ERASMUS UNIVERSITY ROTTERDAM

Erasmus School of Economics

An Improved Estimator for the Systematic Risk Under Extremely Adverse Market Conditions

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Bsc Thesis Econometrics and Operations Research

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Abstract

This paper analyses the estimation of systematic risk. In particular, I attempt to answer the research question whether we can adequately estimate systematic risk, given extremely low values of the market portfolio. The foundation of this research is the estimator proposed in van Oordt and Zhou (2019). A key extension of this estimator is an improved estimator for the tail index and results in an overall better performance. I assess the performance of the improved estimator in a simulation study as well as in an empirical application. In the simulation study, the improved estimator has a lower mean squared error than the original estimator of van Oordt and Zhou (2019). In the empirical application, the improved estimator produces lower root mean squared errors compared to the original estimator and the estimator of the conditional regression approach.

Keywords: extreme value theory, bias correction, tail dependence, systematic tail risk

The views stated in this thesis are those of the author and not necessarily those of the supervisor, second assessor, Erasmus School of Economics or Erasmus University Rotterdam.

Contents

1 Introduction

A widely used method to model the (excess) return of a financial asset is the Capital Asset Pricing Model (CAPM). The (excess) return is modeled as a linear function of the (excess) market return with the slope defined as the sensitivity of the asset to the market risk (Sharpe (1964)). This sensitivity is also called systematic risk. Hence the return of an asset, depends on the market conditions. A good estimation of the systematic risk, under the right conditions, is thus crucial for modelling asset prices correctly.

Since extreme losses often occur in times of financial distress, investors who are concerned about these possible losses may need to analyze their systematic risk under non-favourable market conditions, e.g. market crashes. This raises the question whether we can adequately estimate systematic risk, given extremely adverse market conditions.

Estimating the systematic risk can be done via a standard univariate regression using historical data. However, historical data includes periods where the market conditions were not volatile. Hence an accurate estimate of the systematic risk, in times of financial distress, may not be obtained via simple regression. One might consider to employ data where the market condition is unstable and apply a conditional regression, using only observation corresponding to extremely low market returns. However, the estimator following this approach might suffer from a relatively large variance, since a small number of observations is included.

van Oordt and Zhou (2019) answer the research question by deriving an estimator for the systematic risk, given extremely low values of the market portfolio. This paper replicates their results and further improves the estimator. More specifically, I will improve the estimator of van Oordt and Zhou (2019) by analyzing its subcomponents and improving them individually. An improved estimator of van Oordt and Zhou (2019) will result in more accurate estimates of the systematic risk under extremely adverse market conditions.

Financial asset returns are often fat-tailed (see Campbell, Lo, and MacKinlay (1997), Embrechts, Klüppelberg, and Mikosch (1997) and Mikosh (2003)). More explicitly, a typical feature for the distribution of financial returns and losses is that the distributions have a high peak and that they are longer and heavier in the tail. This feature is also called leptokurtic and it stems from the stylized fact of asset returns (Cont (2001)). Another result of the stylized fact of asset returns is that large negative returns occur more often than large positive ones, implying that asset returns are negatively skewed. Consequently, the tail of the normal distribution is not appropriate for modelling financial returns, in the sense that it decays faster than the tail of a fat-tailed distribution. Thus we can not adequately capture extreme losses in the tail.

A potential solution for this problem is to model the tail of the distribution itself and make statistical inference using only observations in the tail. This is what Extreme Value Theory achieves. The estimator of van Oordt and Zhou (2019) is based on Extreme Value Theory. van Oordt and Zhou (2019) derive an estimator by exploiting the tail dependence imposed by the heavy-tailedness of two variables. Under mild conditions, this estimator is consistent and asymptotically normal. Their estimator results in a lower mean squared error compared to the estimator of a conditional regression on tail observations. In short, the estimator of van Oordt and Zhou (2019) consists of a tail dependence measure between the two heavy-tailed variables, quantile estimates for both variables and a tail index estimator. The estimator for some of these components are biased and sensitive to the number of observations in the tail region. For example, van Oordt and Zhou (2019) use the estimator of Hill (see Hill (1975)) as their estimator for the tail index.

The Hill estimator is widely used by researchers and practitioners, even though the Hill estimator may suffer from an asymptotic bias and is very sensitive to the number of order statistics used in the estimation procedure (Drees, de Haan, and Resnick (2000)). The bias and the sensitivity of the Hill estimator are further discussed by Caeiro, Gomes, and Pestana (2005) and Gomes, Pestana, and Caeiro (2009), respectively. Other estimators of the tail index have been proposed by several researchers, see e.g. Pickand (1975), Smith (1987) and de Haan and Pereira (1999) among others.

Next to other estimators of the tail index, a lot of research focused on improving the Hill estimator itself. Paulauskas and Vaiciulis (2013) use a family of distribution functions which satisfies the so-called second-order regular-variation (SORV) condition. This improved estimator reduces the asymptotic mean squared error of the Hill estimator. Next to the lowest order statistic, Nuyts (2010) suggests to also take the highest order statistic into consideration when estimating the Hill estimator. This alteration of the estimator improves the performance of the Hill estimator in cases where it originally performed badly. To reduce the sensitivity of the Hill estimator to the number of order statistics used, Resnick and Stărică (1997) apply a smoothing procedure. In this procedure, Resnick and Stărică (1997) averages the values of the Hill estimates corresponding to different number of order statistics. However, the bias of the Hill estimator still remains after the smoothing procedure. In order to reduce the bias of the Hill estimator, Huisman et al. (2001) propose a weighted average of a set of Hill estimators each conditioned on different number of tail observations, where the weights are determined with generalized least squares. de Haan, Mercadier, and Zhou (2016) propose an estimation procedure for the second order parameters and use the estimated second order parameters to propose a new, bias corrected estimator of the extreme value index. A drawback of this procedure is that the second order parameters may not always be available, as some requirements in the estimation procedure are not always met.

In this paper, I apply the bias correction procedure of Caeiro, Gomes, and Pestana (2005) to reduce the bias in the Hill estimator. Caeiro, Gomes, and Pestana (2005) estimate the second order parameters in the bias and use these parameters to correct for the bias in the classical Hill estimator. An advantage of this procedure are the requirements for estimating the second order parameters, as they are less restrictive than the requirements of de Haan, Mercadier, and Zhou (2016) for example. The bias corrected Hill estimator of Caeiro, Gomes, and Pestana (2005) is asymptotically normal and results in a lower mean squared error, for all values of k, i.e. the number of observations in the tail. By improving this subcomponent of the tail beta, I aim to improve the overall performance of the tail beta of van Oordt and Zhou (2019).

An improved version of the tail beta estimator of van Oordt and Zhou (2019) will lead to improved estimates of systematic risk under extremely adverse market conditions. Hence this research can be considered as relevant for financial institutions, such as asset managements, banks and insurance companies, who seek to anticipate on extreme losses and prevent them during distress events. Namely, for asset managers, estimating the systematic risk, under extremely adverse market conditions, can be usefull to asses the extreme loss on the stock portfolio in the event of a market crash. Moreover, banks can measure their sensitivity to large shocks in the financial system. van Oordt and Zhou (2016) apply this method to the banking industry. Lastly, this methodology can be applied in the insurance business, where the sensitivity of an insurance company to a large claim can be measured. An improved version of the tail beta also serves as the novel contribution to the literature.

I compare the improved estimator with the original estimator of van Oordt and Zhou (2019) and the estimator of a conditional regression approach in a simulation study, as well as in an empirical study with 48 industry-specific stock portfolios. In the simulation study, the improved estimator has a lower mean squared error than the original estimator due to the reduction of the bias. Comparing the improved estimator with the original estimator in the empirical part, I find that the improved estimator consistently produces lower root mean squared errors in predicting the losses for the different industry portfolios. The reduction in the mean squared error is in six cases statistically significant. In almost all subperiods the improved estimator produces lower root mean squared errors compared to the conditional regression approach. These lower mean squared errors are in ten cases statistically significant. Overall, I conclude that the improved estimator has a better performance than the aforementioned estimators.

The remainder of this paper is organized as follows. Section 2 covers the theory that is needed for the estimation methods and the estimation methods themselves. Next, Section 3 reports the simulation results for the different estimators. Further, Section 4 contains an empirical application and a comparison of the different approaches. Lastly, Section 5 summarizes the main findings of this paper in a concluding form and discusses some limitations of this research as well as further recommendations for prospective studies.

2 Methodology

The methodology of van Oordt and Zhou (2019) results in an estimation of a linear model between two heavy-tailed variables, conditional on the explanatory variable containing extremely low values. The model is as follows

$$Y = \beta^T X + \epsilon \quad \text{for} \;\; X < Q_x(\bar{p}), \tag{1}$$

where ϵ is the noise term independent of X under the condition that X contains extremely low values. $Q_x(\bar{p})$ denotes the quantile function of X and is defined as $Q_x(\bar{p}) = \inf \Big\{ c : P(X \leq c) \geq \bar{p} \Big\}$. Y and X are (excess) returns of the stock portfolio and the market portfolio, respectively. β^T is distinguished from the regular coefficient β in a linear model, since the relationship in Equation (1) only holds for extremely low values of X.

Before I discuss how β^T can be estimated in section 2.2, I will briefly summarize the theory needed for this estimation method.

2.1 Theory

Assume that the random variable X is heavy-tailed, i.e.

$$\lim_{u \to +\infty} \frac{\Pr(X \leq -ux)}{\Pr(X \leq -u)} = x^{-\alpha_x} \qquad \forall x > 0. \tag{2}$$

Equivalently, the tail distribution of X is heavy-tailed if it can be expressed as

$$\Pr(X \leq -u) = u^{-\alpha_x} l_x(u),$$

where $l_x(u)$ is a slowly varying function as $u \to +\infty$. That is,

$$\lim_{u \to \infty} \frac{l_x(tu)}{l_x(u)} = 1 \qquad \forall t > 0.$$

The parameter α_x is the tail index, which will be estimated in section 2.2. I assume that the heavy-tailedness also holds for the variable Y.

As van Oordt and Zhou (2019) assume that the linear model is only valid for extremely low values of X, a measurement for the tail dependency is needed. Since the dependency of these variables relies on the (left) tail of their distributions, the tail dependency measure can be interpreted as the probability of observing an extreme low value of Y given that an extremely low value of X is observed. Hence, the tail dependency measure functions as a correlation coefficient, focusing on the dependence in the tails only. van Oordt and

Zhou (2019) consider the following tail dependence measure from Multivariate Extreme Value Theory

$$\tau := \lim_{p \to 0} \tau(p) := \lim_{p \to 0} \Pr(Y < Q_y(p), X < Q_x(p)) = \lim_{p \to 0} \Pr(Y < Q_y(p) | X < Q_x(p)), \tag{3}$$

where $Q_y(p)$ denotes the quantile function of Y, defined as $Q_y(p) = \inf \left\{ c : P(Y \leq c) \geq p \right\}$. Under mild conditions, van Oordt and Zhou (2019) proof that β^T in Equation (1) is equal to

$$\beta^T = \lim_{p \to 0} (\tau(p))^{1/\alpha_x} \frac{Q_y(p)}{Q_x(p)}. \tag{4}$$

2.2 Estimation

Suppose that we have independent and identically distributed (i.d.d.) observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. van Oordt and Zhou (2019) show that an estimator for β^T in Equation (4) can be obtained via

$$\hat{\beta}^T := \hat{\tau}(k/n)^{1/\hat{\alpha}_x} \frac{\hat{Q}_y(k/n)}{\hat{Q}_x(k/n)}. \tag{5}$$

Here $\hat{\alpha}_x$ is an estimate for the tail index α_x. van Oordt and Zhou (2019) use the k_1 lowest observations of X with the estimator proposed in Hill (1975) as follows. By ranking the observations of X as $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ with $X_{(i)}$ denoted as the i-th order statistic, one can define the Hill estimator as

$$\hat{\alpha}_x := k_1 \left(\sum_{i=1}^{k_1} log \left(\frac{X_{(i)}}{X_{(k+1)}} \right) \right)^{-1}. \tag{6}$$

Further, I estimate the tail dependence measure of Equation (3) as

$$\hat{\tau}(k/n) := \frac{1}{k} \sum_{i=1}^{n} \mathbb{1}_{\left\{ Y_i < Y_{(k+1)}, X_i < X_{(k+1)} \right\}} \tag{7}$$

Lastly, the quantiles of X and Y are estimated by their $(k+1)$-th lowest order statistic, i.e. $X_{(k+1)}$ and $Y_{(k+1)}$, respectively.

van Oordt and Zhou (2019) mention that the estimator $\hat{\beta}^T$ shows similarities to the regression coefficient in a standard regression analysis. In a standard regression analysis, the coefficient is equal to the product of the correlation coefficient and the ratio of standard deviations. In the setting of β^T, the correlation coefficient is replaced by the tail dependence measure and the ratio of standard deviations is mimicked by the tail quantiles.

2.3 Improvement of the tail index

In order to improve the Hill estimator by reducing its bias, I follow the procedure of Caeiro, Gomes, and Pestana (2005). I start with estimating the second order parameters ρ and β_ρ. Note that β_ρ is different from the tail beta described in Equation (4) and Equation (5). By parameterizing ρ, one can estimate it as

$$\hat{\rho}(k) \equiv \hat{\omega}_n^{(\omega)}(k) := - \left| \frac{3(T_n^{(\omega)}(k) - 1)}{T_n^{(\omega)}(k) - 3} \right|, \tag{8}$$

where

$$T_n^{(\omega)}(k) = \begin{cases} \frac{(M_n^{(1)}(k))^{\omega} - (M_n^{(2)}(k)/2)^{\omega/2}}{(M_n^{(2)}(k))^{\omega/2} - (M_n^{(3)}(k)/6)^{\omega/3}} & \text{if } \omega > 0 \\ \\ \frac{log(M_n^{(1)}(k)) - \frac{1}{2}log(M_n^{(2)}(k)/2)}{\frac{1}{2}log(M_n^{(2)}(k)/2) - \frac{1}{3}log(M_n^{(3)}(k)/6)} & \text{if } \omega = 0, \end{cases} \tag{9}$$

with

$$M_n^{(j)}(k) := \frac{1}{k}\sum_{i=1}^{k}\left\{log\left(\frac{X_{(i)}}{X_{(k+1)}}\right)\right\}^j, \quad j \geq 1 \quad \text{[Note that } M_n^{(1)} \equiv \hat{\alpha}_x \text{ in (6)]}.$$

Following the advice of Caeiro, Gomes, and Pestana (2005), I set the tuning parameter ω equal to 0 in this research. Further, by incorporating the scaled log-spacings U_i as

$$U_i = log\left(\left(\frac{X_{(i)}}{X_{(k+1)}}\right)^i\right) \text{ for } 1 \leq i \leq k$$

and using $\hat{\rho}(k)$ of Equation (8), the β_ρ-estimator can be defined as

$$\hat{\beta}_{\hat{\rho}}(k) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{(\frac{1}{k}\sum_{i=1}^{k}(\frac{i}{k})^{-\hat{\rho}})(\frac{1}{k}\sum_{i=1}^{k}U_i) - (\frac{1}{k}\sum_{i=1}^{k}(\frac{i}{k})^{-\hat{\rho}}U_i)}{(\frac{1}{k}\sum_{i=1}^{k}(\frac{i}{k})^{-\hat{\rho}})(\sum_{i=1}^{k}(\frac{i}{k})^{-\hat{\rho}}U_i) - (\frac{1}{k}\sum_{i=1}^{k}(\frac{i}{k})^{-2\hat{\rho}}U_i)}. \tag{10}$$

Finally, given $\hat{\rho}$ and $\hat{\beta}_{\hat{\rho}}$, the improved tail index estimator can be defined as

$$\hat{\alpha}_x^* := \alpha_x\left(1 - \frac{\hat{\beta}_{\hat{\rho}}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right). \tag{11}$$

To summarize, I denote the improved estimator from the tail beta in van Oordt and Zhou (2019) as $\hat{\beta}_E^T$ as follows

$$\hat{\beta}_E^T := \hat{\tau}(k/n)^{1/\hat{\alpha}_x^*}\frac{\hat{Q}_y(k/n)}{\hat{Q}_x(k/n)}. \tag{12}$$

I compare $\hat{\beta}^T$ and $\hat{\beta}_E^T$ to each other, as well as to the estimator of a conditional regression, which serves as the benchmark in this research. The ordinary least squares (OLS) estimator conditional on the observations in the tail can be defined as

$$\hat{\beta}_{OLS}^T = \frac{\sum_{X_i < X_{(k+1)}}\left(Y_i - \bar{Y}^T\right)\left(X_i - \bar{X}^T\right)}{\sum_{X_i < X_{(k+1)}}\left(X_i - \bar{X}^T\right)^2},$$

where $\bar{X}^T = \frac{1}{k}\sum_{X_i<X_{(k+1)}} X_i$ and $\bar{Y}^T = \frac{1}{k}\sum_{Y_i<Y_{(k+1)}} Y_i$ are the sample means of the tail observations of the explanatory variable and dependent variable, respectively.

3 Simulation

Before testing and comparing different estimators in an empirical application, I study the performance of the estimators in a simulation. The generated sample consists of 1250 random observations of X and Y, which corresponds roughly to the length of the estimation window in the empirical setting.

Following van Oordt and Zhou (2019), I consider three global linear models in which the relation is unaffected by the observation of X, i.e. $\beta = \beta^T = 0.5$, 1, 1.5. Further, I consider two segmented linear models, where the slope of the model changes for the observations in the tail. More precisely, if the observation X_i is larger than the third percentile of X, then the observation Y_i will be generated from a linear model with $\beta = 1$. Otherwise, Y_i will be generated from two different linear models. The first one will be a linear model with $\beta^T = 0.5$ and the second one will be a linear model with $\beta^T = 1.5$.

van Oordt and Zhou (2019) consider several data generating processes for X and ϵ. Their simulations are based on draws from a Student's t-distribution with three, four and five degrees of freedom. As the patterns across the simulations in van Oordt and Zhou (2019) are very similar for the Student's t-distribution with the different degrees of freedom, I will only consider the Student's t-distribution with four degrees of freedom in my simulation. More specifically and differently from van Oordt and Zhou (2019), I use a skewed Student's t-distribution with four degrees of freedom in order to capture and fit the stylized facts of asset returns better in this simulation study. I set the skewing parameter equal to the skewness of the market portfolio in the empirical part.

For each model and each data generating process, I generate 10.000 samples and estimate β^T in each sample, for all the different approaches. The real β^T will be used to compare the different estimates in terms of mean squared error (MSE) and its components, i.e. the squared bias and the variance.

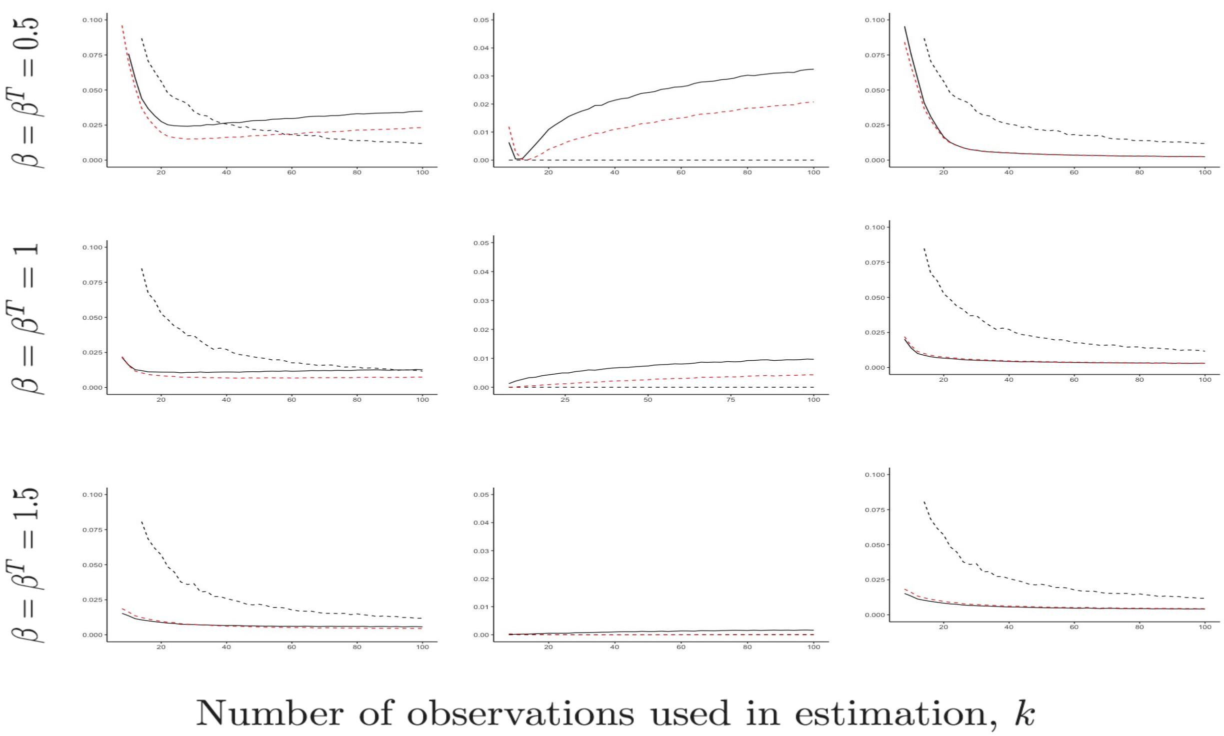

Figure 1: Simulations with a global linear model.

Notes: The red dashes report the simulation results of the improved estimator. The solid lines and the dashed lines report the simulation results for the EVT approach and the conditional regression approach, respectively. The simulations are based on $m = 10.000$ samples with $n = 1250$ observations each. The estimates from the simulations, i.e. $\hat{\beta}^T$s, are compared with the true value. I evaluate the performance of $\hat{\beta}^T$ by its mean squared error (MSE) and its subcomponents. The MSE is calculated as $m^{-1}\sum_i(\beta^T - \hat{\beta}_i^T)^2$, where i refers to the i-th simulated sample. The squared bias and the variance are calculated as $(\beta^T - \bar{\beta}^T)^2$ and $m^{-1}\sum_i(\bar{\beta}^T - \hat{\beta}_i^T)$, respectively. Here $\bar{\beta}^T = m^{-1}\sum_i \hat{\beta}_i^T$.

Figure 1 and 4 show the results of the simulation across different values of the number of observations in the tail, i.e. k. The first column of Figure 1 and 4 compares the MSE between different approaches for the estimation of the tail beta. The second and third column compare the squared bias and the variance, respectively. In this simulation setup, the EVT approaches ($\hat{\beta}^T$ and $\hat{\beta}^T_E$) perform better than the conditional regression approach, especially for low levels of k. The conditional regression approach performs better if the number of observations in the tail increases as it even outperforms the EVT approaches at a certain point, which is line with the results of van Oordt and Zhou (2019). However, even if the EVT approaches are outperformed by the conditional regression approach as k increases, the MSE of the EVT approaches are very insensitive to the increment of k.

Further, a relatively large variance is observed for the conditional regression approach, affecting the mean squared error. On the contrary, the EVT approach of van Oordt and Zhou (2019) has a relatively low variance but a relatively high squared bias, especially as the number of observations in the tail increases.

The improved estimator leads to an overall bias reduction of the tail beta. Even if the variance of the improved estimator increases, as the squared bias decreases, the improved estimator results in a lower mean squared error compared to the estimator of van Oordt and Zhou (2019) for the same value of k. Next, by examining where the mean squared error of the original EVT estimator is minimized, I observe that the improved estimator has a lower mean squared error than the original estimator along each curve. So I conclude that it is the reduction of the bias that decreases the overall mean squared error.

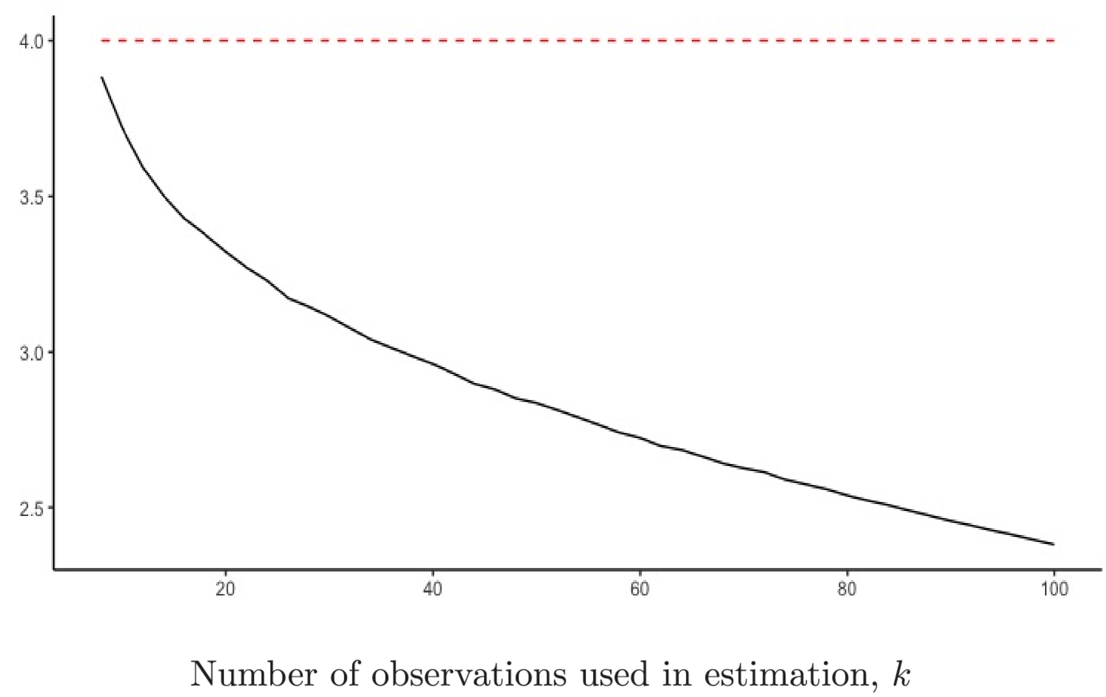

Figure 2: Estimates for the tail index. The solid line reports the estimates of the original Hill estimator in Equation (6), for different values of k. van Oordt and Zhou (2019) mention that the true tail index of the (skewed) Student's t-distribution is equal to its degrees of freedom. Hence, the true tail index should be equal to 4.

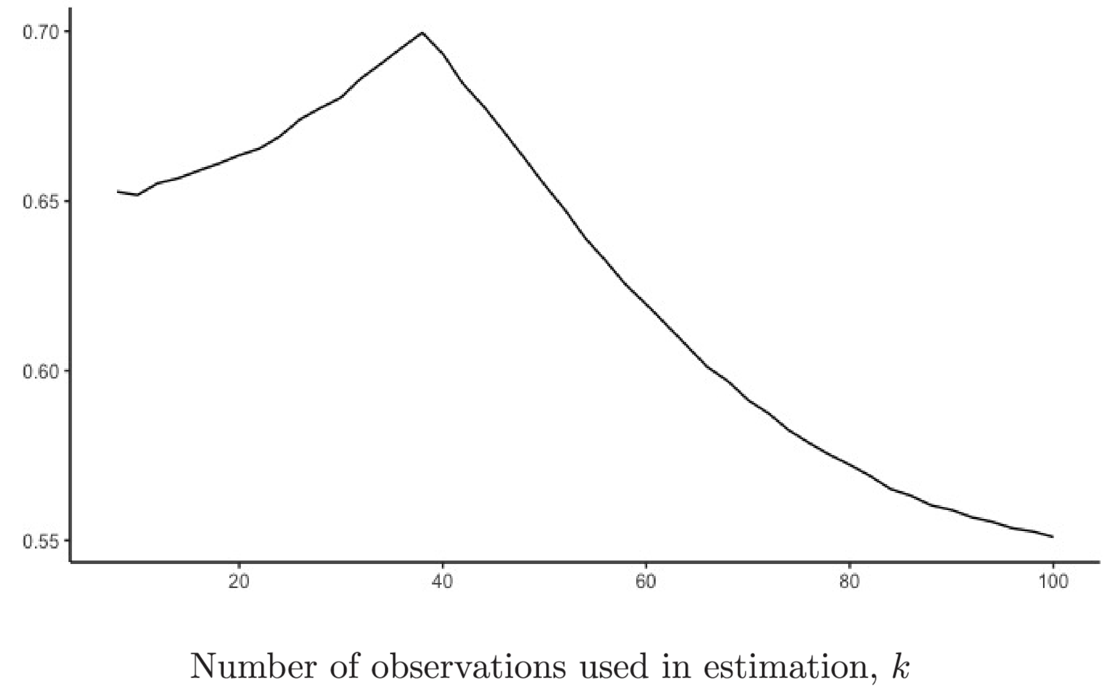

Figure 3: Tail dependency. The solid line reports the estimates for the tail dependency measure of Equation (7) across different values of k.

The segmented linear model with $\beta = 1, \beta^T = 1.5$ is the only case where the improved estimator can not outperform the estimator of van Oordt and Zhou (2019). The fact that the improved estimator performs poorly is due to the stronger tail dependency in this case, as $\beta^T > \beta$, and the bias in the estimator of the tail dependence measure. As Fougères, De Haan, and Mercadier (2015) point out, the estimator for the tail dependence measure may be biased with a bias that deviates more from zero as the number of observations in the tail increases. In this particular case, the estimator of the tail dependence measure is downwards biased as shown in Figure 3.

As the observations in the tail increases, the Hill estimator is also downwards biased, which is empirically justified in Figure 2. Combining these two biases, they cancel each other out in the subcomponent $\hat{\tau}(k/n)^{1/\hat{\alpha}_x}$ of the original estimator. However, by correcting the bias of the tail index, the bias in $\hat{\tau}(k/n)$ appears and remains in the improved estimator and functions as a punishment for the improved estimator, resulting in the poor performance of the improved estimator. Further comparison between the original tail beta of van Oordt and Zhou (2019) and a tail beta estimator where the bias of both components is corrected might be of interest for this particular case.

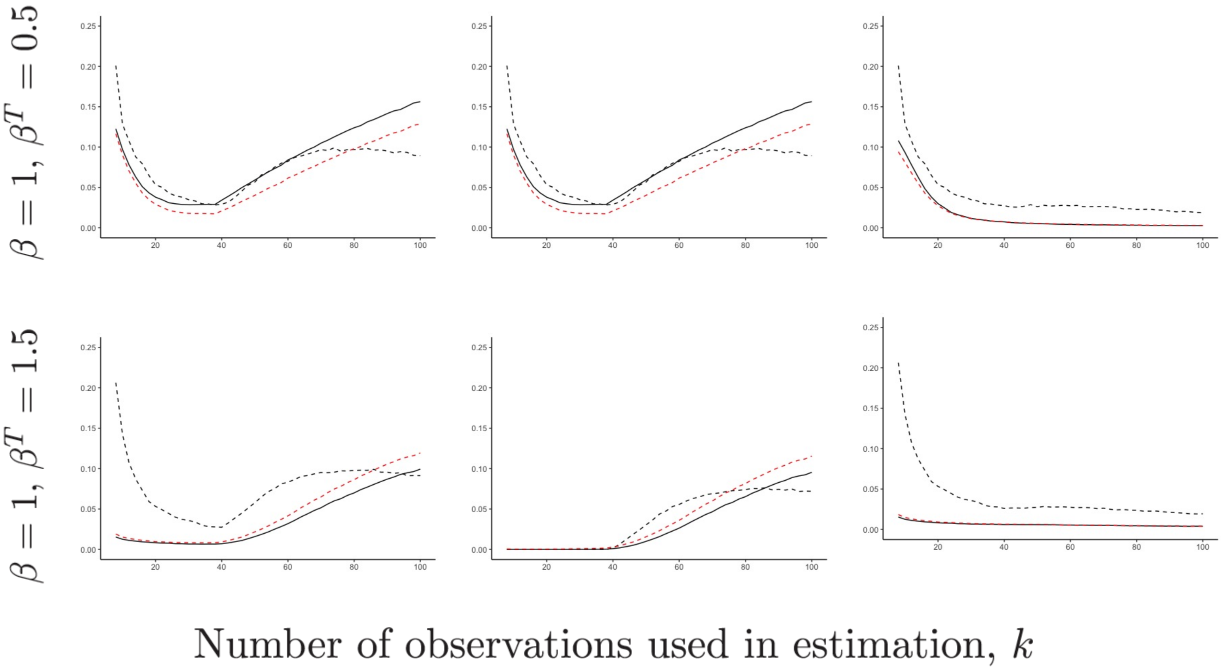

Figure 4: Simulations with a segmented linear model.

Notes: The red dashes report the simulation results of the improved estimator. The solid lines and the dashed lines report the simulation results for the EVT approach and the conditional regression approach, respectively. Whether the observation Y_i is generated from a linear model with slope $\beta = \beta^T$ or a segmented linear model with slope $\beta \neq \beta^T$, depends on whether the observation X_i is above or below its third percentile, respectively. The simulations are based on $m = 10.000$ samples with $n = 1250$ observations each. The expectation is that, on average in each sample, approximately 40 observations will be generated from the segmented linear model. The estimates from the simulations, i.e. $\hat{\beta}^T$s, are compared with the true value. I evaluate the performance of $\hat{\beta}^T$ by its mean squared error (MSE) and its subcomponents. The MSE is calculated as $m^{-1}\sum_i(\beta^T - \hat{\beta}_i^T)^2$, where i refers to the i-th simulated sample. The squared bias and the variance are calculated as $(\beta^T - \bar{\beta}^T)^2$ and $m^{-1}\sum_i(\bar{\beta}^T - \hat{\beta}_i^T)$, respectively. Here $\bar{\beta}^T = m^{-1}\sum_i \hat{\beta}_i^T$.

4 Empirical Application

For the empirical part of this research, I collect data of daily returns on 48 different industry-specific stock portfolios and a general market portfolio from the Kenneth R. French database [1]. The original data runs from January 2, 1931 to April 30, 2020. I divide the data into 18 five-year subperiods and within each subperiod, I estimate the coefficient β_j^T in the linear tail model with the (excess) returns of industry portfolio j as the dependent variable and the (excess) market return as the explanatory variable. The same estimation will be done with the estimator of van Oordt and Zhou (2019) and the conditional regression approach.

I assess the performance of the different methods by projecting the loss of an industry portfolio, using the largest market loss within each subperiod. Therefore, in the estimation procedure, I exclude the day on which the market portfolio suffered its largest loss. The number of observations in the superiods vary from 1089 to 1503. On average, a subperiod contains 1300 observations. van Oordt and Zhou (2019) set k equal to 25 for

[1] The results of this paper are based on data accessed on June 1, 2020. The data can be accessed via `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

all subperiods, such that $k/n \approx 2\%$ on average. Since the number of observations in each subperiod varies substantially, I alter k in each subperiod such that $k/n = 2\%$ for each subperiod. This makes k time-varying, varying from 22 to 31 within the subperiods.

Period	Av. $\hat{\beta}^T_{E,j}$	N	S	Minimum $\hat{\beta}^T_{E,j}$		Maximum $\hat{\beta}^T_{E,j}$	
1931-1935	1.16	41	2	0.56	Apparel	1.69	Coal
1936-1940	1.05	43	0	0.40	Tobacco Products	2.02	Construction
1941-1945	1.15	42	0	0.60	Communication	2.49	Real Estate
1946-1950	1.00	43	0	0.33	Communication	1.73	Construction
1951-1955	1.02	43	0	0.40	Communication	1.58	Aircraft
1956-1960	1.06	43	0	0.45	Utilities	1.73	Electronic Equipment
1961-1965	1.13	43	0	0.54	Utilities	1.76	Agriculture
1966-1970	1.23	47	0	0.56	Utilities	2.05	Other
1971-1975	1.17	48	0	0.71	Utilities	1.78	Entertainment
1976-1980	1.08	48	0	0.59	Utilities	1.86	Precious Metals
1981-1985	1.09	48	0	0.62	Utilities	1.94	Precious Metals
1986-1990	0.94	48	0	0.50	Utilities	1.21	Candy & Soda
1991-1995	1.15	48	0	0.63	Utilities	1.80	Shipbuilding, Railroad Equipment
1996-2000	0.98	48	0	0.42	Utilities	1.86	Coal
2001-2005	1.00	48	0	0.56	Food Products	1.75	Electronic Equipment
2006-2010	1.07	48	0	0.55	Beer & Liquor	2.03	Coal
2011-2015	1.07	48	0	0.63	Beer & Liquor	2.20	Coal
2015-2020	0.94	48	0	0.53	Utilities	1.61	Coal

Table 1: Estimates

Notes : Within each subperiod, the improved estimator is used to estimate β^T for 48 daily excess returns of industry portfolios, while excluding the observation on the day of the largest market loss. In each subperiod, k is set such that $k/n = 2\%$ in that specific subperiod. The column labeled "Av. $\hat{\beta}^T_{E,j}$" reports the average of $\hat{\beta}^T_{E,j}$ of the remaining N portfolios. The column labeled S reports the number of excluded portfolios, based on the criteria $\hat{\alpha}_j \leq \frac{1}{2}\hat{\alpha}^*_m$. The last columns report the minimum and maximum $\hat{\beta}^T_{E,j}$ and their industry name from the data documentation.

In line with the condition $\alpha_y > \frac{1}{2}\alpha_x$ in Theorem 1 of van Oordt and Zhou (2019), which implies that the dependent variable should not be too "heavy-tailed" in comparison to the explanatory variable, I exclude portfolios with $\hat{\alpha}_j \leq \frac{1}{2}\hat{\alpha}_m$ in each subperiod. In the analysis where $\hat{\beta}^T_E$ is involved, I apply the same condition with α^*_m, i.e. I exclude portfolios with $\hat{\alpha}_j \leq \frac{1}{2}\hat{\alpha}^*_m$. In the setting of $\hat{\beta}^T_E$, almost no portfolios are excluded. The number of excluded portfolios are denoted by S, while the remaining portfolios in the analysis are denoted by N. Besides the excluded and remaining portfolios, Table 1 also reports the minimum and maximum $\hat{\beta}^T_{E,j}$ with the corresponding industry name, respectively. This gives an insight on the range of $\hat{\beta}^T_{E,j}$. Most of $\hat{\beta}^T_{E,j}$s fall in the range between 0.5 and 2.0, implying that in a market crash, most portfolios are expected to lose between half and twice as much as the market portfolio. Almost in each sample, the average $\hat{\beta}^T_{E,j}$ is slightly above 1, indicating (if one uses the CAPM-model) that on average an industry will suffer a larger loss than the market portfolio in a market crash.

Based on the $\hat{\beta}^T_j$ from the different approaches in each subperiod, I estimate the loss of each portfolio j on the day that the market portfolios had its largest loss. For each subperiod, the largest market loss, defined as $L_m = -R^m_{(1),t}$, is given in Table 2. The corresponding date of the largest market loss is also reported in Table 2. The actual loss of specific industry portfolio on that date is denoted as $L_j = -R^j_{t^*}$, where t^* corresponds to the day of the largest market loss. Following Equation (1), the projections of the different approaches are denoted as $\hat{L}_{E,j} = L_m \hat{\beta}^T_{E,j}$, $\hat{L}_{EVT,j} = L_m \hat{\beta}^T_{EVT,j}$ and $\hat{L}_{OLS,j} = L_m \hat{\beta}^T_{OLS,j}$, respectively.

The performance of the different approaches are compared by their root mean squared error (RMSE), which is defined as $\sqrt{N^{-1} \sum_j e^2_{Z,j}}$, where $e_{Z,j} = L_j - \hat{L}_{Z,j}$ with Z being the different approaches, i.e. $Z = \left\{BTE, EVT, OLS\right\}$. The best-performing methoud should report a lower RMSE. Further, to test whether the mean squared errors of two different approaches differ significantly, I make use of the Diebold-Mariano test (Diebold and Mariano (1995)) with a small-sample correction proposed by Harvey, Leybourne, and Newbold (1997). The Diebold-Mariano test is based on the loss differential $d_j = e^2_{a,j} - e^2_{b,j}$, where $a, b \in Z$ and $a \neq b$. The test statistic can be calculated as

$$DM = \frac{\bar{d}}{\sqrt{\hat{V}(\bar{d})/(N-1)}} \sim t(N-1), \tag{13}$$

where $\bar{d} = N^{-1} \sum_j d_j$ and

$$V(\hat{d_j}) = \frac{\sum_j (d_j - \bar{d})^2}{N}.$$

Note that model a predicts significantly better than model b if the DM statistic is significantly negative. For a significant positive DM statistic, model b predicts significantly better than model a.

Period	Worst day	Market Loss	RMSE EVT	RMSE BTE	t-stat	p-value
1931-1935	July 21, 1933	9.21	6.36	6.41	1.63	0.110
1936-1940	October 18, 1937	8.20	2.98	2.99	0.19	0.849
1941-1945	December 08, 1941	4.15	1.72	1.75	0.43	0.670
1946-1950	September 3, 1946	6.90	1.44	1.42	-1.24	0.221
1951-1955	September 26, 1955	6.52	1.73	1.65	-2.81	0.007
1956-1960	October 21, 1957	3.04	1.25	1.23	-0.56	0.576
1961-1965	May 28, 1962	7.00	2.99	2.80	-2.53	0.015
1966-1970	May 25, 1970	3.21	1.63	1.62	-0.18	0.859
1971-1975	November 18, 1974	3.57	1.21	1.21	-2.44	0.019
1976-1980	October 9, 1979	3.44	1.09	1.06	-1.04	0.305
1981-1985	October 25, 1982	3.62	1.25	1.14	-3.66	0.001
1986-1990	October 19, 1987	17.44	4.15	3.83	-1.08	0.286
1991-1995	November 15, 1991	3.55	1.82	1.75	-4.02	0.000
1996-2000	April 14, 2000	6.72	3.48	3.21	-2.50	0.016
2001-2005	September 17, 2001	5.03	5.20	5.17	-0.95	0.349
2006-2010	December 1, 2008	8.95	1.14	1.12	-1.11	0.273
2011-2015	August 8, 2011	6.97	1.94	1.85	-1.27	0.209
2015-2020	March 16, 2020	12.00	4.88	4.82	-1.03	0.309

Table 2: Performance Evaluation

Notes: Within each period, β^T is estimated for 48 daily excess returns of industry portfolios using my approach and the approach of van Oordt and Zhou (2019). In each subperiod, k is set such that $k/n = 2\%$ in that specific subperiod. Further, in each subperiod, the observation on the day of the largest market loss is excluded in order to evaluate the performances of the projected losses. For this performance evaluation, I use the difference between the actual loss and the projected loss and calculate its root mean squared error (RMSE) for each approach. The last two columns report the t-statistics from Equation (13) and the corresponding p-values for testing the null hypothesis of equal prediction accuracy of the two approaches. Shaded numbers indicate p-values smaller than 5%.

Table 2 compares the improved estimator with the estimator of van Oordt and Zhou (2019). Further, I report the t-statistics of Equation (13) with the corresponding p-values in the last two columns to test against the null of equal forecast accuracy. In fourteen subperiods, the improved estimator reports a lower RMSE than the estimator of van Oordt and Zhou (2019). However the difference in RMSE is significant in less than the half of these cases. On average, the RMSE of EVT is reduced by 3% when implementing the biased corrected Hill estimator into the tail beta.

Diving deeper into the subperiods, the improved estimator consistently produces a lower mean squared error than the original tail beta estimator of van Oordt and Zhou (2019) after World War II. In the subperiods of during the war and post-war, the improved estimator and the original tail beta perform similarly in terms of mean squared errors.

The third subperiod covers a large portion of the Second World War. One may argue that World War II did not affect the stock market much in this subperiod, as the largest market loss of 4.15 on December 08, 1941 is relatively low compared to other market crashes. With an average $\hat{\beta}^T_{E,j}$ of 1.15, one would expect that the losses of the industry-specific portfolios would also remain stable. However, in 26 out of the remaining 42 portfolios the original tail beta of van Oordt and Zhou (2019) underestimates the losses of the industry-specific portfolios. Because of the high tail dependency for most portfolios in this subperiod, the improved estimator performs poorly compared to the original tail beta of van Oordt and Zhou (2019). In the portfolios where the losses have been underestimates by the original estimator, the improved estimator underestimates those losses even more. This feature results in a higher mean squared error of the improved estimator, skewing the difference of the forecast errors more in the favor of the original tail beta.

Diving deeper into the losses of the portfolio, the Construction industry suffered from a loss of 12.97%, while the projected losses of the improved estimator and the estimator of van Oordt and Zhou (2019) were 8.75% and 9.32%, respectively. The loss of the Construction industry was the largest loss of all the remaining portfolios in this subperiod. An explanation for this might be due to the expendings of the war itself. Since the government is the largest client of the Construction industry, the investments in instructions and constructions will be less and hence this industry suffers from a larger loss than the models predict. On contrary, the investments on machinery, electronic equipment and telecommunications will increase and one might expect that these industries will not suffer from a large loss. The actual losses of these industries for this subperiod are 3.97%, 2.74% and 2.53%, respectively.

Because of the reasoning above, the Construction industry might be more tail dependent with the market for this particular crisis. The tail dependence coefficient for this industry is equal to 0.6 in this subperiod. Note that the actual tail dependency will be different, as the estimator for the tail dependency is biased. A similar phenomena occured during the simulation study in Section 3, which explains the poor performance of the improved estimator. Excluding the Construction industry in this subperiod will lead to a RMSE of 1.64 for the improved estimator, while the RMSE of the original estimator is reduced to 1.65. The same reasoning can be given for the first two subperiods.

To summarize, my bias-corrected estimator works better in all periods after the Second World War. Although this better performance is only significant in six cases, it consistently produces lower mean squared errors compared to the estimator of van Oordt and Zhou (2019).

Table 3 compares the improved estimator with the conditional ordinary least squares estimator. The last two columns of Tabel 3 report the t-statistics of the Diebold-Mariano test with the corresponding p-values to test against the null of equal forecast accuracy. In seventeen subperiods, the improved estimator reports a lower RMSE than the conditional OLS estimator and in ten of these subperiods, the reduction in RMSE is

Period	Worst day	Market Loss	RMSE OLS	RMSE BTE	t-stat	p-value
1931-1935	July 21, 1933	9.21	7.98	6.41	-1.18	0.243
1936-1940	October 18, 1937	8.20	3.94	2.99	-1.93	0.060
1941-1945	December 08, 1941	4.15	2.30	1.75	-2.48	0.017
1946-1950	September 3, 1946	6.90	2.26	1.42	-1.78	0.083
1951-1955	September 26, 1955	6.52	2.82	1.65	-2.95	0.005
1956-1960	October 21, 1957	3.04	1.97	1.23	-2.66	0.011
1961-1965	May 28, 1962	7.00	3.32	2.80	-1.35	0.184
1966-1970	May 25, 1970	3.21	2.30	1.62	-2.20	0.033
1971-1975	November 18, 1974	3.57	2.44	1.21	-3.25	0.002
1976-1980	October 9, 1979	3.44	2.16	1.06	-5.30	0.000
1981-1985	October 25, 1982	3.62	2.32	1.14	-3.67	0.001
1986-1990	October 19, 1987	17.44	6.80	3.83	-1.36	0.180
1991-1995	November 15, 1991	3.55	2.74	1.75	-2.96	0.005
1996-2000	April 14, 2000	6.72	4.35	3.21	-1.18	0.242
2001-2005	September 17, 2001	5.03	5.75	5.17	-1.11	0.274
2006-2010	December 1, 2008	8.95	3.11	1.12	-3.98	0.000
2011-2015	August 8, 2011	6.97	1.51	1.85	0.62	0.540
2015-2020	March 16, 2020	12.00	6.79	4.82	-2.27	0.028

Table 3: Performance Evaluation

Notes: Within each period, β^T is estimated for 48 daily excess returns of industry portfolios using my approach and the conditional ordinary least squares approach. In each subperiod, k is set such that $k/n = 2\%$ in that specific subperiod. Further, in each subperiod, the observation on the day of the largest market loss is excluded in order to evaluate the performances of the projected losses. For this performance evaluation, I use the difference between the actual loss and the projected loss and calculate its root mean squared error (RMSE) for each approach. The last two columns report the t-statistics from Equation (13) and the corresponding p-values for testing the null hypothesis of equal prediction accuracy of the two approaches. Shaded numbers indicate p-values smaller than 5%.

significant. On average, the RMSE is reduced by 31% when swapping the conditional OLS estimator for the improved estimator.

The improved estimator and the estimator of van Oordt and Zhou (2019) show similarities when the estimator of van Oordt and Zhou (2019) is compared to the conditional OLS estimator. Here, the estimator of van Oordt and Zhou (2019) also reports a lower RMSE than the conidtional OLS estimator in seventeen subperiods and the difference is also significant in ten out of these seventeen subperiod. However, the RMSE is, on average, reduced by 29%. Tables 4 and 5 show the results of the estimator of van Oordt and Zhou (2019) and can be found in the Appendix.

Diving deeper into the subperiod where the market suffered from a relatively large loss and into the only subperiod the conditional regression estimator outperformed the improved estimator.

March 16, 2020. A few weeks after the outbreak of COVID-19 and a few days after most states in the United States of America went into a lockdown. The relatively high RMSE of the conditional regression approach is due to a few industry portfolios which actually benefited from a profit during the market crash in this subperiod. For instance, the Precious Metals and Coal industries made a profit of 3.10% and 1.52%, respectively. For the Precious Metals industry, the projected losses of the conditional regression approach was 17.71%, while the improved estimator projected a loss of only 11.17%. Similarly, for the Coal industry the projected losses of the conditional regression approach and my approach were 22.67% and 19.34%, respectively. Another essential fact of this subperiod is the number of observations. The number of observations in this subperiod is the smallest among all the subperiods, as this research took place mid year 2020. Because of a relatively small number of observations, the number of observations in the tail, i.e. k, will also be relatively small. A small number of observations in the tail will increase the variance, and hence the mean squared error, of the conditional regression estimator, while a small k is in the concept of the tail beta, resulting in a better performance. These results are also supported by the simulation part of Section 3.

Lastly, I zoom into the only subperiod in which the conditional regression approach actually outperforms the improved estimator. Augustus 8, 2011 was the worst day of this subperiod, due to US debt crisis. The market suffered from a loss of 6.97%. The only severely misprojected loss of the improved estimator was for the Precious Metals industry. This industry suffered from a loss of 1.38%, while the conditional regression approach projected the loss at 4.06 and the improved estimator projected the loss at 10.90%. The actual loss of the Precious Metals industry was the smallest loss among all portfolios and the reason is due to flight-to-safety. Since uncertainty is high during such an economic crisis, investors may want to sell their risky assets and buy safer investments to hedge their risk. Gold serves as such a hedging instrument for the US stock market (see Baur and McDermott (2010)). More interestingly, Kumar (2010) shows that gold can be used as a hedging instrument especially in times of economic crises.

Due to the hedging possibility of gold, the assumption of the linear model in the tail may not hold for the Precious Metals industry in this subperiod. Hence, this would be the main reason both Extreme Value Theory estimators would misproject the losses of this specific industry. Excluding the Gold industry from the analysis in this subperiod will change the RMSE of the improved estimator and the RMSE of the conditional regression approach to 1.25 and 1.48, respectively. Now, the reduction of the mean squared error from the conditional regression approach to my approach is also statistically significant.

To summarize, besides one subperiod, the improved estimator produces a lower root mean squared error for every subperiod. In ten out of the eighteen subperiods, the lower root mean squared error of the improved estimator is statistically significant. Overall, I can conclude that the improved estimator has better performances than the conditional regression approach, as well as the original Extreme Value Theory approach.

5 Conclusion

This paper investigated whether I could adequately estimate systematic risk, given extremely low values of the market portfolio, by improving the tail beta estimator of van Oordt and Zhou (2019). This tail beta can be interpreted as the regression coefficient in a simple regression model like the Capital Asset Pricing Model. The improved estimator proposed in this paper is compared to the original estimator of van Oordt and Zhou (2019) as well as to the estimator of a conditional regression on tail observations. An extensive simulation study shows that correcting the bias in the Hill estimator results in a lower mean squared error of the improved estimator in four out of the five models. In the same simulation study, I show that both Extreme Value Theory estimators perform better than the conditional regression approach.

In the empirical application, I use all aforementioned estimators to project losses of industry portfolios in times of financial distress. Using data of 48 different industry-specific stock portfolios, I compare the performance of the different methods by their root mean squared errors. The improved estimator consistently produces lower root mean squared errors than the original estimator of van Oordt and Zhou (2019). The reduction in the mean squared error is in six subperiods statistically significant. In almost all subperiods, the improved estimator outperforms the estimator of the conditional regression approach and this performance is in ten subperiods statistically significant.

Lastly, this research had some limitations which were mainly due to the fact that this research was bounded by time. As said in Section 3, the tail dependence measure may suffer from an asymptotically bias and one might further improve $\hat{\beta}^T$ by an unbiased estimator of the tail dependence measure. There are several researchers who tackle the problem of the bias in the tail dependence measure and propose a bias corrected estimator for the tail dependence measure, see for example Abdous and Ghoudi (2015), Beirlant, Dierckx, and Guillou (2011) and Goegebeur and Guillou (2013) among others. Further, the simulation study was kept short and I did not incorporate simulations with different copula models as in van Oordt and Zhou (2019). However, since the data generating process in these models does not follow a linear model in the tail and have different dependence structures, these simulations may suit the empirical part better and help to explain some phenomena seen in the empirical application better.

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Derivations

The Student's t-distribution with ν degrees of freedom has a probability density function given by

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

The distribution function of the *skewed* Student's t-distribution with ν degrees of freedom is given by

$$f_{ST}(t) = \frac{\nu^{1/2}\sqrt{(3\lambda^2+1)(\frac{1}{2\nu-2}) - \frac{4\lambda^2}{\pi}\left(\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\right)^2}\Gamma(\frac{1}{2}+\nu)}{\sigma(\pi\nu)^{1/2}\Gamma(\nu)\left(\frac{|t-\mu+m|^2}{\nu(v\sigma)^2(\lambda sign(x-\mu+m)+1)^2}+1\right)^{\frac{1}{2}+\nu}},$$

where

$$m = \frac{2v\sigma\lambda\nu^{1/2}\Gamma(\nu-\frac{1}{2})}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}$$

and

$$v = \frac{1}{\nu^{1/2}\sqrt{(3\lambda^2+1)(\frac{1}{2\nu-2}) - \frac{4\lambda^2}{\pi}\left(\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\right)^2}}$$

Code

Simulation.R

1. **beta05:** this function draws values of X and ϵ from a skewed Student's t-distribution, generates the linear model with slope $\beta = 0.5$ and returns the estimates for the improved estimator, the original estimator and the conditional ordinary least squares estimator.

2. **beta1:** this function draws values of X and ϵ from a skewed Student's t-distribution, generates the linear model with slope $\beta = 1$ and returns the estimates for the improved estimator, the original estimator and the conditional ordinary least squares estimator.

3. **beta15:** this function draws values of X and ϵ from a skewed Student's t-distribution, generates the linear model with slope $\beta = 1.5$ and returns the estimates for the improved estimator, the original estimator and the conditional ordinary least squares estimator.

4. **seg05:** this function draws values of X and ϵ from a skewed Student's t-distribution, generates the segmented linear model with slope $\beta = 1, \beta^T = 0.5$ and returns the estimates for the improved estimator, the original estimator and the conditional ordinary least squares estimator.

5. **seg15:** this function draws values of X and ϵ from a skewed Student's t-distribution, generates the segmented linear model with slope $\beta = 1, \beta^T = 1.5$ and returns the estimates for the improved estimator, the original estimator and the conditional ordinary least squares estimator.

Plots.R

1. **gather.beta05:** Runs beta05 and computes the mean squared error, squared bias and the variance of all the estimators for different values of k and puts them in a dataframe.

2. **gather.beta1:** Runs beta1 and computes the mean squared error, squared bias and the variance of all the estimators for different values of k and puts them in a dataframe.

3. **gather.beta15:** Runs beta15 and computes the mean squared error, squared bias and the variance of all the estimators for different values of k and puts them in a dataframe.

4. **gather.seg05:** Runs seg05 and computes the mean squared error, squared bias and the variance of all the estimators for different values of k and puts them in a dataframe.

5. **gather.seg15:** Runs seg15 and computes the mean squared error, squared bias and the variance of all the estimators for different values of k and puts them in a dataframe.

6. **plot.beta05:** Plots the mean squared error, squared bias and the variance for different values of k for the linear model with $\beta = 0.5$.

7. **plot.beta1:** Plots the mean squared error, squared bias and the variance for different values of k for the linear model with $\beta = 1$.

8. **plot.beta15:** Plots the mean squared error, squared bias and the variance for different values of k for the linear model with $\beta = 1.5$.

9. **plot.seg05:** Plots the mean squared error, squared bias and the variance for different values of k for the segmented linear model with $\beta = 1, \beta^T = 0.5$.

10. **plot.seg15:** Plots the mean squared error, squared bias and the variance for different values of k for the segmented linear model with $\beta = 1, \beta^T = 1.5$.

Empirical.R

1. **beta:** Collects the returns of the industry-specific portfolios for a subperiod and returns the estimates for the improved estimator, the original estimator and the conditional ordinary least squares estimator.

2. **loss:** Computes the Diebold-Mariano test statistic and its associated p-value.

Additional Results

Period	Av. $\hat{\beta}^T_{EVT,j}$	N	S	Minimum $\hat{\beta}^T_{EVT,j}$		Maximum $\hat{\beta}^T_{EVT,j}$	
1931-1935	1.16	41	2	0.58	Apparel	1.76	Recreation
1936-1940	1.07	42	1	0.41	Tobacco Products	2.05	Construction
1941-1945	1.19	42	0	0.63	Communication	2.70	Real Estate
1946-1950	1.01	43	0	0.33	Communication	1.76	Construction
1951-1955	1.03	43	0	0.40	Communication	1.61	Aircraft
1956-1960	1.10	42	1	0.46	Utilities	1.82	Electronic Equipment
1961-1965	1.18	43	0	0.55	Utilities	1.86	Recreation
1966-1970	1.25	47	0	0.59	Utilities	2.13	Other
1971-1975	1.15	48	0	0.71	Utilities	1.79	Entertainment
1976-1980	1.10	48	0	0.60	Utilities	1.92	Precious Metals
1981-1985	1.13	48	0	0.65	Utilities	2.14	Precious Metals
1986-1990	0.97	48	0	0.53	Utilities	1.29	Candy & Soda
1991-1995	1.16	48	0	0.65	Utilities	1.85	Shipbuilding, Railroad Equipment
1996-2000	1.03	48	0	0.45	Utilities	1.98	Coal
2001-2005	1.04	48	0	0.58	Food Products	1.83	Electronic Equipment
2006-2010	1.08	48	0	0.56	Beer & Liquor	2.05	Coal
2011-2015	1.09	48	0	0.65	Beer & Liquor	2.27	Coal
2015-2020	0.95	48	0	0.54	Utilities	1.67	Coal

Table 4: Estimates

Notes : Within each subperiod, the EVT approach of van Oordt and Zhou (2019) is used to estimate β^T for 48 daily excess returns of industry portfolios, while excluding the observation on the day of the largest market loss. In each subperiod, k is set such that $k/n = 2\%$ in that specific subperiod. The column labeled "Av. $\hat{\beta}^T_{EVT,j}$" reports the average of $\hat{\beta}^T_{EVT,j}$ of the remaining N portfolios. The column labeled S reports the number of excluded portfolios, based on the criteria $\hat{\alpha}_j \leq \frac{1}{2}\hat{\alpha}_m$. The last columns report the minimum and maximum $\hat{\beta}^T_{EVT,j}$ and their industry name from the data documentation.

Period	Worst day	Market Loss	RMSE OLS	RMSE EVT	t-stat	p-value
1931-1935	July 21, 1933	9.21	7.98	6.36	-1.21	0.233
1936-1940	October 18, 1937	8.20	3.94	2.98	-1.97	0.056
1941-1945	December 08, 1941	4.15	2.30	1.72	-2.80	0.008
1946-1950	September 3, 1946	6.90	2.26	1.44	-1.73	0.091
1951-1955	September 26, 1955	6.52	2.82	1.73	-2.75	0.009
1956-1960	October 21, 1957	3.04	1.97	1.25	-2.53	0.015
1961-1965	May 28, 1962	7.00	3.32	2.99	-0.96	0.344
1966-1970	May 25, 1970	3.21	2.30	1.63	-2.22	0.032
1971-1975	November 18, 1974	3.57	2.44	1.21	-3.24	0.002
1976-1980	October 9, 1979	3.44	2.16	1.09	-5.22	0.000
1981-1985	October 25, 1982	3.62	2.32	1.25	-3.46	0.001
1986-1990	October 19, 1987	17.44	6.80	4.15	-1.38	0.173
1991-1995	November 15, 1991	3.55	2.74	1.82	-2.82	0.007
1996-2000	April 14, 2000	6.72	4.35	3.48	-1.01	0.319
2001-2005	September 17, 2001	5.03	5.75	5.20	-1.02	0.311
2006-2010	December 1, 2008	8.95	3.11	1.14	-3.97	0.000
2011-2015	August 8, 2011	6.97	1.51	1.94	0.70	0.487
2015-2020	March 16, 2020	12.00	6.79	4.88	-2.29	0.027

Table 5: Performance Evaluation

Notes: Within each period, β^T is estimated for 48 daily excess returns of industry portfolios using the estimator of van Oordt and Zhou (2019) and the conditional ordinary least squares approach. In each subperiod, k is set such that $k/n = 2\%$ in that specific subperiod. Further, in each subperiod, the observation on the day of the largest market loss is excluded in order to evaluate the performances of the projected losses. For this performance evaluation, I use the difference between the actual loss and the projected loss and calculate its root mean squared error (RMSE) for each approach. The last two columns report the t-statistics from Equation (13) and the corresponding p-values for testing the null hypothesis of equal prediction accuracy of the two approaches. Shaded numbers indicate p-values smaller than 5%.