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A Less Biased Tail Index Estimator for Improving Performance of a Systematic Risk Estimator under Extremely Adverse Market Conditions

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Abstract

In this paper, I consider the method to estimate systematic risk under adverse market conditions proposed by van Oordt and Zhou (2017). I use the bias corrected tail index estimator proposed by Gomes et al. (2007) to tackle the asymptotic bias issue in the Hill estimator which is a component in the tail beta estimator in van Oordt and Zhou (2017). In a simulation study, I show the better performance of the bias corrected tail beta estimator. In an empirical illustration, I show the slightly better performance in projecting industry-specific portfolio losses in a market downturn. However, investors must research the tail dependence of the financial assets to determine if using the bias corrected tail index estimator in the tail beta estimator will lead to better tail beta estimates.

Contents

1 Introduction

During the subprime mortgage crisis that started around 2008, global losses amounted to around 15 trillion dollars (Yoon (2012)). Being able to analyze systematic risk under extremely adverse market conditions, such as in this mortgage crisis, is of great importance to risk managers. By estimating and hedging this systematic risk, these huge losses may possibly be avoided in future distress events.

King and Wadhwani (1990), Longin and Solnik (2001) and Ang and Chen (2002) show that asset returns show stronger correlation than usual during market downturns. This implies that the relation between these asset returns and market risk is different in times of adverse market conditions. So in order to analyze systematic risk in case of a distress event, we need a method that estimates systematic risk differently under adverse market conditions.

Van Oordt and Zhou (2017) estimate a linear model between two heavy-tailed variables, conditional on the explanatory variable having extremely low or high values. To estimate the coefficient in the linear model, named tail beta, they use a tail dependence measure, tail index and quantiles estimated from tail observations. The coefficient that is estimated gives the relation between the systematic risk under extremely adverse market conditions, and asset returns. However, an issue proposed in van Oordt and Zhou (2017) is that the estimator of the tail index exhibits asymptotic bias.

Gomes et al. (2007) propose a bias corrected estimator for the tail index estimator. They achieve this by considering the external estimation of a second order shape and scale parameter for this measure. In this research, I compare the performance of the tail beta estimator using the bias corrected tail index estimator from Gomes et al. (2007), with the tail beta estimator used in van Oordt and Zhou (2017) using the Hill estimator as tail index estimator. This leads to the main research question:

Does using a bias corrected estimator for the tail index in the framework of van Oordt and Zhou (2017), improve the estimator for tail beta?

To evaluate the performance of the bias corrected tail beta estimator, I use a simulation study as in van Oordt and Zhou (2017). I show that the tail beta estimator using the bias corrected tail index estimator performs better than the tail beta estimator used in van Oordt and Zhou (2017) in four out of five simulated models. Furthermore, in an empirical illustration considering industry-specific stock portfolios, I show that using the bias corrected tail beta estimator may lead to different investment decisions. Lastly, I evaluate the performance of the tail beta estimators using the two different tail index estimators by their ability to project losses of industry-specific stock portfolios in a market downturn. I show that the tail beta estimator using the bias corrected tail index estimator performs slightly better than the tail beta estimator of van Oordt and Zhou (2017). The bias corrected tail beta estimator mostly works better in more recent years, which may be of interest to investors looking to use this method the coming years. However, investors must research the tail dependence of the financial assets to determine if the bias corrected tail index estimator will be beneficial to use in the tail beta estimator. This is because downward biased estimation of the tail index by the Hill estimator can counter downward biased estimation of the tail dependence measure used in the estimation of tail beta. This way, worse estimation of the tail index can lead to more accurate tail beta estimates and the Hill estimator might perform better in the tail beta estimator.

In the remainder of this report, I provide a description of the tail beta estimator provided by van Oordt and Zhou (2017) and of the bias corrected tail index estimator provided by Gomes et al. (2007). After this, I describe the simulation study and empirical illustration used to assess the performance of the tail beta estimators using the two different tail index estimators. Lastly, the results and implications are given.

2 Methodology

I build further on the model proposed by van Oordt and Zhou (2017). This model is shown in equation (1),

$$Y = \beta^T X + \epsilon, \qquad \text{for} \quad X < Q_x(\bar{p}), \tag{1}$$

where $\bar{p}$ is a very small probability, $Q_x(\bar{p})$ is the quantile function of X defined as $Q_x(\bar{p}) = sup\{c : Pr(X \leq c) \leq \bar{p}\}$. Furthermore, ϵ is the error term assumed to be independent of X under the condition imposed in equation (1). Similar as in van Oordt and Zhou (2017), the relation only holds for extremely low values of X. The index T is used to distinguish β^T from the coefficient in a global linear model. If we suppose Y to be returns on a stock portfolio and X to be returns of the market portfolio, β^T can be regarded as a measure of systematic risk under extremely adverse market conditions.

Van Oordt and Zhou (2017) propose an estimator for β^T by exploiting the tail dependence imposed by the heavy-tailedness of X and Y. The estimator has a structure similar to a standard regression coefficient. Only instead of consisting of a correlation dependence measure and standard deviations as marginal risk measures, the estimator of β^T consists of a tail dependence measure and quantiles obtained from tail observations. Van Oordt and Zhou (2017) uses the tail dependence measure from multivariate Extreme Value Theory (EVT) as seen in equation (2),

$$\tau = \lim_{p \to 0} \quad \Pr(Y < Q_y(p) | X < Q_x(p)), \tag{2}$$

where $Q_x(p)$ and $Q_y(p)$ are defined similar as in equation (1). Since τ is the limit of a probability procedure, it holds that $0 \leq \tau \leq 1$, where $\tau = 0$ is regarded as tail independence and $\tau = 1$ as complete tail dependence.

Van Oordt and Zhou (2017) show the relation between the tail dependence measure and the coefficient β^T to be as follows:

$$\beta^T = \lim_{p \to 0} \quad (\tau(p))^{1/\alpha_x} \quad \frac{Q_y(p)}{Q_x(p)}, \qquad \text{for} \quad \alpha_y > \frac{1}{2}\alpha_x, \quad \beta^T \geq 0. \tag{3}$$

The restriction $\alpha_y > \frac{1}{2}\alpha_x$ requires Y not to be too heavy-tailed compared to X. If Y is too heavy-tailed, the impact of extreme values of ϵ would overshadow the impact of extreme values of X.

2.1 Estimating tail beta

As in van Oordt and Zhou (2017), to estimate β^T we estimate each component in equation (3). Suppose we observe independent and identically distributed observations (X_1, Y_1),...,(X_n, Y_n). To mimic the limit procedure from equation (**??**), only the lowest k observations in the tail region are considered, where $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$. The low probability in equation (**??**) is set as $\frac{k}{n}$. The estimator for the tail beta from van Oordt and Zhou (2017) is given in equation (4),

$$\hat{\beta}^T = \hat{\tau}(\frac{k}{n})^{1/\hat{\alpha}_x} \quad \frac{\hat{Q}_y(\frac{k}{n})}{\hat{Q}_x(\frac{k}{n})}, \tag{4}$$

where $\hat{\tau}(\frac{k}{n})$ is the estimate of the tail dependence measure considered at $\frac{k}{n}$, $1/\hat{\alpha}_x$ is the estimate of the tail index and $\hat{Q}_y(\frac{k}{n})$ and $\hat{Q}_x(\frac{k}{n})$ are estimates of the quantiles of Y and X respectively, at $\frac{k}{n}$.

2.2 Estimating tail dependence measure

Suppose we order the observations as $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ and $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$. Equation (5) gives the tail dependence measure estimator used in van Oordt and Zhou (2017). This is a nonparametric estimator, provided by Embrechts et al. (1999):

$$\hat{\tau}(\frac{k}{n}) = \frac{1}{k}\sum_{t=1}^{n} \mathbf{1}\{Y_t < Y_{k+1,n}, X_t < X_{k+1,n}\}. \tag{5}$$

2.3 Estimating the quantiles

The quantiles of X and Y at $\frac{k}{n}$, $\hat{Q}_x(\frac{k}{n})$ and $\hat{Q}_y(\frac{k}{n})$ respectively, are estimated by their $(k+1)$-th lowest order statistic. That is, $X_{k+1,n}$ and $Y_{k+1,n}$.

2.4 Estimating the tail index measure (van Oordt and Zhou (2017))

For estimating $\hat{\alpha}_x$, equation (6) gives the so-called Hill estimator used in van Oordt and Zhou (2017),

$$\frac{1}{\hat{\alpha}_x} = \frac{1}{k_1}\sum_{i=1}^{k_1} \log(\frac{X_{i,n}}{X_{k+1,n}}), \tag{6}$$

where k_1 is another number of observations such that $k_1 = k_1(n) \to \infty$ and $k_1/n \to 0$ as $n \to \infty$. As in van Oordt and Zhou (2017), I choose the possibly non-optimal choice $k_1 = k$. This is what is used in the simulation and empirical illustration.

2.5 Estimating the bias corrected tail index measure

Gomes et al. (2007) propose an estimator for the tail index that is less biased than the Hill estimator. The estimator is computed by a weighted combination of log-excesses of the observations of X, where the log-excesses are defined by $V_{ik} = \log(\frac{X_{i,n}}{X_{k+1,n}})$. The relation of the log-excesses to the Hill estimator is given as $\frac{1}{\hat{\alpha_x}} = \frac{1}{k_1}\sum_{i=1}^{k_1} V_{ik}$. The bias corrected estimator for the tail index measure proposed in Gomes et al. (2007), is given in equation (7),

$$WLE_{\hat{\beta},\hat{\rho}}(k_1) = \frac{1}{k_1}\sum_{i=1}^{k_1} p_{ik_1}(\hat{\beta},\hat{\rho})\{\log(\frac{X_{i,n}}{X_{k_1+1}})\}, \tag{7}$$

where $p_{ik_1}(\hat{\beta},\hat{\rho})$ is the set of weights dependent on estimates of scale parameter β and shape parameter ρ. The estimator is based on k_1 observations, again an intermediate sequence as for the Hill estimator in equation (6).

$$p_{ik_1}(\hat{\beta},\hat{\rho}) = \exp\{-\hat{\beta}\,(\frac{n}{k_1})^{\hat{\rho}}\;\hat{\psi}_{ik_1}\} \tag{8}$$

$$\hat{\psi}_{ik_1} = -\frac{(i/k_1)^{-\hat{\rho}}-1}{\hat{\rho}\log(i/k_1)} \qquad 1 \leq i \leq k_1 \tag{9}$$

Gomes et al. (2007) propose the set of weights as in equation (8), and advice for explicit external estimation of $\hat{\beta}$ and $\hat{\rho}$ based on k_2 observations.

Equation (10) to (12) give the estimator of shape parameter ρ as in Gomes et al. (2007):

$$M_n^{(j)}(k_2) = \frac{1}{k_2}\sum_{i=1}^{k_2}\{\log(\frac{X_{i,n}}{X_{k_2+1,n}})\}^j, \tag{10}$$

$$T_n^{(\tau)}(k_2) = \frac{\{M_n^{(1)}(k_2)\}^{\tau} - \{M_n^{(2)}(k_2)/2\}^{\tau/2}}{\{M_n^{(2)}(k_2)/2\}^{\tau/2} - \{M_n^{(3)}(k_2)/6\}^{\tau/3}}, \tag{11}$$

$$\hat{\rho}(k_2;\tau) = -|3\{T_n^{(\tau)}(k_2)-1\}/\{T_n^{(\tau)}(k_2)-3\}|. \tag{12}$$

Equations (13) to (16) give the estimator of scale parameter β used in Gomes et al. (2007):

$$W_i = i\,\{\log(\frac{X_{i,n}}{X_{k_2+1,n}})\}, \tag{13}$$

$$d_{k_2}(\alpha) = \frac{1}{k_2}\sum_{i=1}^{k_2}(\frac{i}{k_2})^{\alpha}, \tag{14}$$

$$D_{k_2}(\alpha) = \frac{1}{k_2}\sum_{i=1}^{k_2}(\frac{i}{k_2})^{\alpha}\,W_i, \tag{15}$$

$$\hat{\beta}(k_2;r) = (\frac{k_2}{n})^r\;\frac{d_{k_2}(-r)\,D_{k_2}(0) - D_{k_2}(-r)}{d_{k_2}(r)\,D_{k_2}(-r) - D_{k_2}(-2r)}. \tag{16}$$

In equation (16), r is a consistent estimate of shape parameter ρ. The estimate in equation (12) is used.

Gomes et al. (2007) advise to estimate the shape and scale parameter based on k_2 observations, where k_2 is of a larger order than k_1. An optimal choice for k_2 is not yet useful in practice. Gomes et al. (2007) propose a possibly non-optimal choice of the type $k_2 = n^{1-\epsilon}$, for small $\epsilon > 0$. Furthermore, the estimation of ρ depends on a tuning parameter τ. The (simple) choice of $\tau = 1$ is used. A robustness check is done for $\tau = 0$. In equation (11), the case of $\tau = 0$ is defined by continuity. These choices for τ are inspired by the algorithm for choosing τ proposed by Gomes et al. (2007).

Now for $k_2 > k_1$ and $\tau \in \{0, 1\}$ the bias corrected estimator for the tail index is given in equation (7).

2.6 Comparing by simulation

To compare the performance of using the bias corrected tail index estimator with the estimator from van Oordt and Zhou (2017) to estimate tail beta, a simulation is set up similar to van Oordt and Zhou (2017). The generated sample consists of 1250 random observations for (X_t, Y_t). This is approximately the amount of observations that will later be used in an empirical illustration. The estimation accuracy of using the two different estimators for the tail index to estimate tail beta are compared, by comparing the estimated and true values for β^T from equation (1). As in van Oordt and Zhou (2017), I consider global linear models and segmented linear models. In the global linear models, the relation between the variables is unaffected by the value of X, that is $\beta = \beta^T$. Three values for the coefficient in the global linear model are considered: $\beta = 0.5$, $\beta = 1$ and $\beta = 1.5$. In the segmented linear models, if the observation of X is larger than the third percentile of X then Y is generated from a linear model with $\beta = 1$. Otherwise it is generated from a linear model with $\beta^T = 0.5$ and $\beta^T = 1.5$. This is in line with the simulation performed in van Oordt and Zhou (2017).

The simulation is performed so that X and ϵ exhibit temporal dependence and are generated from a GARCH(1,1) process. Equations (17) and (18) show the data-generating process for $Z = (X, \epsilon)$, with corresponding parameter choices when using innovations from a standardized Student's tdistribution with eight degrees of freedom:

$$Z_t = \sigma_{Z,t}\zeta_t, \tag{17}$$

$$\sigma_{Z,t} = \psi_0 + \psi_1 Z_{t-1}^2 + \psi_2 \sigma_{Z,t-1}^2, \tag{18}$$

where $(\psi_0, \psi_1, \psi_2) = (0.5, 0.08, 0.91)$. These parameter choices are the same as in van Oordt and Zhou (2017). For the standardized Student's t-distributed innovations, X and ϵ are heavy-tailed with tail index 3.82.

As in van Oordt and Zhou (2017), 10,000 samples will be generated of the three global linear models and the two linear segmented models. In each sample β^T is estimated using the bias corrected tail index estimator and the Hill estimator. Furthermore, β^T is estimated and plotted for multiple values of k, the amount of observations used in the estimation of the tail dependence measure and quantiles. The performance of these two estimators is evaluated by comparing the tail beta estimates with the real value of β^T. The Mean Squared Error (MSE), estimation bias and estimation variance are considered.

3 Empirical illustration

Finally, the performance of the two estimators for tail beta are evaluated in an empirical illustration. Since the real values of tail beta are unobserved, the accuracy of the estimates relative to the real value cannot be evaluated. The impact of using the different tail index estimators on the estimated value of β^T is evaluated. I consider the average, minimum and maximum value of the estimated tail betas.

For the empirical illustration, I use the same data as used in van Oordt and Zhou (2017). Data on 48 value-weighted industry-specific stock portfolios, and a general market index in the United States are used. The data is obtained from the personal website of Kenneth French [1]. The secondary data are based on returns of stocks listed on NYSE, AMEX and NASDAQ in the CRSP database. The data runs from 1931 to 2015 in 17 five-year subperiods. The average number of observations per subperiod is 1312. The performance of the two tail beta estimators are assessed by projecting the losses of industry portfolios on the day of the largest market loss within each subperiod. Within

[1] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

each subperiod the observation of the day on which the largest market losses are observed, are excluded. This observation is used as a pseudo out-of-sample observation, to assess the estimation accuracy of the two estimators.

4 Results

4.1 Simulation

The bias corrected tail index estimator is based on the logarithm of ratios between observations beyond a certain threshold, and the threshold itself. In section 2.5 this is denoted as $\log(\frac{X_{i,n}}{X_{k+1,n}})$, where the threshold is the observation $X_{k+1,n}$. Because of this, the sign of the observations used need to be the same as the threshold. Since we consider left tail observations in this paper, this means the observations used and the threshold need to be negative. With the choice of $k_2 = n^{1-\epsilon}$ for small $\epsilon > 0$ mentioned in section 2.5, the sign of the threshold and the observations used might differ. To solve this issue, I use $k_2 = n_-^{1-\epsilon}$ for small $\epsilon > 0$, where n_- is defined as the number of negative observations in the sample. Furthermore, for the bias corrected tail index estimator we can use a k_1 that is much higher than for the original Hill estimator. This is the benefit received from the bias correction. In the simulation and empirical illustration I use $k_1 = \frac{n_-}{2}$ and $k_2 = n_-^{0.98}$ for the bias corrected tail index estimator. For these values both the observations used and the threshold will be strictly negative and $k_2 > k_1$, while k_1 is higher than for the original Hill estimator. Only after fixing these values for k_1 and k_2, we can estimate β^T for multiple values of k. So here, the amount of observations only varies in the estimation of the tail dependence measure and quantiles. In the simulation and empirical illustration, it holds that $k_2 > k_1 > k$.

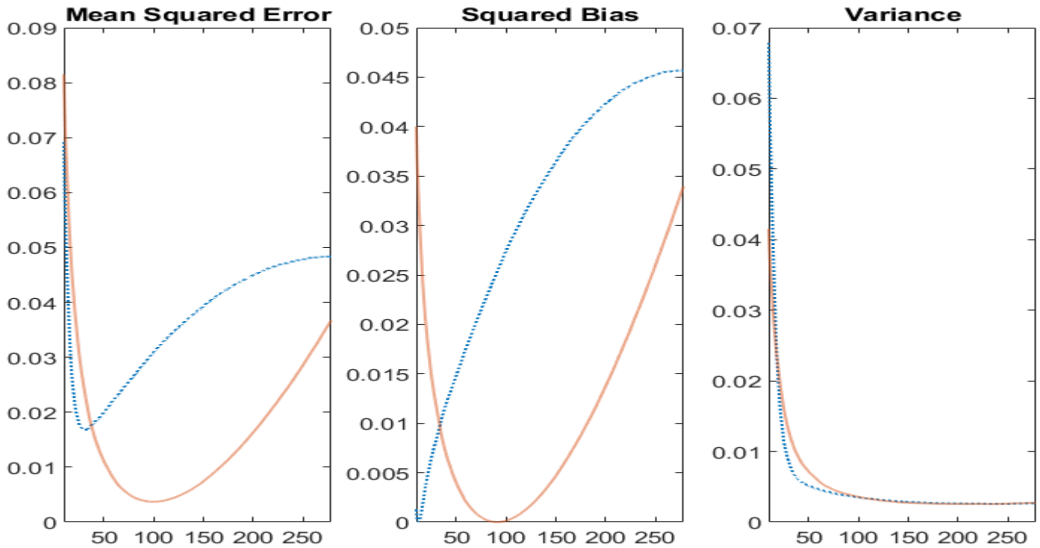

Figure 1. Global linear model, $\beta^T = \beta = 0.5$

Horizontal axis: number of observations used in estimation, k

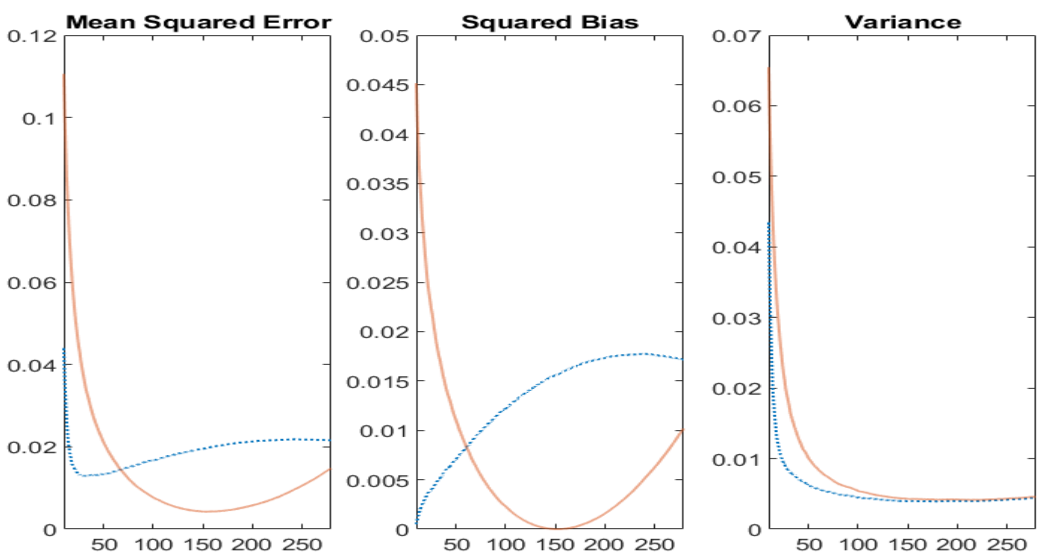

Figure 2. Global linear model, $\beta^T = \beta = 1.0$

Horizontal axis: number of observations used in estimation, k

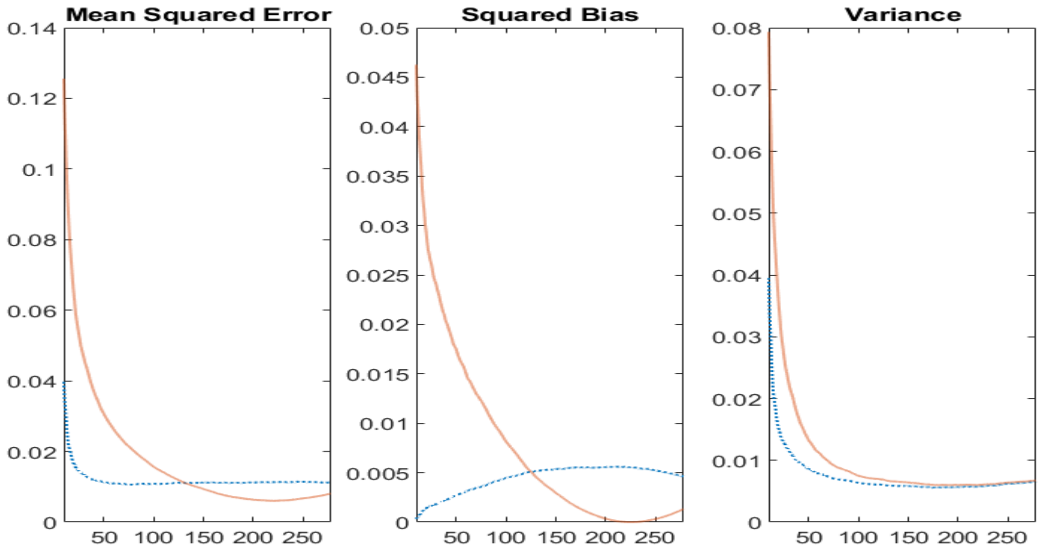

Figure 3. Global linear model, $\beta^T = \beta = 1.5$

Horizontal axis: number of observations used in estimation, k

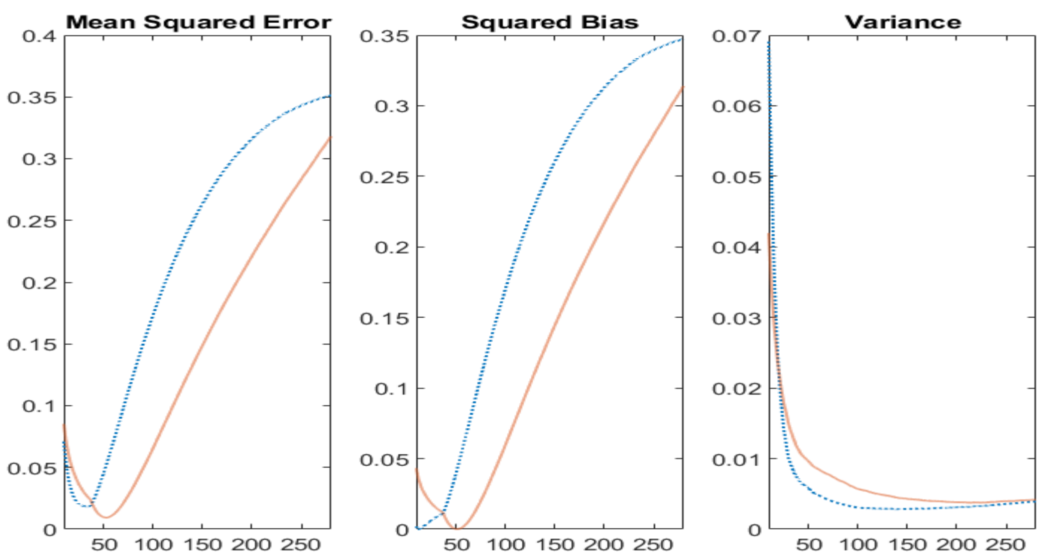

Figure 4. Linear segmented model, $\beta^T = 0.5, \beta = 1$

Horizontal axis: number of observations used in estimation, k

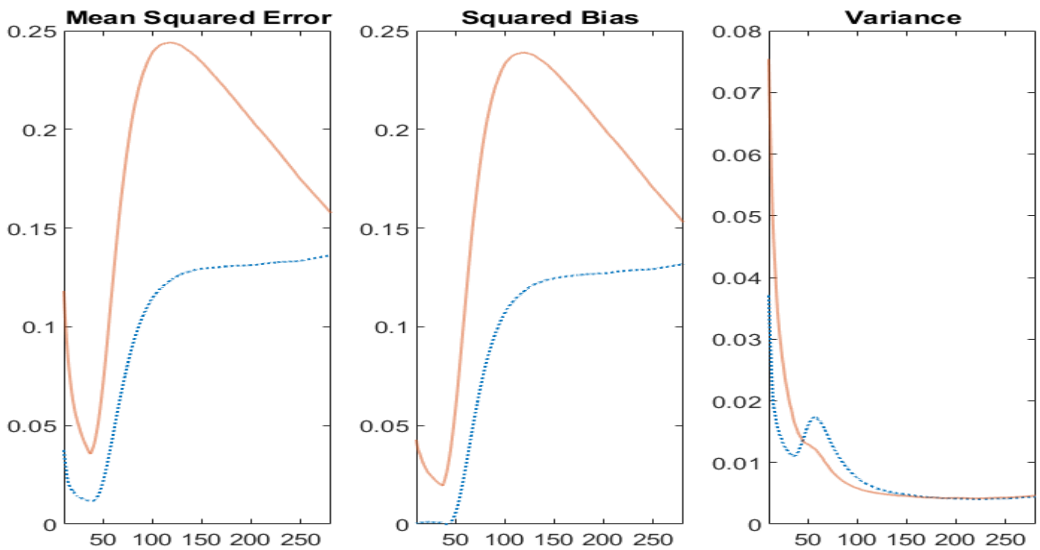

Figure 5. Linear segmented model, $\beta^T = 1.5, \beta = 1$

Horizontal axis: number of observations used in estimation, k

Figure 1 to 5 show the MSE, squared estimation bias and variance of the estimated tail beta when using the bias corrected tail index estimator or the standard Hill estimator. In all figures, the dotted line corresponds to the tail beta estimated using the Hill estimator. The solid line corresponds to the tail beta estimated using the bias corrected tail index estimator. For figure 1 to 5, a tuning parameter of $\tau = 1$ is used in the estimation of shape parameter ρ (see section 2.5).

For the global linear models (figure 1 to 3), we observe that only for small k, using the Hill estimator gives a lower MSE and squared bias for the tail beta estimates. The estimation variance of tail beta is higher when using the bias corrected tail index estimator of equation (7) instead of the Hill estimator in estimating the tail beta, for all k in the global linear models. Furthermore, for all global linear models we observe that the MSE and squared bias of the tail beta estimate is an increasing function in k (up to a certain point) when using the Hill estimator. However, for the bias corrected tail beta estimate we observe that the MSE and squared bias are a decreasing function in k for a larger interval of k. This leads to the 'optimal' value of k (which I will consider as minimal MSE) to be higher for all global linear models. Moreover, for all global linear models the MSE at the optimal value of k is lower for the bias corrected estimator than for the tail beta estimator using the Hill estimator.

For the linear segmented model in the case of $\beta^T = 0.5$, $\beta = 1$, again the Hill estimator only gives lower MSE and squared bias for (very) small k in the estimation of tail beta, but a lower estimation variance. We also observe that for the bias corrected tail beta estimates, the MSE and squared bias are a decreasing function in k for a larger interval of k. This leads to the 'optimal' value of k to be higher. However, the difference between the values for optimal k is noticeably smaller than in the global linear models. For the linear segmented model in the case of $\beta^T = 1.5$, $\beta = 1$, the Hill estimator gives lower MSE and squared bias in the estimation for tail beta for all k. The bias corrected tail beta estimator gives lower estimation variance between values of k of approximately 50 and 150.

One exceptional case is that the MSE and squared bias of the estimated tail beta is higher when using a bias corrected tail index estimator, for the linear segmented model with $\beta^T = 1.5$ and $\beta = 1$.

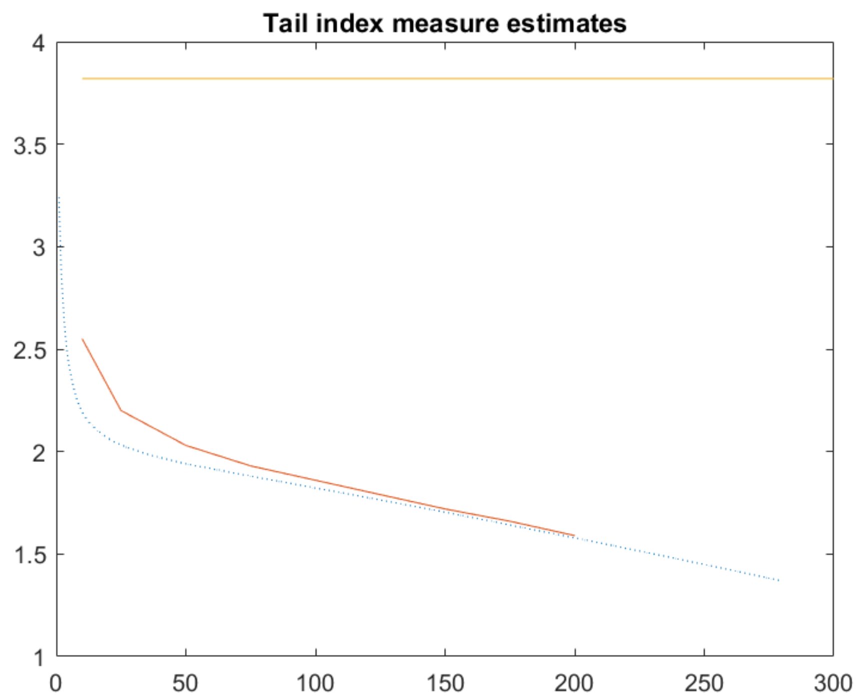

Figure 6. Tail index estimates

Horizontal axis: number of observations used in estimation, k

Figure 6 shows the estimates of the Hill estimator and bias corrected tail index estimator, plotted against the number of observations used in the estimation of the tail index estimator. The dotted line corresponds to the Hill estimator, the solid orange line to the bias corrected tail index estimator. The solid yellow line at 3.82 corresponds to the real value of the tail index, which is known in the simulation (see section 2.6). We observe that the Hill estimator consistently provides lower tail index estimates than the bias corrected tail index estimator. We also observe that the bias corrected tail index estimator gives estimates closer to the real tail index.

For the linear segmented model with $\beta^T = 1.5$ and $\beta = 1$, the tail dependence measure (see equation (5)) is estimated lower when an increasing amount of observations are used in the estimation. This is because when more observations are used, we move into an area where the dependence structure becomes $\beta = 1$, thus lower than the real tail beta $\beta^T = 1.5$. This is shown in figures 7 and 8:

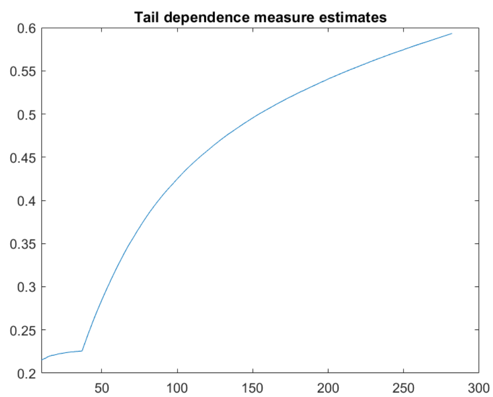

Figure 7. Tail dependence estimates, $\beta^T = 0.5$, $\beta = 1$

Horizontal axis: number of observations used in estimation, k

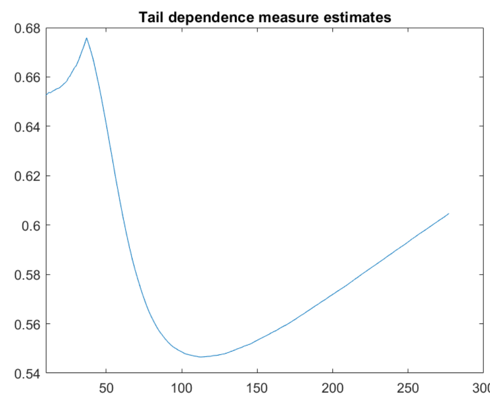

Figure 8. Tail dependence estimates, $\beta^T = 1.5$, $\beta = 1$

Horizontal axis: number of observations used in estimation, k

This downward biased estimation of the tail dependence measure can be compensated by the downward biased estimation of the tail index by the Hill estimator, as seen in figure 6. This way, worse (lower) estimation of the tail index can actually lead to better tail beta estimates. This explains the results shown in figure 5.

To summarize, using the bias corrected tail index estimator proposed by Gomes et al. (2007) to estimate the tail beta, outperforms using the standard Hill estimator used by van Oordt and Zhou (2017). The bias corrected estimator performs better based on mean squared estimation error and squared estimation bias, when using a higher number of observations (k) in the estimation. This holds for three types of global linear models when $\beta^T = 0.5, \beta^T = 1, \beta^T = 1.5$ and for a linear segmented model when $\beta^T = 0.5, \beta = 1$. The bias corrected estimator does not perform better in the linear segmented model when $\beta^T = 1.5, \beta = 1$. This has some serious implications for empirical applications of using the bias corrected tail index estimator to estimate tail beta. In empirical applications, the dependence structure of the underlying assets are unobserved. If the underlying assets approximately follow a global linear model, using the bias corrected estimator is preferred, together with using a relatively high number of observations (k) in the estimation. However, if the underlying assets are likely to follow a linear segmented model where the tail dependence is higher than the dependence in the rest of the observations, using the standard Hill estimator while using a relatively low number of observations in the estimation is preferred. To make a choice on which tail index estimator to use and the amount of observations to use in the estimation, investors must research the dependence structure of the underlying assets. For example, this can be done by making plots as in figure 7 and 8 for the assets. When observing declining tail dependence measure estimates while the number of observation used in the estimation increases (as in figure 8), the investor should choose to use the standard Hill estimator as in van Oordt and Zhou (2017) to estimate tail beta. When this is not the case, the investor should choose to use the bias corrected tail index estimator from Gomes et al. (2007) in the estimation of tail beta.

4.2 Robustness check for tuning parameter τ

As mentioned in section 2.5, a robustness check is done for the tuning parameter τ which is used in the external estimation of shape parameter ρ, for $\tau = 0$.

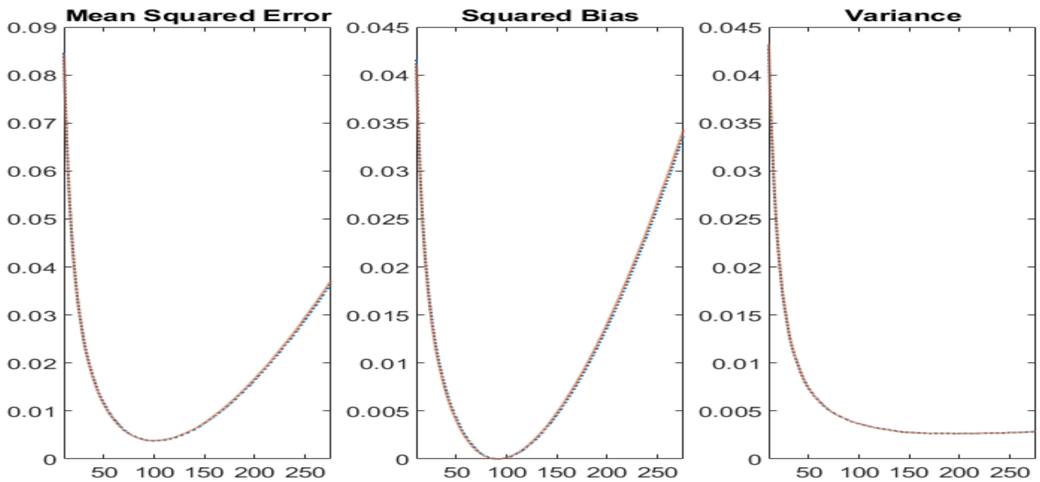

Figure 9. Robustness check for global linear model, $\beta^T = \beta = 0.5$

Horizontal axis: number of observations used in estimation, k

Figure 9 shows the MSE, squared bias and estimation variance of the estimates of tail beta, when tail beta is estimated using the bias corrected tail index estimator. The dotted line corresponds to the case when tuning parameter $\tau = 1$, the solid line to the case when $\tau = 0$. We conclude that both values for τ perform equally, since there is no clear difference between the lines. The performance of the bias corrected tail beta estimator is robust to the change of tuning parameter $\tau = 1$ to $\tau = 0$. Figure 9 shows the performance for the global linear model when $\beta^T = \beta = 0.5$, but the pattern of equal performance holds across the three global linear models and two linear segmented models discussed previously.

4.3 Empirical illustration

Since the real structure of the industry-specific stocks are unobserved, some assumptions have to be made to choose the value of k at which we consider the bias corrected tail index estimator for the empirical illustration. This is because in the simulation we have shown for which values of k the bias corrected tail beta estimator outperforms the tail beta estimator using the Hill estimator. We assume the structure of the industry-specific stocks to be as a global linear model. For this purpose, I use $k_1 = 150$. This follows from the simulation results, and works well in practice. At this value in the simulation, the MSE of the bias corrected tail beta estimator is lower than for the tail beta using the standard Hill estimator considered at its optimal value for k (approximately 25), for the three global linear models. As in the simulation, I use $k_2 = n_-^{0.98}$. The average, minimum and maximum tail betas of the industry-specific stock portfolios are computed for the tail beta using the standard Hill estimator for $k = 25$ observations, and for the bias corrected tail beta estimator for $k = 100$ observations. These values follow from the simulation results.

Table 1

Estimates

Period	Av. $\hat{\beta}_h^T$	Av. $\hat{\beta}_c^T$	N	Min. $\hat{\beta}_h^T$		Min. $\hat{\beta}_c^T$		Max. $\hat{\beta}_h^T$		Max. $\hat{\beta}_c^T$	
1931-1935	1.16	1.01	43	0.58	Clths	0.41	Clths	2.48	Paper	2.32	Paper
1936-1940	1.08	1.03	43	0.42	Smoke	0.42	Smoke	2.02	Toys	2.18	Cnstr
1941-1945	1.11	1.06	43	0	Rubbr	0	Rubbr	2.41	RlEst	2.05	Cnstr
1946-1950	1.05	1.01	43	0.35	Telcm	0.33	Telcm	1.61	Cnstr	1.72	Cnstr
1951-1955	0.96	0.91	43	0.36	Telcm	0.31	Telcm	1.56	Aero	1.59	Aero
1956-1960	1.07	0.92	43	0.52	Food	0.37	Util	1.74	Chips	1.47	Steel
1961-1965	1.13	1.08	43	0.60	Util	0.44	Util	1.93	Toys	1.91	Toys
1966-1970	1.23	1.09	47	0.53	Util	0.47	Util	1.93	Other	1.60	PerSv
1971-1975	1.13	1.02	48	0.64	Util	0.53	Util	1.74	Fun	1.60	Meals
1976-1980	1.07	1.01	48	0.58	Util	0.49	Util	1.61	Health	1.48	Coal
1981-1985	1.06	1.00	48	0.60	Util	0.49	Util	2.19	Gold	1.82	Gold
1986-1990	0.97	0.95	48	0.53	Util	0.52	Util	1.29	Soda	1.36	Soda
1991-1995	1.13	1.01	48	0.63	Util	0.55	Util	1.75	Ships	1.43	Comps
1996-2000	0.98	0.81	48	0.44	Util	0.38	Util	1.76	Coal	1.46	Coal
2001-2005	1.00	0.92	48	0.58	Food	0.49	Food	1.69	Chips	1.87	Chips
2006-2010	1.11	1.06	48	0.56	Beer	0.53	Beer	2.14	Coal	1.84	Coal
2011-2015	1.08	1.06	48	0.61	Beer	0.60	Beer	2.24	Coal	1.87	Coal

Notes. The tail beta estimate when using the Hill estimator is denoted as $\hat{\beta}_h^T$. When using the bias corrected tail index estimator, is denoted as $\hat{\beta}_c^T$. N shows the number of industry-specific stocks used in the estimation. Industry portfolios with missing observations were excluded. In every period the condition $\hat{\alpha}_Y > \hat{\alpha}_X$ was met.

Table 1 shows the average, minimum and maximum tail beta estimates of the 48 industry-specific portfolios in the corresponding five-year subperiods, for the tail beta estimates using the Hill estimator and the bias corrected tail index estimator when estimating tail beta. For every subperiod, the bias corrected tail beta estimator gives a lower or equal average and minimum tail beta. For most subperiods, it also gives a lower maximum tail beta. Furthermore, the bias corrected tail beta estimator leads to a different industry-specific portfolio to have minimum tail beta in one case, and a different maximum tail beta in seven cases. This implies that using the bias corrected tail index estimator to estimate tail beta may lead to different investment decisions, when investors consider the tail beta of industry-specific stock portfolios.

The performance of the two tail beta estimators are also assessed by their ability to project losses of industry-specific portfolios, of the day on which the largest market loss is observed in each subperiod. The projections under the two approaches are,

$$\hat{L}_{Hill} = L_m \hat{\beta}_h^T \tag{19}$$

$$\hat{L}_{BC} = L_m \hat{\beta}_c^T \tag{20}$$

where L_m is the largest market loss observed in the subperiod, the tail beta estimates are as defined in table 1, $\hat{L}_{Hill}$ the projected loss by the tail beta estimator using the Hill estimator and $\hat{L}_{BC}$ the projected loss by the bias corrected tail beta estimator. To evaluate the loss projection of the two tail beta estimators, I compute the Root Mean Squared Error (RMSE),

$$RMSE_{Hill} = \sqrt{\frac{1}{N}\sum_j e^2_{Hill,j}} \tag{21}$$

$$RMSE_{BC} = \sqrt{\frac{1}{N}\sum_j e^2_{BC,j}} \tag{22}$$

where $e^2_{Hill,j}$ is the squared error of the projected loss with the real portfolio loss of portfolio j when using the Hill estimator to estimate tail beta, $e^2_{BC,j}$ is the squared error of the projected loss with the real portfolio loss of portfolio j when using the bias corrected estimator. The tail beta estimator that reports a lower RMSE, performs better. As in van Oordt and Zhou (2017), to test the statistical significance of the difference in RMSE I use a Diebold and Mariano (1995) type test with small-sample correction as proposed by Harvey et al. (1997).

$$d_j = e^2_{BC,j} - e^2_{Hill,j}, \qquad \bar{d} = \frac{1}{N}\sum_j d_j \tag{23}$$

$$t - \text{stat} = \frac{\bar{d}}{\sqrt{\hat{V}(\bar{d})/(N-1)}} \tag{24}$$

$$\hat{V}(\bar{d}) = \frac{\sum_j (d_j - \bar{d})^2}{N} \tag{25}$$

Equation (24) shows the test statistic for each subperiod, where the test statistic follows a Student's t-distribution with (N-1) degrees of freedom (Harvey et al. (1997)). For this test, normality of the differences d_j is assumed. Although we use heavy-tailed data on stock returns, Harvey et al. (1997) show that the test is not very sensitive to the presence of heavy tails.

Table 2

Performance evaluation

Period	Worst day	Market loss	RMSE Hill	RMSE BC	t-Stat	p-Value
1931-1935	July 21, 1933	-9.21	4.71	5.27	1.69	0.049
1936-1940	October 18, 1937	-8.17	3.02	3.12	0.61	0.271
1941-1945	December 8, 1941	-4.15	1.96	1.97	0.14	0.447
1946-1950	September 3, 1946	-6.90	1.40	1.51	0.99	0.164
1951-1955	September 26, 1955	-6.52	1.34	1.32	-0.20	0.422
1956-1960	October 21, 1957	-3.04	1.20	1.38	1.30	0.101
1961-1965	May 28, 1962	-7.00	2.47	2.55	0.87	0.194
1966-1970	May 25, 1970	-3.21	1.49	1.61	1.00	0.160
1971-1975	November 18, 1974	-3.57	0.97	1.02	0.44	0.331
1976-1980	October 9, 1979	-3.44	0.95	1.10	1.74	0.044
1981-1985	October 25, 1982	-3.62	0.95	0.90	-0.62	0.27
1986-1990	October 19, 1987	-17.44	3.97	3.69	-1.26	0.107
1991-1995	November 15, 1991	-3.55	1.88	1.55	-2.43	**0.009**
1996-2000	April 14, 2000	-6.72	3.14	2.59	-2.61	**0.006**
2001-2005	September 17, 2001	-5.03	5.14	5.08	-0.54	0.296
2006-2010	December 1, 2008	-8.95	1.30	1.36	0.63	0.266
2011-2015	August 8, 2011	-6.97	1.77	1.54	-1.14	0.13

Note. The RMSE of the projected loss when using the standard Hill estimator to estimate tail beta is denoted as RMSE Hill. When using the bias corrected tail beta estimator, is denoted as RMSE BC. For the two-sided t-type test, statistically significant p-Values on a 5% level are in bold.

Table 2 shows the results of the performance evaluation of the ability to project large market losses, by the two tail beta estimators. In total, we observe that in two subperiods there is a statistically significant difference in RMSE between the two tail beta estimators. In these two cases, the bias corrected tail beta estimator outperforms the tail beta estimator using the Hill estimator. On average, the bias corrected estimator leads to a 0.73% larger RMSE than the tail beta estimator using the Hill estimator. In general when looking at the errors $e_{Hill,j}$ and $e_{BC,j}$ across all subperiods and industry-specific stock portfolios, we see that the bias corrected estimator makes lower loss projections. This is in line with the expectation, since the simulation showed that the bias corrected tail index estimator makes higher tail index estimates than the Hill estimator. This leads to a lower tail beta, thus lower loss projections. When looking at the errors in periods where the bias corrected estimator outperforms the tail beta estimator using the Hill estimator, such as 1991-1995 and 1996-2000, the projection errors show that this lower loss projection is closer to the real industry portfolio loss in most cases. So for most industry portfolios in these periods, the lower loss projection by the bias corrected estimator is beneficial. However, when looking at periods where the tail beta estimator using the Hill estimator outperforms the bias corrected estimator (albeit not statistically significant) such as 1931-1935, the lower loss projection of the bias corrected estimator is not always beneficial. For some industry portfolios the lower projection causes the loss projection to be further from the real portfolio loss. Most likely, in these cases a similar phenomenon is happening as with the simulated linear segmented model when $\beta^T > \beta$. The tail dependence measure is estimated with a downward bias, which the Hill estimator counters with a downward biased estimation of the tail index. This way, the tail beta is actually estimated more accurately. Because of this phenomenon, investors must be careful when using the bias corrected tail index estimator proposed by Gomes et al. (2007). Careful research into the dependence structure of the industry-specific stock portfolio is necessary to determine if the bias corrected estimator will provide more accurate tail beta estimates than the tail beta estimator using the Hill estimator. For example, investors can make a plot of the tail dependence measure from equation (5) against the number of observations used in the estimation (as in figures 7 and 8), for the considered assets. When investors observe a declining value for the tail dependence measure when the amount of observations used in the estimation increases, using the Hill estimator is advised for estimating tail beta. When this is not the case, using the bias corrected tail index estimator from Gomes et al. (2007) likely gives more accurate tail beta estimates, so this is advised.

We conclude that the bias corrected tail beta estimator performs slightly better than the tail beta estimator using the Hill estimator, in projecting large market losses in a market crash. Although the average RMSE of the loss projections did not improve when using the bias corrected estimator, the only statistically significant differences occurred when the tail beta estimator using the Hill estimator was outperformed. Furthermore, the bias corrected estimator especially seems to work better in more recent years. This might be interesting to investors looking to use this method the coming time.

5 Conclusion

In order to analyze systematic risk in case of a market downturn, a method that estimates systematic risk differently under adverse market conditions is necessary. van Oordt and Zhou (2017) provides such a method, by using a tail dependence measure, tail index measure and quantiles estimated from tail observations. In this paper, I evaluated the performance of a tail beta estimator using a less biased tail index estimator, proposed by Gomes et al. (2007). The performance was evaluated in a simulation study and an empirical illustration. In the simulation study, the bias corrected tail beta estimator performs better in terms of mean squared error and squared estimation bias for three global linear models, and one linear segmented model. It performs worse for one linear segmented model. This uncovered a phenomenon, where downward biased estimation of the tail index by the Hill estimator counters downward biased estimation of the tail dependence measure, to provide more accurate tail beta estimates. This implies that in order to make a correct choice on which tail index estimator and the amount of observations to use in the estimation in an empirical analysis, the tail dependence structure of the financial assets must be researched. For example, investors can make a plot of a tail dependence measure against the amount of observations used in the estimation. When observing a declining value for the tail dependence measure when the amount of observations used in the estimation increases, using the Hill estimator is advised. Else, using the bias corrected tail index estimator in the estimation of tail beta is advised.

Furthermore, an empirical analysis on 48 value-weighted industry-specific stock portfolios was performed. Firstly, I found that using the Hill estimator in the estimation of tail beta led to higher average, minimum and maximum tail beta estimates. The bias corrected tail beta estimator may lead to different investment decisions than when using the Hill estimator in the estimation of tail beta. Secondly, the performance of the two tail beta estimators was evaluated by their ability to project losses of industry-specific stock portfolios on the days that large market losses occurred. The bias corrected tail beta estimator performed slightly better in projecting these losses. In only two of the seventeen considered periods there was a statistically significant difference in the performance of the two estimators. In these two cases, the bias corrected tail beta estimator performed better than the tail beta estimator using the Hill estimator. Another positive note is that it seems to perform better in more recent years. For investors looking to use this method the coming time, this might be of interest.

For future research, one of the biggest issues to tackle is how to determine when to use the bias corrected tail index estimator in an empirical illustration. The simulation study performed in this paper shows that better estimation of the tail index does not always lead to better estimates of tail beta. In this paper I propose to make a plot of the value of a tail dependence measure against the amount of observations used in the estimation, to determine which tail index estimator to use.

How to do this efficiently in an empirical application and how well this method performs, should be researched further. Further research could also look into combining better estimation of the tail dependence measure with better estimation of the tail index, to get more accurate tail beta estimates. For example, one could look at combining the bias corrected tail dependence measure provided by Fougères et al. (2015) with the tail index estimator of Gomes et al. (2007).

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