ERASMUS UNIVERSITY ROTTERDAM

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Optimization in an uncertain world

The impact of uncertainty on portfolio allocation

Abstract

In this paper, we investigate the impact of uncertainty on portfolio allocation and how incorporating stochasticity in the investment strategy improves performance. We use both stochastic programming and robust optimization to maximize return with constrained risk measured by Conditional Value-at-Risk (CVaR), using scenarios generated via Filtered Historical Simulation (FHS). We compare the results based on return, risk, stability of the weights over time, and a newly introduced dispersion measure. We find that incorporating uncertainty only slightly enhances performance. In an expanding window estimation, the effect of incorporating uncertainty in returns disappears and the inclusion of parameter uncertainty has a negative impact. Also, managerial and legislative restrictions have much influence on the optimization outcomes and induce the stochasticity in the risk constraint to have little impact. When we exclude these supplementary restrictions, incorporating uncertainty becomes more effective.

Keywords: Portfolio Optimization, (Parameter) Uncertainty, Stochasticity, Conditional Value-at-Risk, Filtered Historical Simulation, Insurance, Solvency II

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Contents

List of Figures

List of Tables

1 Introduction

This paper studies optimal portfolio allocation under uncertainty for a life insurance company, in this case Nationale-Nederlanden (NN). NN is a Dutch (life) insurer that invests capital received from policyholders. The mix of assets in which they invest can be subdivided into strategic classes, also referred to as the Strategic Asset Allocation (SAA). As the realized return on the total portfolio contributes to the firm's profit, NN puts effort into optimizing its allocation on a long-term basis. The investment policy is subject to the European legislation that aims to unify a single EU insurance market and to enhance policyholder protection. The legislation is mostly captured in the Solvency Capital Requirement (SCR) as defined in the Solvency II directive (European Commission, 2015). It serves as a prudence against the most extreme expected losses over a year. In practice, the SCR plays a major part in determining the SAA. When NN reconsiders its portfolio allocation, experts estimate the future return of the associated asset classes. From here on, these estimates are assumed to be certain and the allocation is optimized in a deterministic manner.

The main issue arising with the current allocation procedure is that uncertainty regarding the returns is not considered. However, returns are uncertain. As a consequence, the chosen allocation based on this deterministic approach might lead to suboptimal results in the light of this uncertainty. Furthermore, the allocation is highly sensitive to the exact estimate done by experts while they may only be sure of this estimate within a certain bandwidth. This causes doubt about which estimate to use for the optimization. Moreover, if these estimates fluctuate drastically over time, the allocation also changes substantially, which is undesirable for a long-term investor such as an insurer. Criticism towards deterministic optimization approaches is also outlined in the literature, by among others Broadie (1993), Chopra and Ziemba (1993), Jobson and Korkie (1980) and Michaud (1989). They show that, for instance, returns' sample moments contain large estimation errors and ignoring this uncertainty leads to suboptimal results.

In this paper, we investigate the impact of uncertainty on portfolio allocation. Firstly, we examine whether incorporating uncertainty in returns leads to different asset mixes and, as a result, enhances portfolio performance. Secondly, we investigate the impact of including parameter uncertainty. Next, we analyze in what way the given information set influences the outcomes. And lastly, we want to know which constraints influence the optimal asset mix and its performance the most.

To answer our research questions, we consider two stochastic approaches and compare these to a deterministic benchmark that represents a simplified version of the current optimization within NN. The deterministic method is defined as a maximization of expected end-of-horizon return under the SCR constraint, which is a quadratic optimization problem. The results are used to put the outcomes of the stochastic methods into perspective. We introduce uncertainty by using stochastic programming to optimize expected return under a Conditional Value-at-Risk (CVaR) constraint. We use the linear approximation as introduced by Rockafellar and Uryasev (2000) that ensures the problem is simplified such that it can be solved using linear programming. The input for this method is a set of scenarios generated using Filtered Historical Simulation (FHS), an algorithm to generate correlated paths for a set of risky assets, introduced by Barone-Adesi, Giannopoulos, and Vosper (1999). The simulation is built on an ARMA-GARCH type return model and uses historic standardized residuals to create a semiparametric distribution around the (volatility of the) returns. Lastly, as robust method, we incorporate uncertainty in the parameters of the ARMA-GARCH model. We again optimize expected return subject to a CVaR constraint but under worst-case conditions. We simulate scenarios under shocked realizations of the parameters and use the set with the worst objective value in our optimization. The result is a min-max problem definition that can still be solved using linear programming.

To get a grasp of how the outcomes vary over time, we implement all three methods using the rolling window approach as in line with both Diris, Palm and Schotman (2014) and DeMiguel, Galappi and Uppal (2007). We use publicly available price and yield indices from March 2005 to May 2019 of the following strategic asset classes: Government Bond, Corporate Bond, Real Estate, Equity, and Mortgages. We evaluate the methods based on four performance measures: return, risk, the distributional dispersion of these two, and stability of the weights over time. The measure of dispersion indicates the uncertainty around the outcomes and has, to the best of our knowledge, not been used in the literature before. Lastly, we compare the results of using an expanding window as opposed to a fixed window for parameter estimation. And, we analyze the effect of excluding some of the constraints in the standard problem definition.

Our main finding is that, in our dataset, incorporating uncertainty leads to only slight improvements in the four performance measures. Moreover, adding parameter uncertainty to the optimization only leads to better performance in a fixed window estimation. When using an expanding window, the effect of incorporating uncertainty in returns disappears and the inclusion of parameter uncertainty has a negative impact. Most importantly, we find that the managerial and legislative restrictions have a much stronger impact on the outcomes than the risk constraint itself. When we exclude these additional restrictions, incorporating uncertainty becomes more effective.

We find that the results of the SCR and CVaR methods are much alike and, in the expanding window case, even equivalent. While the stochastic elements are only embodied in the risk constraint and objective function of the optimization, we find that this constraint is not binding in most cases. Instead, constraints that are added due to legal and managerial restrictions have a much stronger impact on the optimal allocation. This explains why the outcomes of the SCR and CVaR methods are so analogous and hence incorporating stochasticity in this way has little impact. The Worst-Case method generally leads to the best performance, but only in the fixed window case.

When an expanding window estimation is applied, the estimation errors of the volatility model's coefficients become smaller for most assets. The resulting optimal weights are fluctuating less over time, especially due to the more stable underlying scenarios. In this setting, the Worst-Case method does not perform superior anymore. Overall, the SCR and CVaR methods now obtain the best performance based on all four measures. This means that the effect of incorporating uncertainty is only of value in the fixed window case. However, implementation issues might arise as, in this setting, the optimal weights are strongly concentrated in the Mortgage asset class. All in all, we find that the effect of incorporating stochasticity into an optimization method depends on which information set is applied. Furthermore, the choice of information set has a substantial effect on the portfolio results and should thus be taken carefully.

Lastly, as the additional restrictions have such a strong impact on the results, we are interested in the behavior of the different methods with fewer constraints and the general impact of the restrictions on the results. When eliminating constraints supplementary to the risk constraint, we induce the optimal solution to depend stronger on the risk constraint itself. Consequently, we find greater differences between the three methods and, generally, incorporating uncertainty becomes more effective. Under these conditions, the CVaR method overall leads to the best return, whereas the Worst-Case method generates asset mixes with the lowest return but also the lowest risk. Comparing these results to the ones including the standard constraint set, the SCR and CVaR methods generally improve in performance, especially in terms of return. The Worst-Case method, on the other hand, performs worse with fewer constraints. Since, when excluding constraints, the region of feasible asset allocations becomes broader, this method also has more possibility to move into a position that is robust to worst-case circumstances. This position is not necessarily one that performs well in the actual outcome. We notice that the decision of which constraints to include in the problem definition has a strong impact on the optimal asset mixes. We conclude that in practice, the impact of the risk constraint is not that strong and that restrictions on regulative and management levels are more important.

We contribute to the literature in several aspects. Most closely related are the papers that implement uncertainty in the application of portfolio optimization. The bulk of papers within the literature introduce uncertainty by using an explicit distribution (see Hellmich and Kassberger (2011)), whereas we use the empirical return distribution directly. Other papers define a specific uncertainty set around the model parameters, often in the form of a confidence interval around the estimated value (Bertsimas, Brown & Caramanis, 2011; Garlappi, Uppal & Wang, 2006; Hellmich & Kassberger, 2011). Our results are in contrast to those of Barberis (2000), who found that incorporating parameter uncertainty changes the optimal allocation significantly. The author also analyzes a portfolio strategy with optimal rebalancing. He, however, uses a Bayesian approach to incorporate parameter uncertainty, focuses on the allocation to stocks and uses a predictor variable to model returns. Zhu and Fukushima (2009) find that portfolio selection using the worst-case CVaR as risk measure performs robustly in practice. To the best of our knowledge, none of the papers apply the regulation on insurance as directly as we do.

This research is relevant both as an addition to the financial literature regarding portfolio optimization under uncertainty, as well as for practitioners who are seeking to enhance investment performances. Starting with the latter, the choice of investments is one of the key elements in the financial management of a life insurance company. Since nowadays, fixed income assets' returns are declining and thus the traditional return of the insurer via this main asset class is also declining. Therefore, it has become more important to optimize allocation over multiple, alternative asset classes. Moreover, the volatile financial markets of the past few years have induced financial institutions to become more cautious regarding their investments. We notice however that overall, risk is not the leading factor in optimizing the portfolio. The managerial and legislative restrictions have a much stronger impact on the results instead.

On an academic level, this research can be applied to any stochastic optimization problem and is not limited to the practical circumstances here defined. This research adds to the literature by specifically focusing on the application of uncertainty to portfolio optimization for a life insurance company, using the most recent set of asset return data, and taking the practical implementation of the allocation strategy into account. The research includes a practical analysis and implementation of the legislation on insurance policies and their risk management strategies.

The rest of this paper is structured as follows. Section 2 describes the related literature and Section 3 discusses the data. Section 4 outlines the optimization problem definitions, the methods implemented to incorporate uncertainty, and the performance measures used to evaluate the outcomes. Section 5 presents and evaluates the results and Section 6 discusses the findings in a broader context.

2 Literature

2.1 Traditional portfolio optimization

The theory of optimal portfolio selection was first developed by Markowitz (1952), which is nowadays also known as mean-variance optimization. The method has since received much criticism among both academics and practitioners, as discussed by for instance Cornuejols and Tütüncü (2006), DeMiguel and Nogales (2009), Tütüncü and Koenig (2004) and Fliege and Werner (2014). The main criticism is that mean-variance optimization is extremely sensitive to errors in the estimates of the input parameters: the expected return and covariance matrix. Traditionally, the sample mean and covariance matrix have been used to approximate the true mean and covariance matrix. However, due to the estimation error caused using this estimation method, policies constructed based on these estimators are extremely unstable. Moreover, Chopra, Hensel, and Turner (1993) show that minor adjustments to the input parameters can result in substantial changes in the composition of the optimal portfolio. This is especially unfavorable from the perspective of a portfolio manager, who is reluctant to implement policies that recommend such drastic changes in the portfolio composition. Moreover, mean-variance optimization might even lead to suboptimal asset allocation, also referred to as error maximization (Michaud, 1989), and performs poorly out-of-sample, as also shown by Chopra and Ziemba (1993), and Broadie (1993).

Barberis (2000) discusses the relation between parameter uncertainty and the sensitivity of the optimal allocation. He finds that, when parameter uncertainty is incorporated in the optimization strategy, the asset mix becomes less sensitive to changes in the input. Therefore, this strategy leads to more gradual shifts in the portfolio composition over time.

2.2 Risk measure: Conditional Value-at-Risk

To overcome the shortcomings that arise from using variance as a measure of risk, we use Conditional Value-at-Risk (CVaR) instead. Variance is only a useful risk measure for normally (or symmetrically) distributed losses. Since variance is measured in either direction, tail losses arising from skewed loss distributions are not taken into account (Kisiala, 2015).

CVaR, also known as Expected Shortfall, is defined as the expected value of the loss exceeding the Value-at-Risk (VaR). It is known to have better properties than VaR itself (Cornuejols & Tütüncü, 2006). For example, CVaR is considered a more consistent measure of risk than VaR (Rockafellar & Uryasev, 2000; Artzner, Delbaen, Eber & Heath, 1999). Most importantly, it is considered a coherent and convex measure and, therefore, has a theoretical advantage as opposed to VaR. Moreover, Uryasev (2000) shows that CVaR optimal portfolios are near-optimal in VaR terms, and thus optimizing CVaR effectively takes both measures into account.

CVaR is, as opposed to variance, a measure that takes the entire left tail of the return distribution into account. Government regulators already mandate that financial institutions control their holdings in certain ways and place margin requirements for risky positions (Braun, Schmeiser & Schreiber, 2017). Since these regulations become more based on such distributional measures, it may be more applicable to also use these when optimizing the portfolio. Rockafellar and Uryasev (2000), Krokhmal, Palmquist, and Uryasev (2002) and Uryasev (2000) all use CVaR as a risk measure and show how a CVaR constrained problem can be rewritten as a linear, convex problem. This definition also allows for handling portfolios with many instruments and/or scenarios. For these reasons, this paper uses CVaR as risk measure and implements stochastic programming to find a solution to the optimization problem.

2.3 Incorporating uncertainty: Stochastic Programming

Stochastic programming assumes that the uncertain parameters are random variables with (known) probability distributions (Cornuejols & Tütüncü, 2006). This information is then used to transform the stochastic program into a so-called deterministic equivalent. Scenario generation plays an important role within stochastic programming as it determines how accurately the underlying stochastic situation is represented. It refers to the process of describing the actual situation in the form of a set of possible scenarios. Herein, the trade-off between realism and model simplicity should be considered when deciding on the number of scenarios. Once the scenarios are generated in a representative manner, the rest of the problem is straightforward, as shown by Rockafellar and Uryasev (2000). They demonstrate that the CVaR constraint can be replaced by a simple linear approximation that assures that the CVaR values are properly restricted.

Over time, multiple approaches to account for parameter uncertainty in mean and covariance of returns have been examined. It has been shown that there is usually a greater estimation risk in mean returns as opposed to the covariance of returns (Merton, 1980; Best & Grauer, 1991). For this reason, researchers have focused on the minimum-variance portfolio, which ignores the estimation of the mean returns and is thus not as sensitive to estimation error (Chan, Karceski & Lakonishok, 1999; Jagannathan & Ma, 2003). This portfolio however still seems quite vulnerable to the impact of estimation error as discussed by DeMiguel and Nogales (2009). They notice the fact that the sample covariance matrix is the maximum likelihood estimator based on the assumption of normally distributed returns. The efficiency of these estimators is however highly sensitive to deviations from this assumption (Huber, 2011). Extensive evidence shows that the empirical distribution of returns usually deviates from the normal distribution and therefore the maximum likelihood estimator suffers from estimation risk (DeMiguel & Nogales, 2009). As an alternative, Maillard, Roncalli, and Teïletche (2008) propose equally weighted portfolios to avoid relying on the expected average returns.

DeMiguel and Nogales (2009) show the advantages of using robust estimators. Robust estimators are less sensitive to deviations from the distribution assumption. Though they are not as efficient as maximum likelihood estimators when the underlying distribution is correct. Other researchers that used robust estimation techniques to account for parameter uncertainty are Cavadini, Sbuelz, and Trojani (2001), Vaz-de Melo and Camara (2003), Perret-Gentil and Victoria-Feser (2005), and Welsch and Zhou (2007).

Another way of dealing with parameter uncertainty is the use of Bayesian approaches. In a Bayesian approach, one removes the dependence of the optimization problem on the unknown parameters (Brandt, 2010). Bayesian portfolio policies use estimators that are generated by combining the investor's prior beliefs with the evidence obtained from historical return data; see Jorion (1986), Black and Litterman (1992), and Pástor and Stambaugh (2000). For further understanding of the available Bayesian approaches in portfolio optimization, Fabozzi, Huang, and Zhou (2010) provide an overview.

Lastly, several methods to increase the performance of the traditional estimators in terms of estimation error can be interpreted as shrinkage methods. DeMiguel, Garlappi, Nogales, and Uppal (2009) propose to include an additional constraint in the minimum-variance problem that restricts the norm of the weight vector. Ledoit and Wolf (2003) propose to estimate the covariance matrix of returns by an optimally weighted average of the sample covariance matrix and a single-index covariance matrix. Garlappi et al. (2006) recommend a multi-prior model that can be interpreted as a shrinkage of the mean-variance portfolio towards either the risk-free asset or the minimum-variance portfolio. They show that allowing for parameter uncertainty reduces the fluctuation of portfolio weights over time and improves the out-of-sample performance. Jagannathan and Ma (2003) show that imposing short-selling constraints can help to reduce the impact of estimation error on the stability and performance of the minimum-variance portfolio.

2.4 Scenario generation: Filtered Historical Simulation

Barone-Adesi and Giannopoulos (2001) propose Filtered Historical Simulation (FHS) as an alternative to Historical Simulation (HS) to overcome shortcomings such as a limited set of outcomes and unresponsiveness to changes in market volatility. FHS is a generalized HS; it has all the positive properties and overcomes most of its weaknesses. A major advantage of FHS over HS is that it provides a systematic approach to generate extreme simulations that are not present in the historical data, completing the tails of the return distribution. In other words, the FHS algorithm creates a range of outcomes that is broader than the original set of historical observations. As a result, FHS requires a shorter dataset to simulate the entire return distribution. Giannopoulos and Tunaru (2005) specifically show how FHS can be used in estimating CVaR. The proposed methodology is flexible in the sense that it can handle individual securities as well as a portfolio of securities, provided that a robust conditional volatility model is in place.

Barone-Adesi et al. (1999) introduce an FHS algorithm where, at each simulation trial, a value for each asset is generated and all securities in the portfolio are re-priced. After running many simulation trials, a set of portfolio values is generated that forms the empirical distribution for the portfolio values at a certain horizon. FHS is based on a re-sampling approach of past returns, thereby following a non-parametric methodology. Though unlike other methods, the current market conditions are also considered through scaling the historical residuals by that period's conditional volatility forecast.

Many different methods to generate scenarios are used in the literature, we outline a few of the alternatives and their drawbacks as opposed to the method implemented here. Kaut and Wallace (2007) evaluate several scenario generation methods and propose approaches to test their quality within a certain problem definition. As a test case, they discuss moment-matching to generate the scenarios. This method generates scenario trees that match the given first four moments of the marginal distribution and the correlation matrix. Høyland, Kaut and Wallace (2003) present algorithms to produce scenarios according to the moment-matching method in an efficient manner. The main drawback of this method, however, is that all higher moments are completely ignored.

Kouwenberg (2001) uses event trees for scenario generation and compares the results for three methods: random sampling, adjusted random sampling, and fitting the mean and the covariance matrix. They benchmark with a simple fixed mix method for different values of risk aversion. Cariño et al. (1994) generate scenarios in three possible ways: independent scenarios, events dependent over time (time-series), or user-specified outcomes of the stochastic elements. They then use a modular structure including a scenario generator and solve a mean-variance problem.

Hellmich and Kassberger (2011) discuss the fact that empirical asset return distributions are non-normal and thus realistic modeling calls for alternative probability distributions. Recent empirical studies conducted in a multivariate setting make a convincing case for the multivariate generalized hyperbolic distribution and its subclasses (McNeil, Frey & Embrechts, 2005). Lastly, Uryasev (2000) and Krokhmal et al. (2002) use historical portfolio instrument prices and use these, after transforming them to returns, directly as scenarios for the stochastic programming approach. They, however, do not account for current market conditions in their scenario model but simply directly use historical values as scenarios.

2.5 Robust portfolio optimization: Worst-Case CVaR

Robust optimization is used when one wants a solution that behaves well in all possible realizations of the uncertain data (Cornuejols & Tütüncü, 2006). This approach explicitly recognizes that the result of the estimation process is not a single-point estimate, but rather an uncertainty set, where the true parameters lie within a certain confidence interval. The size of the uncertainty set is determined by the level of desired robustness and can be formed in multiple ways, for example by different opinions of future values of certain parameters or via statistical techniques from historical data and/or Bayesian techniques.

We adjust a method as introduced by Hellmich and Kassberger (2011). They suggest a robust optimization approach using Worst-Case Conditional Value-at-Risk (WCVaR) as a risk measure. Zhu and Fukushima (2009) demonstrate that WCVaR inherits subadditivity, positive homogeneity, monotonicity, and translation invariance from CVaR, and therefore - just as CVaR - is a coherent risk measure. Moreover, the authors show that WCVaR is convex in the portfolio weights. The solution to the mean-WCVaR problem will then be the allocation with optimal worst-case properties.

Hellmich and Kassberger (2011) show a numerical example of a robust portfolio optimization problem (R1" in their paper). They specify the uncertainty set by shifting the parameters from the base case estimation either up or down by 10 percent. Thus, the worst-case returns will have lower means, more pronounced negative skewness, and higher variances and covariances, leading to lower expected returns and higher risk of efficient portfolios.

Since such conservatism may not be desirable in some situations, an alternative is to seek robustness in a relative sense, where decisions made by a portfolio manager are considered successful in case the portfolio performs better than a certain benchmark (Cornuejols & Tütüncü, 2006). Gabrel, Murat, and Thiele (2014) also recognize the issues of over-conservatism in robust optimization and emphasize the trade-off between system performance and protection against uncertainty. Also, the Bayesian posterior distribution can be used to determine how likely a certain scenario set is.

Robust optimization distinguishes between constraint robustness and objective robustness (Cornuejols & Tütüncü, 2006; Gabrel et al., 2014). The first case refers to the possibility that data uncertainty puts the feasibility of solutions at risk, whereas the second case concerns the optimality of the generated solution. For more research on robust optimization, see among others Tütüncü and Koenig (2004), Garlappi et al. (2006), Goldfarb and Iyengar (2003), Ben-Tal, El Ghaoui and Nemirovski (2009), Ben-Tal and Nemirovski (2002), and also the overview given by Gabrel et al. (2014).

2.6 Practical background

Lastly, we discuss some relevant aspects of the practical environment of NN as (life) insurer. Currently, NN optimizes its asset allocation every three years. At the beginning of the three years, experts estimate the future return of the associated asset classes. They use the Solvency Capital Requirement (SCR) to determine the amount of risk a certain asset mix contains. The SCR is the percentage of funds that insurance and reinsurance companies are required to hold under the European Union's Solvency II directive (European Commission, 2015). The percentage is set such that the insurer's own funds can absorb losses over a 1-year horizon with a probability of at least 99.5% (Kouwenberg, 2018).

In practice, the Strategic Asset Allocation (SAA) is decided on keeping in mind the efficient use of regulatory capital, stable solvency ratio, attractive earnings, well-managed liquidity requirements, and diversification. Specific for the investment policy of an insurance company is the structure of payment responsibilities to contractors. Cariño et al. (1994) account for decisions in allocation strategies over time using a dynamic asset-liability model. They specifically integrate the uncertainty and movement of the liabilities over time in their model. This research focuses on the asset allocation part of the problem and thus not fully integrates asset-liability management into the optimization methods. Though we include constraints that assure matching of the maturity of liabilities with the timing of investment returns.

3 Data

In this study, we focus on five of the asset classes that Nationale-Nederlanden (NN) usually considers in its Strategic Asset Allocation (SAA), namely: Government Bond, Corporate Bond, Real Estate, Equity, and Mortgages. These asset classes correspond to the character of NN as a non-risky, long-term, buy-and-hold investor. We mostly use European indices from the financial database Bloomberg as a reference for the multiple bonds or assets available in the specific asset class. We use weekly data from 11 March 2005 to 25 May 2019, meaning that we have 742 observations in total. This time frame is a result of the data availability over the most important asset classes that are not too strongly correlated. We transform all indices to realized weekly returns and hereafter refer to these series as (weekly) return. Table A.1 in the Appendix shows an overview of the specifications of the data.

As sample set, we select the first five years of the data (260 observations), from 11 March 2005 to 26 February 2010, and use these observations for the selection of the appropriate volatility models. Although we use the most recent data when optimizing over the parameters, it is still important that the data in this sample set somehow is representative enough to decide on the correct volatility model. However, an important factor leading to the decision of the sample size is also the ability to estimate the volatility model correctly and thus the need for a large enough sample set.

The summary statistics presented in Table 1 show that the mean return in the sample period is rather different from the mean return in the full dataset, whereas the standard deviation is relatively more comparable. We see that Equity, Mortgages, and especially Real Estate give lower returns in the sample period, whereas Government Bonds and Corporate Bonds give higher returns. The correlations as shown in Table 2 are rather low, except for the correlation between Government Bonds and Corporate Bonds, and between Real Estate and Equity. The strongest correlation in the full dataset is 0.76, which we still consider as acceptable. We thus consider all five asset classes to be relevant to include in our model.

Table 1: Summary return statistics full and sample dataset

	Full dataset			*Sample set*		
Asset	Mean	Median	Std.	Mean	Median	Std.
GOV	3.75	4.94	4.63	4.80	4.54	5.37
CB	3.82	4.53	3.29	5.03	5.90	4.08
RE	2.26	16.45	22.70	-5.66	15.22	29.80
EQ	5.90	22.01	20.78	3.35	14.27	24.04
MO	5.82	4.95	3.36	4.27	5.64	3.79

All statistics are annualized values of the weekly return data (in %).

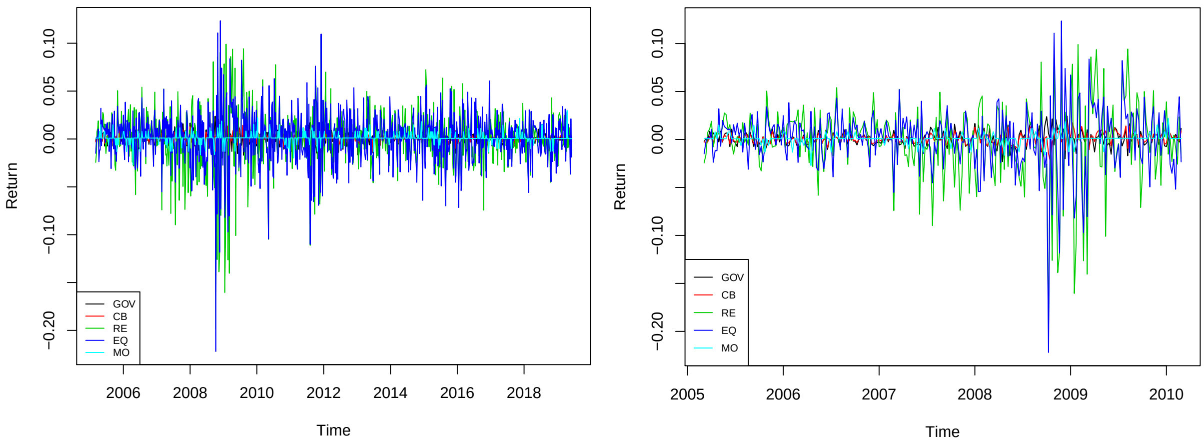
Table 2: Correlation matrix returns full and sample dataset

	GOV	CB	RE	EQ	MO
GOV	1	0.78	-0.28	-0.35	0.01
CB	0.76	1	-0.11	-0.18	0.04
RE	-0.15	0.02	1	0.77	0.04
EQ	-0.31	-0.09	0.74	1	0.04
MO	-0.01	0.01	0.05	0.05	1

The lower triangular shows the correlations in the full dataset and the upper triangular shows the correlations in the sample dataset.

The plots in Figure 1 reveal the presence of heteroscedasticity. We observe periods of high and low volatility that occur in clusters. Around 2007 to 2009 we see some more negative returns as a result of the financial crisis. The sample set is chosen such that this period of higher volatility is included. As a consequence, the sample statistics differ somewhat more from the statistics of the full dataset, especially for Real Estate.

Figure 1: Weekly returns over time



(a) Full dataset **(b)** Sample set

Table 3 shows the results of three statistical tests on the distribution of returns, performed on the return series of both the full dataset and the sample set. For the full dataset, the Jarque-Bera test rejects the null hypothesis of zero skewness and zero excess kurtosis, whereas, for one of the five assets in the sample set, the null hypothesis cannot be rejected on a 5% significance level. The results of the Ljung-Box-Pierce test shows that there is no clear presence of autocorrelation in the data, except for the Corporate Bond and Equity classes in the full dataset. For all classes except Mortgages, the Engle's LM test rejects the null hypothesis that the returns form a random sequence of normal disturbances, hence the presence of heteroscedasticity. Following the features of the majority of the asset classes, we use an ARMA(p,q)-GARCH(1,1) type of model to capture the time-varying volatility of returns.

Table 3: Statistical tests full and sample dataset

	Jarque-Bera test		Ljung-Box Q		Engle's LM test	
	χ^2	p-value	χ^2	p-value	χ^2	p-value
			Full dataset			
GOV	58.62	0.00	0.56	0.45	114.38	0.00
CB	148.85	0.00	11.17	0.00	153.12	0.00
RE	941.64	0.00	1.98	0.16	184.20	0.00
EQ	1209.76	0.00	10.29	0.00	114.35	0.00
MO	1881.33	0.00	2.14	0.14	10.94	0.53
			Sample set			
GOV	5.69	0.06	0.02	0.89	40.39	0.00
CB	22.61	0.00	6.33	0.01	61.92	0.00
RE	184.97	0.00	1.12	0.29	64.42	0.00
EQ	820.25	0.00	5.39	0.02	45.82	0.00
MO	417.39	0.00	1.16	0.28	9.61	0.65

4 Methods

4.1 Portfolio optimization methods

We formulate two problem definitions, one that represents a simplification of the current method used by Nationale-Nederlanden (NN) and one that uses Conditional Value-at-Risk (CVaR) as risk measure. In both cases, the objective of the optimization problem is to maximize the total portfolio end-of-horizon return, as NN is a buy-and-hold investor that does not change its portfolio weights during the horizon of the investment. We define $x_{i,t}$ as the proportion of the total funds invested in security i and $r_{i,t}$ as the average simulated end-of-horizon return of security i, at time t. We calculate $r_{i,t}$ by first taking the product over each simulated weekly return (plus 1) within the investment horizon, and then averaging on all J scenarios at time t. We then define $\boldsymbol{x}_t$ and $\boldsymbol{r}_t$ as the $N \times 1$ vectors that contain the portfolio weights and returns of all N assets respectively. In the same manner, we define $\mu_{i,t}$ as the realized end-of-horizon return of security i at time t and $\boldsymbol{\mu}_t$ the resulting $N \times 1$ vector containing the values of all N assets.

4.1.1 Current method

The problem definition **currently used by NN** is the maximization of future returns subject to several constraints among which most importantly the restriction formed by the Solvency Capital Requirement (SCR). At each time t, we calculate the optimal portfolio mix $\boldsymbol{x}_t$ by solving the following problem definition:

$$
\begin{aligned}
\max_{\boldsymbol{x}_t} \quad & \boldsymbol{x}_t^T \boldsymbol{r}_t \\
\text{s.t.} \quad & A_t \cdot \mathrm{SCR}(\boldsymbol{x}_t) \cdot (1+\lambda) \quad \leq A_t - L_t \\
& \boldsymbol{x}_t \in \mathbb{C}.
\end{aligned}
\tag{1}
$$

Here, $\mathrm{SCR}(\boldsymbol{x}_t)$ is the required percentage of the assets' value to be held as a buffer based on the portfolio with weights $\boldsymbol{x}_t$, λ is the bandwidth to which we want to be close to the legally required amount, A_t and L_t are the current assets' and liabilities' values respectively and $\mathbb{C}$ is the set of feasible solutions of $\boldsymbol{x}_t$ taking into account the appropriate constraints on regulations and liquidity (these are explicitly described in Section B.1 in the Appendix). The right-hand side of the constraint is also referred to as own funds, defined as the difference between the market value of the assets and the liabilities. Therefore, it measures the capital that the company owns to cover future losses and to pay out dividends. Here, A_t is determined by multiplying the realized return over the previous period with the value of the assets in the previous period: $A_t = A_{t-1}(1+\boldsymbol{x}_{t-1}^T \boldsymbol{\mu}_{t-1})$ where $\boldsymbol{x}_{t-1}$ is the optimal allocation in the previous period and $\boldsymbol{\mu}_{t-1}$ are the actual end-of-horizon returns in that period. We determine the value of the liabilities over time in the same manner, correcting the previous value for the current interest rate. Assuming that our Government Bond series are approximately risk-free and have the same duration as our liabilities, we use this curve for adjusting the liabilities to the current market value. When optimizing the allocation at time t, the values of A_t and L_t are fixed and hence $\boldsymbol{x}_t$ is the only variable vector in the problem definition in equation 1. Following the internal guidelines within NN, we set λ equal to 1, such that the capital in own funds is at least 200% of the required amount. We set A_0 and L_0 equal to 100 and 90 respectively.

Next, we define the calculation of $\mathrm{SCR}(\boldsymbol{x}_t)$ more closely. In practice, life insurers are subject to several regulations to safeguard the financial solvency for its contractors, among which the SCR. The SCR is defined as the percentage of own funds to be held to have a 99.5% confidence that the insurer can survive the most extreme expected losses over a year, i.e. the 99.5% VaR level (European Commission, 2015). The Solvency II regulation offers a standard calculation of the SCR that should theoretically hold for all insurers within the EU. As NN is a very large player in the European insurer market, it can use an internal calculation of the SCR (after approval of the DNB). Nonetheless, to make this research more broadly applicable to insurers in general, we apply the standard calculation.

The SCR is divided into modules corresponding to different risk types: non-life underwriting, life underwriting, health underwriting, market, counterparty default, intangible asset, operational. Each module is again divided into multiple sub-modules and results in a standalone amount to be held in own funds to cover for the risk in each respective asset type. The standalone amounts are determined by multiplying the SCR percentage with the total amount of capital. The standalone SCR percentages are determined by the legislation, also referred to as shocks. However, the total risk of a portfolio is lower than the sum of the separate assets' risks thanks to the diversification effect. As this diversification effect is of substantial influence on a portfolio's total risk, the SCR calculation also takes this into account. Each SCR module therefore corrects for the diversification advantage after calculating the standalone amounts of the sub-modules by using a correlation matrix. This correlation matrix is again provided by the supervisory authorities.

Following the reasoning as explained above, we approximate the SCR in two steps. First, the standalone amounts per asset class are determined based on the shocks and duration of each asset class and the absolute amount of capital invested in each respective asset class. We can thus define a vector, $\mathbf{SCR}_s$, containing the standalone SCR percentages per asset class as follows:

$$\mathbf{SCR}_s(\boldsymbol{x}_t) = \boldsymbol{x}_t \odot \boldsymbol{s}, \tag{2}$$

where $\boldsymbol{x}_t$ is the vector containing the portfolio weights per asset class and $\boldsymbol{s}$ the vector containing the shocks per asset class. We use $\odot$ to indicate element-wise multiplication, meaning that the result is an $N \times 1$ vector.

Second, we calculate the actual total percentage of capital to be held taking the diversification advantage into account. We therefore pre- and post-multiply the correlation matrix with the vector $\mathbf{SCR}_s$ as defined above, and take the square root. The total SCR is thus defined as:

$$\begin{aligned} \mathrm{SCR}(\boldsymbol{x}_t) =& \sqrt{\mathbf{SCR}_s^T \cdot \boldsymbol{R} \cdot \mathbf{SCR}_s} \\ =& \sqrt{\boldsymbol{x}_t^T \boldsymbol{\Omega} \boldsymbol{x}_t}, \end{aligned} \tag{3}$$

where $\boldsymbol{R}$ is the correlation matrix, and we define $\boldsymbol{\Omega} = \mathrm{diag}(\boldsymbol{s}) \cdot \boldsymbol{R} \cdot \mathrm{diag}(\boldsymbol{s})$, where $\cdot$ is used to indicate (usual) matrix multiplication.

This means that we can write the SCR constraint as follows:

$$
\begin{aligned}
A_t \cdot \mathrm{SCR}(\boldsymbol{x}_t) \cdot (1+\lambda) &\leq A_t - L_t \\
\mathrm{SCR}(\boldsymbol{x}_t) &\leq \frac{A_t - L_t}{(1+\lambda)A_t} \\
\boldsymbol{x}_t^T \boldsymbol{\Omega} \boldsymbol{x}_t &\leq \left[\frac{A_t - L_t}{(1+\lambda)A_t}\right]^2,
\end{aligned} \tag{4}
$$

which is a quadratic inequality constraint of the portfolio weights $\boldsymbol{x}_t$. This is an important result as this means that the problem given in equation 1 turns into a quadratic program and can be solved analytically. The analytical solution to this problem definition including (only) the SCR constraint is given in Section B.2 in the Appendix.

The values of the shocks as well as the correlation matrix are given in the implementing measures of Solvency II regulation (European Commission, 2015). All asset classes except Mortgages fall within the market risk module as described in Section 5 of this directive, whereas the Mortgages fall within the counterparty default risk module. We find the following shocks for the five considered asset classes:

- *Government Bonds AAA*: 0%, following Article 180.2. We assume that this class falls within the class of bonds issued by the central government, funded in the domestic currency of that central government, as described in Note a) of this article.
- *Corporate Bond*: 7.63%, following Article 176.3. Based on rating 2 and an average duration of 5.9, we calculate the shock for this asset class as $7.0\% + 0.7\% \cdot (5.9 - 5) = 7.63\%$.
- *Real Estate*: 25%, following Article 174.
- *Equity*: 39%, following Article 169.1b and assuming our equity index falls within the Type 1 class as described in Article 168.6.
- *Mortgages*: 0.75%, following Article 189.3. We assume Mortgages fall within the Type 2 exposure within the counterparty default risk module. We calculate the loss-given-default (LGD), following Article 192.4, as $\mathrm{LGD} = \max(\mathrm{loan} - 80\% \cdot \mathrm{mortgage}, 0)$. Assuming that half of our Mortgages has a Loan-to-Value (LTV) lower than 80% and the other half has an average LTV of 90%, our average LGD is $0.5 \cdot 0 + 0.5 \cdot (90\% - 80\%) = 5\%$. Following Article 202 for Type 2 exposures and ignoring the factor for receivables from intermediaries, we calculate the appropriate percentage as $15\% \cdot 5\% = 0.75\%$.

Next, we define the correlation matrix to be used to incorporate the diversification effect. Article 164.3 presents the standard correlations. Following Article 165 and assuming the risk of a decrease in the term structure of interest rates, we set parameter A equal to 0.5; this gives us the following correlation matrix.

Table 4: Correlations within market risk module Solvency II directive

	GOV	CB	RE	EQ
GOV	1	0.5	0.5	0.5
CB	0.5	1	0.5	0.75
RE	0.5	0.5	1	0.75
EQ	0.5	0.75	0.75	1

Values taken from Article 164.3 (European Commission, 2015).

To calculate the SCR over the total portfolio, we use the correlation between the market risk and counterparty default risk modules of 0.25 as given in Annex IV Article 1 to Directive 2009/128/EC (European Commission, 2009). Officially, the SCR is calculated via a two-step procedure in which one first calculates the SCR based on the market risk, $\text{SCR}_{\text{market}}$, using the correlation matrix as shown above, and the standalone amount for the default risk module (hence Mortgages), $\text{SCR}_{\text{default}}$. One then calculates the total SCR as $\text{SCR}_{\text{total}} = \sqrt{\mathbf{SCR}^T \cdot \boldsymbol{R} \cdot \mathbf{SCR}}$, where $\mathbf{SCR}$ is the vector containing $\text{SCR}_{\text{market}}$ and $\text{SCR}_{\text{default}}$, and $\boldsymbol{R}$ contains the correlations between all sub-modules and the default risk module separately. However, to ensure that the SCR constraint remains a quadratic function of the portfolio weights, we simplify this multiplication. We set all correlations of the sub-modules within the market risk module with the counterparty default risk module equal to the correlation between the modules itself, hence 0.25. We thus use the following correlation matrix in our SCR constraint.

Table 5: Full correlation matrix SCR calculation

	GOV	CB	RE	EQ	MO
GOV	1	0.5	0.5	0.5	0.25
CB	0.5	1	0.5	0.75	0.25
RE	0.5	0.5	1	0.75	0.25
EQ	0.5	0.75	0.75	1	0.25
MO	0.25	0.25	0.25	0.25	1

Values taken from Article 164.3 (European Commission, 2015) and Annex IV Article 1 (European Commission, 2009).

4.1.2 Test method

To evaluate the effect of incorporating uncertainty into the optimization method, we define a **test problem definition**. This problem takes the form of the traditional Markowitz formulation but using CVaR as risk measure instead of variance. We define the problem as follows:

$$\begin{aligned} \max_{\boldsymbol{x}_t} \quad & \boldsymbol{x}_t^T \boldsymbol{r}_t \\ \text{s.t.} \quad & \text{CVaR}_\beta(\boldsymbol{x}_t) \leq C_\beta \\ & \boldsymbol{x}_t \in \mathbb{C}, \end{aligned} \tag{5}$$

where C_β represents the maximum tolerable CVaR value at the confidence level β and $\mathbb{C}$ is the same set of feasible solutions as used in equation 1. We set C_β equal to $0.8 \cdot (\text{A}_t - \text{L}_t)/\text{A}_t$ and β equal to 99.5%, as these values are the confidence levels also used in the calculation of the SCR. This implies that the average loss in the 0.5% worst cases cannot exceed 80% of the percentage of own funds that we hold at the beginning of the investment period (i.e. $(\text{A}_t$ - $\text{L}_t)/\text{A}_t$). The 80% is also chosen so that the CVaR constraint is most comparable to the SCR constraint. In the SCR restriction, we multiplied the right-hand side with 50% (implied through the λ of 1). However, the SCR is based on Value-at-Risk (VaR) and therefore, by definition, more restrictive than the constraint based on CVaR. To account for this, we set the percentage higher than the 50% used in the SCR constraint.

CVaR is defined as the expected loss given that the loss exceeds VaR (Cornuejols & Tütüncü, 2006). The loss function $f(\boldsymbol{x}, \boldsymbol{\mu})$ is defined as the negative of the return on a portfolio $\boldsymbol{x}$, which is the sum of the returns on the individual instruments in the portfolio, μ_i, scaled by the proportions x_i. The function is also defined as

$$f(\boldsymbol{x}, \boldsymbol{\mu}) = -[x_1\mu_1 + ... + x_N\mu_N] = -\boldsymbol{x}^T\boldsymbol{\mu}, \tag{6}$$

where N is the number of instruments. We then use this function to define VaR. Given a probability level β, the β-VaR of the loss function is given by

$$\text{VaR}_\beta(\boldsymbol{x}) := \min\{\gamma : P(f(\boldsymbol{x}, \boldsymbol{\mu}) \geq \gamma) \leq 1 - \beta\}. \tag{7}$$

Using this definition of VaR, the β-CVaR of the portfolio with weights $\boldsymbol{x}$ can then be defined as:

$$\text{CVaR}_\beta(\boldsymbol{x}) := \frac{1}{1-\beta} \int_{f(\boldsymbol{x},\boldsymbol{\mu}) \geq \text{VaR}_\beta(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{\mu}) p(\boldsymbol{\mu}) d\boldsymbol{\mu}, \tag{8}$$

where $p(\boldsymbol{\mu})$ is the probability density function of the random return vector $\boldsymbol{\mu}$. It is clear that the calculation of the CVaR is not a linear formula, therefore Section 4.2 shows how CVaR can be approximated so that it can be solved with a linear program.

Krokhmal et al. (2002) show that the problem as defined in equation 5 can be equivalently formulated as minimizing CVaR subject to minimum total return, or maximizing CVaR adjusted total return (see Theorem 3 in their article). Moreover, Rockafellar and Uryasev (2000) compare three similar problem formulations: minimum β-CVaR, minimum β-VaR, and minimum variance, all constrained with a maximum portfolio loss. They show that these problems give the same optimal portfolio $\boldsymbol{x}_t^*$ under the following conditions: the asset returns $\boldsymbol{\mu}$ are normally distributed, $\beta \geq 0.5$ and the constraint on maximum portfolio loss is active (see the Proposition and its proof in Rockafellar and Uryasev (2000)). They show that in case of normally distributed asset returns, and $\beta \geq 0.5$, both the formula for VaR and CVaR become a linear function of the mean and variance of portfolio loss (see Section B.3 in the Appendix for the proof). If then also the constraint on maximum portfolio loss is active, one can replace the inequality constraint by the equality constraint $l(\boldsymbol{x}_t) = -R$ and substitute the mean portfolio loss by the maximum accepted loss. The formulas for VaR and CVaR then simplify to:

$$\text{VaR}_\beta(\boldsymbol{x}) = -R + c_1(\beta)\sigma(\boldsymbol{x}) \quad \text{and} \quad \text{CVaR}_\beta(\boldsymbol{x}) = -R + c_2(\beta)\sigma(\boldsymbol{x}), \tag{9}$$

where the coefficients $c_1(\beta)$ and $c_2(\beta)$ are positive and $\sigma(\boldsymbol{x})$ is the variance of the loss associated with portfolio $\boldsymbol{x}$. Thus, minimizing either of the expressions in equation 9 over $\boldsymbol{x}$ is the same as minimizing $\sigma(\boldsymbol{x})$ over $\boldsymbol{x}$. Section B.4 in the Appendix shows how the problem definition in equation 5 is solved analytically under the normality assumption. These conditions indicate in what case problem 1 and 5 are both equivalents to the traditional mean-variance formulation. This also gives rise to the simplification made in the current problem definition used by NN, as asset returns are in practice usually not normally distributed. The use of the definition given in equation 5 relaxes this assumption and is therefore expected to be a more realistic problem definition. We refer to the Literature Review in Section 2 for more deficiencies in the use of VaR constraints.

From a statistical point of view, the SCR constraint, as used in the current problem definition in equation 1, is equivalent to

$$x_0 = \inf\Big\{x : \Pr\Big(-\frac{A_{t+1} - L_{t+1}}{1 + r_f} \leq x\Big) = \alpha\Big\}, \tag{10}$$

where $A_t - L_t$ are the own funds and r_f the risk-free rate (Zhou, 2018). Hence, x_0 is the VaR of $\dfrac{A_{t+1} - L_{t+1}}{1 + r}$ at the probability level α. Although the actual calculation of SCR is a more complex calculation, it can be considered as a VaR constraint. This interpretation helps comparing the current problem definition given in equation 1 to the test problem definition as given in equation 5. Important to realize here is that the parameters in the current problem definition are taken as a point estimate and therefore this method does not take uncertainty into account.

The optimization problem as shown in equation 5 is solved using both stochastic programming and robust optimization to see the effect of using stochastic optimization methods. To sum up, we compare the performance of two specific methods that take uncertainty into account with the deterministic method currently used by NN. The problems that are to be solved and compared are the following:

i. *SCR method*: deterministic method currently used by NN as specified in equation 1,
ii. *CVaR method*: problem defined in equation 5 solved using stochastic programming,
iii. *Worst-Case method*: problem defined in equation 5 solved using robust optimization.

By comparing the results of both stochastic methods with the performance of the benchmark, we are able to evaluate the effect of taking uncertainty into account via the different methods.

4.2 Stochastic Programming

We use stochastic programming to incorporate uncertainty in returns and solve the problem given in equation 5. Rockafellar and Uryasev (2000) made an important contribution to CVaR optimization by proving that, in the problem definition as given in equation 5, CVaR can be replaced by the approximating function:

$$\tilde{F}_\beta(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^{J} (-\boldsymbol{x}^T \boldsymbol{r}^{(j)} - \alpha)^+, \tag{11}$$

where $(x)^+ = \max\{x, 0\}$ and $\boldsymbol{r}^{(j)}$ is the vector with returns in scenario j. This function converges to the exact CVaR when the number of scenarios, J, approximates infinity. Rockafellar and Uryasev (2000) show how the use of this approximation simplifies the problem to one that can be solved using linear programming, namely

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & l(\boldsymbol{x}) = J^{-1} \sum_{j=1}^{J} -[x_1 r_1^{(j)} + ... + x_N r_N^{(j)}] \\ \text{s.t.} \quad & \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^{J} u^{(j)} \leq C_\beta \\ & u^{(j)} \geq 0 \quad \text{and} \quad \boldsymbol{x}^T \boldsymbol{r}^{(j)} + \alpha + u^{(j)} \geq 0 \quad \text{for } j = 1, ..., J \\ & \boldsymbol{x} \in \mathbb{C}. \end{aligned} \tag{12}$$

Here, the terms $(-\boldsymbol{x}^T\boldsymbol{r}^{(j)}-\alpha)^+$ are replaced by auxiliary variables u_j and additional constraints are imposed. Moreover, they show how to use stochastic programming when there is no explicit probability distribution function available for future returns. Section B.5 in the Appendix shows the full derivation and explanation of this theory.

It is the generation of these J scenarios and their implementation in the constraints given in equation 12 that makes the problem stochastic. The crux of the problem then lies in the manner of scenario generation and the assumptions underlying this choice. The stochasticity is however only embodied in the constraints and not in the objective function of this problem, as the input in the objective function are the sample means of the respective asset classes.

4.3 Filtered Historical Simulation

We generate scenarios following the Filtered Historical Simulation (FHS) method. The main reason for the use of this method is that we can exploit the historical data instead of making a distributional assumption on the assets' return movements. Barone-Adesi et al. (1999) introduce an algorithm to generate correlated pathways for a set of risky assets using FHS. The main steps in this algorithm are summarized below (Giannopoulos & Tunaru, 2005):

i. First, a conditional volatility model is selected based on the fit with the sample dataset. The model should produce i.i.d. residuals and have strong forecasting power in predicting volatility over the investment horizon. As conditional volatility models, we use ARMA(p,q)-GARCH(1,1) types of specifications. The ARMA(0,0)-GARCH(1,1) model including mean is specified as follows

$$\begin{aligned} r_t &= \mu + \eta_t \qquad \eta_t \sim (0, h_t) \\ h_t &= \omega + \alpha\eta_{t-1}^2 + \beta h_{t-1} \end{aligned} \tag{13}$$

with restrictions $\omega, \alpha, \beta > 0$ and $\alpha + \beta < 1$.

ii. The resulted model from step one is fitted on the historical data of each asset class separately to generate volatilities for each day of the sample period. The realized residuals are then standardized by dividing each one of them by the corresponding conditional volatility ($e_t = \hat{\eta}_t \oslash \sqrt{\hat{h}_t}$) and the standardized values should be i.i.d.. We gather the standardized residuals for the entire sample set and all asset classes in a $260 \times N$ matrix and use it together with the estimated coefficients $\hat{\omega}, \hat{\alpha}, \hat{\beta}$ and $\hat{\mu}$ to predict the volatilities and returns in step three and four.

iii. The third step consists of using the last observed volatility h_t and residual η_t from the sample set to forecast the first volatility, in case of the GARCH(1,1) specification, by $\hat{h}_{t+1} =$

$\hat{\omega} + \hat{\alpha}\eta_t^2 + \hat{\beta}h_t$. We then draw a random residual return e_t^* from the set of standardized residuals from the second step (semiparametric distribution implementation) and scale it with the forecasted volatility to obtain $\hat{\eta}_{t+1} = e_t^* \odot \sqrt{\hat{h}_{t+1}}$. The correlation between the asset classes is preserved by selecting one row (referred to as "strip" by Barone-Adesi et al. (1999)) from the matrix of standardized residuals and use the corresponding values for all N assets in this volatility prediction. This way, the pathways for the variance of the multiple asset classes reflect the co-movements between asset returns. Finally we can calculate the one-step-ahead return prediction, in the case of the ARMA(0,0) model, $\hat{r}_{t+1} = \hat{\mu} + \hat{\eta}_{t+1}$.

iv. To create the two-step-ahead return predictions (and further), we use the estimated values from the previous draw to obtain a new volatility forecast, scaled random residual, and return prediction, and repeat these steps throughout the investment horizon. This results in a single return path simulation. We then transform this return path into a single end-of-horizon return. We repeat steps three and four many times to generate a complete set of scenarios at the given moment in time, where we still use the same matrix of standardized residuals and estimated coefficients from step two.

v. Finally, for each moment in time on which we want to make an investment decision, we repeat steps two to four to obtain a set of simulated return paths. We use these compound end-of-horizon returns as input for the optimization algorithms.

In step one, we fit the ARMA(1,1), ARMA(1,0) and ARMA(0,0) (all combined with GARCH(1,1)) models using the semiparametric distribution that directly uses the empirical distribution of historical observations. We select one of these three return models by comparing the Akaike and Bayes information criteria. Empirically, asset returns are proven not to be normally distributed, however, the estimates based on the normal distribution would asymptotically still be valid, whereas this would not be the case with other distributions. Thus, we test whether the normal distribution assumption holds for this model.

4.4 Worst-Case CVaR

By using FHS, the stochastic programming method takes uncertainty around the returns into account. However, the parameters in the GARCH model as described in equation 13 are still taken as a fixed, deterministic point estimate.

As robust optimization, we incorporate, next to the uncertainty around the returns, also parameter uncertainty. We create a distribution around the coefficient estimates by adding a normal distributed shock with corresponding variance to the coefficients. For each set of coefficients realizations we implement the FHS method and determine the objective value of the optimized asset mix. We then select the set of coefficients that corresponds to the lowest objective value and use the corresponding scenario set as Worst-Case realization. The input in the objective function is the mean return on all simulations in the scenario set and is thus dependent on the distribution around the coefficient estimates in the GARCH model. This way, this approach not only incorporates uncertainty in the CVaR constraint but indirectly also in the objective function. At each moment in time, the optimization is implemented in the same way as the standard CVaR optimization, using the approximation given in equation 12. Mathematically written, for each optimization time t we solve the following problem:

$$\begin{aligned} \min_q \quad & \left[\max_{\boldsymbol{x}} \quad \boldsymbol{x}^T \boldsymbol{r}_q\right] \\ \text{s.t.} \quad & \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^{J} u_q^{(j)} \leq C_\beta \\ & u_q^{(j)} \geq 0 \quad \text{and} \quad \boldsymbol{x}^T \boldsymbol{r}_q^{(j)} + \alpha + u_q^{(j)} \geq 0 \quad \text{for } j = 1, ..., J \\ & \boldsymbol{x} \in \mathbb{C}, q \in \mathbb{Q}. \end{aligned} \tag{14}$$

Here, $\boldsymbol{r}_q$ is the vector with average returns of all N assets on the J scenarios in one specific realization q, $\mathbb{C}$ contains the set of feasible asset mixes under the set of constraints as given in equation B.1 and $\mathbb{Q}$ is the set of Q coefficient realizations with subsequent set of J scenarios. Due to computational time limitations, we set Q equal to 25.

The results of this method show whether our outcomes are dependent on the specific coefficients' point estimate used for the scenario generation. The optimal asset mixes are interpreted as the mix that is robust to the worst realization of the asset returns. Therefore, we expect that this method behaves better in years with lower realizations of returns.

4.5 Implementation

Following the methods as used by both Diris et al. (2014) and DeMiguel et al. (2007) we use a rolling-window approach to create a time-series of optimal allocations. We start by fitting the volatility model to a sample set to obtain the coefficient estimates and residuals. We use a fixed window of size S instead of an expanding window as used by Diris et al. (2014). This way, the differences between the resulting estimates are independent of the size of the data window and can be fairly compared. As a sensitivity analysis we will also show the results of using an expanding window. The sample size S is five years, this way we use 260 weekly observations which gives us a dataset that is large enough to appropriately estimate the model. We then use these estimates to forecast a series of $J = 1000$ return paths. The length of each path is equal to the investment horizon (H) of one year, resulting in 52 weekly simulations or one end-of-horizon return. These return paths are the input for the allocation optimizations. Finally, we repeat these steps with the same frequency as the investment horizon, thus every year, such that the return paths are non-overlapping and we can interpret the time-series as a re-investment procedure.

The full process is then as follows. Every year, starting from time $t = S + 1$, we use the data in the previous S weeks to estimate the volatility model, generate J scenarios, and find the optimal portfolio weights. We then use these weights to compute the end-of-horizon return in week $t + H$. This process is continued by adding the return for the next year in the dataset and dropping the earliest returns, until the moment at which we can still calculate the actual return over the full investment horizon, thus $T - H$, where T is the total number of observations in the dataset. The outcome of this rolling-window approach is a series of $\frac{T-S-H}{H}$ yearly out-of-sample returns generated for each optimization method, as the frequency with which we optimize is equal to H.

4.6 Evaluating performance

We evaluate portfolio performance based on return, risk, dispersion, and stability over time and use various metrics to indicate these measures. We compare the SCR, CVaR and Worst-Case methods based both on the average of these measures over time as well as on how the measures evolve over time.

Return

After the optimal allocation $\boldsymbol{x}_t^*$ at time t is determined, we compute the realized end-of-horizon portfolio return as

$$\mathrm{R}_t = \boldsymbol{x}_t^T \boldsymbol{\mu}_t, \tag{15}$$

where $\boldsymbol{\mu}_t = (\mu_{1,t}, ..., \mu_{N,t})$ are the actual end-of-horizon returns within the current investment period t for the N assets.

Risk

We use three metrics to measure risk. Firstly, the portfolio's variance is calculated as $\mathrm{Var}[\boldsymbol{x}_t^T \boldsymbol{\mu}_t] = \boldsymbol{x}_t^T \boldsymbol{Q}_t \boldsymbol{x}_t$, where $\boldsymbol{Q}_t$ is the covariance matrix over the actual (weekly) returns within the same period. We then compute the Sharpe Ratio (SR) as

$$\mathrm{SR}_t := \frac{\boldsymbol{x}_t^T \boldsymbol{\mu}_t}{\sqrt{\boldsymbol{x}_t^T \boldsymbol{Q}_t \boldsymbol{x}_t}}. \tag{16}$$

As we optimize our portfolios based on CVaR, we also calculate the SR based on CVaR (SR^{C}). We estimate CVaR by first taking the 99.5% VaR and then calculating the average over the returns that are lower than the 99.5% VaR. We then calculate SR^{C} as

$$\mathrm{SR}_t^{\mathrm{C}} := \frac{\boldsymbol{x}_t^T \boldsymbol{\mu}_t}{\mathrm{CVaR}_\beta(\boldsymbol{x}_t)}. \tag{17}$$

Lastly, we compute the solvency II ratio as

$$\mathrm{SII}_t = \frac{\mathrm{A}_{t,H} - \mathrm{L}_{t,H}}{\mathrm{SCR}(\boldsymbol{x}_t)\mathrm{A}_{t,H}}, \tag{18}$$

where $\mathrm{A}_{t,H} - \mathrm{L}_{t,H}$ is the value of own funds at the end of the investment horizon H, and $\mathrm{SCR}(\boldsymbol{x}_t)$ is the legally required capital to be held based on portfolio $\boldsymbol{x}_t$ as defined in equation 3. Generally, insurers want this ratio to be at least larger than 2 so that we hold 200% of the required amount in own funds.

Dispersion

To indicate how the portfolio return and risk would have behaved in different scenarios than the reality we use two metrics to indicate dispersion. We generate a specific scenario set that represents a hypothetical distribution around the five assets' return and use it for all three methods and its variations so that we can compare the results accordingly.

First, we use the optimal allocation vector $\boldsymbol{x}_t^*$ to calculate the portfolio's return distribution as

$$\mathrm{R}_t^{(j)} = \boldsymbol{x}_t^T \boldsymbol{r}_t^{(j)} \quad \text{for} \quad j = 1, ..., J, \tag{19}$$

where $\boldsymbol{r}_t^{(j)}$ is the vector of end-of-horizon returns in scenario j within the investment period t. We compute the portfolio's horizon return on all J return scenarios and calculate the 0.5, 50, and 99.5 percent quantiles. We use the same scenario set to calculate these quantiles for all three methods (and variations made in the Sensitivity Analysis) so that we can compare the results accordingly. These quantiles are useful to determine the uncertainty around the portfolio return. They indicate whether the optimal portfolio mix is also performing well in scenarios worse than reality and how robust the results are to changes in asset returns. We specifically evaluate the range between the 99.5 and 0.5 percent quantiles.

Second, we compute the distribution around the SII ratio in the same manner,

$$\mathrm{SII}_t^{(j)} = \frac{\mathrm{A}_{t,H}^{(j)} - \mathrm{L}_{t,H}}{\mathrm{SCR}(\boldsymbol{x}_t)\mathrm{A}_{t,H}^{(j)}} \quad \text{for} \quad j = 1, ..., J, \tag{20}$$

where $A_{t,H}^{(j)}$ is the asset value in return scenario j. For this ratio, we again calculate the value on all return scenarios and calculate the quantiles. As insurers want this ratio to be larger than 2, we specifically focus on the number of times the lowest quantile passes this requirement. This way, we can determine whether we still satisfy the requirement also in scenarios worse than reality.

Stability over time

Lastly, we also compare the change in allocations over the different asset classes over time using portfolio turnover (TO) as defined by DeMiguel et al. (2007)

$$\mathrm{TO}_t := \frac{1}{2}\sum_{i=1}^{N}\left(|x_{i,t} - x_{i,t^-}|\right), \tag{21}$$

where $x_{i,t}$ is the desired portfolio weight in asset i *after* rebalancing at time t, x_{i,t^-} the actual weight in asset i at time t *before* rebalancing, and N is the number of assets. Note that $x_{i,t-1}$ may be different from x_{i,t^-} due to changes in asset prices between time $t-1$ and t. The $\frac{1}{2}$ is added as the wealth is traded from one asset to another and should not be counted to be traded twice. The TO quantity can be interpreted as the average percentage of wealth traded in each period. It thus indicates the transaction costs belonging to the investment strategy as well as the stability of the weights over the investment period. In general, and especially for a long-term investor such as NN, a low TO is preferred over a high TO.

To indicate stability, we also examine the optimal weights as such. The difference between the optimal weights over time is purely caused by the use of a different information set and is thus independent of the realized return and risk in that year. Therefore, the weights indicate how stable the strategy is to the use of different information sets. We compare the optimal weights over time and evaluate the difference when using an expanding window estimation in the Sensitivity Analysis.

5 Results

The analysis of our results is split into two parts. We start with a discussion of the scenarios generated using Filtered Historical Simulation (FHS), after which we discuss the optimal portfolios resulting from the SCR, CVaR and Worst-Case methods. We end with a Sensitivity Analysis in which we show and compare the results using alternative settings.

5.1 Simulation results

We start the FHS method by selecting the most appropriate conditional volatility model from the six considered options: ARMA(0,0)-GARCH(1,1)[1] versus ARMA(1,0)-GARCH(1,1) versus ARMA(1,1)-GARCH(1,1) specification using either the normal distribution or the semi-parametric distribution.

We first select the ARMA order based on the Akaike and the Bayes information criteria. The results in Table 6 show that the ARMA(0,0) specification is the most appropriate model. The values of the AIC are the same for all three specifications, meaning that the gain in log-likelihood is evened out by the loss of using extra parameters in the return model. The BIC penalizes the use of extra parameters more extremely, and therefore favors the model with the least number of parameters, in this case, ARMA(0,0).

Table 6: Information criteria for different ARMA orders

		GOV	CB	RE	EQ	MO	Av.
AIC	(0,0)	-7.07	-7.70	-3.89	-4.22	-7.63	-6.10
	(1,0)	-7.06	-7.71	-3.88	-4.22	-7.63	-6.10
	(1,1)	-7.06	-7.71	-3.88	-4.22	-7.63	-6.10
BIC	(0,0)	-7.01	-7.65	-3.84	-4.17	-7.58	-6.05
	(1,0)	-6.99	-7.64	-3.81	-4.15	-7.56	-6.03
	(1,1)	-6.98	-7.62	-3.79	-4.14	-7.55	-6.02

Results are given for GARCH(1,1) order including mean, non-parametric estimation.

Based on the goodness of fit measures and the data statistics of the full dataset, as shown in Table 7, the normal distribution assumption does not hold and we base our estimations on the semi-parametric distribution. In the sample set, the Jarque-Bera test fails to reject the normal assumption on a 5% significance level for Government bonds. However, the full dataset rejects the normal assumption for all asset classes and we thus follow these directions in selecting our model settings.

Table 7: Tests for normality

	GOV	CB	RE	EQ	MO
	Sample set				
Shapiro stat.	0.99	0.98	0.93	0.91	0.66
Shapiro p-value	0.12	0.01	0.00	0.00	0.00
JB stat.	5.69	22.61	184.97	820.25	417.39
JB p-value	0.06	0.00	0.00	0.00	0.00
	Full dataset				
JB stat.	58.62	148.85	941.64	1209.76	1881.33
JB p-value	0.00	0.00	0.00	0.00	0.00

Results are given for ARMA(0,0)-GARCH(1,1) order including mean, non-parametric estimation. *JB* refers to Jarque-Bera test.

[1] The models are implemented in R using the Rugarch package written by Ghalanos (2019), for more information we refer to the CRAN documentation available via https://cran.r-project.org/package=rugarch.

Table 8 shows the resulting GARCH coefficients' estimates and standard errors over time. Overall, the $\hat{\omega}$ estimate is very small and $\hat{\beta}$ is close to one. The constants $\hat{\omega}$ and $\hat{\mu}$ are varying the most over the different assets. For the Real Estate and Equity classes, the constant $\hat{\omega}$ is higher and more varying over time. Overall, the coefficients for the Mortgage class are most stable over time and we thus expect these scenarios also to be relatively stable over time.

Table 8: Coefficient estimates GARCH model

		1	2	3	4	5	6	7	8	9
GOV	$\hat{\omega}$	0.08	0.11	0.30	0.28	0.16	0.00	0.05	0.06	0.14
		(0.36)	(0.33)	(0.59)	(0.85)	(0.31)	(0.22)	(0.27)	(0.20)	(0.05)
	$\hat{\alpha}$	0.77	0.85	0.88**	0.77**	0.72	0.66	0.88	0.68	0.53
		(0.63)	(0.58)	(0.40)	(0.39)	(0.46)	(0.47)	(0.65)	(0.49)	(0.09)
	$\hat{\beta}$	9.10***	8.98***	8.74***	8.82***	8.96***	9.31***	9.00***	9.07***	8.80***
		(0.66)	(0.62)	(0.48)	(0.52)	(0.55)	(0.42)	(0.62)	(0.50)	(0.23)
	$\hat{\mu}$	0.45	0.51	0.98**	0.96*	0.64	1.07***	0.92***	0.68**	0.37
		(0.40)	(0.41)	(0.50)	(0.51)	(0.42)	(0.31)	(0.32)	(0.29)	(0.28)
CB	$\hat{\omega}$	0.03	0.04	0.10	0.12	0.21*	0.00	0.02	0.04	0.04
		(0.24)	(0.18)	(0.27)	(0.30)	(0.00)	(0.20)	(0.31)	(0.26)	(0.11)
	$\hat{\alpha}$	0.90	1.02**	1.03	0.93	1.08***	0.80	1.33	1.43	0.80*
		(0.62)	(0.47)	(0.64)	(0.68)	(0.23)	(0.62)	(1.49)	(1.50)	(0.42)
	$\hat{\beta}$	9.01***	8.88***	8.76***	8.73***	8.05***	9.17***	8.66***	8.40***	8.84***
		(0.60)	(0.44)	(0.66)	(0.76)	(0.44)	(0.56)	(1.27)	(1.33)	(0.40)
	$\hat{\mu}$	0.54*	0.62**	0.95***	1.25***	1.16***	1.05***	0.95***	0.90***	0.59***
		(0.29)	(0.28)	(0.34)	(0.34)	(0.29)	(0.19)	(0.17)	(0.17)	(0.20)
RE	$\hat{\omega}$	3.70*	5.41*	8.53**	5.19*	5.04**	5.34**	5.68*	2.44	2.03
		(1.97)	(3.08)	(4.27)	(2.77)	(2.35)	(2.52)	(3.26)	(3.57)	(2.02)
	$\hat{\alpha}$	1.70***	1.26***	1.27***	2.06***	1.17**	1.02**	1.06**	0.42	0.51
		(0.48)	(0.40)	(0.47)	(0.74)	(0.49)	(0.45)	(0.48)	(0.37)	(0.33)
	$\hat{\beta}$	8.20***	8.44***	8.30***	7.80***	8.21***	8.20***	8.20***	9.14***	9.11***
		(0.44)	(0.44)	(0.53)	(0.63)	(0.59)	(0.63)	(0.74)	(0.93)	(0.57)
	$\hat{\mu}$	2.37	0.84	-1.05	2.30	2.93*	3.29**	2.42	2.26	1.51
		(1.78)	(2.08)	(2.28)	(1.79)	(1.61)	(1.46)	(1.50)	(1.41)	(1.37)
EQ	$\hat{\omega}$	3.73	8.90	17.04**	17.95**	21.33**	15.66*	5.13	1.83	1.55***
		(2.60)	(5.80)	(8.25)	(8.31)	(9.14)	(8.33)	(4.25)	(1.82)	(0.59)
	$\hat{\alpha}$	1.72**	1.41*	1.96*	1.88*	3.37***	2.73**	1.20**	0.32	0.58***
		(0.73)	(0.85)	(1.16)	(1.11)	(1.14)	(1.26)	(0.58)	(0.24)	(0.10)
	$\hat{\beta}$	8.21***	7.94***	7.02***	7.02***	4.37***	5.44***	8.22***	9.37***	9.15***
		(0.72)	(1.07)	(1.29)	(1.23)	(1.56)	(1.86)	(0.93)	(0.39)	(0.24)
	$\hat{\mu}$	3.16**	2.27	2.27	3.02	4.33***	3.40**	1.83	2.17	2.03
		(1.57)	(1.94)	(2.34)	(2.38)	(1.53)	(1.48)	(1.57)	(1.49)	(1.35)
MO	$\hat{\omega}$	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00
		(0.06)	(0.04)	(0.06)	(0.06)	(0.05)	(0.05)	(0.03)	(0.05)	(0.08)
	$\hat{\alpha}$	0.00	0.00	0.00	0.05**	0.00	0.00***	0.00	0.00	0.00
		(0.01)	(0.01)	(0.00)	(0.02)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)
	$\hat{\beta}$	9.99***	9.99***	9.99***	9.92***	9.99***	9.98***	9.95***	9.99***	9.99***
		(0.01)	(0.01)	(0.00)	(0.03)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)
	$\hat{\mu}$	0.80**	0.67**	0.97***	1.29***	1.53***	1.63***	1.68***	1.57***	1.42***
		(0.33)	(0.33)	(0.33)	(0.30)	(0.27)	(0.26)	(0.24)	(0.23)	(0.25)

$\hat{\omega} \times 10^5$, $\hat{\alpha} \times 10$, $\hat{\beta} \times 10$, $\hat{\mu} \times 10^3$. Coefficient estimates, standard errors in brackets. Superscripts *, **, *** denote rejection of the null-hypothesis with a significance level of 10%, 5% , and 1%.

Using the coefficients as given in Table 8, we generate 1000 simulations per period. We generate one scenario set to calculate the quantile performance measures, referred to as *performance set*, and another to be used as input for the SCR and CVaR methods, referred to as *simulation set*. Table 9 shows the statistics of the simulation scenario set together with the real counterparts. We see that the volatility of all asset classes except for Corporate Bond are matched quite accurately with the real estimates. The volatility of the Corporate Bond is somewhat overestimated. This is mostly caused by years four, five and nine in which the real standard deviation is also high. However, in the other years, the volatility is matched quite accurately. Figure C.1 in the Appendix plots the simulation scenario set and real compound returns over time. Except for Corporate Bonds in 2015 and Mortgages in the last year, the real returns are captured within the 0.5% to 99.5% quantiles of the simulations. Table C.1 and Figure C.2 in the Appendix show the same results for the performance set. These results are very analogous and show the same patterns as the simulation set discussed here.

Table 9: Statistics simulated versus real returns simulation set

			1	2	3	4	5	6	7	8	9	av.
GOV	std.	real	4.61	3.09	20.05	20.84	4.02	7.41	4.49	26.94	29.35	13.42
		sim	4.60	3.52	22.07	23.62	3.74	4.74	3.46	23.22	22.60	12.40
	mean	real	0.59	9.97	6.48	0.58	8.23	1.95	0.38	-2.92	3.11	3.15
		sim	3.21	3.71	5.54	6.17	4.23	5.50	4.94	3.62	1.74	4.30
CB	std.	real	3.21	5.11	3.34	14.38	18.00	1.98	3.66	2.75	15.90	7.59
		sim	3.84	6.21	4.40	27.24	24.81	3.62	5.53	3.75	22.19	11.29
	mean	real	0.86	8.24	6.27	2.99	6.78	-1.64	3.95	0.40	0.15	3.11
		sim	3.30	3.78	5.32	6.62	6.56	5.37	3.53	3.48	2.43	4.49
RE	std.	real	15.24	2.97	2.04	1.20	16.08	17.60	2.85	4.13	2.91	7.22
		sim	25.27	2.76	4.40	3.39	20.07	21.66	2.86	1.83	0.93	9.24
	mean	real	19.24	-11.59	15.87	15.34	31.48	-12.08	1.13	-0.94	-2.73	6.19
		sim	0.53	-5.90	-14.79	-1.84	11.16	12.43	8.63	11.56	7.52	3.25
EQ	std.	real	21.46	21.84	2.62	2.99	2.23	16.91	16.61	2.62	3.21	10.06
		sim	18.94	20.48	2.74	3.80	3.74	20.41	21.88	2.66	3.37	10.89
	mean	real	12.96	-12.36	8.08	22.60	14.51	-15.70	18.72	6.13	-2.81	5.79
		sim	-1.16	-2.29	-5.39	-2.65	8.80	7.62	4.58	9.71	8.35	3.06
MO	std.	real	2.02	12.78	12.65	3.43	2.16	1.54	12.81	13.62	2.72	7.08
		sim	2.43	16.43	16.70	2.54	3.19	2.16	16.44	16.95	2.80	8.85
	mean	real	2.08	9.57	8.79	9.84	12.87	5.78	5.28	4.66	-0.47	6.49
		sim	4.12	3.58	5.29	6.87	8.28	8.69	9.44	8.50	7.88	6.96

Std.: annualized standard deviation in weekly returns within the respective year. *Mean*: compound end-of-horizon return over the respective year. *Av.*: arithmetic average value over the investment period. All values in %, $J = 1000$.

To deal with uncertainty in the parameter estimates of the GARCH model, we implement the Worst-Case method by generating multiple scenario sets. These are based on shocked realizations of the coefficients, dependent on their estimates and standard errors from Table 8. To ensure the appropriate restrictions in the GARCH model still hold, we first transform[2] the coefficients

[2] To ensure $\omega, \alpha, \beta > 0$ and $\alpha + \beta < 1$ also hold for the shocked realizations of the estimates, we first transform the coefficients using $\ln(x)$ and $\ln(x)$ - $\ln(1$ - $x)$ and then add a shock to these altered estimates. The variances of the transformed coefficients are determined using the Delta method.

before adding the shock. Table C.2 in the Appendix shows the worst-case realizations of these estimates. The estimates for $\hat{\omega}$, $\hat{\alpha}$ and $\hat{\beta}$ all change in the positive as well as the negative direction in comparison to the respective estimates from Table 8. The estimates for $\hat{\mu}$ however almost always change in a negative direction. This is understandable as $\hat{\mu}$ has the most direct influence on the objective value of the optimization and we used this as criteria for the selection of the worst-case scenario set.

5.2 Portfolio optimization results

We now discuss the results of optimizing the portfolio allocation using the SCR, CVaR and Worst-Case methods[3] and use the outcomes to answer our research questions as were stated in the Introduction. Table 10 shows the summarized results of the performance measures and optimal weights for all three methods. Detailed results are given in Table 11.

5.2.1 The effect of incorporating uncertainty: SCR vs CVaR

To see how incorporating uncertainty around returns affects portfolio performance, we compare the results of the SCR and CVaR method in Table 10. In terms of average realized return, the two methods only differ by four basis points in favor of CVaR (4.32% as opposed to 4.28%). In terms of risk measures, the CVaR method also performs slightly better than the SCR method. The CVaR method attains values of 12.09, 2.96 and 6.19 for the SR, SR^c, and SII respectively. The SCR method reaches 11.97, 2.27 and 5.77 for the respective measures opposingly. Thus, even though the differences are small, they all point into the direction of the method that incorporates uncertainty. The dispersion in terms of return is somewhat smaller and at a higher level for CVaR as opposed to SCR. In terms of solvency ratio, the range is smaller in the SCR method but also at a lower level. We notice that for both methods, even in the 0.5% quantile, the SII ratio does not fall below the desired minimum value of 2. Lastly, the turnover percentage is approximately the same for both methods, 11.82% and 11.78% for the SCR and CVaR methods respectively, yet again in favor of the CVaR method. This roughly means that we cannot distinguish the two methods based on expected transaction costs. Also, the stochastic method does not lead to significantly more stable weights over the investment period. In general, the optimal weights are much alike and even equivalent in three of the five asset classes. The average asset mixes only

[3] The optimization problems are implemented using the ROI package written by Hornik, Meyer, Theussl and Wuertz (2019), for more information we refer to the CRAN documentation available via https://cran.r-project.org/package=ROI. The solver used for SCR optimization is the Augmented Lagrangian Adaptive Barrier Minimization Algorithm (Alabama) written by Varadhan (2015), intended for optimizing smooth nonlinear objective functions with constraints, for more information we refer to https://cran.r-project.org/package=alabama. The solver used for CVaR optimization is GNU Linear Programming Kit (GLPK) written by Theussl and Hornik (2019), intended for solving large-scale linear programming, for more information we refer to https://cran.r-project.org/package=Rglpk.

differ by a maximum of 1%. This similarity can be explained by the fact that both methods use the same input in the objective function, namely the average simulated return.

Overall, the results indicate strong similarity between the two methods. Yet, the differences on all four measures are in favor of the CVaR method. Together, the results indicate that incorporating uncertainty in returns only slightly enhances performance.

5.2.2 The effect of incorporating parameter uncertainty: CVaR vs Worst-Case

Next, we turn to the effect of incorporating parameter uncertainty on portfolio performance. We compare the results of the CVaR and Worst-Case methods in Table 10. We find that the Worst-Case method has an average realized return that is four basis points higher than the CVaR method, 4.36% versus 4.32% respectively. In this comparison, the risk measures indicate contradictory results. The Sharpe ratio for the Worst-Case method (15.47) is higher than in the case of the CVaR method (12.09). However, in terms of the CVaR Sharpe ratio and the solvency ratio, the CVaR method performs better (2.96 and 6.19 opposing 2.00 and 5.88 respectively). We had yet expected the Worst-Case method to perform better in terms of risk as the method is based on worst-case realizations and should thus turn to safer asset classes to meet its restrictions. The dispersion in returns in the Worst-Case method is approximately three percent points smaller but also at a lower level respective to the CVaR method. For the solvency ratios, the same trend is visible but the differences between the two methods are less strong. In practice, the value of knowing the return and risk with more certainty could be important to a risk-averse investor. Therefore, an investor might be willing to accept somewhat lower returns if the results are more certain in that case. We notice that again none of the solvency ratios fall below the desired minimum value of 2.

Next, we consider the stability of the weights over time. The Worst-Case method has a turnover of 12.01% as opposed to 11.78% in the CVaR method. This means that the CVaR method attains somewhat more stable weights over time. In general, the difference in optimal asset mixes between these two methods is larger than the difference between the SCR and CVaR methods. We specifically invest more in Corporate Bonds and less in the other classes in the Worst-Case method as opposed to the CVaR method. The larger difference is explained by the fact that in this comparison, the input in the objective functions is not equivalent. The CVaR method uses the average returns in the standard simulation scenario set, whereas the Worst-Case method uses another scenario set that is based on the worst-case realization of the GARCH parameters.

Overall, the Worst-Case method leads to somewhat higher return, Sharpe ratio and smaller dispersion of return and solvency ratio. Though the SR^c and SII are lower and the turnover percentage is higher compared to the CVaR method.

Summarized, comparing all three methods in Table 10, the results indicate that the Worst-Case method leads to the highest return. The average return is 4.36% as opposed to 4.28% and 4.32% for the SCR and CVaR method respectively. Based on the traditional Sharpe ratio, the Worst-Case method outperforms the other two methods. And, in terms of dispersion, the ranges of both the return and SII quantiles are the smallest and thus the results the most certain. On average, all methods invest most capital in Mortgages and invest no capital in Equity. All in all, incorporating uncertainty in returns as well as parameter uncertainty leads to slightly enhanced performance.

Table 10: Summary portfolio results

		SCR	CVaR	WC
Return	R (%)	4.28	4.32	4.36
Risk	SR	11.97	12.09	15.47
	SR^c	2.27	2.96	2.00
	SII	5.77	6.19	5.88
Dispersion	$R_{0.5}$ (%)	-1.76	-1.68	-1.90
	R_{50} (%)	6.02	6.03	5.15
	$R_{95.5}$ (%)	14.69	14.66	11.64
	$SII_{0.5}$	4.03	4.44	3.50
	SII_{50}	6.48	6.96	6.00
	$SII_{95.5}$	8.84	9.41	8.04
Stability	TO (%)	11.82	11.78	12.01
Weights	w_{gov}(%)	32	32	30
	w_{cb} (%)	24	23	28
	w_{re} (%)	4	4	2
	w_{eq} (%)	0	0	0
	w_{mo} (%)	39	40	39

All values are the (arithmetic) average values over the investment period. The SR^c is shown in positive sense, according to the convention of displaying risk measures.

5.2.3 Detailed results

To get an idea of how consistent the results are over time, we present the outcomes of the measures over the investment period in the detailed overview in Table 11. As each year uses a different set of information as input for its scenarios and therefore the portfolio optimization, these results also show how dependent the optimization is on its given input.

Table 11a shows the return and risk measures of all three methods over time. In general, we see that the entire period has been fruitful as (almost) each decision leads to positive returns and high SII ratios. The fifth year (February 2014 to February 2015) yields the best annual returns, where year nine (February 2018 to February 2019) yields the worst annual returns. We notice that both the return and the risk measures fluctuate quite strongly over time for all three methods. Nonetheless, also at each moment in time, the difference between the SCR and CVaR methods is again very small and we see the same trends as were visible for the averages. In all years, CVaR slightly outperforms (or equals) SCR in terms of return and risk measures.

Comparing the CVaR to the Worst-Case method, the results point towards different directions each year. In years one, three, four, and six, both the return and all three risk measures point at the Worst-Case method. The Worst-Case method yields the lowest volatility within all years. Furthermore, it is best able to cope with the lower returns in year nine, yielding a return of 0.83%, whereas the other two methods yield a return of 0.37%. However, we had expected the Worst-Case method to persistently outperform the other methods in years with lower realizations of returns, which is not the case in year eight. In this year, the Worst-Case method performs worse than both CVaR and SCR, and even results in negative realized return. The Worst-Case method strongly decreased its weight in Mortgages and increased its weight in Government Bonds in this year. The statistics in Table 9 show that exactly in this year, the returns for these two asset classes were low relative to Mortgages.

The differences between the results over time can be explained by the fact that each year uses a different set of information as input for its scenarios and therefore optimization. To determine whether the results are more stable over time when using a more comparable information set, we evaluate the results when using an expanding window estimation in the Sensitivity Analysis. Overall, the results in Table 11a indicate that the influence of incorporating stochasticity into an optimization method depends on which information set is used.

To see what influence the constraints had on the optimal portfolio mix, we report which constraints were binding in which cases in Table 11b. We notice that the risk constraint was never binding in the SCR method and only once in the CVaR method. This explains even better why the two methods generate such similar results, as the difference between the two is (only) in the risk constraint. Instead, the duration constraint was binding five out of nine times in both methods and the weight constraints were binding 32 and 31 times out of 90 in the SCR and CVaR methods respectively. When we implement the worst-case realization of the scenarios, the risk constraint is binding more often (four out of nine times). As a result, the duration and weight constraints are binding in fewer cases.

We notice that for some years the weights in the Worst-Case method do not sum to 100%, meaning that the imposed constraints do not lead to a feasible solution. This is the case in three

Table 11: Detailed portfolio results

Time		1	2	3	4	5	6	7	8	9	av.
		a) Measures over time									
SCR	R (%)	1.37	**9.15**	6.72	3.85	10.69	1.18	3.25	1.91	0.37	4.28
	SR	3.51	16.38	14.45	10.97	45.59	2.52	8.37	4.81	1.17	11.97
	Vol. (%)	0.39	0.56	0.47	0.35	0.23	0.47	0.39	0.40	0.32	0.40
	SR^c	0.24	2.02	0.89	0.84	11.33	10.25	0.84	-6.71	0.73	2.27
	CVaR (%)	5.75	4.54	7.52	4.60	0.94	0.12	3.88	-0.28	0.51	3.06
	SII	4.03	**3.09**	3.20	4.08	4.09	3.75	5.45	8.60	15.64	5.77
CVaR	R (%)	1.37	9.15	6.75	3.93	**10.76**	1.26	**3.27**	**1.98**	0.37	4.32
	SR	3.51	16.38	14.69	11.27	45.58	2.71	8.41	**5.05**	1.17	12.09
	Vol. (%)	0.39	0.56	0.46	0.35	0.24	0.47	0.39	0.39	0.32	0.40
	SR^c	0.24	2.02	0.91	0.88	**12.51**	13.65	**0.91**	**-5.31**	**0.86**	**2.96**
	CVaR (%)	5.75	4.54	7.43	4.47	**0.86**	0.09	**3.60**	**-0.37**	**0.43**	**2.98**
	SII	4.03	3.09	3.30	4.22	4.23	3.87	5.63	**9.33**	**17.99**	**6.19**
WC	R (%)	**1.72**	8.18	**7.21**	**5.61**	10.19	**2.25**	3.24	-0.02	**0.83**	**4.36**
	SR	**4.72**	**17.91**	**28.36**	**18.06**	**45.65**	**7.80**	**13.06**	-0.05	**3.77**	**15.47**
	Vol. (%)	**0.37**	**0.46**	**0.25**	**0.31**	**0.22**	**0.29**	**0.25**	**0.34**	**0.22**	**0.30**
	SR^c	**0.35**	**2.15**	**1.58**	**2.67**	11.52	**-0.93**	0.38	-0.01	0.30	2.00
	CVaR (%)	**4.90**	**3.80**	**4.56**	**2.10**	0.88	**-2.43**	8.47	3.10	2.76	3.13
	SII	**4.50**	3.06	**6.47**	**5.54**	**7.33**	**5.69**	**6.81**	6.61	6.95	5.88
		b) Constraints and weights over time									
SCR	Risk	no	no	no	no	no	no	no	no	no	0
	Duration	no	no	no	no	yes	yes	yes	yes	yes	5
	Weights	5	4	4	4	3	3	3	3	3	32
	w_{total} (%)	100	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	33	34	44	34	27	31	30	30	26	32
	w_{cb} (%)	32	42	41	41	31	20	9	0	0	24
	w_{re} (%)	1	0	0	0	5	11	10	9	4	4
	w_{eq} (%)	0	0	0	0	1	0	0	0	0	0
	w_{mo} (%)	34	24	14	25	36	39	50	61	70	39
CVaR	Risk	no	no	yes	no	no	no	no	no	no	1
	Duration	no	no	no	no	yes	yes	yes	yes	yes	5
	Weights	5	4	3	4	3	3	3	3	3	31
	w_{total} (%)	100	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	33	34	44	34	27	31	30	29	25	32
	w_{cb} (%)	32	42	40	40	30	19	8	0	0	23
	w_{re} (%)	1	0	0	0	5	11	10	8	3	4
	w_{eq} (%)	0	0	0	0	1	0	0	0	0	0
	w_{mo} (%)	34	24	15	26	37	40	52	63	72	40
WC	Risk	yes	yes	yes	yes	no	no	no	no	no	4
	Duration	no	yes	yes	no	no	yes	no	no	no	3
	Weights	3	1	2	2	4	3	4	4	4	27
	w_{total} (%)	95	100	94	94	100	100	100	100	100	98
	w_{gov}(%)	33	26	27	32	23	22	32	41	30	30
	w_{cb} (%)	24	28	18	16	27	36	35	30	35	28
	w_{re} (%)	3	5	1	0	0	0	0	5	0	2
	w_{eq} (%)	0	0	0	4	0	0	0	0	0	0
	w_{mo} (%)	34	41	48	42	51	42	33	24	35	39

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case).

out of the four instances in which the risk constraint was binding. This is not surprising as this method is based on the worst-case realization of the GARCH parameters. Hence, the simulated returns are also lower, making it more difficult to satisfy the CVaR constraint. In practice, a total weight of less than 100% would imply that we would not fully invest our capital and hold some of it as cash. As a result, we do not have to calculate a value for the SCR or other risk measures over this capital, but we also do not earn any return.

When looking at the weights over time in Table 11b, we recognize a momentum strategy. The objective in each optimization method is to maximize expected end-of-horizon return. The expected return is calculated as the average over the 1000 simulations at that time. As the simulations at each moment in time depend on the movements of the asset classes within the given sample period, the strategies could also be interpreted as momentum. We recognize this strategy in our results, as in case the returns of a specific asset class have been negative in the previous period, the scenario returns in the next year are lower and the optimal weights tend to not go into this asset class and vice versa. For now, we leave the reader with the note to keep this facet in mind when drawing conclusions from the results. The question as to whether this investment strategy is the most appropriate is left for other research.

Overall, we see that also in each separate year, the weights from the SCR and CVaR methods are similar to one another. Both methods first decrease and then increase their weight in Mortgages, while for Government Bonds and Corporate Bonds the trend is the opposite. The Worst-Case method fluctuates its direction in the weight of all classes over the years. All three methods only invest in Equity once, this can be explained by the fact that all constraints relate negatively to this asset class. In particular, it has no duration, its maximum change respective to the previous allocation is 5% and we start with a weight of 4%. In terms of risk constraint, Equity has an SCR shock of 39% (the strongest shock among all asset classes) and the scenarios are highly dispersed, making it harder to satisfy the CVaR constraint with much weight in Equity. In practice, it would not be realistic to not invest in a certain asset class at all, as one would want to diversify its assets over all possible classes.

5.3 Sensitivity analysis: The influence of the information set

In the analyses executed until now, we use a fixed rolling window approach to estimate the GARCH model parameters over time. This choice was made so that we use the same number of observations for each period and hence the estimation results can be compared over time. However, in practice, one usually prefers to exploit all information available at that time instead of only a subset. Therefore, we evaluate here to what extent the simulation, as well as the portfolio optimization results, change if we use an expanding rolling window approach for the estimation of the GARCH model parameters.

5.3.1 Simulation results

We compare the coefficient estimates and standard errors in Table 8 to those using the expanding window estimation in Table C.3 in the Appendix. The results for the first period are naturally equivalent as, in this year, the same sample set is used for estimation. In general, we see that the estimates of the expanding window estimation are rather stable over time. This result is understandable, as the information set in the expanding window case is also more comparable over time as you only add an extra year to the sample set (and not remove the earliest year). Therefore, we expect more stable scenarios and thus asset mixes in this case as well. For Government Bonds, the standard errors are generally larger than in the case of fixed window estimation, whereas for the other four classes it is the other way around. This indicates that for the other four classes, the model estimates still improve when adding information. For Government Bonds, on the other hand, adding more years leads to more uncertainty around these parameters. This observation could indicate model misspecification for this asset class. Another notable result is the standard error of the α coefficient for Government Bonds in the second year, which is approximately 15 times larger than its estimate. Consequently, we see in Table C.5 in the Appendix that the respective Worst-Case realization is the most extreme value among all realizations for α. The rest of the trends in these realizations are the same as in the fixed window case.

Using the coefficients given in Table C.3, we again generate a so-called simulation scenario set. The statistics are given in Table C.4 in the Appendix, which we compare to the results of the simulation set based on fixed window estimation in Table 9. We see that the volatility match of the Corporate Bond and Real Estate classes has improved, while for the other three classes the simulated average volatilities have moved away from their real values. We notice that in terms of mean return, the simulations of the Real Estate and Equity classes strongly underestimate their real values and do not fluctuate as much. Figure C.3 in the Appendix shows that the scenarios are more stable over time, as we had already expected based on the more stable coefficient estimates. We foresee that this trend will again be visible in the optimal asset mixes.

5.3.2 Portfolio optimization results

Next, we show the summary results of the portfolio optimization using an expanding window in Table 12. Note that, to calculate the dispersion measures, we use the same performance scenario set for both tables so that we can appropriately compare the results.

We first compare the three methods within the expanding window setting. We find that, rounded to percentage points, the SCR and CVaR methods obtain equivalent optimal asset mixes. This implies that when using an expanding window estimation, there is no effect of including uncertainty in returns in the portfolio optimization. This observation contrasts with both our expectations and findings in the fixed window case. Furthermore, the SCR and CVaR methods outperform the Worst-Case method in all four measures. The dispersion in solvency ratio is larger for the SCR and CVaR methods. However, the range lies, as a whole, far above the range in the Worst-Case method (21.25-34.69 as opposed to 3.44-8.33) and thus favors the SCR and CVaR results over the outcome of the Worst-Case method. All in all, these results indicate that the effect of incorporating any uncertainty disappears when using an expanding window estimation.

Next, to examine the effect of using an expanding versus a fixed window estimation, we compare the results in Table 12 to the results in Table 10. We find that for the SCR and CVaR methods, in terms of all measures except for the CVaR solvency ratio, the expanding window results are better than their counterparts in the fixed window case. Even though the dispersion of the solvency ratios is larger, the range lies again, as a whole, above the ranges in the fixed window case (21.25-34.68 as opposed to 4.03-8.84 and 4.44-9.41). The higher solvency ratios indicate that the mixes resulting from this method not only gave better returns but also had a lower risk. We specifically notice that the weights are more stable over time, with a turnover percentage of 8.08% as opposed to 11.82% and 11.78% in the fixed window case. This agrees with what we had expected, as the information set that is used as input for the scenario generation is also relatively similar over time. Overall, the weights move from Government Bonds and Corporate Bonds to Mortgages, resulting in an average weight in Mortgages of 65%. In practice, this might cause implementation issues as Mortgages are only limited available. We also notice that there is almost no weight in Real Estate and Equity. This observation is in line with what we found in the fixed window case as well as with our conclusions from the scenario statistics in Table C.4. We saw that the simulations underestimated the actual returns for these two asset classes and therefore make them unattractive in the optimization definition.

For the Worst-Case method, we again find contradictory results. We observe that in terms of realized return and SII, the expanding window performs better than its counterparts in the fixed window case. However, in terms of both Sharpe ratios, dispersion in both return and solvency ratio and stability, the expanding window performs worse than the fixed window case.

Table 12: Summary portfolio results using expanding window

		SCR	CVaR	WC
Return	R (%)	5.33	5.33	4.54
Risk	SR	16.32	16.32	14.84
	SR^c	1.61	1.61	1.14
	SII	26.87	26.87	6.11
Dispersion	$R_{0.5}$ (%)	-0.62	-0.62	-1.82
	R_{50} (%)	6.19	6.19	5.26
	$R_{95.5}$ (%)	13.48	13.48	12.26
	$SII_{0.5}$	21.25	21.25	3.44
	SII_{50}	28.25	28.25	5.99
	$SII_{95.5}$	34.68	34.68	8.33
Dispersion	TO (%)	8.08	8.08	13.10
Weights	w_{gov}(%)	27	27	33
	w_{cb} (%)	8	8	28
	w_{re} (%)	1	1	1
	w_{eq} (%)	0	0	1
	w_{mo} (%)	65	65	36

All values are the (arithmetic) average values over the investment period. The SR^c is shown in positive sense, according to the convention of displaying risk measures.

The results over time can be found in the detailed overview in Table C.6 in the Appendix. We observe that in none of the nine years, the risk constraint in either the SCR or CVaR method is binding. This explains why the two methods come to equivalent optimal asset mixes, as the input in the objective function as well as all the other constraints are also equivalent. Furthermore, we notice that especially the SCR and CVaR methods have a lot of weight in Mortgages in the last five years (up to 77%). The outcomes in the first year are logically equal to the results in the fixed window case as we use the same information set in this year.

To sum up, we state that the choice of information set definitely influences the results. For all asset classes except for Government Bonds, the coefficients have lower standard errors in the expanding window case. For the Corporate Bond and Real Estate classes, it creates scenarios with better-matched volatility. In the portfolio results, the direction of the influence depends on what stochastic method you choose. In case you only include uncertainty in returns, the outcomes improve when expanding the estimation window. Nonetheless, when also including parameter uncertainty in the optimization method, the expanding window estimation does not consistently enhance performance on all metrics.

Within the expanding window estimation, the SCR and CVaR methods come to the same optimal asset mixes. The risk constraints in both these methods are never binding, which explains this equivalence. Together, these two methods outperform the Worst-Case method. This outcome contrasts with what we found in the fixed window case, where the Worst-Case method generally outperformed the SCR and CVaR methods.

5.4 Sensitivity analysis: The influence of the constraints

The optimization results so far showed that the constraints on weights and duration are often binding while the risk constraint is not. However, the uncertainty in returns is only incorporated in the risk constraint and not in the other restrictions. Hence, incorporating stochasticity in this way has little impact on the outcomes. Moreover, the constraints on weights and duration are not hard but rather a preference in practice. For these reasons, we are interested in the behavior of the different methods with more binding risk constraints and the general impact of the restrictions on the results. We expect that once the risk constraint is binding more often, the difference between the three methods also becomes stronger.

To investigate this, we create five alternatives to the standard problem definition (referred to as panel A here). All alternatives eliminate one or more constraints from the set as described in equation B.1. The upper part of Table 13 shows the resulting panels and which constraints they exclude. The full investment and risk constraint, in terms of either SCR or CVaR, are included in all cases. The lower part of Table 13 gives an overview of the summary results of all panels. We added the information about which constraints were binding in the optimization to the summary results as these are particularly interesting in this analysis. The detailed results of each panel are given in Tables C.7 to C.11 in the Appendix. The main goal of this paper was to find the effect of incorporating (parameter) uncertainty in portfolio optimization. We therefore compare the three different optimization methods within each panel. Nonetheless, we are also interested to know what effect (ignoring) each constraint has on the outcome of a certain method and, therefore, compare the results of each method in all panels.

We explicitly pay attention to the implementation of the Worst-Case method in this case. Here, for all five panels, we use the same worst-case scenario set as was selected based on the standard problem definition (thus including all constraints). Because leaving out constraints usually also results in a larger feasible region; the objective values of the different scenario sets could also be different. Therefore, if we would determine the worst-case scenario in each panel separately, we could end up with a different set per panel. However, to ensure that the results are comparable over the different panels, we use the same scenario set as was selected in the standard case but optimize under fewer constraints.

5.4.1 The influence on the effect of uncertainty: comparison methods

We start with a comparison of the three methods within the different panels. The summarized results in Table 13 generally show that the Worst-Case method obtains the lowest realized return among the three methods. In three out of five panels, the CVaR has the highest realized return, whereas, in the other two panels, the SCR method obtains the highest return. The risk measures all give very contradictory results. In terms of dispersion in return, the Worst-Case method has the smallest range in all five panels. However, we note that the levels of the quantiles are overall also lower than in the other two methods. The Worst-Case method outperforms the SCR and CVaR methods on stability over time. This result is even stronger in the more extreme cases, such as in panels D and F. Besides, in these two panels, the risk constraint is binding in most cases. As a result, as was anticipated, the performances also differ most between the three methods in these two panels. However, since panels D and F allow short-selling, the resulting weights become unrealistic in practice. We therefore focus on panels B, C, and E from here on.

Overall, within panel B, the CVaR method leads to the asset mixes with the highest return, whereas the Worst-Case method leads to the lowest risk. The CVaR method leads to a realized average return of 4.60% as opposed to 4.56% and 2.99% for the SCR and Worst-Case methods respectively. Even though all three risk ratios are highest for the CVaR method, the Worst-Case has lower risk in terms of volatility and CVaR as such. The Worst-Case method has a volatility and CVaR of 0.30% and 3.13%, whereas the SCR and CVaR methods come to values of 0.52% and 4.28%, and 0.49% and 3.84% for volatility and CVaR respectively. The dispersion in both returns and solvency ratios is smaller but also at a lower level in the Worst-Case method compared to the CVaR method. The asset mixes in the SCR and CVaR methods are rather similar, whereas the Worst-Case method clearly has lower weight in Mortgages and more in Corporate Bonds. This is not surprising, because in the Worst-Case method the risk constraint is always binding, whereas, in the SCR and CVaR methods it is only binding once and three times respectively. As a consequence, when leaving out the constraint on weight changes, the risk constraint has a stronger effect on the outcomes in the Worst-Case method.

Within panel C, all measures except for the realized average return point into the direction of the Worst-Case method. The realized return is highest for the CVaR method, with a return of 4.32% as opposed to 4.29% and 3.87% for the SCR and Worst-Case methods respectively. The risk ratios, as well as the risk values as such, are superior for the Worst-Case method in comparison to the other two methods. Moreover, the dispersion in return and solvency ratio is smaller and/or at a higher level for the Worst-Case method. Overall, this means that when excluding the restriction on average duration, incorporating parameter uncertainty reduces the risk and dispersion of the results. Comparing the weights independently, the Worst-Case method specifically places more weight in Government Bonds and less in the other four classes compared to the other two methods. In general, panel C has the least number of binding risk constraints and, as a result, the weights are most comparable between the three methods.

Lastly, we compare the three methods when excluding both the constraints on weights and duration (panel E). We find that in this case, the SCR method has the highest realized return (5.45% as opposed to 5.29% and 3.23% for the CVaR and Worst-Case methods). However, the Worst-Case method leads to lower risk, since almost all risk ratios and risks values as such are enhanced compared to the other two methods. Especially the dispersion in returns is also smaller for the Worst-Case method. On top of that, the turnover percentage is lower for the Worst-Case method (43.24% as opposed to 60.10% and 60.81% for SCR and CVaR respectively). The weights alone are rather different comparing the three methods. Furthermore, compared to panels B and C, the differences between the three methods are the largest. This panel also has the most binding risk constraints relative to the other panels, thus this result is not surprising. The Worst-Case method clearly invests less in Mortgages and Real Estate, and more in Corporate Bonds compared to the other two methods. All in all, these results indicate that when loosening the supplementary constraints, incorporating uncertainty in portfolio optimization becomes more effective.

5.4.2 The influence on portfolio optimization: comparison panels

Next, we turn to comparing the five panels within each method. As mentioned above, the results for panels D and F are less realistic and hence less comparable over the different panels. Thus, for these panels, we mostly focus on the differences in the number of binding constraints and the weights as such. We see that panels B and C still do not have many binding risk constraints. Also, the asset mixes are relatively similar to those in the standard case. In panels D, E and F the differences are larger. We notice that again the weights in the Worst-Case method do not always sum to 100%. As ignoring restrictions generally broadens the feasible region of the optimization problem, we had expected the sum of the weights to go closer to 100% over the different panels. However, the opposite trend is found in the results. Naturally, the turnover is higher in panels B to F in comparison to the standard case, as the weights are less restricted. Therefore, even though in some cases the realized return is higher, the transaction costs using these strategies will be higher as well. Hence, the net return might still be lower than in the standard case in the end.

For the return and risk measures, we focus on comparing panels A, B, C and E. In terms of realized return, the SCR and CVaR methods perform best in panel E, whereas the Worst-Case method performs best in the standard panel. The latter result can be explained by the fact that, over the different panels, the region of feasible asset allocations becomes broader. As a consequence, this method tends to move stronger into a position that is robust to worst-case circumstances. This allocation, however, is not necessarily one that performs well in the actual situation. Next, based on the risk measures, the SCR method mainly performs best in panel A. The CVaR method also performs best in panel A based on the CVaR Sharpe ratio but turns out better in panel B for the other two measures. The measures for the Worst-Case method point to different panels.

Comparing panel B to the standard case, in terms of average realized returns, the optimization without constraint on weight changes is only performing better for the SCR and CVaR method. The constraints on weight changes lead to asset mixes with better realized returns for the Worst-Case method. The risk measures are rather comparable to the standard case and including the constraint on weight changes enhances performance for some cases. The dispersion is larger in all methods for the case without constraint on weight changes. The strongest difference between the two panels is in the turnover percentage, as excluding the constraint on weight changes leads to more volatile asset mixes over time (32.40% to 45.58% as opposed to 11.78% to 12.01%). This result is not surprising as this is exactly the intention of this particular constraint. The weights are mostly different for the SCR and CVaR methods, where the weight is moved from Corporate Bonds to Mortgages when comparing panel A to panel B. It is remarkable that for the Worst-Case method, even though the risk constraint has become binding nine instead of four cases, the asset mixes are so similar between panel A and B. We had expected the optimal weights to differ more relatively to panel A.

Next, we compare the results of panel C to those of panel A. The average realized returns are only substantially different for the Worst-Case method, in which it is 3.87% without the constraint on duration while it was 4.36% in the standard case. However, the dispersion in returns is somewhat smaller in the case without constraint on duration. The risk measures are rather different compared to the standard case, and for the SCR and CVaR method they are better including the constraint on duration. In the Worst-Case method, the CVaR and solvency measures are better for panel C relative to panel A. This means that the restriction on average duration leads to less risky asset mixes when only including uncertainty in returns. The turnover is somewhat higher relative to panel A and is similar for all three methods. Overall, the weights of panel C are most similar to those in the standard case. This means that the restriction on average duration broadly has the least impact on the optimization outcomes. This is somewhat striking since the duration constraint was binding in quite some cases within panel A.

To evaluate the effect of ignoring both the constraints on weight changes and duration, we compare the results of panel E to those in panel A. For the SCR and CVaR methods, the realized return increases while for the Worst-Case method, the return decreases when excluding both these restrictions. Based on the risk measures, all methods perform better when the restrictions are included. This implies that leaving out both the constraints on weights and duration does not enhance performance in terms of risk for any of the three methods. This is not surprising as the restrictions generally limit risk. The same holds for dispersion in return and solvency ratio, the ranges become larger comparing panel E to panel A. Relative to panels B and C, the weights in panel E are most different from those in panel A. The mixes are especially more diversified over all five asset classes. Naturally, this also results in higher turnover percentages. We notice, when comparing the weights of panels B and E, that once the constraint on weight changes is left out, the duration restriction has a stronger impact on the optimization outcomes.

Tables C.7 to C.11 in the Appendix show the detailed results of panels B to F. We shortly discuss the most striking results within these tables. Both panels B and E contain a period in which they obtain an optimal asset mix that invests 100% in Government Bonds. As a result, since the SCR charge for Government Bonds is zero, the total SCR value for these periods is equal to zero as well. Because this value is the denominator in the solvency ratio, it causes the ratio to explode. The solvency ratios for these two panels are therefore not correctly interpretable. In all panels except for panel C, the weights are often concentrated in only two or three asset classes. In practice, this would not be desirable as one would prefer to diversify over multiple asset classes.

Summarizing, we state that when excluding additional constraints, the effect of incorporating uncertainty indeed becomes clearer. When ignoring the restrictions on weight changes and/or duration, incorporating uncertainty reduces risk and in some cases also enhances return. Overall, the constraint on weight changes has the greatest impact on the optimization outcomes.

Table 13: Overview of summary results panels constraint sensitivities

Panel	A (standard)			B			C			D			E			F		
	weight	dur.	short	weight	dur.	short	weight	dur.	short	weight	dur.	short	weight	dur.	short	weight	dur.	short
Incl.	×	×	×		×	×	×		×		×				×			
	SCR	CVaR	WC	SCR	CVaR	WC	SCR	CVaR	WC	SCR	CVaR	WC	SCR	CVaR	WC	SCR	CVaR	WC
R (%)	4.28	4.32	4.36	4.56	4.60	2.99	4.29	4.32	3.87	8.04	11.85	1.98	5.45	5.29	3.23	13.18	11.33	2.58
SR	11.97	12.09	15.47	11.81	12.49	11.81	10.62	10.70	13.62	8.00	10.46	11.06	9.48	10.40	13.92	7.32	9.57	9.18
Vol. (%)	0.40	0.40	0.30	0.52	0.49	0.30	0.49	0.49	0.31	1.43	2.09	0.25	0.75	0.76	0.33	2.20	2.08	0.41
SR^c	2.27	2.96	2.00	2.35	2.40	2.25	1.76	1.90	12.47	0.62	0.87	1.11	0.47	0.53	0.55	0.55	0.80	0.54
CVaR (%)	3.06	2.98	3.13	4.28	3.84	3.13	5.05	5.00	2.42	19.91	22.66	4.91	9.50	10.20	5.46	31.41	22.42	9.58
SII	5.77	6.19	5.88	-	26.63	16.20	3.02	3.07	7.26	2.49	1.82	1.07	-	14.67	5.73	2.67	1.73	1.81
$R_{0.5}$ (%)	-1.76	-1.68	-1.90	-3.13	-2.72	-2.86	-3.28	-3.25	-1.39	-15.72	-17.35	-3.56	-7.51	-8.15	-4.03	-25.92	-17.14	-7.08
R_{50} (%)	6.02	6.03	5.15	6.44	6.41	4.04	6.47	6.49	4.97	13.29	25.33	2.59	7.46	7.47	4.95	19.37	25.50	4.35
$R_{95.5}$ (%)	14.69	14.66	11.64	17.27	16.66	10.21	18.74	18.72	11.33	49.10	78.21	7.73	25.90	26.97	13.58	75.01	78.16	13.97
$SII_{0.5}$	4.03	4.44	3.50	-	14.68	6.93	1.30	1.34	4.16	-1.80	-0.10	-2.57	-	4.23	2.16	-2.22	0.07	0.07
SII_{50}	6.48	6.96	6.00	-	27.59	17.36	3.19	3.25	7.61	2.68	1.74	0.63	-	14.36	6.23	2.62	1.84	2.08
$SII_{95.5}$	8.84	9.41	8.04	-	39.84	27.37	5.06	5.12	10.68	5.28	2.98	3.51	-	24.00	9.81	5.13	3.00	3.56
TO (%)	11.82	11.78	12.01	45.58	42.95	32.40	14.70	14.56	14.59	142.06	236.18	51.54	60.10	60.81	43.24	225.00	247.98	105.06
Risk	0	1	4	1	3	9	0	1	7	8	9	6	5	7	8	9	9	8
Dur.	5	5	3	7	8	1	0	0	0	7	6	4	0	0	0	7	3	2
Weights	32	31	27	28	25	17	39	38	31	0	0	0	35	33	25	0	0	0
w_{total} (%)	100	100	98	100	100	82	100	100	88	100	100	58	100	100	94	100	100	89
w_{gov}(%)	32	32	30	35	31	27	22	22	35	85	74	28	14	8	8	23	21	-24
w_{cb} (%)	24	23	28	8	8	27	24	23	11	-131	-400	17	8	10	48	-210	-344	91
w_{re} (%)	4	4	2	4	4	1	7	7	4	-14	26	-3	15	16	4	-18	25	0
w_{eq} (%)	0	0	0	0	0	0	5	5	0	20	-14	1	4	7	0	32	-15	-5
w_{mo} (%)	39	40	39	53	56	27	42	43	39	140	415	15	59	59	34	274	413	27

All values are the (arithmetic) average values over the investment period. The results of panel A are the same as in Table 10 but included here for ease of comparison. Excluding the duration constraint implies a minimum duration of zero. Excluding the constraint on weight changes implies that the weights are bound by a minimum of zero and a maximum of one. Excluding the short-selling constraint implies that the weights are not bound at all. The risk and full investment constraints are always included. SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. In two cases the SII ratios are removed, as in one period, the denominator of the ratio is zero and therefore the ratio explodes.

6 Conclusion

In this paper, we investigate the impact of uncertainty on portfolio allocation. To incorporate uncertainty in returns, we use stochastic programming and solve a Conditional Value-at-Risk (CVaR) constrained optimization problem (CVaR method). As input, we generate scenarios using Filtered Historical Simulation (FHS) based on an ARMA-GARCH volatility model. We implement a worst-case variant to the CVaR method to also capture parameter uncertainty (Worst-Case method). We compare the results of these two stochastic methods to a deterministic benchmark (SCR method) which represents a simplification of the current allocation strategy at Nationale-Nederlanden (NN). The European legislation on insurance plays an important part in this research. To compare performances, we measure realized return, risk, the dispersion of these two and stability of the weights over time.

Our main finding is that, as opposed to what is stated in the literature, the loss of ignoring uncertainty in returns is not substantial in our dataset. Moreover, adding parameter uncertainty to the optimization method only leads to better performance in a fixed window estimation. When using an expanding window, the SCR and CVaR methods obtain the same asset mixes and generally outperform the Worst-Case method. This means that the effect of incorporating uncertainty is only of value in the fixed window case. We conclude that the choice of information set has a substantial effect on the portfolio results and should thus be taken carefully.

Most importantly, we find that the managerial and legislative restrictions have a much stronger impact on the outcomes than the risk constraint itself. This explains why incorporating stochasticity in this way has little impact on the outcomes. When we exclude these supplementary restrictions, incorporating uncertainty becomes more effective.

For future research, it may be beneficial to implement more in-depth studies regarding modeling of returns as such. This paper focuses on the prediction of volatility and does not model returns itself. Moreover, with our data, the GARCH model including an asymmetric component had issues converging. Nonetheless, the asymmetric GARCH is probably more appropriate for some classes, especially for the more volatile assets such as Equity. The optimization outcomes are strongly dependent on the scenarios that are drawn from the applied prediction model and hence it plays an important role in the outcomes of this research.

Furthermore, this paper implements two specific methods to include stochasticity in portfolio optimization. However, many alternative approaches have been introduced in the literature that might lead to different conclusions than the ones drawn here. Specifically, the results and interpretation of the robust optimization method are strongly dependent on the criterium used to select the worst-case realization among a set of scenarios. The current method might be too conservative, and one might come to different conclusions when using another criterium.

In the practice of investing insurance capital, the timing of liabilities plays an important role. This research indirectly embodies matching the timing of assets and liabilities via the minimum constraint on duration. However, implementing asset-liability management more actively may be of great interest to investors within insurance. Lastly, the use of scenarios generated via FHS indirectly induces a momentum investment strategy. It may be beneficial to further analyze the influence of uncertainty under different, possibly more appropriate investment policies.

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A Data

Table A.1: Data specifics

Name	Short	Available from	Bloomberg Index	Proxy	Data type	Duration
Government Bond AAA	GOV	31-12-99	let8yw index	-	Yield	13.5
Corporate Bond	CB	29-03-02	lecfyw index	EU non-financial (includes Snr + Subs)	Yield	5.9
Real Estate	RE	31-12-99	MXEU0RE Index	MSCI Europe ex UK Real Estate Net Total Return index	Last index price	-
Equity	EQ	31-12-99	SX5T Index	Eurostoxx50	Last index price	-
Mortgages	MO	03-03-05	-	NN Bank mortgage rate (non-NHG $\leq$ 90% LTV)	Yield	5.7

Data type *yield* indicates the market priced expected return on the indicated asset (index); data type *last price* indicates the level of the index at the end of the trading day. Yield is transformed to realized weekly returns by the following formula: - delta yield * duration / 100 + accrued interest over past week = -(current yield - previous yield) * duration / 100 + (previous yield / 100) * (7/365). Last index price is transformed by the following formula: (current yield - previous yield)/previous yield.

B Methods

B.1 Constraint set $\mathbb{C}$

The set $\mathbb{C}$ represents the feasible solutions of $\boldsymbol{x}_t$ taking into account the appropriate constraints on regulations and liquidity. It concerns constraints based on legal requirements as well as availability in practice and managerial preferences. The set of constraints is defined as follows:

$$\begin{aligned} \boldsymbol{x}_t \in \mathbb{C} &: \text{for all } t \in \mathbb{R} \\ \boldsymbol{1}^T \boldsymbol{x}_t = 1 &\qquad \text{full investment,} \\ \boldsymbol{d}^T \boldsymbol{x}_t \geq D_{\min} &\qquad \text{minimum duration,} \\ \boldsymbol{x}_{t^-} - \Delta_x \leq \boldsymbol{x}_t \leq \boldsymbol{x}_{t^-} + \Delta_x &\qquad \text{maximum weight changes,} \\ \boldsymbol{0} \leq \boldsymbol{x}_t \leq \boldsymbol{1} &\qquad \text{no short-selling.} \end{aligned} \tag{B.1}$$

Here, $\mathbb{R}$ is the set of moments in time on which we optimize our asset allocation, which is in our case equal to once a year starting after the sample set of five years. The vector $\boldsymbol{d}$ contains the duration of the different assets and D_{min} is the minimum required duration to be able to match the assets with the liabilities. In our case, $\boldsymbol{d} = (13.5, 5.9, 0, 0, 5.7)^T$, where the Real Estate and Equity classes do not have a duration and are thus not included in this constraint, and $D_{\min}$ is equal to 7.5. The vector Δ_x contains the values with which the weights can change respective to its previous weight per asset class, which we set equal to $(10, 10, 5, 5, 10)^T$ (in %). We define $\boldsymbol{x}_{t^-}$ as the actual weights *before* rebalancing at time t and $\boldsymbol{x}_t$ as the desired portfolio weights at time t *after* rebalancing. Note that $\boldsymbol{x}_{t^-}$ may be different from $\boldsymbol{x}_{t-1}$ due to the realized returns in the period $t-1$. We start the optimizations with the current allocation at Nationale-Nederlanden (NN), which is equal to $\boldsymbol{x}_0 = (43, 22, 6, 4, 24)^T$ (in %). We define $\boldsymbol{0}$ and $\boldsymbol{1}$ as the $N \times 1$ vectors containing only zeros and ones respectively. This set of constraints is added to each of the problem definitions as discussed in Section 4.

B.2 Solution SCR optimization

When ignoring the additional set of constraints given by the set $\mathbb{C}$, the optimization problem in equation 1 becomes

$$\begin{aligned} \max_{\boldsymbol{x}_t} &\qquad \boldsymbol{x}_t^T \boldsymbol{r}_t \\ \text{s.t.} &\qquad \boldsymbol{x}_t^T \boldsymbol{\Omega} \boldsymbol{x}_t \leq \left[\frac{A_t - L_t}{(1+\lambda)A_t} \right]^2 = C_{\text{SCR}}. \end{aligned} \tag{B.2}$$

To solve this optimization problem analytically, we set up the Lagrange function and the necessary and sufficient conditions for optimality (we remove the subscript t for clearer notation):

$$\begin{aligned} L &= \boldsymbol{x}^T \boldsymbol{r} - \kappa(\boldsymbol{x}^T \boldsymbol{\Omega} \boldsymbol{x} - C_{\mathrm{SCR}}) \\ \frac{\partial L}{\partial \boldsymbol{x}} &= \boldsymbol{r} - 2\kappa \boldsymbol{\Omega} \boldsymbol{x} = 0 \\ \frac{\partial L}{\partial \kappa} &= \boldsymbol{x}^T \boldsymbol{\Omega} \boldsymbol{x} - C_{\mathrm{SCR}} = 0. \end{aligned} \tag{B.3}$$

Solving the first equation for $\boldsymbol{x}$ gives

$$\boldsymbol{x} = \frac{1}{2\kappa} \boldsymbol{\Omega}^{-1} \boldsymbol{r}. \tag{B.4}$$

Substituting this value for $\boldsymbol{x}$ in the second equation and solving for κ gives

$$\begin{aligned} \Big(\frac{1}{2\kappa} \boldsymbol{\Omega}^{-1} \boldsymbol{r}\Big)^T \boldsymbol{\Omega} \Big(\frac{1}{2\kappa} \boldsymbol{\Omega}^{-1} \boldsymbol{r}\Big) &= C_{\mathrm{SCR}} \\ \frac{1}{4\kappa^2} \boldsymbol{r}^T \boldsymbol{\Omega}^{-1} \boldsymbol{r} &= C_{\mathrm{SCR}} \\ \kappa = \sqrt{\frac{\boldsymbol{r}^T \boldsymbol{\Omega}^{-1} \boldsymbol{r}}{4 C_{\mathrm{SCR}}}} &= \frac{1}{2} \sqrt{\frac{\boldsymbol{r}^T \boldsymbol{\Omega}^{-1} \boldsymbol{r}}{C_{\mathrm{SCR}}}}. \end{aligned} \tag{B.5}$$

Substituting this value for κ in the first equation gives

$$\boldsymbol{x}^* = \frac{1}{2 \cdot \frac{1}{2} \sqrt{\frac{\boldsymbol{r}^T \boldsymbol{\Omega}^{-1} \boldsymbol{r}}{C_{\mathrm{SCR}}}}} \boldsymbol{\Omega}^{-1} \boldsymbol{r} = \frac{\boldsymbol{\Omega}^{-1} \boldsymbol{r}}{\sqrt{\frac{\boldsymbol{r}^T \boldsymbol{\Omega}^{-1} \boldsymbol{r}}{C_{\mathrm{SCR}}}}}, \tag{B.6}$$

where $\boldsymbol{x}^*$ is the optimal solution under the SCR constraint as given in equation B.2.

B.3 VaR and CVaR under distributional assumptions

Let L represent the loss of a portfolio and suppose $L \sim N(\mu, \sigma^2)$. We then calculate Value-at-Risk (VaR) by

$$\alpha = \Pr(L \leq \mathrm{VaR}_\alpha) = \Pr\Big(\frac{L - \mu}{\sigma} \leq \frac{\mathrm{VaR}_\alpha - \mu}{\sigma}\Big). \tag{B.7}$$

The quantile of a standard normal distribution is $\Phi^{-1}(\alpha)$. Hence, under the assumption that L is normally distributed, we have that

$$\mathrm{VaR}_\alpha = \mu + \sigma \Phi^{-1}(\alpha). \tag{B.8}$$

Alternatively, suppose $L \sim t(\nu, \mu, \sigma^2)$, which means that $\dfrac{L-\mu}{\sigma} \sim t(\nu)$. Similar to the case of the normal distribution, we have that

$$\mathrm{VaR}_\alpha = \mu + \sigma t_\nu^{-1}(\alpha). \tag{B.9}$$

See example 2.14 in the book of McNeil et al. (2005).

We show the same for Conditional Value-at-Risk (CVaR), or Expected Shortfall (ES). If $L \sim N(\mu, \sigma^2)$, we calculate ES by

$$\mathrm{ES}_\alpha = \mu + \sigma \mathrm{ES}_\alpha^0 \quad \text{where} \quad \mathrm{ES}_\alpha^0 = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}. \tag{B.10}$$

And if $L \sim t(\nu, \mu, \sigma^2)$, we have

$$\mathrm{ES}_\alpha = \mu + \sigma \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1}\right), \tag{B.11}$$

where g_ν and t_ν are the PDF and CDF of the standard $t(\nu)$ distribution respectively. See examples 2.18 and 2.19 in the book of McNeil et al. (2005). The equations above show that under the assumption of normal or Student's t-distributions, both VaR and CVaR simplify to a linear function of mean and variance of portfolio losses.

B.4 Solution CVaR optimization under normality

We assume that $\boldsymbol{x}_t^T \boldsymbol{\mu}_t = \boldsymbol{x}_t^T \boldsymbol{m}_t$ and $\sigma^2(\boldsymbol{x}_t) = \boldsymbol{x}_t^T \boldsymbol{V} \boldsymbol{x}_t$, where $\boldsymbol{m}_t$ is the vector with normally distributed variables. We can then re-formulate the problem given in equation 5 when we assume normality of the asset returns as:

$$\begin{aligned} \max_{\boldsymbol{x}_t} \quad & \boldsymbol{x}_t^T \boldsymbol{m}_t \\ \text{s.t.} \quad & -\boldsymbol{x}_t^T \boldsymbol{m}_t + c_2(\beta)\sqrt{\boldsymbol{x}_t^T \boldsymbol{V} \boldsymbol{x}_t} \leq C_\beta \\ & \boldsymbol{x}_t \in \mathbb{C}. \end{aligned} \tag{B.12}$$

To solve this optimization problem analytically, we set up the Lagrange function and the necessary and sufficient conditions for optimality (we remove the subscript t for clearer notation):

$$\begin{aligned}
L &= \boldsymbol{x}^T\boldsymbol{m} - \lambda(-\boldsymbol{x}^T\boldsymbol{m} + c_2(\beta)\sqrt{\boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}} - C_\beta) \\
\frac{\partial L}{\partial \boldsymbol{x}} &= \boldsymbol{m} - \lambda(-\boldsymbol{m} + c_2(\beta)\frac{1}{\sqrt{\boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}}}\boldsymbol{V}\boldsymbol{x}) \\
&= \boldsymbol{m} + \lambda\boldsymbol{m} - \lambda c_2(\beta)\frac{1}{\sqrt{\boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}}}\boldsymbol{V}\boldsymbol{x} = 0 \\
\frac{\partial L}{\partial \lambda} &= \boldsymbol{x}^T\boldsymbol{m} - c_2(\beta)\sqrt{\boldsymbol{x}^T\boldsymbol{V}\boldsymbol{x}} + C_\beta = 0
\end{aligned} \tag{B.13}$$

These equations should then be solved for $\boldsymbol{x}$ numerically.

B.5 CVaR optimization according to Rockafellar and Uryasev (2000)

The problem given in equation 5 could be viewed from the perspective of the problem given by Rockafellar and Uryasev (2000). Key to their approach is that the formulation of CVaR, defined by

$$\phi_\beta(\boldsymbol{x}) = (1-\beta)^{-1}\int_{f(\boldsymbol{x},\boldsymbol{y})\geq\alpha(\boldsymbol{x},\beta)} f(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y}, \tag{B.14}$$

can be replaced by a much simpler function

$$F_\beta(\boldsymbol{x},\alpha) = \alpha + (1-\beta)^{-1}\int_{y\in\mathbb{R}^N}[f(\boldsymbol{x},\boldsymbol{y})-\alpha]^+p(\boldsymbol{y})d\boldsymbol{y}. \tag{B.15}$$

Note that here $\alpha(\boldsymbol{x},\beta)$ is the VaR function of portfolio $\boldsymbol{x}$ with confidence level β and $\boldsymbol{y}$ is the vector of uncertain prices. This notation is similar as in the paper by Rockafellar and Uryasev (2000) while the rest of the notation is adjusted to fit with the notation used in this paper (i.e. mean loss is referred to with $l(\boldsymbol{x})$ instead of $\mu(\boldsymbol{x})$ to prevent confusion with mean return).

We define $p(\boldsymbol{y})$ as the probability density function of the random vector $\boldsymbol{y}$. In case this function is not available in analytical form, but rather in the form of many scenarios represented by, for example, historical observations, then the function $F_\beta(\boldsymbol{x},\alpha)$ can be approximated by

$$\tilde{F}_\beta(\boldsymbol{x},\alpha) = \alpha + \frac{1}{J(1-\beta)}\sum_{j=1}^{J}(f(\boldsymbol{x},\boldsymbol{y}^{(j)})-\alpha)^+. \tag{B.16}$$

Here, J represents the previous days/months for which we have historical data and $\boldsymbol{y}^{(j)}$ is the vector containing the observations for all instruments on day j. The mean loss can now be approximated by

$$l(\boldsymbol{x}) = E[f(\boldsymbol{x},\boldsymbol{y})] = J^{-1}\sum_{j=1}^{J} f(\boldsymbol{x},\boldsymbol{y}^{(j)}) = J^{-1}\sum_{j=1}^{J} -[x_1y_1^{(j)} + ... + x_Ny_N^{(j)}]. \tag{B.17}$$

Uryasev (2000) shows that now, the problem of minimizing the mean loss subject to some balance constraints $\boldsymbol{x} \in \mathbb{Z}$ and a CVaR constraint with a constant C_β at confidence level β can be formulated as

$$\begin{aligned} \min \quad & l(\boldsymbol{x}) = J^{-1} \sum_{j=1}^{J} -[x_1 y_1^{(j)} + ... + x_N y_N^{(j)}] \\ \text{s.t.} \quad & \tilde{F}_\beta(\boldsymbol{x}, \alpha) \leq C_\beta \\ & \boldsymbol{x} \in \mathbb{Z}, \end{aligned} \tag{B.18}$$

which can then be equivalently represented by

$$\begin{aligned} \min \quad & l(\boldsymbol{x}) = J^{-1} \sum_{j=1}^{J} -[x_1 y_1^{(j)} + ... + x_N y_N^{(j)}] \\ \text{s.t.} \quad & \alpha + \frac{1}{J(1-\beta)} \sum_{j=1}^{J} u^{(j)} \leq C_\beta \\ & u^{(j)} \geq 0 \quad \text{and} \quad \boldsymbol{x}^T \boldsymbol{y}^{(j)} + \alpha + u^{(j)} \geq 0 \quad \text{for } j = 1, ..., J \\ & \boldsymbol{x} \in \mathbb{Z}. \end{aligned} \tag{B.19}$$

Here, the terms $(f(\boldsymbol{x}, \boldsymbol{y}^{(j)}) - \alpha)^+$ are replaced by auxiliary variables $u^{(j)}$ and additional constraints are imposed. It is the generation of these J scenarios and their implementation in the constraints given in equation B.19 that makes the problem stochastic. For the proof of the equivalence of the last two problems, see Uryasev (2000) and Rockafellar and Uryasev (2000). As minimizing the mean loss is equivalent to maximizing the mean return, the problem given by equation B.19 is equivalent to the problem given by equation 5. The result is that these problems are linear functions subject to linear constraints w.r.t. $\boldsymbol{x}$ and can be solved using linear programming techniques. Alternatively, we can also formulate the problem above by minimizing CVaR while requiring a minimum expected return (Krokhmal et al., 2002). These are equivalent formulations of the same optimization problem in the sense that they produce the same efficient frontier, which is traced by varying the parameter C_β.

The above formulated problem could be compared to the standard mean-variance optimization problem (Markowitz, 1952):

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{\mu}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^T \Sigma \boldsymbol{x} \leq C_\Sigma \\ & \boldsymbol{x} \in \mathbb{Z}. \end{aligned} \tag{B.20}$$

Here, $\boldsymbol{\mu}$ and Σ are the sample expected return and covariance over the historical returns as also used for the scenario generation in the CVaR optimization method.

C Results

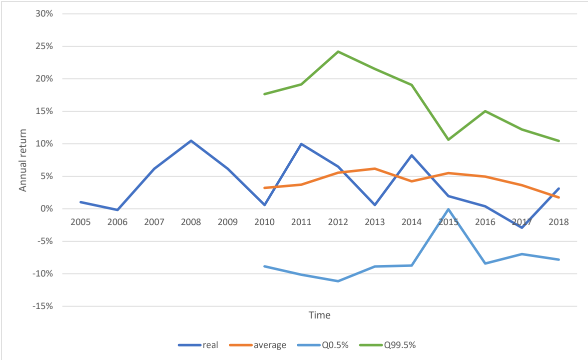
C.1 Simulation results

Table C.1: Statistics simulated versus real returns performance set

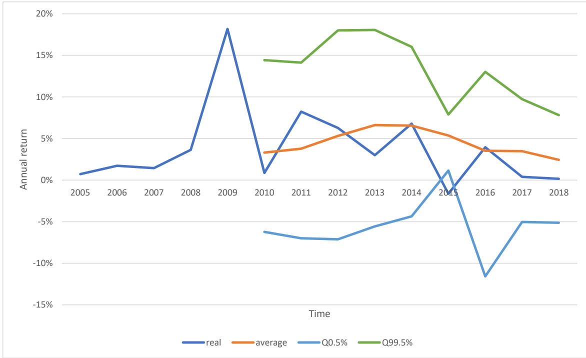
			1	2	3	4	5	6	7	8	9	av.
GOV	std.	real	4.61	3.09	20.05	20.84	4.02	7.41	4.49	26.94	29.35	13.42
		sim	4.53	3.44	21.80	22.71	3.72	4.70	3.47	22.47	22.01	12.10
	mean	real	0.59	9.97	6.48	0.58	8.23	1.95	0.38	-2.92	3.11	3.15
		sim	3.53	3.78	6.21	6.21	4.28	5.59	4.70	3.46	1.66	4.38
CB	std.	real	3.21	5.11	3.34	14.38	18.00	1.98	3.66	2.75	15.90	7.59
		sim	3.80	6.25	4.46	27.96	26.36	3.64	5.53	3.82	21.73	11.51
	mean	real	0.86	8.24	6.27	2.99	6.78	-1.64	3.95	0.40	0.15	3.11
		sim	3.54	3.85	5.47	6.41	6.58	5.45	3.15	3.45	2.33	4.47
RE	std.	real	15.24	2.97	2.04	1.20	16.08	17.60	2.85	4.13	2.91	7.22
		sim	25.24	2.77	4.36	3.36	19.68	21.43	2.86	1.78	0.92	9.15
	mean	real	19.24	-11.59	15.87	15.34	31.48	-12.08	1.13	-0.94	-2.73	6.19
		sim	1.39	-4.45	-15.26	-2.07	12.35	13.80	8.19	11.71	7.76	3.71
EQ	std.	real	21.46	21.84	2.62	2.99	2.23	16.91	16.61	2.62	3.21	10.06
		sim	18.21	19.76	2.75	3.95	3.93	20.89	22.38	2.68	3.38	10.88
	mean	real	12.96	-12.36	8.08	22.60	14.51	-15.70	18.72	6.13	-2.81	5.79
		sim	-0.31	-1.21	-6.81	-2.42	10.71	7.68	3.39	11.04	9.72	3.53
MO	std.	real	2.02	12.78	12.65	3.43	2.16	1.54	12.81	13.62	2.72	7.08
		sim	2.50	16.55	16.80	2.56	3.18	2.16	16.24	16.76	2.82	8.84
	mean	real	2.08	9.57	8.79	9.84	12.87	5.78	5.28	4.66	-0.47	6.49
		sim	4.32	3.57	5.15	6.88	8.09	8.56	9.40	8.63	7.52	6.90

Std.: annualized standard deviation in weekly returns within the respective year. *Mean*: compound end-of-horizon return over the respective year. *Av.*: arithmetic average value over the investment period. All values in %, $J = 1000$.

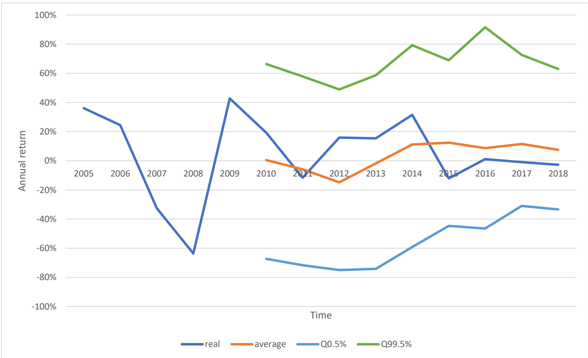
Figure C.1: Simulated versus real returns over time simulation set



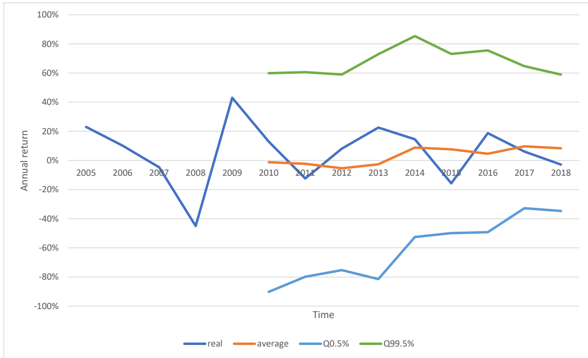
(a) Government Bonds



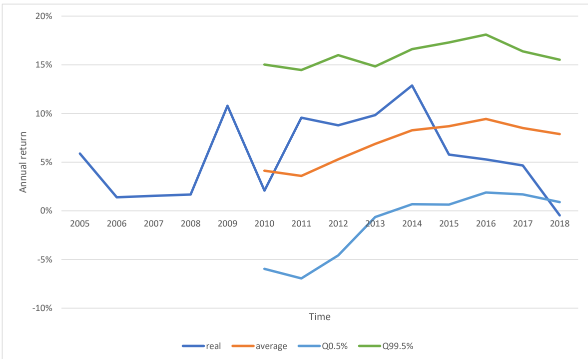
(b) Corporate Bonds



(c) Real Estate

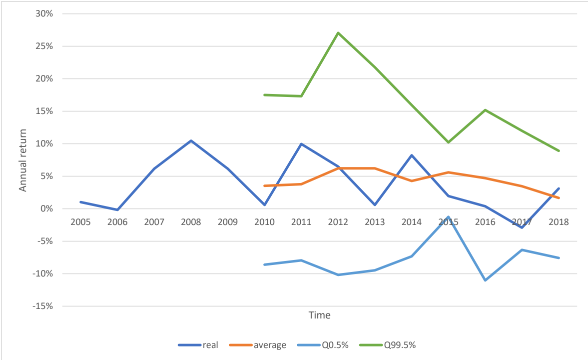


(d) Equity

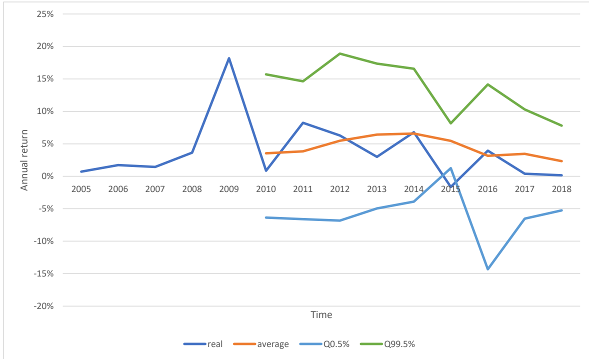


(e) Mortgages

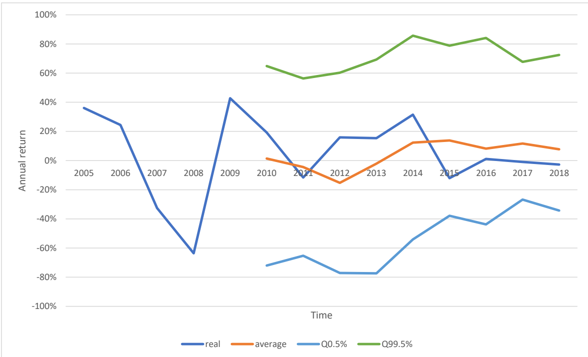
Figure C.2: Simulated versus real returns over time performance set



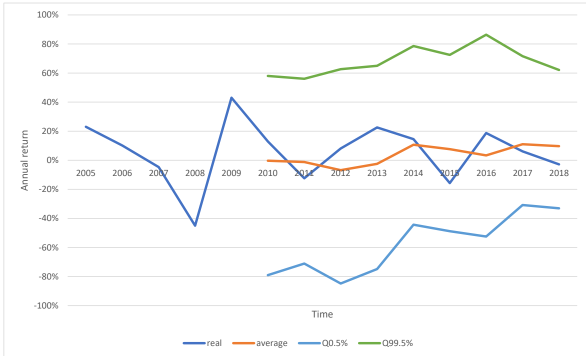
(a) Government Bonds



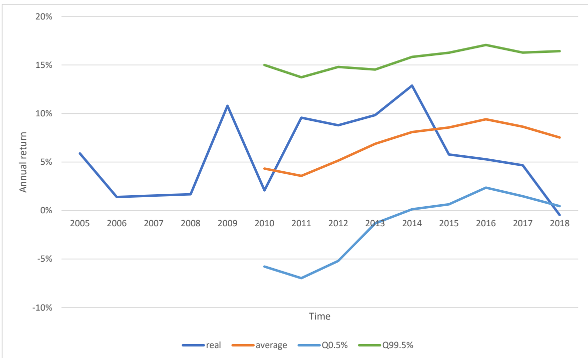
(b) Corporate Bonds



(c) Real Estate



(d) Equity



(e) Mortgages

Table C.2: Coefficient realizations Worst-Case method

		1	2	3	4	5	6	7	8	9
GOV	$\hat{\omega}$	0.18	0.00	0.26	0.06	0.27	0.15	0.01	0.41	0.15
	$\hat{\alpha}$	0.60	0.38	1.18	0.92	0.74	0.89	1.96	0.07	0.43
	$\hat{\beta}$	9.09	9.36	7.66	8.89	9.03	8.91	7.96	9.67	8.95
	$\hat{\mu}$	-0.39	-1.42	-0.74	-0.46	-0.79	-0.45	-0.15	-0.27	-0.46
CB	$\hat{\omega}$	0.02	0.12	0.30	0.21	0.36	0.00	0.04	0.27	0.09
	$\hat{\alpha}$	0.85	0.74	0.48	0.76	1.05	1.06	0.63	0.53	0.69
	$\hat{\beta}$	9.13	9.14	9.43	8.67	7.91	8.60	0.00	0.18	9.04
	$\hat{\mu}$	0.06	-0.09	-0.63	0.28	-0.06	-0.18	-0.17	-0.17	-0.17
RE	$\hat{\omega}$	1.20	6.10	14.04	9.00	0.61	5.72	9.33	3.20	5.32
	$\hat{\alpha}$	2.06	1.24	1.22	2.02	1.60	1.42	0.88	0.97	0.59
	$\hat{\beta}$	6.52	8.35	8.59	7.98	7.84	7.91	8.57	7.48	8.28
	$\hat{\mu}$	-2.17	1.69	1.70	1.20	-0.52	0.44	-1.35	0.20	-0.58
EQ	$\hat{\omega}$	3.07	9.16	18.57	5.71	43.31	11.31	10.11	4.63	2.69
	$\hat{\alpha}$	1.61	1.36	2.17	2.28	3.52	2.73	1.88	0.21	0.41
	$\hat{\beta}$	8.08	8.51	7.08	6.69	3.76	4.96	7.72	9.26	9.39
	$\hat{\mu}$	-1.12	1.62	-3.24	0.83	-0.58	-0.08	0.88	-1.02	-0.42
MO	$\hat{\omega}$	0.04	0.00	0.00	0.10	0.00	0.00	0.00	0.00	0.06
	$\hat{\alpha}$	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.01
	$\hat{\beta}$	9.99	10.00	9.99	9.97	9.99	9.99	9.95	9.98	9.98
	$\hat{\mu}$	-0.31	0.30	-0.01	-0.37	-0.33	-0.58	-0.36	-0.44	-0.09

$\hat{\omega} \times 10^5$, $\hat{\alpha} \times 10$, $\hat{\beta} \times 10$, $\hat{\mu} \times 10^3$.

C.2 Sensitivity analysis: The effect of the information set

Table C.3: Coefficient estimates GARCH model using expanding window

		1	2	3	4	5	6	7	8	9
GOV	$\hat{\omega}$	0.08	0.08	0.09	0.11	0.09	0.01	0.05	0.05	0.05
		(0.36)	(6.15)	(0.55)	(0.30)	(0.49)	(0.44)	(0.35)	(0.24)	(0.28)
	$\hat{\alpha}$	0.77	0.72	0.84	0.75	0.75	0.78	0.82	0.80	0.75
		(0.63)	(11.32)	(1.00)	(0.48)	(0.86)	(0.94)	(0.76)	(0.54)	(0.61)
	$\hat{\beta}$	9.10***	9.13	9.05***	9.07***	9.09***	9.21***	9.09***	9.11***	9.14***
		(0.66)	(12.09)	(1.06)	(0.53)	(0.92)	(0.88)	(0.75)	(0.53)	(0.61)
	$\hat{\mu}$	0.45	0.41	0.51	0.55	0.48	0.84***	0.72***	0.62***	0.47***
		(0.40)	(0.27)	(0.36)	(0.34)	(0.31)	(0.25)	(0.24)	(0.22)	(0.21)
CB	$\hat{\omega}$	0.03	0.04	0.04	0.05	0.04	0.00	0.02	0.02	0.02
		(0.24)	(0.20)	(0.29)	(0.30)	(0.00)	(0.20)	(0.31)	(0.18)	(0.28)
	$\hat{\alpha}$	0.90	0.86*	0.90	0.83	0.85	0.89	1.02	0.97	0.86
		(0.62)	(0.49)	(0.75)	(0.76)	(0.62)	(0.56)	(1.06)	(0.61)	(0.87)
	$\hat{\beta}$	9.01***	9.01***	9.02***	9.01***	8.98***	9.10***	8.94***	8.96***	9.05***
		(0.60)	(0.48)	(0.73)	(0.76)	(0.63)	(0.53)	(0.96)	(0.57)	(0.81)
	$\hat{\mu}$	0.54*	0.50*	0.58**	0.67***	0.65***	0.91***	0.75***	0.73***	0.62***
		(0.29)	(0.26)	(0.25)	(0.23)	(0.21)	(0.17)	(0.11)	(0.14)	(0.17)
RE	$\hat{\omega}$	3.70*	3.51*	5.04**	4.17**	4.00***	3.70***	3.64***	3.74***	3.29***
		(1.97)	(1.83)	(2.08)	(1.64)	(1.48)	(1.33)	(1.31)	(1.31)	(1.13)
	$\hat{\alpha}$	1.70***	1.57***	1.60***	1.58***	1.52***	1.40***	1.32***	1.24***	1.24***
		(0.48)	(0.45)	(0.47)	(0.41)	(0.38)	(0.33)	(0.31)	(0.29)	(0.28)
	$\hat{\beta}$	8.20***	8.30***	8.17***	8.19***	8.21***	8.33***	8.40***	8.42***	8.45***
		(0.44)	(0.42)	(0.45)	(0.39)	(0.37)	(0.34)	(0.32)	(0.32)	(0.30)
	$\hat{\mu}$	2.37	2.90*	2.67*	2.71**	2.89**	3.03***	2.71**	2.44**	2.19**
		(1.78)	(1.58)	(1.54)	(1.33)	(1.21)	(1.11)	(1.08)	(1.03)	(0.95)
EQ	$\hat{\omega}$	3.73	4.34	7.71*	7.06*	6.04*	4.95*	4.25**	4.13**	3.19 *
		(2.60)	(2.95)	(4.00)	(3.81)	(3.52)	(2.70)	(2.13)	(1.99)	(1.49)
	$\hat{\alpha}$	1.72**	1.59**	2.53**	2.17**	1.82**	1.52***	1.30***	1.20***	1.21***
		(0.73)	(0.72)	(1.05)	(0.90)	(0.77)	(0.55)	(0.40)	(0.36)	(0.33)
	$\hat{\beta}$	8.21***	8.18***	7.13***	7.40***	7.72***	8.07***	8.34***	8.40***	8.49***
		(0.72)	(0.78)	(1.06)	(0.99)	(0.94)	(0.70)	(0.50)	(0.47)	(0.40)
	$\hat{\mu}$	3.16**	3.20**	3.59**	3.36**	3.31***	2.93***	2.49**	2.58**	2.48***
		(1.57)	(1.48)	(1.40)	(1.34)	(1.23)	(1.12)	(1.07)	(1.02)	(0.93)
MO	$\hat{\omega}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01
		(0.06)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.03)
	$\hat{\alpha}$	0.00	0.00	0.00***	0.01***	0.00***	0.03***	0.03***	0.04***	0.04***
		(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)
	$\hat{\beta}$	9.99***	9.99***	9.98***	9.97***	9.99***	9.94***	9.96***	9.94***	9.93***
		(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)
	$\hat{\mu}$	0.80**	0.74**	0.82***	0.84***	1.11***	1.19***	1.28***	1.22***	1.19***
		(0.33)	(0.30)	(0.27)	(0.24)	(0.22)	(0.21)	(0.19)	(0.18)	(0.18)

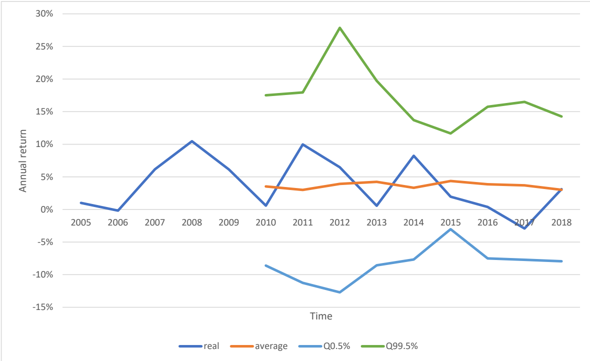
$\hat{\omega} \times 10^5$, $\hat{\alpha} \times 10$, $\hat{\beta} \times 10$, $\hat{\mu} \times 10^3$. Coefficient estimates, standard errors in brackets. Superscripts *, **, *** denote rejection of the null-hypothesis with a significance level of 10%, 5% , and 1%.

Table C.4: Statistics simulated versus real returns simulation set using expanding window

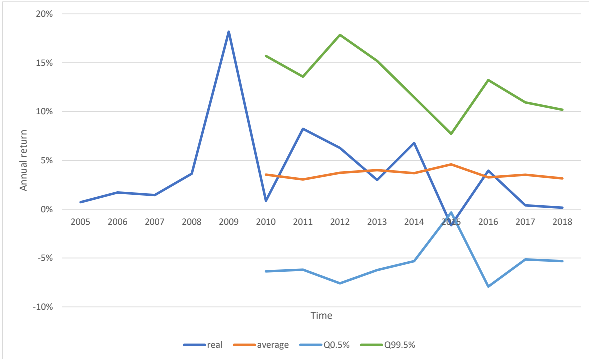
			1	2	3	4	5	6	7	8	9	av.
GOV	std.	real	4.61	3.09	20.05	20.84	4.02	7.41	4.49	26.94	29.35	13.42
		sim	4.53	3.44	21.80	22.71	3.72	4.59	3.35	20.28	19.66	11.56
	mean	real	0.59	9.97	6.48	0.58	8.23	1.95	0.38	-2.92	3.11	3.15
		sim	3.53	3.00	3.93	4.23	3.33	4.37	3.86	3.70	3.01	3.66
CB	std.	real	3.21	5.11	3.34	14.38	18.00	1.98	3.66	2.75	15.90	7.59
		sim	3.77	6.50	4.50	24.16	21.52	3.66	4.89	3.30	20.08	10.26
	mean	real	0.86	8.24	6.27	2.99	6.78	-1.64	3.95	0.40	0.15	3.11
		sim	3.54	3.05	3.74	4.00	3.68	4.59	3.26	3.53	3.16	3.62
RE	std.	real	15.24	2.97	2.04	1.20	16.08	17.60	2.85	4.13	2.91	7.22
		sim	20.59	3.29	4.08	2.85	20.51	20.60	3.22	2.35	1.15	8.74
	mean	real	19.24	-11.59	15.87	15.34	31.48	-12.08	1.13	-0.94	-2.73	6.19
		sim	1.39	3.37	0.36	2.51	4.80	6.29	4.32	5.54	4.10	3.63
EQ	std.	real	21.46	21.84	2.62	2.99	2.23	16.91	16.61	2.62	3.21	10.06
		sim	21.80	20.64	3.08	4.02	3.51	24.11	23.43	2.89	3.59	11.90
	mean	real	12.96	-12.36	8.08	22.60	14.51	-15.70	18.72	6.13	-2.81	5.79
		sim	-0.31	2.31	1.63	2.48	3.07	4.60	0.80	5.27	4.22	2.67
MO	std.	real	2.02	12.78	12.65	3.43	2.16	1.54	12.81	13.62	2.72	7.08
		sim	2.47	19.25	18.26	2.82	3.46	2.35	19.88	20.36	3.01	10.21
	mean	real	2.08	9.57	8.79	9.84	12.87	5.78	5.28	4.66	-0.47	6.49
		sim	4.32	3.84	4.76	5.24	5.80	6.52	6.66	6.43	6.16	5.53

Std.: annualized standard deviation in weekly returns within the respective year. *Mean*: compound end-of-horizon return over the respective year. *Av.*: arithmetic average value over the investment period. All values in %, $J = 1000$.

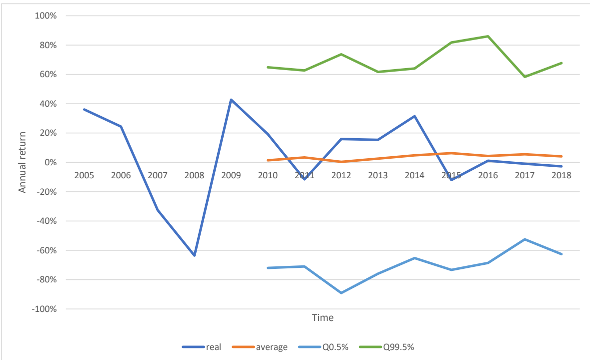
Figure C.3: Simulated versus real returns over time simulation set using expanding window



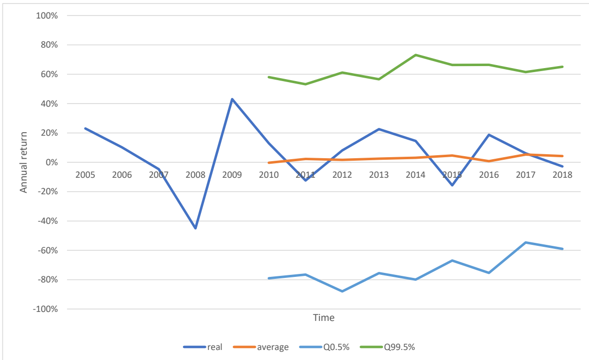
(a) Government Bonds



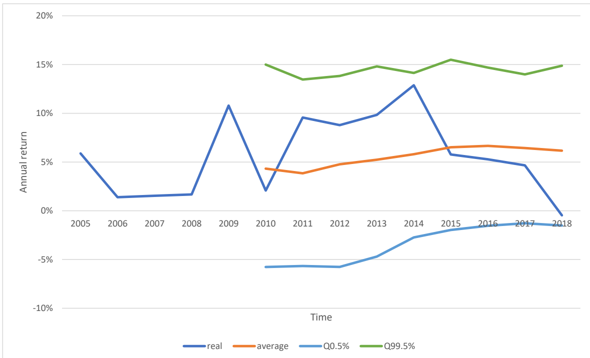
(b) Corporate Bonds



(c) Real Estate



(d) Equity



(e) Mortgages

Table C.5: Coefficient realizations Worst-Case method using expanding window

		1	2	3	4	5	6	7	8	9
GOV	$\hat{\omega}$	0.18	0.70	0.00	0.28	0.00	0.00	0.52	0.42	0.01
	$\hat{\alpha}$	0.60	4.57	0.57	1.15	0.38	1.06	0.64	0.45	1.44
	$\hat{\beta}$	9.09	0.00	0.00	8.69	9.01	0.00	9.35	0.00	8.51
	$\hat{\mu}$	-0.39	-0.15	-0.20	-0.43	-0.37	-0.38	0.12	-0.18	-0.29
CB	$\hat{\omega}$	0.02	0.00	0.00	0.01	0.01	0.00	0.42	0.09	0.00
	$\hat{\alpha}$	0.85	0.15	0.85	1.49	0.69	1.72	0.10	0.01	0.26
	$\hat{\beta}$	9.13	9.64	8.89	8.35	7.69	8.04	9.90	9.99	9.64
	$\hat{\mu}$	0.06	-0.60	0.03	0.07	-0.05	0.15	-0.07	0.02	-0.19
RE	$\hat{\omega}$	1.20	2.88	2.82	5.32	3.50	5.51	3.01	2.86	4.78
	$\hat{\alpha}$	2.06	1.27	1.92	1.90	1.44	1.14	1.30	1.30	1.33
	$\hat{\beta}$	6.52	8.63	7.99	7.49	8.39	8.66	8.57	8.01	8.48
	$\hat{\mu}$	-2.17	0.11	-1.17	0.73	0.32	-0.26	-2.39	-0.69	-0.75
EQ	$\hat{\omega}$	3.07	10.70	5.95	11.23	5.40	11.76	7.86	2.53	2.37
	$\hat{\alpha}$	1.61	2.09	2.54	1.55	0.77	1.46	0.98	1.30	1.34
	$\hat{\beta}$	8.08	2.49	7.28	6.33	8.30	8.32	8.20	8.32	8.41
	$\hat{\mu}$	-1.12	-2.64	0.16	2.59	-2.28	0.65	0.08	-0.84	-0.60
MO	$\hat{\omega}$	0.04	0.00	0.07	0.00	0.00	0.01	0.06	0.03	0.00
	$\hat{\alpha}$	0.00	0.00	0.00	0.01	0.00	0.06	0.03	0.04	0.02
	$\hat{\beta}$	9.99	9.99	9.99	9.97	9.99	9.91	9.92	9.94	9.95
	$\hat{\mu}$	-0.31	-0.53	-0.38	-0.36	-0.29	-0.25	-0.02	-0.40	-0.15

$\hat{\omega} \times 10^5$, $\hat{\alpha} \times 10$, $\hat{\beta} \times 10$, $\hat{\mu} \times 10^3$.

Table C.6: Detailed portfolio results using expanding window

	Time	1	2	3	4	5	6	7	8	9	av.
		a) Measures over time									
SCR	R (%)	1.37	8.07	**7.72**	**6.69**	**11.80**	**4.89**	4.15	**2.91**	0.36	**5.33**
	SR	3.51	**17.66**	**28.42**	**19.87**	38.09	**16.47**	14.13	**7.61**	1.14	**16.32**
	Vol. (%)	0.39	**0.46**	**0.27**	**0.34**	0.31	**0.30**	0.29	0.38	0.31	0.34
	SR^c	0.24	2.01	**1.57**	**3.42**	**18.66**	-6.86	**-3.06**	**-3.02**	**1.51**	**1.61**
	CVaR (%)	5.75	4.01	**4.91**	**1.95**	**0.63**	-0.71	**-1.36**	**-0.97**	**0.24**	**1.61**
	SII	4.03	3.10	**8.89**	**27.04**	**31.24**	**35.23**	**40.22**	**47.76**	**44.31**	**26.87**
CVaR	R (%)	1.37	8.07	7.72	6.69	11.80	4.89	4.15	2.91	0.36	5.33
	SR	3.51	17.66	28.42	19.87	38.09	16.47	14.13	7.61	1.14	16.32
	Vol. (%)	0.39	0.46	0.27	0.34	0.31	0.30	0.29	0.38	0.31	0.34
	SR^c	0.24	2.01	1.57	3.42	18.66	-6.86	-3.06	-3.02	1.51	1.61
	CVaR (%)	5.75	4.01	4.91	1.95	0.63	-0.71	-1.36	-0.97	0.24	1.61
	SII	4.03	3.10	8.89	27.04	31.24	35.23	40.22	47.76	44.31	26.87
WC	R (%)	**1.72**	**9.54**	7.12	5.93	9.23	1.74	**4.17**	0.88	**0.53**	4.54
	SR	**4.72**	17.62	18.02	16.85	**48.34**	5.54	**17.06**	3.09	**2.31**	14.84
	Vol. (%)	**0.37**	0.54	0.39	0.35	**0.19**	0.31	**0.24**	**0.29**	**0.23**	**0.32**
	SR^c	**0.35**	**2.80**	1.08	1.71	3.61	**-0.83**	0.72	0.45	0.34	1.14
	CVaR (%)	**4.90**	**3.41**	6.57	3.47	2.55	**-2.09**	5.77	1.95	1.56	3.12
	SII	**4.50**	**8.24**	5.59	4.32	6.29	5.27	5.55	7.48	7.74	6.11
		b) Constraints and weights over time									
SCR	Risk	no	no	no	no	no	no	no	no	no	0
	Duration	no	yes	no	no	yes	yes	yes	yes	yes	6
	Weights	5	3	4	4	3	3	3	3	3	31
	w_{total} (%)	100	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	33	27	32	32	23	23	23	23	23	27
	w_{cb} (%)	32	23	13	2	0	0	0	0	0	8
	w_{re} (%)	1	6	0	0	0	0	0	0	0	1
	w_{eq} (%)	0	0	0	0	0	0	0	0	0	0
	w_{mo} (%)	34	44	55	65	77	77	77	77	77	65
CVaR	Risk	no	no	no	no	no	no	no	no	no	0
	Duration	no	yes	no	no	yes	yes	yes	yes	yes	6
	Weights	5	3	4	4	3	3	3	3	3	31
	w_{total} (%)	100	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	33	27	32	32	23	23	23	23	23	27
	w_{cb} (%)	32	23	13	2	0	0	0	0	0	8
	w_{re} (%)	1	6	0	0	0	0	0	0	0	1
	w_{eq} (%)	0	0	0	0	0	0	0	0	0	0
	w_{mo} (%)	34	44	55	65	77	77	77	77	77	65
WC	Risk	yes	no	no	yes	no	no	yes	no	no	3
	Duration	no	no	no	no	no	no	yes	no	yes	2
	Weights	3	4	4	2	4	4	3	4	3	31
	w_{total} (%)	95	100	100	100	100	100	100	100	100	99
	w_{gov}(%)	33	44	45	34	39	29	25	28	22	33
	w_{cb} (%)	24	16	25	21	30	40	28	38	33	28
	w_{re} (%)	3	0	0	1	0	0	0	0	0	1
	w_{eq} (%)	0	0	0	5	1	0	4	0	0	1
	w_{mo} (%)	34	40	30	38	30	32	43	33	45	36

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case).

C.3 Sensitivity analysis: The impact of constraints

Table C.7: Detailed portfolio results panel B

	Time	1	2	3	4	5	6	7	8	9
		a) Measures over time								
SCR	R (%)	**1.73**	**8.72**	6.48	**7.70**	**11.80**	-2.78	**4.15**	**2.91**	0.36
	SR	**3.76**	12.86	9.18	**21.95**	38.09	-2.42	**14.13**	**7.61**	1.14
	Vol. (%)	0.46	0.68	0.71	0.35	0.31	1.15	**0.29**	0.38	0.31
	SR^c	**0.32**	1.13	0.53	**5.31**	**18.67**	-0.21	**-3.06**	**-3.02**	**1.51**
	CVaR (%)	5.43	7.72	12.26	**1.45**	0.63	13.09	**-1.36**	**-0.97**	**0.24**
	SII	19.08	1.82	-	27.63	32.29	**1.57**	30.77	38.85	35.16
CVaR	R (%)	1.73	8.65	**7.13**	7.70	11.80	-3.01	4.15	2.91	0.36
	SR	3.76	13.67	14.56	21.95	38.09	-2.55	14.13	7.61	1.14
	Vol. (%)	0.46	0.63	0.49	0.35	0.31	1.18	0.29	0.38	0.31
	SR^c	0.32	1.18	0.88	5.31	18.66	-0.22	-3.06	-3.02	1.51
	CVaR (%)	5.43	7.32	8.12	1.45	0.63	13.66	-1.36	-0.97	0.24
	SII	19.08	1.73	**49.69**	28.43	**33.06**	1.56	**31.22**	**39.27**	**35.59**
WC	R (%)	0.86	7.17	5.34	1.45	7.05	**-0.12**	3.60	0.75	**0.81**
	SR	2.80	**15.15**	**22.80**	7.44	**41.12**	**-0.35**	11.69	2.78	**2.83**
	Vol. (%)	**0.31**	**0.47**	**0.23**	**0.19**	**0.17**	**0.35**	0.31	**0.27**	**0.29**
	SR^c	0.19	**1.84**	**1.28**	0.46	14.78	**0.11**	0.17	0.17	1.22
	CVaR (%)	**4.52**	**3.90**	**4.16**	3.14	**0.48**	**-1.12**	21.16	4.48	0.67
	SII	**32.29**	**2.91**	21.60	**48.04**	17.90	0.87	1.18	2.19	18.83
		b) Constraints and weights over time								
SCR	Risk	no	no	no	no	no	yes	no	no	no
	Duration	yes	no	no	yes	yes	yes	yes	yes	yes
	Weights	3	3	5	3	3	2	3	3	3
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	23	28	100	23	23	50	23	23	23
	w_{cb} (%)	0	72	0	0	0	0	0	0	0
	w_{re} (%)	0	0	0	0	0	37	0	0	0
	w_{eq} (%)	0	0	0	0	0	0	0	0	0
	w_{mo} (%)	77	0	0	77	77	13	77	77	77
CVaR	Risk	no	yes	yes	no	no	yes	no	no	no
	Duration	yes	yes	no	yes	yes	yes	yes	yes	yes
	Weights	3	2	3	3	3	2	3	3	3
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	23	21	72	23	23	51	23	23	23
	w_{cb} (%)	0	75	0	0	0	0	0	0	0
	w_{re} (%)	0	0	0	0	0	38	0	0	0
	w_{eq} (%)	0	0	0	0	0	0	0	0	0
	w_{mo} (%)	77	4	28	77	77	11	77	77	77
WC	Risk	yes	yes	yes	yes	yes	yes	yes	yes	yes
	Duration	no	yes	no	no	no	no	no	no	no
	Weights	2	0	2	2	2	2	2	2	3
	w_{total} (%)	68	90	70	47	63	100	100	100	100
	w_{gov}(%)	41	35	35	35	20	16	11	15	36
	w_{cb} (%)	0	18	0	0	2	71	86	66	0
	w_{re} (%)	0	5	1	0	0	0	0	0	0
	w_{eq} (%)	1	1	0	0	0	0	0	0	0
	w_{mo} (%)	27	31	34	12	41	13	4	20	64

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case). This optimization excludes the constraint on maximum weight changes. In one case, the SII ratio is removed as the denominator of the ratio is zero and therefore the ratio explodes.

Table C.8: Detailed portfolio results panel C

	Time	1	2	3	4	5	6	7	8	9
		a) Measures over time								
SCR	R (%)	**1.37**	**9.15**	6.72	3.85	10.93	-0.30	4.40	**3.55**	-1.10
	SR	3.51	**16.38**	14.45	10.97	37.92	-0.44	8.59	6.15	-1.91
	Vol. (%)	0.39	0.56	0.47	0.35	0.29	0.68	0.51	0.58	0.58
	SR^c	0.24	**2.02**	0.89	0.84	10.42	-0.05	0.84	**0.85**	-0.17
	CVaR (%)	5.75	4.54	7.52	4.60	1.05	5.94	5.24	4.19	6.63
	SII	4.03	**3.09**	3.20	4.08	3.02	1.77	3.07	2.74	2.16
CVaR	R (%)	1.37	9.15	**6.75**	3.93	**11.00**	-0.21	**4.42**	3.55	-1.10
	SR	3.51	16.38	14.69	11.27	37.93	-0.31	8.62	**6.16**	-1.91
	Vol. (%)	0.39	0.56	0.46	0.35	0.29	0.68	0.51	0.58	0.58
	SR^c	0.24	2.02	0.91	0.88	11.50	-0.04	**0.87**	0.85	-0.17
	CVaR (%)	5.75	4.54	7.43	4.47	0.96	5.96	5.08	4.18	6.62
	SII	4.03	3.09	3.30	4.22	3.09	1.81	3.15	2.77	2.19
WC	R (%)	1.14	7.29	5.18	**4.13**	10.70	**2.36**	3.00	0.08	**0.94**
	SR	**3.83**	14.71	**23.46**	**17.07**	**42.75**	**5.85**	**10.94**	0.21	**3.74**
	Vol. (%)	**0.30**	**0.50**	**0.22**	**0.24**	**0.25**	**0.40**	**0.27**	**0.37**	**0.25**
	SR^c	**0.27**	1.66	**1.33**	**2.34**	**108.16**	**-2.74**	0.67	0.04	**0.48**
	CVaR (%)	**4.19**	**4.39**	**3.89**	**1.77**	**0.10**	**-0.86**	**4.46**	**1.89**	**1.96**
	SII	**7.70**	2.82	**6.99**	**7.80**	**8.78**	**4.68**	**9.34**	**7.43**	**9.83**
		b) Constraints and weights over time								
SCR	Risk	no	no	no	no	no	no	no	no	no
	Duration	no	no	no	no	no	no	no	no	no
	Weights	5	4	4	4	5	5	4	4	4
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	33	34	44	34	23	12	13	2	0
	w_{cb} (%)	32	42	41	41	31	20	10	0	0
	w_{re} (%)	1	0	0	0	5	11	15	19	13
	w_{eq} (%)	0	0	0	0	5	10	4	9	14
	w_{mo} (%)	34	24	14	25	36	47	60	70	72
CVaR	Risk	no	no	yes	no	no	no	no	no	no
	Duration	no	no	no	no	no	no	no	no	no
	Weights	5	4	3	4	5	5	4	4	4
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	33	34	44	34	23	12	13	2	0
	w_{cb} (%)	32	42	40	40	30	19	8	0	0
	w_{re} (%)	1	0	0	0	5	11	15	19	13
	w_{eq} (%)	0	0	0	0	5	10	4	9	14
	w_{mo} (%)	34	24	15	26	37	48	61	70	72
WC	Risk	yes	yes	yes	yes	yes	yes	yes	no	no
	Duration	no	no	no	no	no	no	no	no	no
	Weights	3	2	4	3	4	3	4	4	4
	w_{total} (%)	74	93	69	72	86	100	100	100	100
	w_{gov}(%)	33	34	28	30	29	31	41	50	38
	w_{cb} (%)	12	19	11	7	0	10	15	10	15
	w_{re} (%)	1	7	1	0	5	8	2	7	2
	w_{eq} (%)	1	0	0	2	0	0	0	0	0
	w_{mo} (%)	27	33	30	34	52	51	42	33	45

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case). This optimization excludes the constraint on minimum duration.

Table C.9: Detailed portfolio results panel D

	Time	1	2	3	4	5	6	7	8	9
		a) Measures over time								
SCR	R (%)	-1.09	**13.80**	-0.26	5.88	17.89	11.35	**11.31**	15.27	-1.82
	SR	-1.29	5.06	-0.11	**14.19**	26.54	11.30	8.64	**8.71**	-0.99
	Vol. (%)	0.84	2.73	2.29	0.41	0.67	1.00	1.31	1.75	1.83
	SR^c	-0.09	0.38	-0.01	**0.83**	2.06	0.86	**0.47**	1.16	-0.09
	CVaR (%)	12.26	36.06	44.17	7.10	8.68	13.26	24.01	13.19	20.42
	SII	1.69	2.73	0.96	3.70	**3.43**	**2.79**	**2.60**	**2.67**	**1.87**
CVaR	R (%)	**0.47**	11.45	**6.19**	**7.07**	**22.82**	**39.81**	-9.19	**34.28**	-6.29
	SR	**0.70**	**19.82**	13.03	9.90	27.65	**18.58**	-2.22	7.93	-1.27
	Vol. (%)	0.67	0.58	0.48	0.71	0.83	2.14	4.13	4.32	4.97
	SR^c	**0.05**	**1.87**	0.59	0.52	1.88	**1.93**	-0.16	**1.26**	-0.13
	CVaR (%)	9.38	6.12	10.42	13.49	12.15	20.61	56.56	27.16	48.03
	SII	2.76	4.34	**2.34**	1.87	2.05	0.88	0.57	0.99	0.63
WC	R (%)	-0.61	5.39	3.80	0.54	2.53	0.38	3.83	0.93	**1.00**
	SR	-1.54	10.68	**14.01**	8.42	**38.66**	1.12	**14.17**	3.66	**10.41**
	Vol. (%)	**0.39**	**0.50**	**0.27**	**0.06**	**0.07**	**0.34**	**0.27**	**0.26**	**0.10**
	SR^c	-0.08	1.31	**0.75**	0.41	**6.92**	-0.32	0.20	0.14	**0.66**
	CVaR (%)	**7.56**	**4.11**	**5.08**	**1.33**	**0.37**	**-1.21**	**18.98**	**6.47**	**1.50**
	SII	**4.58**	**5.72**	1.06	**6.31**	-7.74	-0.96	-0.18	0.46	0.41
		b) Constraints and weights over time								
SCR	Risk	yes	no	yes	yes	yes	yes	yes	yes	yes
	Duration	yes	no	no	yes	yes	yes	yes	yes	yes
	Weights	0	0	0	0	0	0	0	0	0
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	9	230	274	13	34	45	44	58	57
	w_{cb} (%)	-1	26	20	26	-97	-166	-242	-321	-421
	w_{re} (%)	-15	-21	-31	-10	3	9	-13	-11	-43
	w_{eq} (%)	-4	3	8	-4	9	15	33	48	75
	w_{mo} (%)	111	-138	-172	75	150	195	276	326	432
CVaR	Risk	yes	yes	yes	yes	yes	yes	yes	yes	yes
	Duration	yes	yes	yes	yes	yes	no	no	no	yes
	Weights	0	0	0	0	0	0	0	0	0
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	15	14	10	8	28	226	142	93	125
	w_{cb} (%)	-6	32	14	-10	-127	-577	-745	-869	-1312
	w_{re} (%)	-5	-10	-27	0	20	65	117	13	61
	w_{eq} (%)	-6	0	10	-21	-18	-54	-94	22	33
	w_{mo} (%)	102	64	92	123	197	439	680	842	1193
WC	Risk	yes	yes	yes	yes	yes	no	no	no	yes
	Duration	yes	no	no	no	no	yes	yes	yes	no
	Weights	0	0	0	0	0	0	0	0	0
	w_{total} (%)	62	60	47	14	21	100	100	100	17
	w_{gov}(%)	47	54	39	17	7	21	19	15	34
	w_{cb} (%)	-10	-12	-16	-7	-1	64	75	79	-24
	w_{re} (%)	-12	3	2	-1	1	0	-5	-12	-2
	w_{eq} (%)	7	0	-5	2	-1	-1	2	4	1
	w_{mo} (%)	30	14	25	4	16	16	8	15	8

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case). This optimization excludes the constraint on maximum weight changes and short-selling.

Table C.10: Detailed portfolio results panel E

	Time	1	2	3	4	5	6	7	8	9
		a) Measures over time								
SCR	R (%)	**2.08**	**8.69**	6.48	**9.84**	**19.41**	-3.27	**5.28**	**1.86**	-1.32
	SR	3.73	12.93	9.18	**23.88**	23.07	-2.21	**14.50**	**1.99**	-1.77
	Vol. (%)	0.56	0.67	0.71	0.41	0.84	1.48	0.36	0.93	0.74
	SR^c	0.32	1.12	0.53	**4.87**	1.23	-0.17	**-3.72**	**0.15**	-0.12
	CVaR (%)	6.50	7.77	12.26	**2.02**	15.73	19.62	**-1.42**	12.19	10.81
	SII	**15.08**	1.82	-	23.77	2.86	1.69	33.54	**2.28**	1.78
CVaR	R (%)	2.08	8.32	**7.13**	9.84	18.40	-2.62	5.28	1.14	-1.95
	SR	3.73	14.46	14.53	23.88	25.00	**-1.90**	14.50	1.00	-1.61
	Vol. (%)	0.56	0.58	0.49	0.41	0.74	1.38	0.36	1.13	1.21
	SR^c	0.32	1.07	0.88	4.87	**1.53**	-0.15	-3.72	0.07	-0.09
	CVaR (%)	6.50	7.77	8.14	2.02	12.00	17.84	-1.42	16.94	22.03
	SII	15.08	1.39	**50.00**	**24.06**	**3.31**	**1.82**	**33.62**	1.79	0.99
WC	R (%)	1.47	8.17	6.69	4.55	7.50	**-1.64**	2.77	0.06	**-0.47**
	SR	**4.89**	**21.05**	**29.69**	17.80	**48.57**	-4.26	8.69	0.11	**-1.24**
	Vol. (%)	**0.30**	**0.39**	**0.23**	**0.26**	**0.15**	**0.38**	**0.32**	**0.57**	**0.38**
	SR^c	**0.39**	**1.50**	**1.61**	1.80	1.53	**4.67**	0.14	0.01	**-6.66**
	CVaR (%)	**3.77**	**5.45**	**4.16**	2.53	**4.91**	**-0.35**	19.34	**9.26**	**0.07**
	SII	2.88	**3.43**	21.19	3.10	1.82	1.20	2.20	1.34	**14.37**
		b) Constraints and weights over time								
SCR	Risk	no	yes	no	no	yes	yes	no	yes	yes
	Duration	no	no	no	no	no	no	no	no	no
	Weights	5	3	5	5	3	3	5	3	3
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	0	26	100	0	0	0	0	0	0
	w_{cb} (%)	0	74	0	0	0	0	0	0	0
	w_{re} (%)	0	0	0	0	35	51	0	50	0
	w_{eq} (%)	0	0	0	0	0	0	0	0	36
	w_{mo} (%)	100	0	0	100	65	49	100	50	64
CVaR	Risk	yes	yes	yes	no	yes	yes	no	yes	yes
	Duration	no	no	no	no	no	no	no	no	no
	Weights	5	3	3	5	3	3	5	3	3
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	0	0	72	0	0	0	0	0	0
	w_{cb} (%)	0	94	0	0	0	0	0	0	0
	w_{re} (%)	0	0	0	0	30	47	0	63	0
	w_{eq} (%)	0	0	0	0	0	0	0	0	63
	w_{mo} (%)	100	6	28	100	70	53	100	37	37
WC	Risk	yes	yes	yes	yes	yes	yes	yes	yes	no
	Duration	no	no	no	no	no	no	no	no	no
	Weights	1	2	2	1	3	5	3	3	5
	w_{total} (%)	89	100	84	77	100	100	100	100	100
	w_{gov}(%)	0	0	32	10	0	0	33	0	0
	w_{cb} (%)	46	20	0	38	88	100	67	75	0
	w_{re} (%)	1	5	1	0	0	0	0	25	0
	w_{eq} (%)	0	0	0	4	0	0	0	0	0
	w_{mo} (%)	42	74	51	26	12	0	0	0	100

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case). This optimization excludes the constraint on maximum weight changes and minimum duration. In one case, the SII ratio is removed as the denominator of the ratio is zero and therefore the ratio explodes.

Table C.11: Detailed portfolio results panel F

	Time	1	2	3	4	5	6	7	8	9
		a) Measures over time								
SCR	R (%)	0.34	**14.39**	-1.11	**14.55**	**26.85**	19.60	**20.59**	29.65	-6.28
	SR	0.26	4.74	-0.45	17.23	19.31	9.39	8.26	**9.31**	-2.14
	Vol. (%)	1.34	3.04	2.48	0.84	1.39	2.09	2.49	3.19	2.93
	SR^c	0.02	0.35	-0.02	1.49	1.11	0.65	**0.43**	1.12	-0.18
	CVaR (%)	20.29	41.07	47.94	9.76	24.26	29.95	47.38	26.53	35.52
	SII	1.96	**2.71**	1.00	**5.45**	**3.34**	**2.69**	**2.49**	**2.50**	1.88
CVaR	R (%)	**0.64**	10.52	5.76	8.20	21.98	**39.15**	-9.02	**33.85**	-9.14
	SR	**1.02**	13.62	12.60	13.40	**23.04**	**18.59**	-2.21	7.93	-1.88
	Vol. (%)	0.63	0.77	0.46	0.61	0.95	2.11	4.08	4.27	4.85
	SR^c	**0.08**	**1.14**	0.54	0.76	**1.80**	**1.94**	-0.16	**1.26**	-0.19
	CVaR (%)	8.42	9.23	10.60	10.78	12.22	20.20	55.80	26.77	47.72
	SII	**3.48**	1.34	1.61	3.25	2.81	0.89	0.56	0.99	0.66
WC	R (%)	-0.14	8.84	**7.90**	4.73	4.70	-8.72	4.13	0.70	**1.09**
	SR	-0.29	**14.06**	**22.15**	**18.28**	14.96	-9.06	**12.77**	3.67	**6.05**
	Vol. (%)	**0.47**	**0.63**	**0.36**	**0.26**	**0.31**	**0.96**	**0.32**	**0.19**	**0.18**
	SR^c	-0.02	1.06	**1.08**	**1.66**	0.48	-0.45	0.17	0.13	**0.78**
	CVaR (%)	**7.14**	**8.37**	**7.33**	**2.84**	**9.86**	**19.48**	**24.43**	**5.43**	**1.39**
	SII	1.69	1.41	**2.49**	2.89	0.94	-0.01	0.43	1.26	**5.16**
		b) Constraints and weights over time								
SCR	Risk	yes	yes	yes	yes	yes	yes	yes	yes	yes
	Duration	yes	no	no	yes	yes	yes	yes	yes	yes
	Weights	0	0	0	0	0	0	0	0	0
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	-87	253	293	-85	-54	-35	-38	-18	-23
	w_{cb} (%)	-1	30	23	29	-168	-282	-396	-505	-619
	w_{re} (%)	-15	-24	-35	-11	5	16	-21	-18	-63
	w_{eq} (%)	-4	4	10	-5	15	26	54	76	110
	w_{mo} (%)	207	-163	-190	172	301	375	501	565	696
CVaR	Risk	yes	yes	yes	yes	yes	yes	yes	yes	yes
	Duration	no	yes	no	no	yes	no	no	no	yes
	Weights	0	0	0	0	0	0	0	0	0
	w_{total} (%)	100	100	100	100	100	100	100	100	100
	w_{gov}(%)	-42	-86	-20	-60	-76	222	140	93	19
	w_{cb} (%)	50	122	62	69	-43	-566	-736	-859	-1195
	w_{re} (%)	-4	-11	-26	-1	18	64	115	12	53
	w_{eq} (%)	-7	-3	9	-19	-24	-53	-93	22	31
	w_{mo} (%)	103	78	75	111	225	432	673	832	1191
WC	Risk	yes	yes	yes	yes	yes	yes	no	yes	yes
	Duration	no	yes	no	no	yes	no	no	no	no
	Weights	0	0	0	0	0	0	0	0	0
	w_{total} (%)	93	100	92	79	100	100	100	71	64
	w_{gov}(%)	-48	-78	36	9	-88	-74	-4	-9	38
	w_{cb} (%)	85	92	-13	40	169	275	105	73	-6
	w_{re} (%)	-10	2	4	-1	-10	20	-1	2	-5
	w_{eq} (%)	-1	-6	-10	5	-4	-31	0	-1	2
	w_{mo} (%)	68	89	75	26	33	-90	0	6	35

SR^c and CVaR are shown in positive sense, according to the convention of displaying risk measures. *Risk* and *duration* indicate whether the respective constraints were binding in that year; *weights* indicates how many of the five minimum and maximum weight constraints (thus 10 in total) were binding in that year. For the average column, the presented (de)nominators do not divide to the value of the ratios themselves, as the ratio is a nonlinear transformation. The best performance over the three methods is printed in boldface. When two methods give equal results, we select the simplest method (thus SCR over CVaR and CVaR over Worst-Case). This optimization excludes the constraint on minimum duration, maximum weight changes, and short-selling.