ERASMUS UNIVERSITY ROTTERDAM

ERASMUS SCHOOL OF ECONOMICS

MASTER'S THESIS QUANTITATIVE FINANCE

Multifactor Attribution for long-only investors

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AUGUST 22, 2019

Abstract

This paper provides a comparative analysis of techniques used to improve the quality of multifactor attributions on constrained long-only portfolios. Our research includes a Weighted Least Squares (WLS) approach, the derivation of constrained factor mimicking portfolios as well as a time series- and a non-linear adjustment approach. Based on the multiperiod attribution of a representative portfolio over the timeframe from 1997 to 2019, we find that significant improvements can be achieved by applying both adjustment methods since these better align the attribution with the constrained investment process. In addition, using a shrinkage estimator in the WLS approach enhances the widely used market cap weighting scheme by adding further statistical robustness.

Key Words: Performance Attribution, Factor Models, Factor Mimicking Portfolios.

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1 Introduction

The Quantitative Equities Team of APG Asset Management applies the multifactor attribution methodology to explain the return of their portfolios. By doing so, the aim is to obtain more insights what the main drivers of the past portfolios' performances are. This is especially relevant to APG for monitoring and reporting about the realized portfolio performance as well as for identifying promising directions for further research. With the rising popularity of passive products such as Exchange Traded Funds (ETFs) it becomes increasingly important for active managers like APG to attribute their portfolio performance properly. Using the multifactor attribution approach, active quant managers are able to show the outcome of their systematic strategies and can demonstrate their added value to clients. Moreover, the recent increasing impact of ESG (Environmental, Social and Governance) policies on the investment portfolios has additionally stressed the need for good performance attribution.

In general, conducting the multifactor attribution technique on the realized returns of a portfolio results in a decomposition thereof into a part which is explained by different factors of the used linear risk model (factor contribution) and an unexplained part which represents idiosyncratic returns (residual contribution). Put into a different perspective, we implicitly aim to explain the attributed portfolio as a linear combination of Factor Mimicking Portfolios (FMPs). These are implicitly constructed unconstrained portfolios which replicate and earn the returns of the corresponding factors in the factor model. For the explanation of the performance of unrestricted long-short portfolios this methodology already results in a clean decomposition of the returns, leaving only an insignificant residual portion unexplained if the risk model is correctly specified. However, pension funds like APG usually construct their portfolios subject to different restrictions such as long-only, turnover, liquidity or ESG constraints due to regulations or client related reasons. Since such a restricted portfolio is naturally less diversified than an unrestricted one, a larger residual contribution as a result of the multifactor attribution is in this case to some extend expected. Nevertheless, we usually observe in the attribution of a constrained long-only portfolio a systematical negative residual contribution of a sizeable magnitude which cannot be fully justified by stock-specific returns and hence, must be related to the portfolio construction itself. As an incorrectly overestimated residual contribution reduces the informativeness and quality of the attribution, this represents a significant issue in practice.

In this paper, we investigate and compare different methods that can be used to improve the multifactor attribution when the examined portfolio is constructed subject to real-world constraints. We apply these methods on a portfolio based on a comprehensive dataset which includes stock-level data from January 1997 to February 2019.

Firstly, we investigate the usage of Weighted Least Squares (WLS) instead of OLS in the estimation of the factor returns in the attribution. As described in Litterman (2003) and Barra (2008), this is a commonly known improvement which aims to correct for heteroscedasticity in asset returns in order to obtain more accurately estimated factor returns. In this approach, we examine the costs and benefits of using whether statistically robust or representative regression weights. We find that using the representative market capitalization weights implicitly results in more realistic FMPs which also translates to an on average 10% lower residual contribution. However, our results suggest that by using the shrinkage estimator by Ledoit and Wolf (2003a) we can add favourable statistical properties to this weighting scheme and conclude that this estimator constitutes an improvement on the widely used market cap weights. Our research contributes thereby to the existing academic literature by proposing this enhanced weighting scheme as well as by conducting an in-depth analysis of the implicit FMPs in order to obtain a better understanding of how different regression weights affect the attribution.

Moreover, we investigate an approach in which we directly derive the FMPs subject to real-world constraints in order to obtain more realistic factor returns. As argued in Vandenbussche and Stubbs (2018), the unconstrained standard FMPs are not well able to explain a constrained long-only portfolio which is one reason for the excessively large residual contribution that we observe in practice. Hence, we examine if we can obtain more realistic FMPs with this approach that represent the opportunity set of a constrained long-only investor better than the unconstrained FMPs which we use in the standard attribution. Also, we assess whether the application of these restricted FMPs thereby translates to more informative attribution results. We find that using tight constraints that exactly represent real-world restrictions indeed result in highly realistic FMPs which, however, lead to inaccurate factor estimates and offset the benefits of the more feasible FMPs. We detect significant improvements when using strongly relaxed constraints which suggests a trade-off between deriving the FMPs as realistic as possible by using tight constraints and ensuring that the underlying optimization problem still leads to reasonable solutions close to the standard FMPs by relaxing the constraints strong enough. This leads us to the conclusion that this approach already yields promising results but is still subject to further research in order to find a balance in the mentioned trade-off. With this analysis, we extend the work of Melas, Suryanarayanan, and Cavaglia (2010) by explicitly formulating and investigating real-world constraints as conditions in an optimization problem and by investigating the effect of relaxing them.

A further issue that we analyze is the observation of a large negative correlation between the time series of the residual and factor contributions in the attribution of constrained portfolios. As a large correlation contradicts the property that true idiosyncratic returns are unsystematic, we investigate a time series regression based approach as proposed by Stubbs and Jeet (2016) which aims to mitigate this correlation and to adjust the factor contributions accordingly. Applying this approach in the attribution of our example portfolio, we find that the adjustment indeed results in a substantial decrease of the correlation towards zero, while also reducing the residual contribution itself. In addition to that, we refine this approach by conducting the time series regression over a rolling window and thereby allowing for a time-varying adjustment. This refinement leads to further improvements in terms of a reduction of the magnitude of the residual contribution by 8% and is therefore preferable to the standard adjustment.

Moreover, we study problems that arise with the assumption of linearity of the applied factor models. In general, the linear model is not well able to capture non-linear properties of asset returns which is as stated in De Boer (2019) another source of the overestimated residual contribution. Consequently, he proposes an adjustment approach which re-attributes the residual contribution based on the squared magnitude of each stock's exposure to each factor. In our analysis, we find that this approach only provides little insights into the real drivers of the residual contribution which is why we propose a refinement of this approach by adding further conditions to the adjustment. We conclude that the refined method in which we only take the 10% largest and smallest factor exposures ($\alpha = 0.1$) for the adjustment into account, results in the most substantial improvements, i.e. a reduction of the magnitude of the residual contribution by 35% on average.

In general, we find that combinations of the investigated approaches lead to further improvements in the quality of the attribution in comparison to their single applications. We conclude that the regression with the WLS shrinkage estimator with the consecutive usage of the refined Non-Linear Adjustment ($\alpha = 0.1$) results in the overall smallest residual contribution and the best representation of the properties of true idiosyncratic returns.

Finally, we conduct a simulation study in which we run the attribution based on simulated return data with different empirical characteristics such as heteroscedasticity or nonlinearties. We use this study to obtain a more precise evaluation of the attribution results and to identify possible reasons for the issues that we encounter in the attribution of empirical data. We find that omitted factors in the linear model as well as non-linear properties of asset returns are possible sources for these issues. Moreover, we justify the usage of the investigated approaches by showing their ability to increase the estimation accuracy in the attribution and hence, to increase the quality of the results.

The outcomes of this paper are beneficial for the Quantitative Equities Team of APG in various ways. First of all, the proposed metrics provide an easy and extendable way to measure the quality of an attribution and to assess the improvements obtained by applying different enhancement methods. Furthermore, our research on the FMPs enables APG to get a deeper understanding of how these can be better aligned with the underlying investment process of the attributed portfolio in order to obtain more realistic results in the attribution. The generality of these analyses also facilitates further extensions of the research and the application on different portfolios. Finally, our results show how the examined approaches can improve the attribution on APG's portfolios significantly. This allows a more accurate identification of the main drivers of past portfolio returns which is amongst others essential for performance reviews and for research purposes with the aim to enhance the portfolios' future performances.

The remainder of this paper is organized as follows. We begin with a description of our datasets that we use in the analyses in Section 2. In Section 3 we elaborate on the methodology of the multifactor attribution and the approaches to improve it. Moreover, we describe the setup of the simulation study in Section 4. Afterwards, we analyze the attribution results based on simulated asset return data in Section 5 and on empirical return data in Section 6. Finally, Section 7 concludes.

2 Data

This section describes the data used in the research. The main dataset analyzed in this paper is provided by APG, including a variety of comprehensive stock-level historical datasets of the constituents of the MSCI World Index[1]. This is a broad global equity index that includes approximately 1600 large- and mid-cap constituents of 23 developed markets at each point in time. The available datasets cover a 22-year period of monthly data from January 1997 to February 2019 (overall 266 data points). In the remainder of this section, we described the different types of data that are used in the research paper.

2.1 Return data and market capitalization

Firstly, we utilize monthly return and market capitalization data of the members of the MSCI World. Table 1 shows the time series averages of the cross-sectional summary statistics of these data over the entire period. Here, the separation of the return statistics for larger and smaller cap stocks shows a higher average return volatility and dispersion among small cap returns.

[1]MSCI World Index: `https://www.msci.com/world`

Table 1: Summary Statistics

	Mean	SD	P_{25}	Median	P_{75}
Returns small caps (in %)	0.76	9.78	-4.66	0.51	5.82
Returns large caps (in %)	0.73	7.74	-3.79	0.62	5.13
Size (in bn USD)	12.30	25.35	2.33	4.82	11.29

Notes: This table shows the average summary statistics of the MSCI World dataset including the monthly returns and market capitalizations of the constituents across the time period from January 1997 to February 2019. The monthly return statistics are shown separately for the companies with the 50% smallest and 50% largest cross-sectional market capitalizations.

The market capitalization data of each stock is given in US-Dollars. It indicates a strong positive skewness caused by several index constituents with extremely large market capitalizations. Amongst others, this data is used to calculate the weight of each asset in the capitalization weighted MSCI World Index at each point in time. This index constitutes the benchmark in this research paper to which the considered portfolio is compared.

2.2 Factor Exposures

In order to estimate the factor returns in the first step of the attribution, we use numerical factor exposure data on stock-level which are available in APG's database. These exposures are constructed from financial stock attributes following the standardized definitions in Barra (2008) of the Barra GEM2 and include the commonly used World, Volatility, Momentum, Size, Value, Growth, Liquidity and Leverage factor. Each stock has a unit exposure to the World factor, which economically represents the aggregate up-and-down movement of the global equity market. As shown in Table 2, the exposures to all other factors are standardized to have a mean of 0 and a standard deviation of 1 in order to provide comparability between different factors. Moreover, the cross-sectional exposures are nearly symmetric distributed around the mean as indicated by the 10th and 90th Percentile (P_{10} and P_{90}). Looking further in the tails of the distributions, we find especially for the *Volatility* and *Leverage* factor fat right tails.

Table 2: Summary Statistics

	Mean	SD	P_1	$P_{2.5}$	P_5	P_{10}	Median	P_{90}	P_{95}	$P_{97.5}$	P_{99}
World	1.00	0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Volatility	0.00	0.86	-1.55	-1.39	-1.23	-1.02	-0.10	1.16	1.63	2.01	2.43
Size	0.00	0.75	-1.47	-1.28	-1.11	-0.91	-0.08	1.10	1.40	1.52	1.68
Growth	0.00	0.84	-2.09	-1.61	-1.25	-0.90	-0.07	1.01	1.55	2.07	2.55
Liquidity	0.00	0.69	-1.73	-1.35	-1.08	-0.81	-0.01	0.85	1.15	1.43	1.78
Leverage	0.00	0.96	-1.36	-1.20	-1.12	-1.00	-0.27	1.53	2.08	2.32	2.56
Value	0.00	0.82	-1.81	-1.45	-1.19	-0.93	-0.07	1.04	1.45	1.84	2.29
Momentum	0.00	0.76	-2.16	-1.67	-1.29	-0.92	0.04	0.88	1.16	1.44	1.81

Notes: This table shows the time series average of the cross-sectional mean, standard deviation, i-th Percentile (P_i) and median of the numerical Factor Exposures of the MSCI World constituents in the time period from January 1997 to February 2019.

2.3 Portfolios

A central role in this research paper plays the portfolio on which the attribution is conducted. As common in academic literature, we consider the active weights of each asset in the portfolio which are obtained from a backtest of the portfolio strategy on the given dataset. Here, the active weight is calculated as the difference between the portfolio and the benchmark weight which is in our analysis represented by the MSCI World Index. We investigate a stylized long-only portfolio that is constructed based on a trading signal combining the Barra Momentum and Value factor (Barra, 2008) and whose investable universe covers the point-in-time constituents of the MSCI World Index. This portfolio is rebalanced monthly and is optimized for the following strategy, including several real-world constraints:

min expected trading costs

s.t. Long Only

Target Active Exposure of 22% to a composite signal consisting of
Barra GEM2L Momentum and Value

Active Weight Limit of 0.5%

Active Region, Country, Sector and Industry Limits of 2%

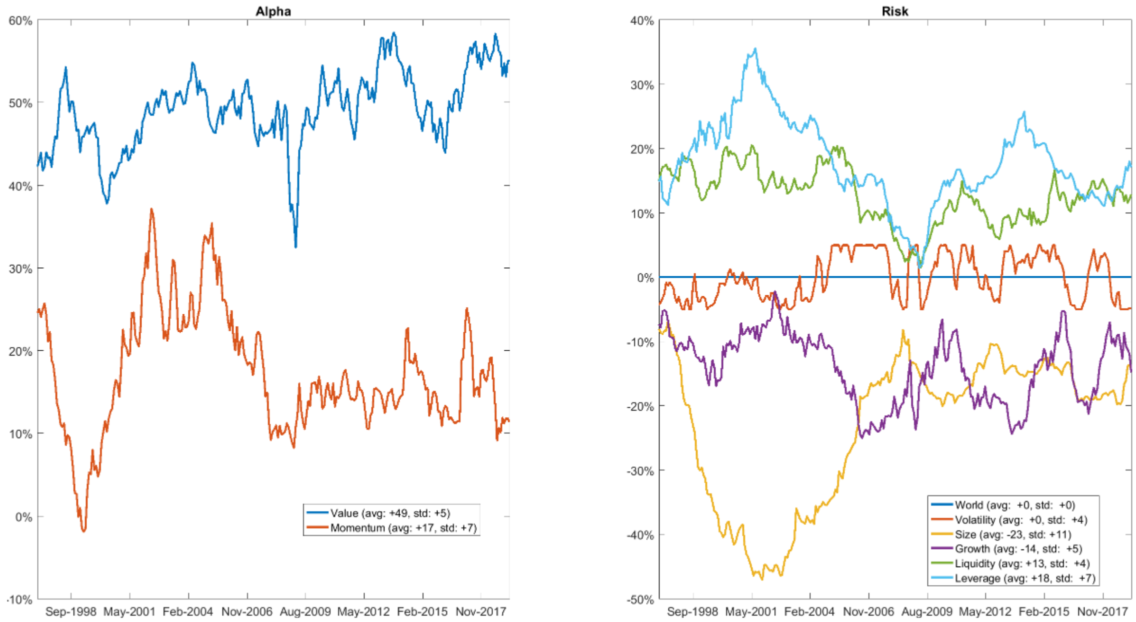
Active Beta Exposure Limit of 2%

Active Volatility Exposure Limit of 5%

Trade Limits of 50% of the Average Daily Volume

in order to resemble the same challenges which are faced in the performance attribution as for the real APG portfolios. Note that due to the country and industry constraints the contribution of these groups on the portfolio's active return are likely to be limited. Therefore, we exclude these groups from the factor model in the later attribution analyses which allows an easier interpretation of the obtained results due to the model's parsimonious structure. As the portfolio is tilted towards the *Momentum* and *Value* factor, we denote these in the further analyses as 'Alpha Factors', whereas the others are summed up as 'Risk Factors'. Figure 1 shows the factor exposures of the active portfolio across the entire time period. It is apparent that the portfolio has its largest exposures to the Alpha factors, but also to *Liquidity* and *Leverage*. Also, the active exposures can be strongly time-varying, as it can be seen at the *Momentum* factor. Furthermore, the negative exposure to the *Size* factor suggests that the portfolio takes many active long positions in small cap stocks as well as short positions in large cap stocks.

Figure 1: Active Portfolio's Exposure to Factors



Notes: This figures shows the exposure of the active portfolio to each of the factors of the risk model over the time period from January 1997 to February 2019.

3 Methodology

In this section, we elaborate on the multifactor attribution technique which is used by APG to attribute their portfolio performance and explain the issue that arises with it. Moreover, we describe the investigated methods used to obtain a more informative attribution.

3.1 Multifactor Attribution

The central technique which is used in this study is the standard multifactor attribution method. This method aims to measure in each single period the effect of the portfolio's exposure to certain factors, such as momentum or value, or groups, such as industries or countries, on the portfolio's realized returns. After specifying a set of factors that are used to explain the cross-section of stock returns in the n dimensional vector $\mathbf{r_t}$, we can formulate a linear factor model as

$$\mathbf{r}_t = \mathbf{X}_t \mathbf{f}_t + \varepsilon_t \quad \text{with} \quad \varepsilon_t \sim (0, \sigma_t^2 \mathbf{I}_n) \quad . \tag{1}$$

Here, $\mathbf{X}_t$ denotes a $n \times m$ matrix with factor exposures of each of the n stocks to the m factors of the model at time t and $\mathbf{I}_n$ represents the identity matrix of size n. Moreover, $\mathbf{f}_t$ is a m dimensional vector of factor returns and ε_t denotes a n dimensional vector of residual asset returns with a constant variance σ_t^2. In the first step of the attribution, we estimate the factor returns with a cross-sectional ordinary least squares (OLS) regression as

$$\hat{\mathbf{f}}_{t,OLS} = (\mathbf{X}_t'\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{r}_t \quad . \tag{2}$$

In order to achieve a realistic estimation that reflects the investment opportunities of the investor at each time t we only include stocks in the regression which have been in the index universe at that point in time. Furthermore, the estimated factor returns depend on the selected factor model as well as the estimation universe. Hence, it is crucial to use a correctly specified factor model and a representative estimation universe in order to obtain informative attribution results. Each of the estimated factor returns can be interpreted as the return of an unconstrained long-short portfolio that has a unit exposure to the corresponding factor and zero exposure to all other factors. These are called factor mimicking portfolios (FMPs) which implicitly consist of each asset with the weights

$$\mathbf{w}_t^{FMP\prime} = (\mathbf{X}_t'\mathbf{X}_t)^{-1}\mathbf{X}_t' \quad . \tag{3}$$

As shown in Melas et al. (2010), the FMPs also have the property of being minimum variance portfolios with a minimized stock specific risk.

After having estimated the factor returns, we calculate the m dimensional vector $\mathbf{E}_t$ which includes the active portfolio's aggregated exposure to each factor with

$$\mathbf{E}_t = \mathbf{X}_t'\mathbf{w}_t^A \quad . \tag{4}$$

Here, $\mathbf{w}^A$ denotes the active weights of the portfolio in which the investor's positions differ from the benchmark weights in each asset. Hence $\mathbf{w}^A$ is calculated as the difference between the portfolio weights $\mathbf{w}^P$ and the benchmark weights $\mathbf{w}^B$

$$\mathbf{w}^A = \mathbf{w}^P - \mathbf{w}^B \quad . \tag{5}$$

We use the active weights in order to explain the additional performance obtained through actively taking diverging positions in comparison to the benchmark. With the obtained active portfolio's factor exposures $\mathbf{E}_t$ and the estimated factor returns $\hat{\mathbf{f}}_t$, we compute the m dimensional vector of factor contributions as the point-wise product

$$\mathbf{FCON}_t = \mathbf{E}_t \odot \hat{\mathbf{f}}_t \quad . \tag{6}$$

The factor contributions $\mathbf{FCON}_t$ represent the parts of the portfolio return that can be explained by each of the m factors of the model. Moreover, we denote the difference between the active portfolio return and the sum of the factor contributions as the residual contribution

$$\begin{aligned} RCON_t &= \mathbf{w}_t^{A\prime}\mathbf{r}_t - \mathbf{E}_t'\hat{\mathbf{f}}_t \\ &= \mathbf{w}_t^{A\prime}\mathbf{r}_t - \mathbf{w}_t^{A\prime}\mathbf{X}_t\hat{\mathbf{f}}_t \\ &= \mathbf{w}_t^{A\prime}(\mathbf{r}_t - \mathbf{X}_t\hat{\mathbf{f}}_t) \\ &= \mathbf{w}_t^{A\prime}\hat{\varepsilon}_t \quad , \end{aligned} \tag{7}$$

which results in the decomposition of the active portfolio return

$$\mathbf{w}_t^{A\prime}\mathbf{r}_t = \mathbf{E}_t'\hat{\mathbf{f}}_t + RCON_t \quad . \tag{8}$$

All calculations are repeated for each time t over the entire period and in the end, the results are compounded for the final attribution results.

In this decomposition, $RCON_t$ represents the stock specific portfolio return which is left unexplained by the attribution model. Alternatively, $RCON_t$ can also be represented as the return of a residual portfolio $\mathbf{w}_t^{res}$ rather than with the active portfolio weights:

$$\begin{aligned} RCON_t &= \mathbf{w}_t^{A\prime}\mathbf{r}_t - \mathbf{E}_t'\hat{\mathbf{f}}_t \\ &= \mathbf{w}_t^{A\prime}\mathbf{r}_t - \mathbf{E}_t'\mathbf{w}_t^{FMP\prime}\mathbf{r}_t \\ &= (\mathbf{w}_t^{A\prime} - \mathbf{E}_t'\mathbf{w}_t^{FMP\prime})\mathbf{r}_t \\ &= (\mathbf{w}_t^{A\prime} - \mathbf{E}_t'\mathbf{w}_t^{FMP\prime})(\mathbf{X}_t\hat{\mathbf{f}}_t + \hat{\varepsilon}_t) \\ &= \underbrace{\mathbf{w}_t^{res\prime}\mathbf{X}_t\hat{\mathbf{f}}_t}_{=0} + \mathbf{w}_t^{res\prime}\hat{\varepsilon}_t \\ &= \mathbf{w}_t^{res\prime}\hat{\varepsilon}_t \quad , \end{aligned} \tag{9}$$

with $\mathbf{w}_t^{res\prime} = \mathbf{w}_t^{A\prime} - \mathbf{E}_t'\mathbf{w}_t^{FMP\prime}$, denoting the weights in which the active portfolio differs from the optimal portfolio as assumed by the FMPs of the underlying factor strategies. This formulation of $RCON_t$ also makes the origin of the large residual we obtain in the attribution of constrained portfolios clearer. If we add more constraints as well as a different optimization objective in the construction process of the active portfolio, it naturally leads to significantly differing positions from an unconstrained minimum variance portfolio such as the FMPs. However, in the multifactor attribution, we try to implicitly explain the active portfolio $\mathbf{w}_t^A$ as a linear combination of the unconstrained FMPs. Therefore, we obtain for a long-only portfolio with real-world constraints a $\mathbf{w}_t^{res}$ with much larger weights and a larger $RCON_t$ than in the attribution of an unconstrained long-short portfolio as mentioned in Section 1.

As described in Clarke, de Silva, and Thorley (2002), $RCON_t$ can be interpreted as a combination of stock specific noise ($\hat{\varepsilon}_t$) and weights which could not be taken by the investor due to investment constraints ($\mathbf{w}_t^{res}$). To obtain more informative attribution results with lower $RCON_t$, it is therefore crucial to investigate the representativity of the FMPs for a restricted investor as well as the two components of the $RCON_t$ which consists of the residual portfolio and the estimated residuals from the cross-sectional regression. For that, we examine different approaches that aim to align the attribution method better with the underlying construction process of the active portfolio and which are described in the following sections.

3.2 Weighted Least Squares

Instead of estimating the factor returns in cross-sectional OLS regressions, it is common in practice to use a Weighted Least Squares (WLS) approach in order to be able to account for heteroscedasticity and thereby obtaining more robust estimates. Using WLS, we assume that the errors ε_{it} of stock i are independently distributed with $\mathbb{E}[\varepsilon_{it}] = 0$ and $\text{Var}(\varepsilon_{it}) = \sigma_{it}^2$ and we estimate the transformed regression model

$$\begin{aligned} \mathbf{W}_t^{\frac{1}{2}}\mathbf{r}_t &= \mathbf{W}_t^{\frac{1}{2}}\mathbf{X}_t\mathbf{f}_t + \mathbf{W}_t^{\frac{1}{2}}\varepsilon_t \quad \text{with} \quad \mathbf{W}_t = diag(w_{1t}, w_{2t}, ..., w_{nt}) \\ \mathbf{r}_t^* &= \mathbf{X}_t^*\mathbf{f}_t + \varepsilon_t^* \quad , \end{aligned} \tag{10}$$

where w_{it} denotes the regression weight on each observation at time t. This transformation in turn leads to the estimated factor return

$$\begin{aligned} \hat{\mathbf{f}}_{t,WLS} &= (\mathbf{X}_t^{*\prime}\mathbf{X}_t^*)^{-1}\mathbf{X}_t^{*\prime}\mathbf{r}_t^* \\ &= (\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{W}_t\mathbf{r}_t \quad . \end{aligned} \tag{11}$$

Consequently, the FMPs implied by the WLS regression are given as

$$\mathbf{w}_t^{FMP\prime} = (\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{W}_t \quad . \tag{12}$$

As derived in Greene (2003), using the inverse of the residual variances as regression weights leads hereby to the most robust estimates of the factor returns $\hat{\mathbf{f}}_{t,WLS}$ with the estimation error

$$\text{Var}(\hat{\mathbf{f}}_{t,WLS}) = (\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1} \quad \text{with} \quad \mathbf{W}_t = diag(\frac{1}{\sigma_{1t}^2}, \frac{1}{\sigma_{2t}^2}, ..., \frac{1}{\sigma_{nt}^2}) \quad . \tag{13}$$

For this approach it is necessary to estimate the variance of the errors ε_t of which the inverses are then used as regression weights. While the standard estimation method as described in Amemiya (1985) is a statistically consistent procedure, one can also use other data such as the market capitalization as a proxy for the error variance in order to put more emphasis on more certain observations. Using company size data has the advantage of obtaining FMPs that better reflect the opportunity set of a restricted investor but comes at the cost of less accurate estimates of the factor returns. Applying WLS therefore leads to the objective to find a weighting scheme that constitutes the best balance in the trade-off between estimation accuracy and representativity. In order to compare the benefits and shortcomings of different regression weights on the multifactor attribution, we examine various statistical values resulting from the estimation. As it is crucial for an informative attribution that the estimated factor model has a high explanatory power for the asset returns, we firstly investigate the R^2 of the cross-sectional regression as a measure of the model fit. Moreover, we also analyze measures to gain more insights in the implied FMPs such as the effective number of stocks EN_t^P of a portfolio P which is defined in Brandes (2004) as

$$EN_t^P = \frac{1}{\sum_{i=1}^n (w_{i,t}^P)^2} \quad . \tag{14}$$

The value of this measure ranges between 1 and the number of stocks n in the estimation universe, where a low EN_t^P refers to a high level of concentration. A high concentration means that a few assets have extremely large positive or negative weights in the portfolio, which suggests a low investability of the FMP in practice due to several restrictions. Furthermore, we also investigate the correlation between the market capitalization of companies and the absolute value of their weights in each FMP. Here, a positive correlation would be more in line with a realistic portfolio, since large-cap companies can typically be stronger underweighted in an active long-only portfolio than small-cap companies.

We examine and compare two commonly used regression weights as well as a combination of both, which are described in the remainder of this section.

3.2.1 Robust estimation

In order to obtain a consistent estimate of the diagonal residual variance matrix, we firstly follow the standard two-stage regression approach as described in Amemiya (1985). In the first step, we obtain the residual returns $\hat{\varepsilon}_{it}$ of each stock i from the cross-sectional OLS regressions of the linear factor model in Equation 1 at each time t. Afterwards, we estimate the residual variances σ_{it}^2 for each stock with the squared residual $\hat{\varepsilon}_{it}^2$ and use their inverse as regression weights

$$\mathbf{W}_t = diag(\frac{1}{\hat{\varepsilon}_{1t}^2}, \frac{1}{\hat{\varepsilon}_{2t}^2}, ..., \frac{1}{\hat{\varepsilon}_{nt}^2}) \quad . \tag{15}$$

As observations which are estimated with a low residual $\hat{\varepsilon}_{it}$ would receive extremely large weights in this approach, we apply a cut-off rule to avoid possible outliers. For that, we firstly define the Interquartile range IQ_t as the difference between the third and the first quartile of the estimated weights in t

$$IQ_t = Q_3(\mathbf{W}_t) - Q_1(\mathbf{W}_t) \quad . \tag{16}$$

We then limit the weights to an upper (ub) and a lower (lb) bound, given as

$$ub_t = Q_3(\mathbf{W}_t) + 2 \cdot IQ_t \quad \text{and} \quad lb_t = Q_1(\mathbf{W}_t) - 2 \cdot IQ_t \quad . \tag{17}$$

The shortcoming of this standard approach is that it only takes the residuals of the considered point in time t into account. However, as we conduct the factor attribution over several periods, we obtain time series of the residuals which we can use to get more accurate estimates of the residual variances σ_{it}^2. As volatility clustering is one of the stylized facts of financial time series (Mandelbrot, 1963), we assume that these variances are only slowly time-varying for each stock. Therefore, we estimate them as the variance of the residuals in a rolling window of the past 60 months. This approach leads to a better incorporation of the relevant information available at each point in time and also has the advantage of only slowly varying regression weights in comparison to the single period estimation approach.

3.2.2 Market Capitalization Weights

As described in Barra (2008), it is common in practice to use the square root of the market capitalization Cap_i of the companies in the estimation universe as a proxy for the inverse of the error variances:

$$\mathbf{W}_t = diag(\sqrt{Cap_{1t}}, \sqrt{Cap_{2t}}, ..., \sqrt{Cap_{nt}}) \quad . \tag{18}$$

This weighting scheme reflects the phenomenon that small-cap stocks tend to have larger error volatilities than large-cap stocks and therefore, should obtain less weight in the cross-sectional regressions. Also, the overweighting of large-cap company returns in the cross-sectional regression leads implicitly to correspondingly larger weights of these companies in the FMPs which is more representative for what restricted investors can implement. Since we consider active portfolios with regards to a capitalization weighted benchmark, a long-only investor can only underweight stocks to the extend in which these stocks are hold in the benchmark portfolio. In addition, large cap stocks tend to be more liquid. Therefore, for a long-only investor who is taking trading costs and liquidity constraints into account, larger long and short positions can be taken in large-cap stocks which is better reflected in the FMPs by using the market capitalizations as regression weights.

3.2.3 Shrinkage Estimator

The previously described regression weights incorporating robust weights and market capitalization data constitute both extremes in the trade-off between statistical consistency and representativity of the portfolio in the estimation of the error covariance matrix. Therefore, we consider a weighting scheme which represents a mixture of these extremes in order to include the favourable properties of both. As the variances in Section 3.2.1 are estimated with an only small sample of residuals at each time t we expect large estimation errors using this approach. In order to overcome this issue, we use the shrinkage estimator proposed by

Ledoit and Wolf (2003a) which is defined as

$$\hat{\mathbf{\Sigma}}_t^{Shrink} = \hat{\delta}\hat{\mathbf{\Sigma}}_t^{Target} + (1-\hat{\delta})\hat{\mathbf{\Sigma}}_t^{Sample} \quad . \tag{19}$$

Here $\hat{\mathbf{\Sigma}}_t^{Sample}$ denotes the diagonal covariance matrix estimated from the residuals $\hat{\varepsilon}$ and $\hat{\mathbf{\Sigma}}_t^{Target}$ represents the shrinkage target towards which we shrink our covariance matrix. As shrinkage target we use the diagonal covariance matrix containing the inverse of the square roots of market capitalizations as it is implied by the weighting scheme in Section 3.2.2

$$\hat{\mathbf{\Sigma}}_t^{Target} = diag\Big(\frac{1}{\sqrt{Cap_{1t}}}, \frac{1}{\sqrt{Cap_{2t}}}, ..., \frac{1}{\sqrt{Cap_{nt}}}\Big) \quad . \tag{20}$$

$\hat{\delta}$ denotes the shrinkage constant which represents the intensity with which we shrink the covariance matrix towards the shrinkage target. A detailed description of the calculation of $\hat{\delta}$ can be found in Appendix A.1. As the shrinkage intensity has a value between 0 and 1, the resulting shrinkage estimator $\hat{\mathbf{\Sigma}}_t^{Shrink}$ can be interpreted as a 'compromise' between both extremes, leading to regression weights that incorporate a balance between the benefits and short comings of both. Finally, we obtain the corresponding regression weights as the inverse of the diagonal shrinkage estimator

$$\mathbf{W}_t = \hat{\mathbf{\Sigma}}_t^{Shrink^{-1}} \quad . \tag{21}$$

3.3 Restricted Factor Mimicking Portfolios

As already mentioned in previous sections, one of the issues that might lead to large **RCON** in the attribution of constrained portfolios are the implied FMPs. As these typically hold large positions in single assets, they are not representative for what investors with real-world constraints can invest in. Therefore, we investigate an approach, in which we derive the FMPs as realistic portfolios satisfying real-world constraints. As shown in De Boer (2019), each FMP is the solution to a variance optimization problem that minimizes the total risk $\mathbf{w}_t^{FMP,k\prime} V \mathbf{w}_t^{FMP,k}$, where

$$V = \text{Cov}(\mathbf{r_t}) = \mathbf{X_t}\Sigma_{f,t}\mathbf{X_t}' + \Sigma_{\varepsilon,t} \tag{22}$$

is the estimated asset covariance matrix, $\Sigma_{f,t}$ is the covariance matrix of the factors included in the linear factor model and $\Sigma_{\varepsilon,t}$ is the matrix of stock-specific variances which is implicitly given as the inverse of the used regression weights $\Sigma_{\varepsilon,t} = \mathbf{W}_t^{-1}$. Melas et al. (2010) showed that the minimization of the total risk is equivalent to the minimization of the stock-specific risk so that we can derive the FMPs with the optimization problem

$$\begin{aligned} \min_{\mathbf{w}_t^{FMP,k}} \quad & \mathbf{w}_t^{FMP,k\prime}\mathbf{W}_t^{-1}\mathbf{w}_t^{FMP,k} \\ \text{s.t.} \quad & \mathbf{X}_t'\mathbf{w}_t^{FMP,k} = e_k \quad , \end{aligned} \tag{23}$$

with $\mathbf{w}_t^{FMP,k}$ being a n dimensional vector with FMP weights of the k-st factor and e_k denoting a vector with a one at the k-st position and zeros elsewhere. Also, the equality constraint ensures that each FMP has a unit exposure to the corresponding factor k and zero to all others. Moreover, this optimization problem emphasizes the fact that the $\mathbf{w}_t^{FMP,k}$ can be calculated separately in single optimizations for each factor k as their dependency among each other is fully described by the exposure condition. As shown in Appendix A.2, without further constraints, this minimization problem leads to equivalent results as the WLS estimation in Section 3.2 with the regression weights $\mathbf{W}_t$. In the case of $\mathbf{W}_t = \mathbf{I}_n$, we obtain the standard OLS FMPs from Section 3.1.

In order to obtain more realistic FMPs, we impose further constraints to the minimization problem in 23. Here, we firstly examine the long-only (LO) constraint in which the FMP weights have to be larger than the negative corresponding weights of each asset in the benchmark, namely

$$\mathbf{w}_t^{FMP,k} \geq -\mathbf{w}_t^B, \; k = 1, \ldots, m. \tag{24}$$

This constraint reflects the negative active positions a long-only investor is able to take in the assets in the FMPs.

Secondly, as one of the real-world constraints under which portfolios are constructed is a maximum weight (MW) limit $w^{max} = 0.5\%$ for each asset, we incorporate this restriction in the optimization constraint

$$w_{i,t}^{FMP,k} \leq w^{max}, \; k = 1, \ldots, m; \; i = 1, \ldots, n. \tag{25}$$

By imposing both constraints separately as well as concurrently to the minimization problem, we obtain more realistic FMPs and factor returns with which we can conduct the attribution. Comparing the resulting $\mathbf{FCON}_k$ of the constrained cases with the standard estimation opens the opportunity to analyze which factors were most affected by the LO and the MW constraint, respectively. Thereby, we also obtain additional information by which factors the **RCON** in the standard case was mostly driven.

Thirdly, we also investigate the imposition of a symmetric upper and lower bound given by a multiple of the benchmark weights $\mathbf{w}_t^B$

$$-d \cdot \mathbf{w}_t^B \leq \mathbf{w}_t^{FMP,k} \leq d \cdot \mathbf{w}_t^B,\ k = 1, \dots, m. \tag{26}$$

This constraint represents first of all the circumstance that an investor with liquidity constraints is also heavier restricted in the possible long positions she can take in small cap stocks. In addition to that, by using a multiple d of the benchmark weights in the constraint, we take into account that the active portfolio is explained by a linear combination of the FMPs. Therefore, restrictions such as the long-only constraint do not have to hold exactly for each of the FMPs. In the further analyses, this constraint is denoted as *Sym 1*.

The last examined constraint combines the aforementioned ones by setting a multiple of the benchmark weights $\mathbf{w}_t^B$ as a symmetric upper and lower bound which is capped by the MW limit of ±0.5%, i.e.

$$\max\{-d \cdot \mathbf{w}_t^B, -0.5\%\} \leq \mathbf{w}_t^{FMP,k} \leq \min\{d \cdot \mathbf{w}_t^B, 0.5\%\},\ k = 1, \dots, m. \tag{27}$$

This restriction incorporates the same considerations as stated before and besides that, ensures that the FMPs are still limited in their position they can take in extreme large cap stocks. In the further analyses, this constraint is denoted as *Sym 2*.

3.4 Adjusted Multifactor Attribution

As $RCON$ represents the unsystematic stock specific part of the active portfolio's return, it is expected to be uncorrelated to the component explained by the factors ($FCON$). Nevertheless, when using regression weights in the cross-sectional regressions which do not exactly coincide with the inverse of the error covariance matrix, we already find ex-ante a factor and residual component of the portfolio whose returns are correlated, as derived in Appendix A.3 and shown in Table B.14. As the true error covariance is usually unknown, this is nearly always the case in practice. The correlation is ex-post even stronger apparent and suggests possible inaccuracies or biases in the estimation of $RCON$. Hence, Stubbs and Jeet (2016) propose an adjustment approach which aims to mitigate this correlation and thereby reducing the residual contribution by its systematic component.

For this purpose, we assume a time-constant linear dependency of the residual contribution on the contributions of each factor

$$\begin{aligned} RCON_t &= \beta' \mathbf{FCON}_t + \tilde{RCON}_t \\ \iff \mathbf{w}_t^{res\prime} \hat{\varepsilon}_t &= \beta' ((\mathbf{X}_t' \mathbf{w}_t^A) \odot \hat{\mathbf{f}}_t) + \tilde{\mathbf{w}}_t^{res\prime} \hat{\varepsilon}_t \quad . \end{aligned} \tag{28}$$

with $\tilde{RCON}_t$ denoting the new uncorrelated residual contribution and β being a m dimensional vector with coefficients estimated in a time series regression. Afterwards, we replace $RCON_t$ in the slightly reformulated standard attribution Equation 8

$$\begin{aligned} \mathbf{w}_t^{A\prime} \mathbf{r}_t &= \mathbf{1}_m' \mathbf{FCON}_t + \beta' \mathbf{FCON}_t + \tilde{RCON}_t \\ &= (\mathbf{1}_m' + \beta') \mathbf{FCON}_t + \tilde{RCON}_t \\ &= (\mathbf{1}_m' + \beta') ((\mathbf{X}_t' \mathbf{w}_t^A) \odot \hat{\mathbf{f}}_t) + \tilde{\mathbf{w}}_t^{res\prime} \hat{\varepsilon}_t \quad , \end{aligned} \tag{29}$$

with $\mathbf{1}_m$ denoting a m dimensional vector of ones. Here, β is referred to as an absolute adjustment to the factor exposures. This representation leads to the interpretation that the negative correlation (i.e. negative β) between $RCON$ and $FCON$ that we usually observe ex-ante and ex-post (Table B.14) is the result of the portfolio's factor exposure being overstated in the attribution. Due to investment constraints, the real exposure of the portfolio to the factors is actually lower which justifies the adjustment.

In the adjusted multifactor attribution, the m dimensional vector β is estimated in a time series regression over the entire time frame following the model in Equation 28. In order to avoid insignificant adjustments, we use an iterative procedure starting with a time series regression of $\mathbf{RCON}$ on the factor contributions $\mathbf{FCON}_k$ including all factors k. In each iteration, factors with a p-value for the estimated β_k-coefficient above a threshold π are removed until only factors with significant β_k^* estimates are left. Afterwards, we conduct a reentry procedure, in which each rejected factor is added to the regression one at a time. In case, the reentered factor has a significant β_k estimate and does not raise the other p-values above π, this factor is added back to the selection. After the reentry procedure is finished, we conduct a final time series regression and use the obtained β^* estimates for the relative adjustment. For each factor k with a significant β_k^* estimate, the adjusted factor contribution $FCON_{k,t}^{Adj}$ at each time t is calculated as

$$\begin{aligned} FCON_{k,t}^{Adj} &= FCON_{k,t}(1+\beta_k^*) \\ &= ((\mathbf{X}_t'\mathbf{w}_t^A)\odot\hat{\mathbf{f}}_t)_k(1+\beta_k^*) \quad , \end{aligned} \tag{30}$$

where the subscript k denotes the k-st element of each vector. The adjusted residual contribution is then correspondingly computed as

$$\begin{aligned} RCON_t^{Adj} &= \mathbf{w}_t^{A\prime}\mathbf{r}_t - \beta^{*\prime}\mathbf{FCON}_t^{Adj} \\ &= \mathbf{w}_t^{A\prime}\mathbf{r}_t - \beta^{*\prime}((\mathbf{X}_t'\mathbf{w}_t^A)\odot\hat{\mathbf{f}}_t) \quad . \end{aligned} \tag{31}$$

The shortcoming of this approach is the assumption of a time-constant linear relationship measured with β_k between $RCON$ and $FCON_k$. In order to drop this assumption, we propose to estimate $\beta_{k,t}$ as time-varying coefficients over a rolling window of the past 60 months at each time t, so that we regress $\mathbf{RCON}_{t-59:t}$ on $\mathbf{FCON}_{k,t-59:t}$ of each factor k. We apply the same selection procedure as in the standard approach in each period in order to avoid insignificant adjustments and obtain time series of the significant $\beta_{k,t}^*$ coefficients. This has also the advantage that linear dependencies, which are only observed over a shorter time period rather than over the entire time frame, are captured in the adjustment. Moreover, in order to obtain less noisy estimates of the $\beta_{k,t}^*$ coefficients, we model them in a local level model which we then estimate with the Kalman Filter as described in Hamilton (1985). Finally, we use this time series of estimated $\beta_{k,t}^*$ coefficients for the adjustments of the factor contributions in Equation 30. We refer to this approach as *Time Series Adjustment* in the following analysis in this paper.

3.5 Non-Linear Factor Attribution

One of the main assumptions of the attribution method in this paper is the linearity of the underlying factor model in Equation 1. As described in De Boer and Jeet (2015), in cases when the returns to a certain factor were particularly strong in its tails, this assumption would lead to the issue that stocks with a large positive exposure to that factor performed much better than the linear factor model would have estimated, leading to a positive residual return $\hat{\varepsilon}_{it}$ for these stocks. Correspondingly, this would also cause large negative residuals for stocks with a strong negative factor exposure. In addition to that, long-only constraints would prevent the investor of being fully able to underweight stocks with large negative exposure as she would like to as assumed by the FMP, leading to positive weights in $\mathbf{w}^{res}$. Similarly, maximum weight constraints restrain her from taking correspondingly large active positions in stocks with large positive exposure which results in negative weights in $\mathbf{w}^{res}$. Overall, these restrictions and assumptions would finally lead to larger negative **RCON** in the attribution as it is calculated as the vector product of $\mathbf{w}^{res}$ and the estimated residual $\hat{\varepsilon}$. However, part of the negative **RCON** can be reasonably re-attributed to the correspondingly overestimated **FCON**, for which De Boer (2019) proposes a non-linear adjustment approach. He defines a $n \times m$ dimensional classification matrix $\mathbf{X}^{classif}$ with the elements

$$\mathbf{X}_{i,k,t}^{classif} = \frac{\mathbf{X}_{i,k,t}^2}{\sum_{k=1}^{m} \mathbf{X}_{i,k,t}^2} \quad \text{with} \quad i \in \{1, ..., n\} \quad \text{and} \quad k \in \{1, ..., m\}, \tag{32}$$

which can be interpreted as the relative importance of each considered factor k for each stock i at time t. Correspondingly to this classification matrix we can then re-attribute the $RCON_{i,t} = \hat{\varepsilon}_{i,t} w_{i,t}^{res}$ of each stock to the factors by which it is most characterized with the adjustment

$$\mathbf{\Delta}_t^{nl} = \hat{\varepsilon}_t' diag(\mathbf{w}_t^{res}) \mathbf{X}_t^{classif} \quad . \tag{33}$$

This leads to the adjusted contribution of factor k

$$FCON_{k,t}^{Adj} = FCON_{k,t} + \Delta_{k,t}^{nl} \quad , \tag{34}$$

where the subscript k denotes the k-st entry of each vector at time t. Analogously, we obtain the adjusted residual contribution

$$RCON_t^{Adj} = RCON_t - \sum_{k=1}^{m} \Delta_{k,t}^{nl} \quad . \tag{35}$$

Depending on which factors are finally included in the re-attribution, we can obtain different interpretations. If we only adjust the contributions of factors that are used in the portfolio construction, we can identify to which extend we could not exploit their return potential due to portfolio constraints. However, as the rows of $\mathbf{X}^{classif}$ sum up to one, we can theoretically also use the non-linear attribution to re-attribute the whole **RCON** by including all factors of the underlying factor model in the adjustment. Through that, we can obtain a clean decomposition of **RCON** and identify the factors by which it was most driven.

Nevertheless, this method has the shortcoming that it does not fully reflect the underlying reasoning that large positive (negative) factor exposures of a stock lead to a negative (positive) residual weight. As it re-attributes **RCON** independently of the magnitude of the factor exposures of each stock and the signs of residual weights and factor exposures, we propose two refinements of the method in order to conduct the adjustments more in line with the aforementioned reasoning. Hence, we firstly include only factor exposures in the calculation of $\mathbf{X}_t^{classif}$ which are larger (smaller) than the $1-\alpha$ (α)-th percentile of the cross-sectional exposures for each factor at time t. By doing so, we only consider factor exposures of stocks with a comparatively large absolute magnitude and which are more likely to be the driver of the residual contributions. Moreover, we only conduct the adjustments, if the weight of a stock i in $\mathbf{w}_t^{res}$ has the opposite sign as the exposure $X_{i,k,t}$ of the stock to the considered factor k. These refinements aim to provide a better distinction between non-linear effects and stock-specific noise in **RCON** in the adjustment. Moreover, by varying the threshold α we can ascertain to what extend the **RCON** is driven by the 'tails' of each factor.

3.6 Quality Measures for the Multifactor Attribution

In order to assess the quality of the attribution results obtained by applying each of the aforementioned approaches, we analyze the corresponding **RCON** which is the main indicator for the informativeness of the attribution. As it is common in related literature such as Vandenbussche and Stubbs (2018), we use different metrics to quantify the quality of the attribution. These include

1) the R^2 of the cross-sectional regression as a measure for the estimation accuracy. As only the WLS estimation in Section 3.2 and the derivation of restricted FMPs in 3.3 affect the factor returns, the R^2 is only relevant for these approaches;

2) the mean absolute **RCON**, the annualized cumulative **RCON** and the volatility of the **RCON** time series as measures for the magnitude of the stock-specific contribution;

3) the ratio of the absolute values of **RCON** and the active portfolio return r^A, i.e. $|\frac{RCON_t}{r_t^A}|$ as a measure for the relative magnitude of the residual contributions. As we noticed that this measure occasionally takes on extremely large values in cases when the active return r^A is near zero, we report the time series median to avoid possible distortions by these outliers;

4) the correlation between the time series of **RCON** and the aggregated **FCON**, since a large correlation suggests a systematic component in **RCON** which is not caused by stock-specific risk and which is therefore likely to be wrongly attributed.

4 Simulation Study Setup

In order to compare the different aforementioned approaches and their ability to improve the informativeness of the attribution, we conduct our analyses on simulated return data next to the empirical return data. For that, we use the underlying linear factor model as data generating process

$$\mathbf{r}_t = \mathbf{X}_t\mathbf{f}_t + \varepsilon_t \quad , \tag{36}$$

where we estimate the factor returns $\hat{\mathbf{f}}_t$ in the first step in cross-sectional OLS regressions for each period t based on the real return data. Afterwards, we use the time series of the residuals $\hat{\varepsilon}$ of these regressions to obtain an estimate of the $n \times n$ residual covariance matrix

$$\hat{\mathbf{\Sigma}}_\varepsilon = \hat{\varepsilon}\hat{\varepsilon}' = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots & \hat{\sigma}_{1n} \\ \hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots & \hat{\sigma}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{n1} & \hat{\sigma}_{n2} & \cdots & \hat{\sigma}_n^2 \end{bmatrix} \quad , \tag{37}$$

with the estimated variances $\hat{\sigma}_i^2$ and covariances $\hat{\sigma}_{ij}$. Afterwards, we sample new errors $\tilde{\varepsilon}_{it}$ with different characteracteristics as described later in this section, and generate the simulated asset returns based on these.

The advantage of using the simulated return data in our attribution is that we know ex ante what portion of the active portfolio's return is driven by the exposure to certain factors or by stock specific return. By analyzing the deviation of the estimated **FCON** and **RCON** from the actual values over time, we therefore obtain more insights on which of the considered attribution approaches leads to the most accurate attribution results. Also, investigating the deviation of the estimated factor returns with the real factor returns in the approaches in Sections 3.2 and 3.3 provides a measure for the accuracy of the cross-sectional regressions and is used as an indicator for the quality of the corresponding attribution.

Moreover, by separately analyzing different error and asset return characteristics, we obtain more insights into which of these properties cause the issues in the attribution that we face in practice and by which of the investigated approaches these issues can be solved.

For the simulation of the return data we consider four different settings which are described in this section.

4.1 Homoscedastic Errors

In the base simulation we draw the new errors $\tilde{\varepsilon}_{it}$ from a normal distribution with $\mathbb{E}[\tilde{\varepsilon}_{it}] = 0$ and a stock- and time constant variance $\mathrm{Var}(\tilde{\varepsilon}_{it}) = \sum_{i=1}^{n} \frac{\hat{\sigma}_i^2}{n} = \overline{\hat{\sigma}}$ with no correlation between the residuals ($\mathrm{Cov}(\tilde{\varepsilon}_{it}, \tilde{\varepsilon}_{jt}) = 0 \quad \forall i, j, i \neq j$). Subsequently, we obtain the simulated returns calculated as

$$\tilde{\mathbf{r}}_t = \mathbf{X}_t \hat{\mathbf{f}}_t + \tilde{\varepsilon}_t \quad . \tag{38}$$

This case represents a scenario in which the data generating process of the asset returns exactly coincides with the assumptions of the cross-sectional OLS regressions in the attribution and enables us to filter purely portfolio construction related issues.

4.2 Heteroscedastic Errors

Secondly, we consider the case when the errors of the stocks are heteroscedastic and uncorrelated, i.e. $\mathrm{Var}(\tilde{\varepsilon}_{it}) = \hat{\sigma}_i^2$ and $\mathrm{Cov}(\tilde{\varepsilon}_{it}, \tilde{\varepsilon}_{jt}) = 0 \quad \forall i, j, i \neq j$. Using the corresponding asset returns generated from Equation 38 especially allows for a more accurate analysis of the implications of applying the different regression weights in Section 3.2 on the attribution and on the accuracy of the estimated factor returns.

4.3 Correlated Errors

As it is aimed in the attribution to use a factor risk model which sufficiently describes asset returns without running the risk of overfitting, it is possible that co-movements of returns are not fully captured by the model. This in turn leads to the estimated residuals exhibiting non-zero correlations between each other which violates one of the Gauss Markov assumptions in the cross-sectional regressions. Therefore, we consider the case in which the errors $\tilde{\varepsilon}_{it}$ are drawn from a Multivariate Normal Distribution with $\mathbb{E}[\tilde{\varepsilon}_{it}] = 0$ and the variance-covariance matrix

$$\mathbf{\Sigma}_{\varepsilon} = \begin{bmatrix} \bar{\hat{\sigma}} & \hat{\sigma}_{12} & \cdots & \hat{\sigma}_{1n} \\ \hat{\sigma}_{21} & \bar{\hat{\sigma}} & \cdots & \hat{\sigma}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{n1} & \hat{\sigma}_{n2} & \cdots & \bar{\hat{\sigma}} \end{bmatrix}, \tag{39}$$

where we again imply a constant variance for each stock's residual in order to isolate the effects of the correlation on the attribution.

4.4 Non-Linear Asset Returns

Finally, we investigate the impact of having a non-linear dependency of the asset returns on the alpha factors instead of the linear relationship as assumed in the factor model. Here, we only include the non-linearities for the alpha factors in order to assess how well the investigated techniques are able to identify which factors are affected. We model these non-linearities as quadratic dependencies so that the influence of the alpha factors on asset returns are especially pronounced for stocks with factor exposures of a large absolute value. However, we do not want to assume a specific "direction" of the non-linearities, i.e. the returns of stocks with a large positive exposure can be whether significantly higher or lower than in a linear dependency. By doing so, we isolate the effect of having non-linearities from a possible bias in one direction which however, may be observable in real return data as described in Section 3.5.

In order to model these non-linearities in the simulated data, we firstly randomly draw from a Bernoulli Distribution with parameter $p = 0.5$ a value d which represents the direction. Depending on the drawn value for d, we then transform the stocks' exposures to the alpha factors as

$$\begin{aligned}
&\text{If } d = 0 \quad X_{i,t}^{transf} = \begin{cases} X_{i,t}^2, & \text{if } X_{i,t} > 0 \\ -X_{i,t}^2, & \text{if } X_{i,t} < 0 \end{cases} \\
&\text{If } d = 1 \quad X_{i,t}^{transf} = \begin{cases} -X_{i,t}^2, & \text{if } X_{i,t} > 0 \\ X_{i,t}^2, & \text{if } X_{i,t} < 0. \end{cases}
\end{aligned} \tag{40}$$

Subsequently, we use the transformed exposures to calculate the simulated returns as previously described.

5 Attribution on Simulated Data

In this section, we show the results of the analyses based on simulated return data in order to provide an overview over the issues that we encounter in the performance attribution of constraint portfolios. We investigate different return characteristics to identify possible causes of these issues and show to what extend the examined techniques solve these problems.

5.1 Homoscedastic Errors

Firstly, we consider the standard case with homoscedatistic errors in which all Gauss-Markov conditions are met. Panel A in Table 3 summarizes the obtained results averaged over 25 simulations for each of the applied attribution techniques in this setting. It shows selected quality measures (annualized cumulative $RCON$ and correlation between $RCON$ and aggregated $FCON$) as well as the average RMSE of the estimated alpha factor returns and contributions with respect to the true factor returns and contributions. Moreover, the table includes the corresponding statistics of the actual simulated data which depicts how the attributed $RCON$ should look like in the ideal case. The complete results as shown in Table B.15 indicate that we obtain in this scenario for all WLS and restricted FMP approaches $RCON$ with a magnitude and volatility close to the true values. Using market cap regression weights (WLS Cap) results in the lowest unexplained $RCON$ on average, which stresses the main advantage of using more representative weights. Moreover, we obtain values near zero for the correlation between $RCON$ and the aggregated $FCON$ when using OLS and WLS as shown in Table 3. This is in line with the ex ante expected correlation shown in Table B.14.

This is not the case when applying the two restricted FMP approaches LO & MW and Sym 1 as we observe negative correlations of -0.23 and -0.29. Appendix A.3 gives more insights in the causes for these larger magnitudes in the correlation.
Regarding the Time Series Adjustments (TS Adj and TS Adj 2), we do not obtain improvements since the standard OLS approach already leads to very low correlations. Therefore, the selection procedure in this approach only rarely results in a significant adjustment so that the obtained quality measures are nearly unchanged from that in the OLS estimation.
When applying the standard Non-Linear Adjustment (NL Adj), we find as expected Quality Measures with zero values since this approach re-attributes the entire $RCON$ to the factor contributions.

Looking at the average RMSE of the estimated factor returns in Panel A of Table 3 it becomes clear that the standard OLS estimation leads to the most accurate estimates. This is not surprising as OLS uses equal regression weights, which exactly correspond to the inverse of the error covariance matrix. On average, the estimated factor returns are per period only 33bp off from the true values for the alpha factors. For WLS Cap we observe a 7bp larger RMSE, which makes the trade-off between representativeness and estimation accuracy clear. Note, that the factor returns in TS Adj and NL Adj do not differ from those obtained by OLS since these adjustments only affect the factor contributions and leave the factor returns unchanged.

We also find a large difference in the alpha factor contributions when applying the NL Adjustments. This is due to the fact that we generated the asset returns from the linear factor model and hence, do not have any non-linearities in the returns which could cause issues. This highlights that the re-attributions in the absence of non-linearities are rather random and do not result in more accurate factor contributions.

In conclusion, if the underlying assumptions of OLS hold, we do not face any issues in the attribution of a constrained long-only portfolio as we do in practice. In this case, the standard attribution with cross-sectional OLS regressions lead to the most accurate results. Nevertheless, by using a WLS approach with market cap weights one can achieve a generally lower $RCON$ which allows a better interpretation of the attribution results at the costs of a lower estimation accuracy.

5.2 Heteroscedastic Errors

When adding heteroscedasticity in the distribution of the errors we obtain in general similar results for the Quality Measures as in the aforementioned case of constant error variances (Panel B in Table 3). The main difference can be found in the RMSE of the alpha factor returns, which suggests that the WLS RW estimation leads to the most accurately estimated factor returns. This stresses the benefit of using WLS weights in the attribution as they take the varying stock specific risk among stocks into account which is a common property of asset returns. Moreover, it is apparent that the correlations between $RCON$ and aggregated $FCON$ are in these cases on average still near zero. This suggests that in this setting the multifactor attribution method is still well able to properly separate the idiosyncratic contribution from that caused by the portfolio's active exposure to the factors.

5.3 Correlated Errors

When adding correlation among the stock-specific errors, we observe two major differences in Panel C in Table 3 in comparison to the previous cases. Firstly, we obtain on average a negative annualized cumulative $RCON$ in the OLS and WLS estimations, which suggests a bias that overestimates the factor returns and $FCON$ on average. As the correlation between residuals would occur in practice if relevant factors are left out in the factor model, this might be a result of an omitted variable bias (OVB). The OVB also explains partly the up to three times larger RMSE of the estimated factor returns of the Alpha factors in comparison to the results in Panels A and B. Therefore, one of the reasons why we obtain a systematically trending $RCON$ in practice can be found in the misspecification of the factor model.

The second difference is the larger negative correlation between $RCON$ and the aggregated $FCON$ of -0.20 to -0.24 when using OLS and WLS. This can also be seen in connection with the negatively biased $RCON$, as $RCON$ and $FCON$ have to add up to the active portfolio's return and hence, trend in the long-run in opposite directions.

Table 3: Attribution on simulated return data

	Sim. Data	OLS	WLS RW	WLS Cap	WLS Shrink	MW & LO	Sym 1	Sym 2	TS Adj	TS Adj 2	NL Adj	NL Adj 2
Panel A: Homoscedastic Errors												
Ann. Cum. RCON (in %)	0.02	0.01	0.00	0.01	0.01	0.06	-0.03	0.01	0.02	0.00	0.00	-0.04
	(0.38)	(0.35)	(0.35)	(0.31)	(0.31)	(0.37)	(0.45)	(0.31)	(0.35)	(0.34)	(0.00)	(0.23)
Corr RCON/FCON	-0.01	0.00	0.00	0.04	0.04	-0.23	-0.29	0.01	0.01	0.00	-0.02	0.06
	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)	(0.05)	(0.05)	(0.07)	(0.06)	(0.05)	(0.05)	(0.06)
RMSE Alpha Factor Returns (in %)		0.33	0.35	0.42	0.39	0.75	0.93	0.44	0.33	0.33	0.33	0.33
RMSE Alpha Factor Contributions		0.17	0.18	0.22	0.21	0.40	0.48	0.23	0.17	0.18	0.24	0.20
Panel B: Heteroscedastic Errors												
Ann. Cum. RCON (in %)	0.00	0.01	0.00	0.00	0.00	0.05	-0.05	0.00	0.02	0.01	0.00	-0.03
	(0.31)	(0.27)	(0.30)	(0.25)	(0.25)	(0.32)	(0.35)	(0.25)	(0.28)	(0.27)	(0.00)	(0.18)
Corr RCON/FCON	0.01	-0.04	-0.02	0.01	0.01	-0.22	-0.25	0.00	-0.03	-0.02	-0.02	0.04
	(0.06)	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)	(0.06)	(0.06)	(0.06)	(0.05)	(0.05)	(0.06)
RMSE Alpha Factor Returns (in %)		0.38	0.32	0.40	0.37	0.65	0.76	0.41	0.38	0.38	0.38	0.38
RMSE Alpha Factor Contributions		0.19	0.16	0.20	0.19	0.34	0.39	0.21	0.19	0.20	0.23	0.21
Panel C: Correlated Errors												
Ann. Cum. RCON (in %)	0.00	-0.07	-0.08	-0.02	-0.05	0.14	0.02	0.01	0.16	0.02	0.00	0.02
	(0.46)	(0.31)	(0.34)	(0.32)	(0.32)	(0.58)	(0.54)	(0.46)	(0.32)	(0.32)	(0.00)	(0.27)
Corr RCON/FCON	0.00	-0.20	-0.24	-0.22	-0.20	-0.54	-0.51	-0.09	-0.07	-0.10	0.00	-0.11
	(0.06)	(0.07)	(0.06)	(0.06)	(0.06)	(0.04)	(0.05)	(0.08)	(0.07)	(0.04)	(0.04)	(0.07)
RMSE Alpha Factor Returns (in %)		1.07	1.16	1.23	1.15	1.73	1.88	0.76	1.07	1.07	1.07	1.07
RMSE Alpha Factor Contributions		0.39	0.42	0.47	0.43	0.80	0.83	0.39	0.35	0.36	0.38	0.40
Panel D: Non-linear Asset Returns												
Ann. Cum. RCON (in %)	0.01	0.03	0.02	0.02	0.02	0.11	-0.05	0.02	0.04	0.02	0.00	-0.04
	(0.38)	(0.46)	(0.46)	(0.38)	(0.38)	(0.55)	(0.49)	(0.36)	(0.36)	(0.39)	(0.00)	(0.26)
Corr RCON/FCON	-0.01	-0.36	-0.34	-0.27	-0.27	-0.52	-0.31	-0.21	0.01	-0.10	0.01	-0.16
	(0.08)	(0.05)	(0.05)	(0.04)	(0.05)	(0.05)	(0.04)	(0.06)	(0.04)	(0.03)	(0.04)	(0.04)
RMSE Alpha Factor Returns (in %)		3.30	3.21	3.14	3.15	3.24	2.94	2.93	3.30	3.30	3.30	3.30
RMSE Alpha Factor Contributions		0.33	0.32	0.30	0.29	0.51	0.51	0.30	0.18	0.24	0.30	0.28

Notes: This table shows the selected average results obtained from 25 multifactor attributions based on simulated return data with different properties in each Panel. These include the mean of the calculated Quality Measures (corresponding Standard deviations are shown in parentheses) and the RMSE of the estimated alpha factor returns and contributions, respectively, with regards to the true alpha factor returns and contributions used in the generation of the return data. The complete results can be found in Tables B.15 - B.18. Abbreviations: *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation, *MW & LO* - Maximum Weight and Long-Only constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW, *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas, *NL Adj* - Standard non-linear Multifactor Attribution, *NL Adj 2* - Refined non-linear Multifactor Attribution with Threshold $\alpha = 0.1$.

Moreover, the negative correlation is an issue which specifically occurs in the attribution of a constrained long-only portfolio since it does not appear for an unconstrained long-short portfolio, as the measures in Table B.19 show. This suggests that the multifactor attribution approach is not fully able to identify idiosyncratic return contributions in this case which leaves room for improvement. Thus, it becomes beneficial to apply the Time Series Adjustment by Stubbs and Jeet (2016) in order to mitigate a large part of the correlation. This also leads to a lower RMSE for the contribution of the alpha factors so that the adjustment also results in more accurate attributions in the case of correlated errors. The need for such an adjustment in the attribution based on real return data can also be seen in the correlations shown in Table B.14. When using regression weights which do not exactly correspond to the inverse of the error covariance, we already find ex ante a certain level of correlation as it is also described in Stubbs and Jeet (2016). In addition to that, further estimation errors in the actual attribution lead to an even higher level of correlation as the ex post values indicate.

Finally, we also find that using the constraint Sym 2 with $d = 10$ to derive the FMP weights leads to the overall most accurately estimated factor returns and the most informative attributions as indicated by the low values of the quality measures. This rather relaxed constraint with multiplier $d = 10$ is less restrictive than the other investigated constraints MW & LO and Sym 1 ($d = 3$) and appears to constitute a good balance between restricting the FMPs and preserving the properties of the statistically correct OLS/WLS estimator.

5.4 Non-Linear Asset Returns

As shown in Panel D of Table 3, a non-linear relationship between the asset and alpha factor returns constitutes another characteristic which causes a negative correlation between $RCON$ and aggregated $FCON$. A reason for that is the impact of the factor covariance as described in Appendix A.3 which already results ex-ante in an expected negative correlation (Table B.14). The more negative observed ex-post correlation can further be explained by larger inaccuracies in the estimation of the alpha factor returns.

Also in this case, the Time Series Adjustment leads to a significant improvement in terms of a lower magnitude and volatility of $RCON$ and a reduced correlation. The lower RMSE of the estimated Alpha $FCON$ of 0.18 indicates that with this adjustment we are also able to correctly identify the factors which caused the larger $RCON$ in the OLS and to re-attribute it accordingly.

We can observe a similar effect when using the Non-Linear Adjustment with $\alpha = 0.1$ (NL Adj 2), for which we also obtain a significant reduction of the correlation and the inaccuracy of the alpha $FCON$ on average. However, this improvement is not as pronounced as for the Time Series Adjustment which may be due to the fact that we only consider one form of a Non-Linearity in this setting, i.e. only a quadratic dependency of the asset returns on the alpha factor returns. This may not be fully representative for real asset returns as the non-linear dependencies can take different functional forms which might be better captured by the Non-Linear Adjustment in practice. However, we do not investigate the further functional forms in this paper as this is out of the scope of our research.

To conclude, this simulation study reveals possible issues we expect to encounter in the attribution based on real return data such as a negatively trending $RCON$ or a highly negative correlation between the time series of $RCON$ and $FCON$. The analyses indicate, that the main drivers for these issues can be found in non-linear properties of asset returns as well as in co-movements of stock returns which are not captured by the factor model. Moreover, the results establish the expectation that the Time Series Adjustment by Stubbs and Jeet is among the investigated approaches best able to correct for these problems.

6 Attribution on Empirical Data

In this section, we describe the results of the attribution based on real asset return data. Moreover, we provide in-depth analyses of the effect the investigated approaches have on the attribution.

6.1 Weighted Least Squares

In order to assess how the usage of different regression weights influences the multifactor attribution, we firstly analyze the corresponding properties of the implied FMPs. Table 4 includes statistics for the FMPs of the alpha factors *Momentum* and *Value*. Those of the risk factors are very similar and therefore, not shown in the table. To allow for a comparison with a realistic portfolio, we also display the corresponding statistics of the attributed active portfolio.

The results shown in Panel A firstly suggest that using the square root of market capitalizations as regression weights lead to FMPs which generally take larger absolute positions in stocks with larger market capitalizations as indicated by the correlation of 0.42. This shift of weights towards large cap companies is also detectable in the portion of absolute FMP weights in the 50% largest companies in Panel B which is 24 percentage points (pp) larger in comparison to the OLS base case. For the statistically robust regression weights WLS SP and WLS RW this shift is significantly weaker which suggests that the assumption that large cap companies have less volatile returns only holds to a limited extend in our data set.

Considering the effective number of stocks of the FMPs in Panel C, it is apparent that the standard OLS estimation leads to the most diversified FMPs which stands in contrast to the rather strongly concentrated active portfolio. Although the active portfolio holds at each point in time stocks of approximately 1600 companies, it has on average the same level of concentration as an equally weighted portfolio with 309 stocks. By using WLS we generally obtain more concentrated FMPs than with OLS, especially for the single period estimation WLS SP. In WLS SP, we face the issue that in cases when single returns are estimated with a low residual $\hat{\varepsilon}_{i,t}$ in the cross-sectional OLS regression in the first stage, they receive correspondingly large weights in the consecutive second-stage WLS regression. This results in these stocks being overrepresented in the FMPs and therefore, also in highly concentrated FMPs. Nevertheless, we overcome this issue with the rolling window estimation in WLS RW which leads to less extreme regression weights as this method incorporates information of several periods instead of just a single one and is therefore less influenced by outliers and estimation inaccuracies.

Table 4: Summary of Statistics of the FMPs

	Active PF	OLS	WLS SP	WLS RW	WLS Cap	WLS Shrink
Panel A: Correlation Market Cap and absolute FMP Weight						
Value	0.56	-0.08	-0.01	0.02	0.42	0.26
	(0.03)	(0.03)	(0.03)	(0.05)	(0.06)	(0.06)
Momentum	0.56	-0.07	0.00	0.03	0.44	0.28
	(0.03)	(0.03)	(0.02)	(0.04)	(0.06)	(0.05)
Panel B: Portion of absolute FMP weights in the 50% largest companies						
Value	0.93	0.45	0.49	0.51	0.69	0.64
	(0.02)	(0.03)	(0.05)	(0.03)	(0.02)	(0.02)
Momentum	0.93	0.46	0.50	0.52	0.69	0.64
	(0.02)	(0.02)	(0.03)	(0.03)	(0.04)	(0.03)
Panel C: Effective Number of Stocks						
Value	309.0	873.4	231.1	555.2	585.7	659.3
	(43.3)	(158.9)	(46.1)	(113.9)	(138.8)	(142.1)
Momentum	309.0	733.8	179.3	458.3	489.3	564.7
	(43.3)	(150.4)	(39.6)	(105.3)	(123.0)	(132.0)
Panel D: Active Share						
Value	0.61	0.51	0.55	0.57	0.57	0.57
	(0.05)	(0.03)	(0.05)	(0.04)	(0.07)	(0.06)
Momentum	0.61	0.57	0.64	0.66	0.61	0.62
	(0.05)	(0.05)	(0.06)	(0.06)	(0.05)	(0.05)
Panel E: 99th percentile of weights in the 50% smallest companies (in %)						
Value	0.13	0.25	0.63	0.34	0.17	0.19
	(0.05)	(0.05)	(0.15)	(0.07)	(0.03)	(0.03)
Momentum	0.13	0.24	0.66	0.33	0.15	0.18
	(0.05)	(0.04)	(0.14)	(0.08)	(0.02)	(0.03)
Panel F: 1st percentile of weights in the 50% smallest companies (in %)						
Value	-0.01	-0.27	-0.62	-0.31	-0.17	-0.19
	(0.00)	(0.06)	(0.15)	(0.09)	(0.04)	(0.04)
Momentum	-0.01	-0.28	-0.75	-0.31	-0.17	-0.19
	(0.00)	(0.05)	(0.16)	(0.08)	(0.03)	(0.03)

Notes: This table shows several statistics of the implied FMPs when using different weighting schemes in the cross-sectional regressions. It shows the average value as well as the standard deviation in parentheses of each statistic over the time period from January 1997 to February 2019. We only include the FMPs for the alpha factors *Momentum* and *Value* for reasons of clarity. For the purpose of comparison, we include the corresponding statistics of the attributed active portfolio. Abbreviations: *SP* - Single Period Estimation, *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation.

Finally, when considering the 1st and the 99th percentile of FMP weights among small cap stocks in Panel E and F, respectively, we notice that using WLS Cap results in FMP weights with the generally smallest magnitude. For the long positions, this corresponds most to what an restricted investor can take in small cap stocks. However, concerning the possible short positions, small cap stocks are in practice much more restricted, which becomes clear in Panel F. In this case, the FMPs take much larger short positions than practically feasible as it can be seen for instance when using WLS Cap in which case the FMP's 1st percentile short position is 17 times larger than for the real active portfolio. Therefore, the long-only restriction can be considered as the constraint which leads to the largest discrepancy between the active portfolio and the implied FMPs.

Regarding the Shrinkage estimator, it can be seen that for most of the analyzed statistics we indeed obtain results with values between those of WLS Cap and WLS RW. However, these values tend to be nearer to the FMPs obtained when using WLS Cap which is due to the estimated Shrinkage Intensity $\hat{\delta}$ with values around 0.75 as depicted in Figure 2. As described in Ledoit and Wolf (2003b), this indicates that there is about three times more estimation error in the sample covariance matrix as there is bias in the target covariance matrix implied by the market capitalization weights.

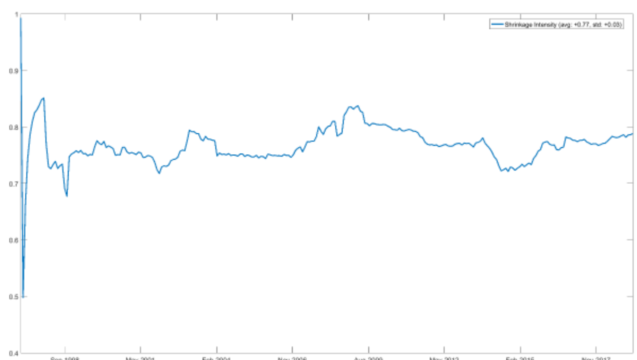


Figure 2: Estimated Shrinkage Intensity $\hat{\delta}$ over time

In general, using market cap weights in the cross-sectional regressions leads to more realistic FMPs which come closest to the properties of the active portfolio. The corresponding FMPs do not only show a high concentration with most of their weight in large cap stocks but also the least extreme positions in small cap stocks. The results of applying the different regression weights in the attribution of the active portfolio are shown in Table 5. Panel

A depicts the decomposition of the realized active return into contributions of each factor and the unexplained residual contribution. Regarding these return attributions we obtain relatively similar annualized cumulative contributions of the factors across the different WLS approaches. In all cases, a large positive part of the return is, not surprisingly, attributed to the alpha factors *Momentum* and *Value*, towards which the portfolio is tilted. However, we find in the OLS estimation that the sum of positive contributions significantly exceeds the portfolio's active return without being offset by negative return contributions of other factors. This leads to the significantly negative annualized cumulative $RCON$ of -0.65% which suggests that the positive $FCON$ were likely to be systematically overestimated in each period.

In comparison, applying WLS Cap leads to generally lower contributions of all factors especially of the alpha factors. This can be seen as a result of implicitly using FMPs in the attribution which are more in line with a constrained portfolio and therefore earn on average lower factor returns.

Considering the investigated quality measures in Panel B, we see that, although using the square root of the market capitalizations as regression weights is statistically incorrect, it does not lead to factor returns with a significantly lower explanatory power than when using statistically correct weights as with WLS RW (both with R^2 of 0.09). Moreover, the obtained $RCON$ for WLS Cap is also of lower magnitude and volatility than for other weighting schemes. This can again be explained by the more realistic FMPs and factor returns which are used in this attribution and which lead to factor contributions that better reflect the investment possibilities of restricted long-only investors. Overall, using WLS Cap leads to a reduction of around 10% (4bp) in comparison to the initial average absolute $RCON$ obtained with OLS. However, as already shown in Stubbs and Jeet (2016), this weighting scheme leads to a larger negative correlation than WLS RW on average which suggests a larger systematic component in $RCON$ additional to the uncorrelated idiosyncratic returns.

Furthermore, WLS Shrink results in quality measures which again constitute a balance between the statistically correct WLS RW and the representative WLS Cap. Not only does it lead to $RCON$ with a magnitude and volatility as low as for WLS Cap, but also it results in having a correlation between $RCON$ and $FCON$ which is nearer to the level of WLS RW and therefore, nearer to zero. Hence, we can conclude in this example that the Shrinkage Estimator for the regression weights indeed incorporates the beneficial properties of the both extremes WLS RW and WLS Cap and can be seen as an improvement on the widely used market cap weighting scheme.

Table 5: Summary Attribution Results

	OLS	WLS SP	WLS RW	WLS Cap	WLS Shrink
Panel A: Summary Performance Attribution (annualized values in %)					
Active Return	1.27	1.27	1.27	1.27	1.27
World	0.00	0.00	0.00	0.00	0.00
Volatility	-0.05	-0.05	-0.03	-0.06	-0.05
Size	0.49	0.48	0.20	0.45	0.40
Growth	-0.02	-0.02	-0.03	-0.04	-0.01
Liquidity	0.13	0.14	0.09	0.12	0.08
Leverage	-0.10	-0.10	-0.13	-0.14	-0.14
Value	0.81	0.87	0.67	0.71	0.79
Momentum	0.65	0.66	0.97	0.58	0.69
RCON	-0.65	-0.71	-0.46	-0.36	-0.48
Panel B: Summary Quality Measures					
R^2	0.10	0.10	0.09	0.09	0.09
	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)
Ratio RCON/r^A	0.75	0.78	0.70	0.67	0.65
	(14.18)	(14.80)	(16.26)	(13.45)	(14.75)
Mean abs. RCON (in %)	0.41	0.41	0.41	0.37	0.38
Std RCON (in %)	0.54	0.53	0.52	0.47	0.48
Corr RCON/FCON	-0.28	-0.26	-0.22	-0.28	-0.22

Notes: This table shows the summary of the annualized attribution results in % as well as of the Quality Measures for the attributions (corresponding standard deviations of the time series are shown in parentheses). Abbreviations: *SP* - Single Period Estimation, *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation.

6.2 Restricted Factor Mimicking Portfolios

In the derivation of the FMPs subject to real-world constraints, we firstly investigate the effect of imposing a maximum active weight constraint (MW) of 0.5% as well as the long-only constraint (LO) on these portfolios. The results in Table 6 firstly show that the imposition of the maximum active weight constraint has only little effect on the FMPs. This is due to the fact that the FMPs only rarely assign weights with a magnitude larger than 0.5% to single stocks so that this constraint alone does not constitute a considerable restriction.

Table 6: Summary of Statistics of the FMPs

	Active PF	OLS	MW	LO	MW & LO	Sym 1	Sym 2
Panel A: Correlation Market Cap and absolute FMP Weight							
Value	0.56	-0.08	-0.08	0.38	0.61	0.68	0.16
	(0.03)	(0.03)	(0.03)	(0.04)	(0.08)	(0.15)	(0.05)
Momentum	0.56	-0.07	-0.07	0.37	0.60	0.69	0.17
	(0.03)	(0.03)	(0.03)	(0.05)	(0.06)	(0.10)	(0.06)
Panel B: Portion of absolute FMP weights in the 50% largest companies							
Value	0.93	0.45	0.45	0.81	0.78	0.88	0.72
	(0.02)	(0.03)	(0.03)	(0.04)	(0.04)	(0.03)	(0.05)
Momentum	0.93	0.46	0.46	0.81	0.76	0.88	0.71
	(0.02)	(0.02)	(0.02)	(0.06)	(0.03)	(0.03)	(0.06)
Panel C: Effective Number of Stocks							
Value	309.0	873.4	873.3	189.4	189.8	198.6	568.7
	(43.3)	(158.90)	(158.9)	(83.6)	(59.0)	(102.8)	(186.8)
Momentum	309.0	733.8	733.7	109.5	144.2	152.7	491.3
	(43.3)	(150.4)	(150.6)	(57.0)	(22.4)	(63.0)	(131.5)
Panel D: Active Share							
Value	0.61	0.51	0.51	0.63	0.79	0.94	0.64
	(0.05)	(0.03)	(0.04)	(0.09)	(0.13)	(0.32)	(0.15)
Momentum	0.61	0.57	0.57	0.72	0.95	0.96	0.67
	(0.05)	(0.05)	(0.05)	(0.08)	(0.10)	(0.19)	(0.09)
Panel E: 99th percentile of weights in the 50% smallest companies (in %)							
Value	0.13	0.25	0.25	0.69	0.51	0.06	0.15
	(0.05)	(0.05)	(0.05)	(0.27)	(0.01)	(0.01)	(0.02)
Momentum	0.13	0.24	0.23	0.73	0.52	0.06	0.15
	(0.05)	(0.04)	(0.04)	(0.37)	(0.01)	(0.01)	(0.02)
Panel F:1st percentile of weights in the 50% smallest companies (in %)							
Value	-0.01	-0.27	-0.27	-0.02	-0.03	-0.06	-0.15
	(0.00)	(0.06)	(0.06)	(0.00)	(0.01)	(0.01)	(0.02)
Momentum	-0.01	-0.28	-0.28	-0.02	-0.04	-0.06	-0.16
	(0.00)	(0.05)	(0.05)	(0.00)	(0.01)	(0.01)	(0.02)

Notes: This table shows several statistics of the implied FMPs when using different weighting schemes in the cross-sectional regressions. It shows the average value of each statistic over the time period from January 1997 to February 2019 (corresponding standard deviations of the time series are shown in parentheses). We only include the FMPs for the factors *Momentum* and *Value* as these are the factors the portfolio is most exposed to. For the purpose of comparison, we include the corresponding statistics of the attributed active portfolio. Abbreviations: *MW* - Maximum Weight Constraint, *LO* - Long Only Constraint, *MW & LO* - MW and LO constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW.

Using the LO constraint on the other hand leads to a significant change in the positions of the alpha factor FMPs. It is noticeable that these FMPs shift a large portion of their short positions towards larger cap stocks in order to satisfy the constraint for the strongly restricted small cap stocks. This is indicated by the larger positive correlation in Panel A of 0.38 for the *Value* FMP in comparison to the -0.08 in the standard OLS estimation as well as the larger portion of weights in large companies in Panel B (45% in OLS and 81% with LO constraint). Furthermore, as the short positions in unattractive stocks are strongly restricted, the FMPs take sizable long positions in single attractive assets independently of their market cap in order to maximize the exposure to the corresponding factor. This results in a high portfolio concentration as shown in Panel C as well as at the large long positions in small cap stocks of more than 0.73%.

By imposing both constraints at once as well as by using a symmetric upper and lower bound in Sym 1 ($d = 3$), the mentioned large long position in small cap stocks are distributed to larger cap stocks, which can be seen at the larger correlation in Panel A. In their properties, the obtained FMPs in these cases come very close to the realistic active portfolio which generally confirms the feasibility of deriving realistic FMPs with this approach.

Finally, when strongly relaxing the symmetric constraints while still limiting the maximum active weight of each stock in the FMPs with Sym 2 ($d = 10$), we obtain FMP statistics with values between those obtained in OLS and Sym 1. These statistics are still close to those of the active portfolio, although taking much larger short positions in small cap companies.

The corresponding attribution results in Panel A of Table 7 firstly show that the single MW constraint only has a minimal effect as it was already suggested by the properties of the FMPs. On the contrary, imposing the LO constraint translates into a significant reduction of the return contributions of the *Size* and the *Momentum* factor by 32 bp and 23 bp, respectively. This indicates that these factors were much weaker drivers of the active portfolio's returns than assumed by the standard attribution method due to the infeasibility of taking many of their assumed short-positions in practice. Using both constraints at once (MW & LO) shows similar results considering the $FCON$.

When looking at the corresponding quality measures in Panel B we find for all constraints except of Sym 2 inferior values in comparison to the standard OLS case. Imposing strict constraints such as in MW & LO and Sym 1 lead to larger and more volatile $RCON$ which suggests that the issue of having a less accurate factor return estimate and thereby larger estimation errors outweighs the benefits of having more realistic FMPs in these cases. However, loosening the constraints in Sym 2 improves on the OLS estimation as we obtain a smaller unexplained $RCON$ while only sacrificing explanatory power of the estimated model to a small extend (R^2 of 0.09). Also, with this constraint we do not obtain a significantly larger negative correlation such as in the other cases which is another favourable property of the obtain $RCON$.

Table 7: Summary Attribution Results

	OLS	MW	LO	MW & LO	Sym 1	Sym 2
Panel A: Summary Performance Attribution (annualized values in %)						
Active Return	1.27	1.27	1.27	1.27	1.27	1.27
World	0.00	0.00	0.00	0.00	0.00	0.00
Volatility	-0.05	-0.05	-0.05	-0.04	-0.07	-0.06
Size	0.49	0.49	0.17	0.24	0.32	0.43
Growth	-0.02	-0.02	-0.02	0.05	-0.04	0.00
Liquidity	0.13	0.13	0.27	0.15	0.03	0.08
Leverage	-0.10	-0.10	-0.07	-0.11	-0.25	-0.13
Value	0.81	0.81	0.72	0.69	0.91	0.71
Momentum	0.65	0.65	0.42	0.43	0.43	0.61
RCON	-0.65	-0.64	-0.31	-0.17	-0.10	-0.38
Panel B: Summary Quality Measures						
R^2	0.10	0.10	-0.04	0.06	0.06	0.09
	(0.08)	(0.08)	(0.17)	(0.09)	(0.09)	(0.08)
Ratio RCON/r^A	0.75	0.75	1.23	0.96	0.81	0.70
	(14.18)	(14.18)	(23.47)	(15.53)	(12.87)	(15.23)
Mean abs. RCON	0.41	0.41	0.76	0.52	0.44	0.38
Std RCON	0.54	0.54	1.10	0.70	0.59	0.49
Corr RCON/FCON	-0.28	-0.28	-0.78	-0.61	-0.58	-0.29

Notes: This table shows the summary of the annualized attribution results in % as well as of the Quality Measures for the attributions (corresponding standard deviations of the time series are shown in parentheses). Abbreviations: *MW* - Maximum Weight Constraint, *LO* - Long Only Constraint, *MW & LO* - MW and LO constraint, *2 * (MW & LO)* - Relaxed MW and LO constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW.

In conclusion, we are indeed able to derive more realistic FMPs by solving a constrained optimization problem. This enables us to identify factors which are most restricted by the long-only constraint, such as *Size* and *Momentum*. However, the imposition of strict constraints (MW & LO and Sym 1) on the FMPs do not directly translate into more informative regression results as the obtained quality measures indicate. Nevertheless, by using more relaxed constraints, we incorporate the circumstance that the active portfolio is explained by a linear combination of FMPs and therefore, that each FMP by itself does not have to strictly satisfy each constraint. This approach leads to an improvement of the obtained quality measures and indicates that the derivation of constrained FMPs can indeed result in a more informative attribution. All in all, this approach leads to promising results but raises the challenge to balance the tightening of the constraints to obtain more realistic FMPs, and the feasibility of the optimization problem in order to ensure that the FMPs still represent reasonable solutions.

6.3 Adjusted Multifactor Attribution

As seen in the previous sections, in all cases the multifactor attribution results in a time series of $RCON$ which shows a negative correlation to the aggregated $FCON$ over time. As this implies that there might still be a significant systematic driver in the $RCON$ apart from stock-specific noise, we apply the adjusted multifactor attribution by Stubbs and Jeet (2016) in the standard OLS attribution to mitigate the correlation.

In Table 8 we show the factors and their beta coefficients as determined by the selection process of the standard Adjusted Multifactor Attribution. The $FCON$ of both alpha factors *Value* and *Momentum* towards which the attributed portfolio is tilted as well as the risk factors *Liquidity* and *Leverage* are here determined to be significantly able to explain the $RCON$ time series. The negative estimated β-coefficients also support the reasoning that the portfolio's actual exposure to certain factors is lower than assumed in the attribution due to the investment constraints and therefore, has to be adjusted to the downside. For example, the β of -0.31 for *Momentum* means that the assumed exposure of the portfolio to this factor was too large and should only be 69% of its original values shown in Figure 1.

Table 8: Beta Coefficients as determined by the standard Adjusted Multifactor Attribution

	β-Coefficient	Standard Error	t-Stat	p-value
Liquidity	-0.63	0.22	-2.79	0.01
Leverage	-0.76	0.27	-2.83	0.00
Value	-0.17	0.07	-2.35	0.02
Momentum	-0.31	0.09	-3.52	0.00

Notes: This table shows the time series regression results including the found significant factors with threshold $\pi = 0.05$.

The corresponding adjustments on the active portfolio's factor exposures as well as the resulting effect on the correlation between the $RCON$ and $FCON$ are presented in Table 9. They show that the standard adjustment method by Stubbs and Jeet (2016) significantly decreases the estimated correlation of the selected factors in Table 8 towards zero.

Table 9: Active Portfolio's Factor Exposures and Correlation between $RCON$ and $FCON$ using Standard (OLS) and Adjusted Attribution

	Active Exposures			Correlation RCON/FCON		
	Standard	TS Adj	TS Adj 2	Standard	TS Adj	TS Adj 2
World	0.00	0.00	0.00	0.00	0.00	0.00
Volatility	0.00	0.00	0.00	-0.05	0.00	0.06
Size	-0.23	-0.23	-0.22	-0.10	-0.01	0.06
Growth	-0.14	-0.14	-0.10	-0.05	-0.07	0.03
Leverage	0.13	0.05	0.12	-0.18	0.00	-0.13
Value	0.18	0.04	0.10	-0.18	0.00	0.03
Momentum	0.49	0.41	0.45	-0.08	0.01	-0.04
RCON	0.17	0.12	0.15	-0.21	0.01	0.02

Notes: This table shows the active portfolio's factor exposures and the correlation between the time series of Residual and Factor Contributions before and after applying the Time Series Adjustments with the threshold $\pi = 0.05$.

Using the adjustment method which allows for time-varying β_t, we also obtain a large, although less clean mitigation of the negative correlation towards zero. This difference can partly be explained by the circumstance that the second approach uses a rolling window of the last available 60 months to estimate the β_t-coefficients so that especially in the first months of the attribution period this approach leads to less accurate adjustments due to the lack of data.

Comparing the results from the adjusted attributions with the standard results in Panel B of Table 10, we see that both adjustments lead to a significant reduction of the correlation of $RCON$ with the aggregated $FCON$ close to zero. In addition, the adjustments also decrease the average magnitude and volatility which suggests a better alignment of attribution results with the underlying constrained investment process. This is most apparent when using the adjustment with time-varying β_t-coefficients. In comparison to the standard adjustment, the time-varying adjustment leads to a further reduction of the average absolute $RCON$ of around 8% (3bp). A reason for this can be that the correlation between $RCON$ and the contribution of certain factors is dependent on the portfolio's exposure to these factors. As this exposure is usually strongly varying over time, the resulting correlation is also not constant across the considered time frame. As shown in Figure B.4 this is indeed already observable for the ex-ante correlation which varies in this example between -0.60 and 0.05. The time variation is then better captured by the time series regression approach based on a rolling window, as this also allows adjustments for factors which only exhibit correlation over a short time frame and not necessarily over the entire time period.

All in all, the time series adjustment proposed by Stubbs and Jeet (2016) can be used to obtain statistically reasonable and significant reductions of $RCON$. This can be especially useful if the attribution is conducted for research purposes in which a high statistical accuracy of the results is needed to draw reliable implications. Applying the refined adjustment provides a better representation of the time-varying properties of the portfolio, however, comes at the cost of statistical accuracy due to the smaller estimation window.

Table 10: Summary Attribution Results

	OLS	TS Adj	TS Adj 2
Panel A: Summary Performance Attribution (annualized values in %)			
Active Return	1.27	1.27	1.27
World	0.00	0.00	0.00
Volatility	-0.05	-0.05	-0.09
Size	0.49	0.49	0.48
Growth	-0.02	-0.02	-0.05
Liquidity	0.13	0.05	0.10
Leverage	-0.10	-0.02	-0.05
Value	0.81	0.68	0.84
Momentum	0.65	0.45	0.43
RCON	-0.65	-0.30	-0.38
Panel B: Summary Quality Measures			
R^2	0.10	0.10	0.10
	(0.08)	(0.08)	(0.08)
Ratio RCON/r^A	0.75	0.73	0.68
	(14.18)	(11.99)	(11.85)
Mean abs. RCON (in %)	0.41	0.40	0.37
Std RCON (in %)	0.54	0.51	0.47
Corr RCON/FCON	-0.28	-0.01	-0.01

Notes: This table shows the summary of the annualized attribution results in % as well as of the Quality Measures for the attributions (corresponding standard deviations of the time series are shown in parentheses). Abbreviations: *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas

6.4 Non-Linear Factor Attribution

In order to obtain more insights, how strong possible Non-Linear effects are apparent in the tails of each factor, we analyze the different re-attributions of $RCON$ when using the standard NL Adjustment as well as the refined NL Adjustment for different values of α. Here, a lower level of the chosen α coincides with the consideration of stocks with factor exposures of large absolute magnitude in the re-attribution. The results in Table 11 show the complete re-attribution of $RCON$ in the standard NL Adjustment in which more than 25% of the annualized cumulative $RCON$ is re-attributed to the *Size* factor. Since the *Size* factor is amongst others constructed by taking short positions in small cap stocks, this factor would be heavily restricted by the long-only constraint in practice. Hence, this large part of the re-attribution reflects well the degree to which the factor is constrained in practice and the resulting inability of the investor in fully emulating the corresponding factor bet. In general, the standard NL Adjustment provides good indicators about which factors might be most affected by constraints in the investment process. However, as this method re-attributes the entire $RCON$ without distinguishing between true stock-specific returns and $RCON$ driven by possible non-linear effects, it does not necessarily lead to more informative attribution results.

Nevertheless, this issue can be encountered by using a more selective classification matrix in the refined NL Adjustment. When choosing $\alpha = 0.1$, which means that at each time t only factor exposures larger than the 90th and smaller than the 10th cross sectional percentile are used in the calculation of the classification matrix, we can still explain a large part of $RCON$. The re-attributed parts of $RCON$ are very similar to that in the standard NL Attribution, although assigning a significantly larger part back to the *Momentum* factor. This stresses the circumstance that also *Momentum* strategies are strongly restricted in practice, especially due to turnover and long only constraints. Moreover, the obtained result that the majority of the re-attributed parts of $RCON$ is negative, is well in line with the reasoning in Section 3.5 and confirms possible non-linear effects in the tails of factor exposures. Also, when putting even more emphasis on the exposure tails with $\alpha = 0.025$ and $\alpha = 0.01$ we obtain similar results in a smaller magnitude which is a further indicator for the non-linearities and justifies the application of this re-attribution method.

Table 11: Summary re-attribution of *RCON*

	NL Adj	NL Adj 2 ($\alpha = 0.1$)	NL Adj 2 ($\alpha = 0.05$)	NL Adj 2 ($\alpha = 0.025$)	NL Adj 2 ($\alpha = 0.01$)
RCON Standard	**-0.65**	**-0.65**	**-0.65**	**-0.65**	**-0.65**
World	0.00	0.00	0.00	0.00	0.00
Volatility	-0.12	0.04	0.01	0.01	0.01
Size	-0.18	-0.25	-0.17	-0.21	-0.06
Growth	-0.08	-0.06	-0.04	-0.09	0.00
Liquidity	-0.06	0.00	-0.05	-0.02	-0.02
Leverage	-0.06	-0.06	-0.01	0.01	-0.03
Value	-0.06	-0.01	0.06	-0.01	-0.04
Momentum	-0.09	-0.15	-0.12	-0.05	-0.03
RCON Adjusted	**0.00**	**-0.15**	**-0.33**	**-0.28**	**-0.48**

Notes: This table shows annualized portions of the Residual Contribution (in %) re-attributed to each factor when using the standard Non-Linear Attribution as well as the refined Non-Linear Attribution for different settings of α. Abbreviations: *NL Adj* - Standard non-linear Attribution, *NL Adj 2* - Refined non-linear Attribution

These non-linear adjustments also translate to attribution results with significantly improved properties of the obtained *RCON*, as it can be seen in Table 12. Since the standard NL Adjustment leads to a complete mitigation of *RCON* by construction, this also translates to zero-values of the corresponding Quality Measures in Panel B. However, as mentioned before, this does not reflect well the idiosyncratic noise the portfolio is expected to show and therefore, cannot necessarily be seen as a more informative attribution. In contrast, the refined NL Adjustment leads to more interpretable results as it aims to distinguish better between true stock-specific returns and more systematic constraint-induced residuals. Considering the case for $\alpha = 0.1$, the adjustment leads to a *RCON* with a significantly lower average magnitude (reduction of 35%/14bp) and volatility than in the *OLS* estimation. The depicted distribution of the *RCON* in Figure 3 also makes clear, that we can explain not only parts of negative *RCON*, but also of positive *RCON*. This suggests that non-linearities in asset returns in combination with portfolio constraints can be seen as the one of the main drivers for excessively large *RCON* of both signs. Also, the low correlation of -0.03 between *RCON* and the aggregated *FCON* in Table 12 implies that the re-attributed *RCON* were indeed rather systematically and likely to be caused by constraints and the portfolio construction itself instead of true stock-specific returns.

The observation that the refined NL Adjustment leads to a significant mitigation of the correlation shows that in our example the main cause for this correlation were indeed non-linearities in the asset returns as it was already suggested by the findings in Section 5.4.

To sum up, the standard NL Adjustment from De Boer (2019) can be used to obtain general insight in possible main drivers of $RCON$ as well as to identify factors which might not have a linear relationship with the asset returns as it is assumed by the factor model. Also, in cases in which the investor wants to fully eliminate the $RCON$, this approach represents an easy and economically justified heuristic to do so. By using the refined NL Adjustment we get a more realistic re-attribution which is justified by the improved statistical properties of $RCON$ such as a lower average magnitude and correlation with the aggregated $FCON$. In general, this technique aligns the multifactor attribution well with the underlying investment process as it takes possible non-linearities into account which have a significant influence on the feasibility of the estimated factor returns for a constrained investor.

Table 12: Summary Attribution Results

	OLS	NL Adj	NL Adj 2 (α = 0.1)	NL Adj 2 (α = 0.05)	NL Adj 2 (α = 0.025)	NL Adj 2 (α = 0.01)
Panel A: Summary Performance Attribution (annualized values in %)						
Active Return	1.27	1.27	1.27	1.27	1.27	1.27
World	0.00	0.00	0.00	0.00	0.00	0.00
Volatility	-0.05	-0.17	-0.01	-0.04	-0.05	-0.04
Size	0.49	0.31	0.24	0.32	0.29	0.44
Growth	-0.02	-0.10	-0.08	-0.06	-0.11	-0.02
Liquidity	0.13	0.07	0.13	0.07	0.11	0.11
Leverage	-0.10	-0.16	-0.16	-0.11	-0.09	-0.13
Value	0.81	0.76	0.80	0.87	0.80	0.77
Momentum	0.65	0.56	0.49	0.53	0.60	0.62
RCON	-0.65	0.00	-0.15	-0.33	-0.28	-0.48
Panel B: Summary Quality Measures						
R^2	0.10	0.10	0.10	0.10	0.10	0.10
	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)
Ratio RCON/r^A	0.75	0.00	0.54	0.59	0.60	0.66
	(14.18)	(0.00)	(8.53)	(8.76)	(12.54)	(13.71)
Mean abs. RCON (in %)	0.41	0.00	0.27	0.31	0.34	0.39
Std RCON (in %)	0.54	0.00	0.36	0.40	0.44	0.51
Corr RCON/FCON	-0.28	-0.02	-0.03	-0.18	-0.23	-0.25

Notes: This table shows the summary of the annualized attribution results in % as well as of the Quality Measures for the attributions (corresponding standard deviations of the time series are shown in parentheses). Abbreviations: *NL Adj* - Standard non-linear Multifactor Attribution, *NL Adj 2* - Refined non-linear Multifactor Attribution with Threshold α

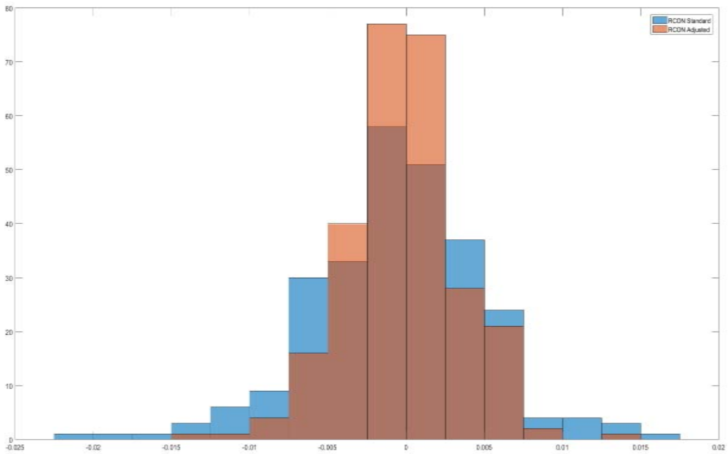


Figure 3: Distributions of the obtained RCON in the standard OLS Attribution (blue) and after adjustment with the refined NL Method (α = 0.1, red)

6.5 Combination of Approaches

Finally, we investigate possible combinations of the introduced estimation (WLS and Restricted FMPs) and adjustment (Time Series and Non-Linear) approaches in order to explore the full potential of these enhancements over the standard OLS approach. We used the most promising techniques of the previous sections, namely WLS Shrink, Sym 2, TS Adj, TS Adj 2 and NL Adj 2 (α = 0.1) for which the results are shown in Table 13. The outcomes shown in Panel A indicate, that all methods lead to an annualized cumulative $RCON$ which is at least 35bp lower in its magnitude than in the standard OLS approach. This stresses the benefit of combining different methods to mitigate the systematically negative trending $RCON$ as these values are less negative than when using them separately.

Looking at the corresponding quality measures in Panel B, we find additional evidence for improvements, especially when using WLS Shrink together with NL Adj 2 (α = 0.1), which results in an 15bp lower mean absolute $RCON$ as well as a correlation between $RCON$ and aggregated $FCON$ of nearly zero. Although NL Adj 2 (α = 0.1) already lead to significant improvements in this regard as depicted in Table 12, this result suggests an attribution of a slightly higher quality. By using the Time Series Adjustments in combination with Sym 2 and WLS Shrink we also obtain $RCON$ of less magnitude and volatility than in the separate analyses, albeit these improvements are not as pronounce as for the aforementioned combinations with NL Adj 2. However, as TS Adj and TS Adj 2 are based on selection procedures with ensure the statistical significance of the adjustments, they provide a more profound statistical justification for the adjustments than NL Adj 2.

All in all, these analyses stress the merits of combining different attribution approaches. We identify the strongest overall improvements when using WLS Shrink in combination with the refined Non-Linear Adjustment with Threshold $\alpha = 0.1$. Using both Time Series Adjustments in addition to Sym 2 and WLS Shrink also lead to strong, but less substantial improvements. Ultimately, it depends on the preferences of the researcher whether she opts for the lowest possible idiosyncratic contribution with a profound economic reasoning (NL Adj 2) or for the statistically more significant, but smaller adjustment (TS Adj).

Table 13: Summary Attribution Results

Estimation Method	OLS	Sym 2			WLS Shrink		
Adjustment Method		NL Adj 2	TS Adj	TS Adj 2	NL Adj 2	TS Adj	TS Adj 2
Panel A: Summary Performance Attribution (annualized values in %)							
Active Return	1.27	1.27	1.27	1.27	1.27	1.27	1.27
World	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Volatility	-0.05	-0.04	-0.06	0.01	-0.06	-0.05	-0.05
Size	0.49	0.29	0.43	0.42	0.33	0.40	0.41
Growth	-0.02	-0.07	0.00	-0.04	-0.06	-0.01	-0.02
Liquidity	0.13	0.06	0.08	0.08	0.04	0.08	0.07
Leverage	-0.10	-0.21	-0.13	-0.15	-0.23	-0.14	-0.11
Value	0.81	0.71	0.55	0.60	0.76	0.68	0.78
Momentum	0.65	0.57	0.48	0.51	0.64	0.53	0.50
RCON	-0.65	-0.04	-0.08	-0.15	-0.14	-0.21	-0.30
Panel B: Summary Quality Measures							
R^2	0.10	0.09	0.09	0.09	0.09	0.09	0.09
	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)
Ratio RCON/r^A	0.75	0.47	0.64	0.65	0.45	0.66	0.64
	(14.18)	(7.67)	(11.88)	(11.59)	(7.3)	(12.4)	(10.76)
Mean abs. RCON (in %)	0.41	0.26	0.36	0.35	0.26	0.37	0.35
Std RCON (in %)	0.54	0.34	0.46	0.44	0.34	0.47	0.45
Corr RCON/FCON	-0.28	-0.10	-0.03	-0.09	-0.01	-0.04	-0.02

Notes: This table shows the summary of the annualized attribution results in % as well as of the Quality Measures for the attributions (corresponding standard deviations of the time series are shown in parentheses). Abbreviations: *Shrink* - WLS Shrinkage Estimation, *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas, *NL Adj 2* - Refined non-linear Attribution ($\alpha = 0.1$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW.

7 Conclusion

In this paper, we investigated different techniques to improve the informativeness of multi-factor performance attributions on constrained long-only portfolios. We applied these approaches on the attribution of a representative equity portfolio and evaluated the results based on several quality measures. Using a comprehensive data set of historic return and factor exposure data of MSCI World Index constituents over a time period of 22 years, we firstly find that the application of market cap regression weights in the WLS approach implicitly leads to more representative FMPs. This translates into a lower unexplained contribution in the attribution while only sacrificing statistical accuracy to a small extend. However, an improvement can be achieved by using a Shrinkage Estimator based on the work of Ledoit and Wolf (2003a) which brings together the favourable properties of a statistically accurate and a representative weighting scheme in the regression. Similar results can be obtained when deriving the FMPs explicitly subject to strongly relaxed real-world constraints. Furthermore, we find that the Time Series Adjustment by Stubbs and Jeet (2016) and the Non-Linear Adjustment by De Boer (2019) are useful methods to ensure the uncorrelatedness of $RCON$ with the aggregated $FCON$ which is a main property of true idiosyncratic return contribution. While the Time Series Adjustment is a re-attribution based on a purely statistical foundation, the Non-Linear Adjustment puts more emphasis on the economic reasons for the occurrence of the excessively large $RCON$. Finally, with the analysis on simulated returns we identify possible reasons, such as omitted factors in the model or non-linear asset returns, for the issues that we face in practice in the attribution of long-only portfolios.

This paper contributes to the recent academic research on multifactor attribution in several ways. Firstly, we introduce possible quality measures to assess the informativeness of attribution results and to evaluate possible improvements through the application of different techniques. Moreover, we propose the usage of a shrinkage estimator in the WLS regression to combine the benefits of the widely used market cap weighting scheme with the properties of the statistically robust weights. In addition, we show in which ways real-world constraints such as the long-only, maximum weight and liquidity constraint can explicitly be incorporated in the attribution in order to align it better with the investment process of a constrained long-only investor. Here, we extend the work of Melas et al. (2010) by investigating possible relaxations or combinations of constraints and by providing an in-depth analysis of their effect on the implied FMPs. Moreover, we expand the approach of Stubbs and Jeet (2016) by allowing time-varying adjustments based on a rolling window estimation instead of the single adjustment as in the standard approach. With that, we take the time-varying properties of the portfolio into account and therefore, obtain more accurately estimated factor contributions in the attribution. Finally, we also extend the Non-Linear Adjustment technique from De Boer (2019) by defining conditions for the adjustment and thereby bring it more in line with the underlying economic reasoning.

All in all, this paper is beneficial for the Quantitative Equities Team of APG as it provides a comprehensive analysis of how different techniques can be used to improve the quality of the multifactor attribution as well as a statistical and economical justification of their application. An advice to which approaches should be used by the researcher to enhance the attribution depends on the purpose of the analysis. If the attribution is conducted for pure research purposes, for instance to identify the effect of a novel alpha signal on a backtested portfolio, one should opt for statistically accurate methods in order to obtain the most reliable results. In these cases, the preferable methods are WLS RW or WLS Shrink as well as the Time Series Adjustments. If the objective of the attribution is of a more illustrative nature, such as in internal performance reviews or client presentations, methods which better incorporate the investment process of the constrained portfolio (WLS Cap, Restricted FMPs, NL Adj 2) and thereby lead to the lowest $RCON$ in general, are better suited. The application of these methods can be economically well justified and the obtained attribution results are easier to interpret and to explain due to the comparatively low level of the unexplained idiosyncratic contribution.

Our research can be extended in various directions. As the approach of deriving the FMPs subject to real-world constraints already lead to promising results, one could investigate whether further improvements are possible by choosing different values for the scaling factor d in order to relax or tighten the restrictions. Also, since we only imposed the constraints on the FMPs of the alpha and risk factors while excluding the industry and country FMPs in the optimization, one can examine the benefits of the derivation of these FMPs subject to the restrictions of our real portfolio. Finally, a more extensive analysis of how the different approaches can be optimally combined and used simultaneously could also significantly enhance the quality of the obtained attribution.

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Acknowledgements

First and foremost, I would like to express my sincere gratitude to my company supervisor Tim Zwinkels for all the useful comments and his exceptional commitment through the entire process of this thesis project. I would also like to thank my colleagues at the Quantitative Equities Team of APG for the knowledge and insights they have shared with me over the five months of the internship which had a large positive impact on my enthusiasm for finance. Furthermore, I thank my university supervisor Dr. Erik Kole for his ideas and remarks which significantly helped me to shape the thesis into its current form. Finally, I also want to thank my parents, sister and friends who supported me throughout my whole studies and who are the main reason for me being able to finish it successfully.

Appendices

A Technical Appendix

A.1 Shrinkage Intensity

The shrinkage method in Section 3.2.3 requires us to estimate the Shrinkage Intensity $\hat{\delta}$. In this appendix section, we provide the formulas for this estimator, in which we closely follow the derivation in Ledoit and Wolf (2003a) and adapt their approach to estimate the residual covariance matrix.

In general, the optimal Shrinkage Intensity δ^* minimizes the quadratic loss function based on the difference between the shrinkage estimator $\hat{\boldsymbol{\Sigma}}_{Shrink}$ and the true residual covariance matrix $\boldsymbol{\Sigma}$, and is defined as

$$\delta^* = \text{argmin}_{\delta\in[0,1]}\mathbb{E}\Big[||(1-\delta)\hat{\boldsymbol{\Sigma}}^{Sample} + \delta\hat{\boldsymbol{\Sigma}}^{Target} - \boldsymbol{\Sigma}||^2\Big] \quad . \tag{41}$$

Ledoit and Wolf (2003b) show that the optimal δ^* asymptotically behaves like a constant over the sample size T, i.e. $\delta^* \approx \kappa/T$ for large T. This constant κ can then be written as

$$\kappa = \frac{\pi - \phi}{\gamma} \quad , \tag{42}$$

where π is a measure of the imprecision of $\hat{\boldsymbol{\Sigma}}^{Sample}$, ϕ quantifies the covariance between the errors in $\hat{\boldsymbol{\Sigma}}^{Sample}$ and $\hat{\boldsymbol{\Sigma}}^{Target}$, and γ is a measure of the discrepancy between $\hat{\boldsymbol{\Sigma}}^{Sample}$ and $\hat{\boldsymbol{\Sigma}}^{Target}$.

As derived in Ledoit and Wolf (2003a) a consistent estimator for π is given as

$$\hat{\pi} = \sum_{i=1}^{n}\sum_{j=1}^{n}\hat{\pi}_{i,j}, \quad \text{with} \quad \hat{\pi}_{i,j} = \frac{1}{T}\sum_{t=1}^{T}\Big((\hat{\varepsilon}_{i,t} - \bar{\varepsilon}_i)(\hat{\varepsilon}_{j,t} - \bar{\varepsilon}_j) - \hat{\boldsymbol{\Sigma}}_{i,j}^{Target}\Big)^2, \tag{43}$$

where the subscript i, j indicate the matrix element in row i and column j and $\bar{\varepsilon}_i$ denotes the average estimated residual return of stock i. Likewise, γ can be consistently estimated as

$$\hat{\gamma} = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\hat{\boldsymbol{\Sigma}}_{i,j}^{Sample} - \hat{\boldsymbol{\Sigma}}_{i,j}^{Target}\Big)^2. \tag{44}$$

Finally, for an equicorrelation shrinkage target, we obtain for ϕ the estimator

$$\hat{\phi} = \sum_{i=1}^{n}\hat{\pi}_{i,i} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{\hat{\rho}}{2}\left(\sqrt{\frac{\hat{\boldsymbol{\Sigma}}_{j,j}^{Sample}}{\hat{\boldsymbol{\Sigma}}_{i,i}^{Sample}}}\hat{\nu}_{ii,ij} + \sqrt{\frac{\hat{\boldsymbol{\Sigma}}_{i,i}^{Sample}}{\hat{\boldsymbol{\Sigma}}_{j,j}^{Sample}}}\hat{\nu}_{jj,ij}\right), \tag{45}$$

where $\hat{\rho}$ denotes the average of the estimated correlation in $\hat{\mathbf{\Sigma}}^{Sample}$ and

$$\begin{aligned}\hat{\nu}_{ii,ij} &= \frac{1}{T}\sum_{t=1}^{T}\Big(\big(\hat{\varepsilon}_{i,t}-\bar{\varepsilon}_i\big)^2-\hat{\mathbf{\Sigma}}_{i,i}^{Sample}\Big)\Big(\big(\hat{\varepsilon}_{i,t}-\bar{\varepsilon}_i\big)\big(\hat{\varepsilon}_{j,t}-\bar{\varepsilon}_j\big)-\hat{\mathbf{\Sigma}}_{i,j}^{Sample}\Big) \quad , \\ \hat{\nu}_{jj,ij} &= \frac{1}{T}\sum_{t=1}^{T}\Big(\big(\hat{\varepsilon}_{j,t}-\bar{\varepsilon}_i\big)^2-\hat{\mathbf{\Sigma}}_{j,j}^{Sample}\Big)\Big(\big(\hat{\varepsilon}_{i,t}-\bar{\varepsilon}_i\big)\big(\hat{\varepsilon}_{j,t}-\bar{\varepsilon}_j\big)-\hat{\mathbf{\Sigma}}_{i,j}^{Sample}\Big) \quad .\end{aligned} \tag{46}$$

Since we use a shrinkage target with zero correlation, we follow the approach in Vlodrop and Lucas (2018) and set $\hat{\rho} = 0$. Bringing these estimates together, we obtain an estimator for κ as

$$\hat{\kappa} = \frac{\hat{\pi} - \hat{\phi}}{\hat{\gamma}} \quad . \tag{47}$$

Finally we can compute the shrinkage intensity estimator $\hat{\delta}$ as

$$\hat{\delta} = \max\left(0, \min\left(\frac{\hat{\kappa}}{T}, 1\right)\right). \tag{48}$$

In our analyses we use a rolling window of the past $T = 60$ months to estimate the Shrinkage Intensity as well as the sample covariance matrix.

A.2 Equivalence between FMPs from OLS/WLS and unconstrained minimization problem

In order to show the equivalence of the FMPs obtained in the unconstrained optimization problem in Section 3.3 with those of the WLS approach in Section 3.2, we firstly formulate the Lagrangian function of the minimization problem

$$\begin{aligned}&\min_{\mathbf{w}_t^{FMP,k}} \quad \mathbf{w}_t^{FMP,k\prime}\mathbf{W}_t^{-1}\mathbf{w}_t^{FMP,k} \\ &\text{s.t.} \qquad \mathbf{X}_t'\mathbf{w}_t^{FMP,k} = e_k \quad ,\end{aligned} \tag{49}$$

as

$$\mathcal{L}(\mathbf{w}_t^{FMP,k}, \lambda) = \frac{1}{2}\mathbf{w}_t^{FMP,k\prime}\mathbf{W}_t^{-1}\mathbf{w}_t^{FMP,k} - \lambda(\mathbf{X}_t'\mathbf{w}_t^{FMP,k} - e_k) \quad . \tag{50}$$

Afterwards, we formulate the corresponding system of equations given by the first order conditions

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}^{FMP,k}} = \mathbf{W}_t^{-1}\mathbf{w}_t^{FMP,k} - \mathbf{X}_t\lambda' = 0 \tag{51}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{X}_t'\mathbf{w}_t^{FMP,k} - e_k = 0 \quad , \tag{52}$$

and solve it by firstly re-arranging Equation 51 to get

$$\mathbf{w}_t^{FMP,k} = \mathbf{W}_t\mathbf{X}_t\lambda' \quad . \tag{53}$$

Then, we fill this formulation in Equation 52 to obtain

$$\begin{aligned} \mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t\lambda' - e_k = 0 \\ \Longleftrightarrow \lambda' = (\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}e_k \quad . \end{aligned} \tag{54}$$

Substitution of this expression in Equation 53 results in

$$\mathbf{w}_t^{FMP,k} = \mathbf{W}_t\mathbf{X}_t(\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}e_k \quad , \tag{55}$$

which is equivalent to the k-st column of the FMP matrix from the standard WLS estimation in Equation 12. Using the Identity Matrix as regression weights correspondingly results in the OLS equivalent.

A.3 Correlation between FCON and RCON

In order to calculate the correlation between the aggregated $FCON$ and $RCON$ we firstly reformulate the decomposition equation of the active portfolio's return in the standard attribution

$$\begin{aligned} \mathbf{w}_t^{A\prime}\mathbf{r}_t &= FCON_t + RCON_t \\ &= \mathbf{w}_t^{A\prime}\mathbf{X}_t\hat{\mathbf{f}}_{t,WLS} + \mathbf{w}_t^{A\prime}\hat{\varepsilon}_t \\ &= \mathbf{w}_t^{A\prime}\mathbf{X}_t(\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{W}_t\mathbf{r}_t + \mathbf{w}_t^{A\prime}(\mathbf{I}_n - \mathbf{X}_t(\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{W}_t)\mathbf{r}_t \\ &= \mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\mathbf{r}_t + \mathbf{w}_t^{A\prime}\mathbf{M}_{t,WLS}\mathbf{r}_t \quad , \end{aligned} \tag{56}$$

with the projection matrix $\mathbf{P}_{t,WLS} = \mathbf{X}_t(\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{W}_t$ and the residual maker $\mathbf{M}_{t,WLS} = \mathbf{I}_n - \mathbf{X}_t(\mathbf{X}_t'\mathbf{W}_t\mathbf{X}_t)^{-1}\mathbf{X}_t'\mathbf{W}_t$ as described in Greene (2003).

Then, the covariance between $FCON_t$ and $RCON_t$ can be calculated as

$$\begin{aligned} \mathrm{cov}(FCON_t, RCON_t) &= \mathrm{cov}(\mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\mathbf{r}_t, \mathbf{w}_t^{A\prime}\mathbf{M}_{t,WLS}\mathbf{r}_t) \\ &= \mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\mathrm{cov}(\mathbf{r}_t, \mathbf{r}_t)\mathbf{M}_{t,WLS}'\mathbf{w}_t^A \\ &= \mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}(\mathbf{X_t}\Sigma_{f,t}\mathbf{X_t}' + \Sigma_{\varepsilon,t})\mathbf{M}_{t,WLS}'\mathbf{w}_t^A \\ &= \underbrace{\mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\mathbf{X_t}\Sigma_{f,t}\mathbf{X_t}'\mathbf{M}_{t,WLS}'\mathbf{w}_t^A}_{=0,\text{ as } \mathbf{X_t}'\mathbf{M}_{t,WLS}' = 0} + \mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\Sigma_{\varepsilon,t}\mathbf{M}_{t,WLS}'\mathbf{w}_t^A \\ &= \mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\Sigma_{\varepsilon,t}\mathbf{M}_{t,WLS}'\mathbf{w}_t^A \quad . \end{aligned} \tag{57}$$

Analogously, we obtain the variances as

$$\begin{aligned} \text{Var}(FCON_t) &= \mathbf{w}_t^{A\prime}\mathbf{P}_{t,WLS}\Sigma_{\varepsilon,t}\mathbf{P}_{t,WLS}^{\prime}\mathbf{w}_t^A \\ \text{Var}(RCON_t) &= \mathbf{w}_t^{A\prime}\mathbf{M}_{t,WLS}\Sigma_{\varepsilon,t}\mathbf{M}_{t,WLS}^{\prime}\mathbf{w}_t^A \quad , \end{aligned} \tag{58}$$

with which we can subsequently compute the correlation

$$\rho(FCON_t, RCON_t) = \frac{\text{cov}(FCON_t, RCON_t)}{\sqrt{\text{Var}(FCON_t) \cdot \text{Var}(RCON_t)}}. \tag{59}$$

Given these formulations, we can identify three possible sources for a high (ex-ante) correlation which we encounter in this paper in several attributions on the simulated and empirical return data:

1) correlation between errors. If the off-diagonal elements of $\Sigma_{\varepsilon,t}$ are non-zero, we correspondingly obtain a larger value for the correlation bettwen $RCON$ and $FCON$;

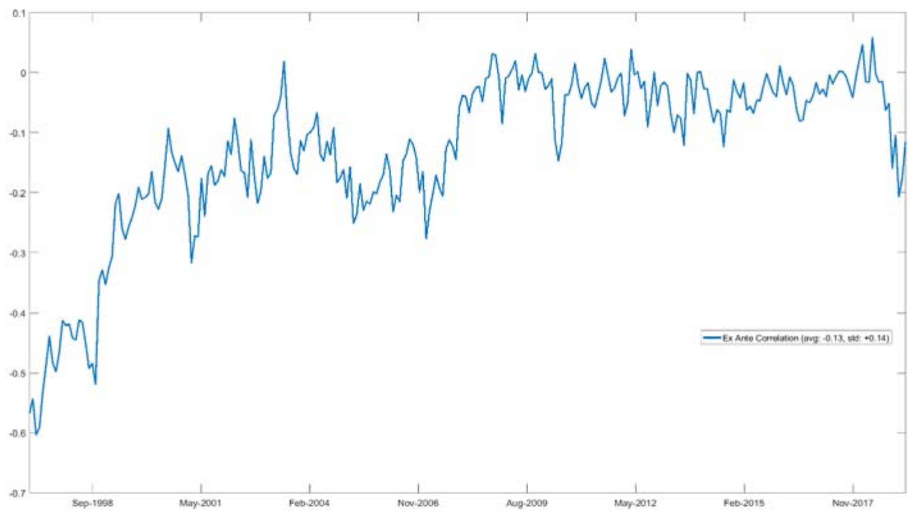
2) a non-linear dependency of asset returns on factor returns as described in Section 4.4. In this case, we use a transformed exposure matrix $\mathbf{X}_t^{transf}$ so that $\mathbf{X}_t^{transf\prime}\mathbf{M}_{t,WLS}^{\prime} \neq 0$. Therefore, the corresponding term in Equation 57 does not cancel out anymore which ultimately results in larger correlations in this setting;

3) the usage of FMPs obtained from the restricted optimization problems in Section 3.3. As the residual maker can also be written in terms of the FMP weights, namely $\mathbf{M}_{t,WLS} = \mathbf{I}_n - \mathbf{X}_t\mathbf{w}_t^{FMP\prime}$, the usage of the derived FMPs also leads to a non-orthogonality of the residual maker to the factor exposures which in turn also results in a larger correlation.

If the covariance matrix of the residuals is known, such as in the simulation in Section 4, we are able to calculate the ex-ante expected correlations by plugging in this covariance matrix for $\Sigma_{\varepsilon,t}$. This allows us to separate the magnitude of correlation which we expect by using different regression weights from the correlation which is caused due to estimation errors and inaccurate regression weights as shown in the results in Table 14.

B Tables and Figures

Figure 4: Time Series of Ex-Ante Correlation



Notes: This figure shows the ex-ante correlation based on equal regression weights and simulated asset returns with correlated errors and a non-linear dependency on the alpha factor exposures over the time period from January 1997 to February 2019.

Table 14: Ex Ante and Ex Post Correlation

	Ex Ante						Ex Post					
	OLS	WLS RW	WLS Cap	MW & LO	Sym 1	Sym 2	OLS	WLS RW	WLS Cap	MW & LO	Sym 1	Sym 2
Homosced. Errors	0.00	0.02	0.09	-0.08	-0.04	0.01	0.00	0.00	0.04	-0.23	-0.29	0.01
Heterosced. Errors	-0.08	0.00	0.02	-0.09	-0.06	0.01	-0.04	-0.02	0.01	-0.22	-0.25	0.00
Correlated Errors	-0.03	-0.01	0.11	-0.11	-0.07	0.01	-0.20	-0.24	-0.22	-0.54	-0.51	-0.09
Non-linear Asset Returns	-0.12	-0.06	-0.06	-0.15	-0.14	-0.04	-0.36	-0.34	-0.27	-0.52	-0.31	-0.21

Notes: This table shows the average ex ante expected correlation between $RCON$ and $FCON$ given different error covariances and regression weights as well as the ex post correlation obtained in the attribution based on simulated asset returns.

Table 15: Attribution on simulated return data with homoscedastic Errors

	Sim. Data	OLS	WLS RW	WLS Cap	WLS Shrink	MW & LO	Sym 1	Sym 2	TS Adj	TS Adj 2	NL Adj	NL Adj 2
Panel A: Quality Measures												
Ann. Cum. RCON (in %)	0.02	0.01	0.00	0.01	0.01	0.06	-0.03	0.01	0.02	0.00	0.00	-0.04
	(0.38)	(0.35)	(0.35)	(0.31)	(0.31)	(0.37)	(0.45)	(0.31)	(0.35)	(0.34)	(0.00)	(0.23)
R^2	0.09	0.10	0.10	0.10	0.10	0.08	0.08	0.10	0.10	0.10	0.10	0.10
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Ratio RCON/r^A	0.66	0.61	0.61	0.55	0.56	0.65	0.64	0.56	0.60	0.60	0.00	0.39
	(0.05)	(0.05)	(0.06)	(0.04)	(0.04)	(0.06)	(0.04)	(0.04)	(0.05)	(0.04)	(0.00)	(0.04)
Mean abs. RCON (in %)	0.43	0.40	0.40	0.36	0.36	0.43	0.43	0.37	0.40	0.39	0.00	0.26
	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.00)	(0.01)
Std RCON (in %)	0.54	0.50	0.50	0.45	0.46	0.54	0.56	0.47	0.50	0.49	0.00	0.32
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.00)	(0.01)
Corr RCON/FCON	-0.01	0.00	0.00	0.04	0.04	-0.23	-0.29	0.01	0.01	0.00	-0.02	0.06
	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)	(0.05)	(0.05)	(0.07)	(0.06)	(0.05)	(0.05)	(0.06)
Panel B: RMSE Factor Returns (in %)												
Risk Factors		0.31	0.32	0.38	0.36	0.61	0.67	0.38	0.31	0.31	0.31	0.31
Alpha Factors		0.33	0.35	0.42	0.39	0.75	0.93	0.44	0.33	0.33	0.33	0.33
Panel C: RMSE Factor Contribution (in %)												
Risk Factors		0.11	0.12	0.15	0.14	0.23	0.26	0.14	0.12	0.14	0.40	0.33
Alpha Factors		0.17	0.18	0.22	0.21	0.40	0.48	0.23	0.17	0.18	0.24	0.20

Notes: This table shows the average results obtained from 25 multifactor attributions based on simulated return data. Panel A shows the mean of the calculated Quality Measures (corresponding Standard deviations are shown in parentheses). Panel B and C include the RMSE of the estimated Factor Returns and Factor Contributions, respectively, with regards to the true factor returns and contributions used in the generation of the return data. Abbreviations: *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation, *MW & LO* - Maximum Weight and Long-Only constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW, *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas, *NL Adj* - Standard non-linear Multifactor Attribution, *NL Adj 2* - Refined non-linear Multifactor Attribution with Threshold $\alpha = 0.1$.

Table 16: Attribution on simulated return data with heteroscedastic Errors

	Sim. Data	OLS	WLS RW	WLS Cap	WLS Shrink	MW & LO	Sym 1	Sym 2	TS Adj	TS Adj 2	NL Adj	NL Adj 2
Panel A: Quality Measures												
Ann. Cum. RCON (in %)	0.00	0.01	0.00	0.00	0.00	0.05	-0.05	0.00	0.02	0.01	0.00	-0.03
	(0.31)	(0.27)	(0.30)	(0.25)	(0.25)	(0.32)	(0.35)	(0.25)	(0.28)	(0.27)	(0.00)	(0.18)
R^2	0.12	0.11	0.10	0.10	0.10	0.09	0.09	0.13	0.11	0.11	0.11	0.11
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Ratio RCON/r^A	0.50	0.56	0.54	0.49	0.50	0.59	0.56	0.44	0.55	0.54	0.00	0.36
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.06)	(0.05)	(0.04)	(0.04)	(0.04)	(0.00)	(0.04)
Mean abs. RCON (in %)	0.35	0.33	0.33	0.30	0.30	0.36	0.35	0.31	0.33	0.32	0.00	0.22
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)
Std RCON (in %)	0.44	0.42	0.41	0.38	0.38	0.46	0.46	0.39	0.42	0.41	0.00	0.27
	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.02)	(0.01)	(0.02)	(0.00)	(0.01)
Corr RCON/FCON	0.01	-0.04	-0.02	0.01	0.01	-0.22	-0.25	0.00	-0.03	-0.02	-0.02	0.04
	(0.06)	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)	(0.06)	(0.06)	(0.06)	(0.05)	(0.05)	(0.06)
Panel B: RMSE Factor Returns (in %)												
Risk Factors		0.31	0.27	0.34	0.32	0.55	0.55	0.34	0.31	0.31	0.31	0.31
Alpha Factors		0.38	0.32	0.40	0.37	0.65	0.76	0.41	0.38	0.38	0.38	0.38
Panel C: RMSE Factor Contribution (in %)												
Risk Factors		0.12	0.10	0.13	0.12	0.20	0.22	0.12	0.13	0.14	0.33	0.26
Alpha Factors		0.19	0.16	0.20	0.19	0.34	0.39	0.21	0.19	0.20	0.23	0.21

Notes: This table shows the average results obtained from 25 multifactor attributions based on simulated return data. Panel A shows the mean of the calculated Quality Measures (corresponding Standard deviations are shown in parentheses). Panel B and C include the RMSE of the estimated Factor Returns and Factor Contributions, respectively, with regards to the true factor returns and contributions used in the generation of the return data. Abbreviations: *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation, *MW & LO* - Maximum Weight and Long-Only constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW, *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas, *NL Adj* - Standard non-linear Multifactor Attribution, *NL Adj 2* - Refined non-linear Multifactor Attribution with Threshold $\alpha = 0.1$.

Table 17: Attribution on simulated return data with correlated Errors

	Sim. Data	OLS	WLS RW	WLS Cap	WLS Shrink	MW & LO	Sym 1	Sym 2	TS Adj	TS Adj 2	NL Adj	NL Adj 2
Panel A: Quality Measures												
Ann. Cum. RCON (in %)	0.00	-0.07	-0.08	-0.02	-0.05	0.14	0.02	0.01	0.16	0.02	0.00	0.02
	(0.46)	(0.31)	(0.34)	(0.32)	(0.32)	(0.58)	(0.54)	(0.46)	(0.32)	(0.32)	(0.00)	(0.27)
R^2	0.16	0.18	0.18	0.17	0.18	0.14	0.14	0.16	0.18	0.18	0.18	0.18
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Ratio RCON/r^A	0.58	0.63	0.64	0.59	0.61	0.77	0.68	0.52	0.63	0.63	0.00	0.42
	(0.04)	(0.06)	(0.05)	(0.05)	(0.05)	(0.09)	(0.07)	(0.04)	(0.05)	(0.06)	(0.00)	(0.05)
Mean abs. RCON (in %)	0.44	0.49	0.50	0.47	0.48	0.62	0.57	0.39	0.49	0.47	0.00	0.33
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)	(0.02)	(0.00)	(0.02)
Std RCON (in %)	0.60	0.69	0.70	0.65	0.67	0.89	0.80	0.53	0.67	0.65	0.00	0.46
	(0.04)	(0.04)	(0.04)	(0.03)	(0.03)	(0.05)	(0.05)	(0.04)	(0.04)	(0.04)	(0.00)	(0.02)
Corr RCON/FCON	0.00	-0.20	-0.24	-0.22	-0.20	-0.54	-0.51	-0.09	-0.07	-0.10	0.00	-0.11
	(0.06)	(0.07)	(0.06)	(0.06)	(0.06)	(0.04)	(0.05)	(0.08)	(0.07)	(0.04)	(0.04)	(0.07)
Panel B: RMSE Factor Returns (in %)												
Risk Factors		0.42	0.47	0.59	0.50	0.98	1.04	0.43	0.42	0.42	0.42	0.42
Alpha Factors		1.07	1.16	1.23	1.15	1.73	1.88	0.76	1.07	1.07	1.07	1.07
Panel C: RMSE Factor Contribution (in %)												
Risk Factors		0.13	0.15	0.22	0.17	0.31	0.41	0.15	0.14	0.20	0.54	0.42
Alpha Factors		0.39	0.42	0.47	0.43	0.80	0.83	0.39	0.35	0.36	0.38	0.40

Notes: This table shows the average results obtained from 25 multifactor attributions based on simulated return data. Panel A shows the mean of the calculated Quality Measures (corresponding Standard deviations are shown in parentheses). Panel B and C include the RMSE of the estimated Factor Returns and Factor Contributions, respectively, with regards to the true factor returns and contributions used in the generation of the return data. Abbreviations: *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation, *MW & LO* - Maximum Weight and Long-Only constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW, *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas, *NL Adj* - Standard non-linear Multifactor Attribution, *NL Adj 2* - Refined non-linear Multifactor Attribution with Threshold $\alpha = 0.1$.

Table 18: Attribution on simulated return data with Non-Linearities

	Sim. Data	OLS	WLS RW	WLS Cap	WLS Shrink	MW & LO	Sym 1	Sym 2	TS Adj	TS Adj 2	NL Adj	NL Adj 2
Panel A: Quality Measures												
Ann. Cum. RCON (in %)	0.01	0.03	0.02	0.02	0.02	0.11	-0.05	0.02	0.04	0.02	0.00	-0.04
	(0.38)	(0.46)	(0.46)	(0.38)	(0.38)	(0.55)	(0.49)	(0.36)	(0.36)	(0.39)	(0.00)	(0.26)
R^2	0.15	0.12	0.11	0.11	0.11	0.12	0.09	0.13	0.12	0.12	0.12	0.12
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Ratio RCON/r^A	0.57	0.63	0.62	0.55	0.56	0.62	0.62	0.51	0.57	0.56	0.00	0.37
	(0.05)	(0.05)	(0.05)	(0.04)	(0.04)	(0.07)	(0.07)	(0.04)	(0.04)	(0.04)	(0.00)	(0.03)
Mean abs. RCON (in %)	0.43	0.44	0.44	0.39	0.39	0.49	0.45	0.40	0.40	0.40	0.00	0.27
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.02)	(0.01)	(0.02)	(0.02)	(0.00)	(0.01)
Std RCON (in %)	0.54	0.56	0.55	0.49	0.50	0.62	0.59	0.51	0.50	0.50	0.00	0.34
	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)	(0.02)	(0.00)	(0.02)
Corr RCON/FCON	-0.01	-0.36	-0.34	-0.27	-0.27	-0.52	-0.31	-0.21	0.01	-0.10	0.01	-0.16
	(0.08)	(0.05)	(0.05)	(0.04)	(0.05)	(0.05)	(0.04)	(0.06)	(0.04)	(0.03)	(0.04)	(0.04)
Panel B: RMSE Factor Returns (in %)												
Risk Factors		0.37	0.37	0.42	0.39	0.63	0.70	0.41	0.37	0.37	0.37	0.37
Alpha Factors		3.30	3.21	3.14	3.15	3.24	2.94	2.93	3.30	3.30	3.30	3.30
Panel C: RMSE Factor Contribution (in %)												
Risk Factors		0.12	0.13	0.16	0.14	0.23	0.27	0.15	0.13	0.16	0.44	0.35
Alpha Factors		0.33	0.32	0.30	0.29	0.51	0.51	0.30	0.18	0.24	0.30	0.28

Notes: This table shows the average results obtained from 25 multifactor attributions based on simulated return data. Panel A shows the mean of the calculated Quality Measures (corresponding Standard deviations are shown in parentheses). Panel B and C include the RMSE of the estimated Factor Returns and Factor Contributions, respectively, with regards to the true factor returns and contributions used in the generation of the return data. Abbreviations: *RW* - Rolling Window Estimation, *Cap* - Market Capitalization Weights, *Shrink* - Shrinkage Estimation, *MW & LO* - Maximum Weight and Long-Only constraint, *Sym 1* - Symmetric Constraint based on Benchmark weights ($d = 3$), *Sym 2* - Relaxed Symmetric Constraint based on Benchmark weights ($d = 10$) and MW, *TS Adj* - Standard Adjusted Multifactor Attribution, *TS Adj 2* - Adjusted Multifactor Attribution with time-varying betas, *NL Adj* - Standard non-linear Multifactor Attribution, *NL Adj 2* - Refined non-linear Multifactor Attribution with Threshold $\alpha = 0.1$.

Table 19: Attribution of a long-short portfolio based on simulated return data (OLS Estimation)

	Homosced. Errors	Heterosced. Errors	Correlated Errors	Non-Linear Asset Returns
Ann. Cum. RCON (in %)	-0.01	-0.02	0.02	-0.03
	(0.04)	(0.05)	(0.05)	(0.04)
R^2	0.09	0.12	0.16	0.11
	(0.00)	(0.00)	(0.01)	(0.00)
Ratio RCON/r^A	0.02	0.01	0.01	0.02
	(0.00)	(0.00)	(0.00)	(0.00)
Mean abs. RCON (in %)	0.05	0.05	0.05	0.05
	(0.00)	(0.00)	(0.00)	(0.00)
Std RCON (in %)	0.07	0.08	0.08	0.07
	(0.00)	(0.00)	(0.01)	(0.00)
Corr RCON/FCON	0.03	0.01	0.03	0.02
	(0.08)	(0.09)	(0.07)	(0.07)

Notes: This table shows the the mean of the calculated Quality Measures (corresponding standard deviations are shown in parentheses) obtained from 25 multifactor attributions based on simulated return data. The attributed portfolio is an unconstrained long-short portfolio which is optimized for the same composite Value-Momentum Signal as the long-only portfolio used in the main analyses. The table shows the results of the attributions when using the standard cross-sectional OLS estimation without further adjustments.