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Conditioning the Covariance Matrix Estimator in a High-Dimensional Portfolio Management Setting

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Abstract

In this paper, I evaluate established and propose novel methods to provide invertible estimates of the covariance matrix when the amount of variables exceeds the amount of observations. This is done in a portfolio management setting, where the covariance estimate is essential to form mean-variance efficient portfolios. The key to derive a feasible estimator in high dimensions is to apply some sort of shrinkage. Ledoit and Wolf (2004, 2012) introduce (non)linear shrinkage methods as a novel class of estimators to establish positive semi-definite estimators. My paper extends the current literature by proposing a new combination of (non)linear shrinkage with the DECO approach of Engle and Kelly (2012) and the application hereof in several high-dimensional portfolio management environments. I test whether these methods improve upon established shrinkage methods in constructing covariance estimates. I find that the extension to DECO models produces stable and invertible covariance matrix estimates, but does not consistently outperform pure (non)linear shrinkage estimators in high dimensions. However, the strength and convergence of the asymptotic results are supported by a simulation study.

Keywords – *Dynamic Conditional Correlation; Dynamic Equicorrelation; Factor Models; (Non)linear Shrinkage; Portfolio Management*

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Contents

1 Introduction

Accurate measures of the covariance matrix and its inverse play a central role in risk and portfolio management. For example, an optimal efficient portfolio allocation from a mean-variance perspective requires solving the mean-variance quadratic optimisation problem of Markowitz (1952). This allocation is based on two inputs: the expected (excess) return for each stock and the associated covariance matrix for the stock universe of interest. It is well known that the expected returns and volatility of financial asset returns are not directly observed and have to be estimated from return data. These estimation problems have been subject in multiple established researches.

For example, Kan and Zhou (2007) estimate both inputs separately, while assuming the known, true value for the other. They find that the loss of estimating the expected returns or covariance matrix separately, with respect to the true optimal value, is a function decreasing in amount of observations and increasing in amount of assets. Furthermore, they show that if both inputs are unknown and have to be estimated, the loss resulting from estimation is increasing in a multiplicative way, leading to exponentially increasing estimation errors. Also, Kan and Zhou (2007) show that when the number of assets is relatively small compared to the sample size of observations, that estimation errors are mainly caused by estimating the expected returns. However, in a more realistic high-dimensional setting, estimation errors in the covariance matrix significantly accumulate and become far more costly than errors in estimated expected returns.

Therefore, in this research I solely focus on the latter case, the estimation of the covariance matrix. For this purpose, I make use of the global minimum variance (GMV) portfolio. The GMV portfolio is the theoretically mean-variance efficient portfolio with the lowest possible variance achievable, while still positioned on the efficient frontier. The estimation of the covariance matrix in the GMV portfolio is a clean problem in terms of evaluating the quality of the estimate, since it does not require estimation of the expected returns simultaneously. Thus, by isolation it reduces estimation error. I investigate the ability of several methods to construct a covariance matrix estimate to ultimately achieve the minimum variance. I follow this procedure while the amount of assets diverges with respect to the amount of return observations.

For the estimation of the covariance matrix, the sample covariance matrix is a natural candidate. Several advantages of the estimator include: simple in construction, unbiased and intuitively appealing as it is the sample estimator of the theoretical second moment. However, in current financial settings, it is often the case that the amount of assets is of much larger magnitude than the amount of daily observations. Then, the estimate of the sample covariance matrix contains a significant amount of sampling error as documented by Jobson and Korkie (1980) among others. They state that the most extreme coefficients in the matrix tend to take on extreme values not because of their true value, but due to extreme amounts of errors. This optimal solution tends to turn into what Michaud (1989) calls error maximisation.

Summarising, the sample covariance matrix has a number of undesirable properties when the dimension of the investment universe is large. First, when the amount of assets is larger than the amount of observations, the sample covariance matrix is not of full rank, so its inverse will not exist. Furthermore, even when the amount of assets is of approximately equal size as the amount of observations, the estimate is unstable. The total amount of parameters to be estimated is of the same order as the total size of the data set and grows quadratically. This leads to the fact that not all parameters can be estimated consistently and inverting this unstable estimate amplifies estimation errors. Making use of this inverted estimator results in weights far from optimal and will most likely not fulfill the task of minimising the variance. This emphasises the need for a fundamentally different research direction. Estimators have to be able to decrease the estimation error for the parameters and to handle high dimensions. Over the past twenty years, the bibliography on decreasing estimation error of the covariance matrix in high-dimensional settings is ever-growing. The extensive amount of data and novel methods pave the way for researchers to develop new methods while using information of only a very limited sample size.

A promising direction to estimate the covariance matrix, is to apply some sort of shrinkage. The main principle of shrinkage is to introduce bias in the estimator, such that the variance of the covariance estimator can eventually be reduced to a minimum. This results in an estimator with the optimal trade-off of minimal combined bias and variance. Shrinkage is often applied as an effective method to decrease estimation error of the covariance matrix. This is frequently done by shrinking the sample covariance matrix to a positive semi-definite, invertible matrix by making use of a structured target matrix. Target matrices are considered to have some sort of supportive structure, associated with statistical or economic theory.

Overall, the amount of parameters that has to be estimated in the target matrix is much less than in the sample covariance matrix. Therefore, it can be estimated more easily and with less estimation error. However, a target matrix can be highly biased if the underlying theories, statistical or economic, are not capturing the structure of the actual, unknown covariance matrix well. Ledoit and Wolf (2004) are among the first to make use of this promising technique. However, the nature of the problem is highly nonlinear and therefore a nonlinear transformation could deliver even better results. Therefore, Ledoit and Wolf (2012) upgrade the covariance estimation to the nonlinear domain. They find that an individualised shrinkage intensity for every sample eigenvalue, instead of a single value for all eigenvalues, reduces estimation error even more. However, the problem is mathematically more challenging, but worth the extra effort.

A different approach is to make use of factor models in order to capture cross-asset correlations. If it is possible to capture a significant part of cross-sectional variability with only a few factors, then the number of parameters in the estimation of the covariance matrix can be significantly reduced as promoted by Fan, Fan, and Lv (2008). The factors in these models can be determined to have strong economical or statistical interpretations, allowing for flexibility of this class of estimators. The implied covariance matrices from factor models can then also be used as a target in linear shrinkage models, combining the prosperous characteristics of both methods.

Aforementioned estimators fall in the class of static models, as they make use a static set of returns (and factors). Another interesting direction is to make use of dynamic models to estimate the time-varying covariance or correlation in a time series setting. One method proven to be useful is the dynamic conditional correlation (DCC) method, as described by Engle (2002). DCC models possess the flexibility of univariate GARCH models, but do not have the complex parameter estimation as in multivariate GARCH. Furthermore, DCC models are able to describe the development of the correlation as varying over time, which is a characteristic of asset returns as shown by Bollerslev (1986) among others. However, the parameters in DCC models are estimated using partial instead of full likelihood function maximisations. Due to this dependence on partial likelihoods, this method in general will be inefficient. In addition, Engle and Kelly (2012) promote dynamic equicorrelation (DECO) models. These models assume that all pairs of returns have the same correlation on a given day, but allow for time-varying correlation.

My paper extends the current literature by combining (non)linear shrinkage techniques with the dynamic equicorrelation approach. I test these methods in a portfolio management setting, in which the goal is the classical example of minimising the out-of-sample variance by constructing global minimum variance portfolios. Furthermore, I compare the out-of-sample performance with current state-of-the-art covariance estimators in extremely high dimensions. This is done in various empirical environs in order to distinguish and evaluate overall performance. Ultimately, I test whether proposed dynamic equicorrelation methods improve upon individually established (non)linear shrinkage methods in constructing covariance estimates. My research question thus states: *"To what extent can dynamic equicorrelation models with (non)linear shrinkage targets improve upon the estimation of the covariance matrix by pure (non)linear shrinkage estimators in high dimensions?"*.

I find that the extension to DECO models produces stable and invertible covariance estimates. Even without imposing a pure shrinkage prior target in the DECO process, the method is able to efficiently shrink the covariance matrix. Although, the estimators do show the capability of minimising the variance, the diverging results in the robustness analyses do not unanimously favour using DECO models above DCC, pure (non)linear shrinkage methods or factor implied shrinkage methods. In the simulation study, I find that the estimates of the inverse covariances based on DECO models converge in increasing dimensions. Also, the deviations itself show very small standard deviations, indicating stability of these methods.

The remaining of this paper is structured as follows, Section 2 lays down the problems with the sample covariance and introduces the global minimum variance portfolio formation rules. Next to that, this section describes the different techniques used to estimate the (inverse) covariance matrix along with the description of the evaluation criteria. Furthermore, this section describes shifted empirical settings for which I perform the analysis. Section 3 describes the data that I use throughout the report. In Section 4, I discuss the convergence of the proposed methods in a theoretical simulation setting. Section 5 comprises the results for the general setting as well as for the robustness analyses. Finally, Section 6 and Section 7 conclude the findings in this paper and discuss the methods and results.

2 Methodology

In this section, I first highlight the general conventional estimator of the covariance matrix. Afterwards, I introduce the portfolio variance minimisation problem, which analytical solution involves the (inverse) covariance matrix estimate. In the sections following, I review recent advanced methods that have the purpose of estimating the covariance matrix with less estimation error in high dimensions. I conclude this section with evaluation criteria and the description of robustness checks.

2.1 General Sample Covariance

Suppose asset returns are captured in some matrix Y. Let $Y_t = (Y_{1t}, \dots, Y_{Nt})'$ be the $N \times 1$ vector of asset returns for $t = 1, \dots, T$. Throughout the report, I denote the amount of assets as N and the amount of observations as T. For example, $Y_{i,t}$ is the return for asset i at time t, for $i = 1, \dots, N$ and $t = 1, \dots, T$. Suppose $E(Y) = \mu$ and $E[(Y - \mu)(Y - \mu)'] = \Sigma$. I assume Σ has a full rank equal to N. The sample mean and sample covariance are then respectively defined as $\bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t$ and $S = \frac{1}{T-1}\sum_{t=1}^{T}(Y_t - \bar{Y})(Y_t - \bar{Y})'$.

2.2 Global Minimum Variance Portfolio

I consider the optimal formation for the portfolios containing N assets to be the global minimum variance portfolio. Without short-sales constraints the problem is formulated as

$$\begin{aligned} &\min_{w} \; w'\Sigma w \\ &s.t. \; w'\mathbb{1} = 1, \end{aligned} \tag{1}$$

where Σ is the $N \times N$ covariance matrix, w is the $N \times 1$ vector of weights in the underlying assets and $\mathbb{1}$ denotes a vector of ones of size $N \times 1$. The analytical solution is given by

$$w = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}'\Sigma^{-1}\mathbb{1}}. \tag{2}$$

However, the true population covariance matrix Σ is often not observed. In that case a natural strategy is to replace the unknown Σ by a plug-in estimator $\hat{\Sigma}$, yielding feasible portfolio weights. The estimation of the GMV portfolio weights is a clean problem in terms of evaluating the quality of the estimate of the covariance matrix, since it does not require estimation of the expected returns simultaneously and hence reduces estimation error. Also, the estimated GMV portfolios have desirable out-of-sample properties according to established research. This holds not only for minimising risk, but also for attaining a relatively high reward-to-risk level as pointed out by Jagannathan and Ma (2003) among others. The ultimate goal of the GMV portfolio, by construction, is to reduce the out-of-sample variance to the achievable minimum while retaining mean-variance efficiency. I evaluate the out-of-sample standard deviation for GMV portfolios using several different methods to estimate the (inverse) covariance matrix and portfolio weights. I also include the equal-weighted $1/N$ portfolio as promoted by DeMiguel, Garlappi, and Uppal (2009) as a benchmark.

2.3 Covariance Matrix Estimators

2.3.1 Static Models

Factor Based Models

The main problem of the estimation of the covariance matrix in high dimensions is the estimation error, which is caused by a high amount of parameters that has to be estimated. As derived by Ross (1976) using the Arbitrage Pricing Theory (APT) and applied in broader economic settings by Chamberlain and Rothschild (1982), a factor model can be used to explain asset returns. Factor models are one of the most frequently used methods to reduce the amount of variables and are widely accepted as a statistical tool for modelling multivariate volatility in finance as done by Fan et al. (2008) among others.

The main idea behind factor models is that, if it is possible to capture a significant part of cross-sectional variability with only a few factors, then the number of parameters in the estimation of the covariance matrix can be significantly reduced. The general notation for such a factor model is given by

$$Y = C + BF + \varepsilon, \tag{3}$$

for an N-dimensional K-factor model where Y is the matrix of returns, C is a matrix of constants and $B = (b_1, \dots, b_N)'$ with $b_i = (b_{i,1}, \dots, b_{i,K})'$, for $i = 1, \dots, N$, are the factor loadings. Furthermore, F is the matrix of factors and ε a matrix of idiosyncratic error terms, which are assumed to be uncorrelated with the factors F. Under this model, the covariance matrix can then be estimated by plugging in the least squares estimates as follows

$$\hat{\Sigma} = \hat{B}\hat{\Sigma}_f\hat{B}' + \hat{\Sigma}_\varepsilon. \tag{4}$$

Here, $\hat{\Sigma}_f$ is the covariance matrix of the factors and $\hat{\Sigma}_\varepsilon$ is the covariance matrix of the residual terms of the factor representation in (3) and is assumed to be diagonal. The matrix of estimated regression coefficients can be written as $\hat{B} = YF'(FF')^{-1}$ and is achieved by making use of a linear regression in the `sklearn` module in Python. Apply the Sherman-Morrison-Woodbury formula to (4) to derive the estimated inverted covariance matrix

$$\hat{\Sigma}^{-1} = \hat{\Sigma}_\varepsilon^{-1} - \hat{\Sigma}_\varepsilon^{-1}\hat{B}[\hat{\Sigma}_f^{-1} + \hat{B}'\hat{\Sigma}_\varepsilon^{-1}B]^{-1}\hat{B}'\hat{\Sigma}_\varepsilon^{-1}. \tag{5}$$

The general form of the factor model is defined without actually specifying the (amount of) underlying factors and the choice for factors remains. Any factors that have statistical or economical explanatory power can be chosen. I consider the well-known one factor CAPM model as introduced by Sharpe (1964) as well as the 3- and 5-factor models as introduced by Fama and French (1993, 2015), all exactly specified. Furthermore, a latent factor model based on sparse Principal Component Analysis (PCA) is discussed in the next section.

POET

As noted in the general formulation of the factor-based approach, the factors are left to be specified. Instead of making use of economic theories underlying the factors, it is possible to approach this problem by inferring factors based on characteristics of the original data by means of PCA. Fan, Liao, and Mincheva (2013) deal with the situation in which the factors are unobserved and must be inferred. Due to the estimation of unobserved factors less, possibly incorrect, structure is assumed, but this comes at the cost of a decrease in the rate of convergence. However, as N gets large enough, this effect becomes negligible and the covariance matrix can be estimated as if the factors are known.

This approach is simple and optimisation free as it uses the data only through the sample covariance matrix. This can be done by firstly running the singular value decomposition on the sample covariance matrix of the returns. Then, isolate the covariance matrix that is formed by the first K principal components and apply a thresholding procedure to the remaining covariance matrix. This results in a principal orthogonal complement thresholding (POET) estimator. By making use of the underlying data the optimal number of common factors K can be determined. The resulting estimated covariance matrix can be decomposed in the spectral domain as

$$\hat{\Sigma} = \sum_{i=1}^{K} \hat{\lambda}_i \hat{\xi}_i \hat{\xi}_i' + \hat{R}^T, \tag{6}$$

where K is the number of (estimated) diverging eigenvalues of Σ, $\hat{\lambda}_i$ are the ordered eigenvalues and $\hat{\xi}_i$ the corresponding eigenvectors. In my applications, I choose K to be the optimal solution to the minimisation problem as described by Bai and Ng (2002), such that

$$\hat{K} = \arg \min_{0 \leq K \leq M} \log \left(\frac{1}{NT} ||Y - \frac{1}{T} Y \hat{P} \hat{P}'||_F^2 \right) + K g(T, N). \tag{7}$$

Here, M is an arbitrary large upper bound smaller or equal to N, Y is the $N \times T$ returns matrix, $\hat{P}$ is a $T \times K$ matrix, whose columns are $\sqrt{T}$ times the eigenvectors corresponding to the K largest eigenvalues of the $T \times T$ matrix $Y'Y$. Furthermore, $g(T, N)$ is a penalty function, for which I use the first information criterion as given by Bai and Ng (2002), $g(T, N) = \frac{N+T}{NT} \log \left(\frac{NT}{N+T} \right)$.

Furthermore, I define $\hat{R}^T$, the principal orthogonal complement after thresholding, as

$$\hat{R}^T = (\hat{r}_{ij}^T)_{N \times N}, \qquad \hat{r}_{ij}^T = \begin{cases} \hat{r}_{ii} & i = j, \\ s_{ij}(\hat{r}_{ij}) I(|\hat{r}_{ij}| \geq \tau_{ij}) & i \neq j, \end{cases} \tag{8}$$

where $s_{ij}(\cdot)$ is the generalised shrinkage function of Antoniadis and Fan (2001) and $\tau_{ij} > 0$ is an entry-dependent threshold parameter. The threshold can be chosen such that it is flexible and can incorporate hard thresholding rules such as $s_{ij}(x) = xI(|x| \geq \tau_{ij})$ and constant thresholding such as $\tau_{ij} = \delta$. However, in practice the most useful application is by making use of adaptive thresholding as follows

$$\tau_{ij} = \tau \sqrt{\hat{r}_{ii} \hat{r}_{jj}}, \ \forall \tau > 0, \tag{9}$$

where $\hat{r}_{ii}$ is the i-th diagonal element of the principal orthogonal complement before thresholding, $\hat{R}$. This is equivalent to applying the threshold with parameter τ to the correlation matrix of $\hat{R}$. Combining the thresholded matrix and the matrix from the first K eigenvalues and eigenvectors results in the POET covariance matrix estimator in (6).

In order to denote the estimated inverse covariance matrix, I write $\hat{\Lambda} = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_K)$ as the first K leading empirical eigenvalues and $\hat{\Gamma} = (\hat{\xi}_1, \ldots, \hat{\xi}_K)$ as the matrix of corresponding leading empirical eigenvectors. Then, the POET estimator for the inverse covariance matrix can be defined, again by using Sherman-Morrison-Woodbury, as

$$\hat{\Sigma}^{-1} = \hat{R}^{T^{-1}} - \hat{R}^{T^{-1}}\hat{\Gamma}(\hat{\Lambda} + \hat{\Gamma}'\hat{R}^T\hat{\Gamma})^{-1}\hat{\Gamma}'\hat{R}^T. \tag{10}$$

Linear Shrinkage

Ledoit and Wolf (2004) argue that, without prior knowledge about the true covariance matrix, an imposed (factor) structure is in general misspecified and can exhibit large amounts of bias. Therefore, a different approach for promising estimators of the covariance matrix in high dimensions, is to combine the structure and interpretation of the pure factor-based methods with the characteristics of the data in the sample covariance matrix. A combined estimator inherits good conditioning properties and can be more accurate than the individual estimators. This class of estimators is generally known as linear shrinkage estimators.

The general idea of shrinkage is to find the optimal trade-off between error due to bias and error due to variance. This optimum is reached by introducing bias to minimise the variance of the estimator. This trade-off was already central to shrinkage techniques described by Stein (1956). Ledoit and Wolf (2004) introduce a class of estimators that have the linear optimal shrinkage solution. In this sense, optimality is meant with respect to some quadratic loss function and asymptotically as the amount of observations and the amount of variables go to infinity together. In its general form, a linear shrinkage estimator of the covariance matrix is given by

$$\hat{\Sigma} = \alpha\hat{\Phi} + (1-\alpha)S, \tag{11}$$

where $\alpha \in [0, 1]$ and denotes the shrinkage intensity, S is the sample covariance matrix and $\hat{\Phi}$ is the shrinkage target, which can be the identity matrix or 1-factor model implied covariance matrix as respectively used by Ledoit and Wolf (2004) and Ledoit and Wolf (2003) or another well-behaved target matrix.

As discussed before, the shrinkage intensity parameter α is chosen such that it is optimal with respect to some loss function, $L(\alpha)$. This loss function is defined as a quadratic measure of distance between the sample and the true covariance matrix. In my research I base the loss function on the Frobenius norm, similar to Ledoit and Wolf (2004). Furthermore, for the target matrix, the identity matrix as well as the 1-factor model implied covariance estimate as in (4) are evaluated. For the formulas and proofs regarding asymptotically optimal shrinkage intensity solutions, I refer to Ledoit and Wolf (2003, 2004).

Nonlinear Shrinkage

Linear shrinkage can heavily improve estimation of covariance matrices by making use of sample eigenvalues. As Ledoit and Wolf (2004) show, sample eigenvalues can be improved in the estimation of the covariance matrix by replacing them with convex linear combinations with all elements shrunk using a single intensity. However, when linear shrinkage does not sufficiently improve the estimation of the covariance matrix, extension to the nonlinear domain might be preferred. The intuition behind nonlinear shrinkage is that it improves upon linear shrinkage by applying different shrinkage intensities for different eigenvalues. By relaxing the estimation of the covariance matrix to the entire parameter space, instead of just the linear part, the nonlinear estimator often performs better than the linear variant as shown by Ledoit and Wolf (2012) among others. In finite-sample, the linear shrinkage estimator rarely outperforms its nonlinear counterpart, only when the linear shrinkage estimator is nearly optimal already.

By proofs following from arguments introduced by Marčenko and Pastur (1967), which relate sample and population eigenvalues, it is possible to show that linear shrinkage is the first-order approximation of a fundamentally nonlinear problem. Furthermore, the findings of Marčenko and Pastur (1967) imply that there exists a limiting sample spectral distribution such that

$$F_N(x) \xrightarrow{\text{a.s.}} F(x), \forall x \in \mathbb{R}\backslash\{0\}. \tag{12}$$

The limit in (12) states that the average number of sample eigenvalues falling in any given interval is asymptotically known. In order to define the relation between sample eigenvalues and population eigenvalues, it is useful to make use of the Stieltjes transform to the complex domain. For any non-decreasing function $G(\cdot)$ on the real line, m_G denotes the Stieltjes transform of $G(\cdot)$ such that

$$m_G(z) = \int \frac{1}{\lambda - z} dG(\lambda), \forall z \in \mathbb{C}^+, \tag{13}$$

where λ denotes the eigenvalues and $\mathbb{C}^+$ denotes the positive half-plane of complex numbers with a strictly positive imaginary part. The empirical distribution function of the population eigenvalues is defined by the limiting spectral distribution function, which is specified as

$$H(t) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[\lambda_i, +\infty)}(t), \forall t \in \mathbb{R}. \tag{14}$$

Here, N denotes the amount of eigenvalues, λ_i denotes the eigenvalues for $i = 1, \ldots, N$ and $\mathbb{1}$ denotes the indicator function of a set. Silverstein (1995) states that the unique solution, relating $F(\cdot)$ to $H(\cdot)$, is the solution to the equation

$$m_F(z) = \int_{\infty}^{+\infty} \frac{1}{\lambda \left[1 - c - czm_F(z)\right] - z} dH(\lambda), \forall z \in \mathbb{C}^+, \tag{15}$$

where $H(\lambda)$ is the limiting spectral distribution function as in (14), $c = \frac{N}{T}$ and $m_F(z)$ is the Stieltjes transform of the sample spectral distribution $F(\lambda)$. Next to that, Silverstein and Choi (1995) show that

$$\lim_{z \in \mathbb{C}^+ \to \lambda} m_F(z) = \breve{m}_F(\lambda), \forall \lambda \in \mathbb{R}\backslash\{0\}. \tag{16}$$

Furthermore, let the linear operator L transform the c.d.f. of $H(\lambda)$ into

$$LH(\lambda) = \int_{-\infty}^{x} \lambda dH(\lambda). \tag{17}$$

Combining the transformed c.d.f. with the Stieltjes transform results in

$$m_{LH}(z) = \int_{-\infty}^{+\infty} \frac{\lambda}{\lambda - z} dH(\lambda) = 1 + z m_H(z). \tag{18}$$

Then, for every λ in the interior of the support function $Supp(F)$, which is defined as in Silverstein and Choi (1995), there exists a unique $v_\lambda \in \mathbb{C}^+$ such that

$$v_\lambda - c v_\lambda m_{LH}(v_\lambda) = \lambda \tag{19}$$

has a unique minimising solution $\hat{v}_\lambda \in \mathbb{C}^+$, which is uniform in λ. Then, it is able to find a consistent estimator of $\breve{m}_F(\lambda)$, also uniformly in λ, given by

$$\breve{m}_F(\lambda) = \frac{1-c}{c\lambda} - \frac{1}{c}\frac{1}{\hat{v}_\lambda}. \tag{20}$$

Using these preliminaries, Ledoit and Wolf (2012) develop an estimator based on the decomposition of the covariance matrix in the spectral domain, which can be represented as

$$\Sigma^* = U L^* U'. \tag{21}$$

Here, $L^* = \text{diag}(l_1^*, \ldots, l_N^*)$ with $l_i^* = u_i' \Sigma u_i$ for $i = 1, \ldots, N$ and U is the matrix of eigenvectors whose i-th column is the sample eigenvector belonging to the i-th sample eigenvalue. The interpretation of l_i^* is that it captures how the i-th sample eigenvector, u_i, relates to the population covariance matrix Σ as a whole. However, this matrix is unobserved and thus has to be estimated. For this purpose Ledoit and Péché (2011) show, by generalising the relationship found by Marčenko and Pastur (1967) between sample and population eigenvalues under large-dimensional asymptotics, that l_i^* can be approximated by the oracle quantity

$$\hat{l}_i^{or} = \frac{\lambda_i}{|1 - c - c\lambda_i \breve{m}_F(\lambda_i)|^2}, \tag{22}$$

where λ_i are the sample eigenvalues for $i = 1, \ldots, N$, $c = \frac{N}{T}$ and $\breve{m}_F(\lambda_i)$ is defined as in (20). Then, the oracle estimator of the covariance matrix is given by

$$\hat{\Sigma}^{or} = U \hat{L}^{or} U', \tag{23}$$

where $\hat{L}^{or} = \text{diag}(\hat{l}_1^{or}, \ldots, \hat{l}_N^{or})$ is the diagonal matrix of shrunk eigenvalues and U is the matrix of accompanying, unchanged eigenvectors. This estimator is feasible and consistent as is does not depend on the unobserved population covariance matrix, but only on the limiting distribution of sample eigenvalues λ_i and the consistent estimator of the unknown quantity $\breve{m}_F(\lambda_i)$. Note that $\hat{\Sigma}^{or}$ constitutes a nonlinear shrinkage estimator, since the value of the denominator of l_i^{or} varies per eigenvalue λ_i. Therefore, the shrunk eigenvalues l_i^{or} are obtained by applying a nonlinear transformation to the sample eigenvalues.

I make use of the derivation of the consistent, so-called bona fide, estimator as in (23) and described by Ledoit and Wolf (2012). This bona fide estimator approaches the oracle estimator and is not dependent on an unknown quantity for $N < T$. Furthermore, Ledoit and Wolf (2015) extend the estimation for the $N > T$ case. For additional details and convergence proofs I refer to their works. I have consulted Michael Wolf's published MATLAB code[1] in order to produce feasible (non)linear shrinkage estimators in Python.

2.3.2 Dynamic Models

DCC Models

Before 1980 most theoretical time series derivations assumed the variance to be fixed over time. However, there is an ever-growing library of research indicating that variance is not fixed but varying over time, hence conditional. Engle (2002) proposes Dynamic Conditional Correlation (DCC) estimators, which greatly simplify multivariate specifications while allowing the conditional correlation to vary over time. DCC estimators can have the flexibility of univariate GARCH models, but not the troublesome complexity of conventional multivariate GARCH models. The main idea behind DCC-type models is to first estimate individual conditional covariance and GARCH model parameters for all of the underlying assets and then use these estimates conditionally to estimate the parameters of the dynamic model. The general formulation of a univariate GARCH conditional variance model, as introduced by Bollerslev (1986), is given by

$$h_{i,t} = \omega_i + a_i y_{i,t-1}^2 + b_i h_{i,t-1}, \tag{24}$$

where $h_{i,t}$ is the conditional variance for asset i at time t, ω_i can be seen as some sort of asset specific mean-reverting variance which has to be estimated. Furthermore, $y_{i,t-1}$ denotes the return of asset i at time $t-1$. The variance parameters in this model (ω_i, a_i, b_i) can be estimated by maximum likelihood, for this purpose I make use of the implemented `arch` module in Python. Once those parameters have been estimated, the next step is to eliminate the conditional volatility from the residuals, sometimes called "de-GARCHing" in literature. The standardised residuals are given by

$$\epsilon_t = D_t^{-1} y_t, \qquad \text{where} \qquad D_t^{-1} = \text{diag}\left(\sqrt{h_{i,t}}\right). \tag{25}$$

In general, changes in correlations appear to be temporary and mean-reverting. A DCC specification that captures this assumption is analogous to the GARCH model in (24) and can be denoted as

$$Q_t = \Omega + \alpha \epsilon_{t-1} \epsilon_{t-1}' + \beta Q_{t-1}, \tag{26}$$

[1] `https://www.econ.uzh.ch/en/people/faculty/wolf/publications.html`

where Q_t is an approximation to the conditional correlation matrix, Ω can be seen as a matrix which Q_t reverts to and ϵ_{t-1} are the standardised residuals as in (25). The DCC parameters (Ω, α, β) are analogous to the variance parameters in (24), except that these are correlation parameters instead of variance parameters. Unfortunately, the amount of model parameters that has to be estimated is quite high ($\frac{1}{2}N(N+1)+2$ for N assets), which can lead to computational issues. To overcome this, an adjustment can be made by fixing the mean-reverting matrix Ω to a pre-specified estimate of the correlation matrix. For example, the unconditional correlation matrix of the standardised residuals can be used. Hence, it is possible to write

$$\bar{R} = \frac{1}{T}\sum_{t=1}^{T} \epsilon_t \epsilon_t' \qquad \text{and} \qquad \hat{\Omega} = (1-\alpha-\beta)\bar{R}. \tag{27}$$

Substitute this into the specification in (26) to write the DCC process as

$$Q_t = \hat{\Omega} + \alpha \epsilon_{t-1}\epsilon_{t-1}' + \beta Q_{t-1}. \tag{28}$$

Note that the amount of parameters that has to be estimated is heavily reduced to only two (α, β). Based on this representation the conditional correlation and conditional covariance matrices are given by

$$R_t^{DCC} = \text{diag}(Q_t)^{-\frac{1}{2}} Q_t \text{diag}(Q_t)^{-\frac{1}{2}}, \tag{29}$$

$$\hat{\Sigma}_t = D_t R_t^{DCC} D_t. \tag{30}$$

DCC models are designed such that they can account for high-dimensional systems, but as the size increases full maximum likelihood estimation becomes cumbersome. A way to overcome this is to look at pairs of assets and make use of the composite likelihood method as described by Pakel, Shephard, Sheppard, and Engle (2020). The composite log-likelihood method is computed by summing up the log-likelihoods of pairs of assets while conditioning on the information from the GARCH estimates in the first step. This possesses great flexibility, but is heavily dependent on a partial likelihood which boasts inefficiency.

The derivation of the conditional likelihood of DCC models is shown in Appendix A. I maximise the likelihood by making use of the differential evolution optimiser in the `SciPy` library. Due to the large amount of likelihood maximisations and (constrained) optimisation combinations, running time easily exceeds days for large moving windows. Therefore, I limit the moving window to $T = 50$ observations for the dynamic models. I consider portfolio formations based on DCC models, which are estimated with the composite likelihood method as in Engle, Ledoit, and Wolf (2019). I make use of methods where the target matrix, $\hat{\Omega}$, is given by the sample correlation matrix of standardised residuals, the linearly and nonlinearly shrunk correlation matrix estimates.

DECO Models

Due to the reliance of the DCC methods on the partial likelihood, the approach will in general be inefficient. Also, the specification of structural assumptions is controversial and prone to errors. Engle and Kelly (2012) solve this trade-off between imposing (unknown) structure and (inefficient) composite likelihood by introducing a system in which all pairs of returns have the same correlation on any given time, but allowing for time-varying correlation. This model is called the Dynamic Equicorrelation (DECO) model and eliminates computational and presentational difficulties of high-dimensional systems. DECO correlations of assets i and j depend on all return histories of all pairs of available assets, whereas DCC models only take the history of those two assets into account. Therefore, DECO models draw more useful information from a broader set of observations and are able to capture a pooling aspect. In a DECO setting, a matrix R_t^{DECO} is an equicorrelation matrix of $N \times 1$ vectors of random variables if it is positive definite and takes the form of

$$R_t^{DECO} = (1 - \rho_t)I + \rho_t J, \qquad \text{where} \qquad \rho_t = \frac{1}{N(N-1)}(\iota' R_t^{DCC} \iota - N). \tag{31}$$

Here, R_t^{DCC} is the conditional correlation matrix of standardised returns as in (29), I is a $N \times N$ identity matrix and J is a $N \times N$ matrix of ones. The main idea is to shrink the off diagonal elements of R_t^{DCC} to their average. The estimate of the covariance matrix in this setting is given by

$$\hat{\Sigma}_t = D_t R_t^{DECO} D_t, \tag{32}$$

where D_t is the diagonal matrix of conditional individual volatilities and R_t^{DECO} is the estimate of the dynamic equicorrelation matrix as in (31). The inverse of R_t^{DECO} is then given by

$$R_t^{DECO^{-1}} = \frac{1}{1 - \rho_t} I - \frac{\rho_t}{(1 - \rho_t)(1 + [N-1]\rho_t)} J. \tag{33}$$

Note that $R_t^{DECO^{-1}}$ only exists if and only if $\rho_t \neq 1$ and $\rho_t \neq \frac{-1}{N-1}$. Furthermore, R_t is positive definite if and only if $\rho_t \in (\frac{-1}{N-1}, 1)$. If these prerequisites hold, the estimate of the inverse covariance matrix can be computed. Again, I consider portfolio formations based on DECO models where the target matrix, $\hat{\Omega}$, is given by the sample correlation matrix of standardised residuals, the linearly and nonlinearly shrunk correlation matrix estimates.

2.4 Overview of Models of Interest

Table 1 shows the portfolio formations and methods to construct the covariance matrix with the objective of minimising the variance as specified in Section 2.2. All calculations and estimations of the covariance matrices are done in Python 3.7.4.

Table 1: Overview of methods to construct the covariance matrix and form the portfolios.

#	Method	Description
1	1/N	The equal weighted portfolio as promoted by DeMiguel et al. (2009). This can be seen as a benchmark portfolio.
2	Sample	The portfolios where $\hat{\Sigma}$ is given by the sample covariance matrix. This portfolio is in general not available when $N > T$, since the sample covariance matrix is not invertible in this case.
Static models		
3	1F	The portfolios where $\hat{\Sigma}^{-1}$ is given by the inverse of the exact one-factor implied covariance as in (5).
4	3F	The portfolios where $\hat{\Sigma}^{-1}$ is given by the inverse of the exact three-factor implied covariance as in (5).
5	5F	The portfolios where $\hat{\Sigma}^{-1}$ is given by the inverse of the exact five-factor implied covariance as in (5).
6	POET	The portfolios where $\hat{\Sigma}^{-1}$ is given by the inverse POET estimator as in (10).
7	Lin1	The portfolios where $\hat{\Sigma}$ is given by the linear shrinkage estimator as in (11) with the identity matrix as target matrix $\hat{\Phi}$.
8	Lin2	The portfolios where $\hat{\Sigma}$ is given by the linear shrinkage estimator as in (11) with the one factor implied covariance matrix as target matrix $\hat{\Phi}$.
9	NL	The portfolios where $\hat{\Sigma}$ is given by the nonlinear shrinkage estimator as in (23).
Dynamic models		
10	DCC-S	The portfolios where $\hat{\Sigma}$ in (30) is obtained with the sample correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
11	DCC-L1	The portfolios where $\hat{\Sigma}$ in (30) is obtained with the pre-processed $Lin1$ correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
12	DCC-L2	The portfolios where $\hat{\Sigma}$ in (30) is obtained with the pre-processed $Lin2$ correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
13	DCC-NL	The portfolios where $\hat{\Sigma}$ in (30) is obtained with the pre-processed NL correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
14	DECO-S	The portfolios where $\hat{\Sigma}$ in (32) is based on the inverse equicorrelation matrix R_t^{-1} as in (33) with the sample correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
15	DECO-L1	The portfolios where $\hat{\Sigma}$ in (32) is based on the inverse equicorrelation matrix R_t^{-1} as in (33) with the pre-processed $Lin1$ correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
16	DECO-L2	The portfolios where $\hat{\Sigma}$ in (32) is based on the inverse equicorrelation matrix R_t^{-1} as in (33) with the pre-processed $Lin2$ correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.
17	DECO-NL	The portfolios where $\hat{\Sigma}$ in (32) is based on the inverse equicorrelation matrix R_t^{-1} as in (33) with the pre-processed NL correlation matrix as target matrix $\hat{\Omega}$ in the DCC process.

2.5 Evaluation Criteria

I report the following out-of-sample performance measures in order to compare the different high-dimensional covariance estimator methods with each other.

- **Average (Avg):** I compute the average of the out-of-sample returns in excess of the risk-free rate, multiplied by 250 to annualise.

- **Standard Deviation (Std):** I compute the standard deviation of the out-of-sample excess returns multiplied by $\sqrt{250}$ in order to annualise. This performance measure is minimised by construction in (1), with lower values indicating the strength of a method to construct a GMV portfolio with the lowest variance.

- **Sharpe Ratio (SR):** The annualised Sharpe ratio is calculated as the annualised average divided by the annualised standard deviation. The Sharpe ratio gives an intuition of the risk-adjusted return per unit of risk, where a higher Sharpe ratio is superior for a mean-variance investor.

- **Utility (Util):** For all methods I compute the out-of-sample expected utility as a performance measure of using portfolio weights $\hat{w}$ by

$$\tilde{U}(\hat{w}) = \hat{w}'\hat{\mu} - \frac{\gamma}{2}\hat{w}'\hat{\Sigma}\hat{w}.$$

  For this calculation I use the conventional value of $\gamma = 3$. The value of utility is ordinal and is used in order to determine the preference for a utility investor to assess the relative indifference for the optimal solutions. I compute the average realised utility for all periods and the relative gain (loss) in utility due to diversifying from the $1/N$ portfolio construction. This can be calculated as using portfolio weights $\hat{w}$ opposed to equal weighting $w^{\frac{1}{N}}$ raising the loss function

$$L(\hat{w}, w^{\frac{1}{N}}) = \tilde{U}(\hat{w}) - \tilde{U}(w^{\frac{1}{N}}).$$

- **Turnover (To):** I compute the turnover as the difference between the weights of the underlying assets in the portfolio at rebalancing dates. Turnover is calculated as

$$\text{Turnover} = \frac{1}{M-T-1}\sum_{t=1}^{M-1}\sum_{j=1}^{N}(|w_{k,j,t+1} - w_{k,j,t+}|).$$

  Here M is the total amount of observations and T is the amount of in-sample observations used for construction. Furthermore, $w_{k,j,t+1}$ is the desired asset weight at $t+1$ and $w_{k,j,t+}$ is the weight in the same asset before rebalancing. Turnover is a proxy for transaction costs, with high turnover indicating large deviations in weights between investment universes. In order to maintain the optimal solution, an investor has to buy or sell stocks and possibly even lever their position, which all lead to an increase in unfavourable transaction costs. Turnover is only calculated and shown for the portfolios which contain assets which have a total return history for the complete out-of-sample period.

2.6 Robustness Analyses

In this section, I describe shifted environments for testing the capabilities of the described methods in order to construct a (high-dimensional) covariance matrix estimate that minimises the out-of-sample variance. The first three robustness analyses are included in Section 5, whereas the latter three are discussed in Appendices B.4, B.5 and B.6

Low Market Capitalisation Habitat

In the general set-up, I choose N assets to construct GMV portfolios based on a random selection from the largest market capitalisation stocks at the investment dates. This selection procedure makes sure that the underlying assets consist of stocks of the largest established companies and those are often frequently traded stocks. However, there resides a well-known size effect, as described by Banz (1981) and Fama and French (2008) among others, for low market capitalisation stocks as their risk-adjusted returns are significantly higher and often unexplained by asset pricing models. Therefore, I also construct GMV portfolios consisting of N assets with the lowest market capitalisation at the investment dates. The performance of the methods in minimising the variance for the formed portfolios is compared with the regular asset selection procedure.

Short-sale Constraints

Some fund managers are bounded to short-sale constraints, due to for example transaction costs of leveraging, or risk exposure. Therefore, I also perform and analyse the performance of described methods while imposing a lower bound of zero on all weights in the portfolio. The non-negative weight constraint is added to the minimisation problem in (1) such that the new minimisation problem is given by

$$\min_{w} \; w'\Sigma w \tag{34}$$
$$s.t. \; w'\mathbb{1} = 1, w \geq 0.$$

This problem can be solved for w with constrained optimisation techniques. As Jagannathan and Ma (2003) state, this optimisation on its own is a different form of shrinkage as imposed from the weights instead of the covariance matrix. I discuss the effects of imposing shrinkage both through the weights and the covariance matrix as imposed by the different methods.

Subperiod Analysis

The out-of-sample period comprises 285 months (or 6,237 days). It might be possible that the performance of certain models is driven by superiority in subperiods, but does not hold universally. I address this concern by dividing the out-of-sample period into three subperiods of 95 months (or 2,079 days) each and repeat the described procedures in each subperiod. Also, a subperiod analysis allows for a more thorough analysis of how the different methods perform during periods of crises.

Longer Formation Period

In the general setup, at any investment date h, the covariance matrix is estimated using the most recent $T = 250$ daily returns, corresponding roughly to one year of past data. As a robustness check, I also use the most recent $T = 500$ daily returns, corresponding roughly to two years of past data, to construct the covariance matrix. This procedure implies that the sample covariance is relatively stable for a higher amount of assets. This method shortens the out-of-sample period by 250 trading days from the start. Also, the sample covariance is now invertible for the asset space consisting of $N = 250$ assets. Besides that, the methods remains equal. The results are discussed in Appendix B.4.

Different Return Frequency

Throughout the report, I use observed daily stock returns as these are widely available from 1962 onward. To measure the ability of the methods of minimising the variance for different frequencies and magnitudes of asset returns, I download and use monthly data from CRSP from 28 June 1985 through 31 December 2019. In this case, I use the $T = 120$ most recent months, equalling ten years of monthly asset returns, to estimate the covariance matrix and subsequently hold the constructed portfolio for one month. Consequently, the out-of-sample investment period yields 295 out-of-sample returns. The remaining details are as before. The results are discussed in Appendix B.5.

Portfolios as Assets

So far, I focused on individual stocks as underlying assets in the formed portfolios. This is assumed to be the most relevant case for fund managers. On the other hand, in academic research often portfolios are used as traded assets and generally the variance of a portfolio is lower than that of individual assets. To check the robustness of the findings for the implemented models, I perform the same analysis with sorted portfolios as assets instead of individual stocks. I consider three sorted portfolio sets of size $N = 100$ from Kenneth French's Data Library[2], the sorted portfolios consist of:

- 100 Portfolios formed on size and book-to-market
- 100 Portfolios formed on size and operating profitability
- 100 Portfolios formed on size and investment

For this implementation I use daily data, the out-of-sample period and the formation period are equal to the general set-up. The results are discussed in Appendix B.6.

[2] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

3 Data

For this research, I make use of daily stock return data from the Center for Research in Security Prices (CRSP). The CRSP database covers historical data on US equities for equities listed on the NYSE, NASDAQ and ARCA. The data set of interest starts from 13 February 1995 up to 17 December 2019, totalling 6,237 trading days. For simplicity, I adopt the common convention that 21 consecutive trading days constitute one month.

At any investment date, h, a covariance matrix is estimated using the most recent $T = 250$ daily returns, corresponding roughly to one year of past data. I construct GMV portfolios using these estimated covariance matrices. The formed portfolios are then held for one month, without adjusting or rebalancing the weights of the underlying assets in the portfolio. Thus, the procedure follows a regular buy-and-hold strategy. The portfolios are updated on a monthly basis, this is done in order to prevent unreasonable amounts of turnover which in turn lead to unfavourably high transaction costs and this is common practice among academics and portfolio managers. As a result, the out-of-sample period ranges from 9 February 1996 through 17 December 2019, resulting in a total of 285 months (or 5,987 days). I consider GMV portfolios of assets of sizes $N \in \{30, 50, 100, 250, 500, 1000\}$. This range covers the majority of the sizes of important stock indexes, such as the Dow Jones Industrial Average, NASDAQ and S&P.

For a given combination (h, N), the investment universe is obtained as follows. Firstly, I determine the 1000 largest stocks, as measured by their market capitalisation on the investment date h, that have a complete return history over the most recent $T = 250$ days as well as a complete return history over the next 21 days. Then, I filter for possible pairs of highly correlated stocks, which exceed a sample correlation of 0.95 over the training observations. I leave out the stocks with the lowest market capitalisation of the pair and iteratively add new stocks until the sample comprises $N = 1000$ stocks again. Then, I select N stocks at random from those 1000 stocks, ensuring for example that the assets for the GMV portfolio containing $N = 30$ assets are also part of the GMV portfolio containing $N = 50$ together with 20 other randomly selected assets. This formation procedure holds for all test sets consisting of increasing amounts of N assets. These N randomly selected stocks constitute the investment universe for the upcoming 21 days. As a result, there are $h = 285$ investment universes possibly containing different assets for different investment dates h over the out-of-sample periods.

Figure 1 shows histograms of the average returns and average standard deviation, for all 285 test sets for $N = 1000$ assets. Table 2 shows the average returns, denoted as $\bar{r}$, and average standard deviations, denoted as $\bar{\sigma}$, over all test set observations for the formed portfolios and for the market. The GMV portfolios for different values of N are for a large part overlapping as the portfolio containing 50 assets contains all assets from the portfolio containing 30 assets and 20 randomly selected other assets. As can be seen in the table, average asset returns and standard deviations are very close to each other for the different dimensions, with returns very close to zero and identical up to 1 basis point for all portfolios of N assets and average standard deviations within a range of around 1% for all N. Furthermore, it can be seen that the market is scoring relatively high from a mean-variance perspective, as for the sample on average it attains higher average daily returns and lower volatility than for any portfolio containing N randomly selected individual assets.

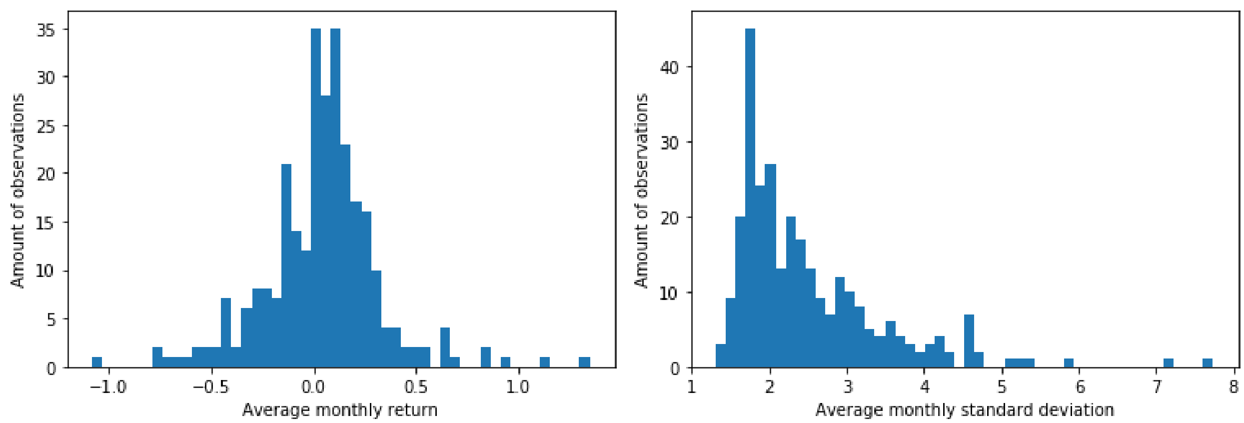

Figure 1: Average returns and average standard deviations in percentages calculated for each of the 285 test sets for the portfolio containing $N = 1000$ assets. The sample period ranges from 13 February 1995 to 17 December 2019.

Table 2: Average returns and average standard deviations in percentages over all test observations for the portfolios of assets. The sample period ranges from 13 February 1995 to 17 December 2019.

N	30	50	100	250	500	750	1000	Mkt
$\bar{r}$	0.0295	0.0342	0.0350	0.0344	0.0394	0.0416	0.0425	0.0466
$\bar{\sigma}$	1.5023	1.6093	1.6850	1.7649	2.2538	2.4202	2.4995	1.0796

4 Simulation

The described methods only focus on empirical asset returns. To check whether the results also hold true in a theoretical setting, I perform a simulation study in Python with returns simulated from a factor model. Therefore, I make use of a simulation similar to the one described by Fan, Liao, and Mincheva (2011). The procedure is split up in a calibration part and a simulation part. I choose to simulate the returns from a 5-factor model, $K = 5$, with the amount of simulated observations equal to one year, $T = 250$, and the same dimensions as in the empirical setting, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$.

Calibration Part

1. First estimate the parameters in the Fama and French (2015) 5-factor model by using roughly two years of daily data (500 observations) from 22 December 2017 to 19 December 2019 of the 100 portfolios sorted on size and book-to-market value. Perform least squares regression $y_t = Bf_t + u_t$ and take the columns of $B = (b_1, b_2, b_3, b_4, b_5)$, such that B is a 100×5 matrix. Gather sample means as μ_B and sample covariance as Σ_B. These sample moments are used for drawing factor loadings from the multivariate normal distribution $N_5(\mu_b, \Sigma_B)$.

2. Let $u_t = y_t - Bf_t$, then for each portfolio i, let $\hat{\sigma}_i$ be the standard deviation of the residuals of the i-th portfolio. Store minimum and maximum values as well as sample mean and standard deviation of $\hat{\sigma}$. Let $V = \text{diag}(\sigma_1^2, \dots, \sigma_N^2)$, where the σ_i's are generated independently from the Gamma distribution $G(\alpha, \beta)$ with mean $\alpha\beta$ and standard deviation $\alpha^{\frac{1}{2}}\beta$ respectively matched to the sample average and sample standard deviation of $\hat{\sigma}$ and obtain α and β. Create a loop that only accepts a value for σ if it is between the minimum and maximum value of $\hat{\sigma}$. Furthermore, generate s_i, where s_i is drawn from $N(0, 1)$ with probability $\frac{0.2}{\sqrt{N log(N)}}$ and $s_i = 0$ otherwise for $i = 1, \dots, N$. For each dimension N, create sparse matrix $\Sigma_u = V + ss' - diag(s_1^2, \dots, s_N^2)$. Create a loop that generates Σ_u until it is positive definite.

3. Assume the factors follow a VAR(1) model $f_t = \mu + \Phi f_{t-1} + \epsilon_t$ for a 5×5 matrix Φ and where ϵ_t is simulated i.i.d. from a $N_5(0, \Sigma_\epsilon)$ distribution for $t = 1, \dots, T$. I estimate Φ, μ and Σ_ϵ from the data and obtain Σ_f.

Simulation Part

For each dimension, N, independently generate return series y_t according to the factor model with calibrated parameters. I generate $T = 250$ returns per asset, roughly equal to one years of daily returns.

1. Generate b_i independently from $N_5(\mu_B, \Sigma_B)$ for $i = 1, \dots, N$ and set $B = (b_1, \dots, b_N)'$.

2. Generate u_t independently from $N_N(0, \Sigma_u)$ for $t = 1, \dots, T$.

3. Generate f_t independently from the VAR(1) model $f_t = \mu + \Phi f_{t-1} + \epsilon_t$ for $t = 1, \dots, T$.

4. Calculate $y_t = Bf_t + u_t$ for $t = 1, \dots, T$.

With this simulated returns, y_t, it is possible to follow the same procedure in forming an estimate of the covariance matrix as through the empirical approach. However, now the true covariance matrix from which the daily returns are simulated from is known. In order to check the capabilities of the different methods to minimise the variance I compute and compare the distance between the true inverse covariance matrix and its estimate, $\frac{1}{\sqrt{N}}||\hat{\Sigma}^{-1} - \Sigma^{-1}||_F$. I make use of 200 simulations and display averages and standard deviations for increasing dimensions with respect to the amount of observations.

Figure 2 and 3 show the average and standard deviation of $\frac{1}{\sqrt{N}}||\hat{\Sigma}^{-1} - \Sigma^{-1}||_F$ for the simulated returns. It shows the deviation between the constructed inverted covariance matrix from the simulated data and the true inverse covariance matrix for 200 simulations. As observed all methods outperform the sample covariance matrix in constructing the true inverted covariance matrix from $N > 50$. It can also be seen that the sample covariance matrix becomes singular and thus very ill-behaved from $N = 250$.

Furthermore and as expected, the 5-factor implied covariance estimator and POET estimators are among the best performing methods. By construction, the data is simulated from a 5-factor model combined with a threshold. The promising POET method of Fan et al. (2013) shows the strong capability of shrinking without imposing the five factors, but to only use information through the eigenvalues and eigenvectors. Also, the linear shrinkage toward the 1-factor model is working extremely well. This shows that allowing bias by linearly shrinking toward the 1-factor implied covariance matrix is a robust method even when the data is not simulated from a 1-factor model. The dynamic models have somewhat inferior results with respect to the factor models, however they do show a steadily decreasing deviation pattern for increasing dimensions. Also, their standard deviation is converging to even lower levels than the factor models. Both these findings indicate that this class of methods are able to produce stable estimators for the (inverted) covariance matrix even in high dimensions.

Figure 3 shows the same results as in Figure 2, except for the fact that the sample covariance is now excluded. All of the methods converge as the dimensionality diverges with respect to the amount of observations. Also, there is a clear visible decrease in the variance of the deviation with the true inverted covariance matrix and from $N > 250$ all estimates remain relatively stable. Despite the good performance of the 5-factor implied covariance matrix and the POET method, they remain among the methods with the highest deviations in performance. However, this deviation with other methods is equal for up to three decimals lacking significance of higher standard deviations.

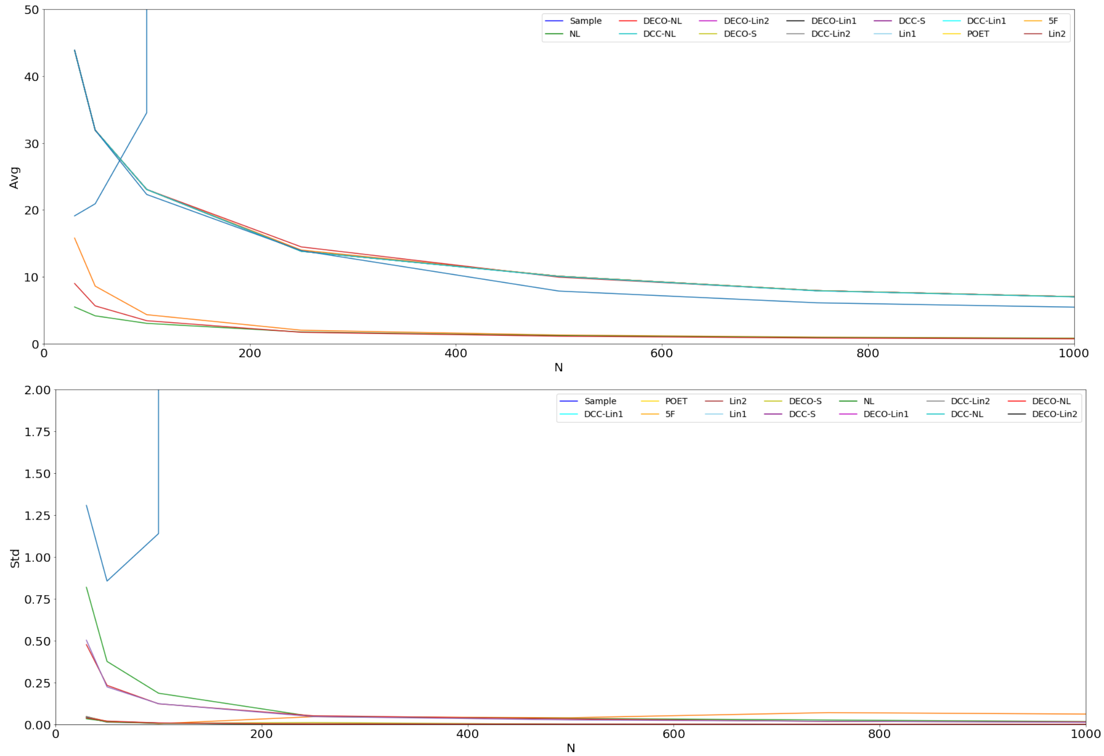

Figure 2: Averages and standard deviations of $\frac{1}{\sqrt{N}}||\hat{\Sigma}^{-1} - \Sigma^{-1}||_F$, the deviation of the inverse covariance matrix constructed from the shrinkage method applied to the simulated data with respect to the true underlying inverse covariance for 200 simulations.

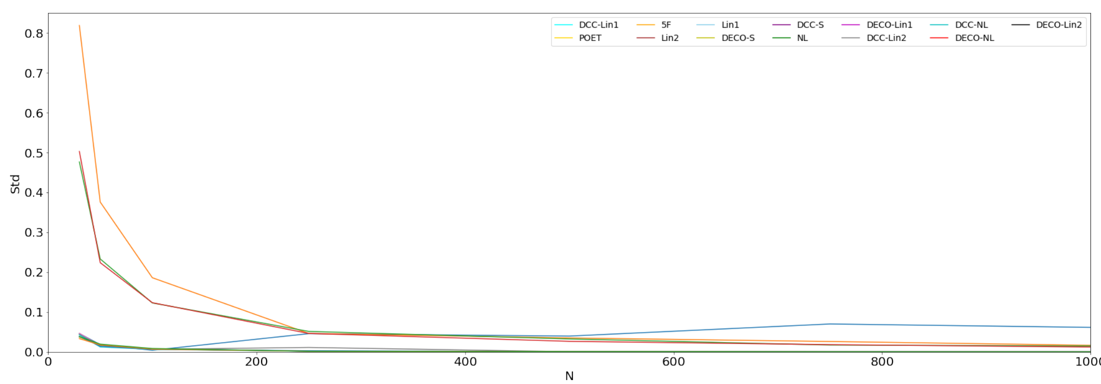

Figure 3: This Figure shows the same results as in Figure 2, excluding the deviations of the performance of the sample covariance matrix.

5 Results

In this section, I present the obtained performance measures for the different methods of interest. Firstly, I describe the capabilities of the methods in minimising the variance by the formed GMV portfolios in the general setting. Then, I discuss the performance of the methods for the shifted empirical environments.

5.1 Performance of the Formed GMV portfolios

Table 3 shows summary statistics for the underlying weights in the assets of the formed GMV portfolios. The table shows average minimum and maximum weights in the formed portfolios and the average standard deviation of the weights. The table gives an indication of how realistic the constructed portfolios are from a practical point of view. High positive or negative weights could indicate instability and imply that, in order to construct GMV portfolios, one has to possibly lever their position by taking large short/long positions in the underlying assets. In general, this is undesirable and impractical due to risk exposure and transaction costs. For all methods, there is an evident indication of decreasing magnitude in both negative and positive weights for increasing dimensions, which is as desired for stable methods. Furthermore and as previously described, the sample covariance becomes ill-behaved as N is roughly equal to or larger than T. This results in very unstable weights and performance. These incomparable results are not shown in the table(s).

Table 4 and Figure 4 show the annualised performance measures for the different methods of interest. For $N > 100$, all shrinkage methods show to be able to produce meaningful, well-behaved covariance matrix estimates. Next to that, almost all methods decrease the variance with respect to the *1/N* portfolio for all values of N. In general, the POET method turns out to be the most robust shrinkage estimator for the sample. This method proves to score among the models with the lowest variance out-of-sample in both low and high dimensions. Next to that, there are no clear preferred methods for the static and dynamic models. Also, there does not seem to be a clear preference for shrinkage targets in the DCC process across dimensions. For this sample, the results do not indicate that the dynamic approach outperforms the static pure (non)linear shrinkage methods in producing estimates that minimise the out-of-sample variance.

For static models, the POET estimator often ranks amongst the models scoring the lowest standard deviation across dimensions. In general, for higher dimensional cases ($N > 250$), the pure 1-factor implied linear shrinkage is also a robust estimator in terms of minimising the out-of-sample variance. On average, the nonlinear shrinkage methods performs relatively poorly for $N > 100$, which is not in line with the findings of Ledoit and Wolf (2017b). Especially when the amount of observations is equal to the amount of assets, $N = 250$, the method is not able to successfully shrink the sample eigenvalues in order to produce stable covariance estimates. This could be due to the fact that there is a large linear dependence between the selected assets and that for $N = 250$ all sample eigenvalues are used and shrunk. In that case eigenvalues could become very close to zero and deteriorate the performance of the shrinkage method. Despite that, the performance restores somewhat for larger dimensions, but it remains the least-performing shrinkage method.

For the dynamic models there is no clear winner among dimensions, although there is a slight indication of DECO models outperforming DCC models for $N < 100$. All dynamic methods perform roughly equal in terms of achieving the minimum variance and do not have extreme turnover values. Therefore, the preference for one shrinkage target over another has to be proven in the time series setting. When comparing the static and dynamic models, it can be seen that the dynamic models have a good and very stable performance across all dimensions. However, lower out-of-sample variances can be achieved in higher dimensions ($N > 100$) by making use of static methods. This does not hold uniformly as most of the static methods perform worse for $N = 750$. Furthermore, the level of turnover is on average lower for the static methods indicating lower weight rebalancing and possible transaction costs. Moreover, the table shows that the *1/N* portfolio attains the highest average annualised returns and that no GMV portfolios based on shrinkage methods outperform this method in terms of Sharpe Ratio.

The shrinkage intensities for the pure linear and nonlinear shrinkage methods are shown in Figure 5. For the linear shrinkage methods, the intensities of both shrinkage methods show a peaked figure, with peaks before the dotcom bubble and in the period from 2012 to 2016. Furthermore, it can be seen that the intensity for the method shrinking toward the 1-factor implied covariance on average attains higher, more extreme, values than the method shrinking toward the identity matrix. During periods of high intensities, the sample covariance is given a relatively low weight, indicating that it is very ill-behaved or singular. In this cases, it is beneficial to deviate from the sample covariance and shrink (fully) towards the shrinkage target. As can be seen in the figure, both shrinkage methods attain relatively low intensities from 2000 to 2012. The global financial crisis is fully captured in this period and it can be seen that the shrinkage targets are given very low weights. This could be due to the fact that the factor models are much less explanatory on average during crisis periods. For example, Bessler and Kurmann (2014) show that stocks of banks had very fluctuating risk exposures and that the common factors were not able to correctly capture the cross-sectional variation well during the financial crisis. Whether this holds in a more broader context is left for further research.

The nonlinear shrinkage intensities figure shows the 5-th and 95-th percentile and the median shrinkage intensity across all shrunk eigenvalues. The figure shows that the median of the intensities is around 0.75 for the sample up to 2008, followed by a small bump following the global financial crisis and a slightly upward sloping trend from 2009 onward. Because the median value is smaller than 1, on average the sample eigenvalues are decreased in order to achieve the optimal solution of the nonlinear shrinkage method. However, there is a large dispersion in deviations for smaller eigenvalues as some smaller eigenvalues are increased up to 3.0 times during the financial crisis. Also, after the global financial crisis, it can be seen that the eigenvalues on the lower end are pulled up more. Whether this is because the eigenvalues tend to be relatively smaller or whether there resides a different structural effect remains for further research.

Table 3: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix. This sample comprises 285 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.3068	0.2587	0.2460	0.2505	0.2236	0.2419	0.3319	0.2717	0.3490	0.3419	0.2391	0.3425	0.2846	0.2845	0.2329	0.2878
Min	0.0333	-0.0576	-0.0355	-0.0290	-0.0291	-0.0352	-0.0469	-0.0761	-0.0475	-0.0331	-0.0296	-0.0114	-0.0274	-0.0114	-0.0113	-0.0054	-0.0117
SD	0.0000	0.0771	0.0726	0.0661	0.0667	0.0613	0.0662	0.0868	0.0692	0.0807	0.0790	0.0554	0.0789	0.0715	0.0715	0.0576	0.0724
									N=50								
Max	0.0200	0.2543	0.1917	0.1888	0.1983	0.1719	0.1866	0.2701	0.1993	0.2781	0.2752	0.1742	0.2772	0.2178	0.2203	0.1773	0.2266
Min	0.0200	-0.0721	-0.0284	-0.0278	-0.0286	-0.0250	-0.0571	-0.0813	-0.0503	-0.0268	-0.0259	-0.0075	-0.0234	-0.0086	-0.0087	-0.0046	-0.0093
SD	0.0000	0.0571	0.0470	0.0444	0.0459	0.0413	0.0472	0.0620	0.0463	0.0531	0.0527	0.0340	0.0530	0.0466	0.0470	0.0375	0.0484
									N=100								
Max	0.0100	0.1907	0.0982	0.1033	0.1087	0.1014	0.1252	0.1859	0.1069	0.2025	0.2010	0.1068	0.2062	0.1372	0.1380	0.1090	0.1464
Min	0.0100	-0.0736	-0.0159	-0.0210	-0.0212	-0.0172	-0.0493	-0.0631	-0.0379	-0.0156	-0.0152	-0.0031	-0.0150	-0.0048	-0.0049	-0.0026	-0.0055
SD	0.0000	0.0395	0.0215	0.0220	0.0230	0.0217	0.0290	0.0374	0.0247	0.0292	0.0290	0.0165	0.0298	0.0241	0.0243	0.0191	0.0258
									N=250								
Max	0.0040	NM	0.0514	0.0567	0.0603	0.0591	0.0703	0.1350	0.0199	0.1218	0.1223	0.0584	0.0645	0.0753	0.0757	0.0600	0.0648
Min	0.0040	NM	-0.0074	-0.0113	-0.0114	-0.0117	-0.0428	-0.0633	-0.0009	-0.0093	-0.0087	-0.0014	-0.0034	-0.0019	-0.0019	-0.0011	-0.0013
SD	0.0000	NM	0.0093	0.0099	0.0104	0.0104	0.0174	0.0242	0.0031	0.0128	0.0128	0.0068	0.0073	0.0101	0.0102	0.0081	0.0088
									N=500								
Max	0.0020	NM	0.2168	0.2131	0.2180	0.2208	0.0265	0.2591	0.0384	0.2595	0.2579	0.2036	0.2747	0.2470	0.2465	0.2238	0.2591
Min	0.0020	NM	-0.0088	-0.0126	-0.0130	-0.0052	-0.0138	-0.0151	-0.0141	-0.0025	-0.0024	-0.0004	-0.0118	-0.0003	-0.0004	-0.0002	-0.0005
SD	0.0000	NM	0.0122	0.0122	0.0125	0.0123	0.0045	0.0146	0.0049	0.0137	0.0136	0.0107	0.0150	0.0130	0.0130	0.0117	0.0138
									N=750								
Max	0.0013	NM	0.2249	0.2170	0.2214	0.2356	0.0164	0.2618	0.0184	0.2660	0.2614	0.2222	0.2808	0.2576	0.2561	0.2401	0.2723
Min	0.0013	NM	-0.0063	-0.0090	-0.0095	-0.0035	-0.0089	-0.0083	-0.0097	-0.0012	-0.0011	-0.0002	-0.0026	-0.0001	-0.0001	-0.0001	-0.0002
SD	0.0000	NM	0.0101	0.0100	0.0102	0.0104	0.0027	0.0116	0.0028	0.0112	0.0110	0.0094	0.0120	0.0109	0.0108	0.0101	0.0116
									N=1000								
Max	0.0010	NM	0.2149	0.2056	0.2092	0.2281	0.0157	0.2936	0.0133	0.2659	0.2587	0.2251	0.2798	0.2612	0.2564	0.2395	0.2709
Min	0.0010	NM	-0.0054	-0.0074	-0.0077	-0.0029	-0.0087	-0.0082	-0.0072	-0.0008	-0.0007	-0.0001	-0.0017	-0.0001	-0.0001	-0.0000	-0.0002
SD	0.0000	NM	0.0083	0.0081	0.0083	0.0087	0.0025	0.0110	0.0020	0.0097	0.0094	0.0082	0.0102	0.0095	0.0093	0.0086	0.0099

Table 4: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per asset. This sample comprises 285 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	9.4957	1.8721	-0.4714	0.1839	0.5845	0.3287	1.9090	0.9666	1.9622	1.2737	0.9729	2.3144	1.0375	-1.1146	-1.3949	1.3056	-1.4927
Std	19.7025	11.7008	12.6382	10.6458	10.9212	**9.8509**	11.7756	13.8535	11.9635	10.5597	10.2102	10.8385	10.0235	9.2358	**8.9419**	9.5526	8.9840
SR	0.4820	0.1600	-0.0373	0.0173	0.0535	0.0334	0.1621	0.0698	0.1640	0.1206	0.0953	0.2135	0.1035	-0.1207	-0.1560	0.1367	-0.1661
To	0.0737	0.3245	0.1537	0.1434	0.1509	0.1692	0.2798	0.3544	0.2835	1.3389	1.2693	0.8506	1.2353	0.7049	0.6821	0.6705	0.6861
Util	-	-0.0737	-0.0969	-0.0903	-0.0863	-0.0888	-0.0733	-0.0829	-0.0728	-0.0797	-0.0826	-0.0693	-0.0820	-0.1032	-0.1060	-0.0792	-0.1070
									N=50								
Avg	9.0571	2.0882	0.0056	0.5036	0.9347	0.6669	1.8728	1.7234	2.1355	1.0425	1.1606	2.9083	1.4380	-1.1684	-1.0606	0.2398	-0.7432
Std	17.8909	10.8196	10.6326	9.0833	9.4688	**8.7524**	10.6940	11.6780	10.0386	10.2337	10.0099	10.1955	9.5131	9.3607	9.2292	9.2192	**8.8971**
SR	0.5062	0.1930	0.0005	0.0554	0.0987	0.0762	0.1751	0.1476	0.2127	0.1019	0.1159	0.2853	0.1512	-0.1248	-0.1149	0.0260	-0.0835
To	0.0712	0.5199	0.1731	0.1782	0.1933	0.2040	0.4185	0.5270	0.3926	1.4669	1.4152	0.8301	1.3552	0.8089	0.7927	0.7487	0.7928
Util	-	-0.0675	-0.0880	-0.0830	-0.0787	-0.0813	-0.0696	-0.0712	-0.0669	-0.0778	-0.0766	-0.0593	-0.0738	-0.0996	-0.0985	-0.0857	-0.0954
									N=100								
Avg	8.4437	-0.0035	-0.2712	-0.2604	-0.1117	-0.8455	-0.3728	-0.3057	0.5309	0.5193	0.3934	2.1238	0.2066	-2.4001	-2.5681	0.2380	-2.8354
Std	16.0370	7.8877	8.7358	7.9069	7.6643	7.3375	7.3974	7.7897	**7.2460**	8.8632	8.8151	9.2381	**8.6471**	8.8862	8.8419	9.0079	8.8835
SR	0.5265	-0.0004	-0.0310	-0.0329	-0.0146	-0.1152	-0.0504	-0.0392	0.0733	0.0586	0.0446	0.2299	0.0239	-0.2701	-0.2904	0.0264	-0.3192
To	0.0653	1.0776	0.1579	0.1928	0.2152	0.2253	0.7155	0.8742	0.5521	1.5790	1.5394	0.7428	1.4947	0.8784	0.8693	0.7680	0.8678
Util	-	-0.0823	-0.085	-0.0848	-0.0833	-0.0906	-0.0859	-0.0853	-0.0769	-0.0773	-0.0785	-0.0613	-0.0803	-0.1061	-0.1078	-0.0800	-0.1105
									N=250								
Avg	8.6737	NM	0.5182	0.0604	0.1349	-0.6136	1.0729	2.6671	11.4165	1.2500	1.6205	2.7990	3.2407	-2.0555	-1.4519	1.2966	0.2905
Std	16.2989	NM	8.7097	8.1177	8.1084	**7.7469**	8.9720	11.9934	21.9450	8.5490	**8.2980**	9.4370	9.7061	9.0491	8.7828	9.7548	9.6855
SR	0.5322	NM	0.0595	0.0074	0.0166	-0.0792	0.1196	0.2224	0.5202	0.1462	0.1953	0.2966	0.3339	-0.2271	-0.1653	0.1329	0.0300
To	0.0657	NM	0.1605	0.2187	0.2523	0.2824	1.6889	2.3089	0.1706	1.6249	1.5972	0.6621	0.8723	0.9389	0.9424	0.7966	0.8442
Util	-	NM	-0.0794	-0.0839	-0.0831	-0.0905	-0.0740	-0.0585	0.1097	-0.0722	-0.0685	-0.0569	-0.0526	-0.1050	-0.0990	-0.0718	-0.0818
									N=500								
Avg	12.9906	NM	1.0532	0.9132	0.7814	0.3853	5.0918	0.6067	5.5495	0.7693	0.6617	1.7721	-0.1501	-1.1371	-1.1708	0.8877	-1.3660
Std	19.8560	NM	6.5484	6.8996	6.9327	6.3034	11.0233	**6.0374**	12.6032	**7.2276**	7.2711	8.5764	7.3591	7.7235	7.8347	8.5116	7.8279
SR	0.6542	NM	0.1608	0.1324	0.1127	0.0611	0.4619	0.1005	0.4403	0.1064	0.0910	0.2066	-0.0204	-0.1472	-0.1494	0.1043	-0.1745
Util	-	NM	-0.1162	-0.1176	-0.1189	-0.1228	-0.0768	-0.1206	-0.0724	-0.1192	-0.1203	-0.1094	-0.1284	-0.1381	-0.1384	-0.1181	-0.1404
									N=750								
Avg	14.6735	NM	2.2835	2.1681	2.1367	0.6205	5.8369	2.4200	6.9305	0.6691	0.6848	1.2934	0.2035	-0.5635	-0.5251	1.0226	-0.7592
Std	21.0688	NM	9.1975	10.0579	10.0827	**5.7662**	11.0954	10.1521	11.6788	**6.8459**	7.1496	7.5694	6.9365	7.1846	7.5735	7.4574	7.6478
SR	0.6965	NM	0.2483	0.2156	0.2119	0.1076	0.5261	0.2384	0.5934	0.0977	0.0958	0.1709	0.0293	-0.0784	-0.0693	0.1371	-0.0993
Util	-	NM	-0.1209	-0.1221	-0.1225	-0.1370	-0.0858	-0.1197	-0.0751	-0.1367	-0.1366	-0.1306	-0.1413	-0.1489	-0.1486	-0.1332	-0.1509
									N=1000								
Avg	15.3933	NM	3.6961	3.2877	3.2751	2.3077	2.6801	1.0648	3.1821	3.5328	3.2360	4.1415	2.5915	2.5757	2.3081	4.1808	1.8779
Std	21.6219	NM	10.9710	11.0598	11.2402	8.9921	6.9079	**4.9983**	7.1147	10.0302	10.0519	10.5177	**9.9275**	10.2020	10.1548	10.4726	10.4177
SR	0.7119	NM	0.3369	0.2973	0.2914	0.2566	0.3880	0.2130	0.4473	0.3522	0.3219	0.3938	0.2610	0.2525	0.2273	0.3992	0.1803
Util	-	NM	-0.1142	-0.1183	-0.1184	-0.1277	0.0263	0.0104	0.0313	-0.1157	-0.1187	-0.1097	-0.1250	-0.1252	-0.1279	-0.1093	-0.1321

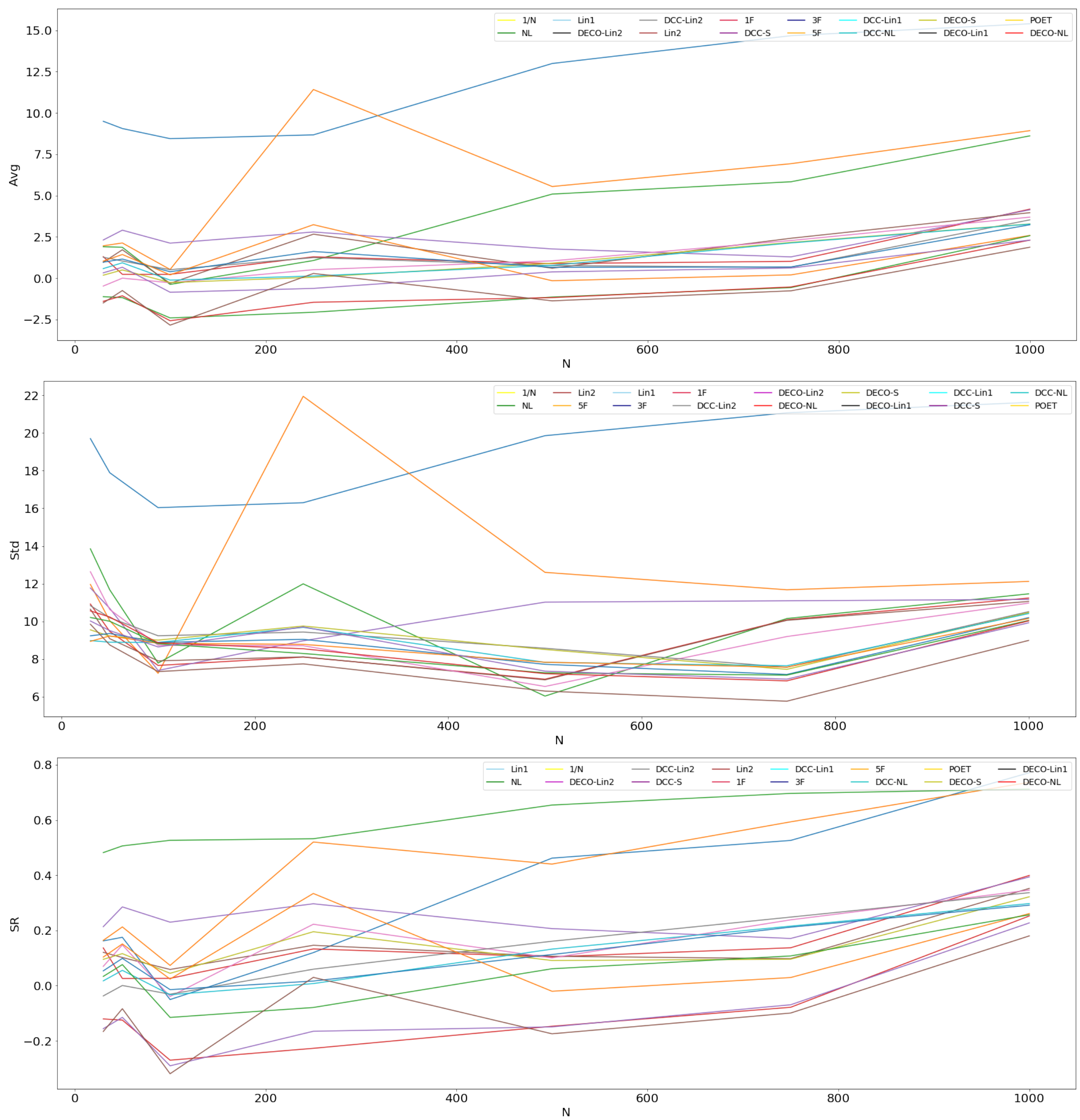

Figure 4: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios as in Table 4, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019. This sample comprises of 285 portfolio formations.

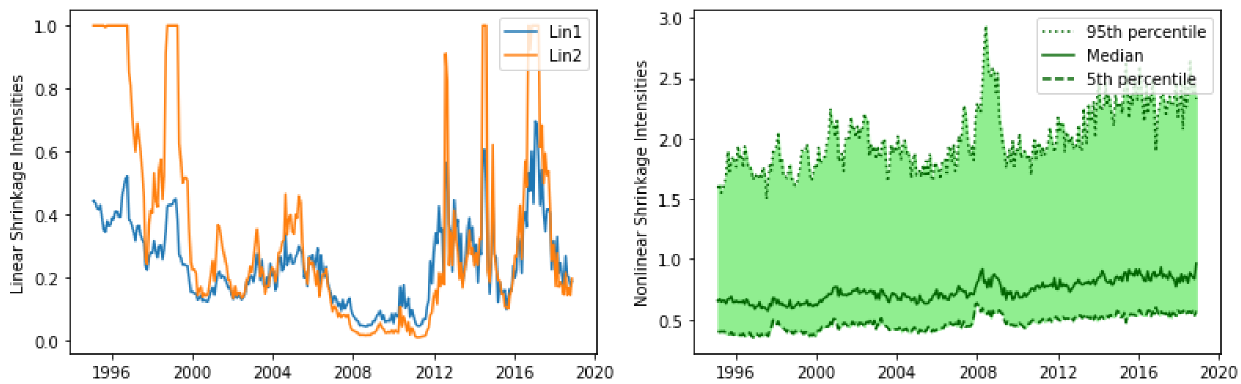

Figure 5: (Non)linear shrinkage intensities for the formed GMV portfolios. The sample period ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per portfolio for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 285 portfolio formations. $N = 1000$.

5.2 Robustness Analyses Results

In this section, I describe the performance of all implemented static and dynamic methods for constructing covariance matrix estimates and the goal of achieving the minimum variance in shifted empirical settings.

5.2.1 Low Market Capitalisation Habitat

In order to assess whether the methods perform equally well in a low market capitalisation environment, I implement the different methods while using daily asset returns of the smallest companies as measured by market capitalisation. Table 5 shows the performance of the methods with respect to achieving the minimum variance. For this setting turnover is not shown, since every investment universe will most likely differ due to the fact that the N lowest market capitalisation assets are varying over time.

On average, the annualised returns and accompanying standard deviations are higher than for the regular case. This is in line with the underlying foundations of the size effect, as discussed by Banz (1981) and Fama and French (2008), which state that risk-adjusted returns of smaller companies outperform larger companies. Furthermore, again the POET shows to be a robust estimator as it performs well in both low and high dimensions. Next to that, it is interesting to note that the pure linear and nonlinear methods do not perform well for the returns of low market capitalisation assets. For this sample, dynamic models tend to achieve slightly lower variance across dimensions, but they do not show a clear winner among shrinkage targets.

Along the lines of Banz (1981) and Fama and French (2008), the performance of the method based on the 1-factor model is inferior in small dimensions comparing with multi-factor models which account for this size effect. However, this effect seems to diminish as the dimensions increase. Also, the sample covariance shows to be fairly competitive in lower dimensions. This could be due to the fact that, according to Reinganum and Smith (1983), risk of small firms is much more identified as idiosyncratic rather than systematic. Idiosyncratic means that it is diversifiable, which can be achieved by holding small firm assets within a relatively large portfolio. For this sample, the pure nonlinear shrinkage estimator is less unstable for $N = 250$. Whether this is due to less dependence between small firm asset returns has to be determined. Regarding utility, nearly no method is able to outperform the $1/N$ method in low dimensions.

Furthermore, the dynamic models are again fairly stable for all shrinkage targets and tend to achieve lower variance for low dimensions up to higher dimensions of $N = 750$ assets. For all methods there is a strong declining variance pattern for increasing dimensionality up to $N = 250$. For higher dimensionalities, the standard deviations remain roughly the same magnitude. Also, these models outperform the $1/N$ and static methods in terms of utility, as almost all methods attain a higher relative value. This indicates that a utility investor prefers these methods over static methods.

The statistics regarding the weights can be found in Table 10 in Appendix B.1. For every method, the weights are decreasing in absolute value for the average minimums and maximums, which is as expected for stable methods. The (non)linear shrinkage intensities are shown in Figure 7 in Appendix B.1. For the linear shrinkage intensities, the figure completely differs with respect to the regular case. In general, the linear shrinkage intensities are a lot higher for the low market capitalisation sample than for the regular sample. This indicates that there is given a heavy weight to the shrinkage target in the optimal solution and thus that the sample covariance is very ill-behaved. The figure clearly shows the impact of the captured periods of crises in the sample. During these periods the weight on the shrinkage target decreases and thus increases on the sample covariance, which also holds for the regular case. Furthermore, the 1-factor implied linear shrinkage intensity again takes on more extreme values, often attaining the upper bound of 1. Next to that, it can be seen that this shrinkage intensity remains at a lower level for a longer period after the global financial crisis around 2008.

The nonlinear shrinkage intensities follow roughly the same pattern as for the regular sample up to 2012, with the median of the nonlinear shrinkage intensities around 0.75 in the period from 1995 to 2012. Also, the bumps in increased eigenvalues during the global financial crisis and during the dotcom bubble are clearly visible. However, after 2012 the nonlinear shrinkage intensities show an upward trend which holds for all eigenvalues as can be seen by the 5-th and 95-th percentiles increasing as well. This pattern is not present for the regular case and the cause of this effect is subject for further research.

Table 5: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using the lowest market capitalisation stocks as assets. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per asset. This sample comprises 285 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	67.3636	1.1790	-10.6391	6.3880	6.1189	5.4427	40.3209	-10.6391	0.5484	2.1167	9.4473	8.9238	5.5210	8.6545	10.2283	10.4588	9.3859
Std	76.8455	22.3729	58.0887	**18.2850**	18.7592	22.1984	55.9122	58.0887	22.8075	24.1890	16.8849	16.8014	**16.4072**	19.6470	20.3865	20.5838	19.9664
SR	0.8766	0.0527	-0.1832	0.3494	0.3262	0.2452	0.7211	-0.1832	0.0240	0.0875	0.5595	0.5311	0.3365	0.4405	0.5017	0.5081	0.4701
Util	-	-0.6659	-0.8031	-0.6116	-0.6144	-0.6221	0.4055	-0.8031	-0.6743	0.0187	0.0963	0.0909	0.0552	0.0875	0.1039	0.1063	0.0950
									N=50								
Avg	63.4854	3.0771	-9.5481	2.5605	2.5701	6.3553	31.6759	-9.5481	-0.2362	0.9902	5.6806	5.7660	2.6676	2.2751	4.6630	4.8793	2.5439
Std	56.7127	15.7842	39.7639	18.1266	18.2695	**11.1212**	37.5766	39.7639	21.4872	14.0960	11.3710	11.7305	11.9152	**9.0699**	9.9980	10.3070	9.4391
SR	1.1194	0.1949	-0.2401	0.1413	0.1407	0.5715	0.8430	-0.2401	-0.0110	0.0702	0.4996	0.4915	0.2239	0.2508	0.4664	0.4734	0.2695
Util	-	-0.6175	-0.7563	-0.6233	-0.6232	-0.5827	-0.3258	-0.7563	-0.6545	0.0090	0.0584	0.0593	0.0269	0.0230	0.0478	0.0500	0.0258
									N=100								
Avg	49.5879	0.0850	-4.5151	-0.8513	-0.9160	1.8006	14.1442	-4.5151	-6.2924	2.3695	2.0303	3.5679	-0.0419	2.6570	2.4614	3.6297	0.8434
Std	37.9932	**7.1560**	21.6739	11.9277	12.1574	10.8444	27.8820	21.6739	23.0550	8.1379	6.7472	6.8152	7.4980	5.9800	5.2609	6.3544	**4.5722**
SR	1.3052	0.0119	-0.2083	-0.0714	-0.0753	0.1660	0.5073	-0.2083	-0.2729	0.2912	0.3009	0.5235	-0.0056	0.4443	0.4679	0.5712	0.1845
Util	-	-0.5103	-0.5604	-0.5205	-0.5211	-0.4927	-0.3683	-0.5604	-0.5807	0.0244	0.0209	0.0369	-0.0008	0.0275	0.0254	0.0376	0.0086
									N=250								
Avg	39.8098	NM	1.6788	1.6957	1.7187	3.0932	9.1560	1.7080	1.7276	2.7164	1.7087	3.2403	3.6881	2.6843	0.5543	3.1568	1.3264
Std	28.2051	NM	4.6973	4.3631	**4.3106**	4.3837	20.9548	4.6940	5.6117	4.5540	3.9540	4.7079	6.0483	4.3958	**3.8081**	4.6351	4.2751
SR	1.4114	NM	0.3574	0.3887	0.3987	0.7056	0.4369	0.3639	0.3079	0.5965	0.4322	0.6883	0.6098	0.6106	0.1456	0.6811	0.3103
Util	-	NM	-0.3939	-0.3937	-0.3934	-0.3791	-0.3190	-0.3936	-0.3935	0.0282	0.0177	0.0336	0.0382	0.0279	0.0057	0.0328	0.0137
									N=500								
Avg	31.2142	NM	1.9153	1.8405	1.8170	2.7878	7.9730	2.1313	6.6462	2.6008	2.7812	2.9349	0.7855	2.2222	1.9044	2.9685	-1.5930
Std	26.0099	NM	5.6632	5.1454	5.0894	**5.0014**	17.0424	5.7167	17.4129	4.7806	4.7662	5.0581	**3.9722**	4.8756	4.9404	5.0304	4.9421
SR	1.2001	NM	0.3382	0.3577	0.3570	0.5574	0.4678	0.3728	0.3817	0.5440	0.5835	0.5802	0.1977	0.4558	0.3855	0.5901	-0.3223
Util	-	NM	-0.3025	-0.3033	-0.3035	-0.2933	-0.2412	-0.3003	-0.2551	0.0270	0.0288	0.0305	0.0081	0.0230	0.0197	0.0308	-0.0167
									N=750								
Avg	26.9790	NM	1.9554	1.8121	1.7291	1.9717	8.2583	1.9806	7.2394	2.3309	2.5758	2.7985	1.5473	1.7069	1.6434	2.3551	-0.6806
Std	25.1824	NM	5.9438	5.4930	**5.4112**	5.5053	14.0891	5.9809	13.8818	5.0804	5.3465	5.4141	**4.2107**	5.2142	5.3969	5.1957	5.7889
SR	1.0713	NM	0.3290	0.3299	0.3195	0.3582	0.5861	0.3312	0.5215	0.4588	0.4818	0.5169	0.3675	0.3274	0.3045	0.4533	-0.1176
Util	-	NM	-0.2578	-0.2592	-0.2601	-0.2576	-0.1932	-0.2575	-0.2038	0.0241	0.0267	0.0290	0.0160	0.0177	0.0170	0.0244	-0.0072
									N=1000								
Avg	26.8931	NM	2.9814	2.9150	2.8985	1.8695	10.3508	2.8388	9.7892	1.7643	1.9452	2.6027	-0.2097	1.2463	1.3705	2.1872	-0.3904
Std	24.7117	NM	6.6716	7.2932	7.4201	**5.9283**	12.9045	6.4611	13.3160	**5.5440**	5.7847	6.3158	12.7919	5.9005	5.8841	6.1305	6.3100
SR	1.0883	NM	0.4469	0.3997	0.3906	0.3153	0.8021	0.4394	0.7351	0.3182	0.3363	0.4121	-0.0164	0.2112	0.2329	0.3568	-0.0619
Util	-	NM	-0.2460	-0.2468	-0.2470	-0.2576	0.1066	0.0294	0.1006	0.0182	0.0201	0.0269	-0.0025	0.0128	0.0141	0.0226	-0.0042

5.2.2 Short-sale Constraints

Table 6 shows the performance measures for the robustness analysis involving non-negative weights in the underlying assets. Only the static methods are considered due to computational constraints, with the constrained optimisation combined with minimisation being cumbersome for dynamic models. As Jagannathan and Ma (2003) argue, constraining the weights to be non-negative can be viewed as implying some form of shrinkage as imposed from the weights. As can be seen in the table, this type of shrinkage conditions the sample covariance matrix such that the obtained estimator is feasible even in high dimensions ($N > 100$). However, its performance is inferior to all shrinkage methods across dimensions.

For this sample there is a clear winner among the static methods. The 1-factor implied linear shrinkage model turns out to consistently achieve the lowest out-of-sample variance for $N > 30$. Also, the general level of standard deviation is much lower across dimensions. Next to that, the factor models and the POET model have very similar results. For all dimensions, except when the amount of observations equals the amount of assets ($N = 250$) the nonlinear shrinkage method is also a good candidate. However, the performance for the special case is a recurring non-stable result. Also, the generated average returns are fairly low or even negative for these two models, indicating that it might not be desired from a return perspective.

Another interesting result is that the covariance matrix from the 1-factor model is performing better than models which include more economic factors. This does not hold for other sub-samples and remains subject to further research. Furthermore, the similarity between the performance of the 3- and 5-factor implied covariance methods and POET methods are striking. Whether this results hold true due to the stopping criteria of the optimiser or whether the optimal results converge to the same solution, remains for further investigation. Nevertheless, this underlines the strength of the POET method without actually specifying the underlying factors. Due to the weight restrictions, there is only put a positive weight in some assets and turnover is lower in general. Despite the promising results, the outperformance from a utility perspective with the $1/N$ strategy does not hold consistently, with all methods attaining lower levels of utility for all utilities.

The statistics regarding the underlying weights in the assets can be found in Table 11 in Appendix B.2. By construction, the weights can not be negative for the different methods, which is reflected in the average minimums being zero or even slightly positive. This constraint also lowers the standard deviations. The comparable results for the $3F, 5F$ and $POET$ estimators are somewhat as expected as the underlying weights are very close to each other. Because this analysis only constraints and adjusts the weights without altering the covariance estimators, the (non)linear shrinkage intensities for the sample eigenvalues remain the same as in the regular case and hence are not shown.

Table 6: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using a constrained optimization enforcing non-negative weights in the underlying assets. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per asset. This sample comprises 285 portfolio formations. Utility is given relative to $1/N$ formation.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL
					N=30				
Avg	6.9095	6.3280	2.2208	6.3237	6.3244	6.3196	1.5082	5.3410	1.5948
Std	15.9493	15.4538	10.1521	15.4480	15.4486	15.4465	9.1608	12.4819	**9.0965**
SR	0.4332	0.4095	0.2188	0.4094	0.4094	0.4091	0.1646	0.4279	0.1753
To	0.0617	0.0635	0.1242	0.0636	0.0636	0.0636	0.1675	0.0881	0.1690
Util[1]	-	-0.0057	-0.0453	-0.0057	-0.0057	-0.0058	-0.0522	-0.0149	-0.0514
					N=50				
Avg	7.0097	6.7172	1.0266	6.7415	6.7415	6.7109	0.7902	4.5468	0.9063
Std	15.2092	14.5472	8.5421	14.5351	14.5360	14.5350	**7.3499**	11.2498	7.3848
SR	0.4609	0.4618	0.1202	0.4638	0.4638	0.4617	0.1075	0.4042	0.1227
To	0.0591	0.0608	0.1177	0.0608	0.0608	0.0609	0.1844	0.0934	0.1884
Util[1]	-	-0.0028	-0.0582	-0.0025	-0.0025	-0.0028	-0.0604	-0.0237	-0.0592
					N=100				
Avg	7.0351	6.0859	0.0664	6.1125	6.1105	6.0757	0.3637	3.3968	0.5168
Std	15.2006	14.0570	7.9782	14.0533	14.0525	14.0516	**7.0347**	9.2233	7.0855
SR	0.4628	0.4329	0.0083	0.4350	0.4348	0.4324	0.0517	0.3683	0.0729
To	0.0608	0.0629	0.1143	0.0629	0.0629	0.0630	0.2427	0.1007	0.2281
Util[1]	-	-0.0092	-0.0679	-0.0089	-0.0090	-0.0093	-0.0648	-0.0350	-0.0633
					N=250				
Avg	8.0658	6.7172	0.4577	6.7501	6.7522	6.6644	0.4674	3.0715	16.1326
Std	16.0234	13.2530	7.6224	13.2216	13.2215	13.2225	**6.3192**	8.4709	27.4323
SR	0.5034	0.5068	0.0600	0.5105	0.5107	0.5040	0.0740	0.3626	0.5881
To	0.0641	0.0745	0.1267	0.0744	0.0744	0.0762	0.2631	0.1022	1.2865
Util[1]	-	-0.0129	-0.0740	-0.0126	-0.0125	-0.0134	-0.0737	-0.0482	0.0778
					N=500				
Avg	8.4419	4.8621	1.2342	4.9952	4.8543	4.8266	-0.2688	1.2638	-0.1928
Std	19.2653	12.4067	8.6105	12.4274	12.4359	12.2618	**4.8068**	7.8511	5.7937
SR	0.4382	0.3919	0.1433	0.4019	0.3903	0.3936	-0.0559	0.1610	-0.0333
Util[1]	-	-0.0344	-0.0697	-0.0331	-0.0345	-0.0348	-0.0842	-0.0693	-0.0836
					N=750				
Avg	8.7544	4.8292	1.3757	4.7567	4.7411	4.5412	0.2182	1.3998	0.2749
Std	20.5645	11.5514	7.9079	11.7073	11.6878	11.4715	**4.6461**	7.3948	5.7258
SR	0.4257	0.4181	0.1740	0.4063	0.4056	0.3959	0.0470	0.1893	0.0480
Util[1]	-	-0.0376	-0.0712	-0.0383	-0.0384	-0.0404	-0.0824	-0.0709	-0.0819
					N=1000				
Avg	8.6297	4.2209	1.5124	4.0785	4.0845	3.9774	0.2215	1.1231	0.0736
Std	21.1416	10.8265	8.1206	10.937	10.9388	10.8201	**4.1261**	7.3867	5.6364
SR	0.4082	0.3899	0.1862	0.3729	0.3734	0.3676	0.0537	0.1520	0.0131
Util[1]	-	-0.0421	-0.0685	-0.0436	-0.0435	-0.0445	-0.0810	-0.0723	-0.0826

5.2.3 Subperiod Analysis

The performance measures for the subperiod analysis can be found in Tables 7, 8 and 9. For both the first two subperiods, the 1-factor implied linear shrinkage model turns out to be a robust minimiser in high dimensions. Only for the last subperiod, the POET method shows to be a more robust candidate. For the dynamic models, there are no clear winners among dimensions and subperiods. Furthermore, the results do not support a logical preference for a shrinkage method as a target in the DCC process across subperiods. The results from the subsamples do not indicate that the dynamic shrinkage approach outperforms its static counterparts.

For the first subperiod, the 3-factor implied covariance matrix performs best in the lower dimensions ($N < 250$) and also has a good performance in the higher dimensions being only consistently outperformed by the 1-factor implied linear shrinkage method. These methods are both favoured in respectively the lower and higher dimensions. Again, the pure nonlinear shrinkage method is unstable for $N = 250$. This holds across all subperiods and also holds in the regular case. This result shows that the found instability is not due to specific crisis periods, but tends to be present over the whole sample. The dynamic models show less clear winners in lower dimensions, but the DCC model with the nonlinear shrinkage as a target is the best estimator in high dimensions. Furthermore, the methods are very competitive in achieving the minimum variance for this sub-sample comparing with the static methods.

For the second subperiod, the magnitudes of average returns and standard deviations are more dispersed among dimensions. Most of the methods have negative average returns during this period, indicating the impact of the global financial crisis. For the static methods, the linear shrinkage methods mainly reach the lowest out-of-sample variance across dimensions. The factor implied covariance methods are remarkably noncompetitive for this sub-sample. This is in line with findings of Bessler and Kurmann (2014), which indicate that factor models have lower explanatory power during periods of crisis. From higher dimensions the 1-factor implied linear shrinkage performs uniformly best. For dynamic models, the overall level of variance out-of-sample is somewhat higher across dimensions than for the other methods. There is no clear winner in lower dimensions for this sub-sample, but for higher dimensions the DECO method with the sample correlation matrix as a target is a relatively robust minimiser.

For the third subperiod, average returns are mainly positive and have the lowest standard deviations across subperiods. The nonlinear shrinkage method performs good in lower dimensions, efficiently shrinking the eigenvalues to reach the lowest variance. However, this method deteriorates for larger dimensions. Also, the POET estimator is a good estimator for semi-high dimensions, but inferior in the highest dimension. Also, for the dynamic models I find relatively higher average returns and lower standard deviations than for other subperiods, except for the highest dimensions $N = 1000$ where the standard deviation is around 12% for every method. What causes this sudden increase remains for more thorough research. In line with the superior performance of the nonlinear shrinkage method, the DCC method with the nonlinear shrinkage covariance matrix as a target has a robust performance along dimensions. Comparing the static and dynamic models with each other, I find that the dynamic models are very competitive across dimensions. However, this performance deteriorates for the extremely high dimensions case of $N = 1000$ as the models do not have the ability of the pure shrinkage methods to decrease the variance to single digit percentages. The cause remains for further research.

Except for the dynamic models in the third sub-period and the pure nonlinear shrinkage method for $N = 250$, almost no method outperforms the $1/N$ method in terms of realised utility over the sub-samples. This indicates that there remain arguments for the utility investor in favour of the $1/N$ method. Regarding turnover, all methods in the three sub-periods show relatively stable results except for the pure linear shrinkage methods. These methods show a present monotonically increasing amount of turnover across dimensions indicating large amounts of rebalancing. Therefore, they are relatively unfavourable from a pragmatic point of view due to the need and importance of rebalancing in order to achieve the minimum variance.

Tables with weights and shrinkage intensities can be found in Tables 12, 13 and 14 and in Figure 12 in Appendix B.3. The weights tables show stable patterns of decreasing magnitudes of the maximum and minimum weights for the first two subperiods. However, for the third sub-sample, this pattern does not uniformly hold. Especially the factor implied covariance methods and the POET method show relatively high values for the average maximum values of the weights for $N > 250$. However, a relatively bad performance of these estimators is only reflected for $N = 1000$. What is underlying these increased weights remains for research. Furthermore, the (non)linear shrinkage intensities are equal to the regular case and hence are only shown for comparison purposes.

Table 7: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using the first subperiod of the whole sample period. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. This subperiod for the assets ranges from 13 February 1995 to 16 January 2004, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 95 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	9.5816	-0.0665	0.3568	0.2334	0.4407	-0.3901	-0.4933	-0.4232	0.1421	0.0793	0.2587	0.7680	0.3661	-1.1354	-0.5949	0.2902	-0.6910
Std	19.9545	8.2161	8.0727	**7.7755**	7.8991	8.2349	8.3534	8.2732	8.2272	8.5907	8.6409	9.0392	8.5378	8.4881	**8.4072**	8.8027	8.4524
SR	0.4802	-0.0081	0.0442	0.0300	0.0558	-0.0474	-0.0591	-0.0512	0.0173	0.0092	0.0299	0.0850	0.0429	-0.1338	-0.0708	0.0330	-0.0818
To	0.0845	0.2061	0.0997	0.0911	0.0949	0.1094	0.1843	0.1997	0.1910	0.9964	0.9029	0.7469	0.9114	0.4836	0.4695	0.4947	0.4693
Util	-	-0.0936	-0.0894	-0.0906	-0.0886	-0.0968	-0.0979	-0.0972	-0.0916	-0.0923	-0.0905	-0.0855	-0.0894	-0.1043	-0.0989	-0.0901	-0.0998
									N=50								
Avg	9.7801	0.9881	1.1520	0.6593	0.7637	0.7831	0.7397	0.6631	0.9590	0.3981	0.8382	1.1015	0.1164	-1.1567	-0.8474	-0.2674	-1.7216
Std	16.9349	7.9756	7.6640	**7.4416**	7.5333	7.6766	7.9199	8.0536	7.8691	8.5278	8.5626	8.5471	**8.3847**	8.6695	8.6300	8.4615	8.5421
SR	0.5775	0.1239	0.1503	0.0886	0.1014	0.1020	0.0934	0.0823	0.1219	0.0467	0.0979	0.1289	0.0139	-0.1334	-0.0982	-0.0316	-0.2015
To	0.0772	0.3268	0.1122	0.1119	0.1180	0.1226	0.2732	0.2835	0.2651	1.1623	1.0313	0.7147	1.0045	0.6339	0.5970	0.5712	0.5937
Util	-	-0.0858	-0.0841	-0.0890	-0.0880	-0.0878	-0.0882	-0.0890	-0.0861	-0.0917	-0.0873	-0.0847	-0.0945	-0.1071	-0.1040	-0.0983	-0.1127
									N=100								
Avg	8.6502	0.9490	2.2503	1.1803	1.3604	1.2397	1.0370	1.0843	1.5710	-0.0245	-0.5181	1.7704	-0.0601	-0.8168	-1.3398	0.4566	-2.5292
Std	15.4687	8.9513	7.9578	**7.6683**	7.6690	7.9047	8.4884	8.4388	8.3301	8.6894	**8.2494**	8.7091	8.3214	8.8670	8.6193	8.6185	8.8359
SR	0.5592	0.1060	0.2828	0.1539	0.1774	0.1568	0.1222	0.1285	0.1886	-0.0028	-0.0628	0.2033	-0.0072	-0.0921	-0.1554	0.0530	-0.2862
To	0.0731	0.7502	0.1454	0.1538	0.1654	0.1697	0.5327	0.4921	0.4046	1.3127	1.2491	0.7053	1.1651	0.7732	0.7380	0.6764	0.7145
Util	-	-0.0753	-0.0623	-0.0729	-0.0711	-0.0723	-0.0744	-0.0739	-0.0691	-0.0850	-0.0899	-0.0672	-0.0853	-0.0928	-0.0980	-0.0802	-0.1098
									N=250								
Avg	9.2391	NM	2.0562	0.8968	1.0439	1.1544	1.7836	2.2147	13.1504	1.6913	2.1718	3.1495	3.3252	-2.8817	-2.4681	0.6856	-0.7770
Std	15.3461	NM	7.5862	7.4126	7.4795	7.4707	7.8148	**7.4059**	23.2304	**7.8538**	7.9024	8.4132	8.7319	8.5093	8.5249	8.7307	8.2155
SR	0.6020	NM	0.2710	0.1210	0.1396	0.1545	0.2282	0.2990	0.5661	0.2154	0.2748	0.3744	0.3808	-0.3387	-0.2895	0.0785	-0.0946
To	0.0759	NM	0.1393	0.1758	0.1935	0.1825	1.3630	1.0082	0.2761	1.3164	1.2859	0.5708	0.8185	0.7887	0.7987	0.6621	0.7398
Util	-	NM	-0.0700	-0.0815	-0.0801	-0.0790	-0.0729	-0.0685	0.0367	-0.0737	-0.0690	-0.0593	-0.0576	-0.1191	-0.1150	-0.0838	-0.0982
									N=500								
Avg	11.7709	NM	2.8720	1.9537	1.9330	1.9483	3.3085	2.4551	4.3302	3.2642	3.0580	4.2569	4.5996	-0.0043	-0.4991	3.1952	1.3890
Std	21.9691	NM	7.2333	6.9785	7.0072	6.9563	8.8012	**6.8031**	8.3422	7.8868	8.0277	8.3509	**6.8361**	8.3862	8.3857	8.6714	8.1298
SR	0.5358	NM	0.3971	0.2800	0.2759	0.2801	0.3759	0.3609	0.5191	0.4139	0.3809	0.5097	0.6728	-0.0005	-0.0595	0.3685	0.1709
Util	-	NM	-0.0865	-0.0956	-0.0958	-0.0957	-0.0823	-0.0906	-0.0722	-0.0827	-0.0848	-0.0729	-0.0694	-0.1151	-0.1201	-0.0834	-0.1013
									N=750								
Avg	10.7571	NM	3.0320	1.8156	1.9945	1.8885	2.3383	2.6556	3.7920	4.7047	2.5529	4.9527	3.9634	3.1279	0.6308	4.8438	1.4352
Std	23.8643	NM	6.9930	6.6886	6.6937	6.7813	8.5190	**6.3839**	8.4803	7.2568	8.0798	8.0703	**6.1508**	7.5311	8.1409	7.9627	7.5725
SR	0.4508	NM	0.4336	0.2714	0.2980	0.2785	0.2745	0.4160	0.4472	0.6483	0.3160	0.6137	0.6444	0.4153	0.0775	0.6083	0.1895
Util	-	NM	-0.0746	-0.0867	-0.0849	-0.0860	-0.0817	-0.0783	-0.0673	-0.0582	-0.0796	-0.0557	-0.0654	-0.0738	-0.0986	-0.0568	-0.0905
									N=1000								
Avg	11.8808	NM	3.2681	1.7300	1.8602	1.6866	3.6871	2.6313	4.9109	4.2875	3.5713	4.8013	2.7257	2.4805	1.2911	4.8519	0.1986
Std	25.3646	NM	6.8346	6.5028	6.5098	6.6659	8.5207	**6.1029**	8.6968	6.4930	7.5088	7.7485	**5.3054**	6.9295	7.7821	7.7055	7.3998
SR	0.4684	NM	0.4782	0.2660	0.2858	0.2530	0.4327	0.4312	0.5647	0.6603	0.4756	0.6196	0.5138	0.3580	0.1659	0.6297	0.0268
Util	-	NM	-0.0833	-0.0986	-0.0973	-0.0990	-0.0793	-0.0896	-0.0672	-0.0733	-0.0810	-0.0683	-0.0887	-0.0913	-0.1031	-0.0678	-0.1138

Table 8: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using the second subperiod of the whole sample period. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. This subperiod for the assets ranges from 19 January 2004 to 3 January 2012, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 95 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	9.9200	0.0871	-0.2333	0.8637	0.4501	-2.0918	0.2056	-0.8458	-0.0527	-6.1814	-6.4697	-3.2965	-5.7192	-4.2461	-4.3912	-3.4695	-4.0027
Std	25.8481	**12.2814**	14.8524	12.4687	12.5323	18.8563	12.3304	14.0095	12.6436	15.2726	15.0773	16.1409	14.6604	13.6947	13.5409	14.5426	**13.5046**
SR	0.3838	0.0071	-0.0157	0.0693	0.0359	-0.1109	0.0167	-0.0604	-0.0042	-0.4047	-0.4291	-0.2042	-0.3901	-0.3101	-0.3243	-0.2386	-0.2964
To	0.0767	0.4105	0.2375	0.2073	0.2188	0.2887	0.3571	0.4918	0.3459	1.4282	1.4092	0.9724	1.3521	0.9012	0.8729	0.8823	0.8693
Util	-	-0.0948	-0.0990	-0.0871	-0.0912	-0.1166	-0.0936	-0.1045	-0.0961	-0.1573	-0.1602	-0.1286	-0.1527	-0.1377	-0.1392	-0.1300	-0.1353
									N=50								
Avg	9.3282	0.0509	0.1210	1.0923	0.1876	-1.0044	0.4576	-0.2989	-0.1206	-2.3409	-2.0205	-1.0415	-2.9845	-3.0960	-2.8302	-2.5787	-3.6214
Std	24.0877	9.5840	12.3028	10.2492	10.3775	12.9373	**9.5385**	10.0891	10.1382	11.4704	11.4062	12.7301	11.105	11.0925	**11.0226**	12.1703	10.925
SR	0.3873	0.0053	0.0098	0.1066	0.0181	-0.0776	0.0480	-0.0296	-0.0119	-0.2041	-0.1771	-0.0818	-0.2687	-0.2791	-0.2568	-0.2119	-0.3315
To	0.0718	0.6331	0.2141	0.2262	0.2429	0.2943	0.5209	0.6584	0.4684	1.5198	1.4692	0.8676	1.393	0.9203	0.9030	0.9074	0.8967
Util	-	-0.0895	-0.0893	-0.0791	-0.0881	-0.0999	-0.0854	-0.0930	-0.0911	-0.1134	-0.1102	-0.1005	-0.1198	-0.1207	-0.1181	-0.1156	-0.1259
									N=100								
Avg	7.6809	0.2301	-0.2458	0.0472	-0.5996	-2.5522	-0.1496	0.1941	-0.1335	-0.4852	-0.4899	0.7412	-1.5075	-3.734	-3.7217	-1.3612	-4.9393
Std	20.6913	8.1593	9.7201	8.7409	8.6845	10.3145	**7.7455**	7.8955	7.7854	11.2579	11.2436	12.4977	**11.0841**	11.3648	11.3524	12.4784	11.2690
SR	0.3712	0.0282	-0.0253	0.0054	-0.0690	-0.2474	-0.0193	0.0246	-0.0172	-0.0431	-0.0436	0.0593	-0.1360	-0.3286	-0.3278	-0.1091	-0.4383
To	0.0665	1.1794	0.1819	0.2194	0.2447	0.3048	0.8493	1.0540	0.6285	1.6378	1.6032	0.809	1.5758	1.0124	1.0097	0.9355	0.9669
Util	-	-0.0714	-0.0765	-0.0733	-0.0797	-0.0991	-0.0751	-0.0717	-0.0749	-0.0788	-0.0788	-0.0668	-0.0889	-0.1110	-0.1109	-0.0876	-0.1230
									N=250								
Avg	8.1045	NM	-0.9886	-0.8287	-1.5532	-2.4931	-3.2229	-2.2164	9.7577	1.0503	1.1414	2.1849	1.6793	-3.1600	-3.1491	0.2415	-1.7085
Std	20.7835	NM	9.2215	8.2788	8.1226	**7.9857**	8.6045	8.2176	27.3923	9.2756	**9.2522**	11.9333	11.7677	9.4314	9.4262	12.0885	11.8025
SR	0.3900	NM	-0.1072	-0.1001	-0.1912	-0.3122	-0.3746	-0.2697	0.3562	0.1132	0.1234	0.1831	0.1427	-0.3350	-0.3341	0.0200	-0.1448
To	0.0679	NM	0.1745	0.2419	0.2769	0.3551	2.3256	3.0067	0.1581	1.7361	1.7277	0.7204	0.8905	1.0640	1.0696	0.9256	1.0305
Util	-	NM	-0.0876	-0.0858	-0.0930	-0.1022	-0.1096	-0.0996	0.0140	-0.0673	-0.0664	-0.0564	-0.0614	-0.1091	-0.1090	-0.0756	-0.0949
									N=500								
Avg	8.5003	NM	-1.4817	-1.5335	-2.325	-1.4314	-0.8055	-0.3218	-0.6889	2.7551	2.5210	2.7288	2.4286	1.8631	1.2520	2.0343	1.8644
Std	22.2805	NM	6.6076	6.1510	5.9268	6.1179	6.5157	**3.8498**	7.0273	7.7982	7.8458	10.8013	7.8863	**7.6035**	7.6984	10.6311	7.8063
SR	0.3815	NM	-0.2242	-0.2493	-0.3923	-0.2340	-0.1236	-0.0836	-0.0980	0.3533	0.3213	0.2526	0.3079	0.2450	0.1626	0.1914	0.2388
Util	-	NM	-0.0956	-0.0961	-0.1039	-0.0950	-0.0889	-0.0839	-0.0878	-0.0537	-0.0560	-0.0543	-0.0569	-0.0626	-0.0686	-0.0612	-0.0625
									N=750								
Avg	8.6215	NM	-0.6852	-1.5719	-2.2996	-0.6324	-0.1894	-0.6173	0.0418	1.598	1.2009	1.5477	4.0214	0.3049	-0.5512	-0.4215	-0.3578
Std	23.0896	NM	5.4228	5.2714	5.1371	5.2648	6.9322	**3.4811**	7.1704	7.6450	7.6075	10.1021	10.4095	7.2053	**7.0476**	9.4397	7.8099
SR	0.3734	NM	-0.1264	-0.2982	-0.4476	-0.1201	-0.0273	-0.1773	0.0058	0.209	0.1579	0.1532	0.3863	0.0423	-0.0782	-0.0446	-0.0458
Util	-	NM	-0.0887	-0.0975	-0.1047	-0.0882	-0.0839	-0.0879	-0.0816	-0.0662	-0.0702	-0.067	-0.0424	-0.0791	-0.0876	-0.0865	-0.0857
									N=1000								
Avg	8.2003	NM	-0.7255	-0.9619	-1.496	0.0187	0.3763	-0.7845	0.2245	0.4056	0.5223	2.0092	-0.0459	-1.6053	-1.3336	0.5399	-1.7793
Std	23.3728	NM	5.3923	5.3877	5.2669	5.2709	7.2995	**3.1670**	7.3817	8.5921	8.8197	9.7437	8.7381	**8.6557**	8.8784	9.1477	8.8594
SR	0.3508	NM	-0.1345	-0.1785	-0.2840	0.0035	0.0516	-0.2477	0.0304	0.0472	0.0592	0.2062	-0.0053	-0.1855	-0.1502	0.0590	-0.2008
Util	-	NM	-0.0848	-0.0871	-0.0924	-0.0774	-0.0740	-0.0852	-0.0756	-0.0739	-0.0727	-0.0581	-0.0010	-0.0938	-0.0912	-0.0726	-0.0956

Table 9: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using the third subperiod of the whole sample period. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. This subperiod for the assets ranges from 4 January 2012 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 95 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	8.7180	1.6010	-1.8240	-0.0878	0.3355	0.5437	2.0466	-0.7555	1.9529	1.1629	0.9803	3.4178	0.8286	1.2413	1.0701	3.0979	0.9153
Std	11.6008	5.9722	6.7277	6.2329	6.0806	6.0303	6.0600	6.2957	**5.8910**	6.1760	6.1502	6.5716	**5.9689**	5.9961	6.0012	5.9995	6.0152
SR	0.7515	0.2681	-0.2711	-0.0141	0.0552	0.0902	0.3377	-0.1200	0.3315	0.1883	0.1594	0.5201	0.1388	0.2070	0.1783	0.5164	0.1522
To	0.0517	0.2949	0.1337	0.1209	0.1290	0.1839	0.2529	0.3212	0.2565	1.2764	1.2131	0.8633	1.1960	0.7189	0.6909	0.6999	0.6934
Util	-	-0.0697	-0.1037	-0.0864	-0.0822	-0.0802	-0.0653	-0.0931	-0.0662	0.0112	0.0094	0.0335	0.0079	0.0120	0.0103	0.0304	0.0088
									N=50								
Avg	9.0278	2.2662	-1.6754	-0.5775	0.0200	0.3197	2.6227	0.3146	2.5645	1.4164	1.3270	4.3391	0.5931	0.3547	0.1539	3.7625	-0.1480
Std	11.8586	5.9197	6.9679	6.4318	6.1902	6.0467	5.9118	6.2329	**5.7733**	7.0583	6.9745	7.2095	6.8776	6.5150	**6.4074**	6.8243	6.5066
SR	0.7613	0.3828	-0.2404	-0.0898	0.0032	0.0529	0.4436	0.0505	0.4442	0.2007	0.1903	0.6019	0.0862	0.0544	0.0240	0.5513	-0.0227
To	0.0532	0.4911	0.1588	0.1595	0.1768	0.2283	0.4107	0.5135	0.3712	1.4622	1.4296	0.8102	1.3765	0.8521	0.8433	0.7336	0.8276
Util	-	-0.0662	-0.1053	-0.0944	-0.0884	-0.0855	-0.0626	-0.0855	-0.0632	0.0137	0.0128	0.0426	0.0055	0.0032	0.0012	0.0370	-0.0018
									N=100								
Avg	7.9174	1.1276	-1.3515	-1.1547	-0.6097	-0.6116	1.0777	0.0807	1.0688	1.0839	1.0488	3.0812	0.0497	-0.5666	-0.5440	1.7940	-1.6071
Std	10.7960	6.1082	6.8133	6.2893	5.9189	5.9107	5.8734	6.1162	**5.5408**	5.9906	5.9885	6.2985	**5.9668**	6.4520	6.4394	6.4860	6.4158
SR	0.7334	0.1846	-0.1984	-0.1836	-0.1030	-0.1035	0.1835	0.0132	0.1929	0.1809	0.1751	0.4892	0.0083	-0.0878	-0.0845	0.2766	-0.2505
To	0.0522	1.1646	0.1386	0.1704	0.1969	0.2373	0.7792	0.9906	0.5951	1.6625	1.6194	0.7551	1.5819	0.9324	0.9275	0.8158	0.9024
Util	-	-0.0666	-0.0912	-0.0892	-0.0838	-0.0838	-0.0671	-0.0770	-0.0671	0.0105	0.0101	0.0302	0.0002	-0.0059	-0.0057	0.0175	-0.0162
									N=250								
Avg	7.4975	NM	0.1179	-0.4523	0.3557	-0.1738	3.3452	5.1634	7.7389	3.1857	3.1372	3.9113	3.4301	1.9593	1.9472	3.0254	3.1673
Std	9.9881	NM	7.3986	7.1200	6.9491	**6.5564**	9.4435	11.2400	13.8031	6.5501	6.5698	6.6131	6.6207	6.6123	6.6163	6.4172	**6.3335**
SR	0.7506	NM	0.0159	-0.0635	0.0512	-0.0265	0.3542	0.4594	0.5607	0.4864	0.4775	0.5915	0.5181	0.2963	0.2943	0.4715	0.5001
To	0.0491	NM	0.1425	0.2115	0.2524	0.3370	1.7381	2.8027	0.2336	1.8726	1.8361	0.7055	0.8476	1.0211	1.0218	0.8537	0.8653
Util	-	NM	-0.0727	-0.0783	-0.0703	-0.0755	-0.0409	-0.0234	0.0016	0.0313	0.0308	0.0385	0.0337	0.0191	0.019	0.0297	0.0311
									N=500								
Avg	8.2117	NM	0.3804	0.1824	0.6445	0.6564	4.0140	0.1226	4.1800	0.9204	1.2623	1.5341	1.1266	0.8229	1.3287	0.9270	0.8389
Std	11.5324	NM	2.8922	3.0508	3.0738	**2.3880**	9.8499	2.7562	8.6211	3.1031	3.0073	3.8601	**2.9233**	3.0772	3.0797	3.5407	3.0757
SR	0.7121	NM	0.1315	0.0598	0.2097	0.2749	0.4075	0.0445	0.4849	0.2966	0.4197	0.3974	0.3854	0.2674	0.4314	0.2618	0.2728
Util	-	NM	-0.0767	-0.0787	-0.0741	-0.0740	-0.0413	-0.0793	-0.0395	0.0091	0.0125	0.0151	0.0111	0.0081	0.0131	0.0091	0.0083
									N=750								
Avg	7.6283	NM	0.3621	0.3483	0.4792	0.4591	5.694	-0.0136	5.3202	0.6229	0.5208	1.4361	0.1593	0.9375	0.8724	1.5735	0.4048
Std	11.9398	NM	2.1377	2.3787	2.2728	1.9982	7.1722	**1.6842**	6.5391	**2.2913**	2.4275	2.9468	2.3445	2.5344	2.6419	2.7903	2.5407
SR	0.6389	NM	0.1694	0.1464	0.2108	0.2297	0.7939	-0.0081	0.8136	0.2718	0.2145	0.4873	0.0679	0.3699	0.3302	0.5639	0.1593
Util	-	NM	-0.0710	-0.0712	-0.0699	-0.0700	-0.01850	-0.0747	-0.0221	0.0061	0.0051	0.0142	0.0015	0.0092	0.0086	0.0155	0.0040
									N=1000								
Avg	7.6757	NM	4.2862	4.3109	4.6037	4.3918	3.9121	4.0370	3.5984	4.9980	5.0949	5.1694	4.6218	5.1628	5.2829	5.1468	4.7036
Std	12.195	NM	12.1486	11.8721	12.2390	12.3120	5.4765	12.1094	**5.1633**	12.8861	12.6666	12.7222	**12.6556**	12.8861	12.6786	12.7267	12.6698
SR	0.6294	NM	0.3528	0.3631	0.3761	0.3567	0.7143	0.3334	0.6969	0.3879	0.4022	0.4063	0.3652	0.4006	0.4167	0.4044	0.3712
Util	-	NM	-0.0335	-0.0332	-0.0304	-0.0325	-0.0364	-0.0360	-0.0395	0.0484	0.0494	0.0501	0.0447	0.0500	0.0513	0.0499	0.0455

6 Conclusion

In this paper, I have summarised and extended different existing estimators of the covariance matrix when the dimensionality does not ensure a well-conditioned sample covariance matrix. My empirical analysis is based on a stylised version of the global minimum variance portfolio formation as introduced by Markowitz (1952) under large-dimensional asymptotics, where the amount of assets tends to go to infinity with respect to the amount of observations.

For this estimation I have used the implied covariance matrix following from 1-, 3- and 5-factor models as initiated by Fan et al. (2008) as well as latent factor models as proposed by Fan et al. (2013). Furthermore, I have used linear shrinkage methods, which usage have arisen due to the works of Ledoit and Wolf (2004). These linear shrinkage methods are extended along the nonlinear domain by optimally shrinking the sample eigenvalues in a nonlinear fashion in the beneficial works of Ledoit and Wolf (2012). Furthermore, I have approached the covariance estimation in the portfolio setting dynamically with time-varying volatility. This is done by first estimating asset returns as individual GARCH series. Then, devolatise the return series and finally model them while allowing for conditional correlations in the DCC process as introduced by Engle (2002). These models are extended by Engle and Kelly (2012), by allowing for pairs of returns to have the same correlation on a given day, but varying over time in DECO models. I combined these two fruitful dynamic methods with the static shrinkage methods, by first applying the (non)linear shrinkage methods to construct a pre-processed correlation matrix and then use this as a target matrix in the DCC process.

My main findings are that DECO models ensure stable weights and have the potential to establish well-conditioned estimates of the (inverse) covariance matrix in high dimensions as well as low dimensions. However, this class of estimators does not consistently outperform the previously suggested factor implied or pure (non)linear shrinkage methods. Also, the results do not indicate a strict preference for DECO models above DCC models. The various and diverging results in the robustness analyses do not point out one uniform winner among those methods and indicate that the prior chosen shrinkage targets are only marginally affecting the performance of the estimate with the goal of attaining the minimum variance.

In contrast to the findings of Ledoit and Wolf (2017b), I do not find that the nonlinear shrinkage method consistently dominates its linear shrinkage counterparts. Especially when the amount of assets equals the amount of observations, the performance heavily deteriorates. This could be due to some negligibly small sample eigenvalues entering the nonlinear optimisation due to the high amount of correlation or linearly dependence between the asset returns. In this case, the optimal nonlinear shrinkage intensities are very unstable and produce inferior estimates of the covariance matrix. It remains for further research how to treat such cases. If one estimator has to be chosen as a winner in my research, it has to be the POET estimator of Fan et al. (2013). This estimator has the capability of being among the most stable estimators across dimensions and different empirical settings. This is achieved without actually imposing factors and by only making use of the principal components of the data.

Next to the empirical study, I performed a theoretical simulation, where I simulate asset returns based on the 5-factor model with thresholding as described in Fan et al. (2011). For these theoretical returns, the only method outperforming the 5-factor implied covariance method in minimising the out-of-sample variance, is the 1-factor implied linear shrinkage model. Also, the POET and dynamic models show very promising results without actually inferring the true model from which the returns are simulated from. This shows the strong underlying theoretical optimal solution and the strength of these class of estimators.

Concluding, I have shown that all shrinkage methods of interest are able to effectively produce conditioned, invertible covariance matrix estimates. These methods are able to handle high dimensions, but could also improve upon common estimators in low dimensions. However, none of the inspected individual methods is able to consistently outperform another in shifted empirical settings or for other samples of asset returns.

7 Discussion

In this research, I focus exclusively on estimating the covariance matrix in high-dimensional settings. The impact of the estimation error of the expected returns is isolated by using global minimum variance portfolios, which are only dependent on the estimate of the covariance matrix. This is common practice for researchers, however an optimal stable estimator, minimising estimation error for both the covariance matrix and expected returns in high dimensions is still lacking and remains an exhaustive topic in finance and beyond.

Regarding considered methods and results, the POET method turns out to be a very robust estimator for the covariance even without imposing observable factors but by only making use of the data through PCA. However, in my research, the choice for the thresholding is somewhat arbitrary. In my research, I have used the adaptive thresholding function as Fan et al. (2013) do, but any other thresholding functions could be employed and analysed for performance. Also, for the factor models, I assumed exactly specified diagonal matrices for the errors. Other initialisations could be employed and used for cross-validation. This is common practice, but the impact of this restriction has yet to be determined.

Also, for nonlinear shrinkage methods, there is still room for improvement. Ledoit and Wolf (2017a) work out a method to directly estimate the inverse covariance matrix instead of inverting the estimate. However, this is still a working paper and the provided code and methods did not result in optimal estimators for my sample. Next to that, Ledoit and Wolf (2017b) prove that their method is the optimal nonlinear shrinkage solution for minimising a certain loss function, but acknowledge that it somewhat lacks intuition or interpretation. Therefore, it is hard to mathematically explain the found instability of this estimator for the samples when the amount of assets is equal to the amount of observations. Applying a correlation filter did not overcome this unstable results and thus this remains for a deeper underlying analysis.

Unfortunately, the estimation of DCC-type models remains computationally heavy due to the presence of multiple likelihood maximisations. Due to this, I have limited the estimation to only include a relatively small amount of observations per moving window. Estimating and optimising the parameters by making use of a longer time series window could possibly improve the performance of this class of estimators. Another extension along the DECO models could be to employ the sorted correlation structures between component in blocks, called Block-DECO. Block-DECO directly models the block correlation structure ex-ante and makes use of it within the estimation procedure. In this way, the estimation structure is allowed to have multiple blocks and employ this structure. A potentially interesting usage of blocks are blocks formed of stock returns with particular intra- and inter-industry correlation dynamics.

Other promising directions for improvements are for example in the field of design-free estimation and graphical lasso. Abadir, Distaso, and Žikeš (2014) introduce design-free estimation by making use of the orthogonal decomposition of the sample covariance matrix. The estimation exploits conditioned orthogonal matrices and improves the estimation of the eigenvalues. The key principle is that the eigenvectors are estimated from only a small fraction of the data. Then, those are used to transform the rest of the data into approximately orthogonal series that deliver well-conditioned estimators by construction. This method for constructing covariance matrices has promising characteristics. First, it is design-free in the sense that no assumptions are made on the densities of the random sample or any underlying parametric model. Second, the method always delivers nonsingular well-conditioned estimators, which are invertible by construction. However, the simple sample covariance estimation that Abadir et al. (2014) use in both steps of their procedure optimises the LS criterion, which is proven to be sub-optimal for estimating the inverse of the covariance matrix.

A different approach would be to estimate the inverted covariance matrix directly, conditional on certain financial or statistical factors. Yuan and Lin (2007) introduce penalised likelihoods for this task, optimised using the graphical lasso (Glasso) algorithm of Friedman, Hastie, and Tibshirani (2008). Graphical lasso regularises the estimate of the inverted covariance matrix by shrinking all entries. The method sets small elements equal to zero and results in a sparse matrix. Also, graphical lasso has the advantage of producing estimates that are symmetric and positive semi-definite. Furthermore, for the calibration it is easy to cross-validate the regularisation for a given specific task, for example minimising portfolio volatility. In this way, the method overcomes the issue of inverting high-dimensional covariance matrices, while remaining computationally feasible. However, one possible drawback is that it is unclear what the factors (should) represent and the heavy dependence on the structure of the factors. Summarising, there is reasonable doubt for some of the found results and potential improvements could be made. One paper is not enough to cover all recent advancements of current state-of-the-art methods and improve upon them in order to deliver an optimal estimate of the covariance matrix in high dimensions. Indications and proposals given here only make up a small domain which can be studied more extensively, but demonstrate that there is ample room for development in high-dimensional covariance estimation.

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Appendices

A Composite Likelihood for DCC

Assume:

$$
\begin{aligned}
y_{t|t-1} &\sim N\left(0, D_t R_t D_t\right), \\
h_{i,t} &= \omega_i + a_i y_{i,t-1}^2 + b_i h_{i,t-1}, \\
D_t &= \text{diag}\left(\sqrt{h_{i,t}}\right), \\
\epsilon_t &= D_t^{-1} y_t, \\
\bar{R} &= \frac{1}{T}\sum_{t=1}^{T} \epsilon_t \epsilon_t', \\
\hat{\Omega} &= (1-\alpha-\beta)\bar{R}, \\
Q_t &= \hat{\Omega} + \alpha \epsilon_{t-1}\epsilon_{t-1}' + \beta Q_{t-1}, \\
R_t &= \text{diag}(Q_t)^{-\frac{1}{2}} Q_t \text{diag}(Q_t)^{-\frac{1}{2}}.
\end{aligned}
$$

By writing $H_t = D_t R_t D_t$, the log likelihood can be written as

$$
\begin{aligned}
L &= -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log|H_t| + y_t' H_t^{-1} y_t\right), \\
&= -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log|D_t R_t D_t| + y_t' D_t^{-1} R_t^{-1} D_t^{-1} y_t\right), \\
&= -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + 2\log|D_t| + \log|R_t| + \epsilon_t' R_t^{-1}\epsilon_t\right), \\
&= -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + 2\log|D_t| + y_t' D_t^{-1} D_t^{-1} y_t - \epsilon_t'\epsilon_t + \log|R_t| + \epsilon_t' R_t^{-1}\epsilon_t\right),
\end{aligned}
$$

which can be split in a volatility part and a correlation part

$$
L(\theta,\phi) = L_V(\theta) + L_C(\theta,\phi),
$$

with the volatility part equal to

$$
L_V(\theta) = -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + 2\log|D_t| + y_t' D_t^{-1} D_t^{-1} y_t\right),
$$

and correlation part equal to

$$
L_C(\theta,\phi) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log|R_t| + \epsilon_t' R_t^{-1}\epsilon_t - \epsilon_t'\epsilon_t\right).
$$

The volatility part of the likelihood function is equal to the sum of individual GARCH likelihoods by writing

$$L_V(\theta) = -\frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(\log(2\pi) + \log\left(h_{i,t}\right) + \frac{y_{i,t}^2}{h_{i,t}} \right).$$

The two-step approach to maximizing the likelihood is to find

$$\hat{\theta} = \arg\max \left\{ L_V(\theta) \right\},$$

subsequently conditional on the optimal value for the volatility likelihood, find the optimal solution to the correlation likelihood

$$\hat{\phi} = \arg\max \left\{ L_C(\phi \mid \hat{\theta}) \right\}.$$

B Robustness Analyses Results

B.1 Low Market Capitalisation Habitat

Table 10: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using the lowest market capitalisation stocks. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix. This sample comprises 285 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.5016	0.5250	0.4836	0.4847	0.4786	0.1199	0.5250	0.4834	0.5422	0.5193	0.5223	0.5274	0.5254	0.5099	0.5077	0.5172
Min	0.0333	-0.0052	-0.0115	0.0002	0.0002	-0.0030	-0.0015	-0.0115	-0.0039	-0.0063	-0.0020	-0.0018	-0.0027	-0.0004	0.0000	0.0001	-0.0001
SD	0.0000	0.0983	0.1045	0.0947	0.0949	0.0938	0.0309	0.1045	0.0949	0.1046	0.0999	0.1005	0.1016	0.1017	0.0985	0.0981	0.1000
									N=50								
Max	0.0200	0.4964	0.5020	0.4671	0.4683	0.4536	0.0774	0.5020	0.4711	0.5306	0.4987	0.4980	0.5146	0.5158	0.4946	0.4920	0.5110
Min	0.0200	-0.0070	-0.0069	-0.0004	-0.0005	-0.0022	-0.0026	-0.0069	-0.0053	-0.0060	-0.0010	-0.0008	-0.0018	-0.0005	-0.0001	-0.0001	-0.0003
SD	0.0000	0.0767	0.0786	0.0724	0.0726	0.0702	0.0189	0.0786	0.0730	0.0807	0.0756	0.0755	0.0781	0.0789	0.0754	0.0750	0.0780
									N=100								
Max	0.0100	0.4948	0.4704	0.4503	0.4514	0.4260	0.0472	0.4704	0.4600	0.5025	0.4871	0.4732	0.5093	0.4932	0.4890	0.4740	0.5169
Min	0.0100	-0.0121	-0.0031	-0.0011	-0.0012	-0.0014	-0.0061	-0.0031	-0.0083	-0.0025	-0.0007	-0.0002	-0.0014	-0.0002	-0.0001	-0.0001	-0.0004
SD	0.0000	0.0548	0.0524	0.0499	0.0500	0.0470	0.0110	0.0524	0.0509	0.0543	0.0525	0.0511	0.0551	0.0535	0.0530	0.0514	0.0561
									N=250								
Max	0.0040	NM	0.3488	0.3453	0.3466	0.3119	0.0209	0.3493	0.3553	0.3593	0.3855	0.3408	0.3620	0.3532	0.3806	0.3415	0.3698
Min	0.0040	NM	-0.0015	-0.0013	-0.0014	-0.0009	-0.0087	-0.0015	-0.0557	-0.0009	-0.0012	-0.0000	-0.0053	-0.0001	-0.0002	-0.0000	-0.0002
SD	0.0000	NM	0.0260	0.0257	0.0257	0.0231	0.0058	0.0260	0.0582	0.0258	0.0276	0.0244	0.0261	0.0255	0.0275	0.0246	0.0267
									N=500								
Max	0.0020	NM	0.2346	0.2387	0.2414	0.2072	0.0115	0.2387	0.0119	0.2469	0.2509	0.2305	0.3077	0.2382	0.2400	0.2283	0.2866
Min	0.0020	NM	-0.0009	-0.0014	-0.0015	-0.0006	-0.0073	-0.0011	-0.0071	-0.0007	-0.0009	-0.0001	-0.0021	-0.0001	-0.0001	-0.0000	-0.0003
SD	0.0000	NM	0.0134	0.0135	0.0137	0.0118	0.0034	0.0136	0.0034	0.0133	0.0135	0.0124	0.0165	0.0130	0.0131	0.0124	0.0156
									N=750								
Max	0.0013	NM	0.1778	0.1852	0.1883	0.1596	0.0096	0.1902	0.0100	0.1937	0.1911	0.1732	0.2512	0.1840	0.1818	0.1753	0.2192
Min	0.0013	NM	-0.0009	-0.0018	-0.0019	-0.0005	-0.0059	-0.0012	-0.0058	-0.0007	-0.0009	-0.0000	-0.0025	-0.0001	-0.0001	-0.0000	-0.0002
SD	0.0000	NM	0.0088	0.0092	0.0093	0.0079	0.0023	0.0093	0.0023	0.0089	0.0088	0.0080	0.0115	0.0085	0.0085	0.0081	0.0102
									N=1000								
Max	0.0010	NM	0.1163	0.1103	0.1129	0.1285	0.0084	0.1296	0.0085	0.1767	0.1671	0.1445	0.2050	0.1605	0.1557	0.1465	0.1741
Min	0.0010	NM	-0.0046	-0.0060	-0.0064	-0.0018	-0.0048	-0.0046	-0.0047	-0.0008	-0.0006	-0.0001	-0.0121	-0.0001	-0.0001	-0.0000	-0.0002
SD	0.0000	NM	0.0054	0.0053	0.0054	0.0057	0.0017	0.0059	0.0016	0.0070	0.0066	0.0058	0.0087	0.0065	0.0063	0.0060	0.0072

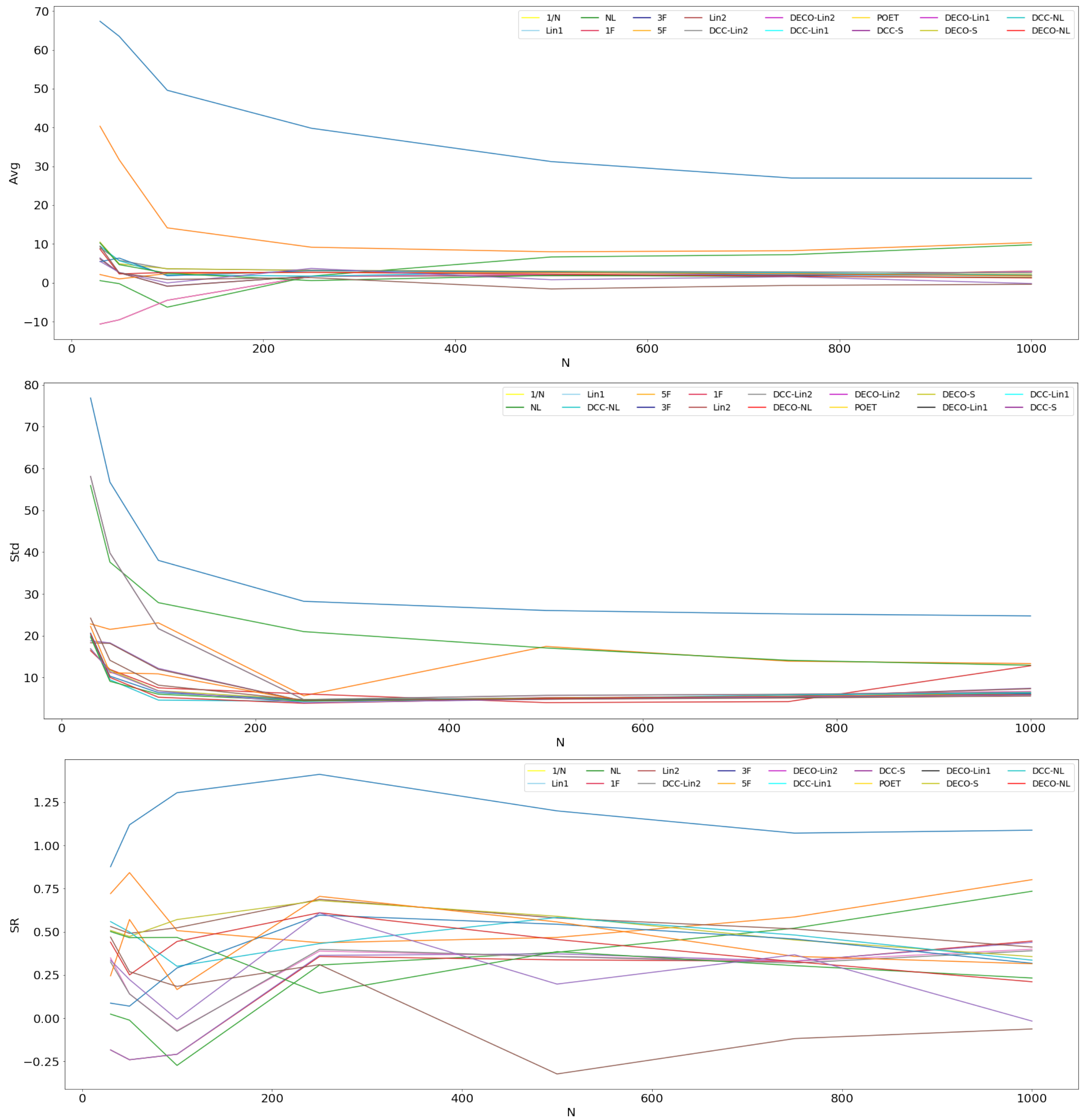

Figure 6: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using the lowest market capitalisation stocks as in Table 5, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019. This sample comprises of 285 portfolio formations.

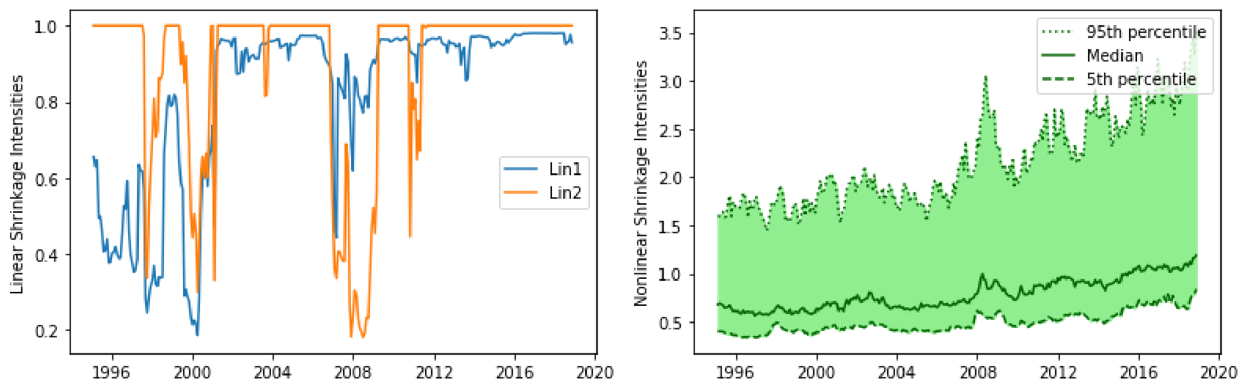

Figure 7: (Non)linear shrinkage intensities for the formed GMV portfolios when using the lowest market capitalisation asset returns for the training period. The sample period ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per portfolio for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 285 portfolio formations. $N = 1000$.

B.2 Short-sale Constraints

Table 11: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using a constrained optimization enforcing non-negative weights in the underlying assets. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix. This sample comprises 285 portfolio formations.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL
					N=30				
Max	0.0333	0.0344	0.1788	0.0344	0.0344	0.0344	0.3006	0.0604	0.3033
Min	0.0333	0.0321	0.0000	0.0321	0.0321	0.0321	0.0000	0.0118	0.0000
SD	0.0000	0.0006	0.0510	0.0006	0.0006	0.0006	0.0708	0.0154	0.0707
					N=50				
Max	0.0200	0.0210	0.1222	0.0210	0.0210	0.0210	0.2272	0.0454	0.2227
Min	0.0200	0.0188	0.0000	0.0188	0.0188	0.0188	0.0000	0.0043	0.0000
SD	0.0000	0.0005	0.0344	0.0005	0.0005	0.0005	0.0476	0.0129	0.0462
					N=100				
Max	0.0100	0.0108	0.0683	0.0108	0.0108	0.0108	0.1640	0.0300	0.1360
Min	0.0100	0.0092	0.0000	0.0092	0.0092	0.0092	0.0000	0.0001	0.0000
SD	0.0000	0.0004	0.0192	0.0004	0.0004	0.0004	0.0279	0.0097	0.0248
					N=250				
Max	0.0040	0.0051	0.0375	0.0051	0.0051	0.0051	0.1548	0.0170	0.0915
Min	0.0040	0.0030	0.0000	0.0030	0.0030	0.0030	0.0000	0.0000	0.0000
SD	0.0000	0.0005	0.0090	0.0005	0.0005	0.0005	0.0170	0.0051	0.0112
					N=500				
Max	0.0020	0.0054	0.0313	0.0053	0.0053	0.0054	0.1761	0.0229	0.0520
Min	0.0020	0.0003	0.0000	0.0003	0.0003	0.0003	0.0000	0.0000	0.0000
SD	0.0000	0.0009	0.0052	0.0009	0.0009	0.0009	0.0126	0.0039	0.0064
					N=750				
Max	0.0013	0.0049	0.0236	0.0048	0.0048	0.0049	0.1433	0.0183	0.0331
Min	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
SD	0.0000	0.0008	0.0038	0.0008	0.0008	0.0008	0.0094	0.0029	0.0042
					N=1000				
Max	0.0010	0.0046	0.0195	0.0045	0.0045	0.0045	0.1240	0.0149	0.0247
Min	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
SD	0.0000	0.0008	0.0030	0.0007	0.0007	0.0008	0.0077	0.0024	0.0032

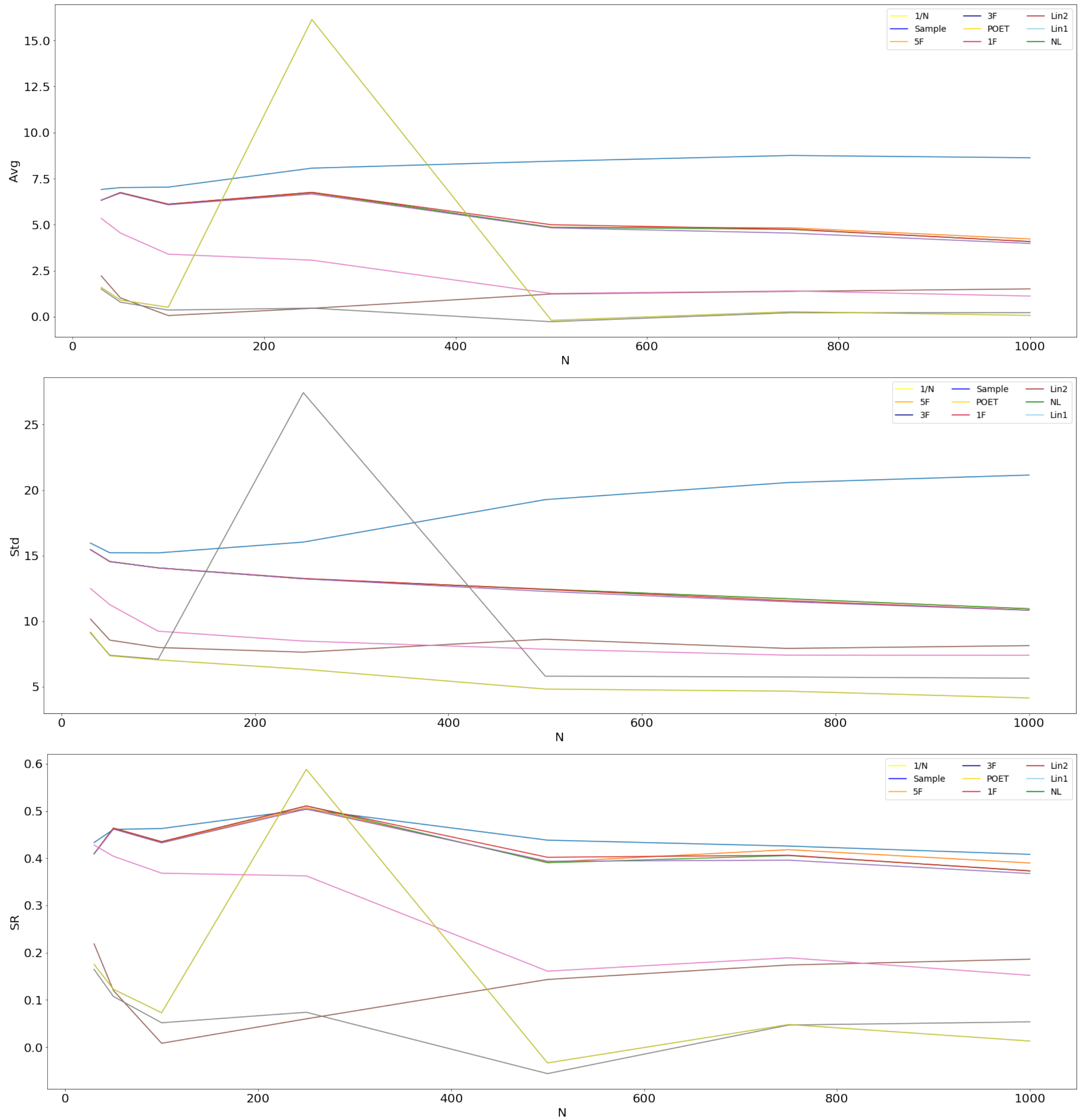

Figure 8: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using a constrained optimization enforcing non-negative weights in the underlying assets as in Table 6, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019. This sample comprises of 285 portfolio formations.

B.3 Subperiod Analysis

Table 12: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using the first subperiod of the sample period. The subperiod for the assets ranges from 13 February 1995 to 16 January 2004, based on 250 training observations used per asset for the construction of the covariance matrix. This sample comprises 95 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.2824	0.2664	0.2584	0.2612	0.2382	0.2408	0.2979	0.2617	0.3565	0.3381	0.2823	0.3472	0.3149	0.3066	0.2565	0.3168
Min	0.0333	-0.0266	-0.0231	-0.0117	-0.0114	-0.0155	-0.0237	-0.0351	-0.0227	-0.0242	-0.0186	-0.0079	-0.0181	-0.0063	-0.0055	-0.0018	-0.0064
SD	0.0000	0.0701	0.0690	0.0652	0.0661	0.0600	0.0634	0.0764	0.0656	0.0790	0.0750	0.0623	0.0770	0.0742	0.0721	0.0600	0.0746
									N=50								
Max	0.0200	0.1890	0.1527	0.1538	0.1544	0.1387	0.1501	0.1887	0.1586	0.2426	0.2309	0.1741	0.2336	0.2003	0.2012	0.1597	0.2089
Min	0.0200	-0.0351	-0.0174	-0.0126	-0.0124	-0.0141	-0.0302	-0.0346	-0.0265	-0.0193	-0.0155	-0.0039	-0.0141	-0.0049	-0.0048	-0.0018	-0.0055
SD	0.0000	0.0421	0.0382	0.0372	0.0377	0.0334	0.0371	0.0434	0.0362	0.0464	0.0445	0.0338	0.0452	0.0426	0.0427	0.0340	0.0444
									N=100								
Max	0.0100	0.1531	0.1031	0.1065	0.1075	0.0996	0.1075	0.1390	0.0904	0.1759	0.1743	0.1101	0.1770	0.1381	0.1430	0.1087	0.1561
Min	0.0100	-0.0542	-0.0155	-0.0159	-0.0155	-0.0108	-0.0368	-0.0408	-0.0262	-0.0131	-0.0128	-0.0023	-0.0117	-0.0037	-0.0038	-0.0019	-0.0047
SD	0.0000	0.0304	0.0214	0.0215	0.0220	0.0203	0.0242	0.0274	0.0198	0.0261	0.0260	0.0168	0.0267	0.0235	0.0242	0.0185	0.0263
									N=250								
Max	0.0040	NM	0.0422	0.0461	0.0468	0.0439	0.0556	0.0811	0.0191	0.0929	0.0932	0.0473	0.0539	0.0630	0.0637	0.0463	0.0549
Min	0.0040	NM	-0.0075	-0.0103	-0.0106	-0.0066	-0.0325	-0.0309	-0.0014	-0.0068	-0.0061	-0.0006	-0.0028	-0.0016	-0.0016	-0.0006	-0.0010
SD	0.0000	NM	0.0083	0.0089	0.0091	0.0086	0.0140	0.0141	0.0035	0.0107	0.0107	0.0062	0.0067	0.0092	0.0093	0.0068	0.0080
									N=500								
Max	0.0020	NM	0.0520	0.0581	0.0613	0.0544	0.0196	0.1097	0.0125	0.094	0.0955	0.0665	0.1226	0.0827	0.0847	0.0682	0.1050
Min	0.0020	NM	-0.0060	-0.0071	-0.0077	-0.0039	-0.0146	-0.0125	-0.0085	-0.0023	-0.0020	-0.0002	-0.0040	-0.0004	-0.0004	-0.0001	-0.0007
SD	0.0000	NM	0.0053	0.0059	0.0061	0.0055	0.0057	0.0084	0.0038	0.0065	0.0066	0.0047	0.0083	0.006	0.0061	0.0049	0.0076
									N=750								
Max	0.0013	NM	0.0453	0.0512	0.0537	0.0489	0.0105	0.0951	0.0076	0.0997	0.0997	0.0771	0.1368	0.0913	0.0917	0.0791	0.1180
Min	0.0013	NM	-0.0051	-0.0060	-0.0071	-0.0032	-0.0089	-0.0090	-0.0062	-0.0013	-0.0015	-0.0001	-0.0029	-0.0002	-0.0002	-0.0001	-0.0004
SD	0.0000	NM	0.0038	0.0043	0.0044	0.0040	0.0033	0.0059	0.0024	0.0052	0.0053	0.0040	0.0070	0.0049	0.0049	0.0042	0.0063
									N=1000								
Max	0.0010	NM	0.0398	0.0450	0.0473	0.0438	0.0073	0.0842	0.0056	0.1055	0.0982	0.0786	0.1429	0.0931	0.0896	0.0785	0.1215
Min	0.0010	NM	-0.0045	-0.0052	-0.0061	-0.0028	-0.0069	-0.0071	-0.0052	-0.0010	-0.0019	-0.0001	-0.0023	-0.0001	-0.0001	-0.0000	-0.0003
SD	0.0000	NM	0.0030	0.0034	0.0035	0.0032	0.0023	0.0047	0.0018	0.0045	0.0043	0.0034	0.0060	0.0041	0.0040	0.0035	0.0053

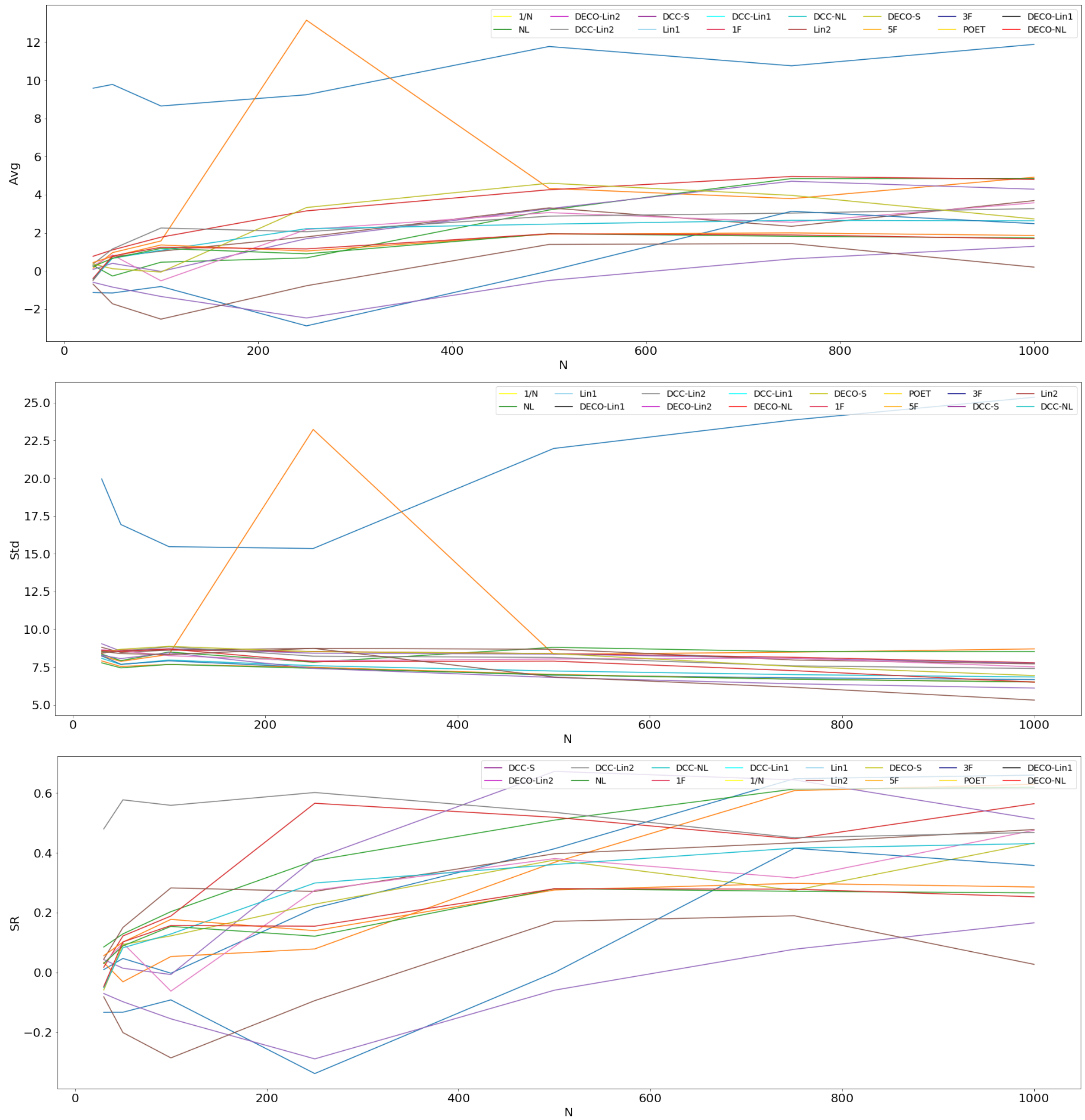

Figure 9: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using the first subperiod of the sample period as in Table 7, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 13 February 1995 to 16 January 2004. This sample comprises of 95 portfolio formations.

Table 13: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using the second subperiod of the sample period. The subperiod for the assets ranges from 19 January 2004 to 3 January 2012, based on 250 training observations used per portfolio for the construction of the covariance matrix. This sample comprises 95 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.3560	0.3444	0.3019	0.3035	0.2900	0.2962	0.4263	0.3034	0.3681	0.3780	0.2835	0.3777	0.3747	0.3856	0.3105	0.3905
Min	0.0333	-0.0726	-0.0490	-0.0455	-0.0424	-0.0553	-0.0647	-0.0963	-0.0619	-0.0398	-0.0411	-0.0179	-0.0369	-0.0183	-0.0195	-0.0115	-0.0201
SD	0.0000	0.0879	0.0932	0.0790	0.0791	0.0785	0.0772	0.1067	0.0766	0.0842	0.0864	0.0616	0.0864	0.0855	0.0882	0.0702	0.0895
									N=50								
Max	0.0200	0.2619	0.2024	0.1966	0.2005	0.1944	0.1984	0.2826	0.2037	0.3008	0.2962	0.2010	0.2967	0.2625	0.2626	0.2131	0.2726
Min	0.0200	-0.0881	-0.0284	-0.0387	-0.0368	-0.0398	-0.0751	-0.0948	-0.0626	-0.0297	-0.0276	-0.0096	-0.0239	-0.0114	-0.0114	-0.0070	-0.0120
SD	0.0000	0.0639	0.0494	0.0472	0.0479	0.0475	0.0543	0.0694	0.0510	0.0560	0.0552	0.0365	0.0553	0.0537	0.0537	0.0433	0.0555
									N=100								
Max	0.0100	0.2252	0.1197	0.1249	0.1310	0.1257	0.1697	0.2260	0.1264	0.2092	0.2073	0.1212	0.2191	0.1555	0.1551	0.1291	0.1658
Min	0.0100	-0.0896	-0.0145	-0.0194	-0.0192	-0.0220	-0.0608	-0.0816	-0.0421	-0.0172	-0.0162	-0.0045	-0.0163	-0.0053	-0.0053	-0.0035	-0.0060
SD	0.0000	0.0449	0.0240	0.0245	0.0253	0.0248	0.0355	0.0440	0.0280	0.0305	0.0302	0.0183	0.0317	0.0267	0.0267	0.0223	0.0285
									N=250								
Max	0.0040	NM	0.0565	0.0629	0.0667	0.0656	0.0937	0.1579	0.0151	0.1316	0.1324	0.0647	0.0707	0.0828	0.0832	0.0679	0.0739
Min	0.0040	NM	-0.0058	-0.0105	-0.0099	-0.0116	-0.0581	-0.0781	-0.0008	-0.0106	-0.0102	-0.0020	-0.0032	-0.0019	-0.0019	-0.0013	-0.0015
SD	0.0000	NM	0.0097	0.0104	0.0108	0.0107	0.0219	0.0288	0.0026	0.0138	0.0139	0.0075	0.0080	0.0110	0.0110	0.0091	0.0099
									N=500								
Max	0.0020	NM	0.1871	0.1953	0.1999	0.1985	0.0451	0.2888	0.0318	0.2074	0.2084	0.1483	0.2219	0.1945	0.1948	0.1688	0.2033
Min	0.0020	NM	-0.0092	-0.0142	-0.0149	-0.0083	-0.0224	-0.0256	-0.0154	-0.0030	-0.0029	-0.0004	-0.0070	-0.0004	-0.0004	-0.0002	-0.0005
SD	0.0000	NM	0.0109	0.0113	0.0116	0.0114	0.0072	0.0168	0.0050	0.0113	0.0113	0.0081	0.0122	0.0107	0.0107	0.0092	0.0113
									N=750								
Max	0.0013	NM	0.2384	0.2418	0.2463	0.2517	0.0253	0.3491	0.0199	0.2463	0.2386	0.1924	0.2782	0.2355	0.2292	0.2093	0.2388
Min	0.0013	NM	-0.0082	-0.0121	-0.0128	-0.0066	-0.0133	-0.0162	-0.0102	-0.0015	-0.0014	-0.0002	-0.0033	-0.0002	-0.0002	-0.0001	-0.0004
SD	0.0000	NM	0.0102	0.0104	0.0106	0.0107	0.0041	0.0148	0.0031	0.0103	0.0099	0.0080	0.0116	0.0098	0.0096	0.0087	0.0101
									N=1000								
Max	0.0010	NM	0.2345	0.2334	0.2385	0.2507	0.0183	0.3507	0.0151	0.2467	0.2382	0.1968	0.2613	0.2416	0.2361	0.2112	0.2475
Min	0.0010	NM	-0.0059	-0.0086	-0.0090	-0.0055	-0.0097	-0.0102	-0.0080	-0.0011	-0.0010	-0.0002	-0.0018	-0.0001	-0.0001	-0.0000	-0.0002
SD	0.0000	NM	0.0086	0.0086	0.0088	0.0091	0.0028	0.0125	0.0023	0.0090	0.0087	0.0071	0.0096	0.0088	0.0086	0.0077	0.0091

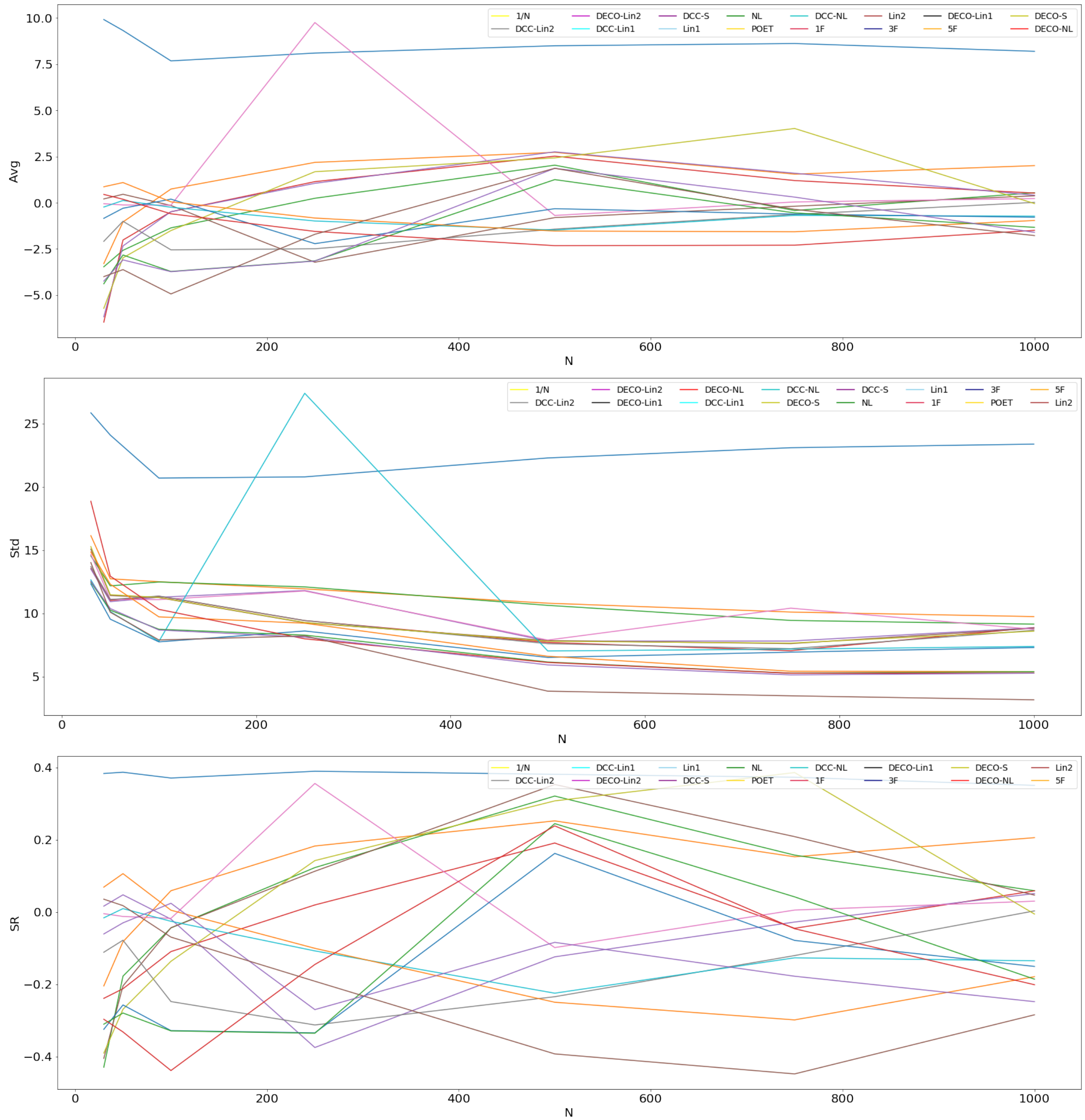

Figure 10: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using the second subperiod of the sample period as in Table 8, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 19 January 2004 to 3 January 2012. This sample comprises of 95 portfolio formations.

Table 14: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using the third subperiod of the sample period. The subperiod for the assets ranges from 4 January 2012 to 17 December 2019, based on 250 training observations used per portfolio for the construction of the covariance matrix. This sample comprises 95 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.4024	0.3518	0.3190	0.3237	0.3161	0.3269	0.4537	0.3545	0.3765	0.3717	0.2699	0.3782	0.3669	0.3665	0.2890	0.3759
Min	0.0333	-0.0518	-0.0343	-0.0262	-0.0226	-0.0361	-0.0452	-0.0624	-0.0461	-0.0371	-0.0342	-0.0136	-0.0332	-0.0134	-0.0132	-0.0061	-0.0142
SD	0.0000	0.0879	0.0851	0.0754	0.0763	0.0762	0.0762	0.1000	0.0792	0.0857	0.0846	0.0605	0.0860	0.0850	0.0849	0.0659	0.0872
									N=50								
Max	0.0200	0.3665	0.2817	0.2679	0.2807	0.2704	0.288	0.4039	0.2896	0.3157	0.3164	0.2026	0.3251	0.2937	0.2969	0.2151	0.3111
Min	0.0200	-0.0618	-0.0355	-0.0321	-0.0256	-0.0309	-0.0525	-0.0687	-0.0451	-0.0271	-0.0260	-0.0086	-0.0244	-0.0104	-0.0105	-0.0045	-0.0116
SD	0.0000	0.0675	0.0565	0.0524	0.0544	0.0536	0.0576	0.0744	0.0551	0.0585	0.0585	0.0367	0.0599	0.0563	0.0568	0.0409	0.0595
									N=100								
Max	0.0100	0.2209	0.1203	0.1225	0.1310	0.1299	0.1536	0.2268	0.1292	0.2095	0.2074	0.1135	0.2139	0.1567	0.1559	0.1198	0.1646
Min	0.0100	-0.0912	-0.0168	-0.0192	-0.0174	-0.0222	-0.0609	-0.0828	-0.0426	-0.0180	-0.0168	-0.0040	-0.0161	-0.0053	-0.0052	-0.0026	-0.0059
SD	0.0000	0.0439	0.0242	0.0242	0.0256	0.0258	0.0335	0.0427	0.0278	0.0310	0.0306	0.0174	0.0315	0.0268	0.0267	0.0204	0.0284
									N=250								
Max	0.0040	NM	0.0477	0.0528	0.0589	0.0643	0.0820	0.1719	0.0355	0.1333	0.1326	0.0581	0.0601	0.0811	0.0807	0.0657	0.0676
Min	0.0040	NM	-0.0075	-0.0106	-0.0108	-0.0163	-0.0459	-0.0805	-0.0031	-0.0117	-0.0110	-0.0019	-0.0028	-0.0022	-0.0022	-0.0013	-0.0015
SD	0.0000	NM	0.0089	0.0095	0.0102	0.0113	0.0189	0.0293	0.0040	0.0138	0.0137	0.0067	0.0069	0.0103	0.0102	0.0084	0.0087
									N=500								
Max	0.0020	NM	0.3506	0.3388	0.3504	0.3750	0.0493	0.4580	0.0422	0.3916	0.4038	0.3132	0.4014	0.3744	0.3840	0.3436	0.3944
Min	0.0020	NM	-0.0110	-0.0221	-0.0202	-0.0054	-0.0222	-0.0211	-0.0189	-0.0020	-0.0021	-0.0002	-0.0017	-0.0002	-0.0003	-0.0001	-0.0003
SD	0.0000	NM	0.0177	0.0174	0.0180	0.0187	0.0058	0.0233	0.0049	0.0191	0.0197	0.0153	0.0197	0.0183	0.0188	0.0167	0.0192
									N=750								
Max	0.0013	NM	0.3746	0.3514	0.3569	0.4041	0.0294	0.4417	0.0258	0.3962	0.3888	0.3363	0.3858	0.3881	0.3826	0.3645	0.3864
Min	0.0013	NM	-0.0068	-0.0122	-0.0117	-0.0025	-0.0139	-0.0103	-0.0122	-0.0011	-0.0010	-0.0001	-0.0007	-0.0001	-0.0001	-0.0000	-0.0001
SD	0.0000	NM	0.0154	0.0146	0.0149	0.0165	0.0033	0.0181	0.0029	0.0161	0.0158	0.0137	0.0158	0.0157	0.0155	0.0147	0.0157
									N=1000								
Max	0.0010	NM	0.3770	0.3509	0.3546	0.4094	0.0203	0.4343	0.0182	0.4415	0.4358	0.3895	0.4294	0.4278	0.4230	0.4024	0.4200
Min	0.0010	NM	-0.0051	-0.0091	-0.0090	-0.0013	-0.0100	-0.0069	-0.0090	-0.0007	-0.0007	-0.0001	-0.0005	-0.0001	-0.0000	-0.0000	-0.0000
SD	0.0000	NM	0.0136	0.0127	0.0129	0.0147	0.0023	0.0156	0.0020	0.0155	0.0152	0.0136	0.0150	0.0150	0.0149	0.0141	0.0148

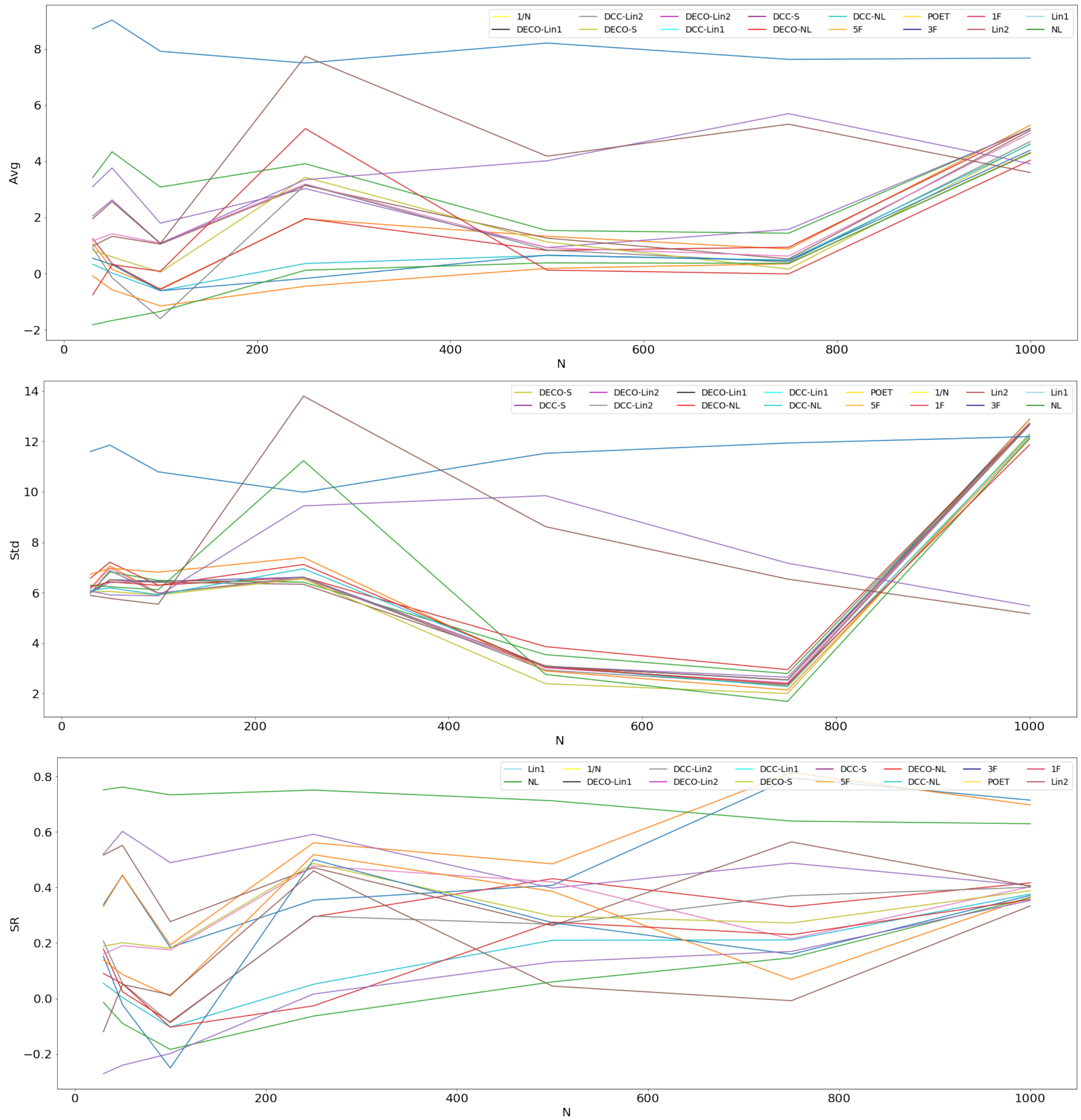

Figure 11: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using the third subperiod of the sample period as in Table 9, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 4 January 2012 to 17 December 2019. This sample comprises of 95 portfolio formations.

(a) 13 February 1995 - 16 January 2004 (b) 19 January 2004 - 3 January 2012 (c) 4 January 2012 - 17 December 2019

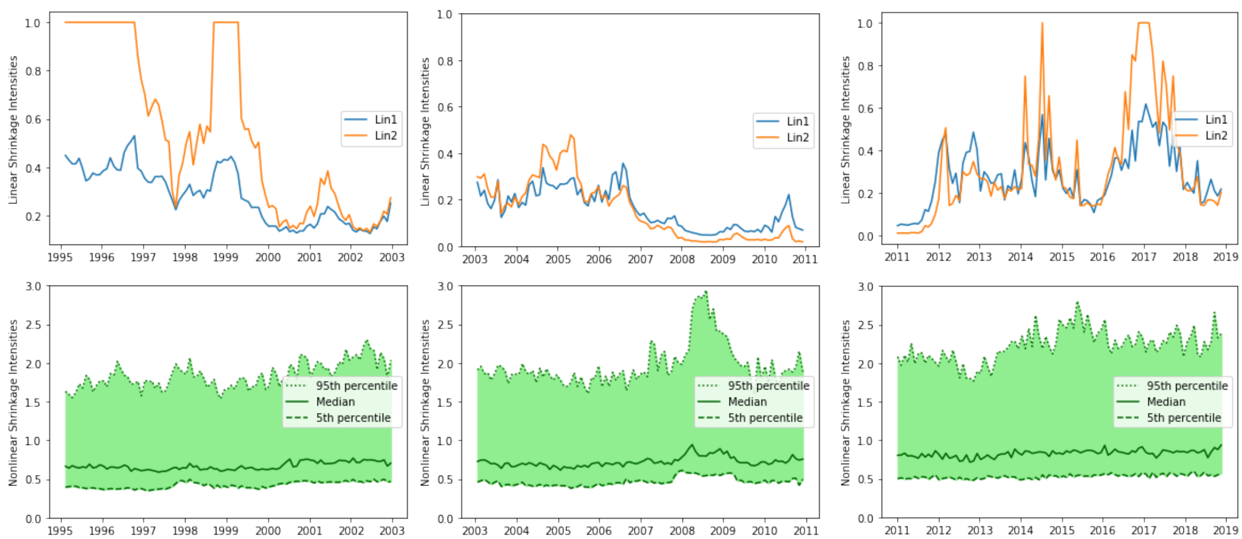

Figure 12: (Non)linear shrinkage intensities for the formed GMV portfolios when using three equally sized subperiods of the sample period. The sample period ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per portfolio for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 285 portfolio formations. $N = 1000$.

B.4 Longer Formation Period

Table 16 shows the performance for the different methods using 500 return observations per asset in the training set. For this sample, the sample covariance is much more competitive in lower dimensions, possibly due to the enlarged training set of returns. Next to that, the POET method again shows to be a robust minimiser, especially in higher dimensions. For the dynamic models there is not a straightforward method to be chosen as most promising target. Also, there is no clear distinction in the performance of static models and dynamic model

Furthermore, in lower dimensions, the shrinkage toward identity is a good estimator for the covariance matrix in order to achieve the lowest out-of-sample variance. Also, the nonlinear shrinkage method performs well in low dimensions. However, both performances do not hold in higher dimensions as performance deteriorates. Again, I find that the performance of the nonlinear shrinkage method is very poor when the amount of assets is equal to the amount of observations. For the dynamic models, the DECO model with linear shrinkage toward identity target is a robust estimator across dimensions. Also, the DCC method with the nonlinear shrinkage covariance matrix as a target performs well on average except for the extremely high dimension $N = 1000$, which is in line with the inferior performance of the pure nonlinear shrinkage method. From a utility perspective, no method achieves a higher utility than the naive $1/N$ diversification, favouring it from the approach of a utility investor. Tables with weights and (non)linear shrinkage intensities can be found in Table 15 and in Figure 14. For factor models, (non)linear shrinkage models and DCC-type models the usual decreasing magnitude of weights over dimensions is observed.

The linear shrinkage intensities in Figure 14 show capricious intensities for both linear shrinkage models. This is partly due to the fact that there are fewer portfolio formations and thus intensities, but this does not explain the jumpy pattern for this sample. Whereas the shrinkage intensity toward the 1-factor implied covariance matrix is above the shrinkage intensity toward identity for the regular case, the opposite is the case when using a longer training observation set. This could be due to the fact that the sample covariance is much less ill-behaved when the amount of assets is only double the amount of return observations instead of four times as large. Next to that, the intensities increase faster and remain higher for a longer period after the global financial crisis indicating heavy reliance on the target matrices. Also, the nonlinear shrinkage intensity shows a different pattern with a higher median of around 1.2 and increased 95-th percentiles intensities around 6-7 for the sample. This shows that on average the eigenvalues are increased and the smaller eigenvalues are generally increased more than for the regular case. Intuition and interpretations around this finding remains for further research.

Table 15: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using a formation period of 500 returns per asset. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 500 training observations used per asset for the construction of the covariance matrix and 21 test observations per portfolio. This sample comprises 273 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.2193	0.1920	0.1796	0.1816	0.1798	0.1894	0.2465	0.2066	0.3195	0.3118	0.2222	0.3141	0.2575	0.2551	0.2048	0.2591
Min	0.0333	-0.0434	-0.0254	-0.0213	-0.0209	-0.0376	-0.0382	-0.0569	-0.0385	-0.0293	-0.0268	-0.0098	-0.0260	-0.0121	-0.0118	-0.0045	-0.0123
SD	0.0000	0.0631	0.0621	0.0564	0.0568	0.0575	0.0573	0.0726	0.0599	0.0739	0.0723	0.0520	0.0729	0.0684	0.0677	0.0531	0.0688
									N=50								
Max	0.0200	0.2001	0.1561	0.1514	0.1541	0.1454	0.1773	0.2195	0.1809	0.2770	0.2729	0.1732	0.2701	0.2102	0.2101	0.1668	0.2140
Min	0.0200	-0.0467	-0.0198	-0.0217	-0.0222	-0.0198	-0.0418	-0.0543	-0.0375	-0.0237	-0.0223	-0.0063	-0.0203	-0.0081	-0.0080	-0.0037	-0.0084
SD	0.0000	0.0489	0.0435	0.0414	0.0420	0.0401	0.0453	0.0542	0.0449	0.0524	0.0518	0.0346	0.0516	0.0482	0.0482	0.0378	0.0491
									N=100								
Max	0.0100	0.1707	0.1101	0.1112	0.1153	0.1073	0.1420	0.1795	0.1325	0.2011	0.2002	0.1105	0.2038	0.1429	0.1436	0.1133	0.1502
Min	0.0100	-0.0577	-0.0135	-0.0178	-0.0180	-0.0154	-0.0480	-0.0590	-0.0386	-0.0162	-0.0161	-0.0034	-0.0156	-0.0045	-0.0046	-0.0024	-0.0050
SD	0.0000	0.0340	0.0245	0.0244	0.0251	0.0239	0.0304	0.0357	0.0276	0.0293	0.0293	0.0175	0.0299	0.0262	0.0263	0.0207	0.0276
									N=250								
Max	0.0040	0.1540	0.0565	0.0622	0.0662	0.0651	0.0818	0.1505	0.0701	0.1241	0.1237	0.0567	0.1298	0.0707	0.0708	0.0567	0.0764
Min	0.0040	-0.0646	-0.0068	-0.0116	-0.0111	-0.0122	-0.0319	-0.0539	-0.0247	-0.0091	-0.0091	-0.0013	-0.0096	-0.0018	-0.0018	-0.0010	-0.0021
SD	0.0000	0.0238	0.0098	0.0106	0.0111	0.0111	0.0148	0.0221	0.0128	0.0128	0.0127	0.0068	0.0135	0.0101	0.0102	0.0081	0.011
									N=500								
Max	0.0020	NM	0.2074	0.2099	0.2172	0.2051	0.0350	0.2884	0.0080	0.2305	0.2293	0.1783	0.1727	0.2189	0.2191	0.1986	0.2093
Min	0.0020	NM	-0.0098	-0.0165	-0.0159	-0.0056	-0.0169	-0.0245	-0.0038	-0.0028	-0.0040	-0.0004	-0.0010	-0.0003	-0.0003	-0.0002	-0.0002
SD	0.0000	NM	0.0118	0.0122	0.0126	0.0116	0.0053	0.0164	0.0015	0.0123	0.0123	0.0095	0.0093	0.0117	0.0117	0.0105	0.0112
									N=750								
Max	0.0013	NM	0.2195	0.2168	0.2217	0.2230	0.0223	0.2831	0.0390	0.2360	0.2356	0.1991	0.2587	0.2299	0.2306	0.2154	0.2408
Min	0.0013	NM	-0.0071	-0.0115	-0.0116	-0.0038	-0.0114	-0.0138	-0.0128	-0.0017	-0.0015	-0.0003	-0.0032	-0.0001	-0.0001	-0.0001	-0.0012
SD	0.0000	NM	0.0100	0.0100	0.0103	0.0100	0.0033	0.0128	0.0040	0.0102	0.0101	0.0086	0.0113	0.0099	0.0099	0.0092	0.0106
									N=1000								
Max	0.0010	NM	0.2229	0.2183	0.2223	0.2296	0.0160	0.2820	0.0305	0.2562	0.2545	0.2171	0.2786	0.2470	0.2480	0.2327	0.2564
Min	0.0010	NM	-0.0052	-0.0084	-0.0089	-0.0027	-0.0087	-0.0087	-0.0094	-0.0012	-0.0012	-0.0002	-0.0257	-0.0001	-0.0001	-0.0000	-0.0004
SD	0.0000	NM	0.0086	0.0086	0.0087	0.0087	0.0024	0.0108	0.0029	0.0094	0.0093	0.0080	0.0110	0.0090	0.0091	0.0085	0.0095

Table 16: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using a formation period of 500 returns per asset. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 500 training observations used per asset for the construction of the covariance matrix and 21 test observations per portfolio. This sample comprises 273 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	7.6458	1.8653	0.0275	0.7159	0.7932	1.0659	1.8914	1.1669	1.9434	1.9566	1.9577	2.1930	1.9511	-0.4049	-0.2144	0.8596	-0.1752
Std	14.2829	6.3867	7.5678	7.0423	7.0302	7.0183	6.4708	6.7480	**6.3781**	7.9837	7.9170	8.2068	7.8721	7.8341	**7.7922**	7.9773	7.8254
SR	0.5353	0.2921	0.0036	0.1017	0.1128	0.1519	0.2923	0.1729	0.3047	0.2451	0.2473	0.2672	0.2479	-0.0517	-0.0275	0.1078	-0.0224
To	0.0544	0.1586	0.0734	0.0695	0.0729	0.1171	0.1387	0.1771	0.1471	1.3023	1.2623	0.8488	1.2594	0.6659	0.6610	0.6291	0.6614
Util	-	-0.0560	-0.0743	-0.0674	-0.0667	-0.0640	-0.0558	-0.0630	-0.0553	-0.0554	-0.0553	-0.0530	-0.0554	-0.0787	-0.0768	-0.0661	-0.0764
									N=50								
Avg	9.3759	1.6270	0.0069	0.3789	0.4854	0.5022	1.6713	0.9570	1.7272	2.2710	2.0249	2.4782	2.1047	-0.4012	-0.5619	0.5467	-0.2128
Std	16.5090	**6.5312**	7.8307	7.4212	7.3518	7.2175	6.5989	6.8055	6.5650	8.4004	8.3278	8.7412	8.1169	7.9464	**7.9095**	8.1400	7.9583
SR	0.5679	0.2491	0.0009	0.0511	0.0660	0.0696	0.2533	0.1406	0.2631	0.2703	0.2431	0.2835	0.2593	-0.0505	-0.0710	0.0672	-0.0267
To	0.0650	0.2296	0.0797	0.0829	0.0883	0.0983	0.2062	0.2450	0.2041	1.3959	1.3571	0.7825	1.3168	0.7031	0.6967	0.6881	0.6932
Util	-	-0.0750	-0.0912	-0.0874	-0.0864	-0.0862	-0.0746	-0.0817	-0.0740	-0.0689	-0.0713	-0.0668	-0.0705	-0.0953	-0.0968	-0.0859	-0.0934
									N=100								
Avg	8.9670	2.3478	0.3107	0.4574	0.6279	-0.0345	2.2762	1.8543	2.2875	1.9291	1.8761	2.2395	1.7149	-0.0561	-0.1386	0.0174	-0.3719
Std	16.3570	7.0048	7.8860	7.5185	7.5034	7.9210	**6.9788**	7.1140	6.9701	8.1337	8.0041	8.9944	**7.8091**	8.1920	8.1705	8.5035	8.1921
SR	0.5482	0.3352	0.0394	0.0608	0.0837	-0.0044	0.3262	0.2606	0.3282	0.2372	0.2344	0.2490	0.2196	-0.0069	-0.0170	0.0020	-0.0454
To	0.0677	0.4334	0.0937	0.1062	0.1166	0.1361	0.3773	0.4307	0.3310	1.5133	1.4911	0.7188	1.4487	0.8632	0.8512	0.7462	0.8348
Util	-	-0.0638	-0.0841	-0.0826	-0.0809	-0.0875	-0.0645	-0.0687	-0.0644	-0.0682	-0.0687	-0.0652	-0.0703	-0.0878	-0.0886	-0.0871	-0.0909
									N=250								
Avg	8.5579	0.8394	0.1935	-0.0355	0.2785	-0.1096	1.3667	0.7104	1.4535	2.3853	2.4375	1.9509	2.1747	0.3848	0.4007	0.0817	-0.1339
Std	16.0397	8.6595	8.3822	7.9023	7.9247	7.4728	**7.0716**	8.3471	7.4773	7.8935	7.8948	8.8156	**7.608**	8.5036	8.4739	8.6193	8.4825
SR	0.5335	0.0969	0.0231	-0.0045	0.0351	-0.0147	0.1933	0.0851	0.1944	0.3022	0.3087	0.2213	0.2858	0.0452	0.0473	0.0095	-0.0158
To	0.0663	1.3768	0.1023	0.1349	0.1535	0.1749	0.7319	1.0998	0.5531	1.5965	1.5833	0.6370	1.6043	0.9738	0.9669	0.8322	0.9791
Util	-	-0.0750	-0.0813	-0.0835	-0.0804	-0.0842	-0.0696	-0.0762	-0.0687	-0.0596	-0.0591	-0.0640	-0.0617	-0.0794	-0.0793	-0.0825	-0.0846
									N=500								
Avg	13.3624	NM	1.4840	1.7835	1.9207	0.9722	4.0472	2.4737	7.9159	2.0190	2.1630	2.1628	0.8834	2.0017	1.5460	1.8750	0.8834
Std	19.0009	NM	7.2027	9.3655	9.3766	**6.7157**	9.4799	9.6338	18.0021	6.5326	**6.5188**	8.1353	6.8769	6.9370	6.7924	7.7285	6.8769
SR	0.7033	NM	0.2060	0.1904	0.2048	0.1448	0.4269	0.2568	0.4397	0.3091	0.3318	0.2659	0.1285	0.2886	0.2276	0.2426	0.1285
Util	-	NM	-0.1156	-0.1129	-0.1115	-0.1206	-0.0905	-0.1061	0.0759	-0.1103	-0.1089	-0.1091	-0.1216	-0.1105	-0.1150	-0.1118	-0.1216
									N=750								
Avg	14.9230	NM	1.3772	2.1797	2.1814	0.9721	5.1832	2.1792	7.7555	2.0707	2.2068	2.0312	1.942	1.8903	2.4929	1.5559	1.0187
Std	19.9301	NM	6.7534	8.7256	8.7408	**6.2829**	10.8754	7.5581	14.7157	6.4886	6.4618	7.9872	**5.9976**	6.3229	6.2317	7.4779	7.1410
SR	0.7488	NM	0.2039	0.2498	0.2496	0.1547	0.4766	0.2883	0.5270	0.3191	0.3415	0.2543	0.3238	0.2990	0.4000	0.2081	0.1426
Util	-	NM	-0.1320	-0.1243	-0.1243	-0.1360	-0.0947	-0.1242	-0.0700	-0.1252	-0.1239	-0.1258	-0.1265	-0.1270	-0.1210	-0.1304	-0.1358
									N=1000								
Avg	16.4641	NM	0.7846	0.8331	0.7902	0.5111	5.2631	0.8336	6.3047	1.2754	1.6202	2.2590	1.1694	0.7917	1.1099	2.2155	0.3543
Std	20.4743	NM	6.0135	5.8169	5.7520	5.7812	11.5098	**4.3261**	13.3420	6.4634	6.2808	7.6629	6.7589	6.4506	**6.0104**	7.0883	6.3774
SR	0.8041	NM	0.1305	0.1432	0.1374	0.0884	0.4573	0.1927	0.4725	0.1973	0.2580	0.2948	0.1730	0.1227	0.1847	0.3126	0.0556
Util	-	NM	-0.1531	-0.1526	-0.1530	-0.1558	-0.1093	-0.1525	-0.0992	-0.1483	-0.1449	-0.1387	-0.1496	-0.1531	-0.1499	-0.1391	-0.1575

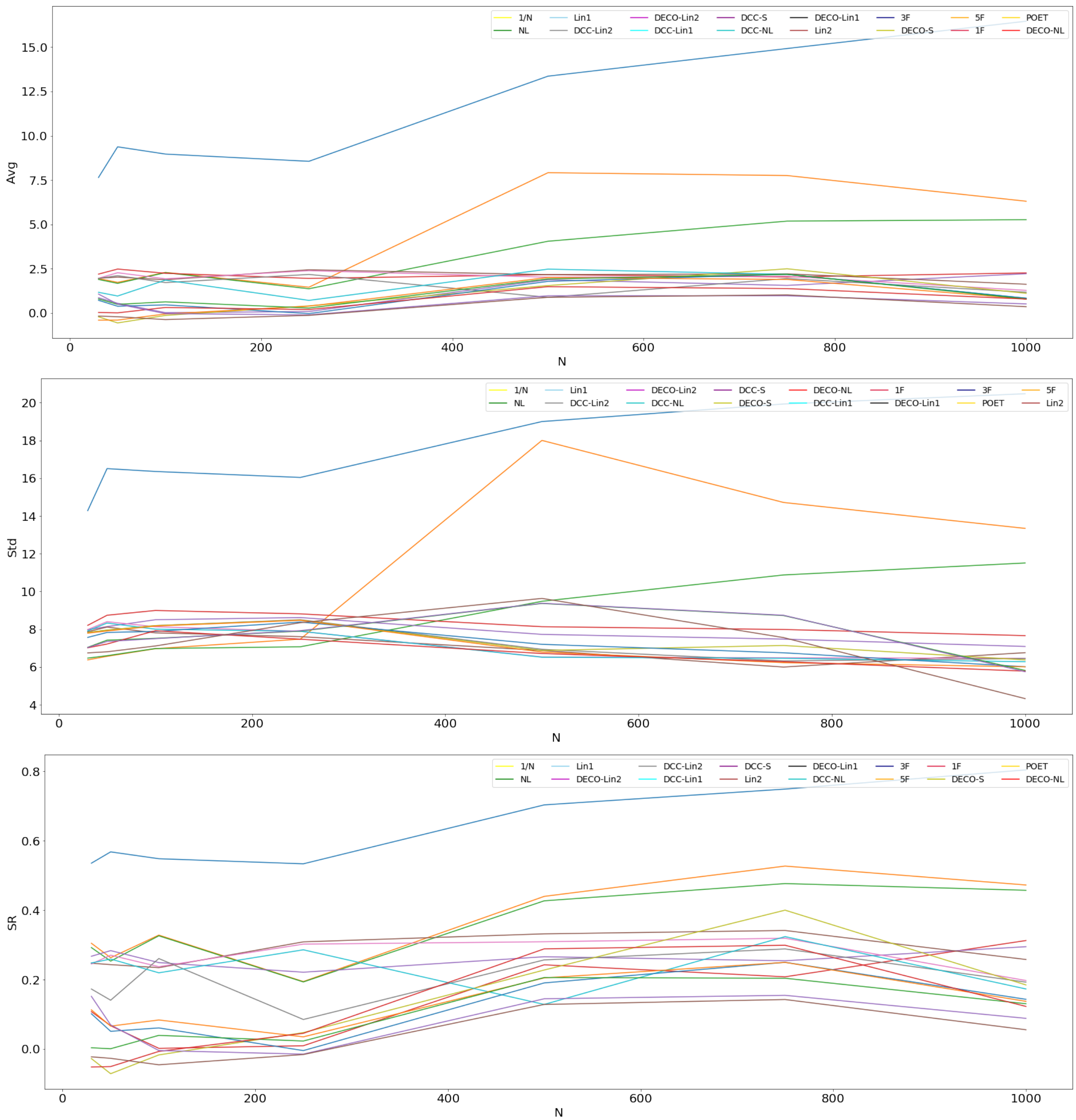

Figure 13: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using 500 returns per asset in the formation period as in Table 16, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019. This sample comprises of 273 portfolio formations.

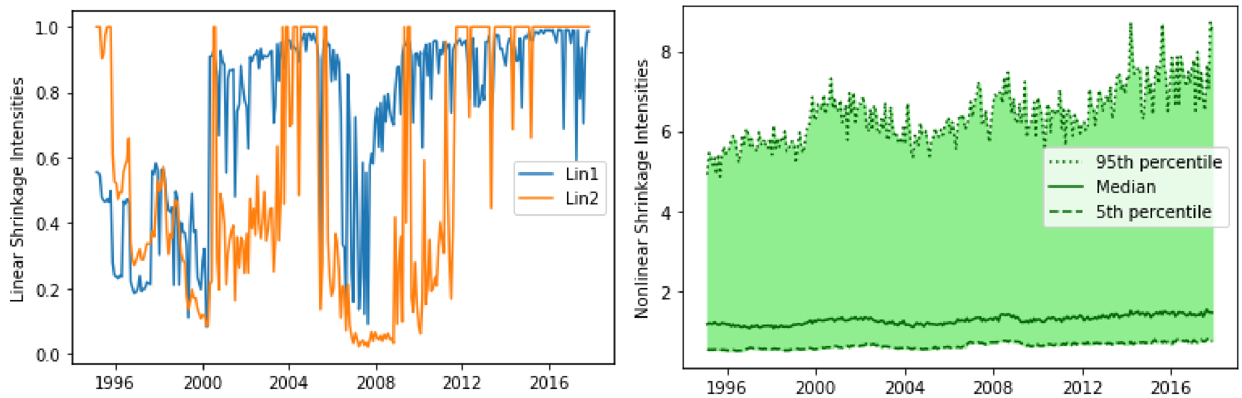

Figure 14: (Non)linear shrinkage intensities for the formed GMV portfolios when using 500 returns per asset in the formation period. The sample period ranges from 13 February 1995 to 17 December 2019, based on 500 training observations used per portfolio for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 273 portfolio formations. $N = 1000$.

B.5 Different Return Frequency

The performance measures for monthly asset returns can be found in Table 18. For the monthly asset returns, the amount of observations is equal to $T = 120$. This means that the factor of dimensionality plays an even bigger role as the largest amount of assets is more than eight times as large as the amount of observations. For this setting, in general, the asset returns are somewhat higher, which is as expected as the returns are accumulated monthly instead of daily. Therefore these returns have increased magnitude, but also increased standard deviations. For this sample, there is a strong indication for the usage of factor models across dimensions. Both the POET and 5-factor model have very robust results. Furthermore, for this sample with monthly returns, the dynamic models seem to achieve superior results for higher dimensions.

Among static methods, the POET estimator is the clear winner in lower dimensions ($N < 250$), being consistently the lowest in terms of out-of-sample variance. For higher dimensions the 5-factor implied covariance method scores best. In terms of out-of-sample realised utility I find that the $1/N$ outperforms all methods. Furthermore, the method shrinking the sample covariance matrix toward the identity matrix is performing the least of the pure shrinkage methods. This holds over all dimensions and this method does not improve upon the $1/N$ method for most dimensions. Next to that, it attains the highest level of turnover, indicating high transaction costs. The dynamic models score substantially lower variances than static methods. Especially in high dimensions ($N > 100$) extremely low out-of-sample covariances smaller than 1% are reached. These results indicate that time-series models are superior when using monthly asset returns in order to construct a portfolio which minimise the variance out-of-sample. For higher dimensions the DCC and DECO models which use the nonlinear shrinkage method as a target perform the best. All dynamic methods have reasonable and stable turnover, but generate an inferior utility with respect to the $1/N$ strategy, indicating that they are not optimal from a utility perspective.

The weights can be found in Table 17, which all show stable and decreasing patterns in magnitude. Figure 16 shows the shrinkage intensities of the different methods. Both due to the fact that monthly returns are used in this robustness analysis and the fact that the dimension of the training set is somewhat different, an exceptional pattern can be seen. The linear shrinkage toward identity seems to be mean-reverting to a level of around 0.96, whereas the method shrinking toward identity puts full weight on the identity matrix over the whole sample. Due to the extremely high dimension of assets with respect to the amount of observations, this could be explained as the covariance matrix is very ill-behaved. Also, for this sample, the shrinkage intensities remain high even during crisis periods.

For the nonlinear shrinkage intensity, a similar pattern arises as in Figure 7, with nonlinear shrinkage identities increasing over the observation sample. Up to around 1995, the nonlinear shrinkage intensities are fairly stable with a median around 0.5, indicating that most eigenvalues are decreased in order to achieve the optimal result. After 1995 the median and 5-th percentile scores are almost monotonically increasing. This indicates that after this period most (smaller) eigenvalues are increased instead of decreased in order to get the optimal nonlinear solution. What causes these effects remains for future research.

Table 17: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using monthly asset returns. The sample period for the portfolios ranges from 28 June 1985 to 31 December 2019, based on 120 training observations used per asset for the construction of the covariance matrix and 1 test observations per portfolio. This sample comprises 295 portfolio formations. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Max	0.0333	0.3094	0.1620	0.1414	0.1524	0.2320	0.3039	0.1620	0.1985	0.3847	0.3549	0.2906	0.3783	0.3318	0.3228	0.2872	0.3430
Min	0.0333	-0.0623	-0.0077	0.0031	0.0021	-0.0557	-0.0604	-0.0077	-0.0439	-0.0351	-0.0258	-0.0116	-0.0272	-0.0092	-0.0082	-0.0034	-0.0106
SD	0.0000	0.0749	0.0370	0.0317	0.0337	0.0617	0.0738	0.0370	0.0500	0.0840	0.0771	0.0613	0.0827	0.0728	0.0707	0.0614	0.0758
										N=50							
Max	0.0200	0.2748	0.1032	0.1000	0.1088	0.1785	0.2631	0.1032	0.1284	0.2869	0.2775	0.1992	0.2940	0.2494	0.2506	0.2110	0.2697
Min	0.0200	-0.0916	-0.0046	-0.0009	-0.0016	-0.0423	-0.0868	-0.0046	-0.0452	-0.0287	-0.0243	-0.0073	-0.0233	-0.0077	-0.0077	-0.0036	-0.0098
SD	0.0000	0.0618	0.0209	0.0195	0.0209	0.0412	0.0598	0.0209	0.0335	0.0529	0.051	0.0355	0.0544	0.0465	0.0468	0.0383	0.0508
										N=100							
Max	0.0100	0.2926	0.0587	0.0572	0.0630	0.1137	0.1901	0.0587	0.0642	0.2829	0.2801	0.2284	0.2944	0.2577	0.2611	0.2357	0.2808
Min	0.0100	-0.2038	-0.0034	-0.0020	-0.0030	-0.0264	-0.1243	-0.0034	-0.0350	-0.0123	-0.0118	-0.0032	-0.0129	-0.0034	-0.0037	-0.0017	-0.0052
SD	0.0000	0.0778	0.0107	0.0103	0.0111	0.0232	0.0525	0.0107	0.0197	0.0359	0.0358	0.0280	0.0386	0.0338	0.0343	0.0300	0.0378
										N=250							
Max	0.0040	NM	0.0799	0.0875	0.0929	0.1083	0.0389	0.0838	0.0255	0.3263	0.3275	0.3125	0.3328	0.3171	0.3178	0.3151	0.3202
Min	0.0040	NM	-0.0020	-0.0026	-0.0032	-0.0058	-0.0275	-0.0024	-0.0175	-0.0014	-0.0012	-0.0000	-0.0010	-0.0000	-0.0000	-0.0000	-0.0000
SD	0.0000	NM	0.0078	0.0084	0.0089	0.0106	0.0115	0.0081	0.0077	0.0262	0.0263	0.0253	0.0266	0.0256	0.0256	0.0255	0.0258
										N=500							
Max	0.0020	NM	0.0706	0.0783	0.0833	0.0818	0.0159	0.0706	0.0128	0.2055	0.2063	0.1962	0.2185	0.1991	0.1995	0.1974	0.2029
Min	0.002	NM	-0.0013	-0.0019	-0.0023	-0.0027	-0.0130	-0.0013	-0.0102	-0.0006	-0.0007	-0.0000	-0.0010	-0.0000	-0.0000	-0.0000	-0.0000
SD	0.0000	NM	0.0048	0.0053	0.0056	0.0055	0.0047	0.0048	0.0038	0.0130	0.0131	0.0126	0.0136	0.0127	0.0127	0.0126	0.0129
										N=750							
Max	0.0013	NM	0.0671	0.0753	0.0799	0.0726	0.0102	0.0671	0.0090	0.1715	0.1735	0.1642	0.1873	0.1665	0.1667	0.1650	0.1705
Min	0.0013	NM	-0.0009	-0.0015	-0.0017	-0.0017	-0.0086	-0.0009	-0.0073	-0.0003	-0.0007	-0.0000	-0.0008	-0.0000	-0.0000	-0.0000	-0.0000
SD	0.0000	NM	0.0037	0.0041	0.0043	0.0039	0.0029	0.0037	0.0025	0.0091	0.0091	0.0088	0.0097	0.0089	0.0089	0.0088	0.0090
										N=1000							
Max	0.0010	NM	0.0660	0.0740	0.0783	0.0708	0.0080	0.0660	0.0073	0.1493	0.1494	0.1421	0.1645	0.1447	0.1441	0.1426	0.1480
Min	0.0010	NM	-0.0007	-0.0012	-0.0014	-0.0014	-0.0064	-0.0007	-0.0057	-0.0002	-0.0008	-0.0000	-0.0007	-0.0000	-0.0000	-0.0000	-0.0000
SD	0.0000	NM	0.0031	0.0034	0.0036	0.0032	0.0020	0.0031	0.0018	0.0070	0.0070	0.0067	0.0075	0.0068	0.0068	0.0068	0.0070

Table 18: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using monthly asset returns. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. The sample period for the portfolios ranges from 28 June 1985 to 31 December 2019, based on 120 training observations used per asset for the construction of the covariance matrix and 1 test observation per portfolio. This sample comprises 295 portfolio formations. Utility is given relative to $1/N$ formation. NM = not meaningful.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
									N=30								
Avg	11.8466	4.0080	8.3807	8.4846	8.3855	3.8554	4.0712	8.3807	5.0595	3.7739	3.5390	5.4483	3.0598	5.1974	5.2334	6.2044	4.7657
Std	16.0584	12.6501	12.4349	12.4697	12.3269	**11.8751**	12.5851	12.4349	12.1239	13.7666	12.7731	11.9330	12.8384	11.2210	11.0862	11.1102	**11.0851**
SR	0.7377	0.3168	0.6740	0.6804	0.6803	0.3247	0.3235	0.6740	0.4173	0.2741	0.2771	0.4566	0.2383	0.4632	0.4721	0.5584	0.4299
To	0.0747	0.1621	0.0915	0.0670	0.0689	0.1115	0.1600	0.0915	0.1182	1.1179	0.9433	0.7127	0.9866	0.4265	0.4238	0.4301	0.4304
Util	-	-1.0855	-0.3966	-0.3461	-0.3528	-1.1809	-1.0724	-0.3966	-0.8932	-1.5791	-1.5956	-1.2499	-1.7085	-1.2437	-1.2288	-1.046	-1.3269
									N=50								
Avg	12.5632	6.1966	8.9036	9.3178	9.1364	6.7796	6.3046	8.9036	7.6707	4.7621	4.7906	6.7767	4.2944	4.2158	4.1724	5.9960	3.0304
Std	16.0393	15.1037	13.2142	13.1598	12.9049	**12.9049**	14.8000	13.2142	13.3889	13.5434	12.8912	12.3318	12.9108	11.6584	11.6320	**11.5712**	11.6875
SR	0.7833	0.4103	0.6738	0.7080	0.7080	0.5254	0.4260	0.6738	0.5729	0.3516	0.3716	0.5495	0.3326	0.3616	0.3587	0.5182	0.2593
To	0.0772	0.2977	0.0931	0.0751	0.0783	0.1274	0.2858	0.0931	0.1623	1.1482	1.0245	0.6404	1.0275	0.4901	0.4944	0.4790	0.4925
Util	-	-0.6998	-0.4578	-0.3454	-0.3614	-0.6788	-0.6773	-0.4578	-0.4399	-1.4310	-1.3970	-1.0831	-1.5052	-1.5828	-1.5787	-1.2196	-1.8215
									N=100								
Avg	12.5761	2.7299	8.7151	9.0691	8.9179	4.2283	4.2513	8.7151	6.1215	3.5142	3.6773	3.9195	2.7842	1.9763	2.5287	3.1308	0.6971
Std	16.4332	25.3797	13.2788	13.4079	13.0798	**11.4357**	21.4238	13.2788	13.2559	11.6914	11.5244	11.7533	10.9590	10.6547	10.4745	10.9136	**10.3446**
SR	0.7653	0.1076	0.6563	0.6764	0.6818	0.3697	0.1984	0.6563	0.4618	0.3006	0.3191	0.3335	0.2541	0.1855	0.2414	0.2869	0.0674
To	0.0788	1.6547	0.0950	0.0869	0.0937	0.1434	0.8975	0.0950	0.3137	0.9482	0.9257	0.4932	0.9424	0.5014	0.5135	0.4658	0.5174
Util	-	-1.2575	-0.4471	-0.3501	-0.3526	-1.1155	-0.9426	-0.4471	-0.6311	-2.4399	-2.3919	-2.4203	-2.5798	-2.8235	-2.7006	-2.5773	-3.1101
									N=250								
Avg	12.5774	NM	5.1888	4.8977	4.6295	2.3548	-4.4907	5.0553	-1.2751	0.1914	0.1915	0.1921	0.1509	0.1906	0.1910	0.1909	0.1734
Std	14.1627	NM	9.0733	9.2002	**8.7637**	8.9641	18.4872	9.0143	11.9011	0.9504	0.9504	0.9504	**0.7561**	0.9505	0.9505	0.9505	0.8694
SR	0.8881	NM	0.5719	0.5323	0.5283	0.2627	-0.2429	0.5608	-0.1071	0.2014	0.2015	0.2021	0.1996	0.2005	0.2010	0.2008	0.1994
Util	-	NM	-0.0892	-0.1175	-0.1486	-0.5722	-1.8399	-0.1078	-29.8481	-0.5925	-0.5907	-0.6127	-0.5979	-0.6117	-0.6118	-0.6118	-0.6151
									N=500								
Avg	14.7184	NM	5.2869	5.1449	4.8259	4.4266	2.6478	5.2869	3.1890	0.1273	0.1234	0.1443	0.0676	0.1360	0.1362	0.1374	0.1162
Std	15.8184	NM	7.4523	7.4097	**7.0277**	7.2803	18.7515	7.4523	16.0419	0.8310	0.8262	0.7991	**0.6489**	0.7953	0.7966	0.7957	0.7378
SR	0.9305	NM	0.7094	0.6943	0.6867	0.6080	0.1412	0.7094	0.1988	0.1532	0.1493	0.1806	0.1042	0.1710	0.1710	0.1727	0.1575
Util	-	NM	-0.9974	-0.9926	-1.0381	-1.1304	-1.3233	-0.9974	-1.2214	-0.3730	-0.3686	-0.3846	-0.3689	-0.3840	-0.3838	-0.3849	-0.3846
									N=750								
Avg	14.8594	NM	5.0041	4.7873	4.4731	4.3630	3.2827	5.0041	2.9780	0.1740	0.1623	0.1770	0.0930	0.1707	0.1535	0.1727	0.1375
Std	13.7850	NM	7.0062	6.8524	**6.4881**	6.9560	15.9166	7.0062	13.5936	0.7430	0.7002	0.7446	0.5815	0.7424	0.6732	0.7431	**0.6445**
SR	1.0779	NM	0.7143	0.6986	0.6894	0.6272	0.2062	0.7143	0.2191	0.2342	0.2318	0.2378	0.1599	0.2299	0.2281	0.2323	0.2134
Util	-	NM	-1.2164	-1.2276	-1.2738	-1.3231	-1.3663	-1.2164	-1.4351	-0.3085	-0.3061	-0.3225	-0.3022	-0.3209	-0.3247	-0.3226	-0.3232
									N=1000								
Avg	14.7680	NM	5.0117	4.8188	4.4692	4.0722	0.3118	5.0117	0.7474	0.3667	0.3675	0.3759	0.2690	0.3674	0.3580	0.3757	0.3342
Std	12.9676	NM	7.0554	6.9362	**6.6170**	6.8408	8.5678	7.0554	8.7892	0.9441	0.9395	0.9504	**0.7882**	0.9463	0.9286	0.9504	0.8906
SR	1.1388	NM	0.7103	0.6947	0.6754	0.5953	0.0364	0.7103	0.0850	0.3884	0.3912	0.3955	0.3413	0.3883	0.3855	0.3953	0.3752
Util	-	NM	-1.2602	-1.2666	-1.3214	-1.4347	0.0629	0.8360	0.1491	-0.2493	-0.2492	-0.2629	-0.2477	-0.2611	-0.2643	-0.2624	-0.2640

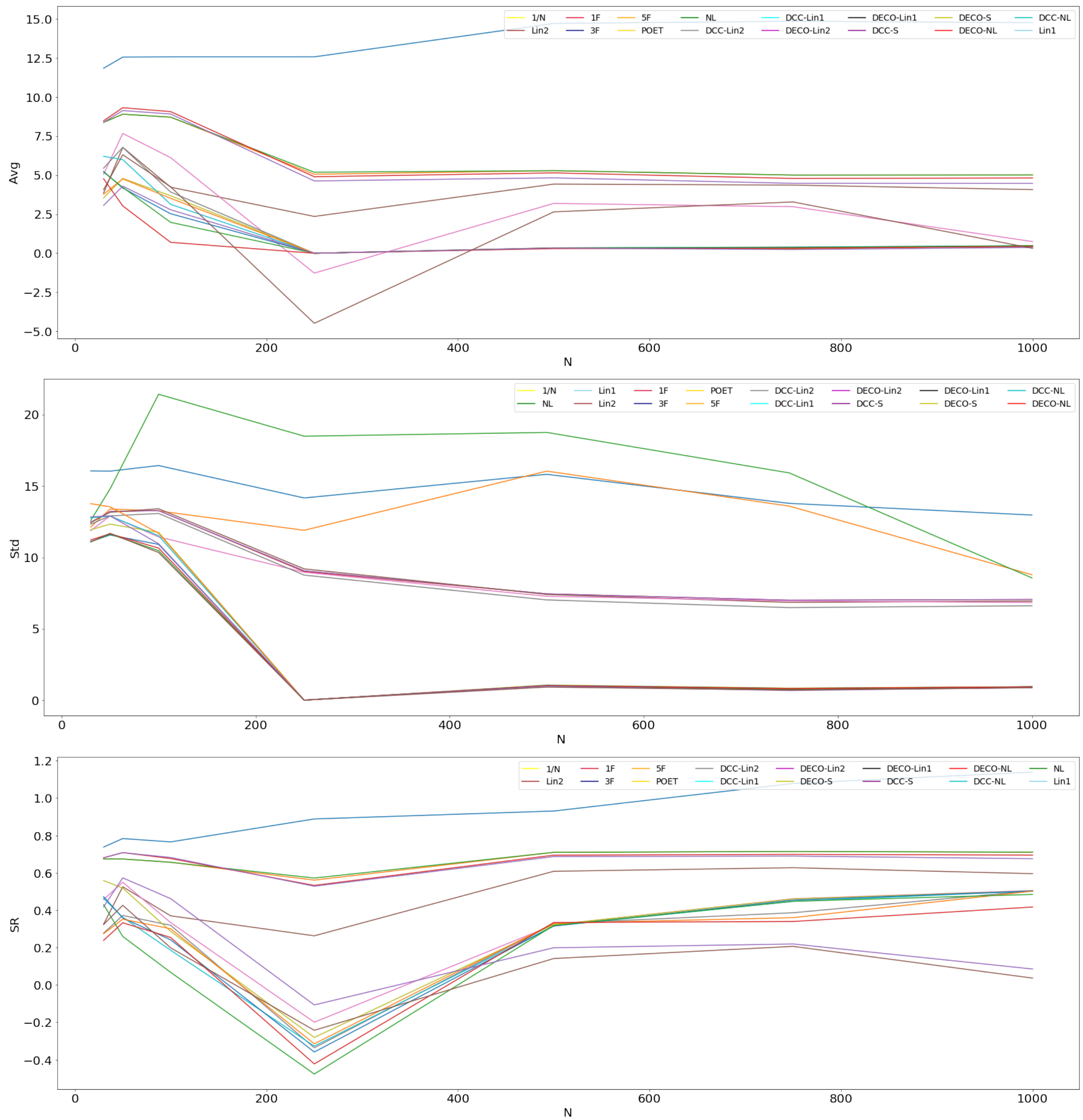

Figure 15: Annualised performance measures (Avg & Std in percentages) for the formed GMV portfolios when using monthly asset returns with 120 returns in the formation period and 1 in the test set as in Table 18, $N \in \{30, 50, 100, 250, 500, 750, 1000\}$. The sample period for the portfolios ranges from 28 June 1985 to 31 December 2019. This sample comprises of 295 portfolio formations.

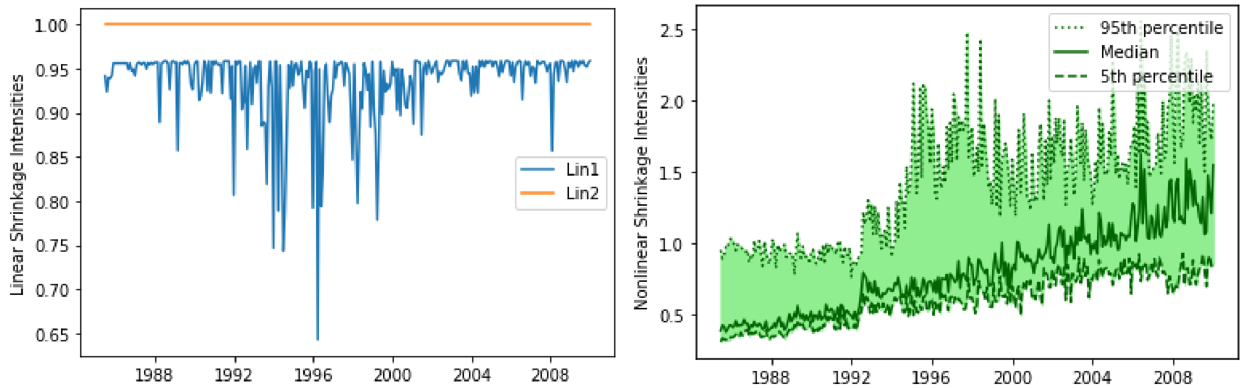

Figure 16: (Non)linear shrinkage intensities for the formed GMV portfolios when using monthly asset returns in the formation period. The sample period ranges from 28 June 1985 to 31 December 2019, based on 120 training observations used per portfolio for the construction of the covariance matrix and 1 test observations per portfolio. The sample comprises 295 portfolio formations. $N = 1000$.

B.6 Portfolios as Assets

In order to assess the robustness of the results from the daily asset returns, I measure the performance of all methods when using daily returns of three different sorted portfolios, each containing 100 portfolios. Table 20 shows the performance of the methods with respect to achieving the minimum variance. For the sorted portfolios on size and book-to-market value and investments I find that the nonlinear shrinkage method of Ledoit and Wolf (2017b) reaches the minimum variance, whereas it is near optimal for the portfolios formed on size and operating profitability. Also, both the linear shrinkage methods perform well for the portfolios. Furthermore, there is a slight indication of static models outperforming dynamic models in achieving the lowest variance for this sample.

The least effective method when using 100 portfolios as assets, both seen from a minimum variance and a utility point of view, is the naive $1/N$ construction. Given the dimensionality of the portfolios, $T = 100$, the sample covariance matrix remains non-singular and thus invertible. However, introducing any form of shrinkage to the sample covariance matrix outperforms the regular sample covariance in attaining the minimum variance out-of-sample. This shows that shrinkage can already be quite useful even when the dimensionality of the data ensures a nonsingular sample covariance matrix. Regarding turnover and feasibility of the formed portfolios, the factor implied methods consistently score among the methods attaining the lowest turnover. Turnover is higher for pure linear shrinkage models. Also, in the conditional correlation models, I find that a DCC model with nonlinear shrinkage target performs best over all portfolios. However, allowing the covariance to be varying over time does not lead to an improvement with respect to minimising the out of sample variance. Also, turnover is relatively high comparing with the factor implied and pure shrinkage methods. All models in this sample attain a higher level of utility with respect to the $1/N$ method, indicating beneficial diversification for the utility investor.

Furthermore, the statistics regarding the weights and the shrinkage intensities for the three sorted portfolios can be found in Table 19 and Figure 17. The linear shrinkage intensities shown are lower than for the other robustness analyses, but this is as expected as only 100 sorted portfolios with 250 training observations are investigated. Hence, the covariance matrix is non-singular and thus invertible. The figures and tables show that introducing only a small deviation from the sample covariance towards the shrinkage target improves the performance of the covariance matrix. The figures also show that the shrinkage methods perform very similar for the different portfolios and that the two different shrinkage methods follow each other somewhat. The figure shows a shrinkage intensity toward the factor model being shifted down with regards to the shrinkage intensity toward the identity matrix. Next to that, the nonlinear shrinkage intensities show that the eigenvalues are increased on average in order to find the optimal nonlinear solution, with the median around 1.2 for all samples. The 5-th percentile is around 0.75 for the samples and the 95-th percentile is around 4.0 for all sorted portfolios. This indicates that there is a large amount of small eigenvalues that are increased in order to obtain the optimal solution of the nonlinear shrinkage method.

Table 19: Average minimum and maximum weights and average standard deviation over all formations for the GMV portfolios when using sorted portfolios as assets. The sample period for the portfolios ranges from 13 February 1995 to 18 December 2019, based on 250 training observations used per portfolio for the construction of the covariance matrix. This sample comprises 285 portfolio formations.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
N=100, portfolios formed on size and book-to-market value																	
Max	0.0100	0.5426	0.2680	0.3474	0.3636	0.3685	0.4360	0.5557	0.2959	0.3564	0.3542	0.1931	0.3499	0.2382	0.2377	0.1962	0.2382
Min	0.0100	-0.2348	-0.1049	-0.1094	-0.1107	-0.1243	-0.1946	-0.2395	-0.1435	-0.0742	-0.0732	-0.0518	-0.0692	-0.0413	-0.0412	-0.0322	-0.0413
SD	0.0000	0.1161	0.0655	0.0717	0.0733	0.0745	0.0974	0.1188	0.0719	0.0618	0.0614	0.0347	0.0604	0.0493	0.0492	0.0403	0.0493
N=100, portfolios formed on size and operating profitability																	
Max	0.0100	0.4232	0.2702	0.2809	0.2923	0.2892	0.3424	0.4328	0.2260	0.3431	0.3417	0.1915	0.3373	0.2239	0.2235	0.1865	0.2238
Min	0.0100	-0.2688	-0.1058	-0.1131	-0.1177	-0.1355	-0.2132	-0.2746	-0.1451	-0.0696	-0.0688	-0.0496	-0.0647	-0.0417	-0.0416	-0.0327	-0.0417
SD	0.0000	0.1104	0.0640	0.0661	0.0686	0.0692	0.0922	0.1130	0.0674	0.0591	0.0588	0.0339	0.0579	0.0476	0.0476	0.0393	0.0476
N=100, portfolios formed on size and investments																	
Max	0.0100	0.3734	0.2314	0.2368	0.2457	0.2493	0.3102	0.3822	0.2110	0.3447	0.3428	0.1924	0.3420	0.2146	0.2142	0.1789	0.2147
Min	0.0100	-0.2704	-0.0920	-0.1126	-0.1194	-0.1396	-0.2183	-0.2759	-0.1520	-0.0771	-0.0760	-0.0497	-0.0717	-0.0429	-0.0428	-0.0337	-0.0429
SD	0.0000	0.1098	0.0610	0.0631	0.0653	0.0690	0.0920	0.1123	0.0668	0.0599	0.0595	0.0345	0.0588	0.0480	0.0479	0.0395	0.0480

Table 20: Annualised performance measures (Avg & Std in percentages) over all formations for the GMV portfolios when using sorted portfolios as assets. Performance measures are given according to a buy-and-hold strategy at each investment date. Lowest standard deviation per static and dynamic models and per N are given in bold. The sample period for the portfolios ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per asset for the construction of the covariance matrix and 21 test observations per portfolio. This sample comprises 285 portfolio formations. Utility is given relative to $1/N$ formation.

	1/N	Sample	1F	3F	5F	POET	Lin1	Lin2	NL	DCC-S	DCC-L1	DCC-L2	DCC-NL	DECO-S	DECO-L1	DECO-L2	DECO-NL
	N=100, portfolios formed on size and book-to-market value																
Avg	11.7819	15.9113	12.9971	14.4488	14.8143	14.8474	15.7459	16.1493	15.6790	13.2933	13.2915	13.1240	13.4069	15.3470	15.3428	15.2685	15.3588
Std	18.1826	12.4777	17.3893	14.0474	13.8373	13.2030	12.1531	12.6035	**12.0305**	13.3418	13.3473	15.1190	**13.3035**	14.3026	14.2935	14.3563	14.3043
SR	0.6480	1.2752	0.7474	1.0286	1.0706	1.1245	1.2956	1.2813	1.3033	0.9964	0.9958	0.8680	1.0078	1.0730	1.0734	1.0635	1.0737
To	0.0474	3.4493	0.7095	0.8218	0.9109	1.1241	2.5822	3.5029	1.4761	3.9779	3.9367	2.4606	3.7932	1.9991	1.9955	1.9412	1.9976
Util	-	0.0430	0.0135	0.0286	0.0322	0.0326	0.0415	0.0454	0.0409	0.01690	0.0169	0.0148	0.0181	0.0373	0.0372	0.0364	0.0374
	N=100, portfolios formed on size and operating profitability																
Avg	11.9994	15.6079	13.1670	15.6908	15.5902	15.7054	15.4637	15.7350	15.5425	13.0792	13.0953	13.5493	12.7996	14.9272	14.9275	14.3298	14.9090
Std	17.7082	12.1979	16.5702	13.4167	13.3891	12.8256	**11.8313**	12.3079	11.8441	12.9060	12.9009	14.2507	**12.8225**	13.3553	13.3510	13.6328	13.3475
SR	0.6776	1.2796	0.7946	1.1695	1.1644	1.2245	1.3070	1.2784	1.3123	1.0134	1.0151	0.9508	0.9982	1.1177	1.1181	1.0511	1.1170
To	0.0462	3.5005	0.6731	0.7597	0.8620	1.0338	2.5967	3.5609	1.4066	3.9225	3.8884	2.4620	3.7227	2.0289	2.0252	1.9521	2.0281
Util	-	0.0378	0.0132	0.0387	0.0377	0.0389	0.0365	0.0391	0.0373	0.0126	0.0127	0.0168	0.0098	0.0309	0.0309	0.0249	0.0308
	N=100, portfolios formed on size and investments																
Avg	12.2882	15.5262	12.9034	13.7411	13.9880	14.6535	15.1985	15.5720	14.2058	12.1709	12.1872	12.1877	12.2263	13.8689	13.8675	12.7158	13.8660
Std	17.7625	12.3317	16.1461	13.6211	13.5939	12.3548	11.8064	12.4525	**11.3716**	12.4653	12.4729	14.1896	**12.4452**	13.9220	13.9157	13.9078	13.9264
SR	0.6918	1.2590	0.7992	1.0088	1.0290	1.1861	1.2873	1.2505	1.2492	0.9764	0.9771	0.8589	0.9824	0.9962	0.9965	0.9143	0.9957
To	0.0458	3.7236	0.6874	0.8354	0.9296	1.2362	2.7838	3.7897	1.4437	4.1273	4.0840	2.5557	3.9304	2.1184	2.1144	2.0522	2.1190
Util	-	0.0340	0.0076	0.0164	0.0189	0.0255	0.0309	0.0345	0.0211	0.0007	0.0009	0.0005	0.0013	0.0175	0.0175	0.0060	0.0175

(a) Size x Book-to-Market (b) Size x Operating Profit (c) Size x Investments

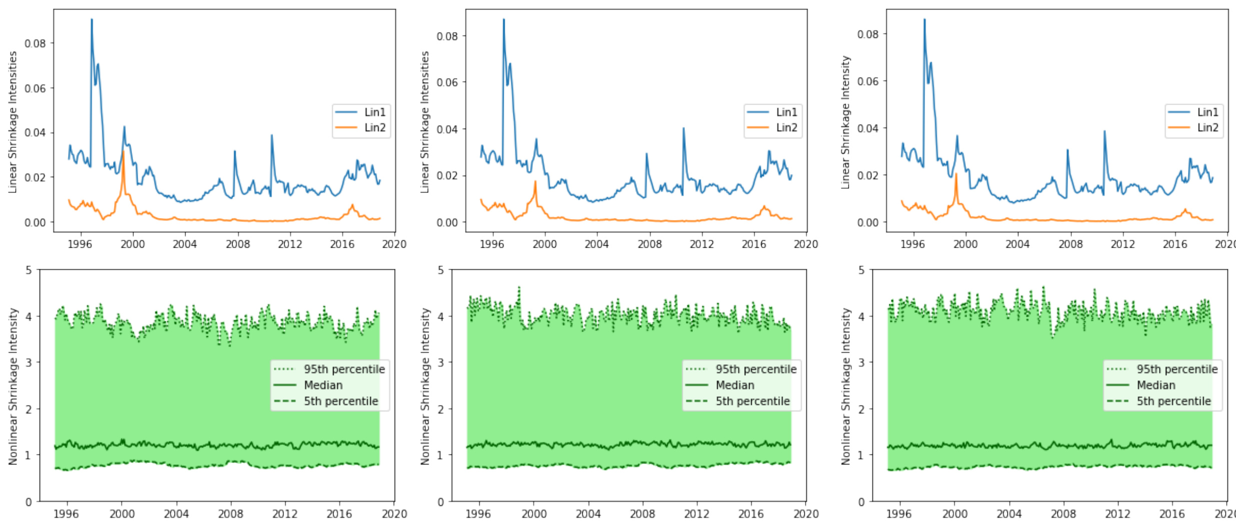

Figure 17: (Non)linear shrinkage intensities for the formed GMV portfolios when using sorted portfolios as assets. The sample period ranges from 13 February 1995 to 17 December 2019, based on 250 training observations used per portfolio for the construction of the covariance matrix and 21 test observations per portfolio. The sample comprises 285 portfolio formations. $N = 100$.