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Optimal time-varying copulas for estimating Value-at-Risk from a risk management perspective

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Abstract

Financial time-series data are of high dimensionality and possess intricate dependence structures. The joint dependence can be effectively modelled with copulas, since these dependence measures do not require the assumption of multivariate normality. Because of this characteristic copulas are a useful tool to estimate the Value-at-risk. There exist different copula families and several methods to allow for time variation. Therefore this research investigates what the best time-varying copula approach is to determine the Value-at-Risk from a risk management perspective. In order to answer this research question, we use daily data on the closing prices of twelve stocks listed on the DAX. As a result we find that regime switching copulas provide the best Value-at-Risk estimates with minimal risk, since this model does not underestimate the Value-at-Risk. This model is based on a time-varying Markov switching approach that switches between the Gaussian- and Clayton copula.

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1 Introduction

Markets interact quickly in the current globalised world, which implies some sort of dependence between stock markets. Financial time-series data is highly dimensional and the multivariate distribution is often assumed to be normal. This assumption rarely holds as proven by Fama (1963). Therefore copulas are a suitable tool to model dependence. Copulas do not require normality assumptions which results into a better empirical fit.

Even though copulas need less assumptions there are many aspects that can influence the correct copula fit. There are different copula families that each have their own dependence characteristics. This means that some copulas better capture the empirical stock returns than others. In addition copulas can be applied in different dynamic time-varying frameworks. A way to asses copula methods is to use them for estimating the Value-at-Risk (VaR). VaR is a popular measure in risk management which indicates the maximum loss over a specific period of time at a certain significance level. VaR estimates that adequately describe reality are paramount for companies and individuals that deal with risk management of their portfolios. A fitting time-varying copula method to accurately estimate the VaR is thus extremely relevant for efficient risk management.

On accounts of this, the main research question that will be answered in this paper is "Which time-varying copula approach provides the best Value-at-Risk estimates from a risk management perspective?" To be able to answer the main research question there are some subquestions that need to be answered. First of all we need to answer what copulas are and which copula family would be the most suitable to apply to financial time-series data. With this information we can decide on which different methods are the most appropriate the estimate the copula parameters. Another subquestion that needs to be answered is how the different copula methods can be implemented to estimate the VaR. Finally there needs to be a procedure to asses the VaR estimates and based on that the main question can be answered.

In order to answer the main question and subquestions the methodology will be partially inspired by the paper *Inhomogeneous Dependence Modeling with Time-Varying Copulae* by Giacomini, Härdle, & Spokoiny (2009). The data that is used differs from this paper since a slightly larger data set will be used that includes the global financial crisis of 2007-2009. The data exists of daily closing prices from January 1, 2005, until December 31, 2010. These prices correspond to twelve German companies listed on the DAX.

With the obtained data four different copula estimation approaches are implemented. These four methods are Riskmetrics, moving window, Local change point (LCP) and regime switching copulas (RSC). With the four procedures the dependence parameters can be estimated and the VaR can be determined. This combination of methods is used since two aspects of the time-varying copulas can be assessed. Firstly it can be determined if the Gaussian copula, Clayton copula or a mixture between these two is optimal to describe dependence between stock returns. Secondly the influence of the estimation window size on the VaR performance can be evaluated.

The main findings are that, from a risk management perspective, the RSC method achieves the best VaR estimates. Although the absolute exceedance errors of the RSC method are not the smallest, it has the best performance since the RSC method overestimates the VaR at all the significance levels. Overestimation of the VaR is preferred from a risk management perspective, since the maximum loss will be less than predetermined. In addition to that the overestimation from the RSC method is not as large as overestimations from other methods, such as the LCP method.

The contribution of this research to the current literature is that the influence of different copula families and estimation window sizes are both assessed. From Giacomini et al. (2009) we know that a dynamic moving window provide VaR measures that overestimate risk. In addition Shamiri et al. (2011) show that mixed copulas perform better than a single copula family. On accounts of the results of this research it seems that dynamic copula families are more suitable to achieve better VaR estimates than dynamic window sizes.

This paper will first provide a literature review in Section 2, which gives some background information and elaborates on existing literature about copulas and the VaR. Then, the used data is presented Section 3. This is followed by the theoretical definition of copulas, the four different copula estimation methods and the application of copulas to VaR estimation in Section 4. Thereafter, in Section 5, we present the results of a simulation study and the empirical VaR estimates for the different methods and compare them. Finally, in Section 6, the conclusion is drawn from our results about the performance of the VaR estimates. Section 6 also indicates possible improvements and proposes suggestions for further research.

2 Literature Review

Recessions and expansions seem to affect almost all markets. An observed characteristic of markets is that it appears that moments of large declines in one market affect other markets. As shown by Kristin et al. (2002), when one market declines sharply this fall is quickly observed in other markets. Their research concluded that there is a high level of market co-movement, in either stable periods or periods of financial distress.

On accounts of market co-movement dependence between stock returns is a field that has been studied frequently. Barkoulas & Baum (1996) studied the linear long- and short term dependence between stock prices. Their result shows no evidence of long term dependence, but does show some cases of short term dependence. Since these results are not convincing Hiremath (2013) looks at non-linear dependence between stock returns. He finds results that suggest strong evidence of a non-linear dependence structure in stock returns. However the non-linear dependence is not consistent throughout the sample period but confined to a few brief periods. These periods are majorly associated with events like uncertainties in international oil prices, political uncertainties and volatile exchange markets.

At the moment still most of the work dealing with dependence relies on Pearson's correlation.

As described by Marti et al. (2016) there are some deficiencies when using Pearson's correlation. For example Pearson's correlation is not robust to outliers and is not able to capture non-linear relationships. Therefore this measure of correlation is a substandard measure of dependence if the series is not jointly Gaussian. Embrechts et al. (2002) and Rémillard et al. (2012) showed that copulas are superior in modelling dependence, since copulas offer more flexibility than the correlation approach.

The copula associated with the Pearson's correlation is the Gaussian copula. The Gaussian copula does not allow for tail dependence as mentioned by Naifar (2011). Consequently the Gaussian copula is not able to capture dependence between extreme events, which is shown by Hartmann et al. (2004). Piccoli et al. (2017) show that stocks react to extreme events. They find that stocks tend to overreact after extreme events, which holds for global and domestic shocks. Furthermore Officer (1972) concludes that stock returns exhibit fat tails compared to the normal distribution. This implies that the Gaussian copula is not always the suitable copula to describe dependence between stock returns. There exist different families of copula, which raises the question which copula is the most appropriate.

In addition to the Gaussian copula there is the student-t copula. This copula implicitly represents the dependence structure of a multivariate t-distribution. Breymann & Embrechts (2003) have shown that the fit of the t-copula is overall better than the Gaussian copula. One cause of this is that the t-copula is able to capture dependent extreme values. Another family of copulas that is often used in finance are the Archimedean copulas. The Archimedean copulas are able to capture lower and upper tail dependence.

The Clayton copula, Gumbel copula and Frank copula are Archimedean copula. McNeil & Neslehov (2009) present some characteristics of the Archimedean copulas. The Clayton copula is able to capture lower tail dependence. Due to the restriction on the dependence parameter, the Clayton copula is unable to account for negative dependence. The Gumbel copula is able to capture upper tail dependence. Similar to the Clayton copula, the Gumbel copula represents only the case of independence and positive dependence. In contrary to the Clayton and the Gumbel copula, the Frank copula represents the maximum range of dependence. Mixing the upper tail dependence and lower tail dependence by using a mixed Clayton-Gumbel copula approach gives promising results in the research by Shamiri et al. (2011). However Manner & Reznikova (2012) show that in the case of time-varying copula approaches Gaussian and Frank copulas fit the data very well. This indicates that there is no real consensus which copula family is the most suitable to model dependence between stock returns.

Since dependence over time is important in risk management copulas can be applied to the VaR estimation. The profits and losses over time determine the VaR. The VaR represents the maximum loss at a certain significance level that will be achieved for a predetermined time period. As there are several copula estimation methods there needs to be decided which method is the most suitable to capture time-varying dependence.

Since dependence changes over time a constant copula approach is not realistic. Manner & Reznikova (2012) studied eight different time-varying copula approaches. Their research includes the local change point (LCP) method as described by Giacomini et al. (2009) which is the research that will be partially replicated in this paper. The LCP method uses dynamic moving windows to estimate the Clayton copula. This procedure appears to overestimate the VaR. Therefore the final recommendation of Manner & Reznikova (2012) is to use the regime switching copula (RSC) method. This method shows good performance in the simulations and it does not require heavy computations. Markov regime switching models are often used because of the implication that they can switch when the underlying state of the economy changes.

Markwat et al. (2012) implement a time-varying mixture copula according to a Markov regime switching model. To accommodate a variety of different dependence structures they select the Gaussian copula and the survival Gumbel copula to constitute the mixture copula. The Gaussian copula is chosen to represent tranquil periods with symmetric dependence. The periods of turmoil with asymmetric dependence and lower tail-dependence are represented by the survival Gumbel copula. Since their research is bivariate the survival Gumbel copula is easy to implement. However in a multivariate situation the Clayton copula is more suitable. The Clayton copula is a appropriate replacement as it captures lower tail dependence. On top of that using a Gaussian-Clayton mixed copula will enable a comparison with the LCP Clayton copula estimates as described by Giacomini et al. (2009).

3 Data

The data which is used to perform a backtesting study of the different VaR copula methods consists of stocks listed on the DAX. The data set is obtained from Yahoo Finance (2020) and is divided into two groups. Stocks from the first group belong to three different industries: automotive (Volkswagen and Daimler AG), insurance (Allianz and Münchener Rück), and chemical (Bayer and BASF). The second group consists of stocks from six industries: electrical (Siemens), energy (E.ON), metallurgical (ThyssenKrupp), airlines (Lufthansa), pharmaceutical (Merck KGAA), and chemical (Henkel). The choice for these particular stocks is derived from Giacomini et al. (2009). In Giacomini et al. (2009) Schering AG is used as the pharmaceutical company but this one was bought by Bayer AG in 2006. On accounts of that Schering AG company is replaced by Merck KGAA, since it is one of the biggest pharmaceutical companies in Germany.

The data includes the full closing price history for the period January 1, 2005, until December 31, 2010 for every stock. Therefore the data includes stable periods and the global financial crisis of 2007-2009. The major stock markets in Germany are open for trading from Monday to Friday, except on holidays. As a consequence the data set exists of $T = 1531$ daily closing prices.

Table 3.1 and 3.2 display some descriptive statistics of the daily logarithmic stock returns for both groups. These statistics are the annualised return, annualised Sharpe ratio, skewness and kurtosis.

The annualised Sharpe ratio represents the risk adjusted return. A high annualised return and Sharpe ratio indicates that a stock performs well. Performing well means that the stock generates maximum returns while at the same time having minimal risk. Based on these two performance measures it can be concluded that both groups contain stronger and weaker stocks.

The kurtosis and skewness indicate if the distribution of the stocks are normally distributed. As remarked in section 2, asset returns are rarely normally distributed. All the stock show excess kurtosis, which implies fat tails and a non-Gaussian distribution.

	Allianz	BASF	Bayer	Daimler AG	Münchener Rück	Volkswagen
Annualized return	-7.97	9.09	9.01	-1.63	-0.20	17.92
Skewness	0.84	0.09	1.05	0.43	0.35	-0.68
Kurtosis	17.36	13.77	22.67	11.51	12.59	13.28
Annualized Sharpe ratio	-8.35	11.29	11.44	-1.65	-0.28	16.25

Table 3.1: Statistical properties of the daily stock returns for group 1

	Siemens	E.ON	ThyssenKrupp	Lufthansa	Merck KGAA	Henkel
Annualized return	0.44	-4.16	2.65	2.62	-0.77	9.47
Skewness	-0.10	0.38	0.06	-0.19	-0.59	0.39
Kurtosis	17.00	15.66	8.58	5.51	11.31	8.23
Annualized Sharpe ratio	0.49	-5.27	2.58	3.33	-1.04	14.00

Table 3.2: Statistical properties of the daily stock returns for group 2

4 Methodology

To determine the dependence structure between different time-series copulas are used. In this paper we implement four different copula methods. These methods are Riskmetrics, moving window, local change point and regime switching copulas. The first method uses a Gaussian copula and the second and the third method use the Clayton copula. The choices for these copula are based on Giacomini et al. (2009). The fourth method uses a mixed Gaussian-Clayton copula. The mixed copula is based on a Markov switching model that switches between the two different copulas.

With these four methods a backtesting study of the portfolio Value-at-Risk (VaR) is performed. The stock returns are modelled similarly for all the methods. Each VaR has its own estimation technique. All the methodology is implemented with MATLAB (2020).

4.1 Copula

Copulas are linking mechanisms that merge the marginal distributions into joint distributions. Therefore copulas are a way of measuring dependence between different variables. A copula is a d-dimensional distribution function $C : [0,1]^n \to [0,1]$ and it has uniform marginals on the interval [0,1]. The multivariate distributions can be modelled according to Sklar's theorem.

Theorem 4.1 (Sklar's theorem) *If F is a d-dimensional distribution function with marginal cdf F_{X_1}, ...,F_{X_d} . Then there exists a copula C $\forall$ $x_1, \ldots, x_d \in \mathbb{R}^d$ satisfying*

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \tag{1}$$

C is unique if F_{X_1}, ...,F_{X_d} are continuous. If C is a copula and F_{X_1}, ...,F_{X_d} are distribution functions, then the function $F(x_1, \ldots, x_d)$ is a joint distribution function with marginal cdf F_{X_1}, ...,F_{X_d}.

For estimating the copula parameter (θ) maximum likelihood estimation is used. Consider a sample $X_t = (x_{t,1}, \ldots, x_{t,d})$ with corresponding copula $C = \{C_\theta, \theta \in \Theta\}$. The canonical maximum likelihood estimator $\hat{\theta}$ maximises the pseudo log likelihood. The pseudo log likelihood is given in equation 2.

$$\tilde{L}(\theta) = \sum_{t=1}^{T} log(c\{\hat{F}_1(x_{t,1}), \ldots, \hat{F}_d(x_{t,d}) : \hat{\theta}\}) \tag{2}$$

Where $\hat{F}_j(s) = \frac{1}{T+1} \sum_{t=1}^{T} 1_{\{x_{t,j} \leq s\}}$, which is the empirical cdf. The denominator $T+1$ avoids infinite values which the copula density may take on the boundary of the unit cube. A global copula can be estimated with the full sample period. This means that a constant copula is assumed over time. However a time-varying copula is more realistic and one way to achieve this is by using a moving window to estimate the dependence parameter. The moving window is set to $T = 250$ days as this equals one investment year.

The copulas that will be used in this paper are the Clayton copula and the Gaussian copula. Mathematical derivations, such as the multidimensional log likelihoods, can be found in section A.1 in the Appendix.

4.2 Local change point copulas

Estimating time-varying copulas with the moving window procedure has some limitations. The choice of a small window results in unstable estimates with large variability. The choice of a large window leads to poor sensitivity of the estimation procedure. Giacomini et al. (2009) implement a local change point (LCP) procedure to solve these limitations. In the LCP procedure a time interval of homogeneity is chosen. This interval is defined as $I_k = [t_0 - m_k, t_0]$, where t_0 is a point in time of the sample period and m_k defines the length of the interval. The idea is to choose an interval that is as large as possible, as long as this interval is still homogeneous. To implement this procedure there needs to be an out-of-sample period to be able to determine the window for the values of the first t_0. This out-of-sample period will be set to $T = 250$ days to keep it consistent with the moving window approach.

The intervals I_k will be screened by $\mathbb{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ to detect if there is a change point in the intervals. Figure 4.1 gives a visual interpretation of this process.

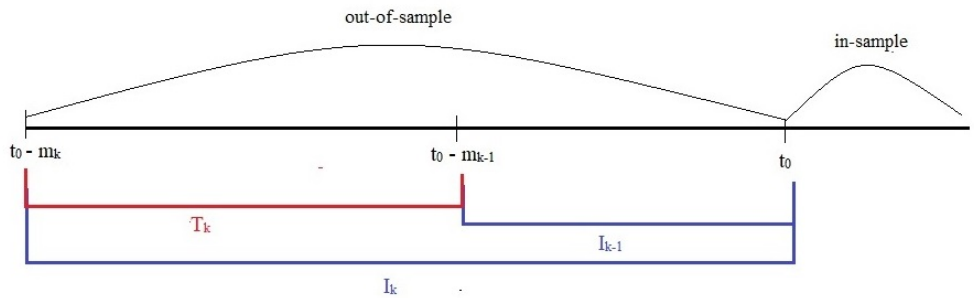


Figure 4.1: Local change point procedure and the choice of interval I_k and $\mathbb{T}_k$

Every $\tau \in \mathbb{T}$ represents a breakpoint and creates two intervals within an interval I. The two new intervals are J and J_c. The likelihood of these two intervals are determined and compared to the full interval by means of a likelihood ratio test as described by Mercurio, Danilo, & Spokoiny (2004). The likelihood ratio test for a single change point τ is given by the following expression: $T_{I,\tau} = L_J(\tilde{\theta}_J) + L_{J_c}(\tilde{\theta}_{J_c}) - L_I(\tilde{\theta}_I)$. Where $L(\tilde{\theta})$ is the likelihood of the corresponding interval. The test statistic is defined as $T_I = \max_{\tau \in \mathbb{T}} T_{I,\tau}$. The null hypothesis is that there is no change point within the interval and the alternative hypothesis that this change point does occur. A visual representation of the likelihood ratio test is given in the Appendix in Figure A.1.

To perform the likelihood ratio tests critical values are required. The critical values $\tilde{\delta}$ are parametrically simulated. The incentive is to avoid that intervals are accepted that are too short, since this causes a larger variability for the estimated copula parameter θ_t. Therefore we want to set the critical values in such a way to minimise the error of choosing an interval that is too short. This is done by means of the following expression as mentioned by Giacomini et al. (2009).

$$E_{\theta^*}|L_{I_k}(\tilde{\theta}_k, \hat{\theta}_k)|^{\frac{1}{2}} \leq \rho * \max_{k \geq 1} E_{\theta^*}|L_{I_k}(\tilde{\theta}_k, \theta^*)|^{\frac{1}{2}} \tag{3}$$

Where θ^* corresponds to real copula value of the full interval, so this is a constant value. $\tilde{\theta}_k$ corresponds to the copula value of the LCP intervals, which means that there are $k = 10$ values. $\hat{\theta}_k$ corresponds to the estimated values of the LCP intervals. This can be equal to the real interval, $\tilde{\theta}_k$, or correspond to a smaller interval. By adhering to equation 3 the chance of an incorrectly estimated interval, which is shorter than the real interval, is minimal. Furthermore $L_{I_k}(\tilde{\theta}_k, \hat{\theta}_k) = L(\tilde{\theta}_k) - L(\hat{\theta}_k)$ and ρ is a predetermined constant value. The critical values are obtained sequentially by means of a grid search.

With the obtained critical values the likelihood ratio test can be performed. Until the test rejects the null hypothesis of homogeneity the interval is accepted. In the accepted intervals a constant copula parameter is assumed. This means that with the accepted intervals the copula parameters can be estimated. For every copula estimation at time t a different amount of observation will constitute the estimation window based on the size of the accepted interval.

To perform the LCP procedure the interval size $I_k = [t_0 - m_k, t_0]$ and $\mathbb{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ need to be determined. The value $m_0 = 20$ is fixed and $m = [m_0 c^k]$ with $c = 1.25$ and $k = 1, \ldots, 10$. These values are derived from Giacomini et al. (2009). This means that the minimum window size equals $T = 20$ days and the maximum window size equals $T = 186$ days. This is important to remark since this means the dynamic windows are always smaller than the constant moving window of $T = 250$ days.

4.3 Regime switching copulas

The value of the dependence parameter depends on the chosen copula. Markwat et al. (2012) suggest that Gaussian copulas are suitable in stable times and Archimedean copula in cases with more extreme returns. Therefore a Markov Switching model that allows switching between the Gaussian copula and the Clayton copula could be able to more accurately capture the joint dependence over a longer period of time. In addition Manner & Reznikova (2012) have shown that regime switching copula perform well compared to other time-varying copula approaches.

The empirical cdf $(\hat{F}_1(x_{t,1}), \ldots, \hat{F}_d(x_{t,d})$ has a certain joint distribution. The joint distribution is captured by the copula distribution $c(\hat{F}_1(x_{t,1}), \ldots, \hat{F}_d(x_{t,d}); \theta)$. The regimes S_t are given by equation 4 and follow from a first order Markov process.

$$u_t \sim \begin{cases} c_{Clay}(u_t; \theta_0) & \text{if } S_t = 0 \\ c_{Gaus}(u_t; \theta_1) & \text{if } S_t = 1 \end{cases} \tag{4}$$

Where the empirical cdf is given by u_t. This means that when the process is in regime $S_t = 0$ the dependence parameter will be estimated by the Clayton copula. When the process is in regime $S_t = 1$ the dependence parameter will be estimated by the Gaussian copula. The probability to switch to the other regime is called the transition probability. The transition probabilities are given by the matrix $\mathbf{P}$.

$$\mathbf{P} = \begin{bmatrix} \mathbb{P}(s_t = 0|s_{t-1} = 0) & \mathbb{P}(s_t = 0|s_{t-1} = 1) \\ \mathbb{P}(s_t = 1|s_{t-1} = 0) & \mathbb{P}(s_t = 1|s_{t-1} = 1) \end{bmatrix} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

We do not observe the complete process S_t, which results in not knowing the initial state S_1. To solve this the parameter ζ is introduced, which is defined by Kole (2019). ζ indicates the probability that the process S_t is in state 0 at time $t = 1$: $\zeta = \mathbb{P}[S_1 = 0]$.

To make an educated guess about the regime S_t at time t the information from the past and current observations are combined with the distributions and the transition probabilities. This allows for an an inference on $\mathbb{P}[S_t = 0|u_t, u_{t-1}, \ldots, u_1]$, which is the probability that the process is in state 0 at time t, given all the information up to and including t.

The vector of inferences at time t is given by:

$$\xi_{t|t} = \begin{bmatrix} \mathbb{P}(S_t = 0|u_t, u_{t-1}, \ldots, u_1) \\ \mathbb{P}(S_t = 1|u_t, u_{t-1}, \ldots, u_1) \end{bmatrix}$$

The following recursion constructs the series of inference and forecast probabilities.

$$\xi_{t|t} = \frac{1}{\xi'_{t|t-1} c_t} \xi_{t|t-1} \odot c_t \qquad\qquad \xi_{t|t+1} = \mathbf{P}\xi_{t|t}$$

Where c denotes the copula pdf. This recursion is known as the Hamilton filter, and was first proposed by Hamilton (1996). The recursion is initialised with $\xi_{1|0} = (\zeta, 1-\zeta)'$.

The Hamilton filter only uses the information up to and including time t. By using all the the information in the complete data set the smoothed inferences can be computed.

$$\xi_{t|T} = \begin{bmatrix} \mathbb{P}(S_t = 0|u_T, u_{T-1}, \ldots, u_1) \\ \mathbb{P}(S_t = 1|u_T, u_{T-1}, \ldots, u_1) \end{bmatrix}.$$

Kim (1994) has shown that these inferences can be computed by the following recursion.

$$\xi_{t|T} = \xi_{t|t} \odot (\mathbf{P}'(\xi_{t+1|T} \div \xi_{t+1|t})) \tag{5}$$

The recursion in Equation 5 is known as the Kim-smoother. The Kim smoother can be initialised by $\xi_{T|T}$, which can be computed by using the Hamilton filter.

The parameters of the RSC model can be estimated by using maximum likelihood. The parameters are $\lambda \equiv (\theta_0, \theta_1)$, $\gamma \equiv (p_{00}, p_{11}, \zeta)$ and $\Delta \equiv (\lambda', \gamma')$. As shown by Kole (2019), the maximum likelihood estimates can be obtained by using the Expectation-Maximization (EM) algorithm. The EM algorithm is based on the notion that we maximise the log likelihood function of the joint observations $(\hat{U}_T, S_T)$, where the set $\hat{U}_T = \{u_1, u_2, \ldots, u_T\}$ contains the empirical cdf's. The set $S_T = \{s_1, s_2, \ldots s_T\}$ contains the realizations of the unobserved process S_t for each time t. However, as mentioned before, we do not actually observe the state the process is in at a time t. Therefore, in order to maximise the log likelihood function of the joint observations $(\hat{U}_T, S_T)$, we need to have the estimates of the values s_t.

On accounts of that, the EM algorithm consists of two steps. First, we compute the expected value of the log likelihood function of the joint observations $(\hat{U}_T, S_T)$. Then, the expected log likelihood function is maximised to obtain updated parameter estimates for θ. These two steps are repeated until the log likelihood converges. The following equation describes the EM algorithm .

$$\Delta^{(k)} = \arg\max_{\Delta} E\left[l_{\hat{U},S}(\hat{U}_T, S_T, \Delta)|\hat{U}_T, \Delta^{(k-1)}\right], \tag{6}$$

Where the log likelihood function is given by equation 7.

$$
\begin{aligned}
l_{\hat{U},S}(\hat{\mathcal{U}}_T, \mathcal{S}_T; \Delta) = &\sum_{t=1}^{T}((1-s_t)logf(\hat{\theta}_t; k-1, k\theta_0) + s_t logf(\hat{\theta}_t; k-1, k\theta_1)) \\
&+ \sum_{t=2}^{T}((1-s_t)(1-s_{t-1})logp_{00} + (1-s_t)s_{t-1}log(1-p_{11}) \\
&+ s_t(1-s_{t-1})log(1-p_{00}) + s_t s_{t-1} logp_{11}) \\
&+ (1-s_1)log\zeta + s_1 log(1-\zeta).
\end{aligned} \tag{7}
$$

The full derivation of the maximisation steps and parameter estimates is given by Kole (2019).

After constructing the regime switching model, the copulas can be estimated. In addition the probabilities that the next observation is in regime $S_t = 0$ or $S_t = 1$ can be obtained. These parameters are estimated with a moving window of $T = 250$ days. With this information the copula is set equal to the estimate of the two states with the highest probability. This results in a time-varying copula that switches between the Gaussian copula and the Clayton copula. The RSC method is implemented with MATLAB (2020), and is partially derived from the Regime Switching Copula (RSC) toolbox (2020).

4.4 Value-at-Risk

Value-at-Risk (VaR) is a broadly accepted measure of risk in finance. The VaR is the maximum loss that will not be exceeded over a period of time. VaR is a quantile of the distribution of the profit and losses function. At time t a portfolio composed of d positions in assets $w = (w_1, \ldots, w_d)'$ with corresponding prices $S_t = (S_{1,t}, S_{2,t}, \ldots, S_{d,t})'$ has the value $V_t = \sum_{j=1}^{d} w_j S_{t,j}$. The profit and losses function is defined as $L_t = (V_t - V_{t-1})$ and the logarithmic losses are defined by $X_{t,j} = \log(S_{t,j}) - \log(S_{t,j-1})$. This means that the profit and losses function can be rewritten as $L_t = \sum_{j=1}^{d} w_j S_{t-1,j}\{\exp(X_{t,j}) - 1\}$. The corresponding cdf of the profit and losses function is given by $F_{t,L_t}(x) = P_t(L_t \leq x)$. This results in the VaR at level α for a specific portfolio, $VaR_t(\alpha) = F^{-1}_{t,L_t}(\alpha)$.

Since the VaR is a quantile of the distribution of the profit and losses function this can be determined by means of a simulation. First of all the distribution of the profits and losses needs to be determined. To achieve this we ought to describe the marginal distribution by means of a model. It is often found that when fitting ARCH models to financial data a high order is required to get a satisfactory fit (Bollerslev (1986)). The GARCH model is an extension of the ARCH family of models and decreases the high order. For example Markwat et al. (2012) use an AR(1)-Threshold GARCH(1,1) model with t-distributed residuals for the marginal distribution of their regime switching copula model. More in accordance with Giacomini et al. (2009) this paper uses a GARCH specification where the log returns, $X_{t,j}$, are devolatized according to univariate riskmetrics volatility with $\lambda = 0.94$ and $\mu = 0$. The residuals for all models are assumed be normally distributed to allow a fair comparison between all the models. This gives the following GARCH(1,1) model, which is partially derived from Sun & Zhou (2014).

$$\begin{aligned} X_{t,j} &= \epsilon_{t,j}\sigma_{t,j} \\ \sigma^2_{t,j} &= \lambda\hat{\sigma}^2_{t-1,j} + (1-\lambda)X^2_{t-1,j} \\ \epsilon_{t,j} &\sim N(0,1) \end{aligned} \tag{8}$$

Where $X_{t,j}$ are the log returns, $\lambda = 0.94$, $\sigma^2_{t,j}$ are the variances of the log returns and $\epsilon_{t,j}$ represent the residuals. The residuals are independent and identically distributed with a standard normal distribution.

The distribution of risk factors increments can be modelled through copulas. To obtain the VaR in this setup we use a moving window approach. Because of the moving window approach the copula parameters are not necessarily constant over time. The size of the moving window is set to one investment year which equals $T = 250$ days.

For a portfolio $w \in \mathbb{R}^d$ and a sample $X_t = (X_{t,1}, \dots, X_{t,d})$ of log-returns, the estimation of the VaR is implemented in the following procedure as mentioned by Giacomini & Härdle (2005).

First model the log returns according to equation 8. Then estimate the residuals $\hat{\epsilon}_t$. With the obtained $\hat{\epsilon}_t$ estimate the empirical marginal cdf's, $F(\hat{\epsilon}_t)$. With the empirical cdf's that are $U \sim (0,1)$ distributed, $u_1, \dots, u_d$, specify the copula $C(u_1, u_2, \dots, u_d; \theta)$ and fit the copula C to obtain the copula parameter $(\hat{\theta})$. Generate a Monte Carlo sample of innovations and losses from $\hat{\theta}$. Finally estimate $V\hat{a}R(\alpha)$ which is the empirical α quantile of the simulated profits and losses, F_L.

The VaR estimates for the Gaussian Riskmetrics framework and Clayton moving window framework are obtained by this procedure with a constant moving window of $T = 250$ days. To recover the conditional Gaussian Riskmetrics framework we use the Gaussian copula with marginals that are described by equation 8. For the moving window framework we use a Clayton copula with marginals modelled according to equation 8.

The VaR for the LCP procedure is obtained in a similar way as the moving window Clayton framework. However, instead of using a window with the constant size of $T = 250$ days the window size changes according to the LCP procedure. This means that every observation at time t is assigned a different window size based on the LCP procedure. Consequently the simulated profits and losses are based on a copula that is estimated with a shorter window size.

The RSC VaR estimations are obtained with a constant moving window of $T = 250$ days. For the RSC VaR every value of the VaR is estimated based on the copula assigned according to the RSC. This means that the profits and losses are either simulated from a Gaussian or a Clayton copula for each VaR at time t.

4.5 Performance measures Value-at-Risk

For every VaR method 100 different portfolios are created. This means there is a set $W = \{w, w_n; n = 1, \ldots 100\}$ where each $w_n = (w_{n,1}, \ldots, w_{n,6})$. Each portfolio weight is $U \sim (0,1)$ distributed with the restriction that the six portfolio weights must add up to one. The equally weighted portfolio, w_*, receives weights $\frac{1}{6}$ for each asset and is therefore the most diversified portfolio. For each portfolio w_n the exceedance ratio $\hat{\alpha}_w$ and the exceedance error e_w can be computed by the following equations: $\hat{\alpha}_w(\alpha) = \frac{1}{T}\sum_{t=1}^{T} 1_{l_t \leq V\hat{a}R_t(\alpha)}$ and $e_w = \frac{\hat{\alpha}_w - \alpha}{\alpha}$. Where α is the predetermined significance level and l_t is the realisation of the corresponding profit and losses function . To compute the absolute exceedance error and the standard deviation all the 100 portfolios are used. The absolute exceedance error is computed by $A_w = \frac{1}{|W|}\sum_{w \in W} e_w$ with the corresponding standard deviation $D_w = \{\frac{1}{|W|}\sum_{w \in W}(e_w - A_w)^2\}^{\frac{1}{2}}$.

5 Results

To asses the performance of the LCP and the RSC procedure a simulation study will be performed. In this simulation study we can compare the estimated copula parameters with the real copula parameters. In addition to a simulation study the VaR of empirical data is determined. The VaR is determined for the Riskmetrics, moving window, LCP and RSC method. The performance measures described in section 4.5 are used to asses the different methods.

5.1 Simulation results

5.1.1 LCP

The simulation study assesses how fast the LCP procedure detects a change in the copula parameter. This is done for both sudden jumps and smooth transitions in copula parameters. Six-dimensional uniform data is generated from Clayton copula with a sudden jump in parameter and with a smooth change in parameter. These representations are given in equation 9.

$$\theta_t \sim \begin{cases} \theta_a & \text{if } -210 < t < 10 \\ \theta_b & \text{if } 10 < t \leq 100 \end{cases} \qquad \theta_t \sim \begin{cases} \theta_a & \text{if } -210 < t < 10 \\ \theta_a + \frac{t-10}{100}(\theta_b - \theta_a) & \text{if } 10 < t < 110 \\ \theta_b & \text{if } 110 < t \leq 190 \end{cases} \tag{9}$$

Where $\theta_a = 0.1$ and $\theta_b = 1.0$ for the upward copula change and $\theta_a = 1.0$ and $\theta_b = 0.1$ for the downward copula change. The critical values are obtained as described in section 4.2 by equation 3. Since these simulations are parametric these critical values apply to any data set. Table 5.1 shows the critical values from Giacomini et al. (2009) and Table 5.2 shows the newly simulated critical values. When the critical values are simulated again the results are quite different from the critical values obtained by Giacomini et al. (2009). The simulated critical values in Table 5.2 are higher than in Table 5.1 which means that the chosen intervals could be too large. To avoid this type 1 error the critical values of Table 5.1 are used.

k	$\rho = 0.2$	$\rho = 0.5$	$\rho = 1$
1	3.69	**3.29**	2.84
2	3.43	**2.91**	2.35
3	3.32	**2.76**	2.21
4	3.04	**2.57**	1.80
5	2.92	**2.22**	1.53
6	2.92	**2.17**	1.19
7	2.64	**1.82**	0.56
8	2.33	**1.39**	0.00
9	2.03	**0.81**	0.00
10	0.82	**0.00**	0.00

Table 5.1: Critical values obtained from 5000 simulations with Clayton copula and $\theta^* = 1$ derived from Giacomini et al. (2009)

k	$\rho = 0.2$	$\rho = 0.5$	$\rho = 1$
1	4.41	4.06	3.73
2	4.06	3.66	3.16
3	3.64	3.27	2.88
4	3.33	2.88	2.46
5	2.98	2.52	2.03
6	2.61	2.26	1.77
7	2.44	1.97	1.35
8	2.17	1.73	0.52
9	1.84	1.36	0.00
10	1.72	1.22	0.00

Table 5.2: Critical values obtained from 5000 simulations with Clayton copula and $\theta^* = 1$ derived from equation 3.

For the simulation study the critical values of $\rho = 0.5$ are used, these values are shown in bold in Table 5.1. The LCP parameters are set to $m_0 = 20$, $c = 1.25$ and $k = 10$. With these values $m = [m_0 c^k]$ and the intervals $I_k = [t_0 - m_k, t_0]$, $\mathbb{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ can be determined. The results of the LCP procedure are based on 10 simulations instead of 100 simulations, which Giacomini et al. (2009) use. A lesser amount of simulations will still be able to asses the effectiveness of the LCP procedure and lightens the computational load. The results of the simulated LCP procedure with sudden jumps in copula parameter is shown in Figure 5.1.

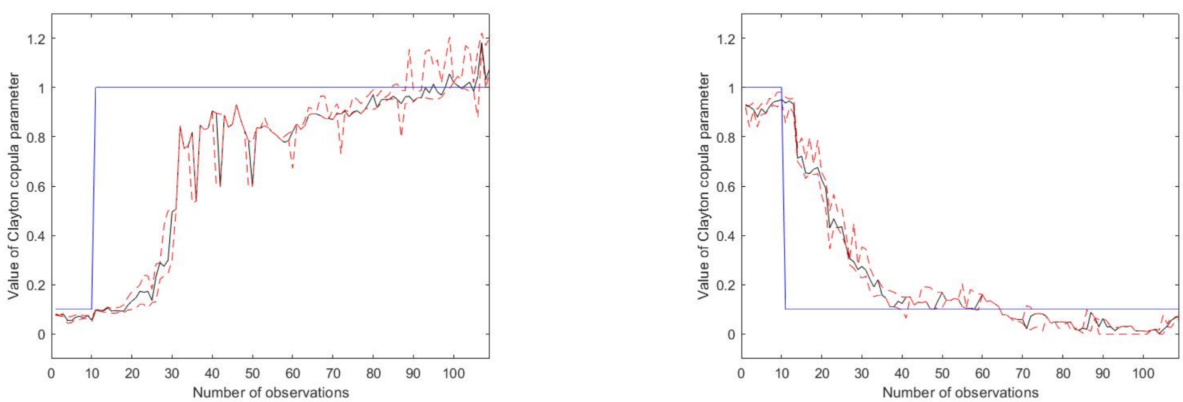


Figure 5.1: LCP and sudden jump in copula parameter. Pointwise median (black), 0.25 and 0.75 quantiles (red dashed) and true parameter θ (blue) with $\theta_a = 0.10$ and $\theta_b = 1.00$ (left), and $\theta_a = 1.00$ and $\theta_b = 0.10$ (right). Based on 10 simulations from Clayton copula, estimated with LCP, $m_0 = 20$, $c = 1.25$, and $\rho = 0.5$.

Figure 5.2 shows the results of the simulated LCP procedure with smooth transitions in copula parameter.

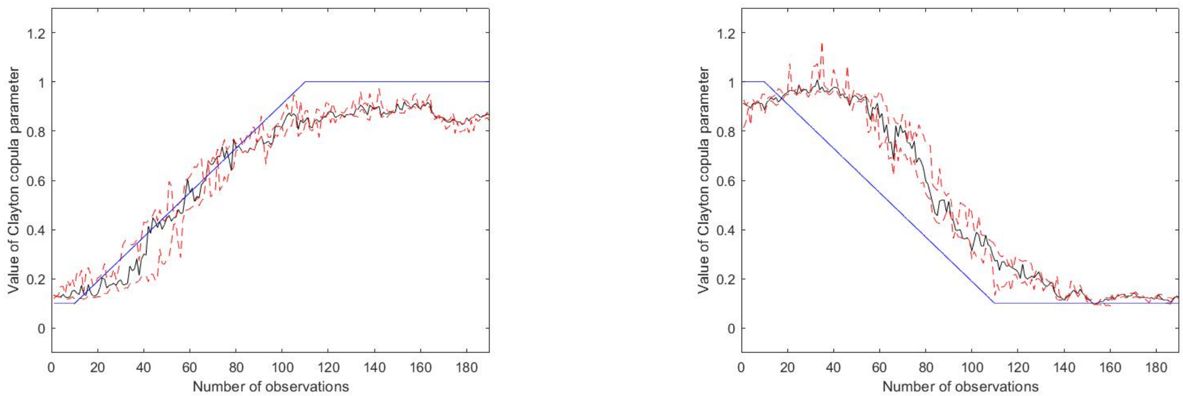


Figure 5.2: LCP and smooth transition in copula parameter. Pointwise median (black), 0.25 and 0.75 quantiles (red dashed) and true parameter θ (blue) with $\theta_a = 0.10$ and $\theta_b = 1.00$ (left), and $\theta_a = 1.00$ and $\theta_b = 0.10$ (right). Based on 10 simulations from Clayton copula, estimated with LCP, $m_0 = 20$, $c = 1.25$, and $\rho = 0.5$.

The detection delay represents the minimum number of steps that are needed to reach a certain fraction $r \in (0, 1)$ of the real copula parameter θ_t. The following equation computes the detection delay δ.

$$\delta(t, r) = min\{u \geq t : \hat{\theta_u} = \theta_{t-1} + r(\theta_t - \theta_{t-1})\} - t; \tag{10}$$

The detection delay statistics are given in Table 5.3.

(θ_a, θ_b)	r	mean	std. dev.	max	min
(0.1, 1)	0.25	14.8	3.55	20	10
	0.50	20.5	1.96	23	18
	0.75	28	3.50	36	26
(1, 0.1)	0.25	6	1.41	7	5
	0.50	14.5	1.52	16	13
	0.75	25	4.24	28	22

Table 5.3: The detection delays calculated according to equation 10, based on 10 simulations.

From the Figures and Table 5.3 we can derive that upward jumps have a higher mean detection delay than downward jumps. This is in accordance with the findings of Giacomini et al. (2009).

5.1.2 RSC

The RSC should switch between the Clayton and Gaussian copula at the appropriate moment. To asses the performance of this procedure the following copula parameters are simulated.

$$\theta_t \sim \begin{cases} \theta_{Clay} & \text{if } -250 < t \leq 100 \\ \theta_{Gauss} & \text{if } 100 < t \leq 200 \end{cases} \tag{11}$$

This means that the process not only switches between values but also between the two copula families. The values of the dependence parameters are set to $\theta_{Clay} = 1$ and $\theta_{Gauss} = 0.1$. There are two different simulation scenarios. In the first scenario there is a downward jump in copula parameter, so from Clayton to Gaussian. In the second scenario this is the other way around. The RSC procedure is performed with a moving window of $T = 250$ days and based on 10 simulations. The median of the estimated dependence parameters, the quantiles, and the real dependence parameters are given in Figure 5.3.

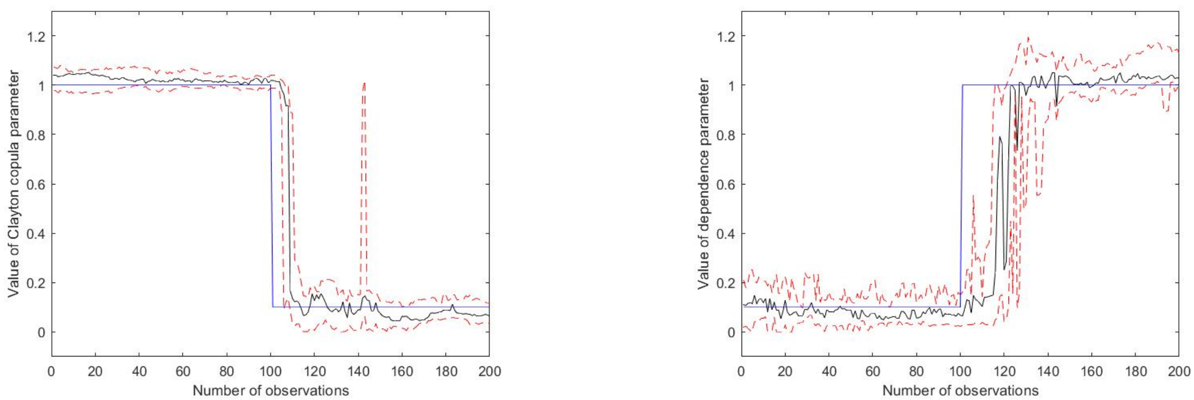


Figure 5.3: Pointwise median of regime switching copula (black), 0.25 and 0.75 quantiles (red dashed) and true parameter θ (blue) with $\theta_{Clay} = 1.00$ and $\theta_{Gauss} = 0.1$. Based on 10 simulations.

The percentage that the regime switches correctly to the Gaussian (Γ_{Gauss}) or Clayton regime (Γ_{Clay}) can be determined by equation 12.

$$\Gamma_{Gauss/Clay} = \frac{\hat{\gamma}}{\gamma} * 100\% \tag{12}$$

Where $\hat{\gamma}$ are the number of correctly estimated regimes and γ the total number of regimes. These statistics are given for the 10 simulations in Table 5.4.

		mean	std. dev.	min	max
downward	Γ_{Clay}	100%	0%	100%	100%
	Γ_{Gauss}	87.2%	6.97%	94%	75%
upward	Γ_{Gauss}	99.3 %	2.21%	87%	100%
	Γ_{Clay}	78.3 %	5.72%	69%	93%

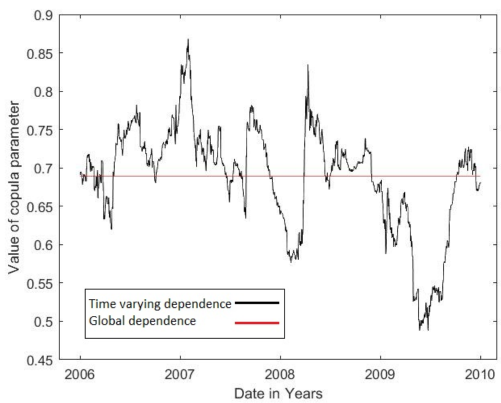
Table 5.4: Percentage of correct regime switches for the Gaussian and Clayton copula. Both for upward and downward jumps and based on 10 simulations.

The starting copula is better captured than the subsequent copula in both cases. This is likely caused by the feature of a moving window. The moving window contains previous values which can influence a delayed regime switch. Overall the RSC procedure switches quite well at the appropriate times and this is slightly better when the first copulas are Clayton copulas.

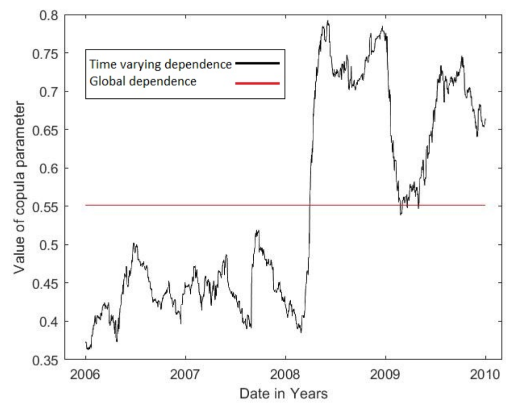
5.2 Empirical Results

For the empirical study the four VaR methods are applied to two groups of six stocks. The exact content of the two groups is described in section 3. For these two groups 100 different portfolios are composed to determine the VaR. How these portfolios are formed is described in section 4.5.

Figure 5.4 shows the estimated time-varying Clayton copula for both groups with a moving window of $T = 250$ and the constant global Clayton copula.



(a) group 1

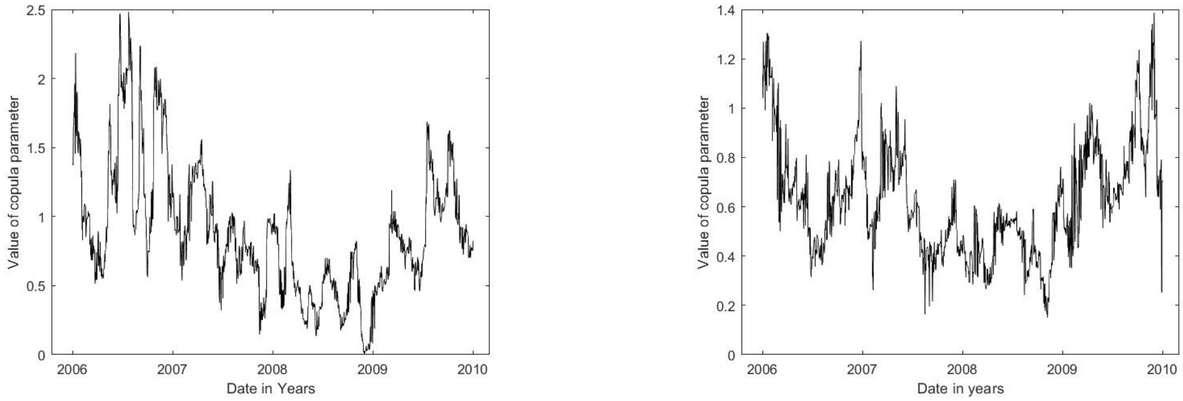


(b) group 2

Figure 5.4: Estimated dependence parameters with Clayton copula and a moving window of $T = 250$, for group 1 and group 2.

Something to remark are the time-varying copula estimates of group 2. The values of the estimates rise sharply in the beginning of 2008, which is during the global financial crisis. This indicates that the dependence between the six different markets in group 2 changed significantly based on the moving window estimates.

The LCP procedure is performed with $m_0 = 20$, $c = 1.25$, $\rho = 0.5$ and $\theta_* = 1$. The critical values for the LCP procedure can be found in Table 5.1. Figure 5.5 shows the estimated Clayton copula parameter for both groups obtained with the LCP procedure.

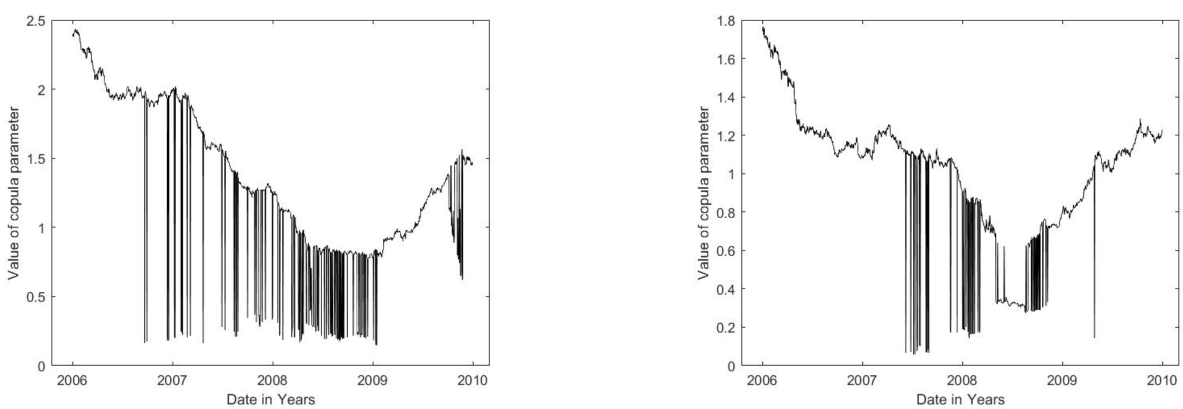


(a) group 1 (b) group 2

Figure 5.5: Estimated dependence parameters with Clayton copula and the LCP procedure and parameters $m_0 = 20$, $c = 1.25$, and $\rho = 0.5$ for group 1 and group 2.

We see that the LCP procedure causes a lot more variability in the estimated copula parameter for both groups of stocks. This is because the windows of the LCP procedure are always smaller than the moving window. This has a significant effect on the sensitivity of the estimates. Furthermore it can be seen that the relative changes between the copula estimates are different too. For example the steep rise that appears in Figure 5.4(b) is not present in Figure 5.5(b) at the same time. Instead we see a steep rise around the end of 2009 in Figure 5.5(b). All in all the LCP procedure results in significantly different estimates of the dependence parameter.

The estimated dependence parameters of the RSC procedure are given in Figure 5.6. The corresponding regime switches and probabilities are given in Figure A.2 and Figure A.3 in the Appendix section A.2.



(a) Estimated copula group 1 (b) Estimated copula group 2

Figure 5.6: Estimated copula parameters with the RSC procedure.

The regime switches cause steep declines in the estimated copula parameter, since the Gaussian copula parameter takes values $\theta \in [-1, 1]$ and the Clayton copula in $\theta \in [-1, \infty)\backslash\{0\}$.

It appears that the switches between the copula families especially occur during the height of the global financial crisis, which is from 2008 till 2009. In stable periods the Clayton copula is more frequently used to estimate the copula parameter. This result is unexpected since we would expect that the Gaussian copula is more prominent in stable times and the Clayton copula in financial crisis. On the other hand unstable financial times could cause quick changes in dependence between markets. Hence this could be a reason why the RSC method switches more frequently during times of financial distress.

The VaR for the Riskmetrics (Risk.) Gaussian copula, moving window (MW) Clayton copula , LCP Clayton copula and RSC mixed copula procedures for group 1 and group 2 are given in Table 5.5 and 5.6 respectively.

	Risk. Gaussian		MW Clayton		LCP Clayton		RSC mixed	
alpha	5	1	5	1	5	1	5	1
$\hat{\alpha_{w_*}}$	6.71	1.17	6.01	0.86	5.15	0.31	4.53	0.94
$\hat{\alpha_{w_1}}$	6.01	1.33	6.48	0.70	4.61	0.23	4.45	0.86
$\hat{\alpha_{w_2}}$	6.40	1.02	6.17	0.94	4.84	0.16	4.99	0.70
A_w	0.24	0.15	0.22	-0.32	-0.04	-0.75	-0.12	-0.22
D_w	0.15	0.23	0.14	0.20	0.15	0.16	0.11	0.18

Table 5.5: VaR performance measures for group 1 based on 100 simulations

	Risk. Gaussian		MW Clayton		LCP Clayton		RSC mixed	
alpha	5	1	5	1	5	1	5	1
$\hat{\alpha_{w_*}}$	5.31	1.02	5.70	0.39	4.22	0.23	4.45	0.55
$\hat{\alpha_{w_1}}$	6.01	0.86	5.47	0.70	4.06	0.08	4.60	0.78
$\hat{\alpha_{w_2}}$	5.86	1.17	4.61	0.78	4.53	0.31	4.14	0.55
A_w	0.13	0.15	0.20	-0.24	-0.16	-0.62	-0.15	-0.36
D_w	0.21	0.35	0.22	0.27	0.20	0.17	0.20	0.19

Table 5.6: VaR performance measures for group 2 based on 100 simulations

w_* represents the equally weighted portfolio and w_1 and w_2 are two portfolios with randomly assigned weights. The VaR plot from the equally weighted portfolio at $\alpha = 5$ for all methods for both groups can be found in section A.2 in the Appendix. In group 1 at a 5% significance level the LCP procedure performs the best since it has the lowest absolute exceedance error. However at a 1% significance level the LCP procedure greatly overestimates the VaR, and has the highest absolute exceedance error. The Riskmetrics method perform the best at a 1% significance level. Overall it can be concluded that not one method outperforms all other methods in group 1. From a risk management perspective it is better to overestimate the VaR than underestimate it, since this means that less risk is taken than preferred. When taking this into account it can be concluded that the RSC method would be preferred for group 1. The RSC method always overestimates the VaR but in contrary to the LCP method this overestimation is not too large. The Riskmetrics method and moving window method sometimes underestimate the VaR, which makes the RSC method superior in comparison.

The VaR results in group 2 show a similar pattern to the results from group 1. Although the Riskmetrics method again underestimates the VaR, the absolute exceedance errors are the lowest compared to the other methods for both significance levels. Based on that criterion the Riskmetrics generates the best VaR performance for group 2. If it is desirable to avoid underestimation of the VaR the RSC method would be the most appropriate for group 2. With the RSC method the VaR is overestimated but less than with the LCP method.

Everything considered the choice for the best copula method to determine the VaR depends on the context. If the aim is to prevent underestimation and still have the lowest possible absolute exceedance errors the RSC method would be optimal. If the main aim is to have the lowest absolute exceedance errors the Riskmetrics method could give better results in some cases.

6 Conclusion and Discussion

The objective of this research was to find an answer to the question:"Which time-varying copula approach provides the best Value-at-Risk estimates from a risk management perspective?" In order to answer this question four different time-varying copula models were implemented and with these four models the VaR was determined. The four time-varying copula methods are Riskmetrics, moving window, LCP and RSC.

The methods are constructed considering two different aspects. One aspect is to asses which copula family fits the data the best and if this changes over time. The two copula families that are used are the Gaussian copula and the Clayton copula. The second aspect involves the technique for estimating the time-varying copula. This is done by using constant moving windows versus an adaptive moving window.

From the results it follows that the RSC method outperforms the other methods from a risk management perspective. The RSC method does not underestimate the VaR in the two groups of data. This means that larger losses than determined by the VaR do not occur. Underestimation does occur in the Riskmetrics and moving window approach, which makes these methods less attractive for managing risk. Just as the RSC method, the LCP method overestimates the VaR in both groups. However the absolute exceedance errors for the LCP are larger than for the RSC which makes the RSC the superior method.

From a different perspective the RSC method is not the most optimal choice. When the only incentive is to obtain accurate VaR estimates the results are not conclusive considering the two groups. For group 2 the Riskmetrics would be the best approach, since it has the lowest absolute exceedance error overall. Group 1 does not have one clear winner when only the absolute exceedance errors are taken into consideration.

All in all this means that the RSC and the Riskmetrics method would be the best time-varying copula methods to estimate the VaR. The choice between these two methods depends on what is considered more important, accurate VaR measurements or optimal risk management. As the research question focuses on the risk management perspective, the answer to the research question would be that the RSC method is the best time-varying copula approach to estimate the VaR.

Since the RSC method shows promising results it could be interesting for further research to extend this method with more regimes and more copula families. This would enable a more specific copula fit overtime which could result in accurate VaR estimates. Also the RSC copula does not choose the Gaussian copula for stable times, but the Clayton copula. Since the simulation study suggests that the RSC method correctly estimates the appropriate copula we could conclude that a combination of the Clayton- and the Gaussian copula is not optimal to describe financial time-series data. Hence, different copula families should be considered that are more fitting to describe the underlying state of the economy. Another point of improvement is that this paper only considers the LCP procedure with the Clayton copula. To be able to make a fairer comparison the Gaussian copula could be implemented in the LCP procedure. A final point of discussion would be to define the marginals more realistically. This paper models individual log returns according to a GARCH(1,1) specification with residuals that follow a standard normal distribution. This model could be analysed in more detail to better fit the data. For example the residuals could be modelled according to a t-distribution instead of the standard normal distribution.

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A Appendix

A.1 Mathematical derivations

A.1.1 Pseudo-loglikelihood Clayton Copula

The density of a multivariate Clayton copula is given by

$$c = \{\prod_{j=1}^{d}(1+(j-1)\theta)\}\{\prod_{j=1}^{d} u_j^{-(\theta+1)}\}\{\sum_{j=1}^{d} u_j^{-\theta} - d + 1\}^{-(\theta^{-1}+d)}$$

which is obtained from Diks, Panchenko, & van Dijk (2010). The variable u_j is the empirical cdf and d represents the number of dimensions. The pseudo-loglikelihood follows from the pdf according to the following equation.

$$L(\hat{\theta}) = \sum_{t=1}^{T}\{\sum_{j=1}^{d}(log(1+(j-1)\hat{\theta}) - \sum_{j=1}^{d}(\hat{\theta}+1)log(u_j) - (\hat{\theta}^{-1}+d)log(\sum_{j=1}^{d} u_j^{-\hat{\theta}} - d + 1)\}$$

Where the maximum likelihood estimator $\hat{\theta}$ maximises $L(\hat{\theta})$.

A.1.2 Generating uniform variables from Clayton Copula

To obtain uniform distributed vector for a given copula the following algorithm can be applied.

- Draw $U' = (U_1,' \ldots, U_d') \sim U(0,1)^{d-1}$
- $U_1 = C^{-1}(U_1'|u_k)$
- $U_2 = C^{-1}(U_2'|U_1, u_k)$
- $\vdots$
- $U_J = C^{-1}(U_J'|U_1, U_2, \ldots, u_k)$

This algorithm gives the following expression for the Clayton copula.

$$C^{(-1)(\theta)}(u_j'|u_1, \ldots, u_{j-1}, u_k) = (1 + (1-(j-1) + \sum_{k=1}^{j-1} u_k^{-\theta})(u_j'^{-\frac{\theta}{1+(j-1)\theta}} - 1))^{-\frac{1}{\theta}}$$

A.1.3 Gaussian copula density

To avoid using the correlation matrix in the pdf of the Gaussian copula a serial autoregressive correlation structure can be used. For details of this derivation see Clemen & Reilly (1999). This gives the following expression for the Gaussian copula density.

$$\frac{1}{\sqrt{(1-\rho^2)^{d-1}}}\exp\{\frac{-\rho}{2(1-\rho^2} \times (2\rho\sum_{i=1}^{d} u_i - \rho(u_1^2+u_d^2) - 2\sum_{i=1}^{d-1} u_i u_{i+1})\}$$

A.2 Figures

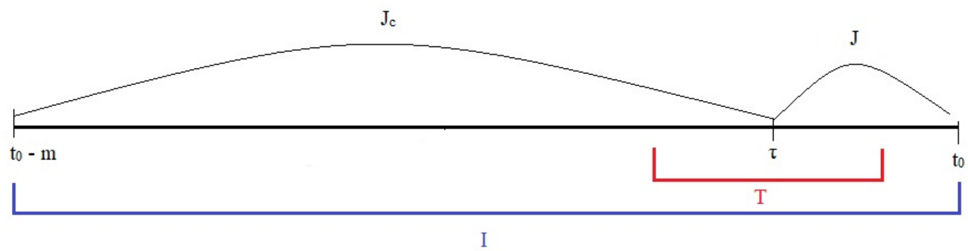
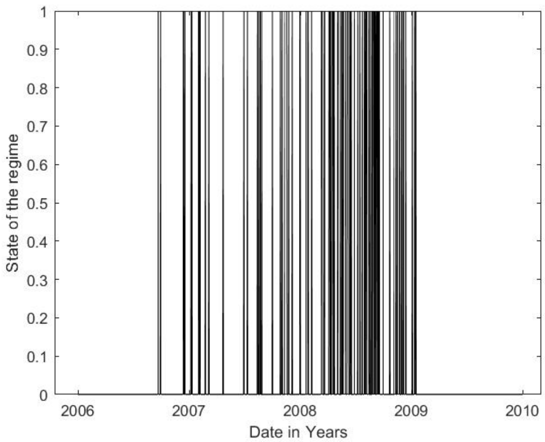
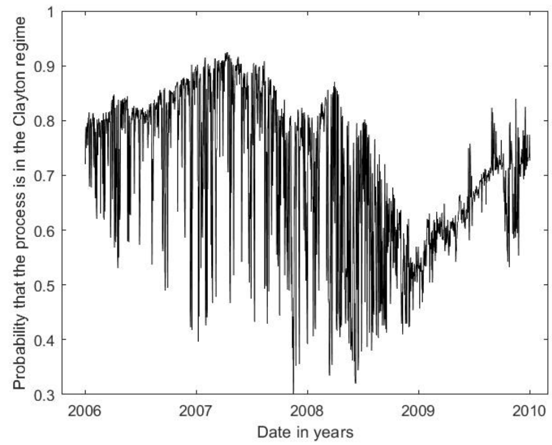


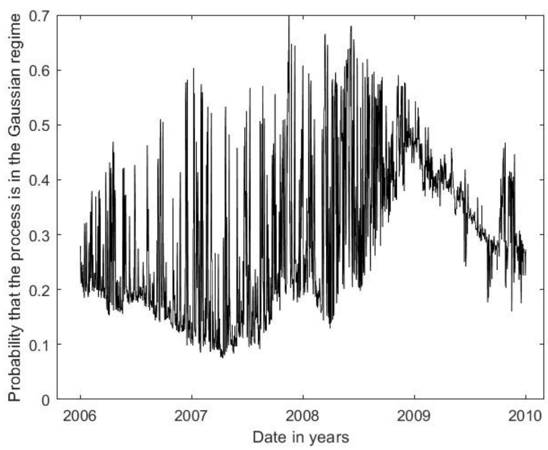
Figure A.1: Homogeneity test. I is the interval that is tested by means of interval $\mathbb{T}$. This creates intervals J and J_c for every breakpoint $\tau \in \mathbb{T}$.



(a) Regime switches

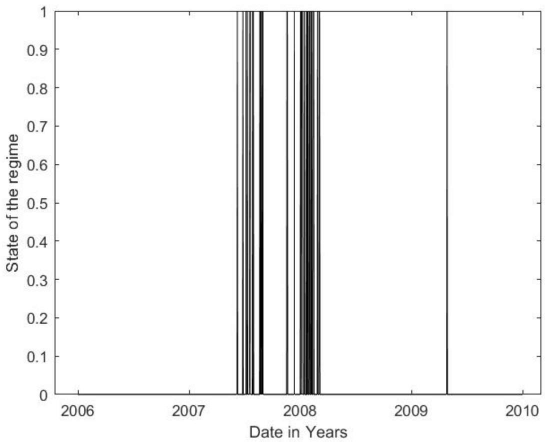


(b) Probability process is in $S_t = 0$

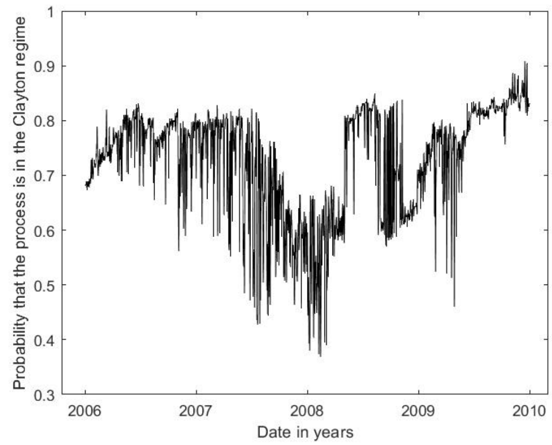


(c) Probability process is in $S_t = 1$

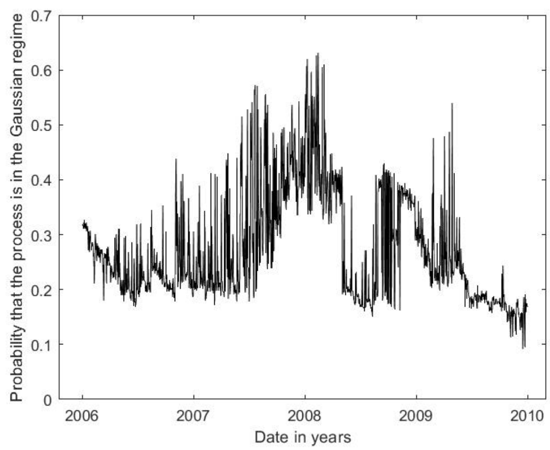
Figure A.2: Regimes for the estimated copula parameters and corresponding regime probability obtained from the RSC procedure for group 1. $S_t = 0$ refers to the Clayton copula and $S_t = 1$ to the Gaussian copula



(a) Regime switches

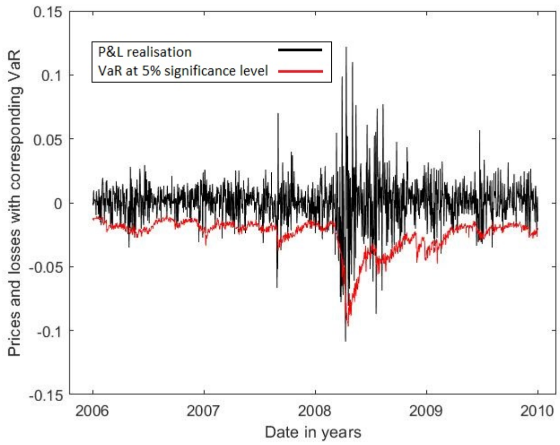


(b) Probability process is in $S_t = 0$

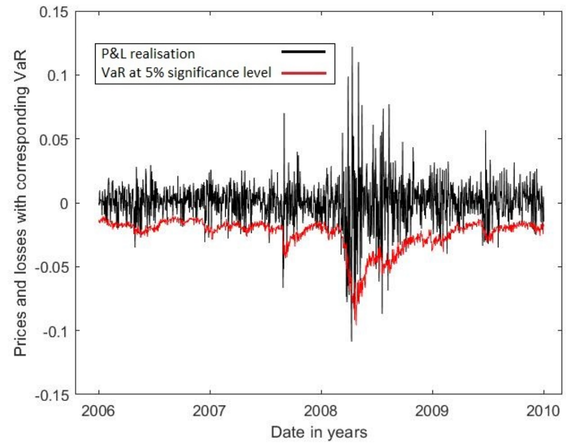


(c) Probability process is in $S_t = 1$

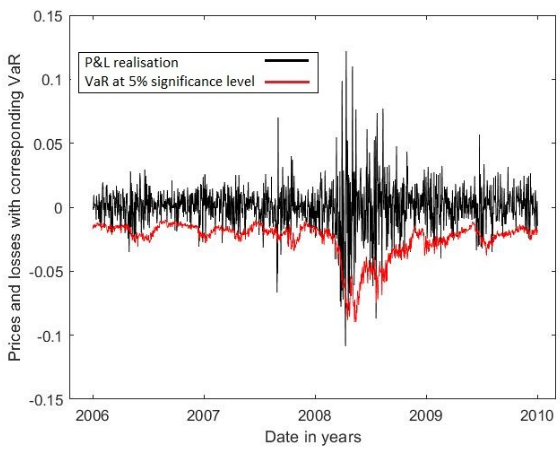
Figure A.3: Regimes for the estimated copula parameters and corresponding regime probability obtained from the RSC procedure for group 2. $S_t = 0$ refers to the Clayton copula and $S_t = 1$ to the Gaussian copula



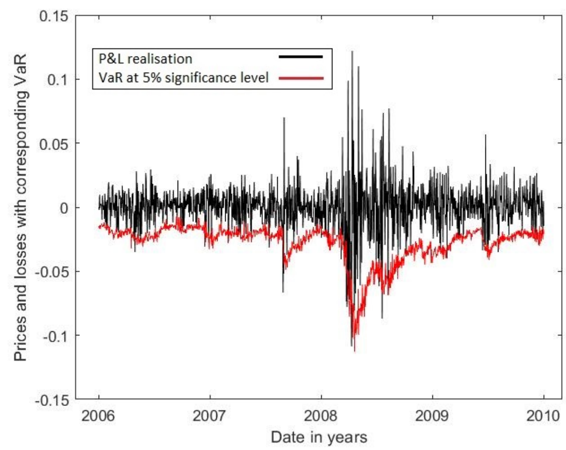
(a) Riskmetrics



(b) Moving Window

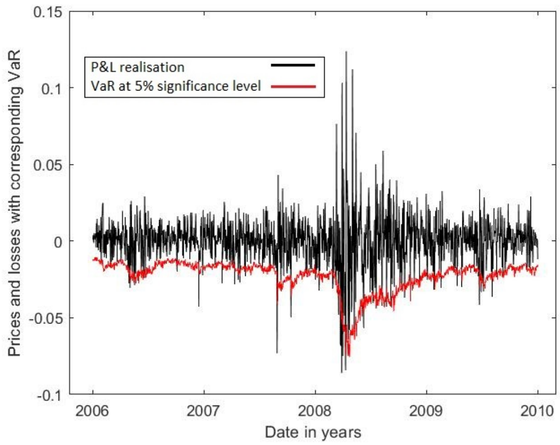


(c) LCP

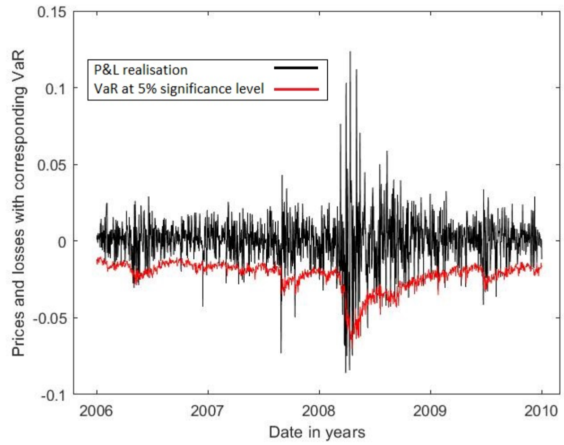


(d) RSC

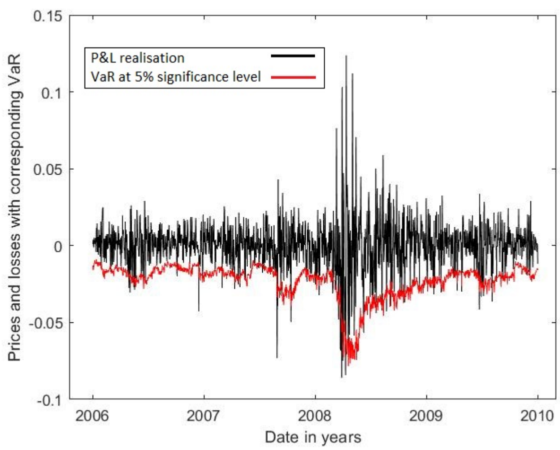
Figure A.4: Realisations of the profits and losses based on the equally weighted portfolio at a significance level of 5% for group 1.



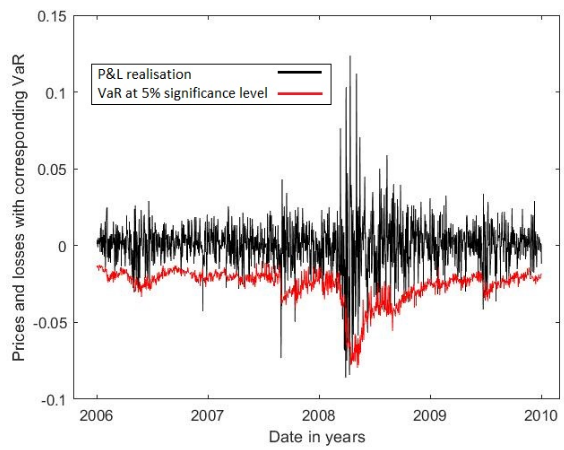
(a) Riskmetrics



(b) Moving Window



(c) LCP



(d) RSC

Figure A.5: Realisations of the profits and losses based on the equally weighted portfolio at a significance level of 5% for group 2.