Incorporating Demand Uncertainty in Vehicle Routing Problems

Master's Thesis Operations Research and Quantitative Logistics

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Abstract

The vehicle routing problem represents one of the most well-known combinatorial NP-hard problems in the field of integer programming. Contrary to its deterministic counterpart, the vehicle routing problem under demand uncertainty allows for uncertain customer demands. In this paper, we address the uncertainty of demand by proposing a column generation heuristic based on robust optimization and stochastic programming. These two approaches model the uncertain parameter of an optimization problem as random variables, and their task is to find the cheapest set of routes which remains feasible in all or most of the demand scenarios. In our suggested branch-and-price method, a naive labelling is adopted with the aim of obtaining close-to-optimal solutions in a reasonable amount of time. The models are tested on three sets of standard literature benchmark instances and compared in terms of additional routing cost and unsatisfied demand. Computational studies demonstrate that the proposed heuristic is able to quickly generate high-quality solutions. Furthermore, a sensitivity analysis provides useful insights concerning the impact of the probability of route failure and the size of the uncertainty support on transportation costs.

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Contents

1 Introduction

Vehicle routing problems have applications in many various domains, particularly in the logistics and freight transportation sectors. They concern the distribution of goods and services between production facilities and end customers, such as courier and food delivery, waste collection, ride service hailing and maintenance engineers.

In deterministic environments, it is assumed that the customer demand is fixed and known in advance. However, in many practical contexts, customer demand is not completely known at the time of determining optimal vehicle routes, and it is only revealed upon the arrival of a vehicle to a customer. As a consequence, in this study we assume that customer demand is subject to uncertainty, that is, we consider a set of possible demand scenarios.

In addition to customers demand, other features of the VRP can be subject to uncertainty. For example, Bent and Van Hentenryck (2004), Hvattum, Lokketangen, and Laporte (2006) and Smith et al. (2010) investigate dynamic problems where the presence of customers is subject to uncertainty. Similarly, Lee, Lee and Park (2012) and Adulyasak and Jaillet (2016) assume uncertain travel times.

The vehicle routing problem is a well known NP-complete problem and even solving the deterministic version to optimality with a standard MIP solver becomes increasingly time consuming as the number of customers grows. Therefore, the aim of this study is to develop performing algorithms that find optimal or sub-optimal solutions for the VRP with demand uncertainty in a reasonable amount of time.

Specifically, we are going to model the uncertainty of the customer demand by means of robust optimization and stochastic programming, which represent two of the most well-known approaches to deal with uncertainty. Furthermore, solutions are going to be constructed through column generation heuristics, where routes are generated based on the definition of feasible routes corresponding to the two approaches.

Finally, we compare the robust solution against the stochastic solution on families of small and medium-sized instances. The comparison investigates how robust and stochastic solutions are able to ensure us against unsatisfied demand at the expense of facing additional costs compared to deterministic optimal routes. The results suggest that robust solutions provide no unmet demand, even though they result in elevated routing costs. On the other hand, stochastic solutions represent a good balance in terms of trade off between the two performance measures.

The remainder of the paper is organized as follows. Section 2 defines the VRP and introduces some notation. Section 3 discusses previous studies concerning ways to deal with demand uncertainty and various solution procedure approaches. In Section 4, our proposed column generation heuristic is explained, and feasibility conditions associated with both robust and stochastic routes are outlined. Section 5 presents measures used for the comparison of the two methods and a description of the data sets. Section 6 is devoted to computational analyses, where the efficiency, the solution quality and the solution sensitivity with respect to changes in the size of the uncertainty set and variation of route reliability level are assessed. We provide some concluding remarks in Section 7.

2 Problem Description

The Vehicle Routing Problem is a combinatorial optimization and integer programming problem whose task is to find the optimal set of routes for a fleet of vehicles traversing in order to service a given set of customers.

A VRP is modeled on a weighted, complete graph $G = (V \cup \{0\}, E)$, where V is a set of customers, 0 represents a depot, and every edge $e \in E$ in the graph has a cost c_e. The VRP has been deeply investigated and many extensions have been proposed in the literature (Toth and Vigo, 2014). One of the most well-known extensions is the so called capacitated VRP, where the vehicles operating from a central depot have a limited capacity Q. Furthermore, only k vehicles are available for servicing the customers. Based on the problem details, it is possible to model the load of the vehicle to be either increasing or decreasing along the route. They correspond to picking up and delivering goods to the customers and this can be done by alternatively thinking of the load as the empty space in the vehicle.

A route is a cycle in G, starting and ending at the central depot, visiting every customer at most once and such that the total demand of the customers on the cycle does not exceed the vehicle capacity. The cost of a route equals the sum of the cost of the arcs traversed in the cycle. **Figure 1** below shows the solution to a hypothetical VRP instance.

As mentioned earlier, the assumption regarding deterministic demand is now relaxed. That is, we allow customer demand to vary within a certain interval of values. The precise amplitude of deviations will be specified later in the paper. This means that, if we want to anticipate the adverse behaviour of nature, which by default adds complexities to the real world, we need to find a set of routes which remains feasible in all or most of the cases.

Specifically, a route is considered to be robust feasible if assuming the highest possible demand for each customers does not cause a violation of the capacity constraint. On the other hand, a route is stochastic feasible if the probability that the total load fits the vehicle capacity is above a specified threshold.

The aim of the VRP under demand uncertainty is indeed to find the cheapest set of routes in terms of routing costs, such that operating vehicles will not have to modify their

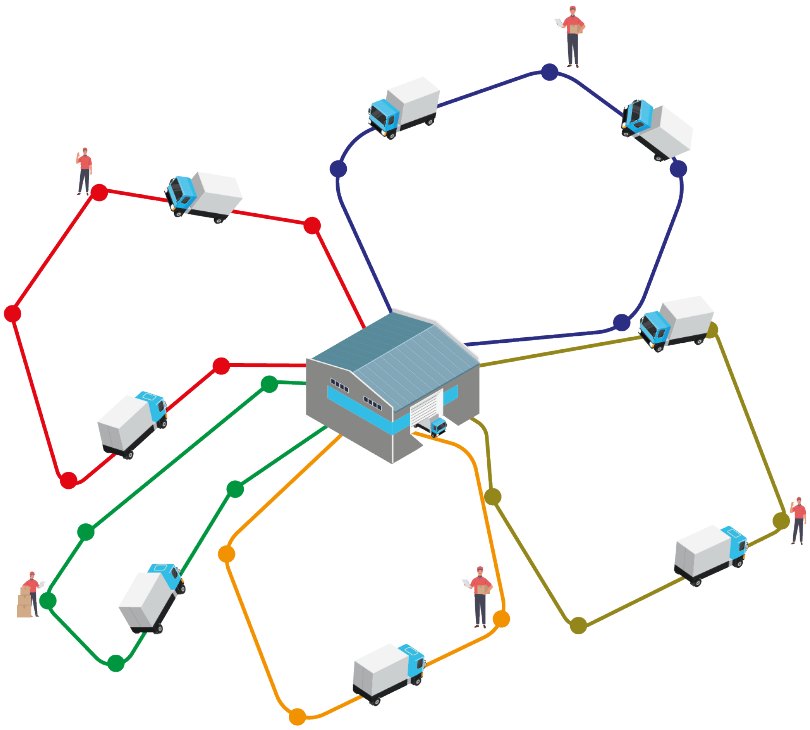

Figure 1: VRP solution

original trip in case of demand load excess. That is, by guaranteeing really high probabilities of success, we will not face any additional routing and back ordering costs.

3 Literature Review

Vehicle routing problems under uncertainty have received much less attention in the literature, compared to the deterministic variants, which have been deeply investigated.

There are several different VRP formulations and according to Ordonez et al. (2007), each of them leads to a different behavior in the observed runtimes of a fixed Integer Program (IP) solver. In particular, the authors verified that solving the so called Miller–Tucker–Zemlin (MTZ) formulation was more efficient than other arc-based formulations considered in their study.

Since the interest is in incorporating uncertainty in demand, the nature of the formulation with respect to the uncertain parameters becomes an important criterion in identifying a suitable formulation for robust optimization frameworks. Firstly, it is commonly known that inequality constraints involving uncertain parameters are preferred to equality constraints. Therefore, in the context of robust optimization, Sungur et al. (2008) show that the MTZ formulation is particularly suited for specific uncertainty sets, as the uncertain demand appears only on one inequality constraint. That is the case for uncertainty sets constructed as linear combination of scenario vectors with weights belonging to the following three types of bounded set: namely convex hull, box and ellipsoidal. Sungur et al. (2008) derive the robust counterpart RVRP for the capacitated VRP and they show that for these three sets the resulting RVRP problem is an instance of the CVRP. However, these types of uncertainty set result to be overly conservative. Furthermore, a CVRP becomes harder for more capacity constrained problems and robust RVRPs are usually more capacity constrained than the corresponding deterministic problem. Therefore, solving RVRPs is likely to be more challenging than obtaining solutions for the deterministic versions.

According to Pecin et al. (2017), the best performing exact algorithms for the capacitated vehicle routing problem are based on the combination of column and cut generation. In their work they propose a new branch-cut-and-price (BCP) algorithm which incorporates and combines for the first time several elements found in previous works. For instance, they make use of route enumeration, strong branching and limited memory subset row cuts. Furthermore, the columns in the BCP are associated with ng-routes instead of elementary routes. In an ng-route, multiple visits to a certain customer i are allowed, given that at least one node j such that $i \notin NG(j)$ is reached between those successive visits. Here $NG(j)$ denote a closest neighborhood of node j. That is, let k indicate the size of the neighborhood, $NG(j)$ represents the k nearest customers to node j. Pecin et al. (2017) were able to solve to optimality instances with up to 199 customers and they could obtain solution for some larger instances with up to 360 customers, which have only been considered before by heuristic methods.

Feillet et al. (2004) propose an exact solution procedure for the Elementary Shortest Path Problem with Resource Constraints (ESPPRC). The authors extend the classical label algorithm, originally developed for the nonelementary path version of this problem, by introducing the concept of unreachable nodes within the label structure. Every partial path is associated with a label indicating the resource consumption. The goal is to eliminate labels by means of dominance rules that incorporate the set of unreachable nodes when labels are compared.

In the context of stochastic programming, Noorizadegan and Chen (2018) propose a flexible branch-and-price method based on column generation. The authors formulate two variants of the capacitated VRP with Stochastic Demand (CVRPSD): a chance-constrained CVRPSD and a distributionally robust chance-constrained CVRPSD, in which probabilistic capacity constraints and distributionally robust probabilistic capacity constraints, respectively, are imposed in order to control the probability of route failure. Noorizadegan and

Chen (2018) also conducted a simulation experiment to assess the solution quality and their results suggest that the chance-constrained CVRPSD outperforms the distributionally robust chance-constrained CVRPSD in terms of expected routing cost. This is because the distributionally robust chance-constrained CVRPSD requires less demand information and, in order to be robust feasible with respect to a family of distributions, rather than a single distribution, it inevitably results in being overly conservative and risk averse. Therefore, even though the distributionally robust chance-constrained CVRPSD results in extremely low failure costs, due to the high reliability, its total expected cost is greater than the cost of the chance-constrained CVRPSD.

4 Methodology

Since we are dealing with customers' demand under uncertainty, it is reasonable to address one of the most significant variants of the VRP, the Capacitated Vehicle Routing Problem (CVRP), in which the vehicle capacity is limited. In this study, we investigate two different approaches which incorporate the uncertain parameter: stochastic programming and robust optimization.

Stochastic programming models the uncertain parameters of an optimization problem as random variables with known probability distributions. To this extent, we study chance-constrained problems, where the decision maker selects here-and-now vehicle routes that satisfy customer demand with a pre-specified probability.

Similar to stochastic programming, robust optimization models the uncertain parameters of an optimization model as random variables. However, in this case the distribution of the random variable is not required, since the aim is to determine a minimum cost delivery plan that remains feasible for all anticipated demand realizations.

Our proposed heuristics are based on column generation, hence the idea is to model the problem using a set partitioning formulation and solve it with a branch-and-price method. We define R_{RO} (resp. R_{SP}) as the set of all robust (resp. stochastic) feasible routes. For every route $r \in R_{RO}(R_{SP})$, define a binary parameter a_{ir} that equals 1 if customer $i \in V \setminus \{0\}$ is visited in route r, and 0 otherwise. Furthermore, define c_r as the cost of route r and introduce a binary decision variable z_r that equals 1 if route r is selected and 0 otherwise. Let k denote the number of available vehicles. The restricted master problem is initialized with a small subset of feasible routes $R'_{RO} \subset R_{RO}$ ($R'_{SP} \subset R_{SP}$). For instance, the restricted master problem of the stochastic model can be formulated as the LP-relaxation of the following Set Partitioning Problem (SPP):

$$
\begin{aligned}
&(\text{SPP}) \quad && \min \ \textstyle\sum_{r \in R'_{SP}} c_r z_r && && (1)\\
&\text{s.t.} && \textstyle\sum_{r \in R'_{SP}} a_{ir} z_r = 1 && \forall i \in V \setminus \{0\} && (2)\\
& && \textstyle\sum_{r \in R'_{SP}} z_r \leq k && && (3)\\
& && z_r \in \mathbb{B} && \forall r \in R'_{SP} && (4)
\end{aligned}
$$

Subsequently, the pricing problem aims at finding new feasible routes that improve the current solution. Therefore, the LP-relaxation of the SPP is solved, given the huge amount of columns, and routes are iteratively generated and added to the master problem. However, it is important to note that there are some differences in the definition of feasible routes between deterministic, robust and stochastic models. Hence, the construction of new routes in the pricing problem differs among the models and in general it holds that $R_{RO} \subset R_{SP} \subset R$, where R denotes the set of deterministic feasible routes.

When the column generation algorithm terminates, the obtained LP solution represents a lower bound for the optimal objective value. Hence, optimizing over all generated columns through an Integer Programming (IP) problem allows us to obtain an upper bound and an integer solution. **Figure 2** below shows an outline of our proposed column-generation heuristics.

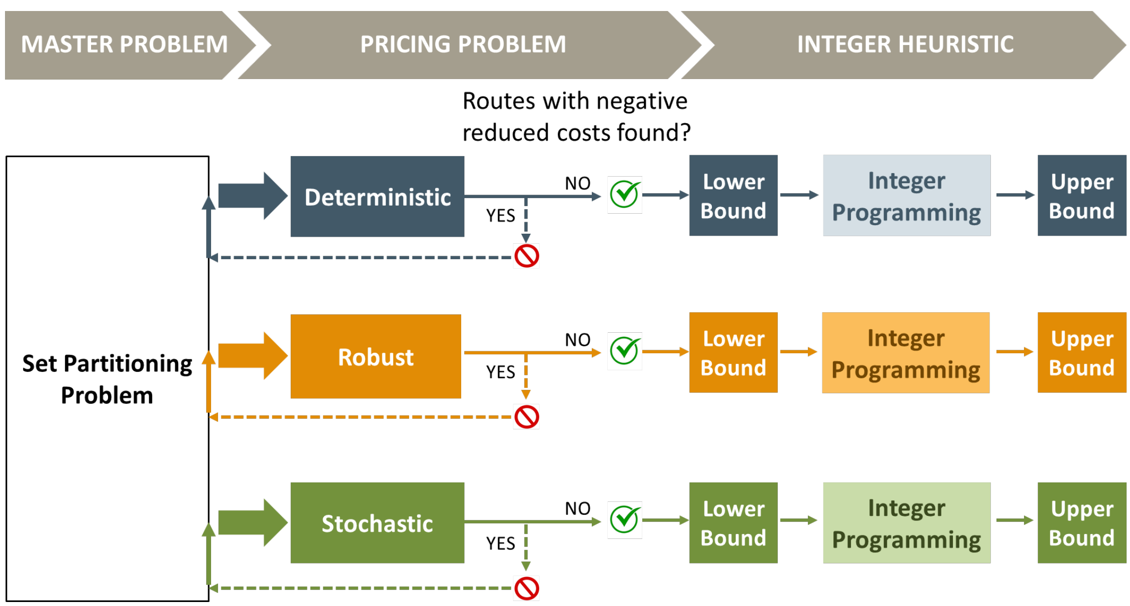

Figure 2: Column Generation Heuristics

4.1 Robust Optimization

It is assumed that only the support $\mathbb{U}$ of the demands is known, and no information about the probability distribution is available. Therefore, it is fundamental to define reasonable uncertainty sets having specific shapes, such that determining whether a route satisfies the capacity requirement for every $q \in \mathbb{U}$ can be easily checked.

First of all, the uncertainty support $\mathbb{U}$ should be a nonempty and closed polyhedron. Additionally, $\mathbb{U}$ cannot be chosen to be very large, because the robust CVRP may become too risk averse or even infeasible. Gounaris et al. (2013) show that the robust CVRP can be reduced to the deterministic CVRP if some conditions are satisfied. For instance, this is the case if the support of the customer demands is rectangular. In order to avoid scenarios where all customer demands attain their worst-case realizations simultaneously, which inevitably results in too conservative solutions, we focus our attention on nonrectangular supports.

We consider one of the two broad classes of demand supports $\mathbb{U}$ proposed by Gounaris et al. (2013), namely budget supports. These supports are partitioned budget polytopes.

4.1.1 Budget Uncertainty Set

We are going to consider budget uncertainty sets of the form

$$\mathbb{U} = \left\{ q \in \mathbb{R}^n_+ : q \in [\underline{q}, \bar{q}], \sum_{i \in B_l} q_i \leq b_l \text{ for } l = 1, 2, .., L \right\} \tag{5}$$

These types of sets constitute the intersection of the n-dimensional hyperrectangle $[\underline{q}, \bar{q}]$, with L budget constraints associated with subsets of customers $B_l \subseteq V$.

Therefore, a route $r \in R_{RO}$ composed by a set of customers S is feasible if the following three conditions are satisfied:

- (a) the route starts and ends at the depot,
- (b) each node (customer) is visited at most once,
- (c) the maximum of $\sum_{i \in S} q_i$ over $\mathbb{U}$ is at most Q.

Condition (a) and (b) can be satisfied within the route construction procedure. On the other hand, condition (c), the load capacity constraint will be checked every time a path is extended to ensure feasibility.

Gounaris et al. (2013) prove that if the partitions of customers are disjoint, then it is possible to efficiently evaluate the corresponding robust rounded capacity inequality constraints. In other words, given a route composed by a set of customers S, the maximum of $\sum_{i \in S} q_i$ over $\mathbb{U}$, which determines whether the route is feasible or not, can be computed in time $O(|S|)$.

It is assumed that the sets $\{B_l\}_{l=1}^{L}$ in (5) are disjoint, that is, $B_l \cap B_{l'} = \varnothing$ for $l \neq l'$. Then for any customer subset $S \subseteq V$, the maximum of $\sum_{i \in S} q_i$ over $\mathbb{U}$ from (5) is calculated by

$$\sum_{i \in S} \underline{q}_i + \sum_{l=1}^{L} \min \left\{ b_l - \sum_{i \in B_l} \underline{q}_i, \sum_{i \in S \cap B_l} (\bar{q}_i - \underline{q}_i) \right\} \tag{6}$$

The above formula allows us to compute the maximum of $\sum_{i \in S} q_i$ quite efficiently for disjoint budget uncertainty sets. Nevertheless, it is possible to further speed up the calculation in case the maximization problem requires to be repeatedly solved for similar sets of customers. That is, given a set of customers S_1, if customers set S_2 is obtained from S_1 by adding or removing a customer, then the maximum of $\sum_{i \in S_2} q_i$ can be derived from the maximum of $\sum_{i \in S_1} q_i$ in constant time $O(1)$. Since feasible routes are going to be constructed iteratively building upon partial solutions via dynamic programming, we will be able to quickly generate multiple robust feasible routes. The procedure is as follows. For a given customer set S, $z = \max_{q \in \mathbb{U}} \sum_{i \in S} q_i$ stores the maximum cumulative customer demands over S. Furthermore, associated with each budget $l \in L$, we have a variable $\rho_l = \sum_{i \in S \cap B_l} (\bar{q}_i - \underline{q}_i)$ which computes the sum of differences between upper and lower demand bounds of those customers in S that belong to the budget B_l. Finally the variable $\pi = \sum_{i \in S} \underline{q}_i$ calculates the sum of lower bounds of all customers in S. The initialization $S = \varnothing$ corresponds to the case $(z, \rho, \pi) = 0$. Then, if a customer $i \notin S$ belonging to budget $l \in L$ needs to be added to S, the values π, ρ_l and z are updated as follows:

- $\pi^{\text{new}} \leftarrow \pi^{\text{old}} + \underline{q}_i$,
- $\rho_l^{\text{new}} \leftarrow \rho_l^{\text{old}} + (\bar{q}_i - \underline{q}_i)$,
- $z^{\text{new}} \leftarrow \pi^{\text{new}} + \sum_{l=1}^{L} \min \left\{ b_l - \sum_{i \in B_l} \underline{q}_i, \rho_l^{\text{new}} \right\}$.

By doing that, for every route composed by a subset of customers S, the maximum of $\sum_{i \in S} q_i$ over $\mathbb{U}$ is the value of z.

4.2 Stochastic Programming

In order to compare the solution quality and solving time of the two approaches, we are going to model the chance-constrained problem in a similar way. We will use the same set partitioning formulation as in the robust case and solve it with a column generation heuristic.

There is only one significant difference regarding the generation of feasible routes compared to the robust model. That is, a route does not need to satisfy the capacity requirement for all demand realizations, but the probability of a failure should be below a fixed threshold.

By 'failure', we refer to the case where the total realized demand of the customers in a route exceeds the vehicle capacity Q.

Therefore, a route $r \in R_{SP}$ is feasible if it satisfies conditions (a), (b) and (d), which is stated below:

- (d) the total realized demand from all customers visited in the route is within the vehicle capacity with probability $(1 - \epsilon)$.

4.2.1 Probabilistic capacity constraints

In line with Noorizadegan and Chen (2018), in order to control the probability of route failure we impose probabilistic capacity constraints as follows .

$$\mathbb{P}\left[\sum_{i \in S} q_i \leq Q\right] \geq 1 - \epsilon \tag{7}$$

Here, S denotes the set of customers visited on the route. From a computational point of view, the Normal distribution is particularly effective and easily tractable, and that is why we are going to make use of it. Although Poisson distribution and scenario-based distributions might result to be more realistic, they are computationally more expensive. If we assume that the demands follow independent normal distributions: $q_i \sim N(\mu_i, \sigma_i^2)$, then the probabilistic constraint of (7) is in the form of:

$$\mathbb{P}\left[z = \frac{\sum_{i \in S}(q_i - \mu_i)}{\sqrt{\sum_{i \in S} \sigma_i^2}} \leq \frac{Q - \sum_{i \in S} \mu_i}{\sqrt{\sum_{i \in S} \sigma_i^2}}\right] \geq 1 - \epsilon \tag{8}$$

Let $\Phi()$ denote the Cumulative Distribution Function (CDF) of the Standard Normal Distribution and $\Phi^{-1}()$ its inverse. This condition implies that if $\frac{Q - \sum_{i \in S} \mu_i}{\sqrt{\sum_{i \in S} \sigma_i^2}} < \Phi^{-1}(1 - \epsilon)$, then the route visiting the set of customers S is not feasible, otherwise the feasibility condition is satisfied, which means that the vehicle load fits capacity with probability at least $(1 - \epsilon)$.

4.3 Column Generation Heuristic

4.3.1 Master Problem

As previously mentioned the algorithm consists in solving a set partitioning formulation through column generation. Following the notation introduced at the beginning of Section 4, consider the robust optimization approach. Let c_r denote the total distance traveled in route r. The restricted master problem can be formulated as follows:

$$
\begin{array}{llll}
\text{(RMP)} & \min \; \sum_{r \in R'_{RO}} c_r z_r & & (9) \\
\text{s.t.} & \sum_{r \in R'_{RO}} a_{ir} z_r = 1 & \forall i \in V \setminus \{0\} & (10) \\
 & \sum_{r \in R'_{RO}} z_r \leq k & & (11) \\
 & z_r \geq 0 & \forall r \in R'_{RO} & (12)
\end{array}
$$

The objective function (9) minimizes the total routing cost. Constraints (10) make sure that each customer is visited exactly once and restriction (11) limits the number of routes/available vehicles. Finally, constraints (12) guarantee that the LP-relaxation is solved, since the aim is to solve the problem by column generation.

Note that the robust model is the most conservative and risk-averse. Therefore, if an instance is robust feasible, then it is certain to be also stochastic and deterministic feasible. On the contrary, if an instance turns out to be infeasible in the robust approach, the constraint limiting the number of available vehicles (11) is relaxed, in order to obtain a solution eventually. In addition, this relaxation is also applied to the deterministic and stochastic scenarios to make the comparison between the models unbiased for that given instance. In our experiment, a given instance is classified as infeasible if either a solution to the IP could not be found or the total running time of the algorithm exceeds one hour. In the results section, instances marked with an $*$ represent the fact that the constraint concerning the number of available vehicles (11) is relaxed.

4.3.2 Pricing Problem

The pricing problem, whose goal is to generate sets of routes that are demand feasible, amounts to solving minimum cost "robust" constrained shortest path problems (Pessoa et al. 2018). Denote by α and β_i the dual variable of the set partitioning formulation corresponding to constraints (11) and (10), respectively. We propose a labeling algorithm based on dynamic programming in order to solve this shortest path problem with respect to the arc reduced costs, where the reduced cost of an arc $a = (i, j) \in A$ is defined as follows.

$$
\bar{c}_a = \begin{cases} \mathrm{c}_a - (\alpha + \beta_j)/2 & \text{if } i = 0 \\ \mathrm{c}_a - (\beta_i + \beta_j)/2 & \text{if } i, j \in V \setminus \{0\} \\ \mathrm{c}_a - (\beta_i + \alpha)/2 & \text{if } j = 0 \end{cases}
$$

Note that a route is feasible if it satisfies conditions (a), (b) and (c) in the robust model and conditions (a), (b) and (d) in the stochastic one.

In the proposed algorithm, a set of labels are defined for each node. Furthermore, some conditions and dominance rules are used in order to manage only useful labels. Let $L(i) = \{L_1(i), L_2(i), ..\}$ be the set of labels at node i. Each label $L_t(i)$ is associated with a path to node i and consists of three components: $L_t(i) = (\bar{c}_t(i), \bar{d}_t(i), p_t(i))$, containing the total reduced cost $\bar{c}_t(i)$ of the path, the total demand load $\bar{d}_t(i)$ of the path and the sequence of nodes $p_t(i)$ of the path. It is important to mention that the definition of demand load $\bar{d}_t(i)$ differs between robust and stochastic settings. The exact distinction will be formalized later in the paper.

Let W indicate the list of all labels in $\bigcup_{i \in V} L(i)$ arranged in lexicographic ascending order, depending on the three label components. The labelling algorithm starts from the central depot 0 and extends the path to its neighborhood $N(0)$. We then add the extended path to customer i to the label set of node i ($L(i)$) and set W if certain conditions are satisfied. More precisely, given two labels at node i $L_1(i)$ and $L_2(i)$, we say that $L_1(i)$ is dominated by $L_2(i)$ if $\bar{c}_2(i) \leq \bar{c}_1(i)$, $\bar{d}_2(i) \leq \bar{d}_1(i)$ and $p_2(i) \subseteq p_1(i)$. This means that any feasible extension of the dominated label is also a feasible extension of the dominant label.

Unreachable Nodes

Feillet et al. (2004) propose an alternative definition of label and dominance rules. That is, in the dynamic programming algorithm each label $L_t(i)$ associated with a path to node i is characterized by the three components described in the previous paragraph and an additional fourth component $u_t(i)$, the set of unreachable nodes. Let us define the meaning of unreachable node. For each path $L_t(i)$ from the origin node to a node i, a node k is said to be unreachable if it is included in $L_t(i)$ ($k \in p_t(i)$) or if extending the path $L_t(i)$ to node k results in a violation of the capacity constraint, meaning the current capacity load prevents the path from reaching node k. When a path is extended, visitation resources corresponding to customers who cannot be visited anymore are consumed. That occurs either due to resource constraints or because they have already been visited. Therefore, every time a new label is created we need to assess feasibility of every outgoing arc from the last considered node.

According to Feillet et al. (2004), this algorithmic modification is computationally attractive because the dominance relation becomes sharper. More precisely, given two labels at node i $L_1(i)$ and $L_2(i)$, we say that $L_1(i)$ is dominated by $L_2(i)$ if $\bar{c}_2(i) \leq \bar{c}_1(i)$, $\bar{d}_2(i) \leq \bar{d}_1(i)$ and $u_2(i) \subseteq u_1(i)$.

Naive Labelling

Section 6.1.1 compares the solution quality and computational time between the dominance rules proposed by Feillet et al.(2004) and a naive labelling, where only the reduced cost of the path $\bar{c}_t(i)$ is considered in the dominance relations, along with the demand load, and no information concerning the actual structure of the route is evaluated. In other words, given two labels at node i $L_1(i)$ and $L_2(i)$, we say that $L_1(i)$ is dominated by $L_2(i)$ if $\bar{c}_2(i) \leq \bar{c}_1(i)$ and $\bar{d}_2(i) \leq \bar{d}_1(i)$.

Because our proposed methods will implement this new dominance relation, it might occur that a current dominated partial path would result in a non-dominated route in later stages, since visited nodes are not considered. Because of that, our algorithms might not find a global optimum, hence the name heuristic. A general form of the proposed algorithm is outlined in **Algorithm 1** below.

Algorithm 1 Labelling algorithm for the pricing problem

```
W ← {}
L_1(0) ← {(0,0,0)}
insert L_1(0) into W
while W ≠ {} do
    l′ ← the first label in W
    remove l′ from W
    i ← node associated with label l′
    for all j ∈ N(i) do
        if extended label l′ to node j holds feasibility conditions then
            reducedCost ← c̄_{l′}(i) + c̄_{ij}
            demandLoad ← updated demand load
            path ← p_{l′}(i) ∪ {j}
            create new label L_*(j)=(reducedCost, demandLoad, path)
            if new label is not dominated then
                if any L_l(j) ∈ L(j) is dominated then
                    remove the dominated labels
                end
                l̂ ← proper index for the new label for node j
                insert L_l̂(j) into W and sort W
                insert L_l̂(j) into L(j)
            end
        end
    end
end
```

As previously mentioned, the demand load $\bar{d}_t(i)$ associated with a path $L_t(i)$ in the robust approach is expressed differently from the stochastic case. That is, in the former approach $\bar{d}_t(i)$ is just a value which is calculated using expression (6) and represents the highest possible demand load. In this case, it can be argued that dominated paths are more likely to lead to either an infeasible path or paths with positive reduced cost. Therefore, even though the naive dominance relations might look too simple, they provide efficient solutions in reasonable amounts of time, as deterministic results suggest in Section 6.1.1.

On the other hand, in the chance-constraint approach there are two parameters to be considered: the mean μ and standard deviation σ. Hence, given two labels at node i $L_1(i)$ and $L_2(i)$, we say that $L_1(i)$ is dominated by $L_2(i)$ in the stochastic approach if $\bar{c}_2(i) \leq \bar{c}_1(i)$, $\bar{d}_2^{\mu}(i) \leq \bar{d}_1^{\mu}(i)$ and $\bar{d}_2^{\sigma}(i) \leq \bar{d}_1^{\sigma}(i)$. Here, $\bar{d}_r^{\mu}(i)$ and $\bar{d}_r^{\sigma}(i)$ for $r = 1, 2$ are computed in this way: $\bar{d}_r^{\mu}(i) = \sum_{j \in p_r(i)} \mu_j$ and $\bar{d}_r^{\sigma}(i) = \sum_{j \in p_r(i)} \sigma_j^2$. This follows from the fact that given two random variables $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$, it is clear that if $\mu_x \geq \mu_y$ and $\sigma_x^2 \geq \sigma_y^2$, then $\mathbb{P}[X > Q] \geq \mathbb{P}[Y > Q]$ for all $Q \geq 0$. Therefore, by taking into account also the reduced cost, extending a path associated with random variable X cannot lead to a better route than Y if $\bar{c}_x \geq \bar{c}_y$, because X is more likely to lead to an infeasible path in later stages.

4.3.3 Integer Heuristic

In both column generation procedures, the LP relaxation of the master problem is solved. When no more columns with negative reduced cost can be found the algorithm terminates and a lower bound LB is obtained. However, the interest lies in obtaining integer solutions. Therefore, the restricted IP is solved by optimizing over the generated columns using a set covering formulation. In other words, constraints (10) are replaced by the following:

$$\sum_{r \in R} a_{ir} z_r \geq 1 \ \forall i \in V \setminus \{0\} \tag{13}$$

The reason behind that is the fact that the set partitioning formulation might not be able to generate feasible solutions, since it assigns each customer to exactly one route. Hence, the procedure is to firstly allow multiple visits to the same customers. Then, once a feasible solution is obtained, if there are customers who belong to multiple routes, they consequently get eliminated from those routes for which the removal of the customer yields the highest cost reduction. A general form of the proposed method is outlined in **Algorithm 2** below.

Algorithm 2 Integer Heuristic - Step 2

Result: Given a set of routes R obtained through IP and the set of customers V, remove customers who are visited more than once by minimizing total routing cost

for ***all*** *customer* $c \in V$ **do**

  $D_c \leftarrow$ Set of routes visiting customer c

  while $|D_c| > 1$ **do**

    Delete c from the routes of D_c which yield the highest cost reduction

    Update R

  end

end

return R , Updated set of routes

This approach gives an upper bound UB on the optimal objective value. Therefore, an ex post optimality gap $\frac{UB-LB}{LB}$ can be derived, which calculates the difference between the integer solution (UB) and the LP value (LB) in percentage.

As explained in Section 4.3.1, it might occur that a solution to the IP (UB) can not be found, due to the size of the uncertainty set. If that is the case, we relax the constraint regarding the number of free vehicles (11) in the master problem of the column generation procedure. In a similar way, the relaxation is consequently applied to the IP.

5 Experimental analysis

5.1 Performance measures

In this section, in line with Sungur et al. (2008), we present performance measures to compare robust, stochastic and deterministic solutions.

The first performance measure, the ratio κ, quantifies the relative additional cost of the LP robust and stochastic solutions with respect to the cost of the LP deterministic solution. It is given by $\kappa_r = (z_r - z_d)/z_d$ and $\kappa_s = (z_s - z_d)/z_d$, where z_d is the optimal LP value of the deterministic CVRP, z_s is the optimal LP value of the stochastic approach and z_r is the optimal LP value of the robust approach (with worst-case demand). By making use of the lower bounds LBs, this ratio highlights the additional cost that would be incurred by implementing the robust or stochastic model to protect against the demand uncertainty, instead of the deterministic approach.

The second performance measure considers the effect of the solutions on the demand since it is subject to uncertainty. The ratio δ quantifies the relative unsatisfied demand that deterministic and stochastic solutions may cause when faced with their worst-case demand.

It is given by $\delta_d = (\gamma_d / \sum_{i \in V} q_i^0)$ and $\delta_s = (\gamma_s / \sum_{i \in V} q_i^0)$, where the numerator γ_d is the maximum amount of lost demand that can take place if deterministic routes are implemented and γ_s is the maximum unsatisfied demand that can occur if obtained stochastic solutions are used. The denominators are the total demand of the deterministic case. To obtain γ_d and γ_s, we fix the routing variables to the deterministic and stochastic optimal solutions respectively, and maximize the unmet demand by varying the demand outcome within the demand uncertainty set. That is, using the obtained routes corresponding to the integer solution (UB), we maximize the demand load of each route using expression (6). Then, if the resulting robust demand load exceeds the vehicle capacity, we store the quantities in excess and repeat the calculations for all routes.

Note that the extra cost is computed using the LP values (LBs), whereas the amount of unsatisfied demand is calculated from the upper bounds (UBs) or IP solutions. By definition, the robust approach gives solutions with zero unmet demand that might have a higher routing cost compared to sub-optimal deterministic and stochastic solutions. On the other hand, these latter solutions may in turn lead to scenarios with unmet demand. Therefore, these two measures, unmet demand and extra routing cost, represent the trade-offs that routing solutions must balance in a CVRP with demand uncertainty.

Lastly, for every instance we will also store the computational time for both the LP and IP values, the total number of generated columns and the number of iterations performed by the three different approaches. By 'iterations', we refer to the number of times the pricing problem manages to find routes with negative reduced cost.

5.2 Dataset

We will test our methods on 71 benchmark instances, originated from standard CVRP benchmark problems introduced in Gounaris et al. (2016). More in detail, 27, 23 and 21 instances of class A (random instances), class B (clustered instances) and class P (modified instances from the literature) have been gathered from Augerat et al. (1995). They correspond to small and medium-sized instances which range from 16 to 80 customers and with up to 15 vehicles. For the majority of the instances the optimal solution corresponding to the deterministic case is also provided.

With respect to the demand values, in the chance constraint approach, the nominal values q^0 specified in the benchmark instances represent the mean μ of the Normal distribution, while the variance σ^2 is calculated as $\alpha \cdot \mu$.

Regarding the budget uncertainty sets, the n-dimensional hyperrectangle $[\underline{q}, \bar{q}]$ is defined by $[(1-\alpha)q^0, (1+\alpha)q^0]$. Furthermore, the decomposition of customers into L disjoint subsets depends on the geographic quadrants defined by the customer coordinates. However, the uncertainty sets' structure is such that the customer demands deviate by at most $\alpha \cdot 100\%$ from their nominal values q^0 and the cumulative demand of each subset does not exceed its nominal value by more than $\beta \cdot 100\%$. That is, we are going to use the following budget uncertainty set:

$$\mathbb{U} = \left\{ q \in [(1-\alpha)q^0, (1+\alpha)q^0] : \sum_{i \in B_l} (q_i - q_i^0) \leq \beta \sum_{i \in B_l} [(1+\alpha)q_i^0 - q_i^0] \text{ for } l = 1, 2, .., L \right\}$$

Finally, due to the large number of parameters to be specified by the modeler, a sensitivity analysis will be performed on some of them. More precisely, we are going to investigate how changes in α and β affect the robust solution using the following uncertainty levels $\alpha = (0.05, 0.1)$ and $\beta = (0.5, 1)$. Moreover, in the stochastic scenario, the effects of variation of the chance-constrained probability failure threshold ϵ will be studied adopting three different levels of failure probability: $\epsilon = (0.1, 0.05, 0.01)$.

6 Results

In this section we report the results of our proposed methods, analysing the obtained solutions in terms of efficiency and solution quality. First, regarding the deterministic model, the dominance rules introduced by Feillet et al. (2004) are tested against our naive relations. Furthermore, we compare our sub-optimal solutions with known global optima. After that, the comparison between the robust and stochastic model is investigated. Finally, the results of the sensitivity analysis are reported, along with the computational times.

All experiments are run on a laptop with a 2.0 GHz Intel Core i3 Processor and 4 GB RAM.

6.1 Deterministic Instances

6.1.1 Labelling methods

In this section, the performance of the dominance rules introduced by Feillet et al. (2004) is evaluated. Therefore, the original label structures are tested against this new definition of dominance rules on a set of modified instances assuming fixed customers demand, or in other words in the deterministic case. **Table 1** below compares the solution of the deterministic model in terms of objective value and computational time. Here, LB represents the LP value obtained through column generation, while UB is the IP value. Furthermore, the number of generated columns and the number of iterations performed by the column generation algorithm are reported.

Table 1: Effects of using unreachable nodes within the route construction

Deterministic	Without unreachable nodes				
Instance	LB	UB	#columns	#iter	Time (s)
test15-2*	635.1	660.92	160	23	1.5
test15-5	587.05	629.05	117	17	0.93
test20-6	938.42	938.42	236	31	2.35
test20-6*	938.42	938.42	183	23	1.69
test20-8	778.95	820.29	132	20	1.69
test20-8*	778.95	820.45	132	21	1.53
P-n16-k8*	442.42	451.94	59	11	0.58
Instance	With unreachable nodes				
test15-2*	635.1	660.92	790	13	29.69
test15-5	587.05	626.31	337	11	4.31
test20-6	938.42	938.42	1158	22	97.74
test20-6*	938.42	938.42	1041	17	83.27
test20-8	778.95	820.29	479	12	7.27
test20-8*	778.95	820.45	459	12	6.31
P-n16-k8*	442.42	451.94	82	11	0.68

First of all, it can be checked that using unreachable nodes does not provide significant improvements in terms of objective function. In fact, the lower and upper bounds obtained with these two approaches are identical in all cases except for the *test15-5* instance. Here, although the LB values are the same, the upper bound (UB) computed with unreachable nodes leads to a decrease of 0.4 %, from 629.05 to 626.31.

On the other hand, the time required to solve the instances increases drastically when we take into account unreachable nodes. This can be seen by looking at the last column of **Table 1**. This is explained by the fact that when unreachable nodes are incorporated within the label structure, the dominance relations become more strict and the number of non-dominated paths remains excessively high. It can be indeed observed that even though the pricing problem is solved fewer times, the amount of generated routes is much higher compared to the case without unreachable nodes.

When a new label associated with a given node k is created, we need to assess the feasibility of an extension through every outgoing arc from k. Hence, the time complexity depends both on the number of nodes and the number of resources. However, it is also strongly related to the tightness of resource constraints. In our case, although there is exactly one resource, the vehicle capacity constraint, it is not tight enough as capacity only plays a role when a route has visited many customers and the load is almost full.

Therefore, the use of unreachable nodes would be more reasonable in the context of highly constrained problems, such as the Vehicle Routing Problem with Time Windows (VRPTW). In addition, since assessing feasibility in the robust and stochastic models requires more calculations than the deterministic scenario, we expect to incur even longer solving times if we decide to consider unreachable nodes in the two models incorporating uncertainty.

6.1.2 Best solutions found

In this section, the solution quality of our proposed branch-and-price method is evaluated by comparing the obtained integer problem value (IP) with the best known deterministic solution on the three sets of instances A, B and P. **Table 2** below provides the comparison regarding the set of instances A, while **Table 20** and **Table 31** in Appendix B.D and P.D refer to set B and P. Here, z^* is the best known solution to the deterministic problem and in line with the previous section, LB and UB are the lower and upper bounds found, respectively. Lastly, Gap^* represents the deviation from the optimal solution ($Gap^* = \frac{UB - z^*}{z^*}$) and the column $Time$ shows the total computational time in seconds.

Overall, it is clear that our proposed method performs considerably well in terms of solution quality, as the average optimality gaps corresponding to set A, B and P are 2%, 4% and 2%, respectively. For what concerns set A in particular, the solution never deviates by more than 6% from the global optimum.

Nevertheless, it is important to mention that the global optima are obtained including all restrictions, while instances marked with * are characterized by the relaxation of constraints (11), that is, there is no limit on the number of available vehicles. Therefore, the comparison is not completely fair. In fact, in **Table 20** and **Table 31** there are three instances with negative optimality gaps, two in set B and one in set P. In other words, the obtained solution corresponds to lower transportation cost, compared to the best solution found. However, because relaxing constraint (11) allows for more routes and since a higher number of routes generally results in longer distances, we can conclude that the comparison is quite unbiased. It can be checked indeed that only 3 out of 55 relaxed instances report negative gaps, among the three sets of instances.

Furthermore, it can be argued that our column generation heuristic generates sub-optimal solutions rather quickly, since small and even medium-sized instances are solved in reasonable amounts of time. In more detail, we can see from **Table 2** that only 2 out of 27 instances were solved in over ten minutes and that more than 75% of the instances could be solved within 3 minutes. Additionally, the set of clustered instances B seems to be the hardest in terms of computational tractability. In fact, the average solving time (374 s) is considerably greater than those associated with sets of instances A and P, 181 and 238, respectively.

Table 2: Set A - Comparison with best known deterministic solution

Instance	z*	LB	UB	Gap*	Time (s)
A-n32-k5	784	760.14	816.25	4%	17.31
A-n33-k5	661	656.33	663.35	0%	12.22
A-n33-k6	742	728.31	744.45	0%	10.21
A-n34-k5*	778	742.72	790.78	2%	13.92
A-n36-k5*	799	776.94	826.54	3%	34.59
A-n37-k5	669	660.17	672.5	1%	26.44
A-n37-k6*	949	930.43	1003.28	6%	17.9
A-n38-k5*	730	706.27	746.12	2%	22.7
A-n39-k5*	822	803.77	833.14	1%	42.18
A-n39-k6	831	805.75	835.56	1%	33.89
A-n44-k6*	937	930.24	938.18	0%	34.71
A-n45-k6*	944	929.18	978.53	4%	42.79
A-n45-k7	1146	1115.71	1170.13	2%	59.93
A-n46-k7	914	904.64	918.95	1%	56.43
A-n48-k7*	1073	1050.48	1121.13	4%	112.48
A-n53-k7*	1010	999.73	1069.88	6%	135.82
A-n54-k7*	1167	1147.31	1196.33	3%	178.53
A-n55-k9*	1073	1059.03	1076.04	0%	54.89
A-n60-k9*	1354	1330.67	1384.56	2%	213.15
A-n61-k9*	1034	1014.61	1059.89	3%	111.7
A-n62-k8*	1288	1260.78	1331.04	3%	672.65
A-n63-k9*	1616	1592.68	1656.6	3%	287.09
A-n63-k10*	1314	1288.16	1328.6	1%	162.41
A-n64-k9*	1401	1376.86	1417.67	1%	436.59
A-n65-k9*	1174	1158.69	1193.44	2%	142.88
A-n69-k9*	1159	1132.35	1172.46	1%	229.73
A-n80-k10*	1763	1732.01	1802.02	2%	1724.82
avg.				2%	181.04

6.2 Robust vs Stochastic approach

In this section the sub-optimal robust and stochastic solutions are compared against each other using the performance measures introduced in Section 5.1. That is, additional routing costs are evaluated in terms of their deviation from obtained deterministic solutions. Furthermore, the proportion of unsatisfied demand that optimal stochastic and deterministic routes may face in the worst-case scenario is computed as fraction of the total customers demand. The ex-post optimality gaps and the computational times are also discussed.

6.2.1 Ex-post optimality gaps

In this section we analyse the ex-post optimality gaps, which measure the deviation of the integer solution (UB) from the LP value (LB). Note that while the lower bound is obtained through a set partitioning formulation (SPP), the first step of the integer heuristic consists in solving a set covering formulation (SCP). Then, the second and last step of the integer heuristic is performed in order to remove customers who are visited multiple times. However, this final step is executed really rarely, because the routes found with the SCP already visit each customer exactly once in almost every instances. **Table 3** below illustrates the average ex-post optimality gap of the column generation heuristic over the three sets of instances for the two cases of the uncertainty support.

Table 3: Average ex-port optimality gap (%) in the robust and stochastic models for the 3 sets of instances for the values of α and β

Gap (%)	$\alpha = 0.1$	$\beta = 0.5$			$\alpha = 0.05$	$\beta = 0.5$		
Instance	Robust	Stochastic			Robust	Stochastic		
Set		$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$		$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$
A	5.63	4.27	4.16	4.06	5.69	4.01	4.50	4.02
B	10.76	10.77	12.76	12.28	11.46	11.15	10.81	11.83
P	4.27	3.68	4.03	3.90	4.24	4.09	4.07	4.33

First of all, it can be checked that there is no significant difference between stochastic and robust models gaps. Additionally, the size of the uncertainty set does not play a role either.

Overall, we can see that our heuristics provide sufficient integer solutions. In other words, the actual obtained routes are slightly more costly than those corresponding to the LP values. The deviations sporadically exceed the order of 10%. In particular, the set of clustered instances B results to have the largest ex-post optimality gaps fluctuating around the level of 11%, which is more than twice the ones associated with sets A and P. Moreover, the gaps do not depend on the amount of transportation cost. In fact, it can be observed from **Table 12** to **Table 19** (in Appendix A.R and A.S) regarding set A and from **Table 23** to **Table 30** (in Appendix B.R and B.S) for set B, that the obtained upper bounds all range between the values of 1000 and 1200. As will be discussed later in Section 6.2.4, the time required to solve the IP for set of instances B also turns out to be remarkably higher compared to the other two sets. Therefore, we can conclude that clustered instances are definitely more problematic to deal with than random generated instances, for example.

6.2.2 Performance measures comparison

Table 4 below reports the performances of the deterministic, stochastic and robust models in the base case ($\alpha = 0.1$ and $\beta = 0.5$) for the set of instances A.

Table 4: Set A - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
A-n32-k5	5.24%	2.43%	2.32%	1.02%	4.96%	0.29%	5.74%	5.93%
A-n33-k5	5.00%	1.77%	1.13%	0.76%	2.13%	0.00%	4.29%	4.51%
A-n33-k6	3.32%	2.51%	4.43%	0.99%	5.77%	0.18%	6.56%	9.68%
A-n34-k5*	5.32%	0.98%	1.84%	0.50%	2.80%	0.26%	3.48%	4.63%
A-n36-k5*	3.71%	1.65%	1.84%	0.77%	4.15%	0.00%	5.36%	6.06%
A-n37-k5	4.67%	2.21%	0.89%	0.00%	2.62%	0.29%	3.96%	4.75%
A-n37-k6*	3.74%	1.21%	2.57%	0.82%	3.83%	0.44%	4.42%	7.47%
A-n38-k5*	4.16%	1.41%	1.53%	0.25%	2.54%	0.25%	3.30%	5.26%
A-n39-k5*	2.16%	2.16%	1.95%	0.00%	3.61%	0.00%	5.64%	6.37%
A-n39-k6	1.50%	1.52%	2.11%	0.95%	3.84%	0.02%	4.75%	5.37%
A-n44-k6*	5.30%	1.37%	1.69%	1.02%	2.72%	0.42%	4.73%	5.44%
A-n45-k6*	4.52%	2.09%	2.65%	0.59%	3.65%	0.22%	4.08%	5.24%
A-n45-k7	4.00%	3.31%	1.73%	1.40%	3.60%	0.41%	6.34%	11.81%
A-n46-k7	6.29%	2.06%	2.98%	0.20%	3.77%	0.22%	4.87%	7.09%
A-n48-k7*	6.10%	2.60%	2.61%	2.06%	3.92%	0.42%	6.24%	7.87%
A-n53-k7*	5.05%	2.36%	2.38%	1.55%	3.78%	0.36%	5.33%	6.87%
A-n54-k7*	3.51%	2.55%	3.24%	1.20%	5.17%	0.19%	6.75%	7.91%
A-n55-k9*	5.98%	2.43%	2.22%	0.98%	3.74%	0.44%	4.35%	6.31%
A-n60-k9*	5.91%	2.17%	2.17%	1.11%	3.94%	0.17%	5.49%	8.10%
A-n61-k9*	5.55%	3.06%	2.82%	1.01%	4.23%	0.17%	5.29%	7.25%
A-n62-k8*	4.42%	2.92%	2.78%	1.09%	4.89%	0.00%	6.28%	7.70%
A-n63-k9*	5.64%	3.34%	3.71%	1.31%	5.10%	0.27%	6.92%	8.67%
A-n63-k10*	4.56%	2.08%	2.72%	1.35%	4.48%	0.27%	6.06%	7.99%
A-n64-k9*	5.13%	2.52%	3.18%	0.68%	4.46%	0.42%	6.48%	7.83%
A-n65-k9*	6.36%	1.04%	4.23%	0.80%	5.43%	0.29%	7.16%	8.48%
A-n69-k9*	5.94%	1.63%	3.07%	0.69%	3.71%	0.28%	5.33%	6.61%
A-n80-k10*	6.03%	2.27%	2.88%	1.23%	4.20%	0.27%	5.95%	7.75%
avg.	4.78%	2.14%	2.51%	0.90%	3.96%	0.24%	5.38%	7.00%

As previously mentioned, δ represents the amount of unsatisfied demand as a fraction of the total demand, while κ shows the additional cost relative to the deterministic case.

For what concerns the stochastic model, three different levels of failure probability are used: $\epsilon = (0.1, 0.05, 0.01)$. For example, $\epsilon = 0.05$ represents the fact that routes are constructed such that the total realized demand of the customers visited in the route is within the vehicle capacity with probability at least 95%. Therefore, lower values of ϵ correspond to higher degrees of robustness.

In order to easily visualize the trade-off between the two competing performance measures **Figure 3** depicts a scatter plot of the values reported in **Table 4**, where each point represents an instance of set A. The x and y-axis indicate the hypothetical unmet demand and the extra cost, respectively.

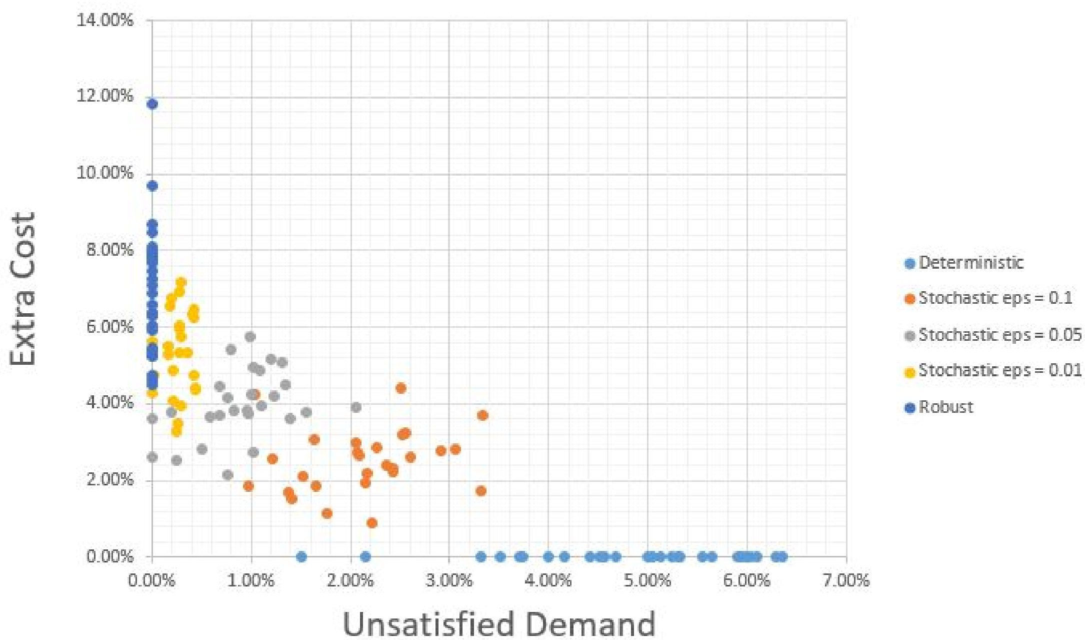

Figure 3: Set A - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

Firstly, it is evident that there is no extra cost associated with the deterministic model, since this measure is calculated in relation to sub-optimal deterministic routes. In fact, the dots representing deterministic solutions in **Figure 3** all lie on the x-axis. However, these solutions face the highest rates of unmet demand as they do not take into account the uncertainty of demand. Specifically, the average lost demand ($\simeq 5\%$) is more than twice the lost demand that is obtained in the stochastic model with $\epsilon = 0.1$, that is, by ensuring at least 90% probability of completing the route without lost demand. In practical contexts, this is a quite unseemly situation, as these frequent route failures are likely to cause additional routing and/or back ordering costs. For instance, the drivers will have to return to the depot before continuing their trip or a different vehicle will have to extend his former route to visit the customer. Therefore, even though the deterministic approach might look cheap, it is plausible to conclude that the actual total cost will be reasonably higher.

On the contrary, robust solutions behave in the opposite way. Because the robust model generates solutions assuming complete adversity by its nature, these latter routes will never run the risk of lost demand as all possible scenarios are considered. Nevertheless, the "price of robustness" (Bertsimas and Sim, 2004) has to be paid. That is, robust optimal solutions are the most costly ones in terms of distance travelled. This can be seen by looking at **Figure 3** and at the last column of **Table 4**. In particular, by selecting these routes, 7% more expensive over deterministic ones, the drivers will always meet the demand of each customer visited on their route. In other words, it is guaranteed that there will not be any additional costs nor unsatisfied demand.

Lastly, stochastic solutions, based on the failure probability threshold ϵ, represents a sort of transit from deterministic routes to robust routes. In fact, as ϵ decreases, the chance-constrained solution becomes more and more risk-averse. That is, while the percentage of unsatisfied demand decreases, the routing cost rises.

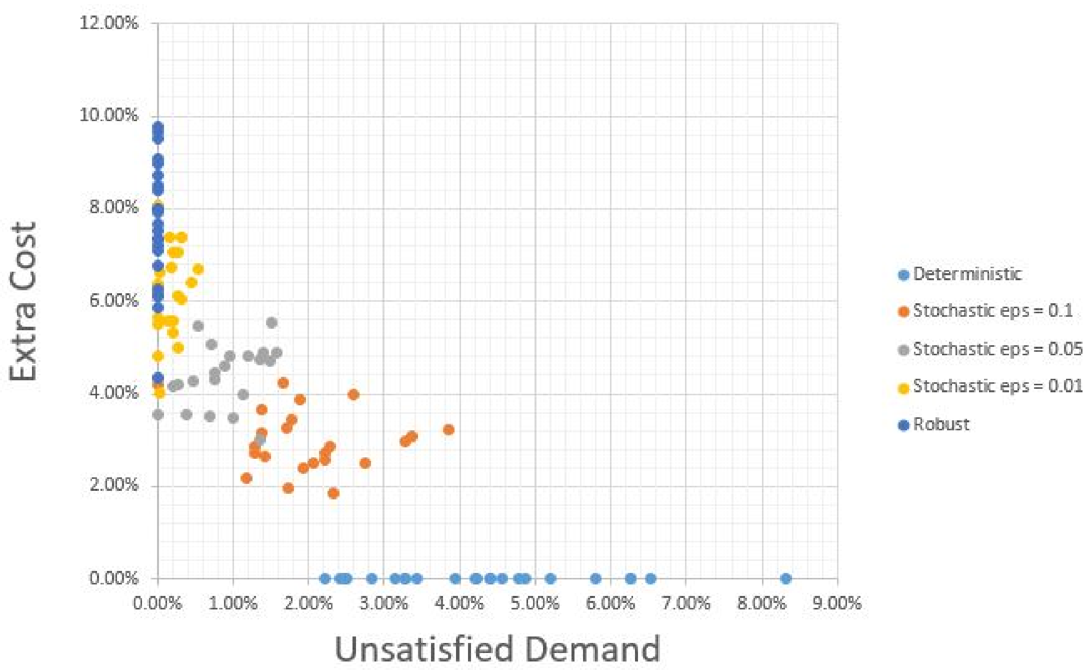

Figure 4: Set B - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

Intermediate values of ϵ such as 5% seem to provide the most balanced and efficient solution. First of all, because extremely high values of ϵ and robust solutions themselves yield too conservative solutions which protect from unlikely scenarios too. On the other hand, higher values of ϵ and deterministic routes, as previously discussed, may have quite noticeable effects on the total cost.

For example, regarding the set of instances A, it can be checked from **Table 4** that using the stochastic model with ϵ set to 5% provides a set of routes which is only 4% more expensive than deterministic ones and with less than 1% of lost demand.

Figure 4 above and **Figure 5** below report the performances of the deterministic, stochastic and robust models in the base case ($\alpha = 0.1$ and $\beta = 0.5$) for the sets of instances B and P, respectively.

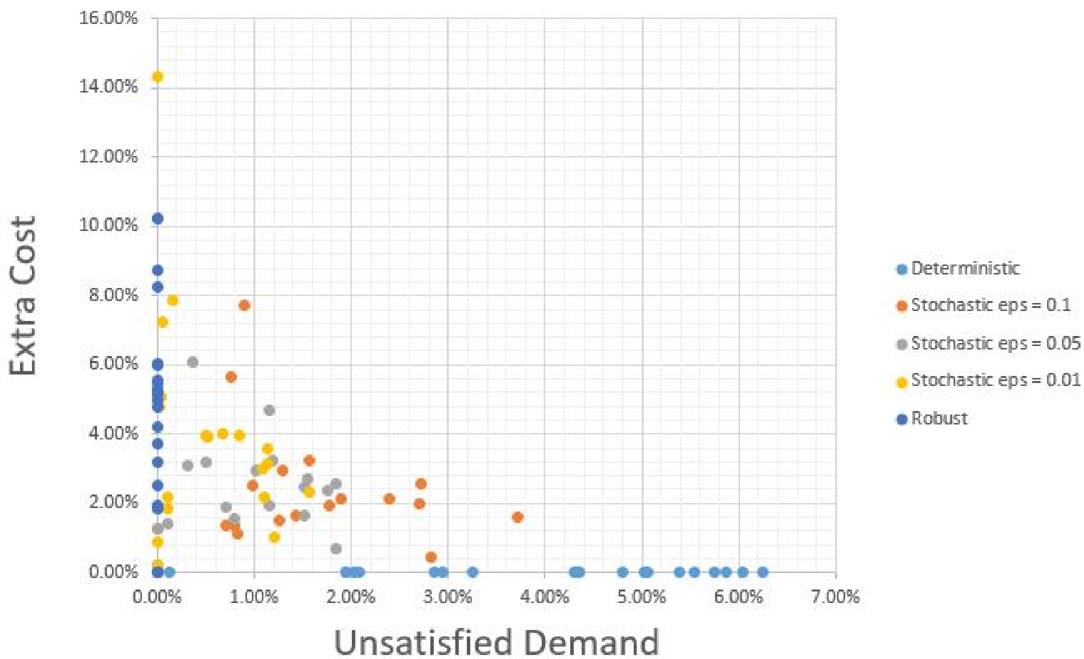

Figure 5: Set P - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

It is clear that while the set of instances B behaves in the same way as set of instances A, set P offers slightly more sparse solutions. This follows from the fact that only 21 instances belong to set P and some of them are characterized by a low number of routes. Therefore, it can be verified by looking at **Tables 32** and **33** in Appendix P.RvS that deterministic, stochastic and robust solutions do not differ at all from each other in some of these instances.

The tables and figures concerning the performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$ are shown in the Appendix. Specifically, **Table 11** and **Figure 8** in Appendix A.RvS regard set A, **Table 22** and **Figure 10** in Appendix B.RvS concern set B, finally **Table 33** and **Figure 12** in Appendix P.RvS represent set P.

Note that the stochastic and robust models are related by the parameter α, which defines both the n-dimensional hyperrectangle $[(1-\alpha)q^0, (1+\alpha)q^0]$ and the variance of the Normal distribution $\sigma^2 = \alpha \cdot q^0$. Nevertheless, for the smaller size of the uncertainty support, the stochastic solutions seem to become more prudent and consequently expensive, as **Figure 6** below depicts for set of instances A. For example, the stochastic routes obtained with $\epsilon = 0.01$ not only result in no unmet demand, but imply even greater routing costs than robust routes.

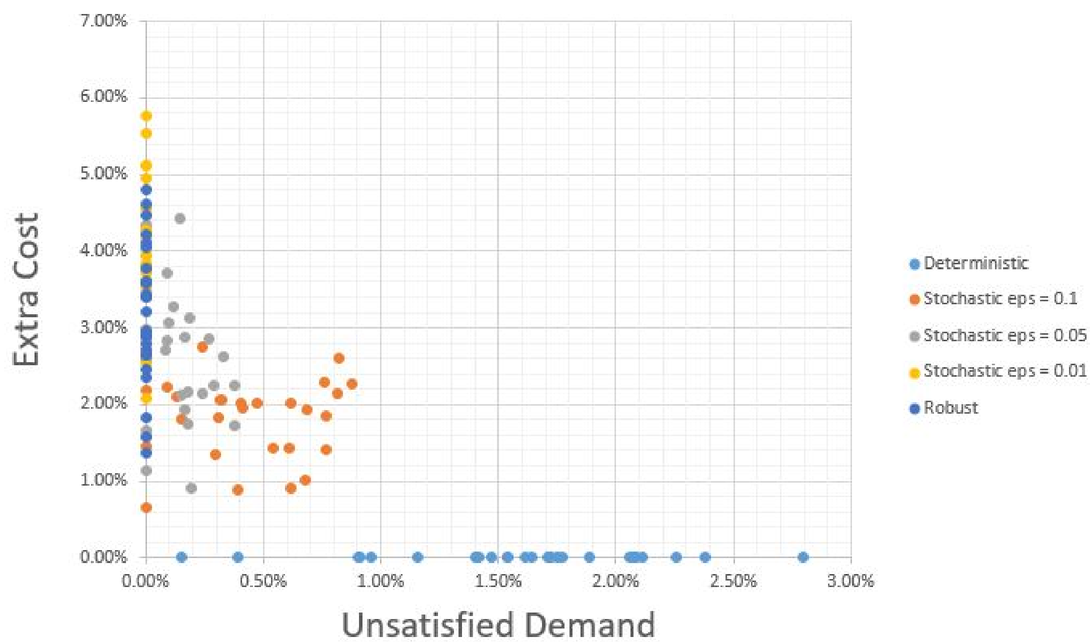

Figure 6: Set A - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

6.2.3 Sensitivity Analysis

The feasibility of the uncertainty set is investigated by performing a sensitivity analysis on the two parameters that define the support, namely α and β. The following uncertainty levels $\alpha = (0.05, 0.1)$ and $\beta = (0.5, 1)$ have been used in this study. Note that the case $\alpha = 0$ is equivalent to the deterministic scenario, independently of the value of β. Moreover, the case $\beta = 1$ corresponds to the rectangular support $[(1-\alpha)q^0, (1+\alpha)q^0]$.

Table 5 below shows the proportion of robust feasible instances for the values of α and β associated with the three sets of instances A, B and P. That is, the fraction of instances which could be solved without relaxing the constraint on the number of vehicles, within one hour of computational time.

It is clear that as the values of α and β increase, the fraction of feasible instances decreases. For example, the deterministic case $\alpha = 0$ corresponds to a degree of feasibility of 100%. Furthermore, rectangular and non-rectangular supports yield different levels of feasibility. In

Table 5: Impact of α and β on the proportion of feasible instances of set A (left panel), set B (central panel) and set P (right panel)

$\alpha \setminus \beta$	0.5	1
0.05	48 %	48 %
0.1	26 %	11 %

$\alpha \setminus \beta$	0.5	1
0.05	61 %	61 %
0.1	30 %	22 %

$\alpha \setminus \beta$	0.5	1
0.05	43 %	33 %
0.1	10 %	5 %

particular, the impact of β becomes more significant for higher values of α.

Table 6 illustrates the results of the sensitivity analysis associated with instance *B-n31-k5*, whose robust model remains feasible in all cases considered. The integer robust objective values are reported on the left panel. Additionally, the increase in terms of routing cost relative to the deterministic scenario is depicted on the right panel of **Table 6**.

Table 6: Integer robust solutions (left panel) and increase in the transportation cost (right panel) relative to the deterministic case ($\alpha = 0$) for instance B-n31-k5

$\alpha \setminus \beta$	0.5	1
0	680.24	680.24
0.05	684.74	693.15
0.1	704.1	714.43

$\alpha \setminus \beta$	0.5	1
0	0.00 %	0.00 %
0.05	0.66 %	1.89 %
0.1	3.51 %	5.03 %

It can be observed that robust solutions are slightly more expensive compared to their deterministic counterparts. In particular, the upper bounds obtained through IP, along with the lower bounds, provides solutions with monotonically increasing values as the size of the uncertainty set grows.

6.2.4 Computational Times

In this section the total running times of our proposed heuristics are investigated. **Table 7** and **Table 8** below report the average solving time of the column generation procedure in seconds over the three sets of instances. The time required to obtain the lower bound is depicted in **Table 7**, whereas **Table 8** shows the computation time associated with the IP heuristic, hence to obtain the upper bound. As mentioned earlier, two size levels of the uncertainty set are considered in the comparison of the total computational time between robust and stochastic models.

Table 7: Average running time (s) to get LBs in the robust and stochastic models for the 3 sets of instances for the values of α and β

Time (s)	$\alpha = 0.1$	$\beta = 0.5$			$\alpha = 0.05$	$\beta = 0.5$		
Instance	Robust	Stochastic			Robust	Stochastic		
Set		$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$		$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$
A	355.23	186.94	173.37	160.94	324.52	202.56	221.27	170.57
B	462.78	389.03	349.80	315.41	637.30	418.49	404.28	579.68
P	396.92	331.41	384.84	392.15	477.89	373.84	402.43	457.62

First, it can be seen that robust models require longer computation times than stochastic approaches, particularly for the set of instances A and B.

Secondly, we can conclude that the tightness of the constraints of the problem is not always directly proportional to the computation time. In fact, we can see that as ϵ decreases, that is, as the problem becomes more constrained, the running time of the stochastic model tend to decrease in some cases.

This can also be observed in the behaviour of the robust model associated with the two levels of uncertainty supports. More specifically, the computations performed with the largest uncertainty sets ($\alpha = 0.1$ and $\beta = 0.5$) need less time compared to the smaller support for set B and P. However, this can be partially explained by the fact that bigger uncertainty sets provide lower levels of feasibility and therefore a relaxation of constraint (11), which means that as the uncertainty set grows, fewer instances will incorporate the restriction.

Overall, it can be argued that there is a trade-off which needs to be balanced in our heuristics. In other words, if the problem is not constrained enough, then the algorithm generates an excessive number of routes. On the other hand, for highly constrained problems, it might take a lot of time to find feasible and consequently acceptable routes.

Table 8: Average running time (s) to get UBs in the robust and stochastic models for the 3 sets of instances for the values of α and β

Time (s)	$\alpha = 0.1$	$\beta = 0.5$			$\alpha = 0.05$	$\beta = 0.5$		
Instance	Robust	Stochastic			Robust	Stochastic		
Set		$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$		$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$
A	4.17	3.49	11.38	4.56	9.17	5.21	27.61	7.21
B	14.29	11.78	16.07	26.65	13.73	26.49	18.62	12.17
P	1.89	1.22	3.25	3.57	3.73	2.11	2.44	4.91

For what concerns the integer heuristics, we observe the lowest computation times in the stochastic model corresponding to $\epsilon = 0.1$ in most of the cases. Then, the running time tends to increase for higher levels of robustness, that is, as ϵ decreases.

Contrary to **Table 7**, **Table 8** proves that obtaining an upper bound in the robust approach does not require significantly higher amounts of time compared to the stochastic ones.

In addition, in line with the computational results of the deterministic scenario described in Section 6.1.2, our results suggest that the set of clustered instances B involves considerably longer running times than sets A and P. This can be explained by the massive number of routes created through the column generation process as illustrated in **Tables 23** to **30** in Appendix B.R and B.S.

7 Conclusion

Previous researches model the vehicle routing problem assuming fixed customer demand. In this paper we propose a column generation heuristic for the capacitated vehicle routing problem under demand uncertainty. In particular, a robust model is developed, where it is assumed that customers require the highest possible amount of goods. Moreover, the solutions of the robust approach are compared to a stochastic model, where the probability of successfully completing a route should be above a certain threshold. The comparison has been conducted by means of two performance measures, namely extra cost and unsatisfied demand. We assume customer demand to follow independent normal distributions in the stochastic model. On the other hand, in line with Gounaris et al. (2013) we consider uncertainty sets of the form of partitioned budget polytopes, where disjoint subsets are created based on the geographic quadrants.

Route feasibility of both approaches can be checked quite efficiently. Furthermore, we were able to further speed up the calculations with the help of naive dominance rules used in the pricing problem of the branch-and-price method. In fact, excluding the actual structure of the paths in the dynamic labelling algorithm can significantly decrease the running time of our algorithm, as deterministic results show. However, the solutions obtained are generally sub-optimal and not globally minimum.

Our heuristics are applied to three sets of small and medium-sized instances originating from standard CVRP benchmark problems. They designate random, clustered and modified instances from the literature (Augerat et al. 1995) with up to 80 customers and 15 vehicles.

The numerical results showed the effectiveness of our methods in terms of solution quality and computational time for what concerns the deterministic scenario. Additionally, the naive dominance relations have been tested against an alternative to the standard rules proposed by Feillet at al. (2004), where the concept of unreachable nodes is introduced. The efficiency of naive labelling is confirmed on a small subset of modified instances.

Regarding the uncertainty models, while the robust approach turned out to be the most conservative, the stochastic model offers more balanced, moderate and quick solutions. Robust solutions are indeed fairly expensive in terms of routing cost, even though they guarantee no unmet demand. On the contrary, stochastic solutions are somewhat halfway between deterministic and robust solutions. That is, as we increase the required success probability, the generated routes becomes more and more costly. Nevertheless, the amount of lost demand gradually decreases, in an opposite way. Depending on the specifics of the problem, more weight should be given to either additional cost or unmet demand. Therefore, stochastic solutions allow to adjust the trade-off of these two competing performance measures, whereas robust routes might seem too extreme.

Finally, a sensitivity analysis is performed on the parameters that define the uncertainty support in order to evaluate the increase in transportation cost and to assess the feasibility corresponding to the robust model. The computational time of the two approaches is also analyzed.

Future work might explore alternative uncertainty sets, such as more generic classes of polyhedra or without the requirement of disjoint budget subsets. For these new types of support it will be an interesting question how to develop efficient robust feasibility verification procedures and effective dominance relations in the route construction.

Our heuristics assume independent normal distributions. Therefore, in the context of stochastic programming, a challenging line of research could be to investigate correlated and/or conditional random demand distributions. For example, discrete random variables and scenario based distributions can represent practical settings in a more realistic way. Nevertheless, it is fundamental to adapt the dominance rules to these new distributions with the aim of obtaining optimal solutions.

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8 Appendix

The Appendix is organized as follows. There are 3 sections: A, B and P, which contain results corresponding to the sets of instances A, B and P. Each of these 3 sections has 4 subsections: D, RvS, R and S. They contain tables and figures associated with the deterministic (D), robust (R) and stochastic (S) models. Furthermore, subsections RvS offer comparisons between robust and stochastic models in terms of the performance measures. Empty cells mean that the given value could not be obtained, either due to infeasibility or because of missing data. For example, if only a lower bound can be found for a given instance, the performance measure corresponding to the unmet demand δ cannot be calculated, as it relies on the upper bound.

8.1 A

8.1.1 D

Table 9: Set A - Comparison with best known deterministic solution

Instance	z*	LB	UB	Gap*	Time (s)
A-n32-k5	784	760.14	816.25	4%	17.31
A-n33-k5	661	656.33	663.35	0%	12.22
A-n33-k6	742	728.31	744.45	0%	10.21
A-n34-k5*	778	742.72	790.78	2%	13.92
A-n36-k5*	799	776.94	826.54	3%	34.59
A-n37-k5	669	660.17	672.5	1%	26.44
A-n37-k6*	949	930.43	1003.28	6%	17.9
A-n38-k5*	730	706.27	746.12	2%	22.7
A-n39-k5*	822	803.77	833.14	1%	42.18
A-n39-k6	831	805.75	835.56	1%	33.89
A-n44-k6*	937	930.24	938.18	0%	34.71
A-n45-k6*	944	929.18	978.53	4%	42.79
A-n45-k7	1146	1115.71	1170.13	2%	59.93
A-n46-k7	914	904.64	918.95	1%	56.43
A-n48-k7*	1073	1050.48	1121.13	4%	112.48
A-n53-k7*	1010	999.73	1069.88	6%	135.82
A-n54-k7*	1167	1147.31	1196.33	3%	178.53
A-n55-k9*	1073	1059.03	1076.04	0%	54.89
A-n60-k9*	1354	1330.67	1384.56	2%	213.15
A-n61-k9*	1034	1014.61	1059.89	3%	111.7
A-n62-k8*	1288	1260.78	1331.04	3%	672.65
A-n63-k9*	1616	1592.68	1656.6	3%	287.09
A-n63-k10*	1314	1288.16	1328.6	1%	162.41
A-n64-k9*	1401	1376.86	1417.67	1%	436.59
A-n65-k9*	1174	1158.69	1193.44	2%	142.88
A-n69-k9*	1159	1132.35	1172.46	1%	229.73
A-n80-k10*	1763	1732.01	1802.02	2%	1724.82
avg.				2%	181.04

8.1.2 RvS

Table 10: Set A - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
A-n32-k5	5.24%	2.43%	2.32%	1.02%	4.96%	0.29%	5.74%	5.93%
A-n33-k5	5.00%	1.77%	1.13%	0.76%	2.13%	0.00%	4.29%	4.51%
A-n33-k6	3.32%	2.51%	4.43%	0.99%	5.77%	0.18%	6.56%	9.68%
A-n34-k5*	5.32%	0.98%	1.84%	0.50%	2.80%	0.26%	3.48%	4.63%
A-n36-k5*	3.71%	1.65%	1.84%	0.77%	4.15%	0.00%	5.36%	6.06%
A-n37-k5	4.67%	2.21%	0.89%	0.00%	2.62%	0.29%	3.96%	4.75%
A-n37-k6*	3.74%	1.21%	2.57%	0.82%	3.83%	0.44%	4.42%	7.47%
A-n38-k5*	4.16%	1.41%	1.53%	0.25%	2.54%	0.25%	3.30%	5.26%
A-n39-k5*	2.16%	2.16%	1.95%	0.00%	3.61%	0.00%	5.64%	6.37%
A-n39-k6	1.50%	1.52%	2.11%	0.95%	3.84%	0.02%	4.75%	5.37%
A-n44-k6*	5.30%	1.37%	1.69%	1.02%	2.72%	0.42%	4.73%	5.44%
A-n45-k6*	4.52%	2.09%	2.65%	0.59%	3.65%	0.22%	4.08%	5.24%
A-n45-k7	4.00%	3.31%	1.73%	1.40%	3.60%	0.41%	6.34%	11.81%
A-n46-k7	6.29%	2.06%	2.98%	0.20%	3.77%	0.22%	4.87%	7.09%
A-n48-k7*	6.10%	2.60%	2.61%	2.06%	3.92%	0.42%	6.24%	7.87%
A-n53-k7*	5.05%	2.36%	2.38%	1.55%	3.78%	0.36%	5.33%	6.87%
A-n54-k7*	3.51%	2.55%	3.24%	1.20%	5.17%	0.19%	6.75%	7.91%
A-n55-k9*	5.98%	2.43%	2.22%	0.98%	3.74%	0.44%	4.35%	6.31%
A-n60-k9*	5.91%	2.17%	2.17%	1.11%	3.94%	0.17%	5.49%	8.10%
A-n61-k9*	5.55%	3.06%	2.82%	1.01%	4.23%	0.17%	5.29%	7.25%
A-n62-k8*	4.42%	2.92%	2.78%	1.09%	4.89%	0.00%	6.28%	7.70%
A-n63-k9*	5.64%	3.34%	3.71%	1.31%	5.10%	0.27%	6.92%	8.67%
A-n63-k10*	4.56%	2.08%	2.72%	1.35%	4.48%	0.27%	6.06%	7.99%
A-n64-k9*	5.13%	2.52%	3.18%	0.68%	4.46%	0.42%	6.48%	7.83%
A-n65-k9*	6.36%	1.04%	4.23%	0.80%	5.43%	0.29%	7.16%	8.48%
A-n69-k9*	5.94%	1.63%	3.07%	0.69%	3.71%	0.28%	5.33%	6.61%
A-n80-k10*	6.03%	2.27%	2.88%	1.23%	4.20%	0.27%	5.95%	7.75%
avg.	4.78%	2.14%	2.51%	0.90%	3.96%	0.24%	5.38%	7.00%

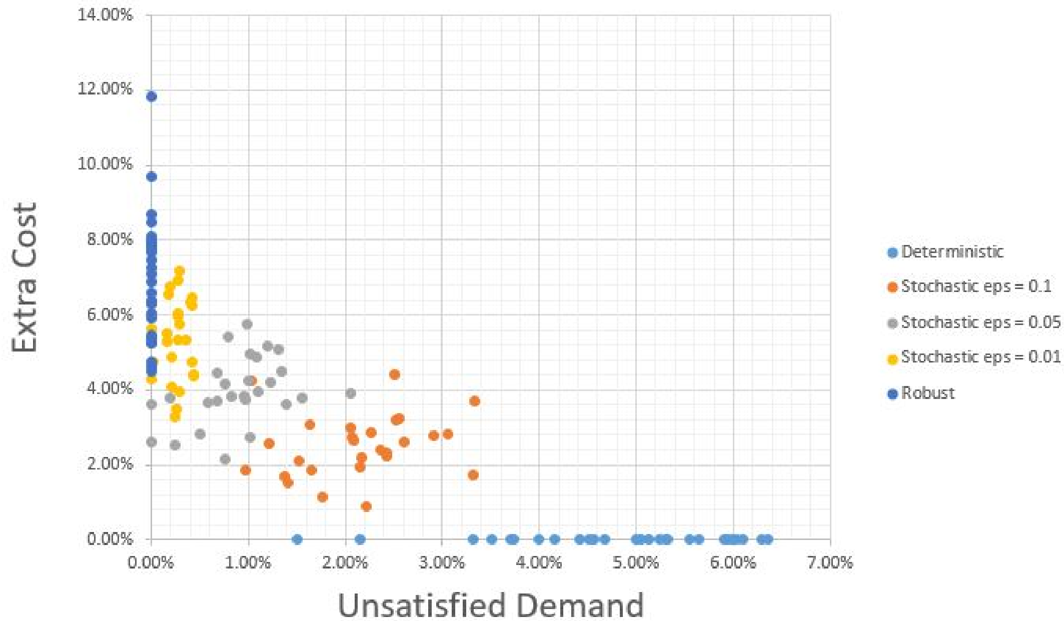

Figure 7: Set A - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

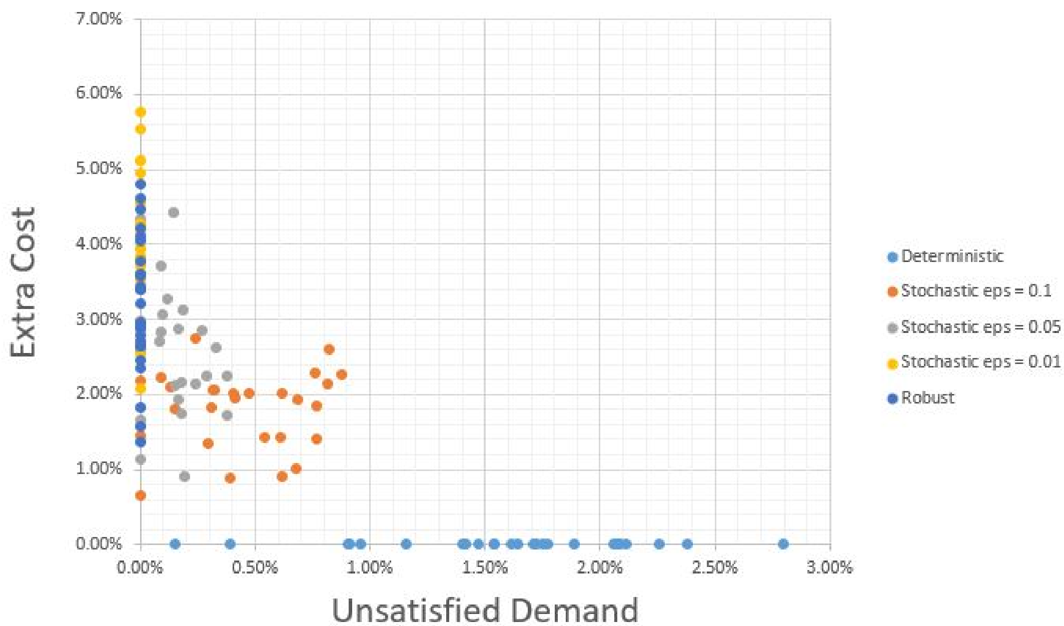

Figure 8: Set A - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

Table 11: Set A - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
A-n32-k5	1.89%	0.62%	2.02%	0.18%	2.16%	0.00%	4.96%	4.11%
A-n33-k5	1.72%	0.00%	0.66%	0.00%	1.13%	0.00%	2.08%	1.38%
A-n33-k6	0.96%	0.68%	1.94%	0.15%	4.43%	0.00%	5.77%	4.79%
A-n34-k5	2.06%	0.40%	2.01%		2.43%		4.44%	4.47%
A-n36-k5	0.90%	0.29%	1.35%	0.18%	1.74%	0.00%	3.94%	2.71%
A-n37-k5	0.91%	0.39%	0.88%	0.20%	0.91%	0.00%	2.94%	2.35%
A-n37-k6*	1.16%	0.32%	2.06%	0.00%	2.57%	0.00%	3.83%	2.64%
A-n38-k5*	2.08%	0.00%	1.44%	0.00%	1.57%	0.00%	2.54%	1.57%
A-n39-k5*	0.39%	0.61%	1.44%	0.17%	1.93%	0.00%	3.62%	2.87%
A-n39-k6	0.15%	0.15%	1.81%	0.15%	2.11%	0.00%	3.80%	2.47%
A-n44-k6*	2.08%	0.61%	0.91%	0.00%	1.67%	0.00%	2.72%	1.84%
A-n45-k6*	1.64%	0.13%	2.10%	0.27%	2.85%	0.00%	3.54%	2.96%
A-n45-k7	1.75%	0.54%	1.44%	0.38%	1.73%	0.00%	3.60%	2.65%
A-n46-k7	2.79%	0.88%	2.27%	0.00%	2.98%	0.00%	3.77%	3.59%
A-n48-k7	1.73%	0.00%	2.18%	0.00%	2.62%	0.00%	3.95%	3.41%
A-n53-k7*	1.47%	0.68%	1.02%	0.24%	2.14%	0.00%	3.75%	2.89%
A-n54-k7*	1.61%	0.24%	2.76%	0.12%	3.28%	0.00%	5.12%	3.77%
A-n55-k9	1.54%	0.41%	1.95%	0.38%	2.24%			3.41%
A-n60-k9	2.26%	0.77%	1.42%	0.29%	2.25%			2.80%
A-n61-k9*	2.11%	0.09%	2.22%	0.09%	2.83%	0.00%	4.25%	2.91%
A-n62-k8	1.41%	0.47%	2.02%	0.33%	2.63%			4.05%
A-n63-k9*	1.54%	0.82%	2.60%	0.09%	3.71%	0.00%	5.12%	4.23%
A-n63-k10*	1.42%	0.77%	1.86%	0.09%	2.71%	0.00%	4.28%	3.22%
A-n64-k9*	1.71%	0.31%	1.82%	0.19%	3.14%	0.00%	4.54%	3.60%
A-n65-k9*	1.77%	0.82%	2.14%	0.00%	4.34%	0.00%	5.53%	4.62%
A-n69-k9*	2.09%	0.31%	2.05%	0.10%	3.06%	0.00%	3.72%	3.45%
A-n80-k10*	2.38%	0.76%	2.30%	0.17%	2.88%	0.00%	4.20%	3.58%
avg.	1.61%	0.45%	1.80%	0.14%	2.52%	0.00%	4.00%	3.20%

8.1.3 R

Table 12: Set A - Robust solutions for the case $\alpha = 0.1$ and $\beta = 0.5$

	Robust						
						Time (s)	
Instance	LB	κ_r	UB	#columns	#iter	IP	Total
A-n32-k5	805.19	5.93%	849.02	1196	61	0.35	21.6
A-n33-k5	685.93	4.51%	686.86	1218	71	0.2	67.33
A-n33-k6	798.84	9.68%	814.21	1332	94	0.15	262.64
A-n34-k5*	777.08	4.63%	824.59	1044	71	0.45	20.4
A-n36-k5*	824.05	6.06%	901.67	1294	68	0.7	49.48
A-n37-k5	691.56	4.75%	724.92	2015	73	0.61	85.82
A-n37-k6*	999.93	7.47%	1033.63	879	56	0.38	28.94
A-n38-k5*	743.44	5.26%	791.66	1163	61	0.59	25.79
A-n39-k5*	854.94	6.37%	870.01	1397	64	0.17	57.82
A-n39-k6	849.04	5.37%	1115.47	1960	84	2.82	153.88
A-n44-k6*	980.81	5.44%	1028.88	1324	72	0.83	58.21
A-n45-k6*	977.87	5.24%	1014.49	1467	85	0.31	55.01
A-n45-k7	1247.51	11.81%	1279.74	2181	158	51.17	2847.22
A-n46-k7	968.81	7.09%	1077.12	2386	95	2.25	136.63
A-n48-k7*	1133.18	7.87%	1198.86	1634	85	0.84	108.44
A-n53-k7*	1068.45	6.87%	1131.66	2206	104	3.6	156.65
A-n54-k7*	1238.09	7.91%	1242.5	2151	117	0.2	267.33
A-n55-k9*	1125.87	6.31%	1158.28	1644	83	0.53	116.32
A-n60-k9*	1438.45	8.10%	1547.54	2903	110	7.67	287.21
A-n61-k9*	1088.12	7.25%	1150.51	1925	101	2.66	213.8
A-n62-k8*	1357.88	7.70%	1421.34	3667	163	4.7	802.97
A-n63-k9*	1730.73	8.67%	1861.26	2696	127	16.09	430.39
A-n63-k10*	1391.14	7.99%	1430.56	2261	98	1.72	234.98
A-n64-k9*	1484.63	7.83%	1564.37	2929	138	6.09	442.4
A-n65-k9*	1256.97	8.48%	1295.57	2202	109	1.49	202.13
A-n69-k9*	1207.21	6.61%	1250.36	2962	122	1.87	393.81
A-n80-k10*	1866.24	7.75%	1930.99	4903	200	4.19	2199.01
avg.	1096.00	7.00%	1155.41	2035	99	4.17	360.23

Table 13: Set A - Robust solutions for the case $\alpha = 0.05$ and $\beta = 0.5$

	Robust						
						Time (s)	
Instance	LB	κ_r	UB	#columns	#iter	IP	Total
A-n32-k5	791.39	4.11%	869.77	1176	59	1.11	33.84
A-n33-k5	665.36	1.38%	684.86	1160	63	0.29	24.29
A-n33-k6	763.22	4.79%	767.51	990	53	0.16	30.9
A-n34-k5	775.91	4.47%	850.05	1261	57	0.21	71.96
A-n36-k5	797.96	2.71%	853.53	1683	80	0.65	61.98
A-n37-k5	675.7	2.35%	712.17	1556	63	0.42	44.42
A-n37-k6*	954.95	2.64%	992.86	932	58	0.17	29.65
A-n38-k5*	717.35	1.57%	756.29	1277	89	0.29	34.86
A-n39-k5*	826.85	2.87%	844.27	1451	70	0.12	63.81
A-n39-k6	825.63	2.47%	859.08	1652	75	0.43	64.37
A-n44-k6*	947.32	1.84%	956.25	1639	98	0.15	76.69
A-n45-k6*	956.69	2.96%	999.33	1517	99	0.58	65.56
A-n45-k7	1145.27	2.65%	1169.73	1840	86	0.39	113.14
A-n46-k7	937.14	3.59%	980.02	2109	87	0.95	80.52
A-n48-k7	1086.29	3.41%	1115.92	2069	106	1.64	151.73
A-n53-k7*	1028.65	2.89%	1081.02	2476	108	1.76	194.35
A-n54-k7*	1190.62	3.77%	1220.98	2454	107	0.89	267.23
A-n55-k9	1095.12	3.41%	1186.17	2559	106	2.69	296.39
A-n60-k9	1367.89	2.80%	1620.42	3493	125	79.45	651.2
A-n61-k9*	1044.17	2.91%	1093.16	2360	112	1.53	216.32
A-n62-k8	1311.88	4.05%	1581.24	5337	169	42.98	1600.25
A-n63-k9*	1659.99	4.23%	1712	2873	110	1.02	449.57
A-n63-k10*	1329.62	3.22%	1354.38	2280	109	0.6	254.92
A-n64-k9*	1426.49	3.60%	1481.95	3101	139	5.49	600.81
A-n65-k9*	1212.22	4.62%	1285.01	2552	110	6.75	270.36
A-n69-k9*	1171.39	3.45%	1254.5	3208	139	11.44	427.01
A-n80-k10*	1794	3.58%	1907.47	5319	216	85.54	2855.87
avg.	1055.52	3.20%	1118.14	2234	100	9.17	334.51

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Table 14: Set A - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.1$

	Stochastic with $\epsilon = 0.1$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
A-n32-k5	777.78	2.32%	853.37	2.43%	1408	70	0.49	19.42
A-n33-k5	663.73	1.13%	680.26	1.77%	1014	58	0.18	13.67
A-n33-k6	760.59	4.43%	801.73	2.51%	906	47	0.31	15.64
A-n34-k5*	756.36	1.84%	818.48	0.98%	1108	75	0.57	16.77
A-n36-k5*	791.27	1.84%	829.88	1.65%	1306	77	0.32	37.03
A-n37-k5	666.06	0.89%	687.63	2.21%	1838	93	0.7	35.02
A-n37-k6*	954.38	2.57%	998.95	1.21%	967	57	0.24	17.54
A-n38-k5*	717.08	1.53%	753.19	1.41%	1234	83	0.27	19.57
A-n39-k5*	819.43	1.95%	838.61	2.16%	1589	80	0.25	48.8
A-n39-k6	822.79	2.11%	859.94	1.52%	1758	79	0.52	46.16
A-n44-k6*	945.96	1.69%	956.25	1.37%	1434	81	0.17	37.67
A-n45-k6*	953.83	2.65%	973.49	2.09%	1445	78	0.34	30.89
A-n45-k7	1135.04	1.73%	1170.33	3.31%	2037	99	0.64	66.7
A-n46-k7	931.63	2.98%	990.58	2.06%	2052	93	2.35	62.38
A-n48-k7*	1077.85	2.61%	1116.79	2.60%	2092	102	0.59	89.77
A-n53-k7*	1023.52	2.38%	1061.78	2.36%	2749	118	1.7	153.48
A-n54-k7*	1184.5	3.24%	1195.99	2.55%	2416	103	0.44	166.89
A-n55-k9*	1082.57	2.22%	1119.06	2.43%	1736	102	0.89	67.8
A-n60-k9*	1359.61	2.17%	1401.98	2.17%	3297	135	1.53	248.79
A-n61-k9*	1043.26	2.82%	1084	3.06%	2129	108	1.6	131.47
A-n62-k8*	1295.88	2.78%	1368.72	2.92%	3556	146	3.93	526.35
A-n63-k9*	1651.82	3.71%	1739.11	3.34%	3182	141	12.51	325.54
A-n63-k10*	1323.24	2.72%	1351.43	2.08%	2229	105	1.16	163.21
A-n64-k9*	1420.67	3.18%	1465.46	2.52%	3407	132	1.98	436.71
A-n65-k9*	1207.69	4.23%	1278.21	1.04%	2549	107	8.41	164.01
A-n69-k9*	1167.07	3.07%	1237.49	1.63%	3396	129	5.06	312.34
A-n80-k10*	1781.87	2.88%	1878.14	2.27%	5571	234	47.09	1901.68
avg.	1048.72	2.51%	1092.99	2.14%	2163	101	3.49	190.94

Table 15: Set A - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.05$

	Stochastic with $\epsilon = 0.05$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
A-n32-k5	797.85	4.96%	798.86	1.02%	1134	54	0.17	15.86
A-n33-k5	670.34	2.13%	682.33	0.76%	1094	63	0.22	16.36
A-n33-k6	770.36	5.77%	775.33	0.99%	1166	54	0.2	23.09
A-n34-k5*	763.49	2.80%	823.99	0.50%	898	73	0.35	13.67
A-n36-k5*	809.18	4.15%	900.59	0.77%	1361	68	1.42	31.77
A-n37-k5	677.46	2.62%	716.65	0.00%	1576	75	0.51	28.49
A-n37-k6*	966.08	3.83%	996.32	0.82%	1002	61	0.12	16.3
A-n38-k5*	724.24	2.54%	765.26	0.25%	1228	80	0.17	18.43
A-n39-k5*	832.82	3.61%	846.54	0.00%	1553	78	0.17	43.28
A-n39-k6	836.69	3.84%	842.92	0.95%	1825	79	0.25	52.77
A-n44-k6*	955.5	2.72%	966.23	1.02%	1455	92	0.17	37.96
A-n45-k6*	963.08	3.65%	1016.79	0.59%	1306	76	0.38	27.23
A-n45-k7	1155.82	3.60%	1190.4	1.40%	2186	90	1.51	93.86
A-n46-k7	938.77	3.77%	985.2	0.20%	2303	106	1.16	72.15
A-n48-k7*	1091.7	3.92%	1114.76	2.06%	1663	96	0.24	75.99
A-n53-k7*	1037.56	3.78%	1100.35	1.55%	3037	119	1.81	170.42
A-n54-k7*	1206.59	5.17%	1228.45	1.20%	2471	106	0.85	169.52
A-n55-k9*	1098.59	3.74%	1119.4	0.98%	1739	92	0.42	63.77
A-n60-k9*	1383.15	3.94%	1448.64	1.11%	2890	126	9.78	216.7
A-n61-k9*	1057.57	4.23%	1107.45	1.01%	2245	104	1.98	132.72
A-n62-k8*	1322.49	4.89%	1413.12	1.09%	3555	147	11.46	500.35
A-n63-k9*	1673.96	5.10%	1741.3	1.31%	2807	156	9.47	291.92
A-n63-k10*	1345.91	4.48%	1372.31	1.35%	2375	117	1.04	167.18
A-n64-k9*	1438.29	4.46%	1531.7	0.68%	3170	150	65.79	465.2
A-n65-k9*	1221.64	5.43%	1288.76	0.80%	2410	103	2.23	142.97
A-n69-k9*	1174.31	3.71%	1255.5	0.69%	3356	130	10.87	388.86
A-n80-k10*	1804.77	4.20%	1910.97	1.23%	5353	220	184.54	1728.14
avg.	1063.64	3.97%	1108.89	0.90%	2117	101	11.38	185.37

Table 16: Set A - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.01$

	Stochastic with $\epsilon = 0.01$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
A-n32-k5	803.75	5.74%	803.75	0.29%	1193	67	0.14	14.78
A-n33-k5	684.47	4.29%	690.93	0.00%	1153	58	0.22	25.93
A-n33-k6	776.08	6.56%	785.79	0.18%	1177	65	0.17	49.26
A-n34-k5*	768.56	3.48%	816.16	0.26%	970	70	0.21	12.75
A-n36-k5*	818.6	5.36%	895.68	0.00%	1331	71	0.68	29.33
A-n37-k5	686.34	3.96%	719.17	0.29%	1490	69	0.45	27.68
A-n37-k6*	971.6	4.42%	997.72	0.44%	997	66	0.14	18.07
A-n38-k5*	729.55	3.30%	764.25	0.25%	1137	79	0.18	15.84
A-n39-k5*	849.11	5.64%	873.38	0.00%	1314	70	0.4	39.91
A-n39-k6	843.99	4.75%	852.99	0.02%	1659	79	0.34	68.74
A-n44-k6*	974.23	4.73%	1005.36	0.42%	1314	79	0.39	32.24
A-n45-k6*	967.12	4.08%	991.14	0.22%	1501	86	0.23	30.89
A-n45-k7	1186.43	6.34%	1214.59	0.41%	2154	96	1.6	225.38
A-n46-k7	948.7	4.87%	981.21	0.22%	2161	121	2.14	89.05
A-n48-k7*	1116.08	6.24%	1144.48	0.42%	1792	82	0.36	71.63
A-n53-k7*	1053	5.33%	1147.94	0.36%	2583	106	1.11	142.15
A-n54-k7*	1224.71	6.75%	1250.24	0.19%	2642	109	0.99	194.22
A-n55-k9*	1105.07	4.35%	1121.96	0.44%	1743	94	0.23	58.07
A-n60-k9*	1403.67	5.49%	1503.26	0.17%	2729	110	2.39	163.08
A-n61-k9*	1068.24	5.29%	1116.01	0.17%	2211	106	1.7	132.09
A-n62-k8*	1340.02	6.28%	1409.32	0.00%	3528	155	11.67	499.22
A-n63-k9*	1702.82	6.92%	1836.52	0.27%	2622	123	60.73	292.03
A-n63-k10*	1366.26	6.06%	1410.46	0.27%	2231	102	1.64	151.11
A-n64-k9*	1466.02	6.48%	1530.54	0.42%	3298	146	2.08	386.06
A-n65-k9*	1241.68	7.16%	1325.53	0.29%	2337	109	14.16	144.81
A-n69-k9*	1192.7	5.33%	1250.45	0.28%	3247	127	3.59	280.99
A-n80-k10*	1835.06	5.95%	1931.93	0.27%	4925	196	15.1	1285.16
avg.	1078.66	5.38%	1124.84	0.24%	2053	98	4.56	165.94

Table 17: Set A - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.1$

	Stochastic with $\epsilon = 0.1$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
A-n32-k5	775.48	2.02%	840.9	0.62%	1161	61	0.54	19.93
A-n33-k5	660.65	0.66%	685.7	0.00%	967	55	0.22	15.24
A-n33-k6	742.41	1.94%	749.99	0.68%	893	49	0.14	13.1
A-n34-k5	765.19	2.01%	791.07	0.40%	1347	71	0.27	36.62
A-n36-k5	788.69	1.35%	814.65	0.29%	1500	73	0.4	50.27
A-n37-k5	665.95	0.88%	681.44	0.39%	1707	87	0.34	40.65
A-n37-k6*	949.63	2.06%	1000.93	0.32%	938	54	0.46	20.52
A-n38-k5*	716.47	1.44%	756.29	0.00%	1257	79	0.29	25.43
A-n39-k5*	815.33	1.44%	840.26	0.61%	1513	77	0.43	58.93
A-n39-k6	820.33	1.81%	835.28	0.15%	1821	73	0.55	50.78
A-n44-k6*	938.66	0.91%	941.61	0.61%	1644	94	0.28	50.7
A-n45-k6*	948.66	2.10%	981.84	0.13%	1554	80	0.39	39.27
A-n45-k7	1131.77	1.44%	1158.14	0.54%	1992	90	0.38	71.93
A-n46-k7	925.22	2.27%	949.34	0.88%	2206	94	0.79	69.85
A-n48-k7	1073.27	2.18%	1116.29	0.00%	2132	106	1.01	112.25
A-n53-k7*	1009.9	1.02%	1043.23	0.68%	2972	133	1.45	156.75
A-n54-k7*	1178.95	2.76%	1205.68	0.24%	2314	100	0.79	143.97
A-n55-k9	1079.47	1.95%	1265.08	0.41%	2669	111	54.55	192.58
A-n60-k9	1349.65	1.42%	1362.19	0.77%	3461	131	1.01	322.04
A-n61-k9*	1037.12	2.22%	1080.4	0.09%	2144	103	1.23	107.57
A-n62-k8	1286.81	2.02%	1352.25	0.47%	5235	201	16.62	963.81
A-n63-k9*	1634.05	2.60%	1676.74	0.82%	2738	132	1.11	287.73
A-n63-k10*	1312.08	1.86%	1334.54	0.77%	2462	103	0.4	185.35
A-n64-k9*	1401.93	1.82%	1500.67	0.31%	3312	138	15.26	409.04
A-n65-k9*	1183.43	2.14%	1236.83	0.82%	2743	127	2.05	180.72
A-n69-k9*	1155.61	2.05%	1201.5	0.31%	3403	130	2.17	287.12
A-n80-k10*	1771.81	2.30%	1851.54	0.76%	4907	209	37.6	1718.99
avg.	1041.43	1.80%	1083.50	0.45%	2259	102	5.21	208.56

Table 18: Set A - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.05$

	Stochastic with $\epsilon = 0.05$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
A-n32-k5	776.58	2.16%	831.64	0.18%	1393	71	0.56	23.85
A-n33-k5	663.73	1.13%	687.72	0.00%	1059	56	0.33	15.36
A-n33-k6	760.58	4.43%	801.73	0.15%	905	47	0.32	16.51
A-n34-k5	768.34	2.43%			1405	68		
A-n36-k5	791.69	1.74%	827.6	0.18%	1525	84	0.53	56.17
A-n37-k5	666.15	0.91%	687.63	0.20%	1788	91	0.36	42.65
A-n37-k6*	954.38	2.57%	998.95	0.00%	969	57	0.3	23.84
A-n38-k5*	717.35	1.57%	753.19	0.00%	1361	88	0.17	24.24
A-n39-k5*	819.32	1.93%	838.61	0.17%	1626	86	0.28	62.51
A-n39-k6	822.79	2.11%	859.94	0.15%	1766	80	0.55	56.55
A-n44-k6*	945.76	1.67%	956.25	0.00%	1498	95	0.38	48.78
A-n45-k6*	955.67	2.85%	985.9	0.27%	1408	84	0.42	37.23
A-n45-k7	1135.04	1.73%	1161.3	0.38%	2137	100	0.64	76.5
A-n46-k7	931.63	2.98%	984.62	0.00%	2085	90	1.97	66.06
A-n48-k7	1077.85	2.62%	1126.16	0.00%	2539	102	0.73	150.11
A-n53-k7*	1021.17	2.14%	1066.16	0.24%	2658	116	1.27	127.94
A-n54-k7*	1184.96	3.28%	1195.99	0.12%	2296	100	0.47	145.61
A-n55-k9	1082.6	2.24%	1118.59	0.38%	2706	121	4.35	167.78
A-n60-k9	1360.69	2.25%	1482.26	0.29%	4076	141	357.15	879.84
A-n61-k9*	1043.35	2.83%	1077	0.09%	2133	108	0.73	107.43
A-n62-k8	1294.53	2.63%	1473.5	0.33%	5218	198	283.65	1355.5
A-n63-k9*	1651.84	3.71%	1727.11	0.09%	2904	126	3.67	274.85
A-n63-k10*	1323.11	2.71%	1348.98	0.09%	2319	120	1.29	157.88
A-n64-k9*	1420.04	3.14%	1467.88	0.19%	3310	149	1.88	421.62
A-n65-k9*	1208.98	4.34%	1265.04	0.00%	2747	118	1.9	144.6
A-n69-k9*	1167	3.06%	1225.24	0.10%	3256	129	2.12	255.63
A-n80-k10*	1781.88	2.88%	1884.07	0.17%	5373	210	51.87	1742
avg.	1049.15	2.52%	1108.96	0.14%	2313	105	27.61	249.27

Table 19: Set A - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.01$

	Stochastic with $\epsilon = 0.01$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
A-n32-k5	797.84	4.96%	798.85	0.00%	1139	55	0.17	18.99
A-n33-k5	670.01	2.08%	681.22	0.00%	1104	62	0.26	17.88
A-n33-k6	770.36	5.77%	775.32	0.00%	1169	54	0.22	27.62
A-n34-k5	783.43	4.44%			1402	74		
A-n36-k5	808.8	3.94%	916.53	0.00%	1682	77	2.29	68.73
A-n37-k5	679.58	2.94%	699.93	0.00%	1583	72	0.39	37.88
A-n37-k6*	966.08	3.83%	996.32	0.00%	1010	61	0.13	18.85
A-n38-k5*	724.24	2.54%	765.26	0.00%	1235	80	0.31	23.33
A-n39-k5*	832.84	3.62%	844.7	0.00%	1553	81	0.62	51.13
A-n39-k6	836.37	3.80%	842.92	0.00%	1667	76	0.36	61.34
A-n44-k6*	955.5	2.72%	966.23	0.00%	1455	91	0.2	43.8
A-n45-k6*	962.03	3.54%	1016.76	0.00%	1341	79	0.75	33.16
A-n45-k7	1155.82	3.60%	1198.03	0.00%	2155	89	0.6	91.06
A-n46-k7	938.77	3.77%	982.8	0.00%	2418	118	1.91	81.47
A-n48-k7	1091.78	3.95%	1114.76	0.00%	2495	123	0.49	151.57
A-n53-k7*	1037.23	3.75%	1097.97	0.00%	2855	112	1.75	135.47
A-n54-k7*	1206.04	5.12%	1228.45	0.00%	2492	113	0.63	150.02
A-n55-k9								
A-n60-k9								
A-n61-k9*	1057.76	4.25%	1116.76	0.00%	2189	103	2.11	105.73
A-n62-k8								
A-n63-k9*	1674.22	5.12%	1741.3	0.00%	2780	142	6.37	288.76
A-n63-k10*	1343.24	4.28%	1370.78	0.00%	2228	101	0.93	122.33
A-n64-k9*	1439.43	4.54%	1548.41	0.00%	3005	141	31.67	362.5
A-n65-k9*	1222.73	5.53%	1287.38	0.00%	2277	105	1.63	115.15
A-n69-k9*	1174.52	3.72%	1259.27	0.00%	3253	133	12.49	324.66
A-n80-k10*	1804.79	4.20%	1912.49	0.00%	5036	206	99.51	1775.73
avg.	1038.89	4.00%	1094.02	0.00%	2063	98	7.21	178.57

8.2 B

8.2.1 D

Table 20: Set B - Comparison with best known deterministic solution

Instance	z*	LB	UB	Gap*	Time (s)
B-n31-k5	672	619.27	680.24	1%	50.54
B-n34-k5*	788	749.14	794.85	1%	59.28
B-n35-k5	955	829.78	986.09	3%	98.42
B-n38-k6	805	719.5	841.95	5%	54.51
B-n39-k5	549	523.26	569.64	4%	130.38
B-n41-k6*	829	801.3	851.65	3%	63.67
B-n43-k6	742	712.11	769.47	4%	119.21
B-n44-k7*	909	867.52	981.53	8%	99.55
B-n45-k5*	751	692.18	774.94	3%	92.53
B-n45-k6*	678	658.25	733.83	8%	74.01
B-n50-k7	741	670.62	764.15	3%	174.61
B-n50-k8*	1312	1259.07	1337.32	2%	315.74
B-n51-k7*	1032	956.9	1021.26	-1%	173.61
B-n52-k7	747	682.44	831.9	11%	472.79
B-n56-k7*	707	639.01	724.11	2%	388.73
B-n57-k7*	1153	1097.07	1143.32	-1%	1035.6
B-n57-k9*	1598	1507.96	1664.52	4%	532.72
B-n63-k10*	1496	1451.75	1585.37	6%	527.21
B-n64-k9*	861	815.38	954.94	11%	300.01
B-n66-k9*	1316	1259.39	1378.45	5%	958.01
B-n67-k10*	1032	993.98	1098.54	6%	434.29
B-n68-k9*	1272	1194.69	1333.57	5%	1059.12
B-n78-k10*	1221	1169.71	1324.67	8%	1378.31
avg.				4%	373.6

8.2.2 RvS

Table 21: Set B - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
B-n31-k5	3.28%	2.21%	2.58%	1.19%	4.80%	0.02%	6.63%	7.52%
B-n34-k5	5.20%	2.07%	2.50%	1.01%	3.48%	0.31%	6.05%	7.07%
B-n35-k5	5.80%	0.00%	4.19%	0.00%	6.33%	0.00%	8.05%	9.66%
B-n38-k6	4.20%	1.37%	3.14%	0.47%	4.27%	0.00%	5.66%	7.66%
B-n39-k5	2.22%	2.32%	1.87%	0.69%	3.52%	0.27%	5.00%	6.13%
B-n41-k6*	4.40%	1.29%	2.73%	0.21%	4.18%	0.21%	5.56%	6.75%
B-n43-k6	3.44%	1.94%	2.40%	0.00%	4.34%	0.27%	6.12%	7.34%
B-n44-k7	6.26%	3.85%	3.23%	0.89%	4.60%	0.19%	6.71%	8.39%
B-n45-k5*	2.50%	1.72%	3.26%	0.00%	3.57%	0.02%	4.03%	4.35%
B-n45-k6*	2.84%	1.74%	1.96%	1.35%	3.02%	0.00%	6.27%	6.26%
B-n50-k7	2.42%	1.17%	2.17%	0.38%	3.55%	0.00%	4.82%	5.86%
B-n50-k8	6.52%	3.36%	3.07%	1.58%	4.89%	0.53%	6.70%	7.91%
B-n51-k7*	4.42%	1.67%	4.24%	0.70%	5.06%	0.00%	6.38%	9.06%
B-n52-k7	2.48%	1.30%	2.86%	0.76%	4.30%	0.00%	5.49%	6.11%
B-n56-k7	8.31%	2.74%	2.50%	1.14%	3.99%	0.19%	5.34%	7.21%
B-n57-k7*	4.78%	1.77%	3.46%	1.48%	4.70%	0.20%	7.05%	8.49%
B-n57-k9	4.87%	2.28%	2.86%	1.35%	4.73%	0.32%	7.39%	8.98%
B-n63-k10	3.29%	2.59%	4.00%	1.51%	5.54%	0.15%	7.39%	9.77%
B-n64-k9*	3.94%	1.42%	2.66%	0.27%	4.20%	0.15%	5.56%	7.36%
B-n66-k9*	3.15%	3.27%	2.97%	0.95%	4.83%	0.45%	6.41%	8.71%
B-n67-k10	4.22%	1.38%	3.66%	1.40%	4.88%	0.26%	7.05%	8.00%
B-n68-k9*	6.26%	1.88%	3.89%	0.53%	5.45%	0.31%	7.37%	9.52%
B-n78-k10*	4.56%	2.23%	2.72%	0.75%	4.45%	0.00%	6.21%	8.01%
avg.	4.32%	1.98%	3.00%	0.81%	4.46%	0.17%	6.23%	7.66%

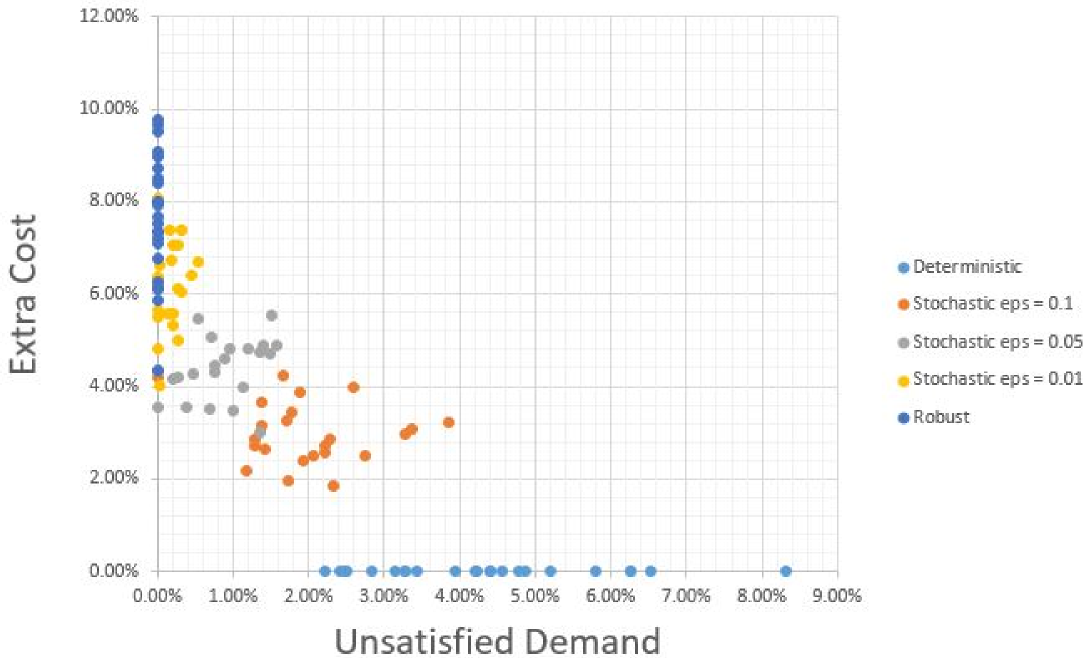

Figure 9: Set B - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

Figure 10: Set B - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

Table 22: Set B - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
B-n31-k5	0.96%	0.19%	1.81%	0.00%	2.58%	0.00%	4.84%	3.06%
B-n34-k5	0.78%	0.18%	2.07%	0.00%	2.50%	0.00%	3.55%	3.09%
B-n35-k5	1.88%	0.42%	2.92%	0.00%	4.16%	0.00%	6.33%	5.05%
B-n38-k6	0.88%	0.16%	2.38%	0.00%	3.35%	0.00%	4.24%	3.32%
B-n39-k5	0.00%	0.42%	1.59%	0.00%	2.21%	0.00%	3.54%	2.27%
B-n41-k6*	1.46%	0.08%	2.51%	0.14%	2.73%	0.00%	4.18%	3.15%
B-n43-k6	1.11%	0.36%	1.94%	0.60%	2.40%	0.00%	4.34%	3.32%
B-n44-k7	1.85%	0.70%	2.37%	0.37%	3.23%	0.00%	4.65%	4.07%
B-n45-k5*	1.18%	0.00%	3.06%	0.00%	2.64%	0.00%	3.57%	2.78%
B-n45-k6*	0.84%	0.48%	1.60%	0.14%	1.98%	0.00%	3.04%	2.34%
B-n50-k7	0.82%	0.13%	1.67%	0.00%	2.23%	0.00%	3.48%	2.55%
B-n50-k8	2.37%	0.86%	2.26%	0.22%	3.04%	0.00%	4.81%	3.55%
B-n51-k7*	1.73%	0.54%	3.19%	0.23%	4.24%	0.00%	5.09%	4.76%
B-n52-k7	1.15%	0.44%	2.18%	0.13%	2.79%	0.00%	4.22%	3.40%
B-n56-k7	3.04%	0.00%	1.97%	0.00%	2.85%	0.00%	3.95%	3.33%
B-n57-k7*	1.79%	0.65%	2.38%	0.23%	3.46%	0.00%	4.71%	3.31%
B-n57-k9	1.65%	0.46%	2.44%	0.20%	3.02%	0.00%	4.66%	4.01%
B-n63-k10	2.37%	0.37%	2.32%	0.26%	3.98%	0.00%	5.54%	4.67%
B-n64-k9*	0.99%	0.00%	2.17%	0.09%	2.65%	0.00%	4.22%	3.20%
B-n66-k9*	1.04%	0.77%	2.24%	0.19%	2.98%	0.00%	4.75%	3.70%
B-n67-k10	1.63%	0.29%	1.53%	0.09%	3.66%	0.00%	4.87%	2.70%
B-n68-k9*	1.87%	0.19%	2.71%	0.19%	3.92%	0.00%	5.44%	4.28%
B-n78-k10*	1.66%	0.17%	2.04%	0.09%	2.73%	0.00%	4.44%	3.39%
avg.	1.44%	0.34%	2.23%	0.14%	3.01%	0.00%	4.45%	3.45%

8.2.3 R

Table 23: Set B - Robust solutions for the case $\alpha = 0.1$ and $\beta = 0.5$

	Robust						
						Time (s)	
Instance	LB	κ_r	UB	#columns	#iter	IP	Total
B-n31-k5	665.81	7.52%	704.1	895	65	0.29	41.6
B-n34-k5*	802.14	7.07%	904.28	1601	88	6.26	92.86
B-n35-k5	909.9	9.66%	1036.83	1507	67	0.5	177.9
B-n38-k6	774.6	7.66%	872.38	1587	67	0.81	89.22
B-n39-k5	555.35	6.13%	618.33	2358	93	0.73	360.54
B-n41-k6*	855.42	6.75%	920.82	1038	65	0.59	67.46
B-n43-k6	764.39	7.34%	832.49	2032	90	2.68	189.97
B-n44-k7*	940.27	8.39%	1068.96	1651	82	3.53	97.42
B-n45-k5*	722.26	4.35%	772.03	2505	132	0.22	141.75
B-n45-k6*	699.44	6.26%	757.91	1420	80	0.74	87.95
B-n50-k7	709.92	5.86%	806.99	3366	129	4.35	522.95
B-n50-k8*	1358.62	7.91%	1405.25	2506	119	7.64	303.95
B-n51-k7*	1043.64	9.06%	1186.26	1735	84	0.9	139.85
B-n52-k7	724.14	6.11%	828.36	3640	156	5.17	824.42
B-n56-k7*	685.06	7.21%	792.05	2214	117	2.3	605.58
B-n57-k7*	1190.21	8.49%	1400.26	3040	138	20.41	1026.55
B-n57-k9*	1643.33	8.98%	1787.52	2038	120	98.13	674.99
B-n63-k10*	1593.65	9.77%	1734	2351	118	29.99	600.14
B-n64-k9*	875.4	7.36%	965.23	2717	132	1.22	455.18
B-n66-k9*	1369.12	8.71%	1468.85	3025	135	17.14	1111.48
B-n67-k10*	1073.47	8.00%	1181.15	2771	120	18.62	510.48
B-n68-k9*	1308.37	9.52%	1500.71	3317	134	103.29	1382.82
B-n78-k10*	1263.35	8.01%	1354.95	3898	151	3.24	1483.84
avg.	979.47	7.66%	1082.59	2313	108	14.29	477.77

Table 24: Set B - Robust solutions for the case $\alpha = 0.05$ and $\beta = 0.5$

	Robust						
						Time (s)	
Instance	LB	κ_r	UB	#columns	#iter	IP	Total
B-n31-k5	638.24	3.06%	684.74	1164	88	0.88	86.03
B-n34-k5	772.31	3.09%	797.3	2349	81	0.3	294.31
B-n35-k5	871.72	5.05%	1036.7	1397	63	0.48	100.13
B-n38-k6	743.36	3.32%	867.51	1495	68	0.45	93.38
B-n39-k5	535.13	2.27%	605.64	2261	87	0.69	221.66
B-n41-k6*	826.53	3.15%	859.87	1063	64	0.31	75.15
B-n43-k6	735.73	3.32%	790.11	2017	86	1.84	177.63
B-n44-k7	902.83	4.07%	1052.29	2132	94	3.46	170.24
B-n45-k5*	711.41	2.78%	773.14	1908	88	0.25	105.36
B-n45-k6*	673.66	2.34%	744.58	1577	88	1.48	102.89
B-n50-k7	687.72	2.55%	781.45	3378	122	0.88	370.19
B-n50-k8	1303.74	3.55%	1357.35	2783	121	3.61	479.19
B-n51-k7*	1002.43	4.76%	1168.09	2016	123	5.84	230.16
B-n52-k7	705.62	3.40%	834.34	3043	150	4.12	697.81
B-n56-k7	660.27	3.33%	824.17	4084	138	11.12	1015.3
B-n57-k7*	1133.38	3.31%	1263.03	3668	143	24.84	1241.07
B-n57-k9	1568.47	4.01%	1673.28	2969	125	3.01	613.99
B-n63-k10	1519.54	4.67%	1623.11	3264	138	1.91	1146.39
B-n64-k9*	841.47	3.20%	954.96	2899	133	3.19	557.46
B-n66-k9*	1306.03	3.70%	1368.3	2960	137	1.72	1512.15
B-n67-k10	1020.81	2.70%	1137.05	4262	176	119.56	1322.3
B-n68-k9*	1245.84	4.28%	1419.45	3453	151	92.83	1969.03
B-n78-k10*	1209.38	3.39%	1345.66	4786	168	33.09	2398.14
avg.	939.8	3.45%	1041.83	2649	114	13.73	651.30

8.2.4 S

Table 25: Set B - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.1$

	Stochastic with $\epsilon = 0.1$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
B-n31-k5	635.26	2.58%	689.86	2.21%	1216	79	0.84	55.73
B-n34-k5*	767.88	2.50%	808.09	2.07%	1705	94	0.5	66.94
B-n35-k5	864.58	4.19%	1022.32	0.00%	1296	60	0.4	68.57
B-n38-k6	742.12	3.14%	840.05	1.37%	1507	69	0.87	69.31
B-n39-k5	533.02	1.87%	565.98	2.32%	2079	82	0.35	144.8
B-n41-k6*	823.15	2.73%	863.92	1.29%	1020	61	0.35	56.07
B-n43-k6	729.21	2.40%	773.25	1.94%	2170	91	0.76	140.69
B-n44-k7*	895.56	3.23%	1007.57	3.85%	1615	89	0.47	87.19
B-n45-k5*	714.73	3.26%	793.24	1.72%	2103	80	0.36	91.14
B-n45-k6*	671.15	1.96%	738.12	1.74%	1628	86	1.16	73.88
B-n50-k7	685.15	2.17%	782.55	1.17%	3385	125	1.04	245.59
B-n50-k8*	1297.7	3.07%	1353.35	3.36%	2077	104	1.57	228.62
B-n51-k7*	997.46	4.24%	1088.76	1.67%	2242	147	1.41	179.41
B-n52-k7	701.99	2.86%	819.63	1.30%	3283	134	7.35	486.23
B-n56-k7*	655	2.50%	773.74	2.74%	2815	114	2.48	542.14
B-n57-k7*	1135.01	3.46%	1229.09	1.77%	3828	148	6.45	817.76
B-n57-k9*	1551.09	2.86%	1670.37	2.28%	2855	129	9.71	459.6
B-n63-k10*	1509.75	4.00%	1600.61	2.59%	2548	117	1.25	451.81
B-n64-k9*	837.05	2.66%	956.82	1.42%	2981	151	5.48	357.91
B-n66-k9*	1296.76	2.97%	1366.82	3.27%	3080	154	8.83	1287.8
B-n67-k10*	1030.35	3.66%	1272.39	1.38%	3063	131	5.26	541.7
B-n68-k9*	1241.13	3.89%	1395.44	1.88%	3940	154	45.41	1180.92
B-n78-k10*	1201.54	2.72%	1340.53	2.23%	4724	191	168.65	1589.82
avg.	935.51	3.00%	1032.72	1.98%	2485	113	11.78	401.03

Table 26: Set B - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.05$

	Stochastic with $\epsilon = 0.05$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
B-n31-k5	648.99	4.80%	703.86	1.19%	995	74	0.99	48.91
B-n34-k5*	775.22	3.48%	890.84	1.01%	1556	89	6.79	67.11
B-n35-k5	882.31	6.33%	1040.92	0.00%	1520	67	1.24	93.3
B-n38-k6	750.23	4.27%	882.84	0.47%	1619	72	1.51	55.9
B-n39-k5	541.7	3.52%	731.58	0.69%	2145	81	2.52	161.66
B-n41-k6*	834.8	4.18%	933.58	0.21%	1072	61	1.51	61.82
B-n43-k6	742.98	4.34%	785.13	0.00%	2144	82	1.86	131.89
B-n44-k7*	907.43	4.60%	1057.42	0.89%	1775	91	7.34	93.22
B-n45-k5*	716.86	3.57%	776.84	0.00%	2151	105	0.53	95.09
B-n45-k6*	678.16	3.02%	742.79	1.35%	1814	96	1.08	76.32
B-n50-k7	694.4	3.55%	782.66	0.38%	3549	119	3.37	274.02
B-n50-k8*	1320.63	4.89%	1357.04	1.58%	2506	120	1.73	258.93
B-n51-k7*	1005.29	5.06%	1193.47	0.70%	1952	102	5.56	120.71
B-n52-k7	711.8	4.30%	836.74	0.76%	3393	162	4.59	545.83
B-n56-k7*	664.49	3.99%	784.09	1.14%	2774	100	2.68	420.41
B-n57-k7*	1148.6	4.70%	1287.19	1.48%	3659	154	45.36	1040.64
B-n57-k9*	1579.29	4.73%	1678.36	1.35%	2537	117	3.38	367.45
B-n63-k10*	1532.13	5.54%	1676.05	1.51%	2317	122	16.86	435
B-n64-k9*	849.64	4.20%	928.47	0.27%	2824	123	1.49	273.61
B-n66-k9*	1320.24	4.83%	1374.93	0.95%	3119	146	1.65	941.86
B-n67-k10*	1042.52	4.88%	1151	1.40%	2684	125	30.56	367.29
B-n68-k9*	1259.78	5.45%	1441.18	0.53%	3757	132	120.36	1069.5
B-n78-k10*	1221.78	4.45%	1366.22	0.75%	4762	174	106.57	1436.04
avg.	949.10	4.46%	1061.01	0.81%	2462	109	16.07	366.80

Table 27: Set B - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.01$

	Stochastic with $\epsilon = 0.01$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
B-n31-k5	660.3	6.63%	697.22	0.02%	1066	69	0.68	36.03
B-n34-k5*	794.43	6.05%	884.86	0.31%	2070	105	13.99	89.82
B-n35-k5	896.61	8.05%	1035.27	0.00%	1505	58	0.5	95
B-n38-k6	760.19	5.66%	895.82	0.00%	1599	73	1.95	53.8
B-n39-k5	549.41	5.00%	615.73	0.27%	2315	84	0.42	143.57
B-n41-k6*	845.83	5.56%	926.68	0.21%	955	63	0.59	47.61
B-n43-k6	755.71	6.12%	839.7	0.27%	2042	81	1.48	131.85
B-n44-k7*	925.76	6.71%	1064.41	0.19%	1698	85	2.5	80.24
B-n45-k5*	720.09	4.03%	789.72	0.02%	2437	109	0.31	103.89
B-n45-k6*	699.49	6.27%	739.75	0.00%	1671	75	1.2	77.7
B-n50-k7	702.93	4.82%	803.24	0.00%	3492	127	3.13	270.52
B-n50-k8*	1343.37	6.70%	1373.58	0.53%	2289	104	2.41	217.78
B-n51-k7*	1017.92	6.38%	1180.63	0.00%	1849	108	1.12	108.9
B-n52-k7	719.88	5.49%	939.23	0.00%	3440	153	85.09	668.53
B-n56-k7*	673.11	5.34%	771.06	0.19%	2638	112	0.76	450.08
B-n57-k7*	1174.38	7.05%	1391.16	0.20%	3023	123	62.89	730.02
B-n57-k9*	1619.44	7.39%	1685.2	0.32%	2346	118	1.83	343.8
B-n63-k10*	1559.04	7.39%	1740.16	0.15%	2499	122	113.23	481.01
B-n64-k9*	860.71	5.56%	972.88	0.15%	2918	122	2.04	266.44
B-n66-k9*	1340.16	6.41%	1468.42	0.45%	3265	164	114.14	985.77
B-n67-k10*	1064.04	7.05%	1177.14	0.26%	2942	121	25.15	360.02
B-n68-k9*	1282.72	7.37%	1471.28	0.31%	3564	135	161.35	1063.86
B-n78-k10*	1242.31	6.21%	1360.51	0.00%	4172	160	16.17	1069.27
avg.	965.56	6.23%	1079.29	0.17%	2426	107	26.65	342.41

Table 28: Set B - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.1$

	Stochastic with $\epsilon = 0.1$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
B-n31-k5	630.49	1.81%	681.44	0.19%	1249	76	0.77	57.49
B-n34-k5	764.64	2.07%	812.37	0.18%	2191	84	0.6	107.95
B-n35-k5	854	2.92%	1031.29	0.42%	1367	65	0.64	81.26
B-n38-k6	736.66	2.38%	850.66	0.16%	1495	70	1.28	71.59
B-n39-k5	531.6	1.59%	604.57	0.42%	2160	82	2.08	178.52
B-n41-k6*	821.38	2.51%	866.87	0.08%	989	51	0.32	65.61
B-n43-k6	725.91	1.94%	771.2	0.36%	2318	99	0.87	175.63
B-n44-k7	888.1	2.37%	1004.21	0.70%	2430	97	2.47	157.3
B-n45-k5*	713.38	3.06%	779.19	0.00%	2305	83	0.44	100.38
B-n45-k6*	668.8	1.60%	739	0.48%	1644	74	1.38	68.74
B-n50-k7	681.8	1.67%	805.31	0.13%	3322	114	3.54	253.17
B-n50-k8	1287.5	2.26%	1339.92	0.86%	2822	121	2.67	318.29
B-n51-k7*	987.4	3.19%	1078.45	0.54%	2165	124	0.79	159.57
B-n52-k7	697.29	2.18%	826.71	0.44%	3053	126	16.17	490.05
B-n56-k7	651.59	1.97%	771.78	0.00%	3682	117	3.19	602.05
B-n57-k7*	1123.17	2.38%	1150.07	0.65%	3382	158	0.93	1111.06
B-n57-k9	1544.68	2.44%	1672.23	0.46%	3049	134	27.76	521.12
B-n63-k10	1485.41	2.32%	1619.67	0.37%	3482	142	7.11	705.88
B-n64-k9*	833.09	2.17%	962.02	0.00%	3079	127	5.28	358.19
B-n66-k9*	1287.66	2.24%	1381.83	0.77%	3060	141	9.06	967.41
B-n67-k10	1009.21	1.53%	1120.48	0.29%	4448	163	58.06	711.76
B-n68-k9*	1227.09	2.71%	1400.68	0.19%	3919	145	168.33	1207.02
B-n78-k10*	1193.62	2.04%	1332.18	0.17%	4729	178	295.55	1776.22
avg.	928.02	2.23%	1026.18	0.34%	2710	112	26.49	445.49

Table 29: Set B - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.05$

	Stochastic with $\epsilon = 0.05$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
B-n31-k5	635.26	2.58%	689.86	0.00%	1215	78	0.86	62
B-n34-k5	767.85	2.50%	802.58	0.00%	2350	84	1.71	192.3
B-n35-k5	864.34	4.16%	1022.52	0.00%	1178	53	0.68	71.71
B-n38-k6	743.61	3.35%	875.14	0.00%	1521	67	1.35	78.95
B-n39-k5	534.83	2.21%	596.42	0.00%	2056	80	0.66	165.33
B-n41-k6*	823.15	2.73%	867.46	0.14%	1003	60	0.56	68.26
B-n43-k6	729.21	2.40%	778.59	0.60%	2189	90	0.84	168.34
B-n44-k7	895.56	3.23%	993.75	0.37%	2249	108	2.14	145.12
B-n45-k5*	710.43	2.64%	782.2	0.00%	2215	91	0.91	116.19
B-n45-k6*	671.28	1.98%	740.56	0.14%	1518	76	0.96	68.93
B-n50-k7	685.57	2.23%	770.21	0.00%	3056	117	0.7	231.76
B-n50-k8	1297.35	3.04%	1349.97	0.22%	2973	125	1.59	340.9
B-n51-k7*	997.48	4.24%	1075.37	0.23%	2091	119	0.53	143.36
B-n52-k7	701.5	2.79%	814.89	0.13%	3154	149	4.98	499.83
B-n56-k7	657.25	2.85%	779.42	0.00%	3745	123	2.28	643.75
B-n57-k7*	1135.04	3.46%	1229.25	0.23%	3849	143	0.9	821.15
B-n57-k9	1553.46	3.02%	1671.89	0.20%	3609	145	7.38	564.53
B-n63-k10	1509.54	3.98%	1664.58	0.26%	3741	139	73.16	847.69
B-n64-k9*	836.95	2.65%	956.62	0.09%	2919	125	6.24	343.98
B-n66-k9*	1296.92	2.98%	1367.83	0.19%	3102	148	4.07	906.75
B-n67-k10	1030.34	3.66%	1177.59	0.09%	4062	145	108.42	702.69
B-n68-k9*	1241.51	3.92%	1417.87	0.19%	3747	141	168.92	1179.2
B-n78-k10*	1201.63	2.73%	1334.39	0.09%	4645	186	38.45	1372.8
avg.	935.65	3.01%	1033.00	0.14%	2704	113	18.62	423.28

Table 30: Set B - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.01$

	Stochastic with $\epsilon = 0.01$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
B-n31-k5	649.26	4.84%	703.86	0.00%	974	71	0.84	49.14
B-n34-k5	775.76	3.55%	812.81	0.00%	2520	100	1.12	3403.3
B-n35-k5	882.31	6.33%	1040.92	0.00%	1521	67	1.28	109.34
B-n38-k6	749.99	4.24%	882.42	0.00%	1676	85	1.6	78.5
B-n39-k5	541.76	3.54%	565.44	0.00%	2298	86	0.48	188.8
B-n41-k6*	834.8	4.18%	862.64	0.00%	1093	66	0.23	72.52
B-n43-k6	742.98	4.34%	813.69	0.00%	2139	83	1.95	155.38
B-n44-k7	907.88	4.65%	1073.85	0.00%	2207	95	3.22	169.78
B-n45-k5*	716.86	3.57%	776.84	0.00%	2149	105	0.65	98.31
B-n45-k6*	678.25	3.04%	740.38	0.00%	1555	81	0.78	74.23
B-n50-k7	693.97	3.48%	790.12	0.00%	3314	116	3.63	260.03
B-n50-k8	1319.6	4.81%	1344.59	0.00%	3101	132	3.77	491.07
B-n51-k7*	1005.65	5.09%	1186.45	0.00%	2114	121	4.74	152.04
B-n52-k7	711.21	4.22%	837.21	0.00%	3745	173	6.09	620.01
B-n56-k7	664.22	3.95%	821.33	0.00%	3845	120	6.47	668.86
B-n57-k7*	1148.77	4.71%	1252.84	0.00%	3672	159	17.54	915.73
B-n57-k9	1578.26	4.66%	1671.91	0.00%	3540	147	4.13	527.61
B-n63-k10	1532.11	5.54%	1851.94	0.00%	3579	146	51.4	1004.92
B-n64-k9*	849.82	4.22%	959.78	0.00%	2736	129	2.66	282.1
B-n66-k9*	1319.17	4.75%	1418.45	0.00%	2934	148	22.07	938.81
B-n67-k10	1042.37	4.87%	1207.82	0.00%	4648	159	70.2	964.68
B-n68-k9*	1259.69	5.44%	1414.78	0.00%	3682	147	34.59	1084.53
B-n78-k10*	1221.69	4.44%	1350.82	0.00%	4618	163	40.58	1321.95
avg.	948.97	4.45%	1060.04	0.00%	2768	117	12.17	592.68

8.3 P

8.3.1 D

Table 31: Set P - Comparison with best known deterministic solution

Instance	z*	LB	UB	Gap*	Time (s)
P-n16-k8*	450	442.42	451.94	0%	0.58
P-n19-k2*	212	205.89	212.65	0%	4.76
P-n20-k2*	216	213.26	220.03	2%	5.26
P-n21-k2	211	212.71	212.71	1%	7.09
P-n22-k2*	216	217.35	224.2	4%	8.07
P-n23-k8*	529	523.54	536.34	1%	1.67
P-n40-k5*	458	452	461.72	1%	37.64
P-n45-k5*	510	503.99	539.51	6%	67.7
P-n50-k7*	554	547.01	571.91	3%	53.32
P-n50-k8*	631	615.41	639.65	1%	30.8
P-n50-k10*	696	689.74	704.45	1%	13.81
P-n51-k10*	741	735.81	758.92	2%	19.12
P-n55-k7*	568	554.66	584.03	3%	111.64
P-n55-k8		567.61	596.11		128.14
P-n55-k10*	694	678.8	700.69	1%	29.98
P-n55-k15*	989	938.24	947.73	-4%	8.15
P-n60-k10*	744	738.82	759.16	2%	46.47
P-n60-k15*	968	962.33	987.01	2%	15.63
P-n65-k10*	792	785.43	831.88	5%	103.12
P-n70-k10*	827	814.53	853.46	3%	197.44
P-n76-k5*	627	622.31	671.25	7%	4105.07
avg.				2%	237.88

8.3.2 RvS

Table 32: Set P - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
P-n16-k8*	1.95%	0.00%	5.14%	0.00%	5.14%			5.98%
P-n19-k2*	4.32%	0.00%	0.20%	0.00%	0.20%	0.00%	0.20%	0.00%
P-n20-k2*	5.03%	0.00%	1.24%	0.00%	1.26%	0.00%	1.84%	1.93%
P-n21-k2	0.12%	0.00%	0.00%	0.00%	0.00%	0.00%	0.88%	0.00%
P-n22-k2*	2.08%	0.80%	1.38%	0.80%	1.57%	0.11%	1.86%	1.86%
P-n23-k8*	6.04%	0.89%	7.73%	0.00%	10.22%	0.00%	14.32%	10.22%
P-n40-k5*	2.03%	0.70%	1.35%	0.70%	1.91%	1.10%	2.17%	3.72%
P-n45-k5*	5.54%	0.83%	1.11%	0.10%	1.39%	0.10%	2.16%	3.19%
P-n50-k7*	2.94%	3.71%	1.61%	1.56%	2.69%	1.15%	3.58%	4.77%
P-n50-k8*	4.33%	2.72%	2.56%	1.02%	2.95%	0.67%	4.02%	5.24%
P-n50-k10*	5.87%	1.89%	2.15%	0.50%	3.17%	0.02%	4.78%	5.47%
P-n51-k10*	4.80%	1.29%	2.94%	1.18%	3.25%	0.04%	5.08%	5.19%
P-n55-k7*	3.25%	1.26%	1.49%	1.52%	1.67%	1.56%	2.33%	4.19%
P-n55-k8	2.86%	2.39%	2.13%	1.77%	2.37%	1.13%	3.17%	4.96%
P-n55-k10*	5.39%	0.99%	2.53%	0.31%	3.11%	0.52%	3.92%	6.03%
P-n55-k15*	6.24%	0.76%	5.67%	0.36%	6.06%	0.15%	7.89%	8.72%
P-n60-k10*	5.06%	1.78%	1.92%	1.52%	2.49%	0.85%	3.95%	5.22%
P-n60-k15*	5.75%	1.57%	3.24%	1.16%	4.69%	0.05%	7.26%	8.27%
P-n65-k10*	4.31%	1.44%	1.63%	1.16%	1.95%	1.09%	3.02%	5.26%
P-n70-k10*	4.35%	2.71%	2.00%	1.85%	2.58%	0.51%	3.98%	5.56%
P-n76-k5*	1.94%	2.82%	0.42%	1.85%	0.68%	1.21%	1.02%	2.54%
avg.	4.01%	1.36%	2.30%	0.83%	2.82%	0.51%	3.87%	4.68%

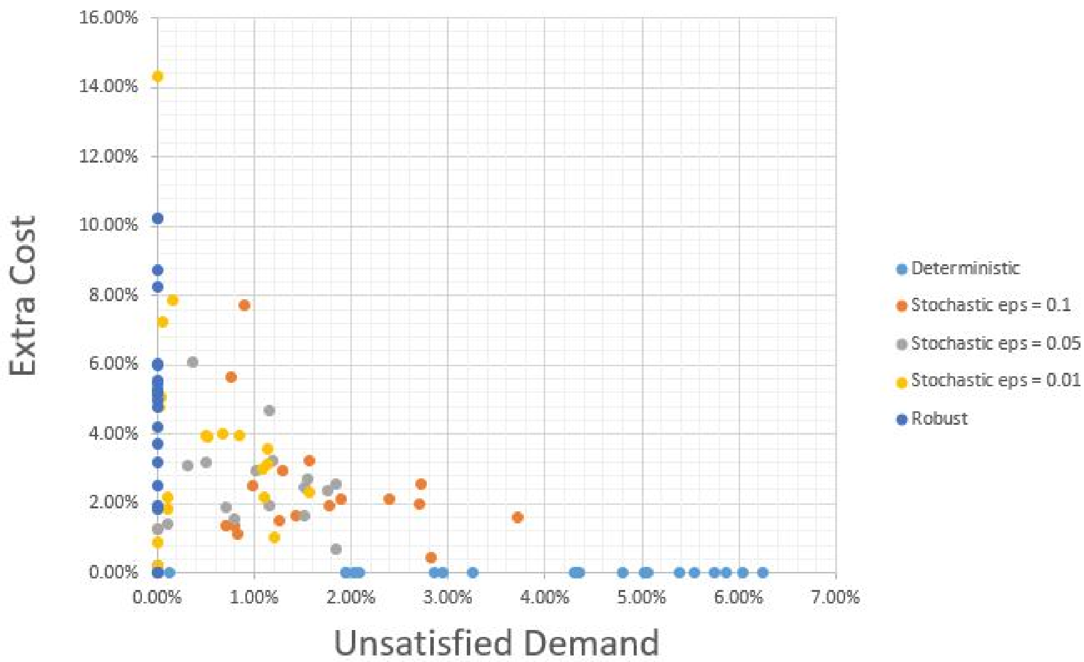

Figure 11: Set P - Performance measures comparison for the case $\alpha = 0.1$ and $\beta = 0.5$

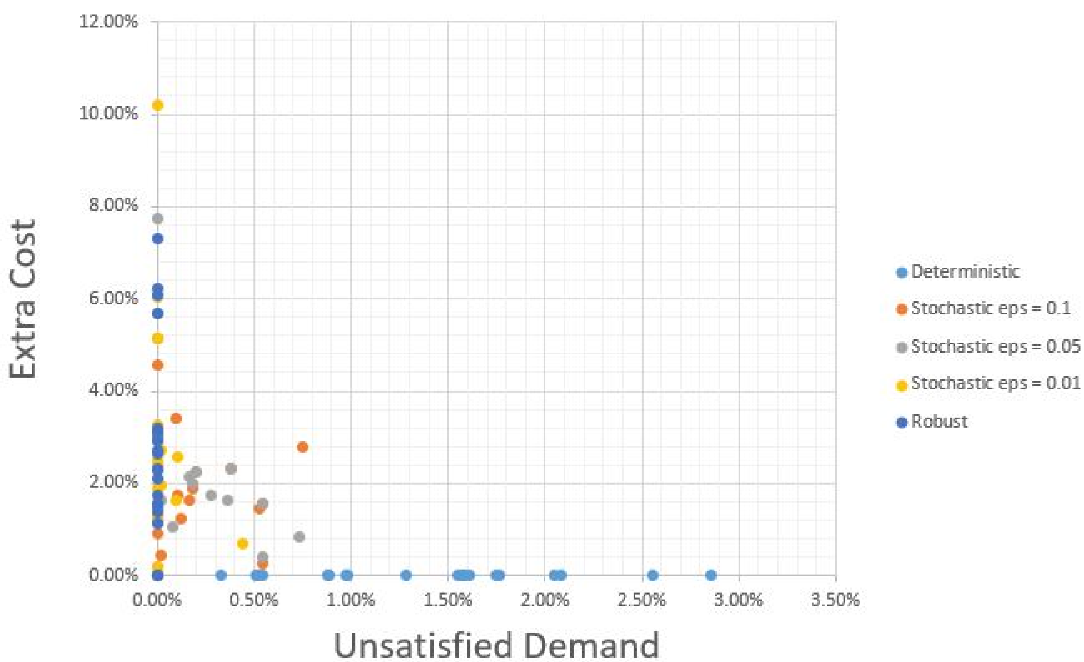

Figure 12: Set P - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

Table 33: Set P - Performance measures comparison for the case $\alpha = 0.05$ and $\beta = 0.5$

	Deterministic	Stochastic						Robust
		$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$		
Instance	δ_d	δ_s	κ_s	δ_s	κ_s	δ_s	κ_s	κ_r
P-n16-k8	1.28%	0.00%	4.57%	0.00%	5.14%	0.00%	5.14%	6.23%
P-n19-k2*	0.88%	0.00%	0.00%	0.00%	0.20%	0.00%	0.20%	0.00%
P-n20-k2*	1.57%	0.00%	1.51%	0.00%	1.27%	0.00%	1.26%	1.57%
P-n21-k2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
P-n22-k2*	0.33%	0.00%	0.90%	0.00%	1.44%	0.00%	1.34%	1.37%
P-n23-k8*	2.86%		7.33%	0.00%	7.73%	0.00%	10.22%	7.33%
P-n40-k5	0.52%	0.00%	1.28%	0.00%	1.34%	0.00%	1.90%	1.73%
P-n45-k5	2.55%		0.72%	0.08%	1.04%		1.39%	1.52%
P-n50-k7	0.97%	0.13%	1.25%	0.36%	1.63%	0.02%	2.72%	3.07%
P-n50-k8*	1.57%	0.38%	2.33%	0.38%	2.33%	0.00%	2.95%	2.95%
P-n50-k10*	2.05%	0.17%	1.64%	0.17%	2.15%	0.00%	3.17%	2.72%
P-n51-k10*	1.55%	0.11%	1.75%	0.00%	2.94%	0.00%	3.25%	2.94%
P-n55-k7	1.75%	0.02%	0.43%	0.73%	0.83%	0.10%	1.65%	2.11%
P-n55-k8	0.51%	0.53%	1.44%	0.27%	1.76%	0.00%	2.38%	2.64%
P-n55-k10	1.59%	0.20%	2.25%	0.20%	2.25%	0.00%	3.11%	3.11%
P-n55-k15*	2.08%	0.10%	3.40%	0.00%	5.67%	0.00%	6.06%	5.67%
P-n60-k10*	0.89%	0.54%	1.58%	0.54%	1.58%	0.00%	2.49%	2.31%
P-n60-k15	1.77%	0.75%	2.78%	0.00%	6.07%			6.07%
P-n65-k10*	0.98%	0.00%	1.16%	0.02%	1.64%	0.02%	1.94%	2.30%
P-n70-k10*	1.61%	0.18%	1.90%	0.18%	2.00%	0.10%	2.58%	3.21%
P-n76-k5*	0.54%	0.54%	0.26%	0.54%	0.41%	0.44%	0.69%	1.13%
avg.	1.33%	0.19%	1.83%	0.17%	2.35%	0.04%	2.72%	2.85%

8.3.3 R

Table 34: Set P - Robust solutions for the case $\alpha = 0.1$ and $\beta = 0.5$

	Robust						
						Time (s)	
Instance	LB	κ_r	UB	#columns	#iter	IP	Total
P-n16-k8*	468.87	5.98%	477.73	72	13	0.02	0.85
P-n19-k2*	205.89	0.00%	220.64	547	32	0.19	8.47
P-n20-k2*	217.38	1.93%	239.53	741	49	0.37	8.89
P-n21-k2	212.71	0.00%	212.71	1179	78	0.1	1843.01
P-n22-k2*	221.39	1.86%	249.72	829	51	0.18	10.95
P-n23-k8*	577.03	10.22%	593.29	146	17	0.35	1.96
P-n40-k5*	468.83	3.72%	502.81	1850	79	0.51	53.5
P-n45-k5*	520.09	3.19%	538.84	2985	101	2.82	129.75
P-n50-k7*	573.08	4.77%	605.43	2287	82	0.38	63.28
P-n50-k8*	647.66	5.24%	662.77	1166	63	0.78	40.63
P-n50-k10*	727.44	5.47%	737.72	808	50	0.41	31.18
P-n51-k10*	774	5.19%	787.06	1068	70	0.29	36.58
P-n55-k7*	577.89	4.19%	610.38	2852	96	5.91	148.21
P-n55-k8	595.75	4.96%	612.3	2253	82	2.04	166
P-n55-k10*	719.71	6.03%	738.64	1851	77	6.29	50.12
P-n55-k15*	1020.01	8.72%	1048.19	516	46	0.18	12.21
P-n60-k10*	777.37	5.22%	799.62	1460	83	1.08	95.57
P-n60-k15*	1041.88	8.27%	1057.16	1497	108	3.24	22.14
P-n65-k10*	826.77	5.26%	875.15	2069	107	2.53	160.82
P-n70-k10*	859.85	5.56%	890.32	2289	109	2.91	274.69
P-n76-k5*	638.11	2.54%	672.74	7195	237	9.17	5218.43
avg.	603.41	4.68%	625.37	1698	78	1.89	398.92

Table 35: Set P - Robust solutions for the case $\alpha = 0.05$ and $\beta = 0.5$

	Robust						
						Time (s)	
Instance	LB	κ_r	UB	#columns	#iter	IP	Total
P-n16-k8	470	6.23%	470	74	14	0.02	0.67
P-n19-k2*	205.89	0.00%	220.64	556	34	0.19	7.37
P-n20-k2*	216.6	1.57%	239.35	743	48	0.37	8.75
P-n21-k2	212.71	0.00%	212.71	1184	77	0.1	86.19
P-n22-k2*	220.33	1.37%	238.47	829	53	0.18	9.85
P-n23-k8*	561.89	7.33%	567.84	144	18	0.35	1.82
P-n40-k5	459.87	1.73%	470.6	1848	80	0.51	81.05
P-n45-k5	511.67	1.52%	533.33	2993	98	2.82	442.45
P-n50-k7	563.83	3.07%	568.11	2293	85	0.38	162.82
P-n50-k8*	633.55	2.95%	653.78	1166	64	0.78	55.12
P-n50-k10*	708.53	2.72%	730.63	806	53	0.41	30.88
P-n51-k10*	757.43	2.94%	766.07	1068	68	0.29	37.61
P-n55-k7	566.68	2.11%	592.97	2855	97	5.91	279.33
P-n55-k8	582.59	2.64%	621.26	2271	82	2.04	168.81
P-n55-k10	699.92	3.11%	794.22	1851	77	6.29	115
P-n55-k15*	991.47	5.67%	1005.32	524	48	0.18	17.61
P-n60-k10*	755.88	2.31%	774.49	1460	85	1.08	112.01
P-n60-k15	1020.73	6.07%	1083.7	1502	109	3.24	192.73
P-n65-k10*	803.5	2.30%	824.67	2079	109	2.53	206.99
P-n70-k10*	840.64	3.21%	880.67	2285	109	2.91	340.41
P-n76-k5*	629.34	1.13%	661.13	8180	248	47.76	7762.39
avg.	591.10	2.85%	614.76	1748	79	3.73	481.89

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Table 36: Set P - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.1$

	Stochastic with $\epsilon = 0.1$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
P-n16-k8*	465.16	5.14%	477.73	0.00%	53	12	0.03	0.7
P-n19-k2*	206.3	0.20%	220.64	0.00%	589	45	0.28	6.68
P-n20-k2*	215.91	1.24%	233.75	0.00%	685	55	0.38	8.78
P-n21-k2	212.71	0.00%	212.71	0.00%	1239	89	0.18	27.64
P-n22-k2*	220.36	1.38%	238.67	0.80%	687	41	0.15	7.58
P-n23-k8*	564.02	7.73%	568.57	0.89%	149	20	0.17	1.66
P-n40-k5*	458.1	1.35%	464.5	0.70%	1399	69	0.17	55.85
P-n45-k5*	509.57	1.11%	536.03	0.83%	2071	86	1.07	114.87
P-n50-k7*	555.82	1.61%	563.94	3.71%	1591	81	0.23	59.23
P-n50-k8*	631.19	2.56%	651.61	2.72%	1144	68	0.35	30.45
P-n50-k10*	704.6	2.15%	728.42	1.89%	838	61	0.39	17.28
P-n51-k10*	757.43	2.94%	766.07	1.29%	1112	72	0.38	21.87
P-n55-k7*	562.92	1.49%	584.47	1.26%	2139	91	1.03	150.26
P-n55-k8	579.71	2.13%	616.92	2.39%	2294	86	2.95	148.01
P-n55-k10*	695.95	2.53%	722.89	0.99%	1271	72	1.12	34.47
P-n55-k15*	991.47	5.67%	1005.32	0.76%	512	45	0.4	9.26
P-n60-k10*	752.99	1.92%	786.45	1.78%	1549	86	2.2	55.66
P-n60-k15*	993.47	3.24%	1010.23	1.57%	805	69	0.32	12.51
P-n65-k10*	798.22	1.63%	832.29	1.44%	1927	96	2.43	109.8
P-n70-k10*	830.83	2.00%	860.33	2.71%	2249	103	2.44	225.17
P-n76-k5*	624.95	0.42%	655.5	2.82%	7859	241	8.96	5903.84
avg.	587.22	2.30%	606.53	1.36%	1532	76	1.22	333.41

Table 37: Set P - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.05$

	Stochastic with $\epsilon = 0.05$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
P-n16-k8*	465.16	5.14%	477.73	0.00%	53	12	0.07	1.03
P-n19-k2*	206.3	0.20%	220.64	0.00%	529	41	0.09	9.17
P-n20-k2*	215.94	1.26%	239.53	0.00%	605	45	0.2	7.07
P-n21-k2	212.71	0.00%	212.71	0.00%	1279	76	0.17	38.19
P-n22-k2*	220.77	1.57%	238.47	0.80%	775	48	0.15	8.05
P-n23-k8*	577.03	10.22%	593.29	0.00%	140	22	0.21	1.55
P-n40-k5*	460.65	1.91%	464.5	0.70%	1675	87	0.16	53.62
P-n45-k5*	510.98	1.39%	535.34	0.10%	1870	78	0.92	106.35
P-n50-k7*	561.74	2.69%	568.11	1.56%	1553	72	0.25	537.7
P-n50-k8*	633.55	2.95%	664.55	1.02%	1111	60	1.02	30.28
P-n50-k10*	711.6	3.17%	729.47	0.50%	850	60	0.4	15.84
P-n51-k10*	759.7	3.25%	772.23	1.18%	1110	71	0.36	23.42
P-n55-k7*	563.94	1.67%	591	1.52%	2189	89	2.04	150.67
P-n55-k8	581.07	2.37%	622.09	1.77%	2441	99	11.6	136.24
P-n55-k10*	699.92	3.11%	731.27	0.31%	1224	73	1.49	35.53
P-n55-k15*	995.11	6.06%	1009.76	0.36%	523	47	0.32	8.47
P-n60-k10*	757.19	2.49%	775.22	1.52%	1465	84	0.9	46.84
P-n60-k15*	1007.43	4.69%	1037.29	1.16%	792	64	0.69	12.5
P-n65-k10*	800.75	1.95%	834.42	1.16%	2098	108	2.58	112.55
P-n70-k10*	835.55	2.58%	865.33	1.85%	2319	104	1.69	203.91
P-n76-k5*	626.56	0.68%	667.24	1.85%	7875	222	43.02	6626.72
avg.	590.65	2.82%	611.91	0.83%	1546	74	3.25	388.84

Table 38: Set P - Stochastic solutions for the case $\alpha = 0.1$ and $\beta = 0.5$ and $\epsilon = 0.01$

	Stochastic with $\epsilon = 0.01$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
P-n16-k8*								
P-n19-k2*	206.3	0.20%	220.64	0.00%	525	42	0.13	6.67
P-n20-k2*	217.19	1.84%	239.53	0.00%	650	44	0.62	7.76
P-n21-k2	214.79	0.88%	214.79	0.00%	1439	91	0.23	399.43
P-n22-k2*	221.4	1.86%	240.83	0.11%	864	64	0.19	10.3
P-n23-k8*	598.52	14.32%	605.24	0.00%	136	19	0.17	1.24
P-n40-k5*	461.83	2.17%	465.7	1.10%	1443	76	0.15	50.3
P-n45-k5*	514.87	2.16%	535.34	0.10%	1787	84	0.84	117.56
P-n50-k7*	566.62	3.58%	569.14	1.15%	1568	78	0.22	52.78
P-n50-k8*	640.15	4.02%	653.05	0.67%	1027	61	0.2	25.93
P-n50-k10*	722.68	4.78%	732.98	0.02%	839	57	0.26	14.3
P-n51-k10*	773.16	5.08%	795.31	0.04%	1031	62	0.48	19.82
P-n55-k7*	567.6	2.33%	601.1	1.56%	2032	87	2.27	125.69
P-n55-k8	585.59	3.17%	612.67	1.13%	2330	92	1.31	151.58
P-n55-k10*	705.44	3.92%	746.56	0.52%	1130	66	1.01	28.47
P-n55-k15*	1012.23	7.89%	1031.22	0.15%	522	47	0.59	7.25
P-n60-k10*	767.99	3.95%	807.96	0.85%	1378	74	4.48	46.73
P-n60-k15*	1032.22	7.26%	1047.97	0.05%	725	59	0.39	12.42
P-n65-k10*	809.13	3.02%	827.87	1.09%	2169	131	0.77	90.84
P-n70-k10*	846.91	3.98%	894.67	0.51%	2235	112	13.44	182.49
P-n76-k5*	628.64	1.02%	669.18	1.21%	8235	274	43.62	6571.4
avg.	604.66	3.87%	625.59	0.51%	1603	81	3.57	396.15

Table 39: Set P - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.1$

	Stochastic with $\epsilon = 0.1$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
P-n16-k8	462.66	4.57%	477.73	0.00%	57	10	0.09	1.05
P-n19-k2*	205.89	0.00%	220.64	0.00%	508	34	0.23	6.95
P-n20-k2*	216.49	1.51%	239.53	0.00%	613	45	0.21	8.6
P-n21-k2	212.71	0.00%	212.71	0.00%	1146	87	0.14	17.47
P-n22-k2*	219.3	0.90%	238.47	0.00%	800	49	0.14	8.67
P-n23-k8*	561.89	7.33%	568.21	0.00%	147	22	0.15	1.61
P-n40-k5	457.82	1.28%	465.18	0.00%	1977	81	0.38	73.93
P-n45-k5	507.61	0.72%			2980	92		
P-n50-k7	553.84	1.25%	565.6	0.13%	2472	88	0.83	113.08
P-n50-k8*	629.75	2.33%	655.71	0.38%	1151	64	0.89	31.07
P-n50-k10*	701.05	1.64%	727.37	0.17%	852	59	0.93	17.15
P-n51-k10*	748.69	1.75%	764.26	0.11%	1210	74	0.44	26.01
P-n55-k7	557.32	0.43%	596.51	0.02%	3072	96	5.83	292.94
P-n55-k8	575.79	1.44%	605.1	0.53%	2287	90	2.27	166.96
P-n55-k10	694.07	2.25%	721.69	0.20%	1805	79	2.06	57.23
P-n55-k15*	970.12	3.40%	991.48	0.10%	598	58	0.39	7.95
P-n60-k10*	750.47	1.58%	756.36	0.54%	1574	84	0.28	62.96
P-n60-k15	989.1	2.78%	1056.07	0.75%	1490	87	9.07	61.66
P-n65-k10*	794.54	1.16%	824.23	0.00%	1897	87	1.14	135.88
P-n70-k10*	830.04	1.90%	852.17	0.18%	2424	107	1.38	211.87
P-n76-k5*	623.94	0.26%	654.23	0.54%	8352	255	15.36	6233.8
avg.	583.96	1.83%	609.66	0.19%	1782	78	2.11	376.84

Table 40: Set P - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.05$

	Stochastic with $\epsilon = 0.05$						Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
P-n16-k8	465.16	5.14%	477.73	0.00%	53	12	0.04	0.94
P-n19-k2*	206.3	0.20%	220.64	0.00%	558	43	0.31	8.56
P-n20-k2*	215.97	1.27%	239.4	0.00%	631	42	0.15	6.86
P-n21-k2	212.71	0.00%	212.71	0.00%	1100	67	0.16	22.59
P-n22-k2*	220.47	1.44%	238.47	0.00%	731	42	0.16	7.55
P-n23-k8*	564.02	7.73%	568.57	0.00%	149	20	0.17	1.61
P-n40-k5	458.1	1.34%	465.18	0.00%	1928	77	0.4	70.51
P-n45-k5	509.26	1.04%	565.84	0.08%	2910	98	3.62	224.16
P-n50-k7	555.9	1.63%	564.43	0.36%	2270	79	0.48	104.88
P-n50-k8*	629.75	2.33%	655.71	0.38%	1151	64	0.89	33.19
P-n50-k10*	704.6	2.15%	728.42	0.17%	838	61	0.33	16.78
P-n51-k10*	757.43	2.94%	765	0.00%	1129	76	0.26	24.5
P-n55-k7	559.54	0.83%	584.71	0.73%	3087	111	2.38	308.09
P-n55-k8	577.58	1.76%	600	0.27%	2546	100	1.36	156.57
P-n55-k10	694.07	2.25%	721.69	0.20%	1805	79	2.07	56.5
P-n55-k15*	991.47	5.67%	1005.32	0.00%	518	45	0.42	7.62
P-n60-k10*	750.47	1.58%	756.36	0.54%	1574	84	0.27	61.99
P-n60-k15	1020.73	6.07%	1076	0.00%	1608	111	1.8	119.62
P-n65-k10*	798.3	1.64%	831.62	0.02%	1946	92	2.18	132.81
P-n70-k10*	830.83	2.00%	852.17	0.18%	2352	105	1.55	243.79
P-n76-k5*	624.88	0.41%	665.51	0.54%	8091	274	32.26	6904.94
avg.	587.98	2.35%	609.31	0.17%	1761	80	2.44	405.43

Table 41: Set P - Stochastic solutions for the case $\alpha = 0.05$ and $\beta = 0.5$ and $\epsilon = 0.01$

	Stochastic with $\epsilon = 0.01$							
							Time (s)	
Instance	LB	κ_s	UB	δ_s	#columns	#iter	IP	Total
P-n16-k8	465.16	5.14%	477.73	0.00%	53	12	0.08	0.85
P-n19-k2*	206.3	0.20%	220.64	0.00%	531	41	0.26	7.48
P-n20-k2*	215.94	1.26%	238.83	0.00%	629	42	0.18	7.6
P-n21-k2	212.71	0.00%	212.71	0.00%	1192	72	0.19	38.12
P-n22-k2*	220.27	1.34%	238.67	0.00%	783	54	0.13	8.55
P-n23-k8*	577.03	10.22%	593.29	0.00%	140	22	0.17	1.37
P-n40-k5	460.65	1.90%	473.44	0.00%	2151	86	0.65	82.36
P-n45-k5	511.03	1.39%			2884			
P-n50-k7	561.89	2.72%	572.71	0.02%	2307	83	0.53	111.31
P-n50-k8*	633.55	2.95%	661.4	0.00%	1106	62	1.54	30.2
P-n50-k10*	711.58	3.17%	729.47	0.00%	852	59	0.38	16.5
P-n51-k10*	759.7	3.25%	767.09	0.00%	1138	79	0.25	26.26
P-n55-k7	564.12	1.65%	589.86	0.10%	2873	100	1.79	205.14
P-n55-k8	581.11	2.38%	620.66	0.00%	2421	95	1.69	122.72
P-n55-k10	699.92	3.11%	758.41	0.00%	2039	87	24.24	115.3
P-n55-k15*	995.11	6.06%	1009.76	0.00%	521	47	0.28	8.32
P-n60-k10*	757.18	2.49%	782.4	0.00%	1465	84	1.3	56.99
P-n60-k15								
P-n65-k10*	800.69	1.94%	820.18	0.02%	2013	99	0.72	125.74
P-n70-k10*	835.55	2.58%	860.98	0.10%	2379	106	1.08	236.1
P-n76-k5*	626.59	0.69%	677.96	0.44%	8181	224	57.83	7588.87
avg.	569.80	2.72%	595.06	0.04%	1783	77	4.91	462.62