Time-Varying Dependence Modelling

Capturing asset dynamics using copula techniques

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Abstract

Coincident downturns are often documented in financial time-series, whereas simultaneous surges are not evident as such. This asymmetrical dependence between assets is however often not accounted for by mainstream financial volatility models, sometimes additionally imposing normality restrictions on the marginals. Copulas, conversely, 'decouple' marginal distributions which allows for analysing dependence separately. In this paper static Gaussian and Clayton copulas, as well as a time-varying Clayton copula are considered to model German and Dutch stock returns. Its Value-at-Risk is compared to *RiskMetrics* by looking at backtesting performance. The Clayton copula techniques yielded the best VaR in terms of average exceedance errors, suggesting asymmetrical dependence in the stock returns.

Contents

1 Introduction

The nature of time series is often assessed in academic literature. This has led to the so called stylised facts observed in asset returns, such as volatility clustering and excess kurtosis, which models based on normality can not capture. There has also been interest in multivariate time-series and their mutual dependency. A natural metric to measure dependency would be the correlation between assets. However, correlation depends on the assumption of Gaussian processes, which financial assets do not show. Therefore a different approach is needed.

Copulas form an interesting alternative to popular dependence modelling techniques. These functions do not impose restrictions on marginal distributions, and even allow for asymmetric dependence. These are functions that 'link' marginal distributions to model a joint distribution. Of course, based on these attributes of copulas, it would be of importance to analyse how well copulas perform, as this has direct implications to portfolio theory and Value-at-Risk. Moreover, in this research the copula will also be time-varying, based on the assumption of local homogeneity. It allows for a dynamic estimation, which is called the Local Change Point (LCP) procedure, and is deemed a relative novel approach. The main research question of this paper thus reads '*how well do time-varying copula models perform compared to traditional methods of determining risk?*' In order to answer this question, Value-at-Risk (VaR) of the LCP procedure (based on the Clayton copula), moving windows (based on Gaussian and Clayton copulas) and *RiskMetrics* is compared.

Historical data from German and Dutch stocks are acquired for the purpose of empirical analysis. Two portfolios consisting of German stocks over the period 2000 to 2005, as well as one portfolio of Dutch stocks over the period 2006 to 2016 are examined to calculate the respective VaR. The copula methods (Gaussian and Clayton) appeared to be performing better than *RiskMetrics* in terms of exceedance errors at different levels. Clayton methods (LCP and moving windows) consistently produced the lowest standard error of exceedances, suggesting lower tail dependence in returns.

This paper is organised as follows. Section 2 discusses relevant literature. Section 3 presents the methodology of this paper, with subsections explaining different steps in the estimation procedure. Simulations are performed with the copula model in Section 4 to assess its dynamics. Following the simulation the empirical data is presented in Section 5, which is used in this paper for estimation and visual purposes. Section 6 presents the estimation results of the time-varying copulas, moving window and *RiskMetrics* estimations. Finally Section 7 concludes.

2 Literature Review

Historically there has been ample interest in financial time series. This has led to the formulation of so called stylised facts of these time series. For example, empirical evidence from a wide selection of financial assets suggest volatility clustering, negative skewness and excess kurtosis. Cont (2001) presents a thorough examination of these stylised statistical facts. This leads to the conclusion that models based on normal distributions often fail to capture non-normal behaviour of financial time series. However, many financial applications such as investment strategies, asset pricing and portfolio management rely on the correct specification of these models. It is therefore of importance to correctly specify financial models that are used in this framework.

Black (1976) found that conditional volatility of asset returns did react differently on positive and negative shocks in asset returns. This asymmetry yields a larger increase in volatility after negative returns compared to positives. Several authors (see for example Kraus and Litzenberger (1976), Friend and Westerfield (1980) and Richardson and Smith (1993)) support this finding of skewness in the distribution of stock returns with empirical evidence.

Recently there has been interest in the (time-varying) multivariate distributions of asset returns. In this new framework, studies also found deviations from normality. An asymmetric dependence between assets manifests itself when in declining markets returns often have greater correlation compared to when a market is expanding. Research from Longin and Solnik (2001) demonstrated that there indeed exists increased correlations between assets in market downturns, but could not conclude the same for bullish markets. Ang and Chen (2002) presented evidence that rejects the notion of multivariate normality at multiple frequencies in asset returns. One possible solution to correct for non-normal behaviour is to apply multivariate ARCH/GARCH models. Various models have been introduced to try to specify some form of conditional heteroskedasticity. One well-known example is the VEC representation of the conditional covariance matrix by Bollerslev et al. (1988).

Many financial models rely on some kind of Gaussian processes (for example the Black-Scholes model implements Brownian motions). However, these well-known multivariate distributions are too rigid in the sense that they cannot account for non-normal tail behaviour in different variables. Sklar (1959) posed a new theorem of copulas, in which an n-dimensional joint distribution of variables can be written as a function of its n marginal distributions. The *copula* then presents the dependence between the n variables. The main motivation for this technique is that it is possible to examine the copula systems under different assumptions and that it does not impose strong restrictions of normality. This has led to the implementation of copulas in many financial applications. For example Cherubini and Luciano (2001) and Embrechts et al. (2002) analyse Value-at-Risk in a risk management setting. Patton (2004) and Boubaker and Sghaier (2013) look at portfolio optimisation using (time-varying) copulas. Finally, in more recent research, there has been interest in high dimensional copulas. Creal and Tsay (2015) evaluate high dimensional copulas by considering them as factor models. Wang et al. (2018) assess complex dependencies when introducing renewable energy by using high dimensional copulas.

Aside from the ability of modelling dependence with a copula, it is also of importance to consider the non-constant nature of conditional volatility frequently observable in economic time series. It follows that models that also account for time-varying volatility may more accurately resemble economic theory. Longin and Solnik (1995) and Engle (2002) show that correlation is not constant over time, which implies dependency between variables is also time dependent. Many time-varying copula models are assessed in literature, see for example Patton (2006), Ausin and Lopes (2010) and So and Yeung (2014), where the researchers modelled volatility with univariate GARCH processes and the dependence structure with copulas. Finally, the dependence structure can also be deemed locally homogeneous. Dias and Embrechts (2010) and Guégan and Zhang (2010), among others, determine whether dependency is locally constant, while Busetti and Harvey (2011) develop a test for changing dependence in the time series.

3 Methodology

3.1 Copulas

Copulas provide an alternative approach to conventional methods of modelling multivariate distribution functions. This technique has the advantage of modelling marginals and dependence structure separately. An n-dimensional copula is defined as a distribution function $\mathcal{C}$ that links standard uniform marginal distributions. This has led to Theorem 3.1 of Sklar (1959):

Theorem 3.1. *Let F be an n-dimensional joint distribution function with marginal cumulative distribution functions $F_1, \ldots, F_n$. Then there exists a copula $\mathcal{C}\colon [0,1]^n \to [0,1]$ such that*

$$F(x_1, \ldots, x_n) = \mathcal{C}\Big(F_1(x_1), \ldots, F_n(x_n)\Big), \tag{3.1.1}$$

for some vector $\boldsymbol{X} = (x_{1t}, \ldots, x_{nt})'$*, with* $t = (0, \ldots, T)$*. Conversely,*

$$\mathcal{C}(u_1, \ldots, u_n) = F\Big(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\Big), \tag{3.1.2}$$

with $u_i = F_i(x_i)$ *and* $F_i^{-1}(x_i)$ *the inverse of* x_i*.*

A proof of the two-dimensional case is given in Schweizer and Sklar (1975). In this research copulas will be used in order to estimate multivariate distribution functions. The scope of this article will cover financial stock returns as considered data. Subsequently, Value-at-Risk estimates are evaluated in order to quantify the copula performance. There are multiple copula families introduced in literature, see for example McNeil et al. (2015). Copulas can either be classified as elliptical or non-elliptical. One popular elliptical copula is the Gaussian copula

$$\mathcal{C}_{\Psi}^{G}(u_1, \ldots, u_n) = F_{\boldsymbol{X}}\Big(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\Big), \tag{3.1.3}$$

with $\Phi(u_i)$ the standard normal CDF and $\boldsymbol{X}$ assumed to be normally distributed around $\mathbf{0}$ with correlation matrix Ψ. Gaussian copulas are convenient to work with. However, the downside of a Gaussian copula is that it assumes normally distributed and therefore symmetric dependence without tail dependence. This means it underestimates kurtosis, as well as the dependence structure in the tails when looking at returns in financial data.

Non-elliptical (or Archimedean) copulas allow for non-elliptical distributions with tail dependencies. Archimedean copulas are defined as $\mathcal{C}(u_1, \ldots, u_n) = \phi^{-1}(\sum \phi(u_i))$, with ϕ^{-1} the inverse of ϕ. The function ϕ is also called the generator of the copula. In this research the Clayton copula

$$\mathcal{C}_{\theta}^{Cl}(u_1, \ldots, u_n) = \Big(\sum_{i=1}^{n} u_i^{-\theta} - (n+1)\Big)^{-\theta^{-1}} \qquad \theta > 0, \tag{3.1.4}$$

is used for modelling tail dependency, where $\mathcal{C}_{\theta}^{Cl}$ is the cumulative distribution function of the copula. To arrive at the density of the Clayton copula, note that for any copula $\mathcal{C}(u_1, \ldots, u_n)$ the PDF $c(u_1, \ldots, u_n)$ can be obtained as $c(u_1, \ldots, u_n) = \partial^n \mathcal{C}((u_1, \ldots, u_n)/\partial u_1, \ldots \partial u_n$. The density for the Clayton copula then is obtained as

$$c_{\theta}(u) = \prod_{k=0}^{d-1}(\theta k + 1)\Big(\prod_{i=1}^{n} u_i\Big)^{-(1+\theta)}\Big(\sum_{i=1}^{n} u_i^{-\theta} - n + 1\Big)^{-(n+1/\theta)}. \tag{3.1.5}$$

See Hofert et al. (2012) for comprehensive proofs of multivariate Archimedean copulas. The density as given in Equation (3.1.5) will be needed when the copula is estimated.

Tail dependency is a measurement of simultaneous movement of the variables in the tails of the distribution function. This concept refers to the dependence in extreme values of the considered variables, which depends mainly on the behaviour in the tails. Variables can exhibit tail dependency when no correlation is present. Lower and upper tail dependency is defined as

$$\lambda_L := \lim_{u \downarrow 0} \frac{\mathcal{C}(u, \dots, u)}{u}; \tag{3.1.6}$$

$$\lambda_U := \lim_{u \downarrow 0} \frac{\hat{\mathcal{C}}(u, \dots, u)}{u}, \tag{3.1.7}$$

respectively, where $u \in (0, 1]$ and $\hat{\mathcal{C}}$ the survival copula of $\mathcal{C}$ (see Joe (1997) for a review of dependency). When $\lambda_L = 0$, the copula does not demonstrate lower tail dependence. When $\lambda_U = 0$, the copula does not demonstrate upper tail dependence. Gaussian copulas are copulas with both independent lower and upper tails ($\lambda_L = \lambda_U = 0$). Clayton copulas exhibit independent upper tail behaviour and dependent lower tail dependency ($\lambda_L = d^{-1/\theta}$, with d the dimension). Figures 1 and 2 present draws from bivariate Clayton and Gaussian copulas, respectively. Notice the lower tail dependency in the Clayton distribution and no upper tail dependency ($\theta = 2.5$). The gaussian distributed variables exhibit constant correlation across the distribution ($\rho = 0.7$). Modelling tail dependency can be beneficial in analysing financial data, since this data often deviate from normality and exhibit greater correlation in market contractions.

Figure 1: Simulated bivariate Clayton copula with $\theta = 2.5$*, 1000 draws.*

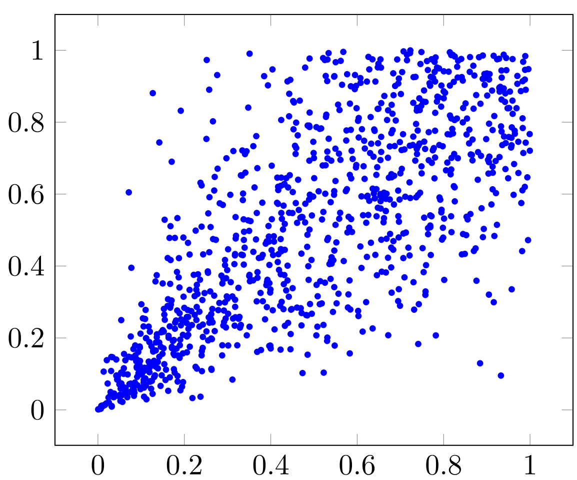
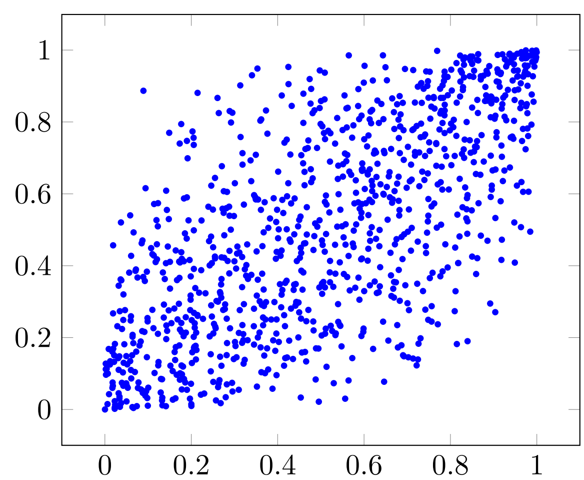


Figure 2: Simulated bivariate Gaussian copula with $\rho = 0.7$*, 1000 draws.*



3.2 Estimation

The next step is to estimate the copula parameter. In this analysis the Gaussian and Clayton copula $\mathcal{C}(\boldsymbol{u})$ (with dependence parameter θ, $\theta > 0$ for the Clayton copula and correlation matrix Ψ for the Gaussian copula) is estimated. This is achieved by means of maximising the pseudo log-likelihood. Let $\{x_{t,i}\}_{t=1}^{T}$, $i = 1, \ldots, n$ be a vector of observations. Then the canonical log-likelihood is given as

$$l(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T; (\theta, \Psi)) = \sum_{t=1}^{T} \log\, c(\boldsymbol{x}_i; (\theta, \Psi)) = \sum_{t=1}^{T} \log\, c\Big(\hat{F}_1(x_{t,1}), \ldots, \hat{F}_n(x_{t,n}); (\theta, \Psi)\Big), \qquad (3.2.1)$$

where c the PDF of the copula, (θ, Ψ) the parameters of either Clayton or Gaussian and moreover assuming that the data is transformed into uniformly distributed variates $\{\hat{u}_{t,i}\}_{t=1}^{T}$. The advantage of working with the canonical maximum likelihood is that it is possible to estimate the likelihood in two steps. The first step is to estimate the nonparametric CDF of the marginal distributions $\hat{F}_i(x_{t,i})$. A variant of the empirical CDF is used for this purpose. This step yields uniform variates

$$\hat{u}_{t,i} = \hat{F}_i(x_{t,i}) = \frac{1}{T+1} \sum_{t=1}^{T} I_{[x_{t,i} \leq x]}. \qquad (3.2.2)$$

Equation (3.2.2) differs from the standard empirical CDF in the scalar $1/(1+T)$, to ensure that the transformed data does not take infinite values on the the boundary of the unit interval [0,1]. Step two then maximises the pseudo log-likelihood which provides maximum likelihood estimator $\hat{\theta}$ or $\hat{\Psi}$,

$$\hat{\theta}, \hat{\Psi} = \arg\max \sum_{t=1}^{T} \log\, c(\hat{u}_{t,1}, \ldots, \hat{u}_{t,n}). \qquad (3.2.3)$$

3.3 Value-at-Risk

In the empirical analysis of this paper the Value-at-Risk will be examined, since the VaR has a direct connection with the dependence between asset returns. Consider $V_t = \sum_{i=1}^{n} w_i a_{t,i}$ the value of a portfolio at time t with n assets, $\boldsymbol{w} = (w_1, \ldots, w_n)'$ the weights of asset i, and $\boldsymbol{A}_t = (a_{t,1}, \ldots, a_{t,n})'$ the prices of asset i ($i \in [1, \ldots, n]$). Then the log returns can be defined as

$$\boldsymbol{R}_t = (\log\, a_{t,1} - \log\, a_{t-1,1}, \ldots, \log\, a_{t,n} - \log\, a_{t-1,n})' = (R_{t,1}, \ldots, R_{t,n})'. \qquad (3.3.1)$$

The profit and loss function can be written using Equation 3.3.1 as

$$L_t = \sum_{i=1}^{n} w_i a_{t-1,i}(e^{R_{t,i}} - 1). \tag{3.3.2}$$

The distribution of L_t is given as $F_{t,L_t}(x) = P_t(L_t \leq x)$ and is determined by the distributions of the returns $R_{t,i}$. It follows that the VaR at confidence level α is written as

$$\text{VaR}_t(\alpha) = F_{t,L_t}^{-1}(\alpha). \tag{3.3.3}$$

This means the unknown CDF $F_{t,L_t}(x)$ has to be specified. For this reason there will be looked at shocks of individual assets.

The shocks $\boldsymbol{\varepsilon_t} = (\varepsilon_{t,1}, \ldots, \varepsilon_{t,n})'$ are obtained to estimate the Gaussian and Clayton copula in this analysis, in order to compare the results with conventional benchmarks which are described in Section 3.4. Therefore the log-returns first have to be demeaned and devolatised. This is achieved by specifying a *RiskMetrics* model for the conditional variance of every asset in the considered portfolio. For asset i the historical volatility is calculated as

$$\hat{\sigma}_{t,i}^2 = (1-\lambda)R_{t-1,i}^2 + \lambda\hat{\sigma}_{t-1,i}^2, \tag{3.3.4}$$

where $\lambda = 0.94$ when operating in the framework of *RiskMetrics*, and R_t^2 the squared asset return of asset i at time t. It allows for estimating the residuals as

$$\hat{\varepsilon}_{t,i} = \frac{R_{t,i} - \mu_{t,i}}{\hat{\sigma}_{t,i}}, \tag{3.3.5}$$

with $\mu_{t,i} = \text{E}(R_{t,i}|\mathcal{F}_{t-1})$ the conditional mean and $\sigma_{t,i}^2 = \text{E}\big((R_{t,i} - \mu_{t,i})^2|\mathcal{F}_{t-1}\big)$ the conditional variance. This yields standardised shocks with unknown joint CDF $F_{\boldsymbol{\varepsilon_t}}$. However, using Theorem 3.1 proposed in Sklar (1959), $F_{\boldsymbol{\varepsilon_t}}$ can be written as

$$F_{\boldsymbol{\varepsilon_t}}(x_1, \ldots, x_n) = \mathcal{C}_\theta\Big(F_{t,1}(x_1), \ldots, F_{t,n}(x_n)\Big). \tag{3.3.6}$$

The dependence parameter θ of copula $\mathcal{C}$ can then be estimated using the marginal CDF's of the shocks $\boldsymbol{\varepsilon_t}$ as explained in Section 3.2. The final step is to generate 1000 Monte Carlo portfolio simulations using the dependence parameter $\hat{\theta}_t$ and CDFs from the residuals. These residuals are then transformed back to the original time-series using its mean and volatility as described in Equation 3.3.5. The ordered simulations subsequently yield estimations of Value-at-Risk.

3.4 Benchmarks and performance measures

The Value-at-Risk obtained with the estimation of the Gaussian and Clayton copula can be compared with traditional techniques. In this analysis the VaR will also be estimated using a *RiskMetrics* model for portfolio log returns with different weights. The specification of the conditional variance is the same as given in Equation (3.3.4). Moreover, the Gaussian and Clayton copula will also be estimated by means of a fixed moving window of 250 days. However, this moving window procedure is dependent on its length. Small windows will produce unstable estimates while at the same time large windows yield delays in the changing dependence parameter. In the following Subsection the LCP procedure is presented which aims to resolve moving window shortcomings.

The performance of the VaR can be quantified using backtesting. At every time t the log returns R_t as defined in Equation (3.3.1) of a portfolio $\boldsymbol{w}$ with given weights, is compared with the estimated VaR at that point. The exceedance ratio $\hat{\alpha}_{\boldsymbol{w}}(\alpha) = \frac{1}{T}\sum_{t=1}^{T} I_{\{R_t < \hat{V}aR_t(\alpha)\}}$ is a measurement of exceedances at confidence level α. The difference between $\hat{\alpha}$ and α subsequently yields the relative exceedance ratio $e_{\boldsymbol{w}} = (\hat{\alpha} - \alpha)/\alpha$. Relative exceedance ratio is computed for confidence level 0.05 and 0.01 for a large portfolio set $W = \{\boldsymbol{w}^*, \boldsymbol{w}_k : k = 1, \ldots, 100\}$. Each $\boldsymbol{w}_k$ is uniformly distributed, and adhering to the constraints $\sum_{i=1}^{n} w_{k,i} = 1$ and $w_{k,i} \geq 0.1$ for asset i in portfolio k. The portfolio set W then produces an average exceedance error and its corresponding standard deviation

$$A_W = \frac{1}{|W|}\sum_{\boldsymbol{w}\in W} e_{\boldsymbol{w}} \tag{3.4.1}$$

$$D_w = \left(\frac{1}{|W|}\sum_{\boldsymbol{w}\in W} (e_{\boldsymbol{w}} - A_W)^2\right)^{1/2} \tag{3.4.2}$$

which will be used to formally compare performance of the VaR estimation.

3.5 Local Change Point procedure

In this Section a technique called the local change point (LCP) procedure will be applied to estimate the Clayton copula parameter. A simulation of changing Clayton copula parameter will also be performed in Section 4 in order to test the performance of the LCP procedure.

Interval of homogeneity Instead of estimating the copula parameter θ_t for the entire sample, or allowing for time variation, the parameter θ_t could be estimated with a smaller interval $I_t = [t - m_t, t]$ for each $t \in (1, \ldots T)$ (thus deemed to be locally homogeneous). Based on the local change point procedure (LCP) of Mercurio and Spokoiny (2005), the null hypothesis of local homogeneity (in other words, $\theta_t = \theta$) is tested against a local change point alternative. The procedure starts for any given $t_0 \in (1, \dots, T)$ with the determination of nested intervals $I_K \supset I_{K-1} \supset \dots \supset I_0$. Each interval I_k, $k \in (1, \dots, K)$ is defined as $I_k = [t_0 - m_k, t_0]$. These intervals are thus defined by m_k. It is chosen such that m_0 is fixed and m_k defined as $m_k = m_0 c^k$ ($c > 1$, rounding to the nearest natural number). Each interval I_k subsequently produces an estimate candidate $\hat{\theta}_k$ for the actual value θ_{t_0}. The LCP procedure then assigns one interval $\hat{I}_k$ with a corresponding estimate $\hat{\theta}_k$ to the value θ_{t_0}. How to choose this interval will be expounded in the following paragraph.

Testing LCP Each interval $I_k = [t_0 - m_k, t_0]$ consists of smaller intervals $W_k = [t_o - m_k, t_0 - m_{k-1}]$. For each point $\omega \in W_k$, the log-likelihood defined in Equation 3.2.1 is computed based on the interval $S = [\omega, t_o]$ and $S^c = [t_0 - m_k, \omega)$. The length of W_k is also determined by the value of m_k. Following Spokoiny (2009), m_0 will be set to 20 and c to 1.25. This procedure yields two log-likelihood for each $\omega \in W_k$. The null hypothesis claims $\theta_t = \theta \ \forall \ t \in I_k$. The alternative hypothesis allows for a change point. A likelihood ratio test with

$$T_{I_k,\,\omega} = \max_{\theta_S,\,\theta_{S^c}} \Big(l_{S^c}(\theta_{S^c}) + l_S(\theta_S) \Big) - \max_{\theta} \ l_{I_k}(\theta) \tag{3.5.1}$$

$$= l_{S^c}(\tilde{\theta}_{S^c}) + l_S(\tilde{\theta}_S) - l_{I_k}(\tilde{\theta}_{I_k}), \tag{3.5.2}$$

can be applied in order to reject the null (with $\tilde{\theta}_i, i \in (S^c, S, I_k)$ the maximum likelihood estimates of the Clayton dependence parameter), with the test statistic defined as $T_{I_k} := \max_{\omega \in W} T_{I_k,\,\omega}$. This statistic will be compared to critical values $\mathcal{V}_k$, which are defined in the following paragraph. Concludingly, the interval I_k is accepted if the test statistic T_{I_k} does not exceed critical value $\mathcal{V}_k$ for each $\omega \in W_k$. If a local change point is detected in W_k, the procedure stops and uses the latest accepted interval I_{k-1} for the selection of the dependence parameter $\hat{\theta}_{I_{k-1}}$ for θ_{t_0}.

Determining critical values Whether the test statistic T_{I_k} has to be rejected depends on the exceedance of some critical value $\mathcal{V}_k$. However, as T_{I_k} is defined as the maximum over the interval W_k, the test statistic does not follow a known distribution function. Then again, it is of significant importance to determine the right critical values.

This methodology considers an approach that is based on the procedure of Spokoiny (2009). Its main goal is to select one (constant) estimate for $\tilde{\theta}_{I_k}$, the maximum likelihood estimator for interval I_k (and focussing on a false positive of an LCP), rather than focussing on the possibility of an LCP. A false positive leads to an estimation of $\hat{\theta}_{I_k}$ (the estimator after k steps of the LCP algorithm) that has a higher variance than $\tilde{\theta}_{I_k}$. It follows that some conditions should be established on both the estimates $\hat{\theta}_{I_k}$ and $\tilde{\theta}_{I_k}$. Spokoiny (2009) looks at the difference of the estimations, normalised by the risk of the global estimator $\tilde{\theta}_{I_k}$. Let $l_{I_k}(\tilde{\theta}_{I_k}, \hat{\theta}_{I_k}) := l_{I_k}(\tilde{\theta}_{I_k}) - l_{I_k}(\hat{\theta}_{I_k})$ be the difference of log-likelihood of estimations, and define the risk of the homogeneous situation be $R(\theta^*) = \max \boldsymbol{E}_{\theta^*}\left(|l_{I_k}(\tilde{\theta}_{I_k}, \hat{\theta}^*)|^{1/2}\right)$, with θ^* the parameter under the null. Then the constraint

$$\boldsymbol{E}_{\theta^*}\left(|l_{I_k}(\tilde{\theta}_{I_k}, \hat{\theta}_{I_k})|^{1/2}\right) \leq \nu R(\theta^*) \tag{3.5.3}$$

will determine the values for $\mathcal{V}_k$, which are selected as minimal values adhering to these conditions. The parameter ν is a scale parameter and determines the sensitivity of the constraints. Critical values are found with Monte Carlo simulation. This simulation is executed under the null with $\theta_t = \theta^*$. Table 1 presents critical values for given θ^* and ν. In this setting both θ^* and ν are unknown parameters. However, based on the findings in the paper of Spokoiny (2009) and on Table 1, both will later be fixed as they have negligible effect on the critical values.

	$\theta^* = 0.5$			$\theta^* = 1.0$			$\theta^* = 1.5$		
k	$\nu = 0.2$	$\nu = 0.5$	$\nu = 1.0$	$\nu = 0.2$	$\nu = 0.5$	$\nu = 1.0$	$\nu = 0.2$	$\nu = 0.5$	$\nu = 1.0$
1	3.64	3.29	2.88	3.69	3.29	2.84	3.95	3.49	2.96
2	3.61	3.14	2.56	3.43	2.91	2.35	3.69	3.02	2.78
3	3.31	2.86	2.29	3.32	2.76	2.21	3.34	2.80	2.09
4	3.19	2.69	2.07	3.04	2.57	1.80	3.14	2.55	1.86
5	3.05	2.53	1.89	2.92	2.22	1.53	2.95	2.65	1.49
6	2.87	2.26	1.48	2.92	2.17	1.19	2.83	2.04	0.94
7	2.51	1.88	1.02	2.64	1.82	0.56	2.62	1.79	0.31
8	2.49	1.72	0.35	2.33	1.39	0.00	2.35	1.33	0.00
9	2.18	1.23	0.00	2.03	0.81	0.00	2.10	0.60	0.00
10	0.92	0.00	0.00	0.82	0.00	0.00	0.79	0.00	0.00

Table 1: Critical values $\mathcal{V}_k$ for given values θ^* and ν, as given in Giacomini et al. (2009).

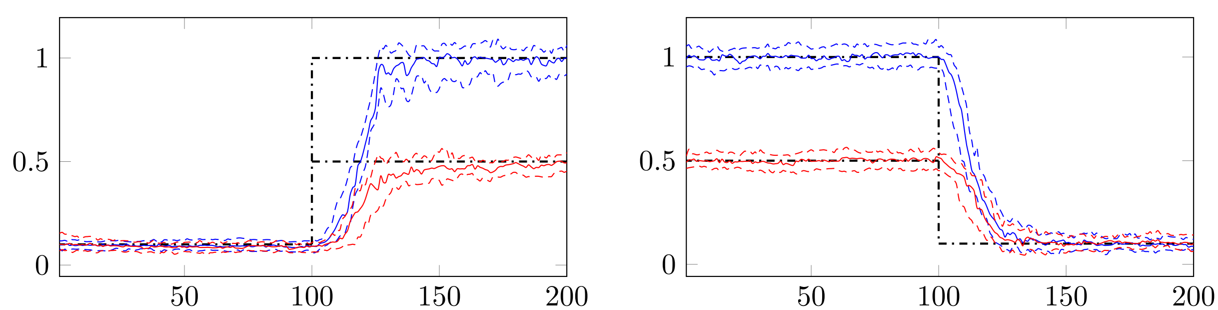
4 Simulation

The local change point procedure as described in Section 3.5 can be applied to simulated data. Sets of 6-dimensional data with a sudden jump in the dependence parameter are simulated for this purpose. The dependence parameter is modelled as

$$f(x) = \begin{cases} \theta_a, & \text{if } 0 \leq t \leq 100 \\ \theta_b, & \text{if } 100 < t \leq 200 \end{cases} \tag{4.0.1}$$

for different values of θ_a and θ_b. The LCP algorithm is calculated with fixed parameters. It can be seen from Table 3 that the choice of θ^* has little effect on the sequence of critical values. Therefore it is set as $\theta^* = 1$ for the simulation examples. Based on Spokoiny (2009) and Giacomini et al. (2009), the intervals I_k will be initiated with parameters $m_0 = 20$ and $c = 1.25$, whereas $\nu = 0.5$. Figure 3 presents simulated examples for different values of θ_a and θ_b, based on the above specification of the LCP procedure. It can be seen that it takes about 20 to 30 observations for the LCP procedure to 'catch up' with the sudden jump in the dependence parameter.

Figure 3: Simulation of data with a sudden jump in dependence parameter. Both figures depict point-wise median (full line) and 0.25 and 0.75 quantiles (dashed) of the parameter $\hat{\theta}_t$ estimated with LCP. The true simulated parameter is shown with the dash-dotted line. In the left figure $\theta_a = 0.1$, $\theta_b = 1$ (estimated in blue), $\theta_b = 0.5$ (estimated in red). In the right figure $\theta_b = 0.1$, $\theta_a = 1$ (estimated in blue), $\theta_a = 0.5$ (estimated in red). Based on 50 simulations where $m_0 = 20$, $c = 1.25$ and $\nu = 0.5$.



It is viable to better understand the detection delay; the time it takes for the algorithm to catch up with the real parameter. It is possible to get more insight of this phenomenon through the expression

$$\delta(t, \gamma, r) = \min\{u \geq t : \hat{\theta}_u = \theta_a + r\gamma\} - t, \tag{4.0.2}$$

where $\gamma = \theta_b - \theta_a$ and r the fraction it takes for the parameter to attain the true value. Table 2 presents statistics for the detection delay of the LCP procedure, based on 50 simulations, $m_0 = 20$, $c = 1.25$ and $\nu = 0.5$. It can be seen that the average detection delay for the different simulation examples is below 30 observations.

(θ_a, θ_b)	(0.1,0.5)			(0.1,1.0)			(0.5,0.1)			(1.0,0.1)		
r	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
Mean	15.30	20.00	26.10	13.98	19.64	24.96	10.94	14.48	20.14	8.30	12.16	18.34
SD	7.27	7.81	12.15	4.01	3.72	4.54	6.18	7.14	10.33	4.05	4.35	5.07
Min	1	2	3	6	11	15	0	0	5	2	4	8
Max	37	48	79	21	27	40	22	30	70	19	27	31

Table 2: Detection delay δ statistics for given values r and (θ_a, θ_b). Based on 50 Clayton copula simulations, implemented with LCP algorithm parameters $m_0 = 20$, $c = 1.25$ and $\nu = 0.5$.

5 Data

The empirical analysis of this research considers two portfolios of six German stocks (obtained from Yahoo Finance), as well as one portfolio consisting of six Dutch stocks. The first German portfolio consists of stocks from automotive (Volkswagen and Daimler), insurance (Allianz and Münchener Rückversicherungs), and chemical (Bayer and BASF) industries. The second German portfolio consists of stocks from electrical (Siemens), energy (E.ON), metallurgical (ThyssenKrupp), airlines (Lufthansa), pharmaceutical (Merck) and chemical (Henkel) industries. All German companies mentioned are traded on the Frankfurt Stock Exchange.

The Dutch portfolio consists of stocks from semiconductors (ASML), brewers (Heineken), telecommunications (KPN), medical equipment (Philips), oil and gas (Royal Dutch Shell) and consumer goods (Unilever). These stocks are all traded on the Amsterdam Stock Exchange.

These data are used to calculate empirical VaR, implementing the Clayton copula based on the LCP procedure. The daily returns of German stock portfolios are observed in the period of January 1, 2000 to December 31, 2004, for a total of 1304 observations. The daily returns of the Dutch stock portfolio is observed from January 1, 2006 to December 31, 2015, for a total of 2561 observations. The selection of intervals as described in Section 3.5 will affect the effective range of observations. This means in the case where $m_0 = 20$ and $c = 1.25$ the first 250 observations will be used for the estimation procedures. Applying *RiskMetrics* to the individual stock data yields residuals $\hat{\varepsilon}_{t,i}$. Table 3 presents p-values from the Ljung-Box test for serial correlation, as well as p-values from the ARCH test for heteroskedasticity effects in the residuals. The assets in j are in alphabetical order.

	German group 1		German group 2		Dutch group 1	
j	Ljung-Box	ARCH	Ljung-Box	ARCH	Ljung-Box	ARCH
1	0.50	0.09	0.00	0.03	0.76	0.84
2	0.06	0.99	0.99	0.98	0.44	0.14
3	0.53	0.00	0.91	0.53	0.92	0.88
4	0.91	0.20	0.72	0.16	0.22	0.89
5	0.11	0.37	0.27	0.00	0.43	0.02
6	0.98	0.53	0.86	0.00	0.00	0.00

Table 3: Reported p-values of serial correlation and heteroskedasticity tests on residuals $\hat{\varepsilon}_{t,j}$, for asset j in alphabetical order.

6 Results

In this section the data described in Section 5 is analysed. There are three methods that are used to estimate VaR for the portfolios. Backtesting is used in order to formally compare the performances of the three estimation procedures:

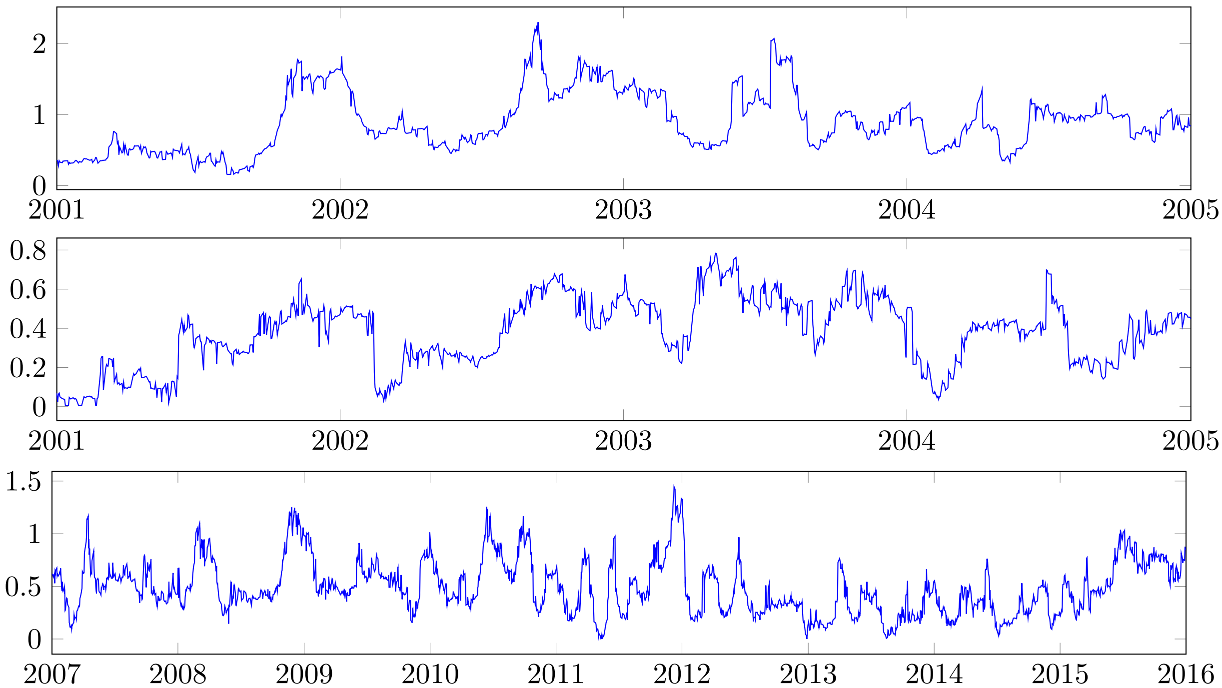
- Apply a *RiskMetrics* specification to the residuals of the portfolio log returns R_t;
- Implement moving window estimation of the dependence parameter of the Clayton copula and Gaussian copula. A fixed moving window of 250 observations is used in this setting;
- Specifying an LCP procedure in order to estimate the time-varying Clayton dependence parameter based on the assumption of locally homogeneous dependence between assets.

6.1 LCP estimations

First the results of the LCP estimation are presented. Figure 4 shows the time-varying dependence parameter $\hat{\theta}_t$ for German portfolio 1, German portfolio 2 and Dutch portfolio respectively. The parameter is estimated with $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$ for each procedure. It can be seen that the dependence parameter is often estimated higher for the first German portfolio than for the second. This can be attributed to the fact that the first German portfolio consists of stocks from three industries, whereas the second is more diversified with six industries.

When looking at the time-varying dependence in the Dutch portfolio, as seen in Figure 4, the level is of about the same level as the second German portfolio. The start of the Financial Crisis at the end of 2008 is also clearly visible, with an extended period of high dependence. It is interesting

Figure 4: Clayton copula dependence parameter $\hat{\theta}_t$ of German group 1 (top), German group 2 (middle) and Dutch group (bottom) estimated with the LCP algorithm.

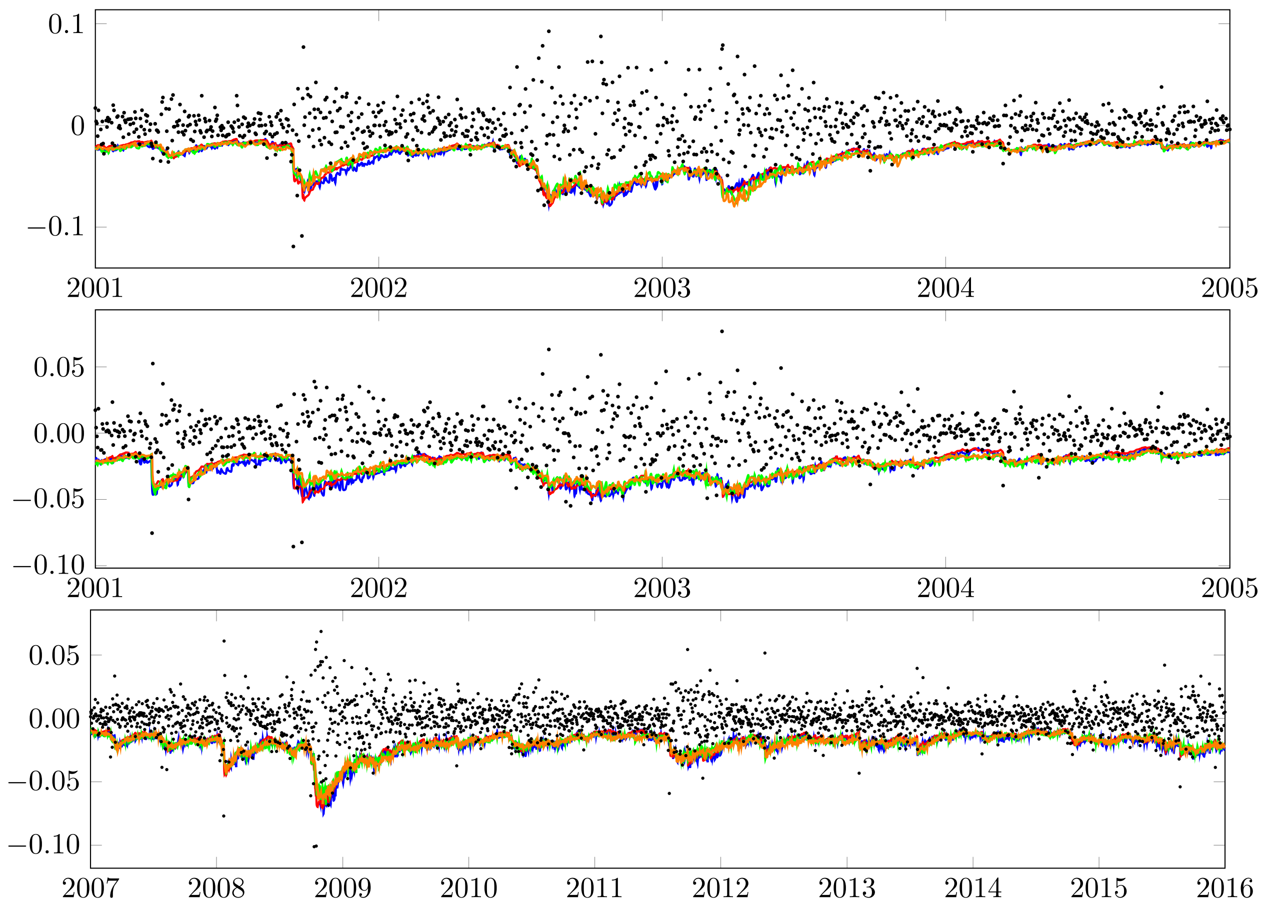


to note that by the end of 2011 the considered Dutch stocks exhibited even higher dependence. After that period dependency is low, with an increasing time-varying dependence parameter in 2015.

6.2 VaR results

Figure 5 present the VaR estimation for the different portfolios. Each figure shows four different VaR based on the different estimation techniques. It can be seen from the figures that the moving window VaR (in green) is less sensitive than the other two estimation techniques. This can be due to the fact that the moving window is 250 observations long, thus having a lagged effect in the portfolio returns. It can also be observed that after periods of high volatility in the returns the VaR based on LCP (in blue) is lagging after the VaR from moving windows and *RiskMetrics*. Otherwise the VaR seems to be similar across the different approaches. The next section will compare the obtained VaR in terms of exceedance ratios.

Figure 5: Value-at-Risk calculated with LCP (blue), RiskMetrics (red) and moving window (Clayton green, Gaussian orange) at level $\alpha = 0.05$. Top figure German group 1, middle figure German group 2, bottom figure Dutch group. Black dots indicate the log returns for the equally weighted portfolio.



6.3 Exceedance ratios

Table 4 presents the exceedance ratios for the equally weighted portfolio $\mathbf{w}^*$ and portfolio with random uniformly distributed weights $\mathbf{w}_1$, for German portfolios 1 and 2 and the Dutch portfolio. It also reports average exceedance A_w and standard errors D_w for the different estimation techniques.

Considering group 1 at confidence level $\alpha = 5.00$, it can be seen that the copula methods (LCP, moving windows) perform better than *RiskMetrics*, both in terms of average exceedance and its errors. LCP has the lowest average exceedance for about the same error. At confidence level $\alpha = 1.00$ in group 1, it can be seen that the copula methods overestimate the Value-at-Risk in this case. However, from an investor's point of view, it means that the considered confidence interval is ensured. It also shows relatively low errors D_w for both LCP and moving windows based on Clayton. This means in this case the Clayton copula is more consistent at estimating the VaR.

Group 2 at confidence level $\alpha = 5.00$ shows that moving windows based on the Clayton copula perform best in terms of average exceedance and its standard error. The Gaussian estimation technique has higher absolute average exceedance than the Clayton copula methods, and *RiskMetrics* performs worst. At confidence level $\alpha = 1.00$, Gaussian estimation (moving windows) produces the best VaR when looking at absolute average exceedance, next to Clayton VaR (LCP and moving windows). However, the Clayton copula yields far lower errors, again pointing at more consistent estimations of VaR.

When looking at the final group at confidence level $\alpha = 5.00$, it can be seen that the moving windows estimation technique based on the Gaussian copula performs best. It shows very low average exceedance for about the same errors across estimation techniques. However, when considering a confidence level $\alpha = 1.00$, it can be seen that the Clayton VaR produces lowest (absolute) average exceedance and lowest errors. In this case *RiskMetrics* also supplies least consistent VaR with high underestimation.

Portfolio	Estimation technique	$\alpha = 5.00$				$\alpha = 1.00$			
		$\hat{\alpha}_{\mathbf{w}^*}$	$\hat{\alpha}_{\mathbf{w}_1}$	A_w	D_w	$\hat{\alpha}_{\mathbf{w}^*}$	$\hat{\alpha}_{\mathbf{w}_1}$	A_w	D_w
Group 1	LCP	5.75	6.23	0.138	0.071	0.48	0.57	-0.486	0.075
	M.W. (Clayton)	5.65	5.94	0.173	0.076	0.38	0.57	-0.597	0.083
	M.W. (Gaussian)	5.75	5.17	0.190	0.082	0.86	1.15	-0.083	0.176
	RiskMetrics	6.32	6.51	0.292	0.066	1.44	1.25	0.511	0.269
Group 2	LCP	4.50	5.75	-0.041	0.139	0.57	0.67	-0.390	0.107
	M.W. (Clayton)	4.98	5.17	-0.017	0.137	0.57	0.38	-0.496	0.058
	M.W. (Gaussian)	5.36	4.69	0.090	0.139	1.15	1.34	0.186	0.242
	RiskMetrics	5.75	5.27	0.162	0.152	1.72	1.92	0.774	0.359
Dutch group	LCP	4.65	5.17	-0.020	0.081	0.87	0.78	-0.125	0.133
	M.W. (Clayton)	4.82	5.35	0.029	0.092	0.87	0.91	-0.104	0.099
	M.W. (Gaussian)	4.74	5.08	-0.002	0.082	1.22	1.17	0.260	0.163
	RiskMetrics	5.95	5.78	0.234	0.091	2.04	2.30	1.269	0.189

Table 4: Exceedance ratios and respective errors of different estimation techniques on German group 1, 2 and Dutch portfolio group, based on 1000 uniformly distributed random portfolios. M.W. moving windows. Performed with confidence level $\alpha = 5.00$ and $\alpha = 1.00$.

These observations based on average exceedance and errors show that *RiskMetrics* produces the least reliable VaR compared to the copula methods. VaR based on Gaussian moving windows sometimes yields average exceedances, but in each case showing high errors. Clayton copula estimation techniques have the lowest standard error (one exception) of all instances. This means that Clayton copula methods capture the underlying information available in the stock returns to a higher degree, thus pointing at dependence dynamics in the lower tail.

The difference between exceedance of LCP VaR and the Clayton moving windows VaR is less evident. It can be seen that the average exceedance and its respective standard errors are about the same for the two estimation methods, which also has been reported in Giacomini et al. (2009). The fixed LCP parameters could play a role in this regard, as they may have a considerable influence on the selection of estimation intervals and their estimators. Nevertheless has LCP the advantage over moving windows in terms of adaptive estimation, which does not impose restrictions on the dynamics of the dependence parameter.

7 Conclusion

The main research question of this paper asked how well time-varying copula models perform. The benchmark that was used to compare estimates was *RiskMetrics*. Value-at-Risk and backtesting were assessed in order to formally compare performances. Both Clayton and Gaussian copulas are used in this paper to estimate time-varying (asymmetric) dependence between assets. One advantage of the Clayton copula is that it does not impose any restrictions on the marginal distributions of the data. *RiskMetrics* does require multivariate normal distributions.

Copula estimation was implemented in a static and dynamic framework. The static framework considered a moving window of 250 observations. The dynamic framework made use of a technique called the Local Change Point procedure. This procedure assumes a locally homogeneous dependence parameter, which is estimated by comparing likelihoods of given intervals. It has an extra advantage over moving windows of not specifying the dependence dynamics. Based on backtesting, the copula methods (LCP, Gaussian and Clayton copula moving windows) performed better than *RiskMetrics*: average exceedance and its errors from the copula methods were reported to be smaller than those from *RiskMetrics*. The Gaussian copula sometimes produced similar results compared to the Clayton copula. However, these results also coincided with higher variability, pointing at a less consistent VaR. VaR based on LCP did not perform notably better than the VaR based on Clayton moving windows, which could depend on the specification of LCP parameters.

These results can be due to the fact that the *RiskMetrics* nor Gaussian copula techniques fully capture the underlying dependence dynamics. It has been proven that stock returns exhibit higher dependence in market contractions, see for example Longin and Solnik (2001); the specification of the Clayton copula makes it possible to capture these non-normal dynamics. It follows that the results from this paper support the findings from e.g. Longin and Solnik (2001) and Ang and Chen (2002). The implication is that (time-varying) copula models should also be considered in financial decision making.

One interesting subject for further research is the examination of different copula models. The Clayton copula model only allows for lower tail dependency. However, there exists multiple families of copulas that allow for different magnitudes of both lower and upper tail dependency. Moreover, this research considered given parameters for the LCP procedure, which might impede copula VaR performance in different time frames. Further investigation of copulas and LCP parameters in a financial framework could result in a fruitful continuation of research on this topic.

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