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Extensions to SCAR Models: Modeling the Dependence Structure in Equity Returns

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Abstract

Despite their drawbacks at higher dimensions, SCAR models are currently being used at large and they perform quite well as compared to their competitors. This paper investigates possible extensions on SCAR models and aims to ultimately find a extension which outperforms the original specification. We empirically apply SCAR to real-world data using daily returns from two popular indices, the Dow Jones Industrial Average and NASDAQ. First, we relax the assumption of Gaussian marginal distributions errors. The Student's t-distribution better describes the stylized facts of asset returns such as the leptokurtosis and we find out that the performance of SCAR significantly increases in this scenario. Another extension we consider is to combine the best-performing copulas into a mixed copula model which captures the dynamics of stock returns better. The mixed copula SCAR only outperformed the original specification during times of significant market occurrences where the combined copula takes into account greater dependence for losses. This research provides the current literature with two extensions of the SCAR model which predominantly outperforms the original model.

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1 Introduction

Modeling volatility and dependencies is of crucial importance in portfolio management, asset allocation, derivative pricing and many other areas of finance. Correctly estimating the correlation structure that exists between different stocks would help investors to allocate high-performing portfolios through hedging and diversification. Moreover, the pricing of derivatives is based on the price dynamics of their underlying assets, which in turn are largely affected by their volatility and these common dependencies. Volatility plays a crucial role in explaining the uncertainty that underlies the returns of financial assets, therefore research on the modeling and forecasting of volatility has been extensive for many years now. Ever since the introduction of the autoregressive conditional heteroskedastic (ARCH) model by Engle (1982) and the generalised ARCH (GARCH) model by Bollersev (1986), there has been a plethora of research and extensions on these volatility models. The literature on multivariate GARCH models (Bauwens, Laurent and Rombouts, 2006) made the transition from univariate models which allowed for no correlation between variables, to a multivariate setting of codependence.

The more recent multivariate stochastic volatility (SV) models proposed by Harvey, Ruiz and Shephard (1994) have become very popular in finance. Even though GARCH-type models continue to appeal because of their computational simplicity, SV models have shown a higher flexibility and goodness-of-fit. Yu and Meyer (2006) introduce stochastic correlations to SV models by using a Gaussian AR(1) process to model correlation. Although highly flexible, a major drawback of these specifications is that they only describe linear dependencies between variables by assuming a multivariate normal distribution. In practice, non-elliptical distributions and asymmetries in the tail dependencies are quite common, especially in equity returns. The joint distribution of returns is non-elliptical because of asymmetries in the tail dependence. The lower tail dependence is usually higher than the upper tail dependence. One way to account for these nonlinear dependencies is through copulas.

Copulas capture the dependence among groups of random variables by decoupling it from the marginal distributions. An important issue that arose in the early implementation of copulas in finance was the unrealistic assumption of constant dependence parameters. Copula models were criticized severely during the financial crisis of 2008, mostly for not being able to adequately capture the risk in portfolios of credit default swaps and mortgages (Creal and Tsay, 2015). Joe (1997) gives a good introduction to copulas, dependence measures, the different parametric copula families, their estimation and inference. A remarkable extension of copula theory has been made by Patton (2006) who introduced time-varying parameters to copula models. Manner and Reznikova (2011) give a review of bivariate time-varying copula models. For an application of copulas in econometrics see Patton (2009) and for their use in the field of risk management see McNeil, Frey and Embrechts (2005). Ever since, a bundle of stochastic copula models have emerged, from Markov-switching models (Pelletier, 2006) to observation-driven (Creal, Koopman and Lucas, 2013) and parameter-driven models (Hafner and Manner, 2012).

SV models are the most common parametric-type specification. In such models, parameters are stochastic processes on their own and as such they are not perfectly observable or predictable given past information. Their parameter estimation is also not as trivial as compared to observation-driven models, such as GARCH-type specifications. In most cases, numerical or analytical approaches for estimation are impossible or highly inefficient. Instead, simulation techniques such as the efficient importance sampling (EIS) by Liesenfeld and Richard (2003) and Markov chain Monte Carlo (MCMC) are necessary. What makes parametric-models appealing is the high flexibility and goodness-of-fit. Harvey, Ruiz and Shephard (1994) were the first to introduce SV models in a multivariate setting, while Yu and Mayer (2006) were the first to introduce a stochastic correlation model following a Gaussian AR(1) specification. Asai and Mcleer (2009) have introduced a DCC model with stochastic correlations driven by a VAR(1) process. By assuming a multivariate normal distribution, these specifications can only describe linear dependencies between variables and thus fail to accurately capture the dependence structure.

Accounting for both time-varying dependence and non-elliptical distributions, Hafner and Manner (2012) propose the stochastic copula autoregressive (SCAR) model. The SCAR model is a dynamic copula model in which the dependence parameters follow a latent Gaussian AR(1) process. First, it fits the marginal distribution of the variables using a Gaussian AR(1) Stochastic Volatility (SV) model. Hafner and Manner (2012) compare the marginal SV model with a GARCH(1,1) model, and they find that the SV model provides a better fit and higher flexibility. This is mainly because the SV specification depends on two error processes instead of one, unlike GARCH. Second, they introduce copulas to decouple the dependence structure from the marginal distributions. The SCAR model can be seen as a generalized SV model which introduces a dynamic copula model to model the dependencies between variables. Copulas model the multivariate distribution and the dependence between two or more variables, regardless of the underlying univariate distributions. Introducing copulas to capture the dependence among groups of random variables regardless of the underlying marginal distributions, simplifies the estimation of the parameters into two steps. First we estimate the parameters of the marginal distributions (which are by definition independent of the dependence parameters), and then the parameter estimates of the copula model. Combining dynamic copulas with the SV model is what makes the SCAR model extremely valuable.

SCAR is found to capture the correlation dynamics quite well under a variety of deterministic and stochastic data-generating processes. Hafner and Manner (2012) compare SCAR with the DCC-GARCH(1,1) model of Engle (2002) and the conditional copula model of Patton (2006), and find that it largely outperforms both models even under misspecified data-generating processes, which one would expect to favour the other (observation-driven) models instead. Ever since it was introduced, the SCAR model has gained popularity in the current literature. On the other hand, there are drawbacks to this model. Multivariate SV models are estimated using techniques such as the efficient importance sampling (EIS) of Liesenfeld and Richard (2003) and Richard and Zhang (2007) or Markov chain Monte Carlo (MCMC) simulations and these require the evaluation of a high-dimensional integral. Hence, extending the SCAR model to larger dimensions other than bivariate would be extremely challenging and inefficient. This is the main reason why the literature on GARCH is more extensive and why GARCH-type models are preferred empirically. Computational complexity aside, SCAR is found to outperforms most of its competitors. Hence, we focus our research in modeling the dependence structure for the bivariate case only. Therefore, we decide to restrict our analysis to the bivariate SCAR model and encourage future research to look further on the matter.

As previously mentioned, the SCAR specification adequately models the multivariate distribution and the dependence structure of different variables. Thereby, it is worthwhile to look into potential extensions that can be associated with SCAR. Our goal is to ultimately find a modified SCAR specification that performs better than the original model. This research focuses on two main fronts. First, the SCAR model assumes Gaussian marginal distributions, which in practice is an unrealistic assumption to make. Typically, financial asset returns show significant evidence of serial correlation, leptokurtosis and volatility clustering. Volatility spillover effects and asymmetries on the conditional variance are also found to be present. Hence, choosing a heavy-tailed distribution instead of a normal one seems more than appropriate for the modeling the marginal distribution of financial data. Second, Hafner and Manner (2012) conclude that the Gaussian copula outperforms its non-elliptical competitors. This result is surprising as we expect returns to have a higher lower-tail dependence. As the Gaussian copula does not account for the tail dependence, it should underestimate potential losses during an adverse economic climate. Hence, non-elliptical distributions should empirically be a better fit in uncertain times of high-volatility. Gaussian copulas, on the other hand, are known to perform better during "tranquil" market periods. One possibility is to combine the best-fitting copulas into a single mixed copula model which potentially captures the dynamics of stock returns better.

This study contributes to the previous literature in several fronts. In the first part of our analysis, we look into the SCAR model and empirically apply it to real-world data. Similarly to Hafner and Manner (2012), we use daily returns from two popular indices, the Dow Jones Industrial Average and NASDAQ. As the sample from Hafner and Manner (2012) is limited to the period from 1990 until 2000, including recent data tests the consistency of SCAR and whether the conclusions from the paper are still valid nowadays. However, we obtain the same results. The Gaussian copula outperforms all its competitor copulas, regardless of time. Secondly, we relax the assumption of normal (univariate) marginal distributions errors, which can better describe the stylized facts of asset returns such as the heavy tailed distribution. Instead of assuming the same distribution for all returns, we determine the degrees of freedom which result in the best-fitting t-distribution for each specific dataset. We find out that the normal distribution is a very poor approximation of univariate return data and the t-distribution has a greater goodness-of-fit for equity return data. Another extension we consider is to combine the best-performing copulas into a single mixed copula model which captures the dynamics of stock returns better. Generally, the mixed copula SCAR model outperforms the original SCAR only for high-volatile periods where the mixed copula takes into account greater dependence for losses.

The paper is structured as follows. As copulas are essential in understanding the methodology behind the SCAR model, Section 2 begins with an introduction to copulas, their dependence measures and a brief survey of some of the most popular copula models. Section 3 describes the specification, estimation and the distributions of margins for the SCAR model, as introduced by Hafner and Manner (2012). In Section 4, we provide the methodology behind the extensions for the SCAR model that we propose. In the first extension, we relax the normality assumption by assuming a Student's t distribution for returns. In the second extension, we introduce mixed copula models to SCAR, to better distinguish between low and highly-volatile market periods. Once the methodology is established, we want to empirically apply these models to real-world data. A detailed description of the data that we use and how we make use of them can be found in Section 5. Our aim is to find an extension that performs better than the original SCAR. Section 6 starts with an overview of SCAR and the the main results by Hafner and Manner (2012). Later on, we obtain the main results for our own extensions and we compare with the original model. Suggestions for further research and some limitations we encountered are presented in Section 7. Finally, we conclude this paper with our closing remarks in Section 8.

2 Copulas

Copulas have been quite popular in quantitative finance, in particular within risk management, portfolio optimization and derivative pricing. They are widely used to model and manage downside risk and perform stress-tests to determine the value of potential losses one could occur during times of uncertainty. Before delving into the dynamic stochastic copula model, a detailed description of copulas is needed to better understand the latter methodology. Hence, we dedicate the next section to copulas, the most common copula families and their dependence measures.

2.1 Definition

Let x and y be two random variables with continuous marginal distributions F and G. Using the probability integral transform, any random variable with a continuous distribution can be converted to a random variable with a standard uniform distribution, U(0,1). Hence, we can define $u = F(x)$ and $v = G(y)$, such that $u, v \sim U(0, 1)$.

A copula is a multivariate cumulative distribution function with uniform marginals which captures the dependence structure between variables irrespective of their marginal distribution. Assuming $H(x, y)$ is the joint distribution function of x and y, Sklar (1959)'s theorem states that there exists a unique copula C such that

$$H(x, y) = C(F(x), G(y)) = C(u, v) \tag{2.1}$$

where $C : [0, 1]^2 \to [0, 1]$. In statistical terms, a copula of (x, y) is defined as the joint cumulative density function (CDF) of (u, v). Hence,

$$\begin{aligned} C(u, v) &= P[U \leq u, V \leq v] \\ &= P[X \leq F^{-1}(u), Y \leq F^{-1}(v)]. \end{aligned} \tag{2.2}$$

Sklar (1959) introduces an important property of copulas in which they decouple the dependence between variables from their marginal distributions. Hence,

$$h(x, y) = f(x) \cdot g(y) \cdot c(F(x), G(y)) \tag{2.3}$$

where f and g are the marginal density functions of x and y respectively, while c is the density function of the copula. Note that this can also be generalized for the multivariate case with more than two variables, but our research only focuses on the bivariate case. A remarkable extension of copula theory has been made by Patton (2006) who introduced time-varying parameters to copula models. A time-varying bivariate copula model is defined as follows

$$(u_t, v_t) \sim C(u, v|\theta_t) \tag{2.4}$$

where $\theta_t \in \Theta \subset \mathbb{R}$ is a stochastic random parameter which captures the dependence between variables. Suppose there is an underlying process λ_t, such that $\theta_t = \Psi(\lambda_t)$, where $\Psi : \mathbb{R} \to \Theta$ is a functional form dependent on the copula. This transformation restricts θ_t to remain in the appropriate domain, Θ.

2.2 Families of Copulas

In this section, we discuss some popular copula models which have a single parameter. Gaussian copulas construct a multivariate normal distribution from marginal univariate normal distributions. Archimedean copulas are an associative class of copulas that can model the dependence in high-dimensions with very few parameters. A copula C is Archimedean if it has the following form

$$C(u, v; \theta) = \psi^{[-1]}(\psi(u; \theta) + \psi(v; \theta)) \tag{2.5}$$

where $\psi : [0, 1] \times \Theta \to [0, \infty)$ and $\psi(1; \theta) = 0$. The function $\psi^{[-1]}$ is its pseudo-inverse specified as

$$\psi^{[-1]}(t; \theta) = \begin{cases} \psi^{-1}(t; \theta) & \text{if} \quad 0 \leq t \leq \psi(0; \theta) \\ 0 & \text{if} \quad \psi(0; \theta) \leq t \leq \infty. \end{cases} \tag{2.6}$$

Some Archimedean copulas used in this paper are the Frank, Clayton and Gumbel copulas. These models will be further explained in the sections below. Figure 1 gives a visual representation of the probability density functions (PDFs) of the different copulas.

2.2.1 Gaussian Copula

Let $x = \Phi^{-1}(u)$ and $y = \Phi^{-1}(v)$ be two random variables where $\Phi(\cdot)$ denotes the standard normal cumulative density function. The Gaussian copula is defined as

$$C^{Gauss}(u, v; \theta) = \Phi(\Phi^{-1}(u), \Phi^{-1}(v); \theta) \tag{2.7}$$

with density function

$$c^{Gauss}(u, v; \theta) = \frac{1}{\sqrt{1 - \theta^2}} \exp\left(\frac{2\theta xy - x^2 - y^2}{2(1 - \theta^2)} + \frac{x^2 + y^2}{2} \right) \tag{2.8}$$

where $\theta \in (-1, 1)$ is the copula parameter. The appropriate transformation function for Gaussian copulas is $\theta_t = \Psi(\lambda_t) = (\exp(2x) - 1)/(\exp(2x) + 1)$.

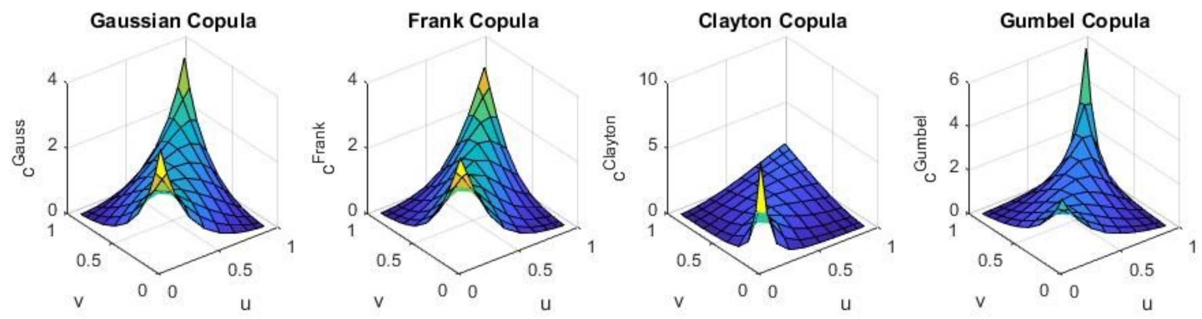

Figure 1: Probability density functions (PDFs) of copulas

2.2.2 Frank Copula

The density function of the Frank copula is

$$c^{Frank}(u, v; \theta) = \frac{\exp((1 + u + v)\theta)(\exp(\theta) - 1)\theta}{\{\exp(\theta) + \exp((u + v)\theta) - \exp(\theta + u\theta) - \exp(\theta + v\theta)\}^2} \tag{2.9}$$

where $\theta \in \mathbb{R} \setminus \{0\}$. The transformation function for Frank copulas is then $\theta_t = \Psi(\lambda_t) = \lambda_t$.

2.2.3 Clayton Copula

The density function of the Clayton copula is

$$c^{Clayton}(u, v; \theta) = u^{(-1-\theta)}v^{(-1-\theta)}(u^{-\theta} + v^{-\theta} - 1)^{(-2-1/\theta)}(1 + \theta) \tag{2.10}$$

where $\theta \in (0, \infty)$. For the copula parameter to remain in its appropriate domain, the transformation function for the Clayton copula is $\theta_t = \Psi(\lambda_t) = \exp(\lambda_t)$.

2.2.4 Gumbel Copula

The density function of the Gumbel copula is

$$\begin{aligned} c^{Gumbel}(u, v; \theta) = &\frac{\{\log(u)\log(v)\}^{(\theta-1)}\{[(-\log(u))^{\theta} + (-\log(v))^{\theta}]^{1/\theta} + \theta - 1\}}{[(-\log(u))^{\theta} + (-\log(v))^{\theta}]^{(2-1/\theta)}uv} \times \\ &\exp\{[(-\log(u))^{\theta} + (-\log(v))^{\theta}]^{1/\theta}\} \end{aligned} \tag{2.11}$$

where $\theta \in [1, \infty)$. The transformation function for Gumbel copulas is $\theta_t = \Psi(\lambda_t) = \exp(\lambda_t) + 1$.

2.3 Dependence Measures: Kendall's τ and Tail Dependence

Comparing copula parameters with each other is not straightforward because they are usually scaled differently for every type of copula. Instead, there exist some copula-based coefficients, which do not depend on the scale or the marginal distributions, and measure the dependence between variables. A well-known rank correlation coefficient that measures the overall dependence is the Kendall's τ, defined as

$$\tau = 4E(C(U, V)) - 1 \tag{2.12}$$

where $\tau \in [-1, 1]$. Other important copula-based coefficients measure the tail dependence, i.e. in the extremes of the distribution. The lower and upper tail dependence are measured as

$$\lambda_L = \lim_{u \to 0} p[U < u | V < v] = \lim_{u \to 0} \frac{C(u, u)}{u} \tag{2.13}$$

$$\lambda_U = \lim_{u \to 1} p[U > u | V > v] = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u} \tag{2.14}$$

Both dependence measures can often be expressed in terms of the copula parameters. See Table 1 for a summary of these relationships between the dependence measures and parameters, as well as the transformation equations for each type of copula and the corresponding domain. Figure 2 shows contour plots for distribution functions using the copulas mentioned previously with a Gaussian marginal distribution and a Kendall's τ of 0.5. Both the Gaussian and the Frank copula are symmetric along the 45° line, but the Frank copula weights heavier along the tails. Financial returns usually depend more on the left tail than on the right, hence symmetry is not a desirable property. The Gumbel and Clayton copula, on the other hand, should be better suited for financial returns as they are asymmetric. The Clayton copula has only lower tail dependence, while the Gumbel copula has only upper tail dependence. As financial returns typically have a higher lower tail dependence, the Gumbel copula is used in its reversed version instead, also known as the survival Gumbel copula.

Table 1: The dependence, transformation process and domain of different copula

	Overall Dependence	Tail Dependence		Transformation	Domain
Copula	Kendall's τ	Lower λ_L	Upper λ_U	$\Psi(\lambda_t)$	θ_t
Gaussian	$\tau = \frac{2}{\pi} \arcsin(\theta)$	-	-	$\Psi(\lambda_t) = \frac{\exp(2\lambda_t)-1}{\exp(2\lambda_t)+1}$	$\theta_t \in (-1, 1)$
Frank	$\tau = 1 + \frac{4(D_1(\theta)-1)}{\theta}$	-	-	$\Psi(\lambda_t) = \lambda_t$	$\theta_t \in \mathbb{R} \setminus \{0\}$
Clayton	$\tau = \frac{\theta}{\theta+2}$	$\lambda_L = 2^{-1/\theta}$	$\lambda_U = 0$	$\Psi(\lambda_t) = \exp(\lambda_t)$	$\theta_t \in (0, \infty)$
Gumbel	$\tau = 1 - \frac{1}{\theta}$	$\lambda_L = 0$	$\lambda_U = 2 - 2^{1/\theta}$	$\Psi(\lambda_t) = \exp(\lambda_t) + 1$	$\theta_t \in [1, \infty)$

For the Frank copula $D_k(x)$ denotes the Debye function defined as $D_k(x) = \frac{k}{x^k} \int_0^x \frac{t^k}{e^t - 1} dt$. Note that the Gaussian and Frank copula do not exhibit any tail dependence.

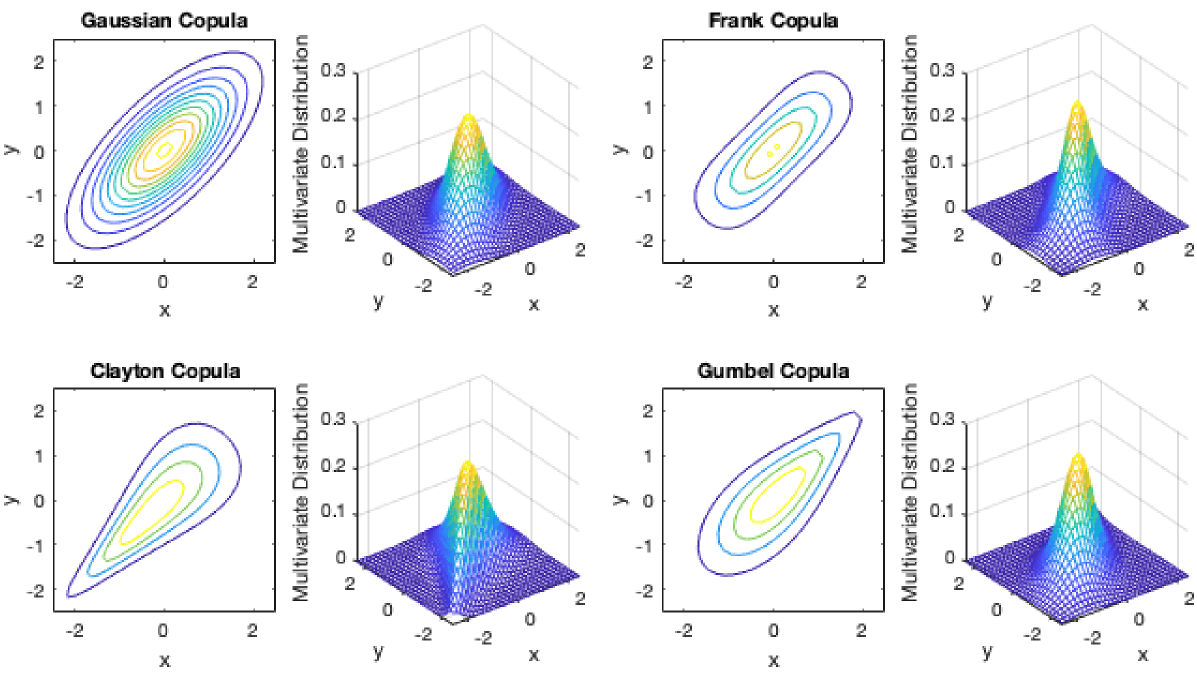

Figure 2: Contour plots of the copulas and their multivariate distributions, assuming standard normal marginal distributions and Kendall's $\tau = 0.5$.

3 The Stochastic Copula Autoregressive (SCAR) Model

The Stochastic Copula Autoregressive (SCAR) model was introduced by Hafner and Manner (2012) as a generalized multivariate stochastic volatility (SV) model which introduces a dynamic copula model with an underlying dependence parameter that follows a Gaussian autoregressive process. The benefit of using a copula model is that it decouples the marginals from the dependence parameters, which makes the estimation of SV models computationally easier. The estimation of SCAR models can be done in two steps, first estimating the parameters of the marginals, then those of the copula model. In this section, we describe in detail each component that makes up the SCAR model, as well as the estimation procedure used for its underlying processes.

3.1 Marginals

Given that we are dealing with financial data, it is crucial to find appropriate marginal distribution models to accurately represent the data. In particular, stock market returns exhibit certain properties such as time-varying volatility and leptokurtosis that must be captured by the underlying model. An adequate candidate for this is the stochastic volatility (SV) model introduced by Clark (1973) and Taylor (1986). The returns for the ith stock and $t = 1, \ldots, T$, are then modeled as

$$\begin{aligned} r_{it} &= \exp{(h_{it}/2)}\varepsilon_{it} \\ h_{it} &= \delta_i + \gamma_i h_{it-1} + \sigma_i \eta_{it} \end{aligned} \tag{3.1}$$

where ε_{it} and η_{it} are assumed to be two independently and identically distributed standard normal random variables. Given the SV model and that the innovations $\varepsilon_{it} = \exp(-h_{it}/2)r_{it}$ are normally distributed with zero mean and unit variance, according to Sklar's theorem, the observation equation becomes

$$(r_{it}, r_{jt}) \mid \lambda_t, h_{it}, h_{jt} \sim C(\Phi(\varepsilon_{it}), \Phi(\varepsilon_{jt})) \tag{3.2}$$

where $\Phi(\cdot)$ denotes the CDF of the standard normal distribution and λ_t the dependence parameter, which is decoupled from the marginal distributions by the copula C and needs to be specified. Using the probability integral transform, the data is converted into U(0,1) random variables and it is ready to be used for estimating the stochastic copula model.

3.2 Stochastic Copula Model

Consider again the bivariate stochastic copula model in (2.4),

$$(u_t, v_t) \sim C(u, v|\theta_t)$$

where θ_t captures the dependence between variables. Assume that θ_t is driven by an underlying process λ_t through a transformation function Ψ, such that $\theta_t = \Psi(\lambda_t)$. This stochastic process is unobserved, hence it is of crucial importance to choose an appropriate specification to model it. In the dynamic copula model presented by Hafner & Manner (2012), the underlying dependence parameter λ_t is assumed to follow an autoregressive process of order one, AR(1),

$$\lambda_t = \alpha + \beta\lambda_{t-1} + \nu\varepsilon_t \tag{3.3}$$

where $\varepsilon \sim N(0,1)$. Furthermore, $|\beta| < 1$ is set as a stationary condition. Let $X = \{x_t\}_{t=1}^T$ and $Y = \{y_t\}_{t=1}^T$ denote two processes with marginal distributions $F(X;\delta_X)$ and $G(Y;\delta_Y)$ where δ_X and δ_Y denote the parameter vectors for the respective marginals of X and Y. Given ω is the parameter vector for the copula model, the joint log-likelihood function becomes a sum of the marginal log-likelihood functions and the copula log-likelihood function.

$$\mathcal{L}(\delta_X, \delta_Y, \omega; X, Y) = \mathcal{L}_X(\delta_X; X) + \mathcal{L}_Y(\delta_Y; Y) + \mathcal{L}_C(\omega; F(X; \delta_X), G(Y; \delta_Y)) \tag{3.4}$$

Then, a two-step estimation procedure can be used to estimate the parameters to reduce the computational complexity. First, we estimate the parameters of the marginals, δ_X and δ_Y, then the parameters of the copula model, ω. This procedure is known as the inference function for the margins (IFM) estimator, see Joe (1997).

3.3 Estimation

Let $\omega = (\alpha, \beta, \nu)$ be the parameter vector of the transition equation. As the latent parameter λ_t is assumed to be unobserved, we want to estimate its underlying dependence process as described in Equation (3.3). Define the observable variables $U = \{u_t\}_{t=1}^T$, $V = \{v_t\}_{t=1}^T$ and the latent process $\Lambda = \{\lambda_t\}_{t=1}^T$. If $f(U, V, \Lambda; \omega)$ denotes the joint probability density function (PDF) of (U, V) and Λ, then the parameter vector ω has the following likelihood function:

$$L(\omega; U, V) = \int f(U, V, \Lambda; \omega) d\Lambda. \tag{3.5}$$

By expressing the joint density function of (U, V) and Λ as a factorization of the conditional densities and defining $U_t = \{u_\tau\}_{\tau=1}^t$ (analogously for V_t and Λ_t), we obtain

$$L(\omega; U, V) = \int \prod_{t=1}^T f(u_t, v_t, \lambda_t | U_{t-1}, V_{t-1}, \Lambda_{t-1}, \omega) d\Lambda \tag{3.6}$$

The high-dimensional nature of this integral requires quite the computational effort and as such it is very inefficient to be evaluated numerically or analytically. Liesenfield and Richard (2003) and Richard and Zhang (2007) propose a simulation technique to avoid this problem, known as the Efficient Importance Sampling (EIS). This method constructs an importance sampler $m(\lambda_t|\Lambda_{t-1}, a_t)$ that exploits information on Λ contained in U and V through some auxiliary parameter a_t. Using the importance sampler, the likelihood function becomes

$$L(\omega; U, V) = \int \prod_{t=1}^T \left[\frac{f(u_t, v_t, \lambda_t | U_{t-1}, V_{t-1}, \Lambda_{t-1}, \omega)}{m(\lambda_t | \Lambda_{t-1}, a_t)} \right] \prod_{t=1}^T m(\lambda_t | \Lambda_{t-1}, a_t) d\Lambda. \tag{3.7}$$

Using Monte Carlo simulation and drawing N trajectories $\{\tilde{\lambda}_t^{(i)}(a_t)\}_{t=1}^T$ from the importance sampler m, we can evaluate the likelihood function as

$$\tilde{L}_N(\omega; U, V) = \frac{1}{N} \sum_{i=1}^N \left(\prod_{t=1}^T \left[\frac{f(u_t, v_t, \tilde{\lambda}_t^{(i)}(a_t) | U_{t-1}, V_{t-1}, \Lambda_{t-1}, \omega)}{m(\tilde{\lambda}_t^{(i)}(a_t) | \tilde{\Lambda}_{t-1}^{(i)}(a_{t-1}), a_t)} \right] \right) \tag{3.8}$$

Of crucial importance are the specification of the importance sampler $m(\lambda_t|\Lambda_{t-1}, a_t)$ and the estimation of the auxiliary parameters a_t, where $a_t = (a_{1,t}, a_{2,t})$ in the bivariate case. To reduce the simulation variance of the likelihood function, Liesenfeld and Richard (2003) make use of a functional approximation $k(\Lambda_t; a_t)$ for f such that

$$m(\lambda_t | \Lambda_{t-1}, a_t) = \frac{k(\Lambda_t; a_t)}{\chi(\Lambda_{t-1}; a_t)} \tag{3.9}$$

where $\chi(\Lambda_{t-1}; a_t) = \int k(\Lambda_t; a_t) d\lambda_t$. Furthermore, they introduce the following decomposition

$$k(\Lambda_t; a_t) = p(\lambda_t | \lambda_{t-1}, \omega) \zeta(\lambda_t, a_t) \tag{3.10}$$

where $\zeta(\lambda_t, a_t)$ is a Gaussian kernel. The SV model assumes that the conditional density of λ_t given it's past values follows a Gaussian distribution. Hence,

$$p(\lambda_t|\lambda_{t-1}, \omega) \propto \exp\left\{ -\frac{1}{2\nu^2}(\lambda_t - \gamma - \delta\lambda_{t-1})^2 \right\}. \tag{3.11}$$

For each period $t = T, \ldots, 1$, given ω and $\chi(\Lambda_T; a_{T+1}) \equiv 1$, we solve the following minimization problem

$$\begin{aligned} \hat{a}_t = \ \arg\min_{a_t} \sum_{i=1}^{N} \big(& \log[f(u_t, v_t, \tilde{\lambda}_t^{(i)}(\omega)|U_{t-1}, V_{t-1}, \tilde{\Lambda}_{t-1}^{(i)}(\omega), \omega) \cdot \chi(\tilde{\Lambda}_t^{(i)}(\omega); \hat{a}_{t+1})] \\ & - c_t - \log k(\tilde{\Lambda}_t^{(i)}(\omega); a_t)\big)^2 \end{aligned} \tag{3.12}$$

If we choose a Gaussian kernel equal to $\zeta(\lambda_t, a_t) = \exp(a_{1,t}\lambda_t + a_{2,t}\lambda_t^2)$, then $k(\Lambda; a_t)$ has the following exponential functional form

$$k(\Lambda_t; a_t) \propto \exp\left\{ -\frac{1}{2}\left[\left(\frac{\gamma + \delta\lambda_{t-1}}{\nu}\right)^2 - 2\left(\frac{\gamma + \delta\lambda_{t-1}}{\nu^2} + a_{1,t}\right)\lambda_t + \left(\frac{1}{\nu^2} - 2a_{2,t}\right)\lambda_t^2 \right] \right\} \tag{3.13}$$

The conditional mean and variance of the importance sampler $m(\lambda_t|\Lambda_{t-1}, a_t)$ then become

$$\mu_t = \sigma_t^2\left(\frac{\gamma + \delta_{t-1}}{\nu^2} + a_{1,t}\right), \quad \sigma_t^2 = \frac{\nu^2}{1 - 2\nu^2 a_{2,t}}. \tag{3.14}$$

Furthermore, integrating $k(\Lambda_t; a_t)$ with respect to λ_t and omitting multiplicative factors for simplicity results in

$$\chi(\lambda_{t-1}, a_t) \propto \exp\left\{ \frac{\mu_t^2}{2\sigma_t^2} - \frac{(\gamma + \delta_{t-1})^2}{2\nu^2} \right\} \tag{3.15}$$

Ultimately, the minimization problem becomes the linear least squares problem in Equation (3.16) below. This facilitates estimation and greatly reduces the computational complexity of the optimization problem. Unfortunately, this is not the case for non-Gaussian latent processes.

$$\log c(u_t, v_t|\theta_t(\omega)) + \log \chi(\tilde{\lambda}_t^{(i)}(\omega); \hat{a}_{t+1}) = c_t + a_{1,t}\tilde{\lambda}_t^{(i)}(\omega) + a_{2,t}[\tilde{\lambda}_t^{(i)}(\omega)]^2 + \eta_t^{(i)} \tag{3.16}$$

This linear problem can be solved by Ordinary Least Squares (OLS) with c_t the regression constant and $\eta_t^{(i)}$ the error term. The EIS procedure then works as follows:

1. Draw N trajectories $\{\tilde{\lambda}_t^{(i)}(\omega)\}_{t=1}^T$ from the natural sampler $p(\lambda_t|\lambda_{t-1}, \omega)$.
2. Estimate the auxiliary parameters $\hat{a}_t = (\hat{a}_{1,t}, \hat{a}_{2,t})$ recursively for $t = T, \ldots, 1$ by using OLS on (3.16). Recall that $\chi(\Lambda_T; a_{T=1}) \equiv 1$.
3. Draw N trajectories $\{\tilde{\lambda}_t^{(i)}(\omega)\}_{t=1}^T$ from the importance sampler $m(\lambda_t|\Lambda_{t-1}, a_t)$ and re-estimate the auxiliary parameters $\{\hat{a}_t\}_{t=1}^T$, similarly as in Step 2. Iterate until convergence is reached.
4. Draw N trajectories $\{\tilde{\lambda}_t^{(i)}(\omega)\}_{t=1}^T$ from the importance sampler $m(\lambda_t|\Lambda_{t-1}, a_t)$ and estimate the likelihood function in (3.8).
5. Obtain the parameter vector $\hat{\omega} = (\hat{\alpha}, \hat{\beta}, \hat{\nu})$ of the transition equation in (3.3) by maximizing the likelihood function computed in Step 4.

Note that at each iteration, generating the same random numbers is required in order to ensure the convergence of $\{\hat{a}_t\}_{t=1}^T$ and hence the smoothness of the likelihood function. The majority of times, convergence is reached at 5 iterations or less for a sample of around 2500 observations. Furthermore, we decide to use $N = 200$ trajectories because at this number the simulation variation is negligible, while the computational time remains reasonable.

4 Extensions to SCAR Models

Hafner and Manner (2012) find that SCAR outperforms popular competitors such as the DCC model of Engle (2002) with GARCH(1,1) marginals as well as the model of Patton (2006). In particular, SV models are currently being used at large by academics and, despite their drawbacks at higher dimensions, they perform quite well in the bivariate case. Hence, in the next section we look into some extensions for the SCAR model that might improve its overall performance even more. First, we relax the assumption of normality that the marginal distributions assume in SCAR. Second, we look into mixed stochastic copula models by combining some of the highest performing (single) copula models. Last, the SCAR model assumes a Gaussian AR(1) process for its underlying dependence parameter. We extend the AR(1) model by including exogenous variables to the specification. Possible exogenous variables are trends, dummies and variables that might explain dependencies between stock returns.

4.1 Relaxing the Normality Assumption

Consider again the SV model for the marginals in Equation (3.1)

$$\begin{aligned} r_{it} &= \exp{(h_{it}/2)}\varepsilon_{it} \\ h_{it} &= \delta_i + \gamma_i h_{it-1} + \sigma_i \eta_{it}. \end{aligned}$$

In their paper, Hafner and Manner (2012) make the assumption of conditional normality of the marginal distributions of the returns. Ultimately, this means that the innovations terms ε_{it} have a standard normal distribution with zero mean and unit variance. Hence, $r_{it}|h_{it}$ is normally distributed because $r_{it} = \exp(h_{it}/2)\varepsilon_{it}$ from the SV model and $\varepsilon_{it} \sim N(0,1)$. However, financial returns are found to exhibit properties such as leptokurtosis (i.e. fat-tailed distributions) and assuming that returns are normally distributed might not be realistic. Student's t distribution is symmetric and bell-shaped like the normal distribution, but has heavier tails which account for more values in the extremes. Heavy-tailed distributions have a greater goodness-of-fit representing equity return data.

Therefore, as a first extension, we relax the assumption of normality that SCAR makes and instead assume that the returns are Student's t-distributed. Hence, $\varepsilon_{it} \sim t(df)$ where df are the degrees of freedom. Note that the degrees of freedom are to be estimated for every sample. In other words, we determine the degrees of freedom which result in the best-fitting t-distribution for that specific dataset. Compared to the normal case, we expect this distribution to represent the data quite for two main reasons. First, the t-distribution is known to represent financial returns better than the normal distribution and second, because it is data-specific through estimating the degrees of freedom, the t-distribution should provide with a better fit. Thereby, it is also expected to see an increase in the performance of SCAR. Given the SV model and that the innovations are t-distributed, the observation equation becomes

$$(r_{it}, r_{jt}) \mid \lambda_t, h_{it}, h_{jt} \sim C(F(\varepsilon_{it}), F(\varepsilon_{jt})) \tag{4.1}$$

according to Sklar's theorem, where $F(\cdot)$ denotes the CDF of the Student's t distribution and λ_t the dependence parameter, which is decoupled from the marginal distributions by the copula C, and is modeled as a Gaussian AR(1) process in the original SCAR as in Equation (3.3). We then use the probability integral transform to convert the data into U(0,1) random variables and estimate the stochastic copula model.

The stochastic copula C models the multivariate distribution of the returns given the marginal univariate distributions and the transition equation for the dependence parameter λ_t. Relaxing the assumption of normality of the marginals will result in different copula structures for the multivariate distributions. Figure 3 provides contour plots of the different selected copulas assuming a marginal Student's t-distribution. In comparison to the normally distributed case, once again we observe that the t-distribution accounts for heavier tails (compare to Figure 2). Dealing with stock returns, we expect the SCAR model using the t-distributed marginals to capture better the leptokurtosis and outperform the SCAR model under the normality assumption. Another interesting point of analysis would be whether the Gaussian copula continues to outperform the other non-elliptical copulas in the case of t-distributed returns.

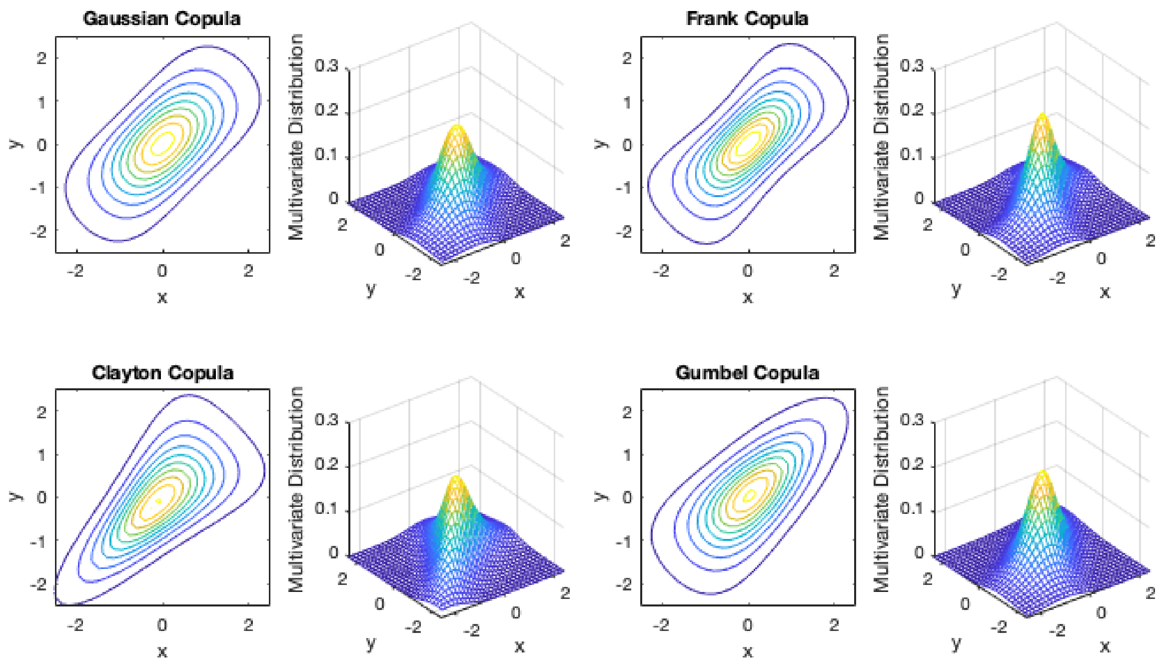

Figure 3: Contour plots of the copulas and their multivariate distributions, assuming marginal standard t-distributions with $\nu = 3$ degrees of freedom and Kendall's $\tau = 0.5$.

Once the SV model and the stochastic copula model have been established, we use the two-step procedure to estimate the SCAR model by performing EIS to each of the models separately. The EIS estimation procedure for this extension remains the same as the original SCAR under the normality condition. A detailed description of the EIS estimation procedure is given in Section 3.3.

4.2 Mixed Stochastic Copula Model

Another main result of Hafner and Manner (2012) is that the SCAR model using a Gaussian copula outperforms the other copula models that assume non-elliptical multivariate distributions. Gaussian copulas perform well, especially in times of "tranquility", where the markets are not excessively volatile and there are not many extreme values of returns. However, if we incorporate these models using return data during periods that include highly volatile market events or financial crises, Gaussian copulas might not capture the excessive extreme returns accurately. Instead, the rotated Gumbel, Frank and Clayton copula with their heavy-tailed distributions might perform better. Non-elliptical copulas are also found to better capture the dependencies between assets during bad times. In particular, stock prices have a tendency to fall together in bad times and they exhibit higher dependencies during such periods.

As our second extension, we look into mixed stochastic copula models by combining some of the highest performing (single) copula models. Consider the following specification

$$c_{mix}(u, v; \theta_1, \theta_2, \kappa_t) = \kappa_t c_1(u, v; \theta_1) + (1 - \kappa_t) c_2(u, v; \theta_2) \tag{4.2}$$

where κ_t is the mixing parameter defined as $\kappa_t = \Psi(\lambda_t) = 1/(1 + \exp(\lambda_t))$ so that $\kappa_t \in (0, 1)$. As an illustration, let's suppose the mixed copula model combines a Gaussian copula together

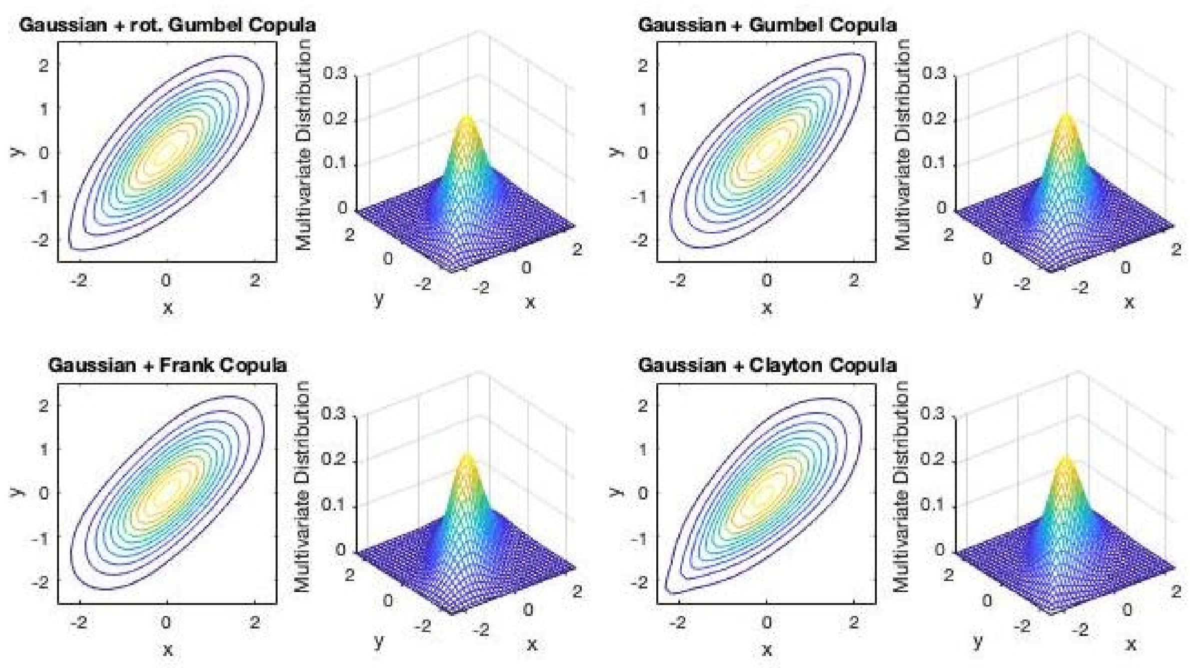

Figure 4: Contour plots of the mixed copulas and their multivariate distributions, assuming marginal normal distribution with a mixing parameter $\kappa = 0.7$ and Kendall's $\tau = 0.5$.

with the Frank copula. The two copulas have different transformation equations. The Gaussian copula has $\Psi^{Gauss}(\lambda_t) = (\exp(2\lambda_t) - 1)/(\exp(2\lambda_t) + 1)$, while the Frank copula has $\Psi^{Frank}(\lambda_t) = \lambda_t$ as transformation equation. Hence, we compute the mixed copula model as a sum of the two individual copulas by using $\theta_1 = \Psi^{Gauss}(\lambda_t)$ and $\theta_2 = \Psi^{Frank}(\lambda_t)$, respectively. Given the stochastic mixed copula model, we use the two-step procedure to estimate the SCAR model by performing EIS to each of the models separately. The EIS estimation procedure for this extension remains the same as the original SCAR, with the minor adjustment that now we incorporate a combined copula model made up of two single ones.

Figuere 4 gives the contour plots of the mixed copulas and their multivariate distributions, assuming marginal normal distribution with a mixing parameter $\kappa = 0.7$ and Kendall's $\tau = 0.5$. It is interesting to see how different combinations result in very different contour plots combining characteristics from both individual copulas. As the Gaussian copula is found to perform best in Hafner and Manner (2012), we will always consider it as the first copula. In our analysis, we will consider the following mixed copulas: Gaussian + rotated Gumbel, Gaussian + Gumbel, Gaussian + Frank and Gaussian + Clayton.

5 Data

Now that the methodology and the estimation procedure for the SCAR model and its extensions have been established, we want to empirically apply these models to real-world data. Daily returns of two well-known indices are considered: Dow Jones Industrial Average (DJ) and the NASDAQ Composite (NQ). The data used by Hafner and Manner (2012) ranges within the timeframe between March 26th, 1990 and March 23rd, 2000. The same dataset is retrieved from the Journal of Applied Econometrics Data Archive. To incorporate recent data and observe whether the conclusions from Hafner and Manner (2012) are still consistent, we expand our dataset from March 26th, 1990 until March 25th, 2019, including a total of 7363 observations from 29 years of data. The remaining data was collected from the Datastream database of historical prices for both the Dow Jones Industrial index and the NASDAQ Composite index. Note that returns are calculated by taking the first difference of the natural logarithm and multiplying by 100.

Table 2: Descriptive Statistics Dow Jones and NASDAQ

	Full Sample: 1990 - 2019								
	Mean	St. Dev.	Minimum	Maximum	Skewness	Kurtosis	JB Test	p-Value	Obs.
DJ	0.0308	1.0566	-8.20	10.51	-0.18	11.21	20,735	0.000	7,363
NQ	0.0392	1.4393	-10.17	13.25	-0.11	9.40	12,590	0.000	7,363

Table 2 summarizes the descriptive statistics for both the DJ and NQ returns. The Jarque Bera (JB) test is a goodness-of-fit test of whether the data is normally distributed. The JB test statistics and the corresponding p-values are given above.

The descriptive statistics for these returns can be found in Table 2. The average return for both indexes is around 0.03 for this particular sample, however NQ returns are somewhat more volatile as their standard deviation is higher than the DJ returns. Both index returns have a negative skewness indicative of a heavy-tailed distribution from the left, very typical of return data. Furthermore, they exhibit excess kurtosis, also known as leptokurtosis, which shows that these returns have highly-peaked distributions. When the distribution exhibits skewness and a high kurtosis (larger than 3), it means that the data does not follow a normal distribution. Another way to test if the returns are normally distributed is to make use of the Jarque-Bera (JB) test, which also depends on the skewness and kurtosis. The JB test statistics are indeed found to be quite high and the p-values are very small, close to zero, thus rejecting the hypothesis of normally distributed returns at a 10%, 5% and 1% significance level. These statistics are in line with the stylized properties of returns.

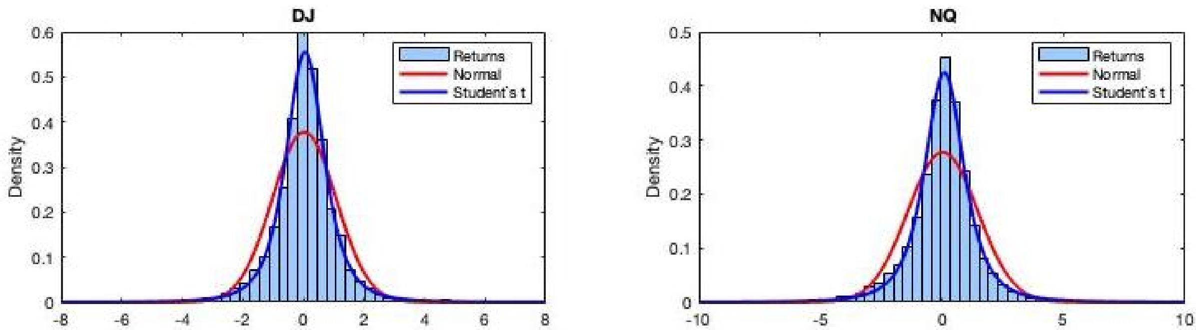

Figure 5: Normal against Student's t-distribution fit for NQ returns

Figure 5 plots a histogram of the DJ and NQ daily returns, as well as its normal and Student's t-distribution fits. Visually, the t-distribution seems to better represent the data as compared to the normal distribution. The estimated location and scale parameters for both the normal and t-distribution fits are provided in Table 3. In the normal case, the location and scale parameters correspond to the mean and standard deviation. This is a special case and as such it is not generally true for other distributions. For the t-distribution, there is another parameter to be estimated, the degrees of freedom (*df*). The DJ index returns seem to fit best to a t-distribution with 2.99 degrees of freedom, while the NQ returns are represented best by using *df* of 2.65. Furthermore, we perform a Kolmogorov-Smirnov (KS) test to empirically compare the two distribution fits. From the KS statistics, we find out that the normal distribution is a very poor representation for the returns. The KS test rejects the normality assumption at the 10%, 5% and 1% significance level for both indexes. For the t-distribution, on the other hand, we see that the KS test cannot reject the null hypothesis of t-distributed returns for DJ, while for NQ it rejects it at a 5% significance level, but not at 1%. Hence, the KS test confirms our initial expectations. The Student's t-distribution is a better fit for return data than the normal distribution.

Table 3: Distribution Fit Parameters and Test

	Full Sample: 1990 - 2019								
	Normal Distribution				Student's t Distribution				
	Parameters				Parameters				
	Location	Scale	KS Test	p-Value	Location	Scale	*df*	KS Test	p-Value
DJ	0.0308	1.0566	0.0817	0.000	0.0576	0.6614	2.99	0.0128	0.175
NQ	0.0392	1.4393	0.0561	0.000	0.1018	0.8565	2.65	0.0227	0.001

For both the DJ and NQ returns, the location and scale parameters for two different distribution fits are given above. The *df* denotes the degrees of freedom for the Student's t-distribution. The Kolmogorov–Smirnov (KS) test is a non-parametric test that can be used to compare a sample with a reference probability distribution, such as the normal and the t-distribution. The KS Test statistics and the coresponding p-values are given in the table above.

One issue to be addressed is that by estimating the SCAR model for the entire period from 1990 until 2019, the possibility that structural breaks exist is quite high. Hence, we divide the data into three subperiods, taking as breakpoints two financial crises which resulted from the internet bubble in the early 2000s and the housing bubble of 2008. The subperiods we consider are 1990-2000, 2000-2010 and 2010-2019 and they are split up in such a way to make sure we have intervals of similar length, hence approximately 10 years of data each. Table 4 gives a summary of the descriptive statistics and JB tests in Panel A, as well as the distribution fit parameters and KS tests in Panel B, for each subsample. What is important to notice here is that the general properties discussed for the whole sample are consistent through all the subsamples. Negative skewness, leptokurtosis, the JB test rejecting the normality assumption and the KS test confirming that the t-distribution is a good fit for the returns, are properties that are once again evident, regardless of the chosen time period. Another thing to note is regarding the degrees of freedom for the t-distribution fit. In general, a $t(3)$ distribution is a very good fit for all subsamples, with the only exception in 1990-2000, where the data resembles more a $t(4)$.

Table 4: Parameter Estimates of the SCAR Model for Dow Jones and NASDAQ for the period from 1990 until 2019

Subsample 1: 1990 - 2000

Panel A: Descriptive Statistics

	Mean	St. Dev.	Minimum	Maximum	Skewness	Kurtosis	JB Test	p-Value	Obs.
DJ	0.0567	0.8934	-7.45	4.86	-0.41	8.28	2942	0.000	2471
NQ	0.0884	1.1267	-8.95	5.85	-0.58	7.78	2497	0.000	2471

Panel B: Distribution Fit Parameters and Test

	Normal Distribution				Student's t Distribution				
	Parameters				Parameters				
	Location	Scale	KS Test	p-Value	Location	Scale	*df*	KS Test	p-Value
DJ	0.0567	0.8934	0.0969	0.000	0.0688	0.6513	4.15	0.0125	0.828
NQ	0.0884	1.1267	0.0811	0.000	0.1340	0.7794	3.58	0.0234	0.133

Subsample 2: 2000 - 2010

Panel A: Descriptive Statistics

	Mean	St. Dev.	Minimum	Maximum	Skewness	Kurtosis	JB Test	p-Value	Obs.
DJ	0.0021	1.2902	-8.20	10.51	-0.01	10.52	6524	0.000	2766
NQ	-0.0134	1.8726	-10.17	13.25	-0.10	7.31	2143	0.000	2766

Panel B: Distribution Fit Parameters and Test

	Normal Distribution				Student's t Distribution				
	Parameters				Parameters				
	Location	Scale	KS Test	p-Value	Location	Scale	*df*	KS Test	p-Value
DJ	0.0021	1.2902	0.0405	0.000	0.0288	0.7915	2.88	0.0190	0.267
NQ	-0.0134	1.8726	0.1043	0.000	0.0359	1.2133	3.04	0.0256	0.053

Subsample 3: 2010 - 2019

Panel A: Descriptive Statistics

	Mean	St. Dev.	Minimum	Maximum	Skewness	Kurtosis	JB Test	p-Value	Obs.
DJ	0.0379	0.8785	-5.71	4.86	-0.50	7.34	1760	0.000	2126
NQ	0.0503	1.0639	-7.15	5.67	-0.47	6.72	1302	0.000	2126

Panel B: Distribution Fit Parameters and Test

	Normal Distribution				Student's t Distribution				
	Parameters				Parameters				
	Location	Scale	KS Test	p-Value	Location	Scale	*df*	KS Test	p-Value
DJ	0.0379	0.8785	0.1255	0.000	0.0718	0.5620	2.97	0.0199	0.365
NQ	0.0503	1.0639	0.0946	0.000	0.1024	0.7068	3.18	0.0211	0.295

6 Results

The Results section will be structured as follows. In Section 6.1, we replicate the results from Hafner and Manner (2012) using the same timeframe that they use in their paper, from 1990 until 2000. The first thing we want to check is if their conclusions are consistent through time, hence we consider extending this dataset and incorporating data until 2019. To avoid issues such as structural breakpoints within this large sample, we divide the dataset into three subperiods 1990-2000, 2000-2010 and 2010-2019. More information on how we divide these subperiods is given in Section 5. We run the SCAR model for each of these subperiods and we compare the results we get with the ones from Hafner and Manner (2012). In the later sections, we switch from the traditional SCAR model towards several extensions that we expect to outperform the original framework. In Section 6.2, we relax the normality assumption that the SV model of SCAR makes for the marginal distributions. In general, return data are found to exhibit leptokurtosis and we take this into account by introducing a heavy-tailed distribution, instead of the normality assumption that Hafner and Manner (2012) make. In Section 6.3, we consider a mixed copula model which combines two copula models into one. It might be the case that one copula better represents the dependence dynamics between returns for a particular subperiod, while another fits best for the rest of the sample.

6.1 The Original SCAR Model

The SCAR model is estimated using the two-step inference function for margins (IFM) estimator introduced by Joe (1997). As a first step, the marginal distributions of returns for both DJ and NQ are fitted using the SV model in Equation (3.1). The EIS procedure used to estimate the SV parameters is described in Section 3.3. Note that we use demeaned returns for the SV model. To compare results with Hafner and Manner (2012), we will first focus on their sample period from 1990 until 2000. The parameter estimates and the negative log-likelihood function values are given in Table 5. We should observe a high level of persistence in volatility, as the γ coefficient of around 0.98 indicates that it depends a lot on its past values. The resulting volatility paths are plotted in Figure 6. The volatility estimates of DJ and NQ stock indices move in a similar fashion through time, hence we expect there to be a high dependence relation between the two. We will see if this is indeed the case in the second step of the estimation, where introducing copulas decouples the dependence structure from the marginal distribution of returns.

Table 5: Estimates of SV Model for Dow Jones and NASDAQ

	SV Model	
Parameters	DJ	NQ
δ	-0.0112	-0.0054
	(0.0051)	(0.0042)
γ	0.9785	0.9731
	(0.0077)	(0.0079)
σ	0.1574	0.2078
	(0.0256)	(0.0268)
Log-Likelihood	-3137.5	-3638.2

Parameter estimates for the SV model of the marginals are given above with the corresponding standard errors in parenthesis. The data consists of 2608 observations from March 26th, 1990 until March 23rd, 2000.

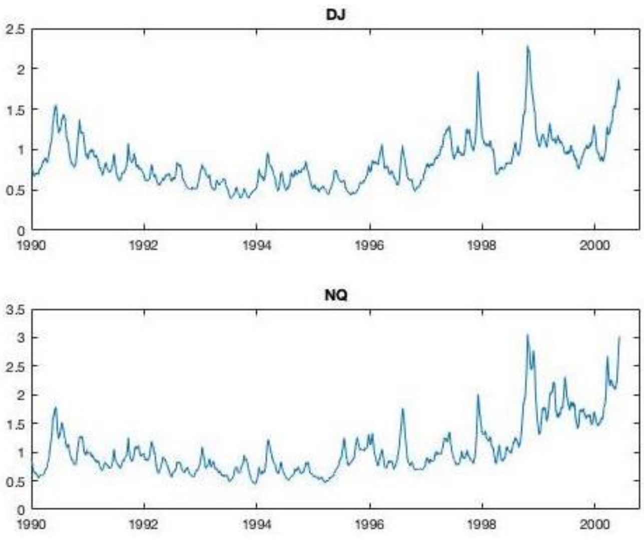

Figure 6: Volatility Estimates: DJ and NQ

Using the probability integral transform, we convert the return data into U(0,1) random variables. Note that Hafner and Manner (2012) assume that returns are normally distributed. We will later relax this assumption. In the second estimation step, we estimate the stochastic copula model using the EIS procedure once again. Several copula models are considered such as the Gaussian, Gumbel, Clayton and Frank. In addition, two survival or rotated copulas for Gumbel and Clayton are also selected. Refer to Section 2.2 for a detailed description of these copulas and their dependence measures. Assuming the normality of the returns, the parameter estimates of the SCAR model are given in Table 6. We obtain similar results as Hafner and Manner (2012). Note that in order to compare between different types of models and to select the best-fitting specifications, we can use the values of the log-likelihood function. The model with the largest log-likelihood value is the best-performing model. Typically, information criteria, such as AIC and BIC, are suitable to compare the fit of the models, but since we are comparing each specification for the time-variation and the same number of parameters is being used, it is equivalent to use the log-likelihood value (Manner and Reznikova, 2010).

Table 6: Parameter Estimates of the SCAR Model for Dow Jones and NASDAQ (assuming standard normal distributed error terms)

	Copula Model					
Parameters	Gaussian	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0330	0.0174	0.0150	0.1505	0.0482	0.0438
	(0.0103)	(0.0067)	(0.0062)	(0.0521)	(0.0164)	(0.0164)
β	0.9789	0.9818	0.9844	0.9871	0.9661	0.9691
	(0.0066)	(0.0070)	(0.0065)	(0.0045)	(0.0114)	(0.0115)
ν	0.0487	0.0589	0.0557	0.4055	0.0879	0.0925
	(0.0087)	(0.0125)	(0.0124)	(0.0793)	(0.0181)	(0.0203)
Log-Likelihood	1570.9	1525.1	1519.3	1466.7	1351.6	1337.1

The SCAR model is estimated using the two-step inference function for margins (IFM) estimator by Joe (1997). First, we estimate the parameters of the marginal distributions, then the parameters of the copula model.

(1) The marginal distribution of the returns r_{it} is specified as a Stochastic Volatility (SV) model

$r_{it} = \exp(h_{it}/2)\varepsilon_{it}$

$h_{it} = \delta_i + \gamma_i h_{it-1} + \sigma_i \eta_{it}$

where $i = 1, 2$ for the bivariate case. The innovations ε_{it} and η_{it} are assumed to follow a **standard normal** distribution. Using the probability integral transform, the data is converted into U(0,1) random variables and the observation equation becomes

$(r_{it}, r_{jt}) \mid \lambda_t, h_{it}, h_{jt} \sim C(\Phi(\varepsilon_{it}), \Phi(\varepsilon_{jt}))$ for $i, j = 1, 2$ and $i \neq j$

in which $\Phi(\cdot)$ denotes the CDF of the standard normal distribution and λ_t the dependence parameter, which is decoupled from the marginal distributions by the copula C.

(2) Using the efficient importance sampling (EIS) algorithm of Liesenfeld and Richard (2003), we estimate the stochastic copula model, and obtain a sequence of the dependence variable λ_t with a Gaussian AR(1) underlying process

$\lambda_t = \alpha + \beta\lambda_{t-1} + \nu\varepsilon_t$

where ε_t is an i.i.d standard normal error term.

Looking at the log-likelihood values, the Gaussian copula model performs best, followed by the rotated Gumbel, Frank and finally Clayton. This result is surprising, as we did not expect the Gaussian copula to outperform the other non-elliptical copula models. In general, returns have a higher lower-tail dependence. As the Gaussian copula does not account for the tail dependence, it underestimates potential losses during an adverse economic climate. Hence, non-elliptical distributions should empirically be a better fit. In contrast, this could be the case for two main reasons. First, the sample period 1990-2000 does not include some of the most adverse economic events, such as the internet bubble, the 9/11 terrorist attacks or the financial crisis of 2008. Extending the dataset to include these important market events, could significant change the results. Considering the fact that the paper was published in 2012, it comes as a surprise that this is not the case. Second, these results could be a consequence of assuming a normal distribution fit for the marginal returns. Hence, relaxing the assumption of normally distributed returns could lead to very interesting results.

Figure 7 gives the dependence time paths estimated by SCAR using the different copula models. Note that the Gumbel and rotated Clayton were quite similar to the rotated Gumbel and Clayton respectively, so their dependence paths are not shown because of redundancy and space constraints. Although scaled differently, the different copula models give resembling results. Notice that the Frank copula gives the smoothest dependence path, while Clayton results in the noisiest. It is interesting to see how these dependence paths differ when we relax the assumption of normality.

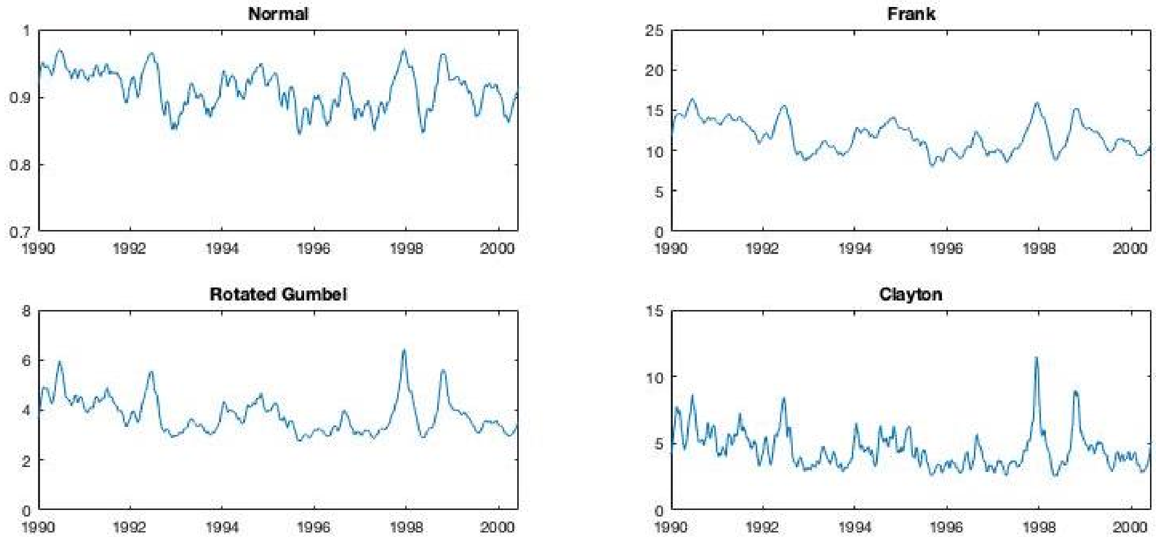

Figure 7: The dependence path between Dow Jones and NASDAQ estimated by SCAR using different copula models

Hafner and Manner (2012) evaluate the performance of the SCAR model relative to several other specifications. They compare the marginal SV model with a GARCH(1,1) model, and they find that the SV model provides a better fit and higher flexibility than GARCH. Furthermore, SCAR is found to outperform other well-known time-varying dependence specifications such as the DCC model and the model of Patton (2006). Hence, the SCAR model performs quite well as compared to its competitors. Although it has some disadvantages in higher dimensions, for the bivariate case it is quite a worthy contender. For this reason, we look into possible extensions of the original SCAR and hope to find a better representation of the dependence structure between variables by delving into the assumptions and specification of the SCAR model itself.

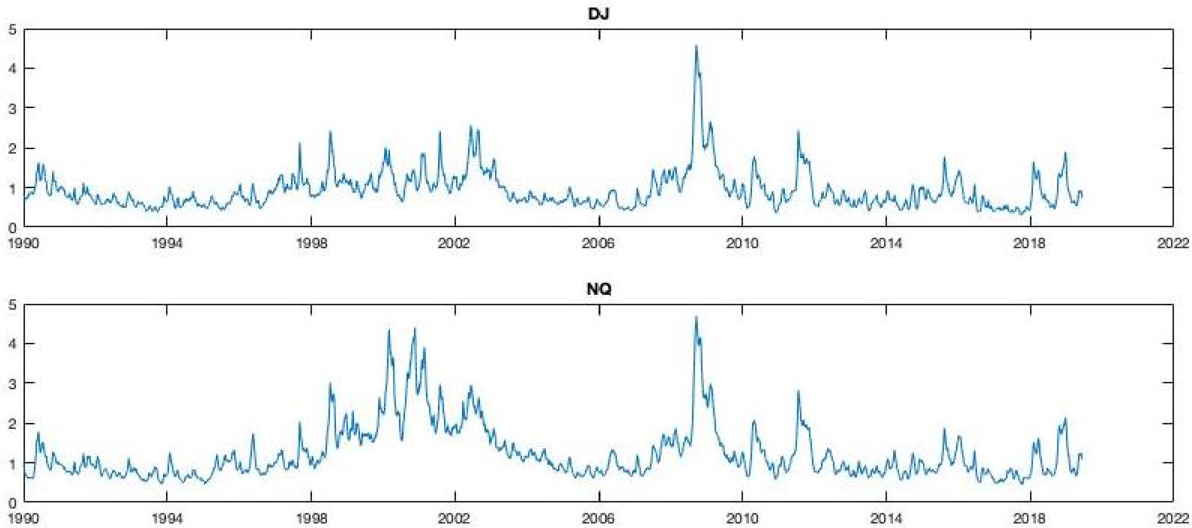

Figure 8: Volatility Estimates for 1990 - 2019: DJ and NQ

The first thing we want to check is if the conclusions from Hafner and Manner (2012) are consistent through time, hence we extend their dataset and incorporate data until March 25th,

2019. This new dataset has 29 years of data which includes a total of 7363 observations. A summary of the descriptive statistics can be found in Table 2. Fitting the marginal distribution of returns using the SV model, we obtain the volatility estimates for the full period from 1990 until 2019 in Figure 8. The volatility dynamics for both indices are somewhat similar, however they differ substantially during the internet bubble of the early 2000s. As the NASDAQ Composite contains mostly companies that are technology and internet-related, the dot-com bubble affected this index more than the Dow Jones Industrial Average. As for the financial crisis of 2008, there are slight differences indeed, however we observe an overall coherence in both volatility dynamics.

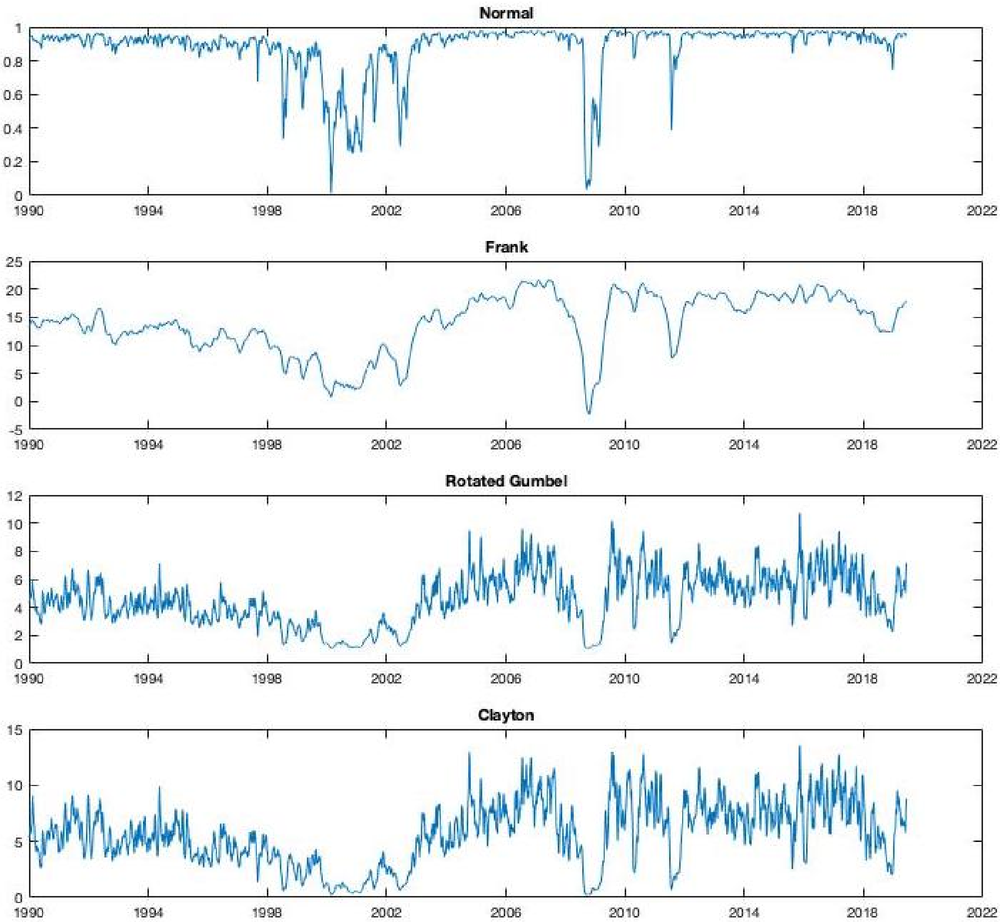

Figure 9: The dependence path between Dow Jones and NASDAQ estimated by SCAR using different copula models for 1990-2019

As our dataset is quite large, we want to avoid issues such as structural breakpoints within the full sample. For this reason, we divide the dataset into three subperiods 1990-2000, 2000-2010 and 2010-2019, as previously discussed in Section 5. The parameter estimates as well as the corresponding standard deviations and log-likelihood values of SCAR for the full sample and the subsamples we consider according to the two breakpoints in 2000 and 2010, across the different copula models, can be found in Table 7. The first thing to observe is that, according to the log-likelihood value, the following results from Hafner and Manner (2012) still hold. The Gaussian copula performs best, followed by the rotated Gumbel, Frank and Clayton. This is surprising as equity returns usually fit best to non-elliptical distributions because of the asymmetry in the tails. One would expect that this would be the case at least for the period from 2000-2009 which includes the financial crisis, but this is not the case. Hence, the Gaussian copula outperforms the other copulas regardless of time of market volatility.

Table 7: Parameter Estimates of the SCAR Model for Dow Jones and NASDAQ for the period from 1990 until 2019

	Full Sample: 1990 - 2019					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0849	0.0101	0.0316	0.0362	0.0710	0.0714
	(0.0145)	(0.0032)	(0.0124)	(0.0467)	(0.0139)	(0.0216)
β	0.9482	0.9849	0.9757	0.9973	0.9560	0.9588
	(0.0092)	(0.0055)	(0.0052)	(0.0031)	(0.0050)	(0.0109)
ν	0.1231	0.1596	0.1769	0.4312	0.1436	0.1509
	(0.0163)	(0.0179)	(0.0189)	(0.0972)	(0.0083)	(0.0124)
Log-Likelihood	4811.78	4655.17	4604.21	4951.36	4108.54	4054.00
	Subsample 1: 1990 - 1999					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0502	0.0222	0.0245	0.2453	0.0631	0.0648
	(0.0159)	(0.0094)	(0.0097)	(0.0850)	(0.0246)	(0.0233)
β	0.9685	0.9780	0.9753	0.9795	0.9571	0.9553
	(0.0099)	(0.0093)	(0.0097)	(0.0071)	(0.0165)	(0.0159)
ν	0.0592	0.0607	0.0656	0.5161	0.1025	0.1013
	(0.0115)	(0.0152)	(0.0153)	(0.1096)	(0.0249)	(0.0226)
Log-Likelihood	1431.69	1394.29	1390.01	1339.59	1238.90	1232.19
	Subsample 2: 2000 - 2009					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.1063	0.0798	0.0801	0.1442	0.0710	0.0713
	(0.0320)	(0.0267)	(0.0234)	(0.0887)	(0.0351)	(0.0324)
β	0.9496	0.9346	0.9345	0.9853	0.9194	0.9194
	(0.0127)	(0.0123)	(0.0127)	(0.0099)	(0.0193)	(0.0185)
ν	0.1189	0.1137	0.1138	0.9321	0.1091	0.1090
	(0.0107)	(0.0103)	(0.0099)	(0.0863)	(0.0243)	(0.0259)
Log-Likelihood	2313.44	2190.54	2164.88	2193.36	1565.55	1511.22
	Subsample 3: 2010 - 2019					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0915	0.0405	0.0309	0.3811	0.0904	0.0541
	(0.0256)	(0.0133)	(0.0107)	(0.0935)	(0.0267)	(0.0179)
β	0.9550	0.9736	0.9803	0.9813	0.9541	0.9734
	(0.0125)	(0.0085)	(0.0068)	(0.0047)	(0.0134)	(0.0087)
ν	0.0919	0.0745	0.0579	1.0000	0.1101	0.0681
	(0.0143)	(0.0134)	(0.0108)	(0.0000)	(0.0191)	(0.0132)
Log-Likelihood	2263.83	2225.35	2236.33	2188.49	2054.54	2059.35

Figure 9 gives the estimated dependence paths for the entire period from 1990 until 2019. The Frank copula appears to result in the smoothest dependence dynamics among all other copula models, while the rotated Gumbel and Clayton result in extremely volatile dependence structures. The Gaussian copula is somewhat in between, which is what probably makes it a better fit than its competitors.

6.2 Relaxing the Normality Assumption

In their paper, Hafner and Manner (2012) make the assumption of conditional normality of the marginal distributions. As financial returns often posses properties such as leptokurtosis (i.e. fat-tailed distributions), this assumption might not be a realistic one to make. Hence, relaxing this assumption might be an interesting extension to the original SCAR model. One possibility is to assume that the marginal distribution errors follow a Student's t-distribution. Compared to the normal case, we expect this distribution to represent the data quite for two main reasons. First, the t-distribution is known to represent financial returns better than the normal distribution and second, because it is data-specific through estimating the degrees of freedom, the t-distribution should provide with a better fit. Thereby, it is also expected to see an increase in the performance of SCAR.

In other words, we determine the degrees of freedom which result in the best-fitting t-distribution for that specific dataset. In Section 5, we compared the normal distribution against the t-distribution fit for the index returns. The t-distribution provided a better representation of the data and this was confirmed by both the histograms of returns and the KS tests. For the estimated location, scale and the degrees of freedom for the t-distribution fit on the whole sample, refer again to Table 3. Of importance here are the degrees of freedom. Recall that the DJ returns best fitted into a t-distribution with 2.99 degrees of freedom, while the NQ returns were best represented by using 2.65 degrees of freedom. The same procedure is applied to each of the subsamples as well.

The parameter estimates for the SCAR model assuming Student's t-distributed error terms of the marginal SV model, are given in Table 8. The first thing to notice is that the log-likelihood values are much higher than the normal distribution case, which asserts our initial hypothesis that the t-distribution better captures the heavy tails of stock returns. Refer back to Table 7 for the overview of the parameter estimates and the log-likelihood values of SCAR under the assumption of normality. Second, the Gaussian copula continues to outperform the other copula models, again followed by the rotated Gumbel, Frank and then the Clayton copula. We certainly did not expect this to change, as the copula models estimate the dependence structure irrespective of the marginal distribution. Hence, relaxing the assumption of normality should not make a significant impact in the performance of the copula models. Third, we observe that the dependence parameter depends even more on its past value as there is an overall increase in the value of β around 0.99. Another important coefficient to notice is that of the error term for the transition equation, ν, which has decreased significantly compared to the normal distribution. This coefficient can be interpreted as the standard deviation of the dependence parameter λ_t. Hence, Student's t-distribution fit for the marginals results in more accurate estimates of the dependence process.

The estimated time paths of the dependence parameters resulting from the Student's t conditional distribution are given in Figure 10. An interesting thing to observe is how the dependence path fluctuates around the financial crisis of 2008. In all the copula models, we notice a spike around that particular year indicative that the dynamics of the DJ and NQ index were extremely dependent on each other's movements. This phenomenon, typically referred to as "together in bad times", is commonly observed during highly-volatile times in equity returns. In a more general note, we see that the paths are smoother compared to the normal distribution case, which is also confirmed by the smaller standard deviations (compare to Figure 9 for the dependence paths under the normality assumption). The Frank copula again appears to be the smoothest, while the Clayton is the most volatile.

Table 8: Parameter Estimates of the SCAR Model for Dow Jones and NASDAQ (assuming Student's t-distributed error terms)

	Full Sample: 1990 - 2019					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0242	0.0109	0.0097	0.1031	0.0214	0.0212
	(0.0076)	(0.0056)	(0.0056)	(0.0399)	(0.0074)	(0.0082)
β	0.9868	0.9918	0.9928	0.9931	0.9882	0.9883
	(0.0041)	(0.0043)	(0.0042)	(0.0027)	(0.0041)	(0.0045)
ν	0.0334	0.0321	0.0319	0.3316	0.0353	0.0406
	(0.0058)	(0.0077)	(0.0080)	(0.0661)	(0.0068)	(0.0084)
Log-Likelihood	6304.5	6170.9	6141.1	5927.9	5592.7	5550.1
	Subsample 1: 1990 - 2000					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0215	0.0096	0.0083	0.0964	0.0219	0.0250
	(0.0074)	(0.0054)	(0.0056)	(0.0379)	(0.0086)	(0.0113)
β	0.9871	0.9914	0.9927	0.9925	0.9861	0.9841
	(0.0045)	(0.0050)	(0.0052)	(0.0031)	(0.0055)	(0.0072)
ν	0.0343	0.0354	0.0352	0.3137	0.0422	0.0552
	(0.0064)	(0.0092)	(0.0102)	(0.0637)	(0.0093)	(0.0137)
Log-Likelihood	2761.8	2720.8	2711.4	2635.5	2554.7	2528.6
	Subsample 2: 2000 - 2010					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0207	0.0091	0.0004	0.0941	0.0229	0.0305
	(0.0075)	(0.0052)	(0.0000)	(0.0373)	(0.0095)	(0.0148)
β	0.9870	0.9911	0.9997	0.9922	0.9845	0.9792
	(0.0047)	(0.0053)	(0.0000)	(0.0032)	(0.0065)	(0.0102)
ν	0.0351	0.0373	0.0225	0.3091	0.0470	0.0691
	(0.0067)	(0.0099)	(0.0062)	(0.0635)	(0.0111)	(0.0191)
Log-Likelihood	3363.2	3290.1	3279.6	3214.4	3119.3	2993.5
	Subsample 3: 2010 - 2019					
Parameters	Normal	Rot. Gumbel	Gumbel	Frank	Clayton	Rot. Clayton
α	0.0203	0.0087	0.0073	0.0923	0.0252	0.0467
	(0.0076)	(0.0050)	(0.0054)	(0.0369)	(0.0111)	(0.0244)
β	0.9868	0.9907	0.9923	0.9919	0.9818	0.9657
	(0.0050)	(0.0055)	(0.0060)	(0.0033)	(0.0080)	(0.0177)
ν	0.0362	0.0394	0.0391	0.3065	0.0542	0.0998
	(0.0071)	(0.0106)	(0.0125)	(0.0635)	(0.0140)	(0.0309)
Log-Likelihood	2.5253	2.4805	2.4684	2.4182	2.3056	1.2805

The SCAR model is estimated using the two-step inference function for margins (IFM) estimator by Joe (1997). First, we estimate the parameters of the marginal distributions, then the parameters of the copula model.

(1) The marginal distribution of the returns r_{it} is specified as a Stochastic Volatility (SV) model

$r_{it} = \exp(h_{it}/2)\varepsilon_{it}$

$h_{it} = \delta_i + \gamma_i h_{it-1} + \sigma_i \eta_{it}$

where $i = 1, 2$ for the bivariate case. The innovations ε_{it} and η_{it} are assumed to follow a **Student's t** distribution. Using the probability integral transform, the data is converted into U(0,1) random variables and the observation equation becomes

$(r_{it}, r_{jt}) \mid \lambda_t, h_{it}, h_{jt} \sim C(F(\varepsilon_{it}), F(\varepsilon_{jt}))$

in which $F(\cdot)$ denotes the CDF of the Student's t-distribution and λ_t the dependence parameter, which is decoupled from the marginal distributions by the copula C.

(2) Using the efficient importance sampling (EIS) algorithm of Liesenfeld and Richard (2003), we estimate the stochastic copula model, and obtain a sequence of the dependence variable λ_t with a Gaussian AR(1) underlying process

$\lambda_t = \alpha + \beta\lambda_{t-1} + \nu\varepsilon_t$

where ε_t is an i.i.d standard normal error term.

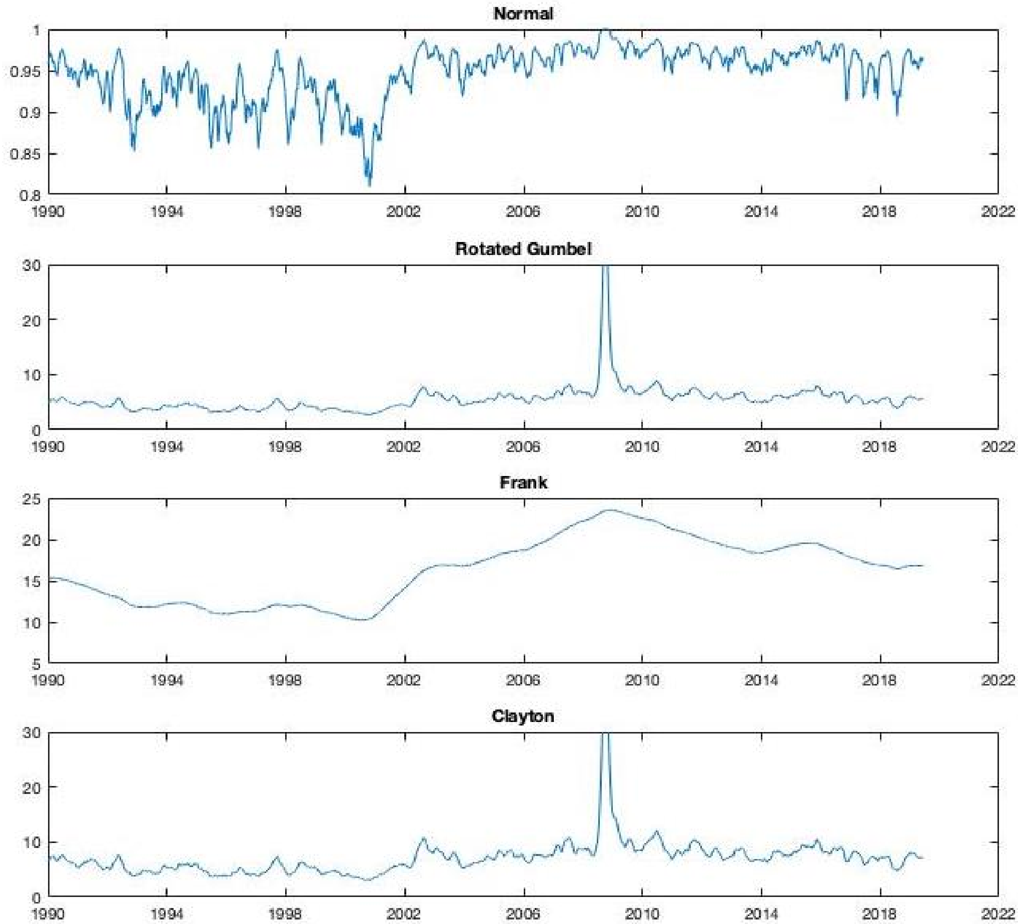

Figure 10: The dependence path between Dow Jones and NASDAQ estimated by SCAR assuming marginal Student t-distributed errors

6.3 Mixed Stochastic Copula Models

Hafner and Manner (2012) use the timeframe 1990-2000 for their return data, excluding important market events such as the internet bubble of the early 2000s and the housing bubble of 2008. By extending the dataset and using a mixed copula model, we can model highly volatile periods using a non-elliptical copula and non volatile periods using a Gaussian copula.

As the Gaussian and the rotated Gumbel were the best performing copula models, a possibility would be to mix these two copulas together. Such a specification could model periods of tranquility using the Gaussian copula and highly volatile periods, during crises for example, with the rotated Gumbel copula where we take into account a greater dependence for losses. The mixed copula combinations we consider are Gaussian and Gumbel, Gaussian and Frank as well as Gumbel and Frank. Table 9 gives the estimates, the standard deviations and the log-likelihood values of SCAR models with a mixed copula during different time periods. We can compare these values to the ones of Table 7 and 8.

Overall, we observe that the log-likelihood values using mixed copulas are not as high as the values when using the Gaussian or Gumbel copula alone. Only for the subperiod ranging from 2000-2010 the log-likelihood values in the mixed copula case is higher than the individual copula model. This might be due to the fact that the other subperiods are characterized as decades of tranquility, hence there are no large fluctuations in volatility. The subperiod 2000-2010 includes the financial crisis of 2008, which makes it extremely volatile as compared to other timeframes. This indicates that the mixed copula SCAR model outperforms the original

SCAR for high-volatile periods where the mixed copula takes into account greater dependence for losses.

Table 9: Parameter Estimates of the SCAR Model for Dow Jones and NASDAQ (assuming Student's t-distributed error terms)

	Full Sample: 1990 - 2019			
Parameters	Gaussian + rot. Gumbel	Gaussian + Gumbel	Gaussian + Frank	Gaussian + Clayton
α	0.0379	0.0218	0.1002	0.0298
	(0.0124)	(0.0122)	(0.0321)	(0.0113)
β	0.9548	0.9558	0.9891	0.9234
	(0.0083)	(0.0077)	(0.0009)	(0.0072)
ν	0.0634	0.0721	0.0814	0.0807
	(0.0069)	(0.0885)	(0.0230)	(0.0209)
Log-Likelihood	4753.6	4365.2	4247.2	4025.9
	Subsample 1: 1990 - 2000			
Parameters	Gaussian + rot. Gumbel	Gaussian + Gumbel	Gaussian + Frank	Gaussian + Clayton
α	0.0215	0.0227	0.3703	0.0381
	(0.0084)	(0.0086)	(0.0196)	(0.0135)
β	0.9823	0.9811	0.9184	0.9748
	(0.0070)	(0.0072)	(0.0011)	(0.0089)
ν	0.0403	0.0421	0.0010	0.0622
	(0.0093)	(0.0095)	(0.0001)	(0.0132)
Log-Likelihood	1352.5	1349.0	798.24	1344.0
	Subsample 2: 2000 - 2010			
Parameters	Gaussian + rot. Gumbel	Gaussian + Gumbel	Gaussian + Frank	Gaussian + Clayton
α	0.0879	0.0878	0.1737	0.0698
	(0.0053)	(0.052)	(0.0134)	(0.0129)
β	0.9506	0.9499	0.9861	0.9196
	(0.0034)	(0.0059)	(0.066)	(0.0127)
ν	0.1105	0.1112	0.1245	0.1098
	(0.0084)	(0.0090)	(0.0022)	(0.0087)
Log-Likelihood	2548.4	2561.0	2247.4	1725.7
	Subsample 3: 2010 - 2019			
Parameters	Gaussian + rot. Gumbel	Gaussian + Gumbel	Gaussian + Frank	Gaussian + Clayton
α	0.0327	0.0432	0.2194	0.0471
	(0.0047)	(0.0069)	(0.0256)	(0.0319)
β	0.9854	0.9832	0.9199	0.9716
	(0.0092)	(0.0060)	(0.0032)	(0.0082)
ν	0.0566	0.0444	0.0022	0.0832
	(0.0139)	(0.0152)	(0.0032)	(0.0097)
Log-Likelihood	1732.9	1721.7	998.42	1544.1

Across mixed copulas, on the other hand, we observe that the Gaussian + rotated Gumbel mixed copula performs best. This results as both individual copulas perform quite well on their own, with the Gaussian copula performing the best out of all individual copulas. We see that this is the case for the full period as well as for each subperiod individually.

7 Further Extensions and Research

One of the main contributions of this paper is to introduce some potential extensions to the original SCAR model presented by Hafner and Manner (2012). Future researchers are encouraged to search for additional extensions as SCAR models are found to perform quite well compared to their competitors. Incorporating copulas to SV models facilitates largely the complexity of their estimation, which makes the SCAR model even more appealing. One possibility for extension could be to introduce exogenous variables in the SCAR framework.

The SCAR model is based on the assumption that the underlying process for the dependence variable λ_t follows a Gaussian AR(1) specification. Although computationally convenient, an AR(1) process might be too simplistic to accurately fit the actual data. Hence, we consider adding other exogenous variables which may explain part of the correlation and perform better than the AR(1) model. Equation (3.3) then becomes

$$\lambda_t = \alpha + \beta\lambda_{t-1} + \rho x_t + \nu\varepsilon_t \tag{7.1}$$

where x_t denotes the exogenous variable and $\varepsilon_t \sim N(0, 1)$. Some potential choices could be to add a trend, a dummy or a proxy for the effect of volatility on correlation such as the trading volume.

On the other hand, SCAR models are not flawless either. One major drawback that SCAR models have is when they are applied to higher dimensions. Like Hafner and Manner (2012), our paper only considers the bivariate SCAR model, in which only two stocks or indices are considered. A solution to this dimensionality problem remains unsolved by the current literature and requires further research and analysis.

8 Conclusion

Despite their drawbacks at higher dimensions, SCAR models are currently being used at large and they perform quite well as compared to their competitors. This paper investigates possible extensions on SCAR models. In the first part of our analysis, we look into the SCAR model and empirically apply it to real-world data. Similarly to Hafner and Manner (2012), we use daily returns from two popular indices, the Dow Jones Industrial Average and NASDAQ. As the sample from Hafner and Manner (2012) is limited to the period from 1990 until 2000, including recent data tests whether the conclusions from the paper are consistent through time. We find that the Gaussian copula continues to outperform all other copula models, regardless of time. This is surprising as equity returns usually have asymmetric tail distributions, hence a non-elliptical copula such as the rotated Gumbel was expected to have a better goodness-of-fit as compared to the Gaussian one.

Secondly, we relax the assumption of normal (univariate) marginal distributions errors, which can better describe the stylized facts of asset returns such as the heavy tailed distribution. Instead of assuming the same distribution for all returns, we determine the degrees of freedom which result in the best-fitting t-distribution for each specific dataset. In general, the t-distribution fits for the index returns across different subsamples required approximately 3 degrees of freedom. Hence, one might imply that $t(3)$ is a very good representation of equity return data in general. Moreover, the JB test, the histograms of returns, the KS test as well as the significant increase in the log-likelihood value of the SCAR model under the Student's t-distributed marginals, all confirmed the same thing. The normal distribution is a very poor approximation of univariate return data and the t-distribution greatly outperforms it. Ultimately, this implies that relaxing the assumption of normality creates a dominating extension of SCAR as compared to the original.

Another extension we consider is to combine the best-performing copulas into a mixed copula model which captures the dynamics of stock returns better. Generally, the mixed copulas did not outperform the Student's t-distributed SCAR model, nor the Gaussian SCAR. However, it was noted that the log-likelihood values in the mixed copula case for 2000-2010 were slightly higher than the individual copula model under the Gaussian SCAR. As the other subperiods are considered as periods of market "tranquility", this might indicate that the mixed copula SCAR model outperforms the original SCAR only for high-volatile periods where the mixed copula takes into account greater dependence for losses.

Our goal was to ultimately find a extension which outperforms the original SCAR and does so in multiple aspects. The mixed copula SCAR only outperformed the original specification during times of significant market occurrences. On the other hand, the t-distributed marginals highly increased the goodness-of-fit of SCAR models regardless of time or market volatility. Hence, we can conclude that we provide the current literature with an extension of the SCAR model which predominantly outperforms the SCAR model of Hafner and Manner (2012).

9 Appendix

9.1 Dependence Paths for the Subperiods

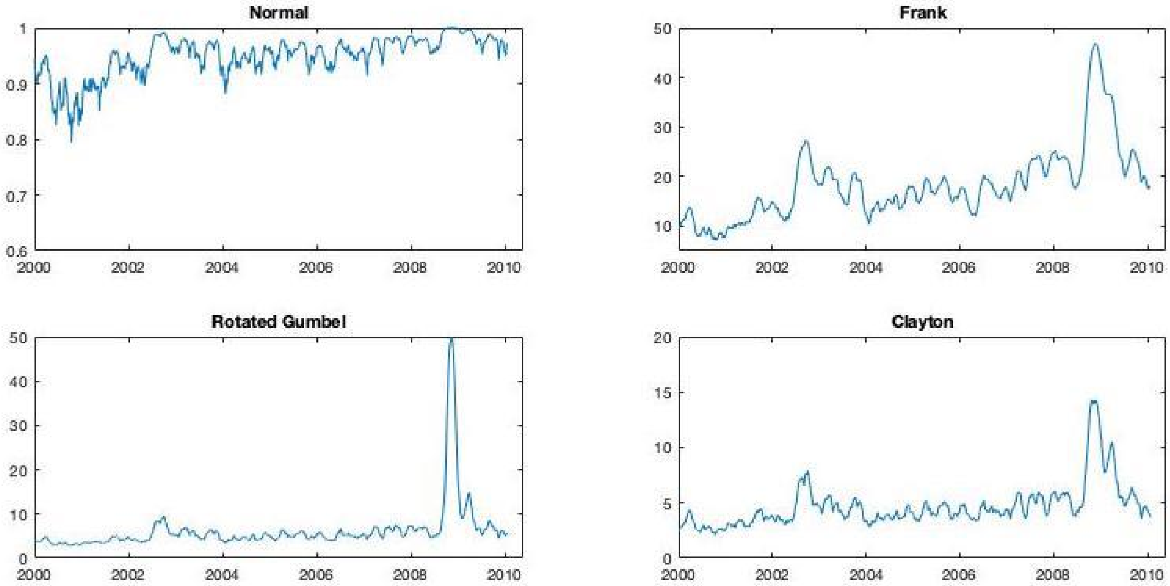

Figure 11: The dependence path for Dow Jones and NASDAQ for the period 2000 - 2009

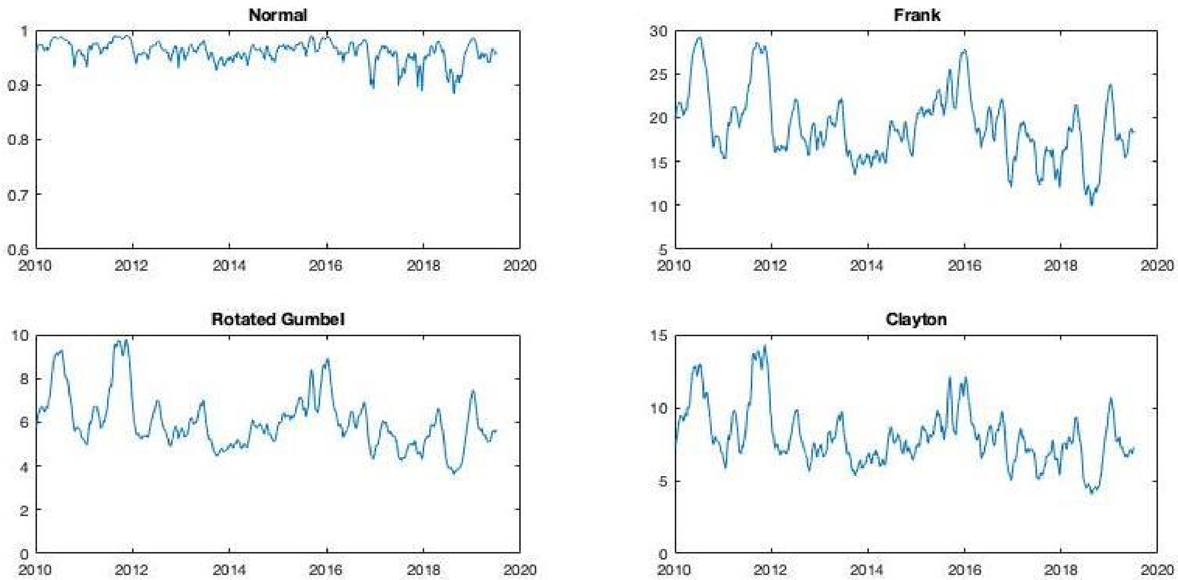

Figure 12: The dependence path for Dow Jones and NASDAQ for the period 2010 - 2019

9.2 Dependence Paths for Mixed Copulas 2000-2009

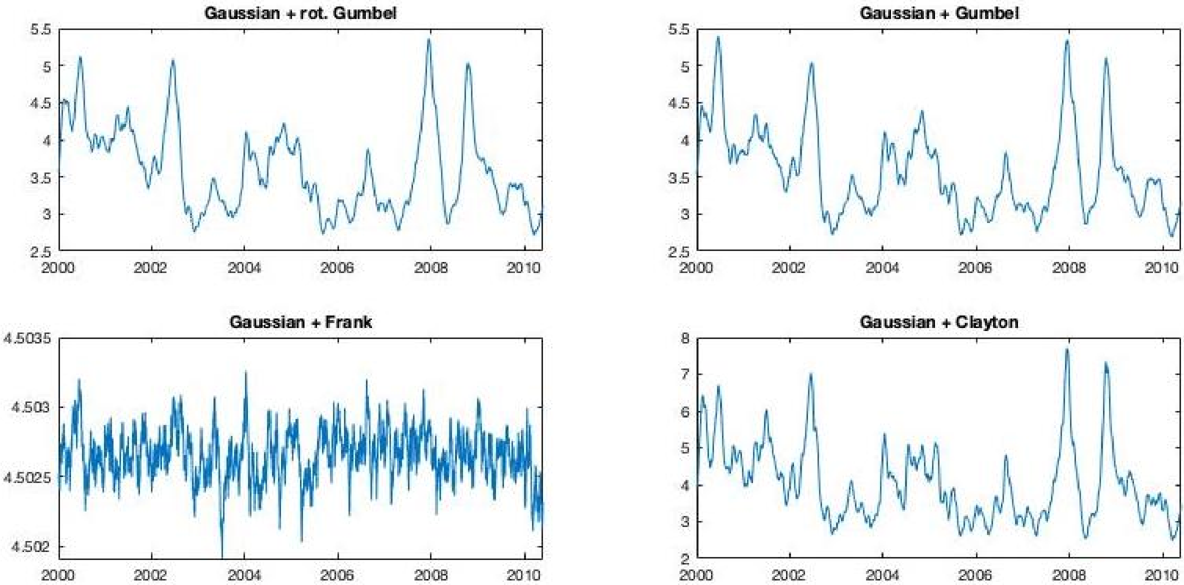

Figure 13: The dependence path for Mixed Copulas for the period 2000 - 2009

9.3 MATLAB Codes and Files

The methodology of the SCAR model and its extensions is based on the paper by Hafner and Manner (2012), published by the Journal of Applied Econometrics. The codes they use to estimate their SCAR model are available online at the Journal of Applied Econometrics website.[1] We extend these codes and adapt them for our own extensions accordingly, without claiming rights for any duplicate of the original work by Hafner and Manner (2012). A list and a description of the codes that have been used to obtain the results of this thesis can be found below. The codes themselves will be attached in a Zip file.

`main.m`	Displays the results, figures and tables for SCAR.
`main_Student.m`	Displays the results, figures and tables for the t-distributed SCAR.
`main_Mixed.m`	Displays the results, figures and tables for the mixed copula SCAR.
`Stochastic_Copula_MLE.m`	Estimates bivariate SCAR and the dependence structure of variables.
`SV_MLE_EIS.m`	Estimates the SV model by efficient importance sampling (EIS).
`LL_SV_EIS.m`	Computes the negative log-likelihood of a Gaussian AR(1) SV model.
`LL_SV_STUDENT_EIS.m`	Computes the negative log-likelihood of a Student's t AR(1) SV model.
`LL_Normal_EIS.m`	Computes the log-likelihood of a stochastic Gaussian copula.
`LL_Gumbel_EIS.m`	Computes the log-likelihood of a stochastic Gumbel copula.
`LL_Frank_EIS.m`	Computes the log-likelihood of a stochastic Frank copula.
`LL_Clayton_EIS.m`	Computes the log-likelihood of a stochastic Clayton copula.
`LL_Mixed_EIS.m`	Computes the log-likelihood of a mixed copula.
`PdfNormal.m`	Returns the PDF of the Normal copula.
`PdfGumbel.m`	Returns the PDF of the Gumbel copula.
`PdfFrank.m`	Returns the PDF of the Frank copula.
`PdfClayton.m`	Returns the PDF of the Clayton copula.

[1] The codes of Hafner and Manner (2012) are provided in the Journal of Econometrics website through the following link: `http://qed.econ.queensu.ca/jae/2012-v27.2/hafner-manner/`

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