	43.82
	BS-VS	27.76	41.96	61.37	168.67	52.81
	BS-CIR	34.21	38.75	39.85	59.76	39.15
	H	29.73	35.79	31.40	39.08	32.47
	H-VS	28.86	38.57	34.43	36.72	32.43
	H-CIR	29.29	49.11	32.92	50.93	32.45
1.00≤ S/K <1.03	*BS*	23.56	36.21	50.30	117.69	33.22
	BS-VS	23.76	37.85	48.08	75.48	30.74
	BS-CIR	23.27	37.00	47.94	54.04	28.82
	H	21.71	32.68	37.48	41.43	25.64
	H-VS	21.83	34.56	39.85	42.19	26.21
	H-CIR	21.77	44.51	41.15	45.62	34.47
S/K>1.06	*BS*	21.44	49.57	91.25	187.77	38.82
	BS-VS	30.80	62.67	69.15	54.45	38.99
	BS-CIR	26.23	60.81	83.78	95.35	38.38
	H	14.73	26.66	29.28	29.40	18.15
	H-VS	14.88	28.02	30.45	30.31	18.58
	H-CIR	19.72	36.22	39.63	29.86	20.41
S/K >1.06	*BS*	13.84	48.05	106.22	277.99	60.62
	BS-VS	16.04	53.55	101.82	183.32	51.29
	BS-CIR	14.82	52.32	106.68	219.99	55.14
	H	7.73	11.10	12.16	15.49	9.76
	H-VS	5.24	9.82	12.09	15.99	8.09
	H-CIR	6.48	14.16	12.13	15.74	12.18
Total Sample	*BS*	19.97	46.64	91.07	216.27	50.73
	BS-VS	21.63	51.72	90.85	166.38	47.54
	BS-CIR	20.74	49.65	90.93	167.49	46.68
	H	14.17	19.47	20.25	24.89	16.77
	H-VS	12.93	19.57	21.21	24.84	16.07
	H-CIR	13.55	26.01	21.73	24.87	18.01

This table presents the 1-day out-of-sample $MSE of put options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins. Underlined values are the smallest $MSE values within their respective bins.

Table 16: 5-Day Out-Of-Sample average $MSE on Call options on the S&P 500 index (2010-2015)

		D <60	60 ≤ D <120	120 ≤ D <180	D ≥ 180	Total Sample
S/K<0.94	*BS*	26.90	69.02	128.75	167.05	88.10
	BS-VS	19.31	50.23	107.96	178.00	80.17
	BS-CIR	26.67	64.70	131.95	228.86	102.80
	H	5.70	14.71	28.95	68.66	27.31
	H-VS	7.07	17.05	33.64	68.42	28.94
	H-CIR	6.59	20.49	35.99	79.06	33.58
0.94≤ S/K <0.97	*BS*	52.00	123.48	147.36	161.99	78.68
	BS-VS	34.80	88.91	124.84	135.04	57.06
	BS-CIR	48.35	109.16	158.22	183.50	75.36
	H	14.89	46.44	76.54	120.58	31.60
	H-VS	17.75	48.66	68.67	94.90	31.67
	H-CIR	20.59	49.99	92.70	111.57	33.81
0.97≤ S/K <1.00	*BS*	70.31	100.05	108.78	184.02	85.60
	BS-VS	49.30	81.78	91.66	102.99	60.56
	BS-CIR	61.05	94.25	108.30	134.93	74.18
	H	45.96	80.18	91.89	108.81	58.36
	H-VS	46.19	70.45	80.60	95.06	55.31
	H-CIR	58.42	93.88	112.49	133.19	61.82
1.00≤ S/K <1.03	*BS*	53.69	90.34	112.58	227.27	75.30
	BS-VS	52.41	98.44	105.85	134.19	69.38
	BS-CIR	52.74	92.62	108.22	132.75	68.47
	H	63.02	86.11	94.22	106.50	71.89
	H-VS	63.71	79.32	78.56	97.47	69.68
	H-CIR	82.36	91.03	110.54	103.67	73.25
S/K>1.06	*BS*	30.03	93.28	104.83	276.87	70.02
	BS-VS	40.68	125.95	133.14	227.27	79.74
	BS-CIR	34.45	102.71	119.42	169.09	64.98
	H	40.16	72.77	61.20	82.25	51.31
	H-VS	41.91	70.09	69.13	89.47	53.21
	H-CIR	46.35	77.58	66.85	93.39	53.61
S/K >1.06	*BS*	17.79	62.13	111.32	346.02	101.54
	BS-VS	24.72	92.24	178.12	394.49	128.64
	BS-CIR	19.25	69.62	119.05	249.26	86.49
	H	16.21	41.20	33.92	44.92	29.94
	H-VS	16.74	25.57	27.56	39.98	24.57
	H-CIR	17.67	39.99	34.18	52.35	35.80
Total Sample	*BS*	55.49	94.35	125.17	186.37	82.89
	BS-VS	41.61	78.16	111.60	161.49	67.30
	BS-CIR	50.74	88.71	128.97	191.24	79.59
	H	34.49	52.19	59.23	88.65	45.90
	H-VS	35.72	49.17	55.54	80.35	44.79
	H-CIR	35.25	54.63	67.46	91.46	48.13

This table presents the 5-day out-of-sample $MSE of call options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins. Underlined values are the smallest $MSE values within their respective bins.

Table 17: 5-Day Out-Of-Sample average $MSE on Put options on the S&P 500 index (2010-2015)

		D <60	60 ≤ D <120	120 ≤ D <180	D ≥ 180	Total Sample
S/K<0.94	*BS*	15.31	47.54	160.00	197.52	81.05
	BS-VS	16.72	65.86	260.16	497.04	162.31
	BS-CIR	16.32	46.10	171.49	275.10	98.84
	H	17.08	42.71	60.45	117.00	50.27
	H-VS	17.42	54.54	48.19	94.70	46.86
	H-CIR	18.52	54.15	61.20	124.98	54.75
0.94≤ S/K <0.97	*BS*	55.68	121.89	190.76	231.87	115.00
	BS-VS	59.52	119.09	225.08	369.24	144.79
	BS-CIR	61.08	119.49	211.07	246.91	122.08
	H	58.56	130.55	178.62	186.95	108.87
	H-VS	66.42	93.65	110.71	140.41	89.12
	H-CIR	73.69	114.30	189.14	223.75	102.49
0.97≤ S/K <1.00	*BS*	69.70	106.97	111.70	152.36	92.09
	BS-VS	63.82	104.25	138.92	250.05	103.53
	BS-CIR	67.58	106.11	118.45	163.48	92.68
	H	76.18	99.97	95.93	124.00	89.32
	H-VS	64.16	79.82	83.44	100.31	73.81
	H-CIR	76.56	120.84	100.28	121.03	79.63
1.00≤ S/K <1.03	*BS*	48.52	89.49	143.05	210.17	70.54
	BS-VS	53.45	91.77	134.52	158.75	70.69
	BS-CIR	51.90	93.00	141.07	143.99	69.12
	H	57.48	86.56	120.40	122.70	69.65
	H-VS	53.98	70.73	90.52	98.38	61.36
	H-CIR	72.68	88.15	134.88	112.72	70.93
S/K>1.06	*BS*	34.14	85.00	136.31	269.31	60.52
	BS-VS	45.10	99.84	117.20	142.56	62.71
	BS-CIR	40.00	95.82	128.52	176.32	60.78
	H	37.03	66.41	85.08	118.68	48.61
	H-VS	37.51	62.86	74.69	102.94	46.94
	H-CIR	41.78	75.03	105.69	147.04	50.62
S/K >1.06	*BS*	16.33	59.26	111.16	308.46	68.41
	BS-VS	18.64	65.57	106.48	211.12	59.05
	BS-CIR	17.46	63.63	111.65	245.71	62.51
	H	15.16	28.27	33.53	49.53	23.63
	H-VS	12.42	25.88	33.71	47.39	21.26
	H-CIR	17.90	30.42	38.40	59.92	23.10
Total Sample	*BS*	31.51	72.92	120.35	269.59	70.67
	BS-VS	35.23	78.70	119.44	213.02	68.20
	BS-CIR	33.73	77.25	120.92	217.81	67.56
	H	33.92	51.15	59.29	78.27	43.73
	H-VS	30.87	44.13	51.44	67.82	38.74
	H-CIR	34.17	52.12	59.26	83.03	46.36

This table presents the 5-day out-of-sample $MSE of put options on the S&P 500 index using the Black-Scholes and Heston models and their stochastic interest rate extensions. The sample period is from 01/01/2010 to 31/12/2015. *BS* stands for the Black-Scholes (1973) model and *H* for the Heston (1993) model. *VS* and *CIR* stands for the stochastic interest rate models of Vasicek and Cox-Ingersroll-Rox, respectively. Moneyness is defined as S/K where S stands for the spot price and K is the strike price. D is the duration in days and represents the time to maturity. The sample is divided into six moneyness bins and four maturity bins. Underlined values are the smallest $MSE values within their respective bins.