Modeling for Exchange Rate Hedging

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Abstract

This paper focuses on constructing an optimal hedge ratio for exchange rate hedging by using future contracts. We use the returns of British and American stock markets and the returns of their respective domestic currencies to construct hedge ratios. We consider an extension of Gaussian copula theory by allowing its correlation parameter to vary over the time, thus enabling us to observe how the correlation parameter moves over the time. Patton's evolution equation for Gaussian copula and the Dynamic Conditional Correlation models are used in order to model the correlation, allowing it to vary over the time. In the end they will be compared as to which method produces hedge ratio with higher utility.

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I. Introduction

As an international investor, it is important to be able to overcome the exchange rate risk. Each time he makes an investment on a foreign stock market, the exchange rate and its risk play an important role. To cover this risk, the investor can hedge his risk. For example, the investor can take position in a future contract to hedge their exchange rate risk. A future contract is a standardized contract, to buy or sell a currency at a certain date in the future, at a market determined price which is called the future price. In this paper, we focus only on buying foreign currency using future contract to hedge the exchange rate risk.

Exchange rate hedging requires estimates of the correlation between the returns of the assets in the hedge, in our case the returns of the foreign stock market and exchange rate. The correlations between the returns of stock market and exchange rate are thus an essential part for the construction of an optimum hedge ratio. Should the correlations change, then the hedge ratio should be adjusted according to the most recent information.

The problem now lies in how to model the dependencies between the return of the foreign stock market and the return of exchange rate, since the correlation might be varying over the time. In this paper, we use copula as an attempt to model the correlation. A copula is used as a general way of formulating a multivariate distribution in such a way that various general types of dependence can be represented. We reconstruct the copula in order to model possible time variation in the correlation by using the evolution equations proposed by Patton(2006).

We also use the Dynamic Conditional Correlation (DCC) model to construct reliable estimates of correlations as an alternate method to model the correlation and its possible time variation. DCC estimators have the flexibility of univariate GARCH but not the complexity of conventional multivariate GARCH (Engle (2002)). These models, which parameterize the conditional correlations directly, are estimated in 2 steps, a series of univariate GARCH estimates and the correlation estimates. These methods have clear computational advantages over multivariate GARCH models in that the number of parameters to be estimated in the correlation process is independent of the number of series to be correlated. We can thus estimate potentially very large correlation matrices.

After the models have been constructed and their parameters estimated, we are going to evaluate the performance of the models, by constructing a hedge ratio strategy based on the models, and observe which model produces the hedge ratio with the highest utility value for an investor. Chapter 2 gives explanations of methods that are used, chapter 3 presents the results, and chapter 4 concludes.

II. Methods

Our data consists of 5000 observations (from 1 January 1990 until 25 February 2009). We have as observations the price of the stock at specific stock markets on a given date, the exchange rates, and weekly and monthly interests. In this paper we observe the price of the American and British stock market (S&P 500 and FTSE respectively), the exchange rate between dollar and pound, and American and British weekly and monthly interest rates. We use the first 4000 observations to estimate the models, and we use the last 1000 observations to construct the hedge ratios.

II.1 Copula

The method we use to model the correlation and its possible time variation is the copula. If F and G are the cumulative distribution function of respectively X and Y, the model for the copula is defined as:

$$\Pr[X \leq x, Y \leq y] = C(\Pr[X \leq x], \Pr[Y \leq y]) = C\big(F_X(x), F_Y(y)\big) = F_{XY}(x,y)$$

In terms of the bivariate density:

$$f_{xy}(x,y) = f_x(x) \cdot f_y(y) \cdot c(F_X(x), F_Y(y))$$

In this paper, we estimate the Gaussian (or the normal) copula to model the correlation, since we want to observe how the correlation varies over the time, and the Gaussian copula only has 1 parameter to be estimated, which is the correlation parameter. However, the Gaussian copula cannot capture asymmetric correlation, since the Gaussian copula is a symmetric copula. The Gaussian copula is the dependence function associated with bivariate normality and is given by:

$$C^{\phi}(u,v;\rho) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v);\rho)$$

with u and v the marginal probabilities, Φ_2 the distribution function for the bivariate standard normal distribution with correlation coefficient ρ, and Φ^{-1} is the inverse of the univariate standard normal distribution. The PDF of the Gaussian copula is given by:

$$c^{GA}(u,v;\rho) = \frac{1}{\sqrt{(1-\rho^2)}} \exp\left\{\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2) + \frac{1}{2}(x^2 + y^2)\right\}$$

with $x = \phi^{-1}(u)$ and $y = \phi^{-1}(v)$.

In order to use the copula, marginal distributions of the data must be estimated first. The estimated marginal distributions should explain the data as good as possible. We construct our marginal distributions using an AR(p), GARCH models. The problem now lies in how to model the time variance in the correlation parameter of the Gaussian copula. A standard Gaussian copula estimates the correlation over the whole dataset, not over every point in the time. This paper considers 2 methods to model the time variation in the correlation, the evolution equation for Gaussian copula proposed by Patton(2006), and the DCC model.

II.2 Time varying copula

Patton (2006) constructed marginal distributions using the AR(p), GARCH(1,1) model with student-t distributed error terms. We use this idea to construct the marginal distributions of our dataset. An AR(p), *t*-GARCH(1,1) model is specified as follow:

$$
\begin{aligned}
r_t &= \mu + \phi r_{t-p} + \varepsilon_t \\
\varepsilon_t &= \sqrt{h_t}\eta_t \\
h_t &= \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1} \\
\eta_t &\sim t_v
\end{aligned}
$$

with r_t is the returns, and h_t is the conditional volatility. We restrict our lagged variable $p \leq 5$ to ensure that the correlations that we observe are not older than 1 week. This restriction is made under the assumption that correlations between daily returns are not significant anymore after 1 week.

We use the innovation part of the GARCH model:

$$\eta_t = \frac{\varepsilon_t}{\sqrt{h_t}}$$

in order to construct the marginal probabilities. The marginal probabilities are calculated as follow: we calculate the distribution CDF value (standard normal or student-t) of the η_t series using corresponding degree of freedom value. Marginal probabilities are used as inputs for copulas.

Patton assumes that the functional form of the copula remains fixed over the sample whereas the parameters vary according to some evolution equation (Patton (2006)). Defining the forcing variable for the evolution equation is complicated, because it is difficult to specify how the parameters evolve over time. Unless the parameter has some interpretation, as the parameters of the Gaussian copula do, it is very difficult to know what might influence it to change.

Patton (2006) proposes the following evolution equation for ρ_t:

$$\rho_t = \widetilde{\Lambda}\left(w_p + \beta_p.\widetilde{\Lambda}^{-1}(\rho_{t-1}) + \alpha.\frac{1}{10}\sum_{j=1}^{10} \phi^{-1}(u_{t-j}).\phi^{-1}(v_{t-j}) \right)$$

where $\widetilde{\Lambda}(x) \equiv (1 - e^{-x})(1 + e^{-x})^{-1}$ is the modified logistic transformation, designed to keep ρ_t in (-1,1) at all times, and ϕ^{-1} is the inverse of the univariate standard normal distribution. The regressor ρ_{t-1} is included to capture any persistence in the dependence parameter, and the mean of the product of the last 10 observations of the transformed variables $\phi^{-1}(u_{t-j})$ and $\phi^{-1}(v_{t-j})$ to capture any variation in the dependence. Setting $w = \alpha = 0$, and setting $\beta = 1$, the equation then becomes $\rho_t = \rho_{t-1}$, thus allowing the correlation at time t to be fully explained by correlation at time *t-1* in case the correlation is static (does not vary over time).

II.3 DCC model

We also use the Dynamic Conditional Correlations (DCC) to model the correlation parameter of the Gaussian copula and its possible time variation. As mentioned above, DCC model has the advantage of having the flexibility of univariate GARCH but not the complexity of conventional multivariate GARCH. The DCC model requires the assumption of normality. The DCC model can be formulated as the following statistical specification (according to Engle (2002)):

$$
\begin{aligned}
& r_t|\zeta_{t-1} \sim N(0, D_t R_t D_t) \\
& D_t^2 = \text{diag}\{\omega_i\} + \text{diag}\{\kappa_i\} \circ r_{t-1}r_{t-1}' + \text{diag}\{\lambda_i\} \circ D_{t-1}^2 \\
& \varepsilon_t = D_t^{-1} r_t \\
& Q_t = S \cdot (1 - \alpha - \beta) + \alpha \cdot \varepsilon_{t-1}\varepsilon_{t-1}' + \beta \cdot Q_{t-1} \\
& R_t = \text{diag}\{Q_t\}^{-1/2}\, Q_t\, \text{diag}\{Q_t\}^{-1/2}
\end{aligned}
$$

with R is the unconditional correlation matrix of the standardized residuals, S is the unconditional correlation matrix of the ε_t series, and $\circ$ is the Hadamard product of two identically sized matrices. Since the assumption of normality is required in the DCC model, the GARCH models for the marginal distributions of the data that are constructed should have error terms that are normally distributed. The assumption of normality in the first equation gives rise to a likelihood function, and the second equation simply expresses the assumption that each asset follows a univariate GARCH process. The second equation is the volatility part of 2 GARCH models (marginal distributions) rewritten in a matrix form. The fourth equation simply calculates the covariance matrix at time t and the last equation calculates the correlation matrix at time t.

It is allowed to use a GARCH model with student-t distributed error terms as marginal distributions for Patton's evolution equation, but it does not hold for the DCC model. A DCC model requires the assumption of normality, thus it is critical to reconstruct the marginal distributions, such that the error terms are standard normal distributed (we reconstruct the marginal distribution, such that $\eta_t \sim N(0,1)$).

II.4 Estimation

The Gaussian copula parameter is estimated using the inference for margins method (Kole et al. (2007), Joe (1997), and Joe and Xu (1996)). This method is a two-step approach, where the parameters for the marginal models are estimated first. In the second step, the copula parameters are estimated with the marginal distributions parameters treated as given. The estimation is done by using the maximum likelihood function.

The inference function for margins uses the maximum likelihood to estimate the copula parameter. Let the parameters for the marginal models be denoted θ(volatility part), and the parameter for the time varying Gaussian copula (the correlation component) be denoted δ. The log likelihood can be written as the sum of a volatility part of the GARCH models and a correlation part:

$$L(\theta,\delta) = L_V(\theta) + L_C\big(\delta;\hat{\theta},u,v\big)$$

The volatility term is:

$$L_v(\theta) = \sum\nolimits_t \left(-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(h_t) - \frac{(r_t-\mu)^2}{2h_t}\right)$$

and the correlation component is:

$$L_c\big(\delta;\hat{\theta},u,v\big) = \log\left(\frac{1}{\sqrt{(1-\rho^2)}}\exp\left\{\frac{-1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)+\frac{1}{2}(x^2+y^2)\right\}\right)$$

with $x = \phi^{-1}(u)$ and $y = \phi^{-1}(v)$, and u and v the marginal probabilities constructed from the marginal models.

The first step of the inference function for margins is to estimate the marginal models:

$$\hat{\theta} = \max\{L_v(\theta)\}$$

and then take this value as given in the second step:

$$\max\{L_C\big(\delta;\hat{\theta},u,v\big)\}$$

The log likelihood for the DCC model can be expressed as:

$$\begin{aligned}
L &= -\frac{1}{2}\sum_{t=1}^{T}(n\log(2\pi) + \log|H_t| + r_t' H_t^{-1} r_t) \\
&= -\frac{1}{2}\sum_{t=1}^{T}(n\log(2\pi) + \log|D_t R_t D_t| + r_t' D_t^{-1} R_t^{-1} D_t^{-1} r_t) \\
&= -\frac{1}{2}\sum_{t=1}^{T}(n\log(2\pi) + 2\log|D_t| + \log|R_t| + \varepsilon_t' R_t^{-1}\varepsilon_t) \\
&= -\frac{1}{2}\sum_{t=1}^{T}(n\log(2\pi) \ + 2\log|D_t| + r_t' D_t^{-1} D_t^{-1} r_t - \varepsilon_t'\varepsilon_t + \log|R_t| + \varepsilon_t' R_t^{-1}\varepsilon_t)
\end{aligned}$$

which can be maximized over the parameters of the model. To make it simpler, we can rewrite the log likelihood function as follow: let the parameters in D be denoted θ (parameters of the GARCH models) and the additional parameters in R be denoted δ(the correlation parameter). The log likelihood can be written as the sum of a volatility part of the GARCH models and a correlation part:

$$L(\theta, \delta) = L_V(\theta) + L_C(\hat{\theta}, \delta)$$

The volatility term is

$$L_V(\theta) = -\frac{1}{2}\sum_t (n \log(2\pi) + \log |D_t|^2 + r_t^{'} D_t^{-2} r_t)$$

and the correlation component is

$$L_C(\theta, \delta) = -\frac{1}{2}\sum_t (\log|R_t| + \varepsilon_t^{'} R_t^{-1} \varepsilon_t - \varepsilon_t^{'} \varepsilon_t)$$

The two step approach to maximizing the likelihood is to find:

$$\hat{\theta} = \arg\max\{L_V(\theta)\}$$

and then take this value as given in the second stage:

$$\max_{\theta}\{L_C(\hat{\theta}, \delta)\}$$

Note that this is exactly the same as the inference for margins method we used for the time varying copula, except that the likelihood function is different. This implies that we can estimate the parameters for both methods by using the inference for margin method.

In order to estimate the marginal distributions, we can use the Akaike and Schwarz Information Criterion. Since we have a large dataset, the Schwarz Information Criterion (SIC) is more important than the Akaike Information Criterion (AIC), thus implying that the SIC should be used. The model with the lowest SIC value should be chosen. After the marginal distributions have been estimated, they are coupled by using the copula.

II.5 Models performance measure

We evaluate the marginal models statistically by evaluating the forecasts of the models. The forecasts of the marginal distributions can be evaluated by using the Mincer-Zarnowitz regression. We have constructed the models using the first 4000 observations of the data, and we construct forecasts for the GARCH variance over the forecast sample 4001-5000. The GARCH variance forecasts can then be evaluated by using the MZ regression:

$$R_{t+1}^2 = b_0 + b_1 \hat{\sigma}_{t+1}^2 + u_{t+1}$$

with R_t the return of the stock market/exchange rate, and $\hat{\sigma}_t^2$ the forecasted GARCH variance. Good volatility forecasts should have $b_0 \approx 0$, $b_1 \approx 1$ and a high R^2 value. According to Andersen and Bollerslev (1998), if $\hat{\sigma}_t^2$ follows a standard GARCH(1,1) process, then $R^2 < {}^1/_3$.

To evaluate the economic performance, we can construct a hedge ratio strategy for the exchange rate based on the models and evaluate the hedge ratios, to observe which model constructed ratio that produces the highest utility value for the investor. The exchange rate risk can be hedged by using a financial tool called the futures contract, which basically sets the exchange rate price at a given date in the future.

We denote the return of the stock market as R_{t+1} and we denote the simulated return of the exchange rate as Z_{t+1}. We also denote W_t as the capital invested in foreign stock market. We can use the returns to calculate the wealth of the investor in the future:

$$W_{t+1} = W_t \left(R_{t+1} Z_{t+1} - \xi \left(Z_{t+1} - \frac{1 + i_t^d}{1 + i_t^f} \right) \right)$$

where i_t^d and i_t^f the domestic interest rate and the foreign interest rate respectively, and ξ is the fraction of the foreign investment that is being hedged using the future contract. ξ has a negative sign to ensure that the investor takes short position, and we assume that $0 < \xi < 1$.

We use the power utility function to calculate the optimum utility for the investor:

$$U(W) = \frac{W^{1-\gamma} - 1}{1 - \gamma}$$

where γ is the coefficient of the risk aversion of the investor. The investor will choose ξ that maximizes his expected utility. To prevent the investor from speculating, we limit the hedge ratio between the interval [0, 1]. This implies that the investor chooses the optimum ξ^*:

$$\xi^* = \arg \max_{0 \leq \xi \leq 1} E_t[U(W_{t+1})]$$

To construct the hedge ratio, we need to simulate returns of both stock market and exchange rate using our estimated models at time $t+1$ (t is the last observation from our sample set) and maximize the investor's expected utility based on the simulated returns. We use simulation because it is difficult to calculate the joint distribution of the stock market and exchange rate returns. We calculate the returns R_{t+1} and Z_{t+1} by using simulation.

III. Results

III.1 Marginal models for copulas

In order to construct a copula, we need to construct the marginal distributions of the returns of the data first. Patton (2006) constructed marginal distributions using the AR(p), GARCH(1,1) model. We use this idea to construct the marginal distributions of our dataset. We restrict our lagged variable $p \leq 5$ to ensure that the correlations that we observe are not older than 1 week. This restriction is made under the assumption that correlations between daily returns are not significant anymore after 1 week.

As mentioned above, we construct GARCH models with student-t distributed error terms as the marginal models for the time varying copula proposed by Patton(2006) and GARCH models with standard normal distributed error terms as the marginal models for the time varying copula whose time variance is modeled using the DCC model.

After the marginal models have been constructed, we can do some evaluation of the forecasts of the marginal distribution by using the Mincer-Zarnowitz regression. A good specified model should have a high R^2 value (according to Andersen and Bollerslev (1998)) and coefficients $b_0 \approx 0$ and $b_1 \approx 1$.

Returns Data	**AR term**	μ	ϕ	ω	α	β	v
FTSE	AR(0)	0.00035 (0.00013)	-	$9.45*10^{-7}$ ($2.64*10^{-7}$)	0.068 (0.0081)	0.923 (0.0089)	11.56
DollarPound	AR(1)	-0.00014 ($7.57*10^{-5}$)	0.04 (0.015)	$2.31*10^{-7}$ ($7.79*10^{-8}$)	0.045 (0.0064)	0.949 (0.0072)	6.27
S&P	AR(3)	0.00052 (0.00011)	-0.047 (0.016)	$3.04*10^{-7}$ ($1.24*10^{-7}$)	0.047 (0.006)	0.952 (0.0058)	6.35

Table 1: marginal distributions of the dataset for the time varying copula proposed by Patton(2006), standard errors in brackets

Returns Data	R^2	b_0	b_1	**Log likelihood value**
FTSE	0.206820	$1.43*10^{-5}$ ($2.31*10^{-5}$)	1.003203 (0.062314)	5935.683
DollarPound	0.159736	$4.71*10^{-7}$ ($5.06*10^{-6}$)	1.079553 (0.078534)	7585.226
S&P	0.225020	$2.11*10^{-5}$ ($2.72*10^{-5}$)	0.946942 (0.055740)	5738.302

Table 2: MZ regression results for the marginal distributions in table 1, standard errors in brackets

Returns Data	AR term	μ	ϕ	ω	α	β
FTSE	AR(0)	0.00031 (0.00013)	-	$1.41*10^{-6}$ ($2.33*10^{-7}$)	0.083 (0.0061)	0.903 (0.0072)
DollarPound	AR(1)	-0.00013 ($8.01*10^{-5}$)	0.053 (0.016)	$6.23*10^{-7}$ ($5.28*10^{-8}$)	0.064 (0.00409)	0.917 (0.0049)
S&P	AR(0)	0.00047 (0.00013)	-	$7.41*10^{-7}$ ($1.00*10^{-7}$)	0.069 (0.00407)	0.927 (0.0043)

Table 3: marginal distributions of the dataset for the time varying copula whose correlation is modeled using the DCC model, standard errors in brackets

Returns Data	R^2	b_0	b_1	Log likelihood value
FTSE	0.209484	$1.37*10^{-5}$ ($2.30*10^{-5}$)	1.011052 (0.062265)	5943.802
DollarPound	0.160626	$-1.43*10^{-6}$ ($5.15*10^{-6}$)	1.168438 (0.084677)	7593.440
S&P	0.232085	$1.72*10^{-5}$ ($2.70*10^{-5}$)	0.994061 (0.057324)	5749.119

Table 4: MZ regression results for the marginal distributions in table 3, standard errors in brackets

Note: In tables 1, 2, 3, and 4, the value of the parameters of PoundDollar is exactly the same as the parameters of DollarPound. The only exception is that the value of PoundDollar's μ in tables 1 and 3 is the negative of DollarPound's μ. Since they are almost identical, the parameters of PoundDollar are not reported.

We observe from tables 2 and 4 that the MZ regression has $b_0 \approx 0$ and $b_1 \approx 1$, and we also observe relatively high R^2 values. This implies "good" volatility forecasts. A model with good volatility forecasts implies a good specified model.

From tables 1 and 3, we can observe that in the GARCH model, the constant (μ) almost doesn't have any effects. The AR term (ϕ) has stronger effect, which implies that lagged returns have stronger effect on the marginal models than the constant. In the conditional volatility part of the GARCH model, we observe that constant (ω) and lagged squared error terms (α) don't have much effect. The lagged conditional volatility (β) however, has a strong effect. This implies that lagged returns and lagged conditional volatility influences the marginal models the most.

As mentioned above, we use marginal probabilities constructed from marginal distributions as input for copulas. In this paper we are only going to work with the Gaussian copula. A Gaussian copula has as inputs: inverse of the standard normal PDF of the marginal probabilities and the correlation coefficient ρ, which is the correlation over the whole sample. The correlation coefficient ρ is chosen such as the sum of the log-likelihood of the PDF of the Gaussian copula is maximized. If the correlation coefficient has been correctly estimated, then it should not deviate much from the empirical correlation. Thus this can be used as an indicator to see whether the correlation coefficient has been correctly estimated.

III.2 Correlation coefficient estimated using the evolution equation proposed by Patton(2006)

A standard Gaussian copula's estimated correlation parameter is the correlation over the whole sample period. There is a possibility that the correlation varies over the time, thus we are going to estimate the correlation parameter using the 2 methods stated in chapter II.1.

According to the evolution equation proposed by Patton(2006), the correlation coefficient depends on a constant, lagged ρ_{t-1} and mean of the product of the transformed last 10 observations. We need to estimate the constant, lagged ρ_{t-1}, mean of the product of the transformed last 10 observations, and a begin value of ρ_t (needed to start the iteration) such as they maximize the sum of the log-likelihood of the PDF of the Gaussian copula over the whole sample. The estimated coefficients are presented below:

Data	w	β	α	**Begin value of** ρ_t	**Log likelihood value**
FTSE, DollarPound	0.5792	-0.3255	0.7550	0.8395	96.67
S&P, PoundDollar	-0.0015	0.9822	0.0372	0.2570	18.34

Table 5: Parameters of the time varying Gaussian copula, using the evolution equation proposed by Patton(2006)

We observe from table 5 that for the equation for FTSE and DollarPound, the constant has quite a strong effect, the average of the last 10 observations has the strongest effect, and a negative parameter for the lagged correlation. This could cause the correlation to be volatile. We also observe that for the equation for S&P and PoundDollar, the constant almost has no effect at all, and the average of the last 10 observations has a weak effect. The lagged correlation however, has a very strong effect on the equation.

We compare the correlation coefficient estimated using the evolution equation of the time varying Gaussian copula proposed by Patton with the empirical correlation. Since the correlation coefficient now varies over time, we use the moving window technique on the empirical correlation. We use a moving window of 100 days. Thus we start comparing the correlations from time t = 100 (since the first 100 observations are needed to construct the first empirical coefficient).

We can see from figure 1 that the correlation coefficient of S&P and PoundDollar fits the empirical correlation. Unfortunately it is not the same for the correlation coefficient of FTSE and DollarPound. The correlation coefficient moves erratically over the time. The cause of this phenomenon could be the negative parameter for the lagged ρ_{t-1}, causing the correlation coefficient to be extremely volatile. I didn't succeed in finding the reason behind this. I suspect this could be caused by the data.

Note: The results have been confirmed by using a simulation technique, where datas are generated based on the estimated parameters. The maximum log likelihood value of the generated data is almost the same as the maximum log likelihood value of the original data.

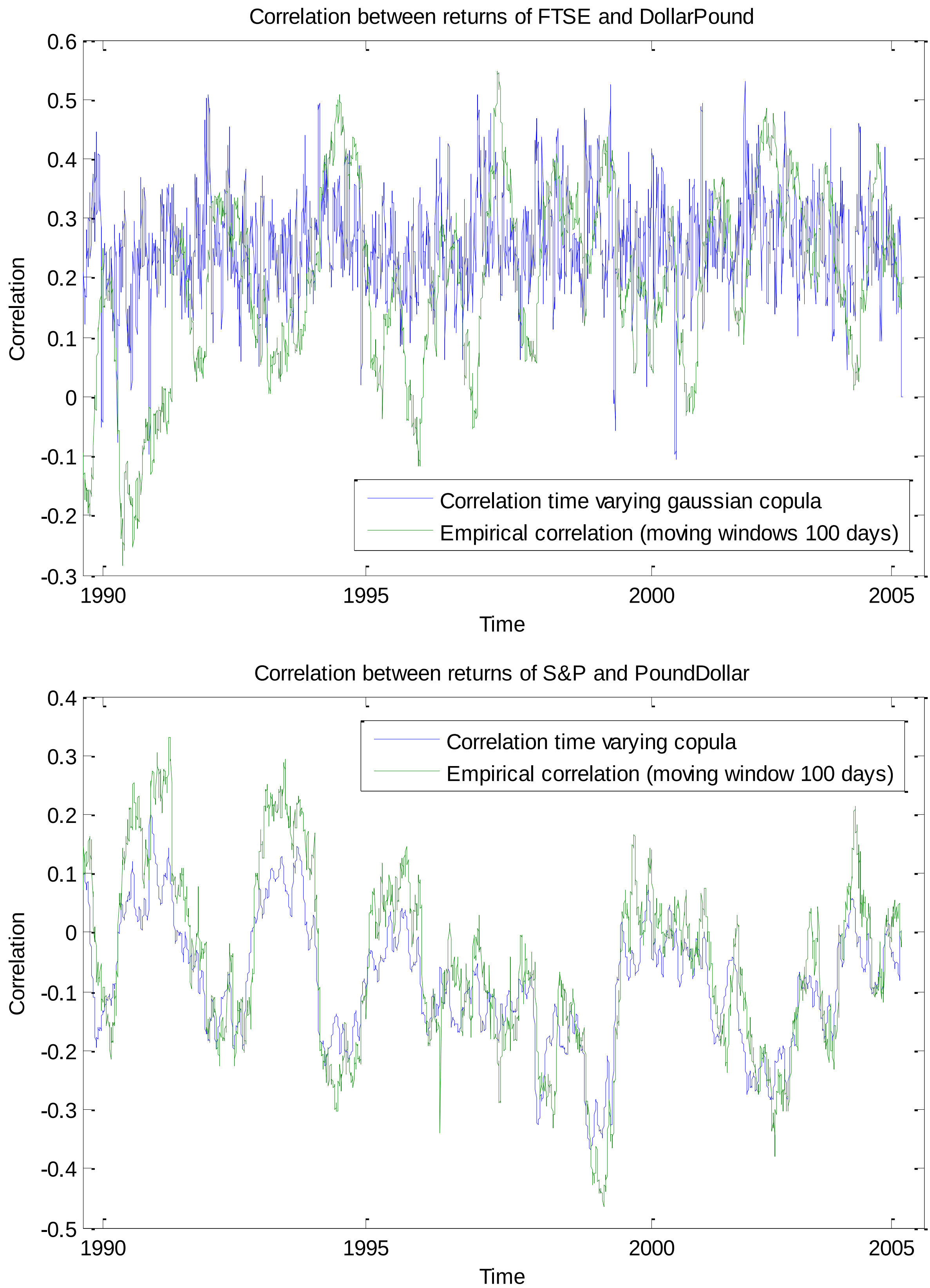


Figure 1: Comparison of the correlation coefficient estimates of the time varying Gaussian copula estimated using the evolution equation proposed by Patton and the empirical correlation. The time on the x-axis is in day units, from 17-5-1990 until 27-4-2005 (not including weekends).

III.3 Correlation coefficient estimated using the DCC model

The first equation of the DCC model uses volatility estimates part of 2 marginal (GARCH) models and they are expressed together in a matrix form. After the first equation has been constructed, the standardized residuals of the GARCH models can also be constructed. The standardized residuals are used as input for the second equation of the DCC model. The second equation estimates the covariance matrix for the standardized residuals. It requires as inputs: the standardized residuals, and the unconditional correlation matrix of the standardized residuals.

We can use the unconditional covariance matrix of the standardized residuals as begin value of Q_t (needed to start the iteration). We need to estimate the parameters α and β such that the sum of the likelihood function stated in chapter II.2 is maximized under the constraints that $\alpha \geq 0, \beta \geq 0$, and $\alpha + \beta < 1$. The estimated parameters are presented below.

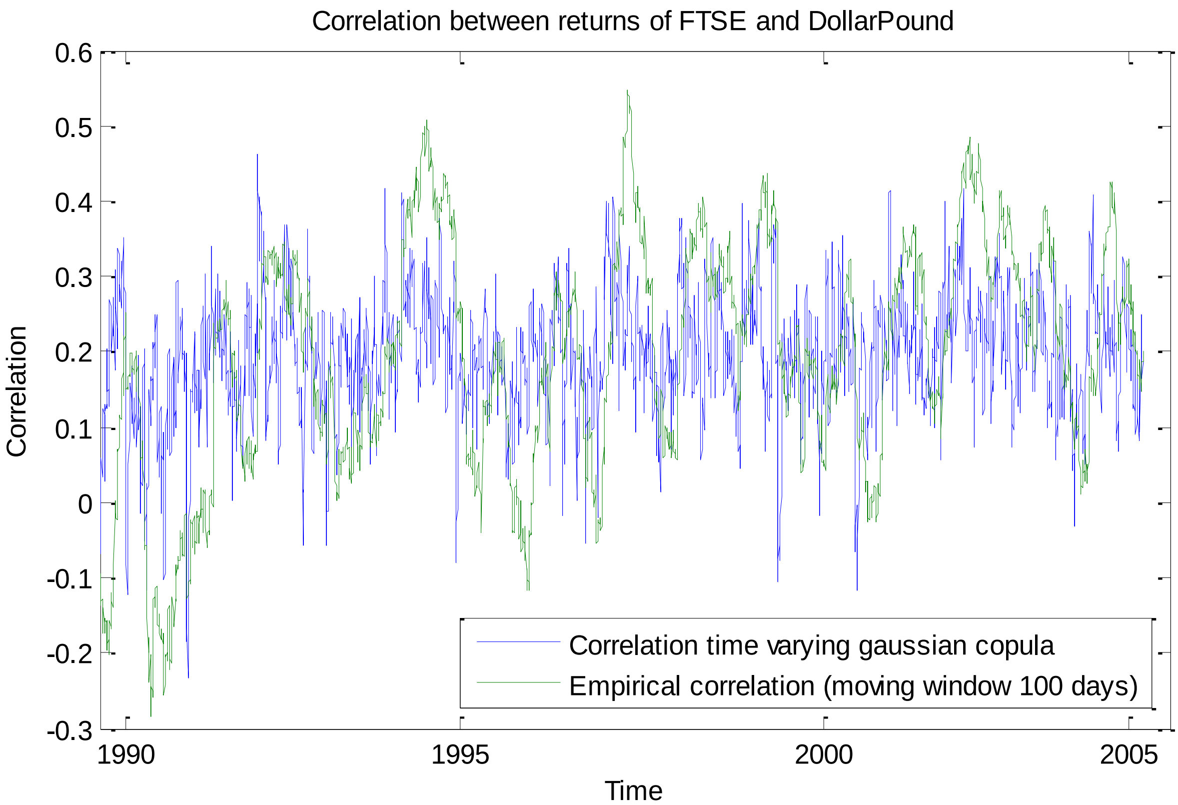
Data	**α DCC**	**β DCC**	**Log likelihood value**
FTSE, DollarPound	0.03825	0.85342	28619.9533
S&P, PoundDollar	0.00869	0.98302	28587.1971

Table 6: Parameters of the DCC model

We observe from table 6 that for the DCC model, the product of lagged standardized error terms parameter (α DCC) for both combinations have weak effect. However, for FTSE and DollarPound combination, the effect of the product of lagged standardized error terms is stronger than for S&P and PoundDollar combination. The parameter of the lagged covariance (β DCC) however has a very strong effect on the model. For S&P and PoundDollar combination, the effect of the lagged covariance is stronger than for FTSE and DollarPound combination.

Again, we compare the correlation coefficient estimated using the DCC model with the empirical correlation. Since the correlation coefficient varies over time, we use the moving window technique on the empirical correlation. We use a moving window of 100 days. Thus we start comparing the correlations from time t = 100 (since the first 100 observations are needed to construct the first empirical coefficient).

Again, we see from figure 2 that the correlation coefficient of S&P and PoundDollar fits the empirical correlation. Unfortunately it is not the same for the correlation coefficient of FTSE and DollarPound. The correlation coefficient moves erratically over the time, implying a very volatile correlation. We also observe from figures 1 and 2 that the movement of the correlation estimated using the evolution equation moves rather identically with the correlation estimated using the DCC model. The correlation coefficient of FTSE and DollarPound moves erratically for both models, implying that the correlation coefficient's extreme volatility might be caused by the data.



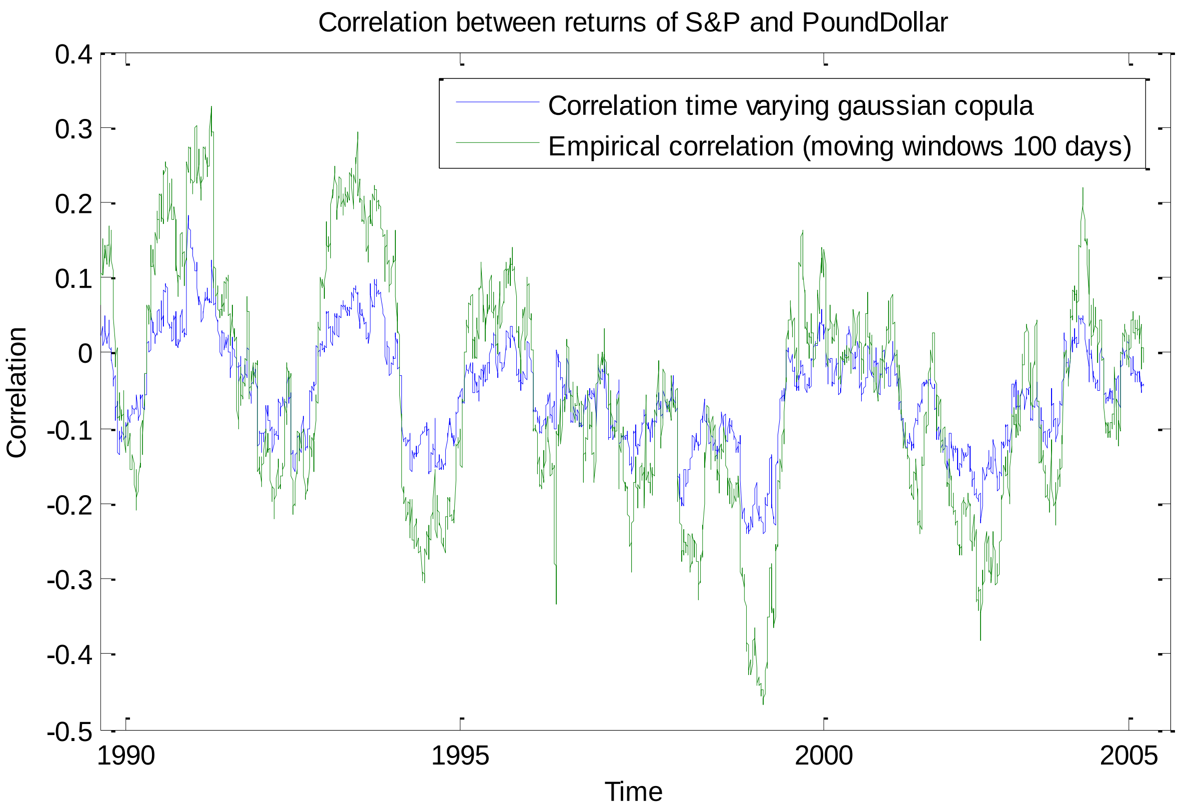


Figure 2: Comparison of the correlation coefficient estimates of the time varying Gaussian copula estimated using the DCC model and the empirical correlation. The time on the x-axis is in day units, from 17-5-1990 until 27-4-2005 (not including weekends).

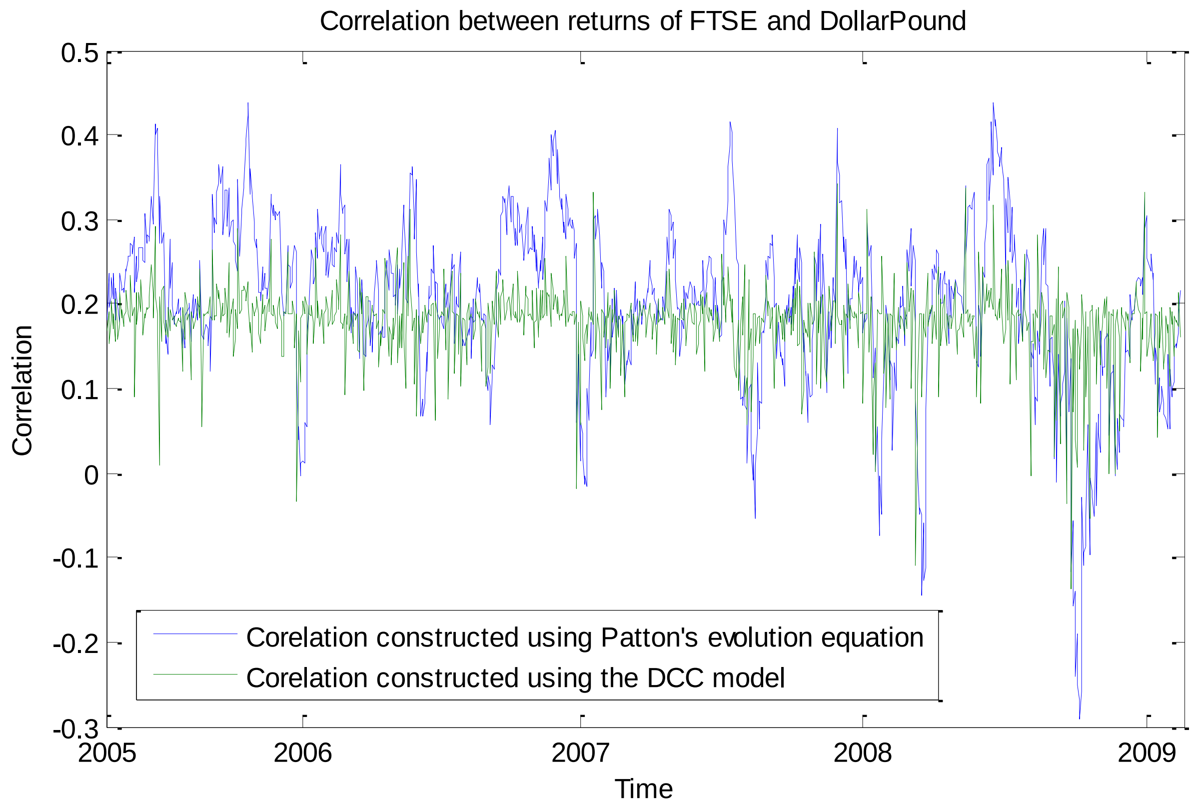
III.4 Hedge ratio

We evaluate our models economically by constructing hedge ratios based on the constructed model for different values of the coefficient of risk aversion. First, we estimate the correlation of the data over time 4001 until 4997 by using both estimated models, Patton's evolution equation and the DCC model. The estimated correlations are presented in figure 3. After the correlations have been estimated, we estimate the values of both the stock market return and the exchange rate return. It is difficult to estimate them using their joint distribution function, so they are estimated by using the simulation technique, by simply simulating the marginal probabilities, and reconstruct them back into the returns data.

We can see from figure 3 that for both combinations, that the correlation coefficients implied by the DCC model are more volatile. We also see that the correlation coefficient of FTSE and DollarPound moves rather identically for both models, though it isn't so for S&P and PoundDollar combination. Correlation constructed by Patton's evolution equation varies more, but less volatile, while correlation constructed by the DCC model does not vary much but very volatile. The reason behind this could be that in the DCC model we only use the last 1 observation, while we calculate the mean of the last 10 observations in Patton's evolution equation. Since we only use the last 1 observation in DCC model, shocks have greater impact, and also stay very short, causing high volatility. Patton's equation includes the average of the last 10 observations, thus shocks have less impact and stay longer, causing less volatility. The parameter values of α in Patton's equation are also higher than parameter values of α in DCC model, which implies that lagged observations have more effect in Patton's equation, thus allowing correlation series it constructed to vary more.

We construct an optimal hedge ratio (ξ) and utility the hedge ratio brings for the investor for each combination by using simulation and formulas that are stated in chapter II.2 based on the estimated correlations stated in figure 3. The formula depends on the investor's risk aversion level γ. In this paper we consider setting $\gamma = 2$ and $\gamma = 5$. When an investor has an aversion risk level of 2, then he is not very risk averse, while if he has an aversion risk level of 5, he is very risk averse. The optimal hedge ratios for every t constructed by using Patton's evolution equation are presented in figures 4 and 5, and hedge ratios constructed by using the DCC model are presented in figures 6 and 7.

We observe from figures 4, 5, 6 and 7 that the hedge ratios constructed by using Patton's evolution equation have similar movements over the time with the hedge ratios constructed by using the DCC model. Even though the movements are similar, the volatility of the hedge ratios does not. We see that hedge ratios constructed by using the DCC model are a lot more volatile than hedge ratios constructed by using Patton's evolution equation. The cause could be because the correlation coefficients estimated by using the DCC model are more volatile than the correlation coefficients estimated by using Patton's evolution equation (see figure 3). The reason for the similar movements could be caused by the GARCH models (the marginal models). The coefficient estimates for the marginal models for Patton's evolution equation don't differ much with the coefficient estimates for the marginal models for the DCC model.



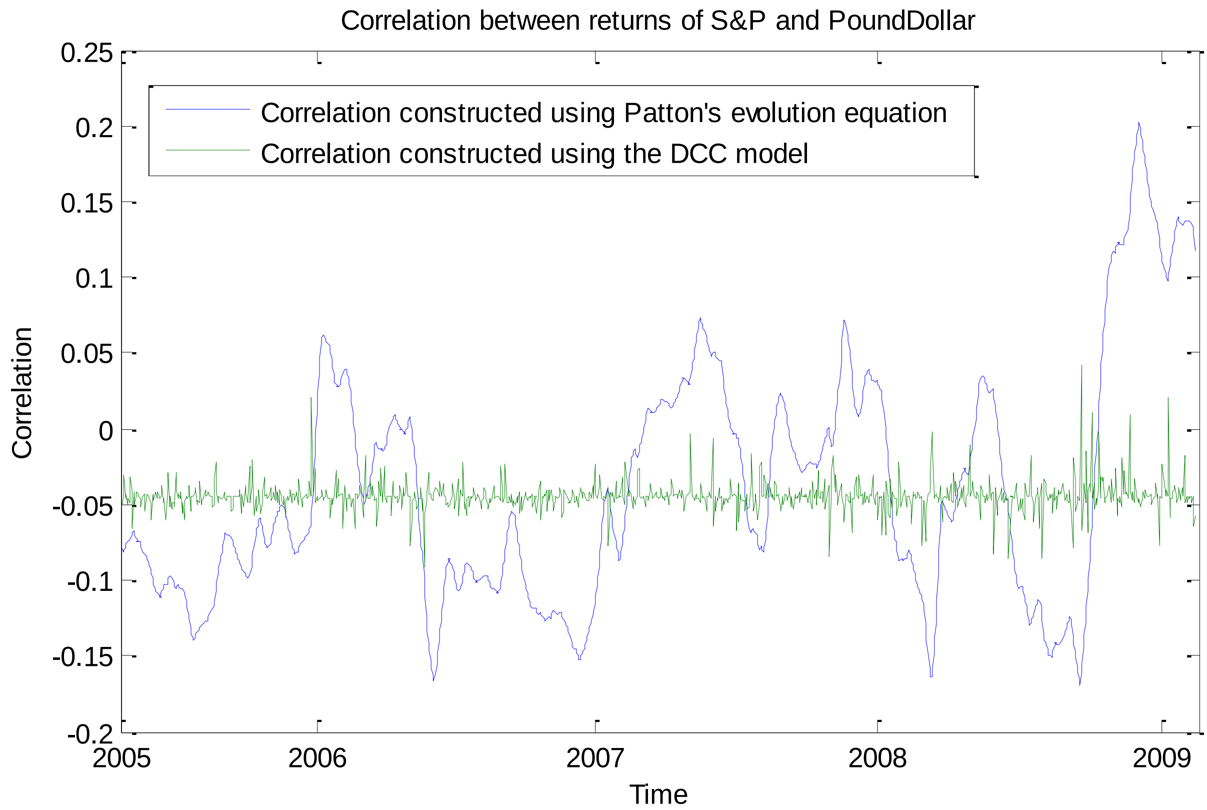
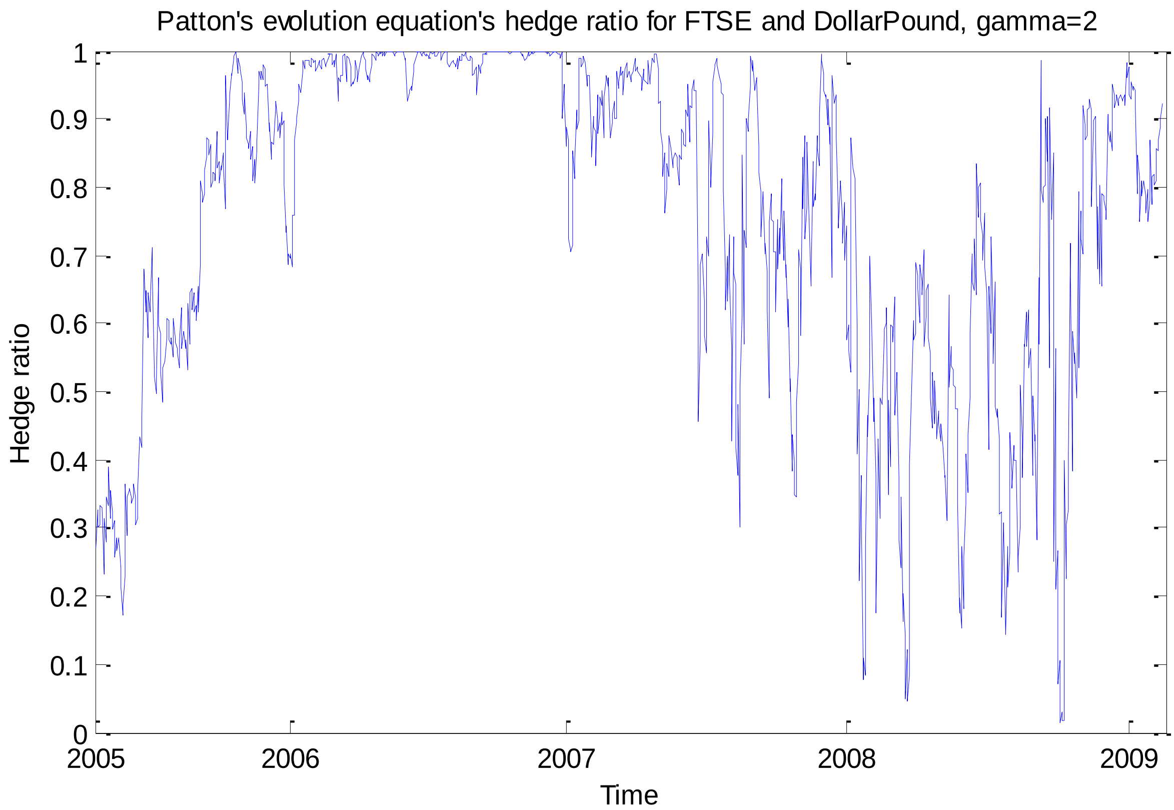


Figure 3: Comparison of the correlation coefficient estimated using Patton's evolution equation and the DCC model. The time on the x-axis is in day unit, from 28-4-2005 until 25-2-2009 (not including weekends).



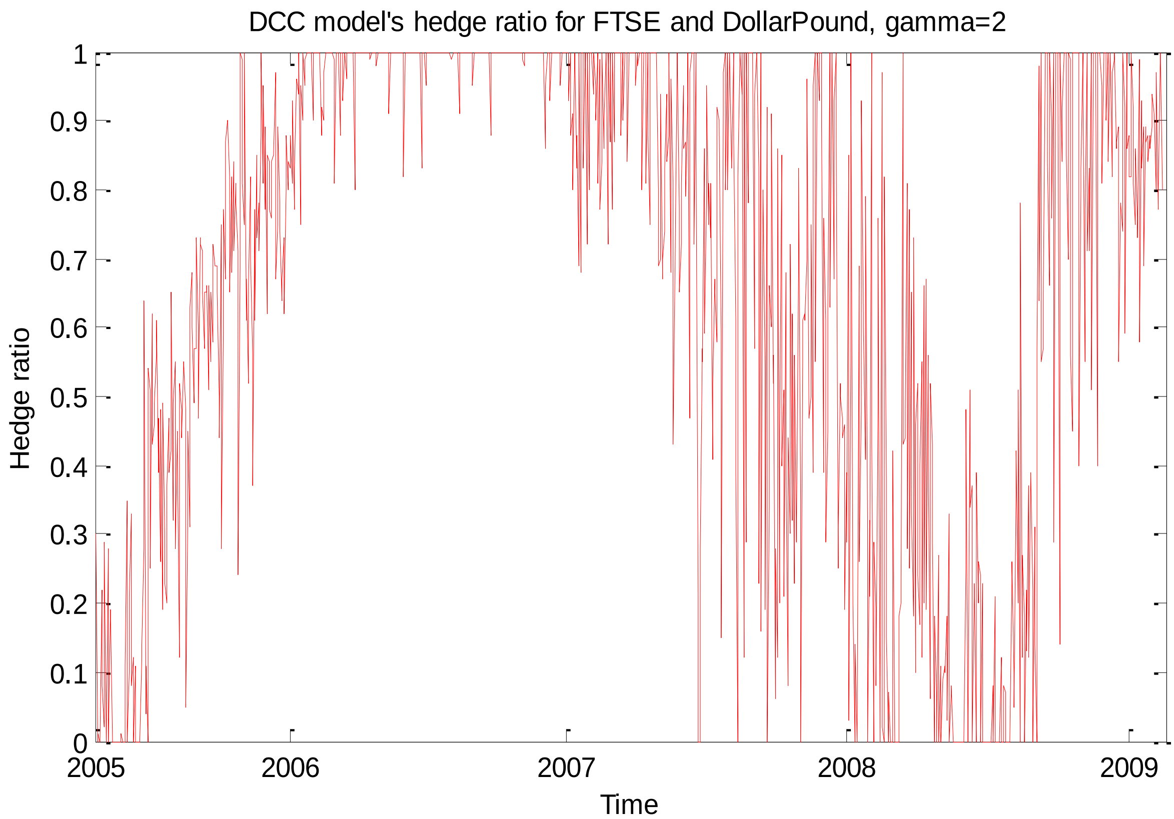
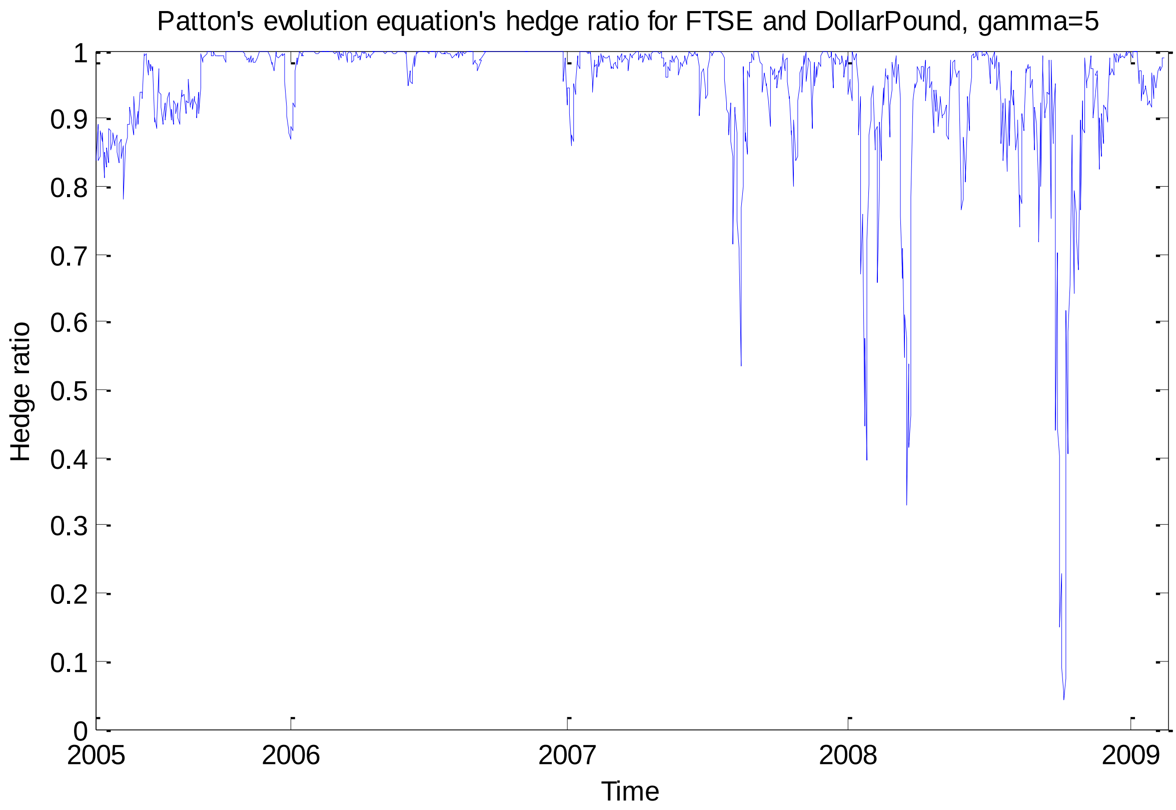


Figure 4: Optimal hedge ratios for FTSE and DollarPound constructed using Patton's evolution equation and the DCC model for $\gamma = 2$. The time on the x-axis is in day unit, from 28-4-2005 until 25-2-2009 (not including weekends).



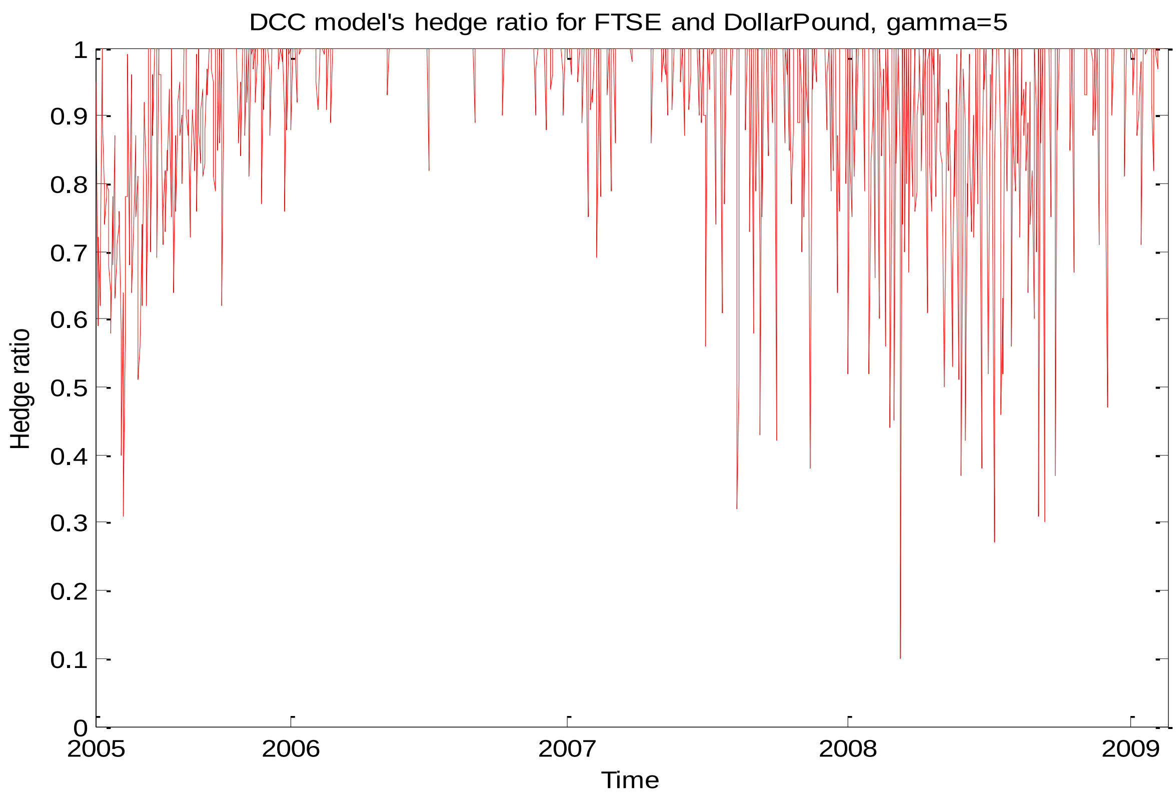


Figure 5: Optimal hedge ratios for FTSE and DollarPound constructed using Patton's evolution equation and the DCC model for $\gamma = 5$. The time on the x-axis is in day unit, from 28-4-2005 until 25-2-2009 (not including weekends).

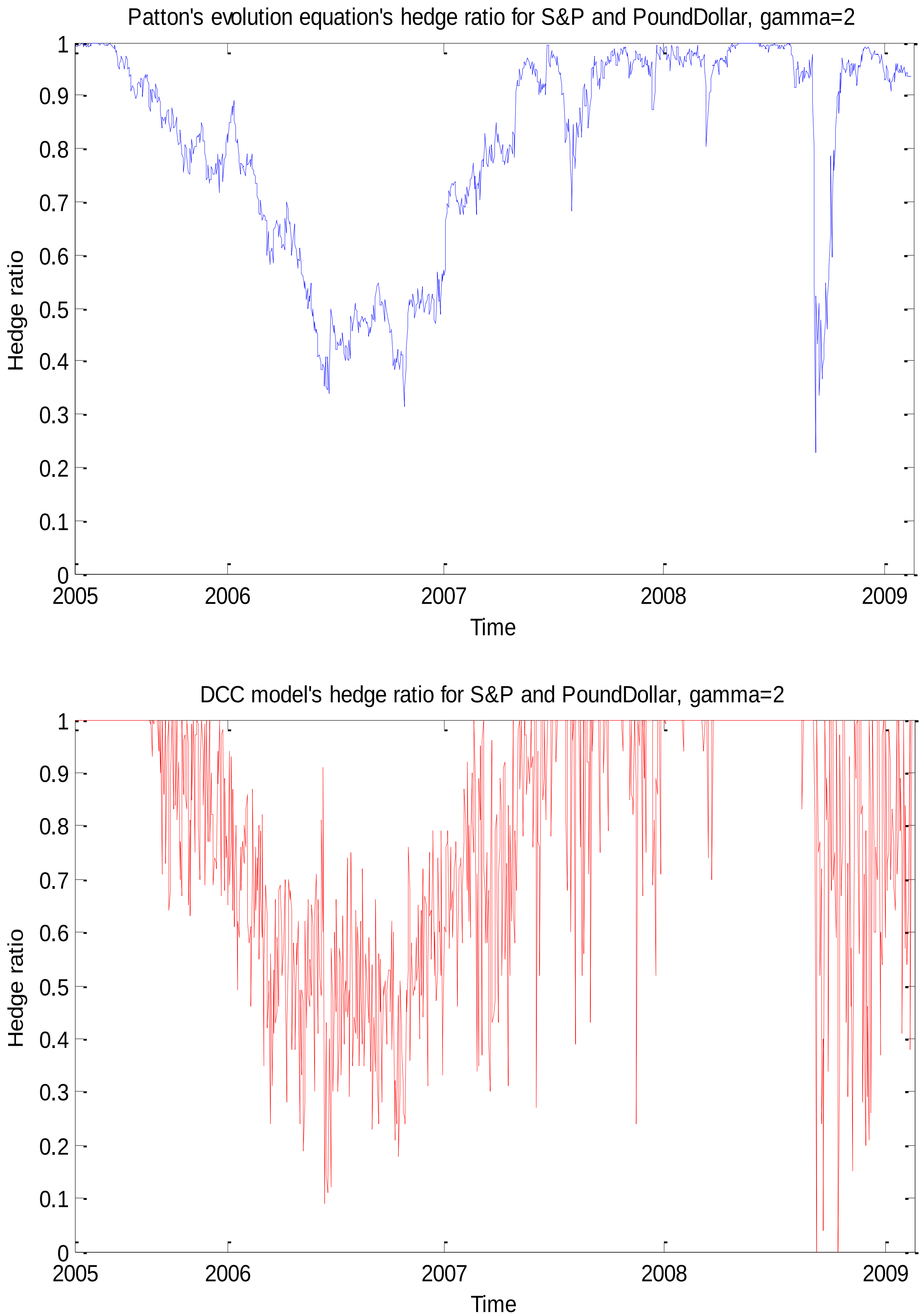
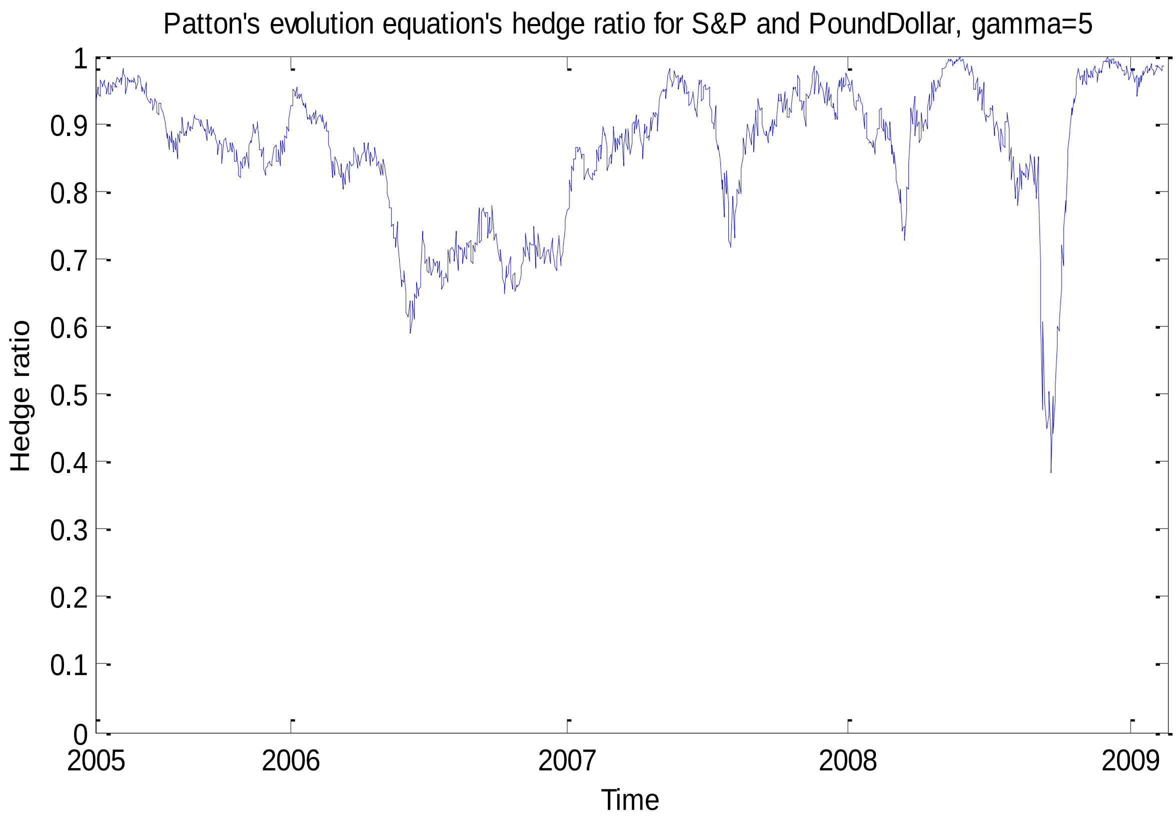


Figure 6: Optimal hedge ratios for S&P and PoundDollar constructed using Patton's evolution equation and the DCC model for $\gamma = 2$. The time on the x-axis is in day unit, from 28-4-2005 until 25-2-2009 (not including weekends).



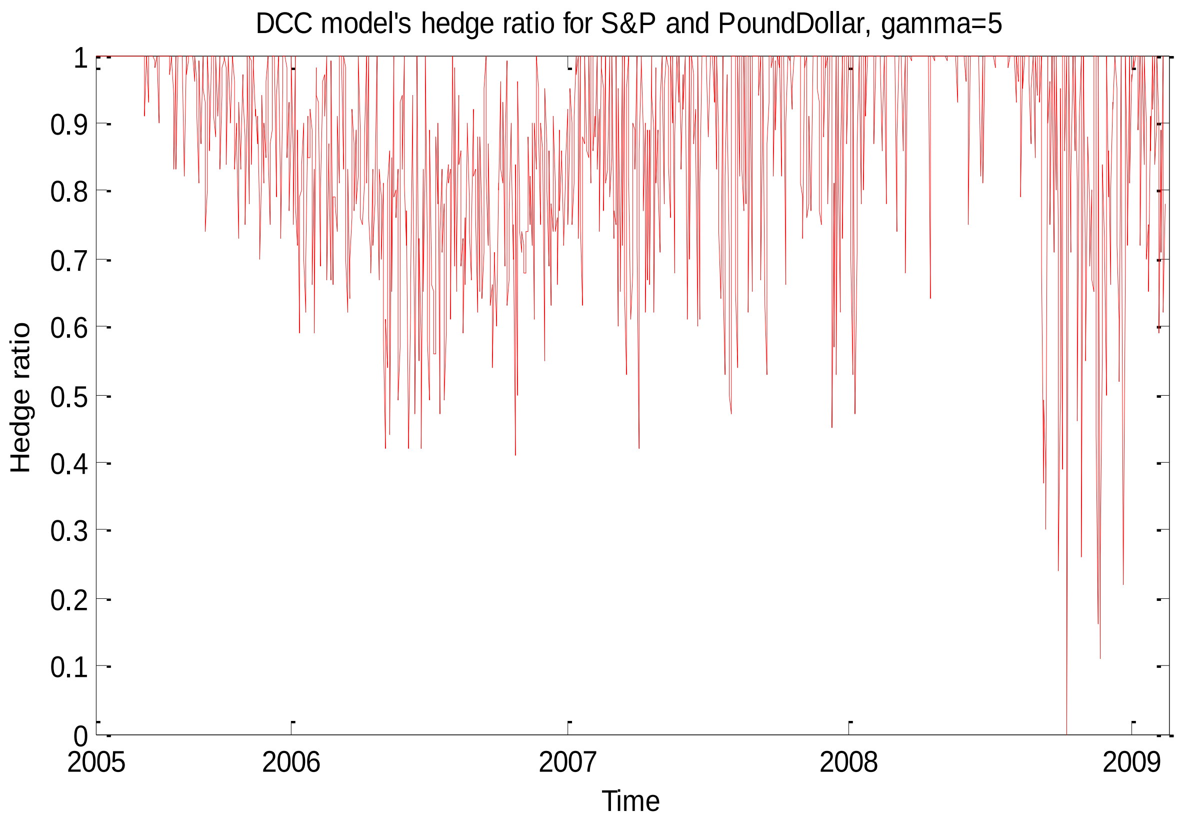


Figure 7: Optimal hedge ratios for S&P and PoundDollar constructed using Patton's evolution equation and the DCC model for $\gamma = 5$. The time on the x-axis is in day unit, from 28-4-2005 until 25-2-2009 (not including weekends).

We observe from figures 4 and 5 that the hedge ratios for FTSE and DollarPound increase towards 1 (full hedge) in the beginning, and they stay around there for a while. From mid 2007 until the end of our sample period, the hedge ratios become very volatile, that they keep moving between 0 (no hedge) and 1 (full hedge) erratically. We observe from figures 6 and 7 that the hedge ratios for S&P and PoundDollar stay around 1 (full hedge) in the beginning, become lower in 2006, and they move back to around 1 (full hedge) in mid 2007. They reach their lowest peak around the end of 2008 and they move again to around 1(full hedge) in begin 2009.

We compare now the average of the expected utility brought by these hedge ratios for the investor based on the simulated returns. We calculate the average of the expected utility over the whole 1000 days period, and we observe which model produces the best expected utility for the investor. The average expected utilities are presented in tables 8 and 9.

Model	γ	Average Expected Utility FTSE and DollarPound
Patton's evolution equation	2	$2.4031*10^{-4}$
	5	$-1.1051*10^{-4}$
DCC model	2	$2.0345*10^{-4}$
	5	$-1.5470*10^{-4}$

Table 7: Expected utility of FTSE and DollarPound for both models

Model	γ	Average Expected Utility S&P and PoundDollar
Patton's evolution equation	2	$4.0886*10^{-4}$
	5	$-8.5445*10^{-5}$
DCC model	2	$3.8433*10^{-4}$
	5	$2.1737*10^{-5}$

Table 8: Expected utility of S&P and PoundDollar for both models

Note: Expected utilities are calculated using the simulated returns R_{t+1} and Z_{t+1} . In order to construct the hedge ratios, we maximized the expected utility of the investor based on the simulated returns.

We observe from tables 7 and 8 that a time varying Gaussian copula with correlation coefficient estimated using Patton's evolution equation overall (3 cases out of 4) produces more expected utility to the investor than a time varying Gaussian copula with correlation coefficient estimated using the DCC model.

We calculate the total profit/loss and the investor's average utility over the whole period to evaluate our hedge ratios. We assume that the investor invests 1 wealth at every beginning of time *t*. We also calculate the total profit/loss and the investor's utility in case of no hedge (ξ = 0) and full hedge (ξ = 1). In order to investigate whether the investor has lower risk if he follows our hedge ratio, we calculate the volatility of his wealth at the end of every period *t* (his wealth after his investments). The total profit/loss and utility for FTSE if the investor invested using Dollar and the volatility of his returns are presented in table 9. The total profit/loss and utility for S&P if the investor invested using Pound and the volatility of his returns are presented in table 10.

Hedge ratio for FTSE	γ	Total profit/loss	Average Utility	Volatility
Patton's evolution equation	2	-0.3002	$-4.9353*10^{-4}$	0.0139
	5	-0.5833	-0.0011	0.0140
DCC model	2	-0.2499	$-4.5901*10^{-4}$	0.0140
	5	-0.6231	-0.0011	0.0142
No hedge	2	0.1636	$-3.6310*10^{-5}$	0.0142
	5		$-3.3907*10^{-5}$	
Full hedge	2	-0.7032	$-9.1300*10^{-4}$	0.0144
	5		-0.0012	

Table 9: Total profit/loss, average utility, and the volatility of returns for FTSE if the investor invests using Dollar for every risk aversion level

Hedge ratio for S&P	γ	Total profit/loss	Average Utility	Volatility
Patton's evolution equation	2	0.2648	$2.6357*10^{-5}$	0.0155
	5	0.1985	$-3.9777*10^{-4}$	0.0154
DCC model	2	0.1604	$-8.6119*10^{-5}$	0.0157
	5	0.2532	$-3.4255*10^{-4}$	
No hedge	2	-0.5484	$-8.6231*10^{-4}$	0.0176
	5		-0.0013	
Full hedge	2	0.2726	$4.3080*10^{-5}$	0.0152
	5		$-3.0536*10^{-4}$	

Table 10: Total profit/loss, average utility, and the volatility of returns for S&P if the investor invests using Pound for every risk aversion level

Note: Utilities are calculated using the real returns R_{t+1} and Z_{t+1}. They are different from the expected utilities, since expected utilities are calculated using the simulated returns.

We see from table 9 that if the investor follows our hedge ratio, he will do better than if he fully hedges his risk (less loss and higher utility), but he will do worse than if he doesn't hedge his risk at all (less profit and lower utility). However, the investor has lower risk since the volatility of his returns is lower when he follows our hedge ratios (volatilities of hedge ratios are lower than both full hedge and no hedge strategies).

We see from table 10 that if the investor follows our hedge ratio, he will do better than if he doesn't hedge his risk at all (more profit and higher utility), but he will do worse than if he fully hedges his risk (less profit and lower utility). The investor also has lower risk than if he doesn't hedge his risk at all (lower utility). However, if the investor follows our hedge ratio, than he has higher risk (higher volatility) than if he fully hedges his risk. It is in the investor's best interest if he just fully hedges his risk if he invests in S&P using Pound.

We can conclude that our hedge ratios perform adequately. The investor does better than if he fully hedges his risk, if he invests in FTSE. He also has lower risk than fully hedging his risk or not hedging his risk at all if he invests in FTSE. He also does better and has lower risk than if he doesn't hedge his risk at all if he invests in S&P. But our hedge ratios are not good enough to produce better results than both no hedge and full hedge strategies. They also can't make the investor's risk lower than the risk of both full hedge and no hedge strategies (our hedge ratio produced lower risk than both strategy for FTSE, but they didn't for S&P). Perhaps there is an asymmetric correlation in our data, that correlations for downside moves are much greater than for upside moves (Ang and Chen (2002)). If that is the case, than we can use other copulas to construct our correlation series and to simulate the returns needed to construct the hedge ratio. In this paper, we limited the investor's hedge ratio between 0 and 1. Perhaps a better result can be obtained if we allow the investor to go short or long on the future contracts.

IV. Conclusion

In this paper we used the Gaussian copula in order to model the correlation between the returns of stock market and the returns of exchange rate. But a standard Gaussian copula only estimates the correlation between these returns over the whole sample period. In this paper we wanted to let the correlation to be able to vary over the time. In order to do this, we used 2 methods to model the correlation coefficient of the Gaussian copula. We used the evolution equation for the Gaussian copula proposed by Patton(2006) and the DCC model.

We used both methods to estimate a model for the correlation over the first 4000 samples. Based on the estimated model, we constructed correlation coefficient for sample 4001 until 5000 and we used these constructed correlation to construct an optimal hedge ratio for an investor based on a general utility function. We compared the utility produced by the hedge ratios constructed using both methods to see which method produces higher utility.

From the expected utility values, we saw that Patton's evolution equation produces better hedge ratios (3 cases out of 4 have higher utility) than the DCC model. This implies that Patton's evolution equation is more suitable to be used to estimate the correlation coefficient of a Gaussian copula in order to maximize investors' expected utility.

The hedge ratios that we constructed perform adequately, that they do better than a full hedge strategy if the investor invests in FTSE, and they do better than a no hedge strategy if the investor invests in S&P, but it does not do better than both of them. The investor also has lower risk than both full hedge and no hedge strategies if he invests in FTSE. He has lower risk than the no hedge strategy if he invests in S&P. We can use other copula in case of asymmetric correlation. We can also allow the investor to go short or long in the future contracts to try obtaining a better result.

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