ERASMUS UNIVERSITY ROTTERDAM
Erasmus School of Economics

Master Thesis Quantitative Finance

FEM61008

Constraining market capitalization weighted indices to reduce downside risk

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August 3, 2021

In collaboration with

AF Advisors

Abstract

This research investigates whether imposing a constraint on the market capitalization index can reduce downside risk and what disadvantages applying these constraints have. For this, we look at naive constraining using restrictions on sectors or stocks taking into account their fundamentals. Applying constraints leads, in most cases, to statistically significantly decrease index volatility and smaller average losses in the 5% worst cases. Strict constraints lead to the best risk-return ratios but have the disadvantage that they incur substantial additional trading costs and lead to tracking error. Dynamic constraints add value if they are based on stable decision variables. The decision variables analyzed are the VIX and the Buffett Indicator. The first in this list is available daily and leads to high transaction costs without the expected risk reduction. The Buffett Indicator is available on a quarterly basis and leads to significant risk reduction. Combining (un)constrained indices does not automatically lead to more risk reduction in the long term. Indices are combined in this study based on volatility and expected shortfall. Combining indices does appear to have added value in more volatile periods. However, constraining a market cap index leads most of the time to a higher turnover, which cancels out the positive consequences of the constraint.

Contents

1 Introduction

Passive investment solutions have become increasingly popular in recent years, as shown in Figure 1. Passive portfolios aim to replicate a chosen benchmark and, unlike many active strategies, offer exposure to a large number of stocks in combination with low transaction costs and management fees.

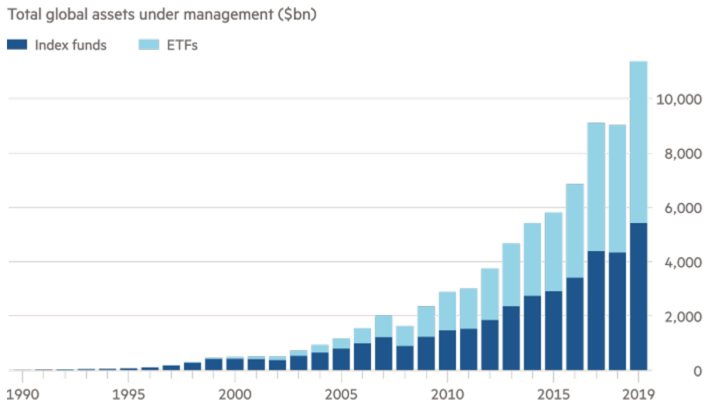

Figure 1: Increasing demand for passive investment solutions[1].

The most common way to compile a passive portfolio or index is to use the market capitalization of the stocks in the index. The market capitalization-weighted index, hereafter referred to as the market cap index, is the market standard to which asset managers and pension funds compare their equity strategies. Researches like Haugen and Baker (1991) and Hsu (2004) show that creating indices based on market capitalization has some major drawbacks. Where the lack of diversification in market cap indices, and the inefficiency of these indices are the two main ones.

When an index is compiled based on market capitalization, the exposure to large companies is substantial, which results in a concentrated index. The ten largest companies in the index as of December 31, 2020 account for 29% of the total market capitalization[2]. Passive investors expect to invest in a diversified portfolio of 500 stocks but are mainly exposed to the largest stocks in the index. This could allow investors to run unnecessary risks which could be avoided by better diversification across all 500 stocks. Another way to look at diversification is to evaluate the sector weights. Historically, the largest companies in the market cap index are often part of the same industry. Therefore, investors are mainly exposed to one or more sectors within the index.

Haugen and Baker (1991) and Grinold (1992) show that creating an index based on market capitalization is inefficient, because more weight is assigned to overvalued stocks and less weight to undervalued stocks within this index. The inefficiency of an index means that it does not provide the highest expected return for a given level of risk. This led academics and professional investors to look for an alternative and more efficient way to build indices. Often stock fundamentals are used for this, such as price-to-earnings ratio or price-to-book ratio. Most alternative

[1]Source: Financial Times - Index funds break through $10tn-in-assets mark amid active exodus
[2]Figure 5 in Appendix A

weighted indices outperform the market cap indices in the longer-term (Livingston, 2010; Bender et al., 2019), but have major drawbacks. The main disadvantage being that the weights do not vary with returns, which means that they have to be rebalanced regularly, making them not completely passive. Frequent rebalancing automatically leads to higher transaction costs. Because the advantage of passive strategies is to keep transaction costs to a minimum, this action is therefore undesirable. Therefore, alternative weighting schemes have not been picked up by investors and the market cap index remains the market standard to date.

From an investment industry perspective, the question remains whether indices can be created which have the advantages of market cap indices, but can prevent or reduce their drawbacks. Introducing weight constraints could be an intuitive simultaneous solution to the inefficiency and diversification problems of the market cap index. Weight constraints ensure better diversification of the index in most cases, as the weights that are restricted are redistributed among the other stocks in the index. Due to better diversification, the exposure towards mega-companies decreases, which could prevent downside risk.

As a result of the weight constraint, the return of the constrained index will be lower compared to the market cap index when the value of a constrained stock rises sharply and vice versa. Since deviating from the market standard is not ideal, staying as close to the index as possible is preferable. On the other hand, investors might want to deviate from the market in times of crisis, as different portfolios will react differently to certain market conditions (Amenc et al., 2012). It will be investigated whether a dynamic constraint leads to less deviation from the market standard while reducing downside risk. Macroeconomic variables will be used to determine whether the business cycle is in or close to an economic recession.

Combining different stocks in a portfolio is known to provide diversification benefits. Combining different covariance estimation matrices has been shown to promote volatility reduction or provide a more stable portfolio in different market conditions (Caldeira et al., 2017). A leading portfolio combination example is the optimal mean-variance portfolio from Markowitz (1952), which is formed by combining risky holdings based on expected returns and volatilities. Kan and Zhou (2007) also concluded that combining different funds, via their three-fund separation example, can lead to better results in terms of volatility and returns. It will be investigated whether combining the weights of multiple (un)restricted indices has benefits in the context of volatility reduction and diversification benefits. If combining different portfolio weights does not result in an absolute decrease in volatility, it is investigated whether this method is robust in different volatility regimes.

To analyze different aspects of a market cap index, the following main question will be central:

"Can you prevent downside risk in a market capitalization-weighted index by imposing static and/or dynamic weight constraints on the stocks in the index?"

To dissect the problem proposed in the main research question, the following *sub-questions* are considered:

1. *Which constraints can improve the market cap index in terms of diversification, returns and volatility?*
2. *What are the consequences of the weight constraints on turnover and tracking error compared to the market cap index?*
3. *Could a dynamic constraint add value in economic recessions in terms of diversification, returns and volatility?*
4. *Does combining multiple (un)restricted indices have added value in terms of diversification, returns and volatility?*

To answer all these sub-questions, a constant, stock fundamentals and sector constraint will be applied to the market cap index. Applying these constraints will cause tracking error with the market cap index, which is undesirable. Another side effect is the additional transaction costs, which wants to be kept to a minimum to keep the index as passive as possible. The continuous application of constraints could lead to high tracking error and turnover. This while applying a constraint may only have added value during economic recessions. To investigate this, a dynamic constraint will be tested. Finally, it will be examined whether combining multiple (un)constrained indices reduces tracking error and turnover while reducing downside risk.

Overall it has been found that constraining the market cap index has a positive influence when the aim is to reduce downside risk. The diversification created by weight constraints does not prevent highly negative returns but does ensure a faster recovery. However, the downward risk reduction does not outweigh the higher turnover that results from constraining weights.

This thesis is structured as follows. In the next chapter, an overview will be given of the available literature and the contribution of this thesis to the current literature will be described. Subsequently, the methods, and the implementation of the methods, will be discussed. They will be used during the analysis of the portfolios. In the results chapter, the results that follow from the analysis will be discussed. Finally, there will be a conclusion and a recommendation for further research.

2 Literature review

2.1 Market cap inefficiency and alternative weighting

The innovative idea of passively investing assets started with Renshaw and Feldstein (1960), which denounced the high costs of advice and monitoring and divergent returns of active managers in their paper. They concluded that many investors, both in terms of costs and returns, would be better off with a passive solution that provides exposure to the market portfolio. According to the capital asset pricing model (CAPM), the market portfolio is mean-variance optimal. Under the assumptions that the CAPM hold, it is concluded that passive investors are best off holding the market portfolio. Many large institutional investment companies, with inspiration from Markowitz (1952), Markowitz (1959) and Sharpe (1965), build a network of passive investment solutions in an attempt to replicate the market portfolio.

Passive investment solutions are often created based on market capitalization. Hsu (2004) shows that this way of indexing has many advantages. Most importantly, this portfolio rarely needs to be rebalanced. This greatly reduces costs, which benefits the investor. However, Haugen and Baker (1991) and Grinold (1992) already showed the disadvantages of indexing based on market capitalization even before the passive investment market grew in popularity. Haugen and Baker (1991) tested the market cap portfolio in a mean-variance environment and Grinold (1992) tested different market cap indices for efficiency using a GRS test (Gibbons et al., 1989). Investing in the market cap portfolio is inefficient as it allocated additional weight to overvalued stocks. Raza et al. (2020) called this phenomenon performance drag and it has a negative influence on market cap portfolio returns. This mispricing, and the associated mean reversion effect, can be caused by the irrational behaviour of noise traders, resulting in stock prices that can move far away from their fundamental price (Poterba and Summers, 1988). The irrational behaviour of investors can have many causes. Examples include trends (Summers, 1986; McQueen, 1992), overreaction to recent news or the opportunism of investors.

The inefficiency of the market cap portfolio, which Haugen and Baker (1991) and Grinold (1992) describe, is a conclusion with potentially serious consequences. It implicitly means that there must be a better, alternative, way to determine portfolio weights. Many papers, like Arnott et al. (2005), Treynor (2005), Perold (2007), Choueifaty and Coignard (2008), Maillard et al. (2010), Amenc et al. (2011), Amenc et al. (2012), Madhogarhia and Lam (2015) and Madhogarhia (2019) propose various alternative weighting methods to determine portfolio weights in a market efficient way. They propose weights based on stock fundamentals and/or diversification characteristics. Examples include: book value, cash flow, sales, revenues, dividends and employment. They assume that market cap portfolio is inefficient and therefore try to determine portfolio weights via a different route. Although alternative weighted portfolios do show more efficient results compared to market cap portfolios, they do not take advantage of the benefits of market cap portfolios. Determining portfolio weights in an alternative way expose investors to additional risks, where model selection risk and relative performance risk are the most important. Amenc et al. (2012) show that different market conditions favour different alternative weighting strategies. He concludes that investors are most of the time best off with an equal-weighted portfolio (1/N portfolio), but in recessions might better switch to the minimum volatility portfolio. The differences in return between the different methods are significant, with a 15% return difference in a six month recession period. The outperformance of the alternatively weighted indices could be due to additional exposure to recognized factors. However, this exposure to external factors can cause greater tracking error with the market cap portfolio, which can yield negative returns compared to the market cap portfolio. Amenc et al. (2012) show that this relative performance risk is mainly present for shorter horizons.

2.2 Portfolio constraints

The most naive way to constrain portfolio weights is to give them all the same weight, resulting in the 1/N portfolio. DeMiguel et al. (2009) shows in his research that the results of this naive way of constraining are hard to beat. This portfolio is often not found to be mean-variance efficient, as it is not based on underlying data. However, the mean-variance portfolio will outperform the 1/N portfolio in presence of a limited number of assets and an extensive estimation window as shown by DeMiguel et al. (2009).

Constraining weights in mean-variance efficient portfolios have a proven positive impact on performance (Frost and Savarino, 1988). Jagannathan and Ma (2003) show that certain constraints on portfolio weights can be interpreted as a form of shrinkage. The best known and most used methods to restrict weights in the covariance matrix are linear shrinkage (Ledoit and Wolf, 2003) and sparsity methods (Bien and Tibshirani, 2011). Weight constraints ensure that portfolio weights are not heavily driven by the sampling error, which often leads to highly concentrated portfolios (Green and Hollifield, 1992). Shrinkage will reduce the relationship between two assets that are highly correlated, causing a trade-off between two types of errors. If the estimation error is larger than the specification error, restricting weights would produce better results. Because the imposition of constraints causes a form of shrinkage, a bias-variance trade-off is created. A bias is created by imposing a weight restriction to reduce the variance and ultimately reduce the mean square error of the optimal portfolio.

According to Jagannathan and Ma (2003), the performance of a portfolio improves when short-selling is prohibited. They also conclude that, for mean-variance efficient portfolios, the introduction of upper bound restrictions can have a positive influence on performance. But this effect would be minimal if a short-sell constraint had already been applied to the portfolio. According to them, even imposing the wrong constraints improve portfolio performance as it shrinks large elements of the covariance matrix and therefore reduces risk. However, Chiou et al. (2009) show that constraints that are too restrictive could work counteracting and lead to worse portfolio performance. If this is the case, the specification error will be greater than the estimation error and a weight restriction will be undesirable.

Although the restriction of weights in a covariance matrix has been extensively researched, the application of constraints to weights of an inefficient estimation-free portfolio, like the market cap portfolio, remains unexposed. This thesis is most similar to the Jagannathan and Ma (2003) and DeMiguel et al. (2009) papers. The influence of portfolio constraints will be tested and compared with other portfolios. This thesis contributes to the current literature by testing constraints on the inefficient market cap portfolio, where the current literature mainly applies constraints to the efficient minimum variance portfolios.

2.3 Portfolio diversification

Diversification ensures risk spreading within a portfolio since an investor is less dependent on the results of one stock, sector, region or model as he spreads his risk over several stocks and sectors. Mao (1970), Woerheide and Persson (1992) and Goetzmann and Kumar (2008) did extensive research on portfolio diversification across individual stocks. However, they do on how many assets an average investor should hold to own a well-diversified portfolio that displays risk-return characteristics of a 1/N portfolio, instead of looking at the diversification of a portfolio itself.

Sector concentration is another way of looking at diversification within a portfolio. Within a sector, stocks are often highly correlated, which can pose a potential risk. When one sector shows good results, the market capitalization of the entire sector increases. If this continues for a longer period, a sector will emerge that dominates the market. When several stocks within this sector lose value, the entire index will lose value. The most recent example of such an event is the dot-com bubble, where many stocks within the IT sector gained and lost value (DeLong and Magin, 2006; Goodnight and Green, 2010). Again, investors thought they were investing in a widely spread portfolio of hundreds of assets, where they invested mainly in the largest assets in the index, which also invested in the same sector. As Figure 5 in Appendix A shows, the concentration of the largest five stocks in the current S&P 500, all of which are classified in the IT sector, is currently higher than the concentration during the time of the dot-com bubble (late 90s - early 00s). Tracking a market cap index thus seems to implicitly expose investors to very specific sectors and stocks. Intuitively, diversification among sectors would be best achieved through an equal-sector weighted portfolio. Conconi and Demidow (2011) have researched this portfolio, which appears to have added value in more volatile periods but lags in return in economic expansion periods.

2.4 Portfolio combinations

Diversification can be achieved by differentiating based on assets or their characteristics, but can also be achieved by combining different methods. Combining methods is beneficial in a variety of research areas, such as regression (Mendes-Moreira et al., 2012; Brown et al., 2005) and time series forecasting (Armstrong, 2001; Timmermann, 2006; Thomson et al., 2019). Combining different portfolio strategies, or alternative weight measures can reduce model selection risk by diversifying across different models, as shown in the three-fund-separation example of Kan and Zhou (2007). Individual methods contain valuable unique information that adds value to the combination. When multiple methods are combined, the valuable information of each method is included, which would lead to more efficient and robust results. In this thesis, the theory can be applied to the different restricted portfolios. When combining results from several methods, a result whose probability law of error will be more rapidly decreasing is observed (Kang et al., 2020). In the portfolio combination, the question is still based on which criteria the different portfolios should be weighted. Even in the current literature, they are not sure which criteria provide the best performing combined method (Hsiao and Wan, 2014; Claeskens et al., 2016).

3 Methodology

3.1 Market capitalization-weighted index

The market cap portfolio has a great advantage in that there is no estimation necessary to calculate the weights. As a result, these portfolios do not suffer from estimation error and parameter uncertainty as in portfolios that are based on the covariance matrix. Market capitalization refers to the total market value of a company's outstanding shares of a stock, and is calculated in the following way:

$$MC_{i,t} = O_{i,t} \cdot P_{i,t},$$

where $MC_{i,t}$ is the i^{th} company's market capitalization (in \$) at time t, $O_{i,t}$ is the number of outstanding shares for company i at time t and $P_{i,t}$ is the i^{th} company's price per share (in \$) at time t. The initial weights of the market cap portfolio are determined by selecting the 500, or any other number of stocks that the index must consist of, stocks with the largest market capitalization on the first trading day of the year. The weight assigned to an individual stock in the portfolio is the percentage of the stocks market cap compared to the total market cap. This automatically ensures that all weights add up to one. The portfolio weights are thus determined as follows:

$$w_{i,t} = \frac{MC_{i,t}}{\sum_{j=1}^{N} MC_{j,t}}, \tag{1}$$

where $w_{i,t}$ is the portfolio weight belonging to the i^{th} company for period t and N is the total number of stocks of which the index consist. This portfolio is rebalanced annually, where at that time new constituents can enter the index.

3.2 Static constraints

Weight constraints can be applied in many different ways and areas. In this thesis, three different algorithms will be used to impose weight constraints. All three algorithms will be discussed below.

3.2.1 Constant constraint

Algorithm 1 uses a constant constraint α. At rebalancing moments, all weights above this threshold α are constrained to that threshold. Outside the rebalancing moments, the portfolio is held via a buy-and-hold strategy and no adjustments are made to the portfolio. It was decided to constrain the weights to α to make as few adjustments as possible to the index such that turnover could be kept to a minimum. As a consequence of constraining weights, the total portfolio weight will not sum up to one. Therefore, normalization will have to take place. Two methods have been developed for this, a general normalization where all weights will be adjusted so that they sum up to one and a sector-neutral normalization where weights are adjusted so that they have the same sector exposure as before applying the constraints. Algorithm 1 is executed with α from 0.5% to 5.0% in steps of 0.5%. This grid is reasonable because there are enough stocks to exceed this threshold, but not enough for the turnover to increase extremely. Table 1 shows the average percentage of stocks that exceed the threshold α at rebalancing moments.

Algorithm 1: Calculation of weights under constant constraints[a]

Input: Price matrix $\boldsymbol{P}$, outstanding shares matrix $\boldsymbol{O}$, return matrix $\boldsymbol{R}$ and sector matrix $\boldsymbol{S}$;

Result: Performance, diversification and comparison measures;

1 Introduce rebalancing moments t_r;
2 Calculate market cap weights $w_{i,t}$ based on $\boldsymbol{P}$ and $\boldsymbol{O}$, for $i \in \{1, ..., N\}$ and $t \in \{1, ..., T\}$;
3 Calculate sector weights s_t;
4 **for** $t \in t_r$ **do**
5   **if** $w_{i,t} \geq \alpha$ **then**
6   Constraint portfolio weight $w_{i,t} \leftarrow \alpha$;
7   **end**
8 **end**
9 Normalize weights $w_{i,t} \leftarrow \frac{w_{i,t}}{\sum_{i=1}^{N} w_{i,t}}$, to assure $\sum_{i=1}^{N} w_{i,t} = 1$;
10 $s_t \leftarrow$ sector of $w_{i,t}$;
11 Normalize weights such that sector weights are equal to s_t, to assure $\sum_{i=1}^{N} w_{i,t} = 1$;
12 Return r_p, σ_p and SR using calculated weights and return matrix $\boldsymbol{R}$;

[a] Note that line 9 is relevant for the general normalization method and lines 10 and 11 are relevant for a sector neutral normalization method.

Table 1: Average percentage of stocks that exceed threshold α at rebalancing moments.

	5.0%	4.5%	4.0%	3.5%	3.0%	2.5%	2.0%	1.5%	1.0%	0.5%
S&P 100	1.00	1.67	2.19	3.10	4.14	6.62	11.57	18.38	30.76	64.14
S&P 500	0.03	0.07	0.09	0.18	0.39	0.60	0.87	1.86	3.73	8.21

Applying a constant stock constraint is a naive way of constraining where no conditional information about the constrained stocks is included. To explore the added value of this conditional information a stock fundamental constraint is introduced.

3.2.2 Stock fundamental constraint

Algorithm 2 can be read in **two different ways**. In the first approach large exposures above a threshold α are restricted if these stocks also have an extreme stock fundamental value. This makes this algorithm similar to Algorithm 1, but not all large exposures are restricted. This would theoretically require fewer adjustments to the market cap index, resulting in lower turnover and tracking error compared to the constant constraint in Algorithm 1. The first approach is based on **lines 6 to 10** in the algorithm. The second approach evaluates all stocks that have an extreme stock fundamental value. Because the exposure of these stocks in the index can vary from large to small, the weights cannot be restricted to one point. That is why it was decided to scale the weights of these stocks with a factor δ. δ will range from 0.9 to 0.1 in 0.1 increments. The second approach is based on **lines 11 to 13** in the algorithm.

To determine whether a stock fundamental value is extreme, the 90% quantitative of the fundamental is determined by sector. This is because stock fundamentals can differ per sector. The quantile value has been set at 90% to make as few adjustments as possible to the market cap index but targeted to restrict the deviating stocks. As with Algorithm 1, after the rebalancing moments, the portfolio is held via a buy-and-hold strategy and two different normalization methods are calculated, the general normalization method and the sector-neutral normalization method.

Algorithm 2: Calculation of weights under fundamental stock data constraints[a]

Input: Price matrix $\boldsymbol{P}$, outstanding shares matrix $\boldsymbol{O}$ and return matrix $\boldsymbol{R}$;
Stock characteristics matrix $\boldsymbol{C}$, sector matrix $\boldsymbol{S}$;
Result: Performance, diversification and comparison measures;
1 Introduce rebalancing moments t_r;
2 Calculate market cap weights $w_{i,t}$ based on $\boldsymbol{P}$ and $\boldsymbol{O}$, for $i \in \{1, ..., N\}$, and $t \in \{1, ..., T\}$;
3 Calculate sector weights s_t;
4 **for** $t \in t_r$ **do**
5 $\quad \psi_t \leftarrow$ 90% quantile of fundamental stock parameter $\boldsymbol{c}_t$ in sector s_t;
6 $\quad$ **if** $w_{i,t} \geq \alpha$ **then**
7 $\quad\quad$ **if** $\boldsymbol{C}_{i,t} \geq \psi_t$ **then**
8 $\quad\quad\quad$ Constraint portfolio weight $w_{i,t} \leftarrow \alpha$;
9 $\quad\quad$ **end**
10 $\quad$ **end**
11 $\quad$ **if** $\boldsymbol{C}_{i,t} \geq \psi_t$ **then**
12 $\quad\quad$ Constraint portfolio weight $w_{i,t} \leftarrow \delta w_{i,t}$;
13 $\quad$ **end**
14 **end**
15 Normalize weights $w_{i,t} \leftarrow \frac{w_{i,t}}{\sum_{i=1}^{N} w_{i,t}}$, to assure $\sum_{i=1}^{N} w_{i,t} = 1$;
16 $s_t \leftarrow$ sector of $w_{i,t}$;
17 Normalize weights such that sector weights are equal to s_t, to assure $\sum_{i=1}^{N} w_{i,t} = 1$;
18 Return r_p, σ_p and SR using calculated weights and return matrix $\boldsymbol{R}$;

[a] There are two constrained indices calculated. Lines 6-10 are only constraining stock weights that have a significant influence on the index, lines 11-13 apply a naive constraint which is computed for some fundamentals. This constraint applies to all stocks, even stocks that have little influence on the index. Note that line 15 is relevant for the general normalization method and lines 16 and 17 are relevant for a sector-neutral normalization method. When one wants to constrain the lower values, the 10% quantile is determined and the inequality sign in lines 7 and 11 changes direction.

Constant and stock fundamental constraints lead to adjustments to specific stocks, while a group of stocks may have to be constrained. Therefore, a sector constraint is evaluated.

3.2.3 Sector constraint

Algorithm 3 is similar to Algorithm 1, only the thresholds are determined differently. In this algorithm, the weights of the three largest sectors are added together at rebalancing moments. If the sum of these three sectors exceeds the threshold β, the sector exposure of the three sectors is restricted to β. β will be varied between 71% and 80% in increments of 1% for the S&P 500 and between 76% and 85% for the S&P 100. Again, after rebalancing the portfolio, the portfolio is held by a buy-and-hold strategy. Normalization only takes place under the general normalization method. Table 2 shows how many percent of the time the threshold β is exceeded at rebalancing moments. And Figure 2 shows the sum of the three largest sector weights. This gives a sense of how strict the sector constraint is.

Algorithm 3: Calculation of weights under sector constraints.

Input: Price matrix $\boldsymbol{P}$, outstanding shares matrix $\boldsymbol{O}$ and the return matrix $\boldsymbol{R}$;
Sector weights matrix $\boldsymbol{S}$;
Result: Performance, diversification and comparison measures;
Introduce rebalancing moments t_r;
Calculate market cap weights $w_{i,t}$ based on $\boldsymbol{P}$ and $\boldsymbol{O}$, $i \in \{1, ..., N\}$, for $t \in \{1, ..., T\}$;
for $t \in t_r$ **do**
  Calculate sector exposures $\boldsymbol{w}'\boldsymbol{S}$;
  Calculate top 3 sector exposures $\boldsymbol{s}_t$;
  if $\boldsymbol{s}_t \geq \beta$ **then**
    Constraint portfolio sector weights $\beta \leftarrow \boldsymbol{s}_t$;
    Calculate constraint portfolio weights $w_{i,t}$ using sector weights $\boldsymbol{s}_t$;
  end
end
Normalize weights $w_{i,t} \leftarrow \frac{w_{i,t}}{\sum_{i=1}^{N} w_{i,t}}$, to assure $\sum_{i=1}^{N} w_{i,t} = 1$;
Return r_p, σ_p and SR using calculated weights and return matrix $\boldsymbol{R}$;

Table 2: Percent of the time that the constraint β is exceeded at rebalancing moments. For the S&P 100 the grid will variate between 85% and 76% in steps of 1%.

	80%	79%	78%	77%	76%	75%	74%	73%	72%	71%
S&P 100	4.76	4.76	23.81	42.86	61.90	71.43	80.95	80.95	85.71	90.48
S&P 500	4.76	19.05	38.10	38.10	61.90	85.71	85.71	100	100	100

Figure 2: Sum of three largest sector weights.

The above constraints are applied continuously while this may not add value in some parts of the economic cycle and only causes more tracking error and turnover. To be able to evaluate this, a dynamic constraint will be looked at that may offer a solution to this problem.

3.3 Dynamic constraints

Following a market cap index has several advantages. The biggest advantage is the seldom if ever rebalancing of the portfolio weights, which is to the advantage of the investor. But, as Amenc et al. (2012) shows, there may be different portfolios that perform well during different stages of the economic cycle. This will be tested using a dynamic constraint. For example, the market cap index is followed, but deviations are made in highly volatile periods. To determine whether you are in an economic recession, several *variables* can be used. The first variable that will be used is the *VIX*. The Chicago Board Options Exchange's CBOE Volatility Index (VIX) is a 30-day expectation of volatility based on S&P 500 out-of-the-money options. This variable measures the volatility of the S&P 500 and thus this constraint is applied when the US stock market has reached a certain level of volatility. Since the S&P 500 and the VIX are strongly negatively correlated, this restriction may come too late.

Therefore, another decision variable will be used as well, the *Buffett indicator*. The Buffett indicator (BI), introduced by Warren Buffett, is a valuation multiple that measures how expensive or cheap the stock market is at a given point in time. This variable compares the valuation of the US stock market with the US gross domestic product (GDP), dividing the market capitalization of the Wilshire 5000 index by the GDP. As a result, it could be a simplified variable reflecting the over- or undervaluation of the US stock market. Should the BI exceed a certain threshold, one can choose to impose a constraint to suffer fewer losses during a possible future correction.

The dynamic constraint is applied according to Algorithm 4. The three different algorithms from Section 3.2 are applied if the decision variable exceeds a certain threshold γ. If after a while the decision variable falls below the threshold again, the switch is made from the constrained portfolio to the market cap portfolio. The value for γ differs for the different decision variables. For the *VIX* the thresholds 36 to 40 have been tested and for the *BI* the thresholds 1.2 to 1.5.

Algorithm 4: Calculation of weights under dynamic constraints.

Input: Price matrix $\boldsymbol{P}$, outstanding shares matrix $\boldsymbol{O}$ and the return matrix $\boldsymbol{R}$;
Recession variable $\boldsymbol{X}$, fundamental stock matrix $\boldsymbol{F}$, sector matrix $\boldsymbol{S}$;
Result: Performance, diversification and comparison measures;
1 Calculate market cap weights $w_{i,t}$ based on $\boldsymbol{P}$ and $\boldsymbol{O}$, $i \in \{1, ..., N\}$, for $t \in \{1, ..., T\}$;
2 **if** $\boldsymbol{X}_t < \gamma$ **then**
3 | Calculate market cap weights based on Equation 1;
4 **else**
5 | Calculate index weights based on Algorithm 1, 2 or 3, buy and hold until $\boldsymbol{X}_t < \gamma$;
6 | **return** to line 2;
7 **end**
8 Return r_p, σ_p and SR using calculated weights and return matrix $\boldsymbol{R}$;

3.4 Index combinations

Timmermann (2006) gives several reasons why combining different methods may be favourable. He stated that combining different methods would lead to better results since valuable information of individual methods can be combined. This would create a robust method that may lag slightly behind the best method in good times, but falls less sharply during recession periods. Combining different methods reduces the influence of possible individual model misspecifications, as less weight is given to the individual models. Determining the weight given to the different methods can be done in many ways. This thesis will specifically look at combining risk-based methods through volatility and expected shortfall. For Y different (un)constrained indices, the combined index weight will be as follows:

$$\boldsymbol{w}_t^{\mathrm{C},m} = \pi_{1,t}\boldsymbol{w}_t^{1,m} + \pi_{2,t}\boldsymbol{w}_t^{2,m} + \cdots + \pi_{Y,t}\boldsymbol{w}_t^{Y,m},$$

where $\boldsymbol{w}_t^{y,m}$ denotes the weight given to the m^{th} asset by method y for period t and $\pi_{y,t}$ denotes the combination weight for method y for period t, such that $\sum_{i=1}^{Y} \pi_{i,t} = 1$ and $\pi_{i,t} > 0$. It is intuitive that the index combination approach should assign higher weights to methods that produce lower (tail)risk while penalising those that produce higher (tail)risk by assigning lower weights to them. Therefore, the inverse rank approach is used as suggested by Aiolfi and Timmermann (2006), where $\pi_{y,t}$ for period t is obtained from:

$$\pi_{y,t} = \frac{\gamma_{y,t-1}^{-1}}{\sum_{i=1}^{Y} \gamma_{i,t-1}^{-1}},$$

where $\gamma_{y,t-1}^{-1}$ is the inverse volatility or expected shortfall of model y up to time $t-1$.

3.5 Downside risk

By imposing the constraints one tries to reduce large exposures to certain stocks, fundamentals or sectors, which can potentially reduce downside risk. The most commonly used measures to quantify downside risk are value at risk (VaR) and expected shortfall (ES). Below is a brief description of how these metrics are calculated and how they will be estimated in this thesis. Throughout the thesis, the 95% VaR and ES will be used.

3.5.1 Value-at-Risk

The VaR is a quantile that indicates the expected negative return in the worst $(1-\alpha)\cdot 100\%$ of the cases. The value for VaR can be determined in different ways and gives different results (Jorion, 2002; Alexander, 2009; Kourouma et al., 2010). Each method has its advantages and disadvantages and the reason for switching between different methods can have different causes. In the world of risk management, it is often calculated with losses instead of returns, which are calculated by:

$$L_i = -\frac{(V_i - V_{i-1})}{V_{i-1}} = -r_{index,i-1},$$

where L_i represent index loss at time i and V_{i-1} and V_i represent the value of the index at time $i-1$ and i. The probability of having a loss below the VaR is quantified by $1-\alpha$, as shown in Equation 2.

$$VaR_\alpha = \inf\left\{l_\alpha \in \mathbb{R} : \mathbb{P}\left(L_{i+1} \leq l_\alpha\right) \geq \alpha\right\} = \inf\left\{l_\alpha \in \mathbb{R} : \mathbb{P}\left(L_{i+1} > l_\alpha\right) \leq 1-\alpha\right\}. \quad (2)$$

In this thesis two methods will be used to estimate the VaR. One of these methods is independent of distribution and for the other, a distribution is estimated.

3.5.1.1 Historical simulation

The big advantage of the historical simulation (HS) method is that no parametric assumptions are made at all and therefore it is computationally easy to calculate. This method is based on an empirical distribution function of the losses L. That is, all losses form a distribution that takes value L_1,...,L_k with probability 1/k each:

$$F_k(l) = \frac{1}{k}\sum_{i=1}^{k} I_{\{L_i \leq l\}}$$

Then the VaR estimate will be:

$$\widehat{VaR}_\alpha = \inf\{x : F(x) \geq \alpha\} = L_{(k\alpha)},$$

where $L_{(k\alpha)}$ is the $k\alpha$ order statistic of the losses L.

3.5.1.2 Normal mixture model

Financial return (loss) data is often leptokurtic, meaning left (right) skewed and often has a high concentration around the mean resulting in a high kurtosis. As a result, a normal distribution is not suitable as a representation of the financial data. Although a t-distribution with a low degree of freedom can be leptokurtic, this distribution cannot incorporate the skewness. A mixture of different normal distributions can introduce both skewness and excess kurtosis. In this thesis the two mixture model is used, where the two separate models are a representation of a 'good' and 'bad' economy. The probabilities for each of these scenarios are λ and $1-\lambda$ respectively. Which ultimately results in a distribution:

$$F(x) = \lambda F_1(x;\mu_1,\sigma_1^2) + (1-\lambda)F_2(x;\mu_2,\sigma_2^2),$$

where F_1 and F_2 are normally distributed.

The parameters of these distributions must be estimated to estimate the VaR. The parameters are estimated using the Expectation-Maximization (EM) algorithm, elaborated in Hamilton (1990). In the case of a two-point normal mixture model (NMM), these five parameters have to be estimated: $\lambda, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ by the maximum likelihood estimation (MLE). These five parameters are denoted as the $\boldsymbol{\theta}$ parameter vector.

The density function of the model, that will be maximized by MLE, is as follows:

$$f(x) = \lambda f_1(x) + (1-\lambda)f_2(x) = \frac{\lambda}{\sigma_1}\phi\left(\frac{x-\mu_1}{\sigma_1}\right) + \frac{1-\lambda}{\sigma_2}\phi\left(\frac{x-\mu_2}{\sigma_2}\right).$$

These parameters are estimated by maximizing:

$$\log L = \sum_{i=1}^{n} \log f(x_i).$$

In this optimization problem, an observation w_i will be drawn from a Bernoulli distribution with probability λ, which will be initialized on the first loop, to determine which scenario one is in, resulting in a joint 'density' function of the model:

$$f(x,w) = (\lambda f_1(x))^{\mathbb{1}_{w=1}} \left((1-\lambda)f_2(x)\right)^{\mathbb{1}_{w=2}} = \begin{cases} \lambda f_1(x) & \text{if } w=1 \\ (1-\lambda)f_2(x) & \text{if } w=2 \end{cases},$$

where $\mathbb{1}_A$ is the indicator function: 1 if A is true, 0 is A is false.
This results in the following log-likelihood function:

$$\log L = \sum_{i=1}^{n} \mathbb{1}_{w_i=1}\left(\log\lambda + \log f_1(x_i)\right) + \mathbb{1}_{w_i=2}\left(\log(1-\lambda) + \log f_2(x_i)\right).$$

The only problem is that w_i is not observable and therefore has to be guessed. An educated guess would be to take the proportion of observations origination from density function 1:

$\frac{\lambda f_1(x_i)}{\lambda f_1(x_i)+(1-\lambda)f_2(x_i)}$. This estimate will be improved iteratively until the log-likelihood is maximized. All in all, this leads to a log-likelihood function of:

$$\log L = \frac{\lambda f_1(x_i)}{\lambda f_1(x_i) + (1-\lambda)f_2(x_i)} (\log \lambda + \log f_1(x_i)) + \frac{(1-\lambda)f_2(x_i)}{\lambda f_1(x_i) + (1-\lambda)f_2(x_i)} (\log(1-\lambda) + \log f_2(x_i)) .$$

The parameters are estimated iteratively by the Expectation (E) and the Maximization (M) steps elaborated in Hamilton (1990). The exposition of the parameters can be found in Appendix B.

After estimating the parameters, a value for the VaR must be estimated. This will be done in this thesis by the Monte Carlo approach. And will consist of the following steps:

1. Simulate k, which will be 100.000.000 throughout this thesis, observations $(B_1, \ldots, B_k)$ from a Bernoulli distribution with $\mathbb{P}(B_i = 1) = \widehat{\lambda}$ and $\mathbb{P}(B_i = 2) = 1 - \widehat{\lambda}$

2. Simulate k i.i.d. standard normal observations $(Z_1, \ldots, Z_k)$

3. If $B_i = 1, L_i = \widehat{\mu}_1 + \widehat{\sigma}_1 Z_i$; If $B_i = 2, L_i = \widehat{\mu}_2 + \widehat{\sigma}_2 Z_i$

4. Then the losses are sorted in ascending order $L_{(1)}, \ldots, L_{(k)}$, where $L_{(i)}$ is the i^{th} order statistic.

5. The VaR estimation will be the same as the HS method in Section 3.5.1.1:

$$\widehat{VaR}_\alpha = L_{(k\alpha)}$$

The advantage of creating a distribution around the data is that it is easy to simulate additional data points from the same distribution. A disadvantage is that the wrong distribution is assumed, which leads to model specification error.

3.5.2 Expected Shortfall

Since the global financial crisis of 2008, many regulators have decided to switch from VaR to expected shortfall ES as the main risk measure. The most important reason for this is that VaR is not a coherent risk measure. This allowed banks to hold extremely risky portfolios without this being directly visible in the reported risk figures. A risk measure is coherent if it satisfies the four axioms mentioned in Artzner et al. (1999). These four axioms consist of monotonicity, positive homogeneity, translation invariance and sub-additivity. VaR does not meet the latter. As a result, it might be better to diversify less from a VaR point of view to reduce risk. As a solution to this shortfall, the ES has been introduced, which represents an "average" loss when moving into the region below the VaR. In contrast to the VaR, the ES is a coherent risk measure (Acerbi and Tasche, 2002). This means that this measure is a better representation of diversification, is a better measure on which to base regulations. The expected shortfall is calculated by the following equation:

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 \text{VaR}_u(r_{index})du.$$

And the ES will be estimated by the following equation:

$$\widehat{ES}_\alpha = \frac{1}{k(1-\alpha)}\sum_{i=[k\alpha]+1}^{k} L_{(i)}$$

3.6 Significance tests

The goal of this thesis is to find a constraint that ensures minimal adjustments to the market cap index, such that a significantly higher Sharpe ratio and/or significantly lower volatility will be obtained. In addition, the point estimation of the ES, the turnover and tracking error will also be central to the analysis. To measure whether the differences in Sharpe ratios and volatilities are significant, the constrained index must be compared with the market cap index. A simple t-test would suffice provided the observations were individual and identical distributed (iid) (Jobson and Korkie, 1981). However, we work with financial data that show serial correlation and volatility clustering. Ledoit and Wolf (2008) provide a solution to this problem in their paper when the Sharpe ratios and volatilities of two portfolios need to be compared. They recommend heteroscedasticity-consistent (HAC) standard errors when calculating this significance. This method works asymptotically but could show undesirable behaviour in finite samples. An alternative they use for this is a studentized times series bootstrap. There is no general test known in the literature for significant differences between ESs. Examples of literature close to this topic are: Acerbi and Szekely (2014); Righi and Ceretta (2015); Wimmerstedt (2015); Novales and Garcia-Jorcano (2019); Ziegel et al. (2020).

3.6.1 HAC Inference

The Sharpe ratio or volatility difference test need to be performed using heteroskedastic and autocorrelation consistent (HAC) estimators, since economic data typically contains autocorrelation and/or heteroskedasticity of unknown form. The test, using HAC inference, is as follows. It is assumed that there are two portfolios with excess returns $r_{t,i}$ and $r_{t,j}$ for $t = 1, \dots, N$ with:

$$\mu = \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \sigma_i^2 & \sigma_{i,j} \\ \sigma_{i,j} & \sigma_j^2 \end{pmatrix}.$$

The difference in Sharpe ratios and volatilities are given by:

$$\Delta = SR_i - SR_j = \frac{\mu_i}{\sigma_i} - \frac{\mu_j}{\sigma_j}, \text{ and } \Delta = \sigma_i - \sigma_j \quad (3)$$

with their estimators:

$$\widehat{\Delta} = \widehat{SR}_i - \widehat{SR}_j = \frac{\widehat{\mu}_i}{\widehat{\sigma}_i} - \frac{\widehat{\mu}_j}{\widehat{\sigma}_j}, \text{ and } \widehat{\Delta} = \widehat{\sigma}_i - \widehat{\sigma}_j. \quad (4)$$

The notation of the equation can be changed so that $\Delta = f(v)$ and $\widehat{\Delta} = f(\widehat{v})$ with,

$$f(a,b,c,d) = \frac{a}{\sqrt{c-a^2}} - \frac{b}{\sqrt{d-b^2}} \text{ for SR, and } f(a,b,c,d) = \sqrt{c-a^2} - \sqrt{d-b^2} \text{ for } \sigma,$$

where a and b are the respectable means, $\mathbb{E}(r_{1,i})$ and $\mathbb{E}(r_{1,j})$, and c and d are the second moments, $\mathbb{E}\left(r_{1,i}^2\right)$ and $\mathbb{E}\left(r_{1,j}^2\right)$, of portfolio i and j.
When using the Delta method in combination with mild regularity conditions on the higher moments of excess returns, the following convergence is achieved:

$$\sqrt{N}\left(\widehat{\Delta} - \Delta\right) \xrightarrow{d} \mathcal{N}\left(0, \nabla' f(v) \Psi \nabla f(v)\right), \tag{5}$$

where Ψ is an unknown symmetric positive semi-definite matrix that represents the limiting covariance matrix and ∇ corresponds to the first derivative with respect to all individual variables a, b, c and d. If a consistent estimator $\widehat{\Psi}$ is available, the standard error for $\widehat{\Delta}$ is given by:

$$s\left(\widehat{\Delta}\right) = \sqrt{\frac{\nabla' f(\widehat{v}) \widehat{\Psi} \nabla f(\widehat{v})}{N}}. \tag{6}$$

Finally, the two-sided p-value for the null hypothesis $H_0 : \Delta = 0$ is given by:

$$\widehat{p} = 2\Phi\left(\frac{-|\widehat{\Delta}|}{s\left(\widehat{\Delta}\right)}\right), \tag{7}$$

where $s\left(\widehat{\Delta}\right)$ is obtained in Equation 6. With this p-value calculation a $1-\alpha$ confidence interval can be created:

$$\widehat{\Delta} \pm z_{1-\alpha/2}\, s\left(\widehat{\Delta}\right),$$

where z_α is the α quantile of the standard normal distribution. The full exposition can be found in Appendix C.

3.6.2 Studentized time series bootstrap

The biggest disadvantage of HAC inference is that it uses asymptotic normality, which does not necessarily apply. The inference accuracy could be improved by letting go of this asymptotic normality assumption and switch to a studentized bootstrap method (Lahiri, 2003). This method works with the original data and does not assume an underlying distribution. The bootstrap method uses an indirect approach, which means that the observed data is resampled. A confidence interval can be compiled with this resampled data. Bootstrapping can mainly offer added value in smaller sample sizes because the asymptotic normality certainly does not have to apply here. Bootstrapping will be used in this thesis for completeness in all cases, but will be necessary for the smaller volatility regimes. The studentized statistic is approximated via the bootstrap as follows:

$$\mathcal{L}\left(\frac{|\widehat{\Delta}-\Delta|}{s\left(\widehat{\Delta}\right)}\right) \approx \mathcal{L}\left(\frac{|\widehat{\Delta}^{*}-\widehat{\Delta}|}{s\left(\widehat{\Delta}^{*}\right)}\right),$$

where Δ and $\widehat{\Delta}$ are the same as defined in Equations 3 and 4, and Δ^{*} is the estimated difference computed from bootstrap data. $s\left(\widehat{\Delta}\right)$ and $s\left(\widehat{\Delta}^{*}\right)$ are the standard error of $\widehat{\Delta}$ and $\widehat{\Delta}^{*}$ respectively. $\mathcal{L}(X)$ denotes the distribution of random variable X.
Then $z^{*}_{|\cdot|,\alpha}$ is introduced as the α quantile of $\mathcal{L}\left(\frac{|\widehat{\Delta}^{*}-\widehat{\Delta}|}{s(\widehat{\Delta}^{*})}\right)$, leading to a $1-\alpha$ confidence interval for Δ of:

$$\widehat{\Delta} \pm z^{*}_{|\cdot|,\alpha}\, s\left(\widehat{\Delta}\right). \tag{8}$$

When data is heavy-tailed, which is usually the case with financial data, the value for $z^{*}_{|\cdot|,\alpha}$ will be higher than z_{α}, introduced in the HAC inference section, and thus generate a wider interval. The bootstrap data is generated by a circular block-bootstrap, introduced by Politis and Romano (1992). Block-bootstrapping is necessary to ensure the accuracy of the bootstrap. The data is divided into blocks so that the time series structure of the original data is preserved. The standard errors are calculated according to Equation 6. The circular bootstrap method aims to remove the effect of the uneven weighting of observations at the beginning and the end of the dataset. A block structure would look like this for example:

- $B_1 = (x_1, \ldots, x_l)$
- $B_2 = (x_2, \ldots, x_{l+1})$
- $B_N = (x_{N-l+1}, \ldots, x_n)$

For example, with a block length of five, the observation x_5 is included five times in the entire structure, while observation x_1 returns only once. To prevent this from happening, an end-to-start wrap around of the data is performed to make additional blocks. The only question remains which value should be chosen for the block length. When Algorithm 3.1 from Ledoit and Wolf (2008) is followed, an optimal block length of 8 is obtained.

3.7 Performance metrics

The different indices can be evaluated using different metrics that can be observed in Table 3.

Table 3: Variable description and calculation

Variable	*Description*	*Calculation*
	Risk and return	
r_p	Annualized portfolio return (in %)	$((1+r_{cum})^{1/\#\text{ of years}}-1)\cdot 100\%$
σ_p	Annualized portfolio volatility (in %)	$\sqrt{\frac{\sum_{i=1}^{T}(r_i-\bar{r})^2}{T-1}}\cdot\sqrt{T/\#\text{ of years}}$
SR	Annualized Sharpe ratio	$\frac{\sum_{i=1}^{T}(r_i-r_i^f)}{\sigma_{excess}}\cdot\sqrt{T/\#\text{ of years}}$
min and max	Minimum and maximum return	-
S	Skewness	$\sum_{i=1}^{T}\left(\frac{r_i-\bar{r}}{\sigma_{index}}\right)^3$
K	Kurtosis	$\sum_{i=1}^{T}\left(\frac{r_i-\bar{r}}{\sigma_{index}}\right)^4$
TO	Portfolio turnover	$\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{N}\lvert w_{i+1,j}-w_{i+,j}\rvert\cdot 10^{-4}$
r_{cum}	Cumulative return for period T	-
$\bar{r}$	Mean return	-
r_i^f	Risk-free rate for period i	-
σ_{index}	Daily index volatility	$\sqrt{\frac{\sum_{i=1}^{T}(r_i-\bar{r})^2}{T-1}}$
	Diversification and comparison	
AS	Average effective number of securities	$\frac{1}{\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{N}w_{i,j}^2}$
TE	Annualized tracking error	$\sqrt{\sum_{i=1}^{T}\left(r_{i,b}-r_{i,index}\right)^2}\cdot\sqrt{T/\#\text{ of years}}$
$r_{i,b}$	Benchmark return at time i	-
	Significance Testing	
$p_{H,\sigma}$	HAC Inference p-value for σ difference	Equation 7
$p_{B,\sigma}$	Bootstrapping p-value for σ difference	Equation 8
$p_{H,SR}$	HAC Inference p-value for SR difference	Equation 7
$p_{B,SR}$	Bootstrapping p-value for SR difference	Equation 8

The average effective number of securities (AS) is calculated by the inverse Herfindahl-Hirschman index (Rhoades, 1993), which is a statistical measure of concentration. For example, a portfolio with a Herfindahl-Hirschman index of 0.02 is equivalent to a portfolio with $AS = \frac{1}{0.02} = 50$ equal-weighted securities. The annualized tracking error identifies the level of consistency in which a portfolio tracks the performance of a chosen benchmark. The tracking error, therefore, can be seen as an indicator of how actively a fund is managed. In this thesis, the S&P 500 and S&P 100 will be used as benchmark indices.

4 Data

4.1 Stock data - risk and return

Daily US stock data was used for this research. A period of 21 years is used for the analysis, from the beginning of January 2000 to the end of December 2020. This period contains several financial business cycles, which makes this a realistic representation of recent history. The S&P Global website[3] has been consulted to obtain a list of current and historical S&P 500 stocks from January 2000 - December 2020.

The S&P 500 index is created by annually selecting the 500 companies with the largest market capitalization on the first trading day of the year. After selecting the index constituents, the stocks are held by a buy-and-hold strategy. As a result, transactions will only take place in the index when a new year starts, when a company decides to issue more shares and when a company in the current index goes bankrupt or private. To analyze the effect of diversification, a more concentrated index will be included in the analysis. This will consist of the largest 100 companies in the index, the S&P 100. In addition, both indices will be included equal weighted to form a naive 1/N benchmark. To find out the stock prices and outstanding shares of any company at any time, the Center for Research in Security Prices (CRSP) was consulted. A total of 723 unique stocks were found for the analyzed period. The CRSP database was accessed via the data platform Wharton Research Data Services (WRDS).

Table 4: Annualized summary statistics of S&P indices.

	r_p	σ_p	SR	min	max	S	K	TO	TE	AS	V_{HS}	V_{NM}	ES_{HS}	ES_{NM}
S&P 100	6.4	19.7	0.33	-11.9	11.5	-0.1	13.5	2.2	-	51	1.87	1.94	2.98	3.21
S&P 500	7.6	19.8	0.39	-12.0	11.5	-0.2	13.7	0.6	-	107	1.87	1.91	3.00	3.22
S&P 100 EW	7.2	19.7	0.37	-12.3	11.9	-0.2	14.3	12.0	2.1	97	1.86	1.88	2.97	3.18
S&P 500 EW	12.0	20.4	0.58	-12.4	11.1	-0.3	13.6	9.7	4.4	477	1.91	1.86	3.07	3.25

The summary statistics can be observed in Table 4. In this table, all returns are expressed in percentages. A negative skewness (S) and excess kurtosis (K) is observed, which is common for financial data. Financial return data is known to be non-normal. Therefore, it was decided to use a normal mixture model to represent the data. The S&P index returns, with associated normal mixture components, are shown in Figures 6 and 7 that can be found in Appendix D. Figure 8 in Appendix E shows that the equal-weighted S&P 500 has achieved the best results over the past 21 years. There appears to be a correlation between diversification and return. As can be seen in Table 4, the S&P 500 and S&P 100 EW have about the same AS which gives a comparable result in terms of returns. The VaR and ES are measured at a 95% significance level. The point estimations can be observed in Table 4 and were estimated with the historical simulation method and a Monte Carlo method for a normal mixture model, for which 100,000,000 simulations were

[3]Source: https://www.spglobal.com/spdji/en/indices/equity/sp-500/#overview

run.

4.2 Stock fundamentals data

Stock fundamentals have an influence on determining the price of a stock. Stock fundamentals are also the basis of many empirical studies, of which the research of Fama and French (1992) is perhaps best known. In this thesis, stock fundamentals will be used to create constraints. The CRSP/Compustat link in WRDS has been used to retrieve stock fundamental data since it is not included in the CRSP database itself. Stock fundamentals of 603 of the 723 companies were found in this database. The data that was missing was often from the smaller constituents in the index. Table 21 in Appendix H gives an overview of all fundamentals that will be investigated and Table 22 in Appendix H displays their respective mean.

4.3 Sector data

There are many different ways to divide stocks into sectors. The best-known variable used for classification is the Standard Industry Classification (SIC). This method was developed in 1930 by the US government and has long been the industry standard for categorizing companies. However, this method has several drawbacks. The industry has changed a lot since the early years. Companies today are mainly service-based, while in the twentieth century, this was mainly manufacturing-based. SIC is slow to recognize new and emerging sectors such as information technology sectors. The figure in Appendix I provide an overview of the S&P index sectors classified based on SIC, which were retrieved from the CRSP database.

4.4 Volatility regimes

To analyze the effect of different business cycles, a distinction must be made between 'normal' periods and highly volatile periods. To make this distinction, the National Bureau of Economic Research[4] (NBER) is consulted. The highly volatile periods between January 2000 and December 2020 are indicated in Table 5. Figure 3 plots volatility over time, where high volatility periods are also highlighted.

Table 5: High volatility regimes between January 2000 and December 2020. Source: NBER

Name	Regime reference	Period	Observations
Early 2000 recession	R_1	March 2001 - November 2001	208
Global financial crisis	R_2	December 2007 - June 2009	397
COVID-19 pandemic	R_3	February 2020 - Ongoing	232

[4]Source: https://www.nber.org/research/data/us-business-cycle-expansions-and-contractions

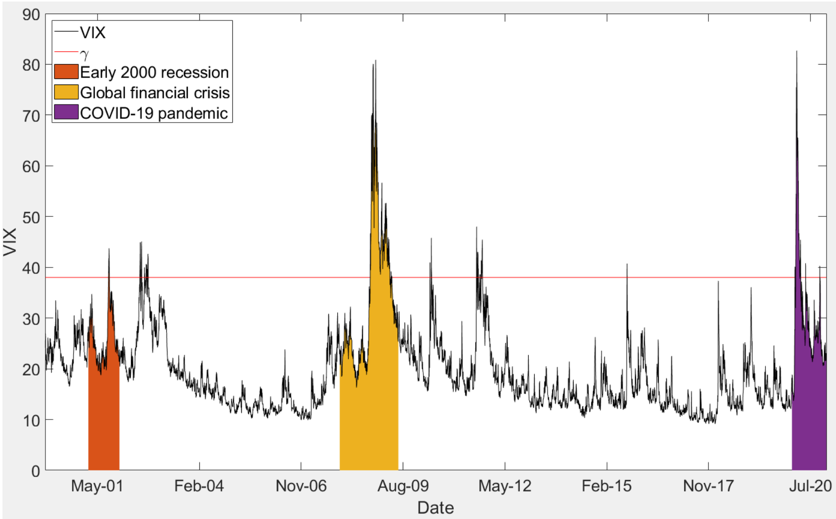

Figure 3: Graph of the CBOE Volatility Index (VIX) with a value for γ displayed (38). Source: CBOE

As can be seen in Figure 8 in Appendix E, the most diversified index has achieved the highest returns over the past 21 years. Diversification benefits appear to pay off mainly in non-volatile periods, where higher excess returns are achieved compared to the market cap index. However, this diversification can be valuable during economic recessions as well, as can be seen in the figures in Appendix F.

Table 6: VaR and ES estimations on a 95% significance level for highly volatile periods. Estimations are based on the historical simulation and on 100,000,000 Monte Carlo simulated normal mixture model random variables with initial parameters $\boldsymbol{\theta}^{(0)} = (\lambda^{(0)}; \mu_1^{(0)}; \mu_2^{(0)}; \sigma_1^{(0)}; \sigma_2^{(0)}) = (0.75;0;0;1;2)$ for regime 1, (0.9;0;-7;1;4) for regime 2 and (0.9;0.5;-5;1;4) for regime 3.

	R_1				R_2				R_3			
	V_{HS}	V_{NM}	ES_{HS}	ES_{NM}	V_{HS}	V_{NM}	ES_{HS}	ES_{NM}	V_{HS}	V_{NM}	ES_{HS}	ES_{NM}
S&P 100	2.44	2.29	3.10	3.20	3.63	3.89	5.63	5.74	3.45	4.00	5.68	6.13
S&P 500	2.16	2.18	2.94	3.04	4.12	4.13	5.79	5.92	3.29	4.05	5.82	6.21
S&P 100 EW	2.33	2.21	2.94	3.07	3.98	4.06	5.79	5.90	3.27	4.00	5.75	6.21
S&P 500 EW	2.02	1.98	2.63	2.75	4.42	4.51	6.12	6.17	3.26	4.12	6.26	6.51

In Table 6, you can see the results of VaR and ES estimations based on historical simulation and a 100 million Monte Carlo simulated normal mixture model. As can be seen, diversification through equal portfolio weights is not the solution to reduce downside risk. As seen in Table 4, the 1/N index takes on more volatility compared to the market cap index. A possible explanation for this additional risk, and the associated downside risks, could be the greater exposure to smaller-cap stocks. Smaller-cap stocks are generally riskier and this effect is reflected in the statistics. The size effect has already been investigated by Fama and French (1992), who draw the same conclusion about smaller-cap stocks.

4.5 Dynamic constraint data

For the dynamic constraint methods, based on the VIX and the Buffett Indicator (BI), 5 different values for γ are tested. The consequences of the value for γ can be seen in Table 7 below, and in Figures 3 and 12 in Appendix G. The VIX data was taken from the Chicago Board Option Exchange (CBOE) website[5], this data is available daily where the close price is used for analysis. The Buffett Indicator itself is calculated by dividing the market capitalization of the Wilshire 5000 index by the US GDP. Both variables can be retrieved from the Federal Reserve Economic Data (FRED) website[6]. US GDP data is available quarterly, therefore the Buffett Indicator is also available quarterly. This automatically leads to less rebalancing, which can be seen in Table 7.

Table 7: Number of trading days that the constraint γ is exceeded and the corresponding number of rebalance moments (regular rebalance moments at first day of the year included).

	VIX					BI			
γ	36	37	38	39	40	1.2	1.3	1.4	1.5
Days	271	244	223	206	188	1238	784	197	64
Rebalance	88	76	78	76	74	24	25	26	22

[5]https://www.cboe.com/tradable_products/vix/vix_historical_data/
[6]https://fred.stlouisfed.org/

5 Results

5.1 Static constraints

5.1.1 Constant constraint

In Tables 8 and 9, the performance of two constant constrained S&P 500 indices (Algorithm 1), one with a general normalization method and one with the sector-neutral normalization method, can be observed. In the tables, we observe that in most cases, imposing a constraint helps to reduce risk. This applies to the volatility of the index σ_p as well as to the downside risk, which is measured by the ES. This conclusion ties in with the one that Jagannathan and Ma (2003) made. In almost all cases, the 1/N portfolio proves hard to beat in terms of Sharpe ratios (DeMiguel et al., 2009).

In Table 8 it is observed that the volatility of the index can be reduced by constraining the index. However, there appears to be a point where a constraint that is too strict leads to higher index volatility, this is mainly observed with a constraint of 1.0% and 0.5%. This is in line with the conclusion of Chiou et al. (2009). However, these strictly constrained indices show significantly better results in terms of Sharpe ratios compared to the market cap index. This improvement can be explained by the fact that the index is pushed more towards the equal-weighted index, which often has a higher Sharpe ratio. The push towards the equal-weighted portfolio can be observed when looking at the AS, which in the strictest constraint indices more than doubles compared to the market cap index. However, a major difference with the equal-weighted index is that the downside risk does not increase, or in some cases is even smaller, compared to the market cap index.

Logically, an increase is seen in the TE and TO as the index is more strictly constrained. Where the sector-neutral normalization method, introduced in Section 3.2, seems to ensure that less TE is incurred. However, this does result in a higher TO, which can be observed in Table 9. Smaller exposures that were barely adjusted in a general normalization method can be significantly adjusted by the sector-neutral normalization method. The general normalization method does allow for more diversification in the index, as each stock in the index is increased by a certain percentage instead of just several stocks, which is the case in a sector-neutral normalization method.

Table 8: Annualized summary statistics of constrained S&P 500 using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation. Based on Algorithm 1.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.74	19.74	0.00	0.00	0.40	0.02	0.02	2.52	0.28	106	2.99	3.21
α of 4.5%	7.75	19.74	0.00	0.00	0.40	0.02	0.03	2.54	0.29	107	2.99	3.21
α of 4.0%	7.77	19.73	0.00	0.00	0.40	0.03	0.04	2.59	0.33	108	2.99	3.20
α of 3.5%	7.78	19.71	0.00	0.00	0.40	0.04	0.05	2.67	0.39	110	2.99	3.20
α of 3.0%	7.78	19.70	0.00	0.00	0.40	0.11	0.12	2.77	0.51	115	2.98	3.20
α of 2.5%	7.78	19.68	0.00	0.00	0.40	0.21	0.22	2.86	0.66	122	2.98	3.20
α of 2.0%	7.81	19.65	0.00	0.00	0.40	0.26	0.26	2.97	0.86	132	2.98	3.19
α of 1.5%	7.91	19.65	0.00	0.00	0.41	0.21	0.21	3.20	1.12	146	2.98	3.19
α of 1.0%	8.17	19.69	0.01	0.00	0.42	0.10	0.10	3.48	1.44	177	2.98	3.19
α of 0.5%	8.71	19.86	0.57	0.57	0.44	0.03	0.03	3.88	2.03	245	3.00	3.21

Table 9: Annualized summary statistics of constrained S&P 500 using a constant and the sector-neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.70	19.77	0.00	0.00	0.40	0.06	0.07	2.49	0.22	106	3.00	3.21
α of 4.5%	7.70	19.77	0.00	0.00	0.40	0.06	0.09	2.52	0.24	107	3.00	3.21
α of 4.0%	7.71	19.76	0.00	0.00	0.40	0.08	0.10	2.58	0.27	108	2.99	3.21
α of 3.5%	7.72	19.75	0.00	0.00	0.40	0.10	0.11	2.66	0.33	110	2.99	3.21
α of 3.0%	7.72	19.73	0.00	0.00	0.40	0.21	0.22	2.77	0.43	114	2.99	3.20
α of 2.5%	7.72	19.71	0.00	0.00	0.40	0.33	0.34	2.86	0.56	121	2.99	3.20
α of 2.0%	7.75	19.68	0.00	0.00	0.40	0.33	0.33	2.99	0.74	130	2.98	3.19
α of 1.5%	7.85	19.66	0.00	0.00	0.40	0.23	0.22	3.24	0.97	144	2.98	3.19
α of 1.0%	8.08	19.69	0.00	0.00	0.41	0.11	0.11	3.55	1.26	172	2.98	3.19
α of 0.5%	8.57	19.85	0.65	0.65	0.44	0.04	0.04	3.98	1.85	235	3.00	3.21

Table 10 and Table 25 in Appendix J show the results of the constant constrained S&P 100 indices. The benefits of constraining a concentrated index appear to be less significant than a more diversified one. Although imposing a constraint reduces index volatility, the change in the Sharpe ratio is not significant. Constraining concentrated indices mainly results in a higher TE and TO.

Table 10: Annualized summary statistics of constrained S&P 100 using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12.00	2.13	97	2.97	3.18
α of 5.0%	6.54	19.60	0.00	0.00	0.34	0.09	0.11	4.82	0.45	53	2.96	3.19
α of 4.5%	6.51	19.58	0.00	0.00	0.34	0.24	0.24	4.92	0.55	55	2.96	3.18
α of 4.0%	6.48	19.55	0.00	0.00	0.34	0.43	0.43	4.99	0.68	56	2.96	3.18
α of 3.5%	6.46	19.53	0.00	0.00	0.34	0.60	0.61	5.09	0.83	58	2.96	3.18
α of 3.0%	6.44	19.51	0.00	0.00	0.34	0.71	0.72	5.25	1.01	61	2.95	3.17
α of 2.5%	6.47	19.48	0.00	0.00	0.34	0.68	0.69	5.49	1.23	64	2.95	3.17
α of 2.0%	6.53	19.47	0.00	0.00	0.34	0.60	0.60	5.90	1.48	69	2.95	3.16
α of 1.5%	6.68	19.48	0.00	0.00	0.35	0.41	0.41	6.53	1.73	75	2.95	3.16
α of 1.0%	6.90	19.57	0.01	0.01	0.36	0.28	0.28	7.50	2.06	84	2.96	3.17
α of 0.5%	7.14	19.70	0.76	0.75	0.37	0.24	0.23	10.00	2.64	95	2.98	3.19

5.1.2 Stock fundamentals constraint

In the stock fundamentals constraints (Algorithm 2) , seen in Tables 28 - 35 in Appendix J, only those stocks that show extreme fundamental values within their respective sector are adjusted. It is assumed that, in the long term, stocks in the same sector show similar stock fundamentals, which will grow over time at approximately the same rate. Stocks that deviate from this pattern are assumed to revert to the mean, exposing investors to greater downside risks. A major advantage of this method is that it leads to minimal adjustments in the index and therefore minimally deviates from the market cap index. This is reflected in the TE, which remains limited, even in the strictest of constraints. Because so few adjustments are made to the index, the TO is lower than that of the constant constraint. Nearly every stock fundamental has been able to achieve a significant reduction in index volatility and thereby reduce downside risk. However, strict constraints are often required to observe a significant improvement in the Sharpe ratio. An income-based constraint (price-to-earnings and price-to-earnings-to-growth) appears to achieve the best Sharpe ratio improvement.

A naive stock fundamentals constraint which can be seen in Table 11 shows promising results. However, a higher TO is observed because 10% of the index weights are adjusted at rebalancing. What stands out is the low TE relative to the market cap index, this can be explained by the fact that mainly the smaller weights in the index are adjusted, which ensures that the rest of the index continues to move along with the market cap index. Because smaller weights in the index are also affected by this constraint, the diversification in this index decreases since the weight of these smaller exposures mainly ends up with the larger exposures in the index after rebalancing. Ultimately, it appears that the index's historical downside risk has diminished. However, applying this constraint creates a potentially greater risk in the future by increasing exposure to the larger companies in the index.

Table 11: Annualized summary statistics of naive constrained S&P 500 using price-to-earnings ratio and the general normalization method, where a constraint of 0.1 corresponds to δ of 0.1. Based on Algorithm 2.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
δ of 0.9	7.73	19.72	0.00	0.00	0.40	0.02	0.02	2.70	0.28	106	2.99	3.18
δ of 0.8	7.77	19.69	0.00	0.00	0.40	0.01	0.01	3.03	0.33	105	2.98	3.18
δ of 0.7	7.81	19.67	0.00	0.00	0.40	0.01	0.01	3.40	0.40	105	2.98	3.17
δ of 0.6	7.85	19.64	0.00	0.00	0.40	0.01	0.01	3.79	0.47	104	2.97	3.17
δ of 0.5	7.89	19.61	0.00	0.00	0.41	0.01	0.01	4.23	0.55	103	2.97	3.16
δ of 0.4	7.94	19.59	0.00	0.00	0.41	0.01	0.01	4.68	0.64	102	2.97	3.15
δ of 0.3	7.98	19.56	0.00	0.00	0.41	0.01	0.01	5.16	0.73	101	2.96	3.15
δ of 0.2	8.03	19.53	0.00	0.00	0.41	0.01	0.01	5.65	0.83	99	2.96	3.14
δ of 0.1	8.08	19.50	0.00	0.00	0.42	0.01	0.01	6.16	0.93	98	2.95	3.13

5.1.3 Sector constraint

In Table 12, and Table 36 in Appendix J, the results of the sector constrained indices (Algorithm 3) can be observed. In this method, the three largest sectors are constrained to β and the remaining sector weights are adjusted accordingly. As with the stock fundamentals constrained indices, the sector constrained indices can achieve a reduction in index volatility. In the sector constrained indices it is hard to observe a significant improvement in the Sharpe ratio. Although the sector constraint threshold is regularly exceeded, there is almost no difference between the different constrained indices. Multiple grid sizes and intervals were tested but hardly changed the results. The sector constraint appears to be able to marginally improve risk at a low tracking error. However, this decrease in risk does not outweigh the portfolio turnover that is quadrupled compared to the market cap portfolio.

Table 12: Annualized summary statistics of constrained S&P 500 using a sector constraint, where a constraint of 80% corresponds to max weight of 80% of the top 3 sectors at the moment of index creation. Based on Algorithm 3.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
β of 80%	7.69	19.74	0.00	0.00	0.40	0.08	0.09	2.50	0.26	106	2.99	3.19
β of 79%	7.70	19.74	0.00	0.00	0.40	0.08	0.09	2.50	0.26	106	2.99	3.19
β of 78%	7.70	19.74	0.00	0.00	0.40	0.09	0.09	2.51	0.27	106	2.99	3.19
β of 77%	7.70	19.74	0.00	0.00	0.40	0.09	0.09	2.51	0.27	106	2.99	3.19
β of 76%	7.70	19.75	0.00	0.00	0.40	0.11	0.11	2.51	0.28	106	2.99	3.19
β of 75%	7.69	19.75	0.00	0.00	0.40	0.13	0.15	2.52	0.28	106	2.99	3.19
β of 74%	7.69	19.76	0.00	0.00	0.40	0.16	0.16	2.53	0.29	107	2.99	3.19
β of 73%	7.69	19.77	0.00	0.00	0.40	0.19	0.18	2.54	0.30	107	2.99	3.19
β of 72%	7.69	19.77	0.00	0.00	0.39	0.22	0.23	2.54	0.31	107	3.00	3.19
β of 71%	7.69	19.78	0.00	0.00	0.39	0.26	0.27	2.55	0.32	107	3.00	3.19

5.1.4 Additional results

The static constrained portfolios, counter-intuitively, show lower returns than the S&P 500. However, due to diversification benefits, the constrained indices recover more quickly from these poor returns. In the end, the constrained indices show better returns, on average taking fewer losses in the worst 5% of cases, resulting in a lower ES. Despite index volatility and ES are estimated in different ways, there seems to be a certain relationship between the two variables. It is observed that the two variables move in the same direction. That is when the volatility of the index decreases, a decrease in the ES is observed. To reduce downside risk, it is probably more convenient to look for a significant decrease in volatility without sacrificing risk-adjusted returns.

In particular, the effect of the static constrained indices in the different volatility regimes has been analyzed, which does not appear to yield the desired result. In many cases, the index volatility is not significantly lower compared to the market cap index and in some cases, it is even higher. An explanation for this is that the moments at which the constraint is applied come too late and the damage has already been done. As a result, the applied constraints do even hurt performance and have too little time to prove their value.

In Appendix J the results of quarterly rebalanced constant constrained indices can be found. This was investigated to see whether the more frequent application of constraints is conducive to limiting downside risk. The effect of interim rebalancing and the more frequent application of constraints appears to be minimal when seeking downward risk reduction. The risk appears to be greater if the index is rebalanced in between years. However, more frequent rebalancing provides better diversification within the index, but this improvement is minimal. All this, in combination with the higher TO and TE, makes the more frequent application of constraints undesirable.

The application of multiple constraints has been tested. However, this turned out to yield no improvement compared to the single constraint. This is because the strictest constraint will dominate the less strict constraints.

5.2 Dynamic constraints

From Section 5.1 it appears that a naive constant constraint works best when one wants to increase the Sharpe ratio of the index or when (downside) risk need to be limited. The sector constraint proved to be best able to achieve minimal change to the index while reducing volatility and downside risk. Therefore, these two methods will be tested in a dynamic environment.

In Tables 13 and 14 the results of two constant dynamic constrained indices can be observed. The other tables can be found in Appendix K. Additionally, Figure 4 has been added to plot the change in the value of the constrained indices against the S&P 500. At first glance, the dynamic constraint seems to do its job. By applying the dynamic constraint, the volatility of the index is significantly reduced. However, it seems that the added value of a dynamic constraint relative to a static constraint is marginal.

Table 13: Annualized summary statistics of dynamic VIX constrained S&P 500 for $\gamma = 40$ using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%. Based on Algorithm 4.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.69	19.78	0.00	0.00	0.39	0.09	0.17	3.19	0.22	106	3.00	3.19
α of 4.5%	7.69	19.78	0.00	0.00	0.39	0.11	0.15	3.77	0.22	106	3.00	3.19
α of 4.0%	7.68	19.77	0.00	0.00	0.39	0.15	0.18	4.48	0.23	106	3.00	3.19
α of 3.5%	7.66	19.77	0.00	0.00	0.39	0.31	0.33	5.51	0.24	106	3.00	3.19
α of 3.0%	7.64	19.78	0.00	0.00	0.39	0.54	0.54	7.26	0.26	107	3.00	3.19
α of 2.5%	7.62	19.78	0.00	0.00	0.39	0.83	0.84	9.59	0.29	107	3.00	3.20
α of 2.0%	7.59	19.79	0.05	0.08	0.39	0.86	0.87	12.73	0.33	107	3.00	3.20
α of 1.5%	7.58	19.81	0.70	0.72	0.39	0.69	0.69	18.09	0.40	107	3.01	3.20
α of 1.0%	7.57	19.85	0.31	0.34	0.39	0.65	0.66	28.15	0.50	108	3.01	3.21
α of 0.5%	7.53	19.94	0.01	0.02	0.39	0.52	0.54	48.47	0.75	109	3.03	3.22

Table 14: Annualized summary statistics of dynamic BI constrained S&P 500 for $\gamma = 1.2$ using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%. Based on Algorithm 4.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.73	19.75	0.00	0.00	0.40	0.03	0.03	2.54	0.26	106	2.99	3.19
α of 4.5%	7.73	19.75	0.00	0.00	0.40	0.04	0.05	2.57	0.28	107	2.99	3.19
α of 4.0%	7.73	19.74	0.00	0.00	0.40	0.07	0.08	2.61	0.31	107	2.99	3.19
α of 3.5%	7.74	19.73	0.00	0.00	0.40	0.11	0.12	2.68	0.37	109	2.99	3.18
α of 3.0%	7.73	19.72	0.00	0.00	0.40	0.22	0.23	2.77	0.47	110	2.99	3.18
α of 2.5%	7.72	19.71	0.00	0.00	0.40	0.35	0.36	2.89	0.60	113	2.98	3.18
α of 2.0%	7.70	19.69	0.00	0.00	0.40	0.51	0.52	3.07	0.76	115	2.98	3.17
α of 1.5%	7.69	19.67	0.00	0.00	0.40	0.63	0.63	3.44	0.95	118	2.98	3.17
α of 1.0%	7.70	19.66	0.00	0.00	0.40	0.67	0.67	4.15	1.16	121	2.97	3.17
α of 0.5%	7.81	19.66	0.00	0.00	0.40	0.51	0.50	5.80	1.47	127	2.97	3.16

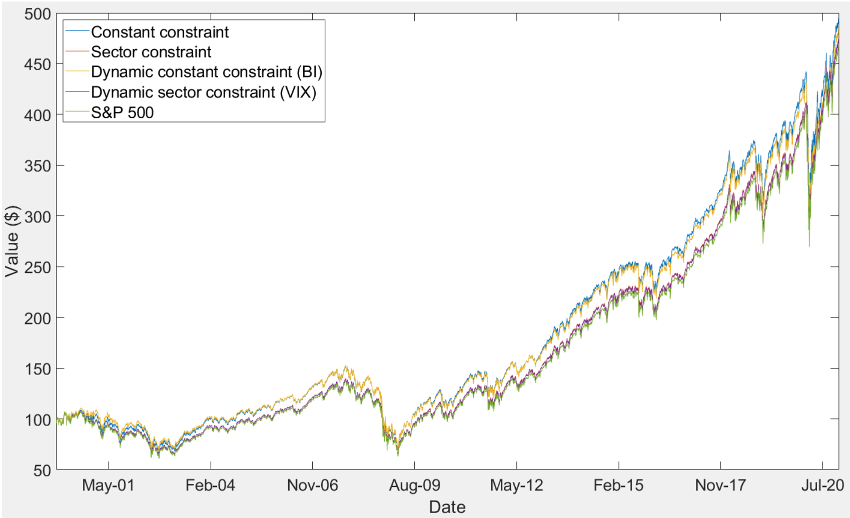

Figure 4: Value comparison of different constrained indices

A dynamic constraint has more advantages for more concentrated indices as can be seen in Table 15. When the threshold value of the dynamic constraint is exceeded, more weights are restricted in a more concentrated index. This makes the impact of the constraint greater and its benefits more clearly visible. However, restricting more weights in an index does result in a higher TO and TE.

Table 15: Annualized summary statistics of dynamic constrained S&P 100 for $\gamma = 1.2$ using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.50	19.62	0.00	0.00	0.34	0.16	0.16	4.82	0.42	52	2.97	3.17
α of 4.5%	6.48	19.61	0.00	0.00	0.34	0.31	0.32	4.90	0.50	53	2.96	3.17
α of 4.0%	6.47	19.60	0.00	0.00	0.34	0.47	0.47	5.01	0.61	54	2.96	3.17
α of 3.5%	6.45	19.57	0.00	0.00	0.34	0.62	0.62	5.14	0.73	54	2.96	3.17
α of 3.0%	6.42	19.56	0.00	0.00	0.34	0.79	0.79	5.31	0.87	55	2.95	3.16
α of 2.5%	6.39	19.54	0.00	0.00	0.34	0.90	0.90	5.60	1.04	55	2.95	3.16
α of 2.0%	6.37	19.51	0.00	0.00	0.33	0.96	0.96	6.10	1.23	56	2.95	3.15
α of 1.5%	6.35	19.50	0.00	0.00	0.33	0.99	0.99	6.82	1.39	57	2.94	3.15
α of 1.0%	6.44	19.50	0.00	0.00	0.34	0.80	0.80	8.02	1.57	58	2.94	3.14
α of 0.5%	6.56	19.52	0.00	0.00	0.34	0.62	0.62	10.07	1.86	59	2.94	3.15

What is striking about the dynamic restriction are the differences between the historical simulation and the normal mixture model approach for calculating the ES. Where the historical simulation approach barely manages to benefit from the dynamic constraints, the normal mixture model approach does seem to benefit from this. The dynamic constraint has little impact on the lowest 5% returns. It is observed, however, that the mean return is more positive and that the volatility of the normal mixture model with dynamic constraint is lower. This results in a lower ES_{NM}, which can yield up to 2 to 3 times more improvement compared to the historical simulation method.

The dynamic constraint is applied based on two variables, the VIX and the Buffett Indicator. The VIX is a variable that varies daily. This allows the investor to choose whether or not to constrain the index daily. The active constrain of the index mainly leads to a much higher TO without countering a major decrease in downside risk. In fact, in some cases, the risk reduction is not significant. These effects are mainly observed within the constant constrained indices. The sector constraint is generally less applied, which automatically results in a lower TO. However, the dynamic sector constraint does not show better results than the static one. In general, the VIX does not seem to be a convenient variable to determine the moment of application of the constraints. The VIX and S&P 500 are heavily negatively correlated (Vodenska and Chambers, 2013). As a result, an increase in the VIX will often lead to negative index returns. The constraint is applied after this event, while one probably would have wanted to apply the constraint a day earlier to limit the loss of the index.

The other variable, on which the moment of applying the constraint was determined, is the Buffett Indicator. This variable is determined once a quarter, which automatically results in fewer rebalancing moments. This results in a much lower TO compared to the dynamic constraints based on the VIX. A greater decrease in volatility is observed with the BI constraint than with the VIX. However, this could be because the index is mainly constrained during the COVID-19 pandemic. This method has not been able to limit downside risk during the great financial crisis, since the threshold value was not exceeded at that moment. Strict constant dynamic constraints based on the BI appear to be quite capable of promoting diversification within the index. The Buffett Indicator could therefore be seen as a convenient variable to determine the moment of applying the constraints to the index. A lower TE compared to the static constraints is observed because there are many moments when the dynamic index follows the market cap index. When looking at diversification, however, one would still be better off with the static constraint. The return achieved by that index is also almost one percent higher. But when looking for a constraint to limit downside risk against a small deviation from the market standard, the Buffett Indicator can offer a favourable solution.

A similar counter-intuitive pattern is observed with the dynamic constraints as with the static constraints. This confirms the find that Amenc et al. (2012) made. Namely, that different portfolios react differently to various economic scenarios. The dynamic constraints do not result in less negative returns, but they do realize a faster recovery compared to the S&P 500. This statement is supported by Table 16, which shows the annualized returns, volatilities and ESs in the different volatility regimes. The subscripts in the variables, therefore, indicate the different regimes, defined in Section 4.4. The diversification benefits appear to be mainly positive in the period after the decline. The diversification of the portfolio ensures a faster recovery from the peak downwards. Ultimately, this leads to a lower value for ES.

Table 16: Comparison between the dynamic 0.5% BI constrained S&P 500 for $\gamma = 1.2$ and the dynamic 5.0% VIX constrained S&P 500 for $\gamma = 40$ using a constant and the general normalization method during volatile periods, where a constraint of 5.0% corresponds to a max index weight of 5.0%. The subscripts in the variables indicate the different volatility regimes. Based on Algorithm 4.

	$r_{p,1}$	$r_{p,2}$	$r_{p,3}$	$\sigma_{p,1}$	$\sigma_{p,2}$	$\sigma_{p,3}$	$ES_{H,1}$	$ES_{H,2}$	$ES_{H,3}$	$ES_{N,1}$	$ES_{N,2}$	$ES_{N,3}$
S&P 500	-5.25	-23.34	21.11	21.59	37.97	35.54	2.94	5.79	5.82	3.04	5.92	6.21
S&P 500 EW	6.00	-22.11	14.14	20.08	40.36	37.84	2.63	6.12	6.26	2.75	6.17	6.51
BI	-4.97	-23.15	17.25	21.53	37.76	35.93	2.93	5.75	5.91	3.04	5.89	6.27
VIX	-4.76	-23.21	21.12	21.43	37.89	35.53	2.92	5.77	5.82	3.02	5.91	6.22

5.3 Index combinations

In Tables 17 and 18 the results of a daily combined index can be observed. The difference between the two tables is the variable on which both indices are combined, in Table 17 this is volatility and in Table 18 historical simulated ES. The three indices with the lowest ES and/or volatility from Sections 5.1 and 5.2, in combination with the market cap index, have been chosen to include in the combined index. The constrained index that showed the best results, the dynamic constant constrained index, is included in all index combinations. The constant constrained index is added in the index which consists of both 3 and 4 indices, and the sector constrained index is only included in the largest combined index. As can be seen, combining indices automatically results in a lower TE. This can be explained by the fact that the market cap index itself is included in the index combination and the constrained index is therefore only partially included, leading to a lower TE. Both methods can significantly reduce volatility and thereby achieve a lower ES. However, this effect is also only a few basis points here. Both methods lead to roughly the same results, confirming the presumption of a correlation between volatility and downside risk. In contrast to naive constraints, the conscious pursuit of volatility and expected shortfall reduction leads to a lower return and a lower Sharpe ratio against marginal risk reduction. Continuously combining different portfolios based on volatility and ES would therefore not offer a favourable solution.

Table 17: Annualized summary statistics of the combined S&P 500 index where the combined index will exist out of 1. market cap index, 2. dynamic constant constrained index, 3. constant constrained index and 4. sector constrained index. A training time of 1 year is applied and the weight calculation is based on volatility.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TE	AS	ES_{HS}	ES_{NM}
S&P 500	8.42	19.63	-	-	0.44	-	-	-	110	3.00	3.20
Comb 4 indices	8.32	19.56	0.00	0.00	0.44	0.82	0.82	0.48	120	2.99	3.19
Comb 3 indices	8.28	19.55	0.00	0.00	0.43	0.76	0.76	0.62	141	2.99	3.19
Comb 2 indices	8.23	19.58	0.00	0.00	0.43	0.16	0.16	0.55	133	2.99	3.19

Table 18: Annualized summary statistics of the combined S&P 500 index where the combined index will exist out of 1. market cap index, 2. dynamic constant constrained index, 3. constant constrained index and 4. sector constrained index. A training time of 1 year is applied and the weight calculation is based on the expected shortfall.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TE	AS	ES_{HS}	ES_{NM}
S&P 500	8.42	19.63	-	-	0.44	-	-	-	110	3.00	3.20
Comb 4 indices	8.32	19.56	0.00	0.01	0.44	0.82	0.82	0.48	120	2.99	3.19
Comb 3 indices	8.28	19.55	0.03	0.04	0.43	0.76	0.77	0.62	141	2.99	3.18
Comb 2 indices	8.23	19.58	0.03	0.05	0.43	0.16	0.16	0.55	133	2.99	3.19

What is interesting to see is how the different combined indexes react to volatile periods, which can be observed in Tables 19 and 20. Combined indices based on volatility tend to lean more towards the constrained indices in more volatile times, which we saw in Section 5.2 show more negative returns during volatile periods. As seen and discussed before, constrained indices benefit in the period after a major decline. Because the volatility regimes are too small, there is not enough time to show these diversification benefits.

The combined index based on historical simulated ES shows promising results. Unlike the combined index based on volatility, it relies on the market cap index in more volatile times. After large negative returns, the exposure to the market cap index is reduced and more is allocated to the constrained indices that outperform the market cap, as can be seen in Sections 5.1 and 5.2. This results in more positive results causing the VaR to be lower, which often results in lower values for the ES.

Table 19: Comparison between combined S&P 500 indices based on volatility during volatile periods. Where the subscripts of the variables indicate the volatility regime introduced in Section 4.4.

	$r_{p,1}$	$r_{p,2}$	$r_{p,3}$	$\sigma_{p,1}$	$\sigma_{p,2}$	$\sigma_{p,3}$	$ES_{H,1}$	$ES_{H,2}$	$ES_{H,3}$	$ES_{N,1}$	$ES_{N,2}$	$ES_{N,3}$
S&P 500	-5.25	-23.34	21.11	21.59	37.97	35.54	2.94	5.79	5.82	3.04	5.92	6.21
Comb 4 indices	-5.38	-23.62	18.80	21.34	37.87	35.52	2.90	5.77	5.81	3.01	5.90	6.22
Comb 3 indices	-5.44	-23.62	18.17	21.33	37.91	35.55	2.90	5.77	5.81	3.01	5.90	6.23
Comb 2 indices	-5.57	-23.72	18.95	21.55	37.87	35.60	2.93	5.77	5.83	3.04	5.90	6.24

Table 20: Comparison between combined S&P 500 indices based on historical simulated expected shortfall during volatile periods. Where the subscripts of the variables indicate the volatility regime introduced in Section 4.4.

	$r_{p,1}$	$r_{p,2}$	$r_{p,3}$	$\sigma_{p,1}$	$\sigma_{p,2}$	$\sigma_{p,3}$	$ES_{H,1}$	$ES_{H,2}$	$ES_{H,3}$	$ES_{N,1}$	$ES_{N,2}$	$ES_{N,3}$
S&P 500	-5.25	-23.34	21.11	21.59	37.97	35.54	2.94	5.79	5.82	3.04	5.92	6.21
Comb 4 indices	-4.98	-22.04	21.55	21.35	37.90	35.51	2.88	5.72	5.78	2.99	5.87	6.20
Comb 3 indices	-5.44	-23.62	21.33	21.33	37.91	35.55	2.90	5.77	5.81	3.01	5.90	6.23
Comb 2 indices	-5.57	-23.72	21.55	21.55	37.87	35.60	2.93	5.77	5.83	3.04	5.90	6.24

6 Conclusion

In this thesis, it is shown that the downside risk of a market cap index can be reduced by applying constraints. Different constraints have been tested in combination with different rebalancing methods. The constraints tested are a constant constraint, a stock fundamentals constraint, a sector constraint and a dynamic constraint. All methods were found to be able to reduce downside risk. This was mainly achieved by significantly reducing the volatility of the portfolio, which seems strongly connected to downside risk reduction. However, applying constraints do not seem to reduce negative returns. But the extra diversification resulting from the constraints ensures faster recovery from these negative returns. Ultimately, this leads to better returns in the long run and the average loss in the 5% worst cases will be less than with a regular market cap index.

Strict constraints allow for broad diversification by forcing the portfolio towards a 1/N portfolio. This results in higher returns with less downside risk. However, like the conclusion of Chiou et al. (2009), a point was found when a restriction turned out to be too strict. A constraint that is too strict exposes the investor to the smaller, riskier stocks in the index. As a result, constraining the index can lead to more risk. Mild constraints have the advantage that few adjustments are made to the market cap index. This limits tracking error and turnover. Even a mild weight constraint reduces downside risk, albeit marginally.

The stock fundamentals constraint produced mixed results. While all variables used were able to reduce downside risk, the variables analyzed based on earnings appear to give the best results. Two ways of applying the constraints were tested, a naive way where all stocks in the top 10% quantile were constrained, and a method where only those stocks that fell in the quantile and make a significant contribution to the index are constrained. The naive method appears to give the best results based on downside risk but has the major disadvantage that this is offset by a lot of turnover and tracking error. This method also allows for a more concentrated index as the smaller exposures in the index are adjusted. When normalizing the index, this weight will mainly end up with the larger stocks in the index.

The main disadvantage of constraining a market cap index is that the weights no longer vary with returns and therefore rebalancing has to take place. This effect is logically greater when stricter constraints are imposed on the index. Indices consisting of a small number of stocks suffer the most from these rebalancing disadvantages since threshold values are exceeded more quickly. One way to avoid this would be to loosen the grid, then similar results to the less concentrated indices would be observed. Another disadvantage is incurring tracking error with the market cap index, which is the market standard. This effect seems to be limited if a sector-neutral rebalancing technique is used. However, applying this rebalancing technique does result in a higher turnover.

To limit these disadvantages, a dynamic constraining method was considered. This constraint is only applied when a decision variable is exceeded and, therefore, will limit turnover. However, a good choice must be made for the decision variable and threshold. In this thesis, two decision variables are analyzed, being the VIX and the Buffett Indicator. The frequency at which the decision variable is available and the limit at which you decide to rebalance appear to be essential for the success of this method. The Buffett Indicator is available quarterly and therefore automatically leads to fewer rebalancing moments and therefore a lower turnover. Compared to the VIX, this variable is less volatile and also ensures less rebalancing. Ultimately, reducing downside risk in a market cap index appears to work well within a dynamic environment, but the intended goal of limiting turnover is not achieved.

Jagannathan and Ma (2003) concluded in their research that applying an upper-bound constraint would add little value from a risk-return standpoint if a short-sell constraint were already in place. I have to agree with their conclusion. In this study, even with strict constraints, a significant increase in the Sharpe ratio was rarely observed. However, by imposing these constraints risk reduction can be achieved, without a negative impact on returns. The second point they make in their paper, that even applying the wrong constraints works, is confirmed by this research. Imposing constraints more often has a positive effect on index performance than a negative one. This research shows that investors are often best off by using a naive way of constraining. That is, constrain anything above a certain threshold without looking at underlying numbers.

In addition to constraining the market cap index, the potential benefits of combining different (un)constrained indices based on volatility and expected shortfall were examined. Combining indices significantly reduces volatility, but comes at the cost of returns. In the different volatility regimes, combining indices based on expected shortfall can provide added value.

All in all, applying constraints to a market cap index has marginal benefits. While risk in the index decreases, it comes at the cost of high turnover. As a result, in most cases, the extra return or the reduction of the marginal risk will not outweigh the extra costs that have to be incurred. If the investor aims for a higher risk-adjusted return, he is better off investing in the riskier equal-weighted index (DeMiguel et al., 2009). Although higher turnover is observed here as well, he will get enough additional return in return.

7 Discussion

From the results section, some neat and relevant conclusions are drawn. However, several subjects need further investigation. The assumptions that were made throughout this thesis yield results that do not necessarily apply in general. This section describes what research needs to follow to complete this study.

First of all, the main question needs to be considered. This thesis looked at how to diversify a market cap index to reduce downside risk. The philosophical question that arises is whether investors would like to invest in a broadly diversified US stock index, with diversification across different companies and sectors, or whether they would like to invest in a reflection of the US economy. When the latter is the case, a concentrated index towards the largest companies and sectors is not a disadvantage as they are over-represented in the US economy. These investors will always exist and so the constrained indices will always be next to the market cap index.

There are several routes that I have followed throughout the thesis which could be followed in several ways. One of the main variables to be questioned is the way the sectors are defined. In this thesis, I chose SIC sectors, which were easy to extract from the CRSP database. The conclusion about sector constraints is therefore partly based on the definition of these sectors. It is possible that if one had used GSCI sectors or NAICS sectors in the analysis, a completely different conclusion would have been reached. The effect of the sector choice could be further investigated in a follow-up study.

The same case can be made for the decision variable in the dynamic models. Two decision variables were investigated in this thesis, the VIX and BI. However, one could use different variables which can explain market overvaluation or volatility. Examples include the treasury bill rate, house prices, business confidence index, business climate indicator, etc.
In this thesis, two different ways are analyzed to estimate the VaR and the ES. However, there are dozens of other ways in which this is possible. As can be seen, the results for the different methods can vary widely. In this thesis, the difference between the historical simulation and normal mixture model approach was often several tens of basis points. For the normal mixture model, two normal distributions with different initial starting points were used. There is a chance that the EM algorithm has converged to a local minimum in some cases, resulting in different results. One solution to this would be to use multiple initial starting points for each analysis. A possible model selection error is also insurmountable. There is no guarantee that a two normal mixture model is a good representation of the underlying data. Perhaps multiple underlying distributions could have improved the estimation of the VaR.

Determining significant differences between ESs is something to be explored in a follow-up study. There are several ways to estimate the ES, but there is no generic method to determine whether the differences between two ESs are significant or not. Precisely because ES can be determined in different ways, this is a difficult issue for which a PhD could be spent.

In this thesis, it was concluded that there may be a correlation between the volatility and ES of an index. However, the magnitude of this correlation needs to be further investigated. A question related to the previous point is, for example, whether a significant reduction in volatility also results in a significant reduction in ES.

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Appendix

A Company concentration

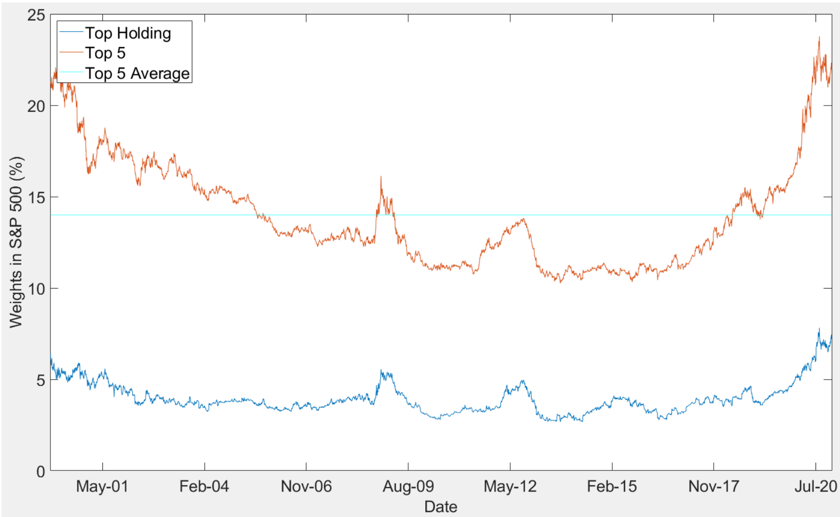

Figure 5: Mega stock weights in the S&P500 over the last 21 years.

B Expectation Maximization parameter derivation

The log likelihood function, mentioned below and elaborated in Section 3.5.1.2, is estimated according to the EM algorithm as follows:

$$\log L = \frac{\lambda f_1(x_i)}{\lambda f_1(x_i) + (1-\lambda) f_2(x_i)} \left(\log \lambda + \log f_1(x_i)\right) + \frac{(1-\lambda) f_2(x_i)}{\lambda f_1(x_i) + (1-\lambda) f_2(x_i)} \left(\log(1-\lambda) + \log f_2(x_i)\right).$$

- E-step:

$$\mathbb{P}(w_i = 1 | X_i = x_i; \boldsymbol{\theta}^{(t)}) = p_i^{(t)} = \frac{\lambda^{(t)} f_1(x_i; \mu_1^{(t)}, \sigma_1^{(t)})}{\lambda^{(t)} f_1(x_i; \mu_1^{(t)}, \sigma_1^{(t)}) + (1-\lambda^{(t)}) f_2(x_i; \mu_2^{(t)}, \sigma_2^{(t)})},$$

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) = \sum_{i=1}^{n} p_i^{(t)} \left(\log \lambda + \log f_1(x_i; \mu_1, \sigma_1)\right) + \sum_{i=1}^{n} (1-p_i^{(t)}) \left(\log(1-\lambda) + \log f_2(x_i; \mu_2, \sigma_2)\right).$$

- M-step $\frac{\partial Q}{\partial \boldsymbol{\theta}} = 0$, leading to:

$$\frac{\partial Q}{\partial \lambda} = \frac{\sum_{i=1}^n p_i^{(t)}}{\lambda} - \frac{\sum_{i=1}^n (1-p_i^{(t)})}{1-\lambda} \to \widehat{\lambda}^{(t+1)} = \frac{\sum_{i=1}^n p_i^{(t)}}{n},$$

$$\widehat{\mu}_1 = \frac{\sum_{i=1}^n X_i p_i^{(t)}}{\sum_{i=1}^n p_i^{(t)}}, \widehat{\sigma}_1^2 = \frac{\sum_{i=1}^n (X_i - \widehat{\mu}_1)^2 p_i^{(t)}}{\sum_{i=1}^n p_i^{(t)}},$$

$$\widehat{\mu}_2 = \frac{\sum_{i=1}^n X_i (1-p_i^{(t)})}{\sum_{i=1}^n (1-p_i^{(t)})}, \widehat{\sigma}_2^2 = \frac{\sum_{i=1}^n (X_i - \widehat{\mu}_2)^2 (1-p_i^{(t)})}{\sum_{i=1}^n (1-p_i^{(t)})}.$$

This process is started with an initial guess of the parameters $\boldsymbol{\theta}^{(0)}$ and is iterated until it converges: $|\log L^{(t+1)} - \log L^{(t)}| < \varepsilon$. This MLE estimation converges to the true MLE: $\lim_{t\to\infty} \boldsymbol{\theta}^{(t)} = \widehat{\boldsymbol{\theta}}_{MLE}$. A disadvantage of the EM method is that it could converge to a local maximum. A solution to this is to provide multiple initial parameters $\boldsymbol{\theta}^{(0)}$ and choose the solution with the highest likelihood.

C HAC Inference derivation

Given the function f, the normalization conditions and the corresponding standard error $s\left(\widehat{\Delta}\right)$ mentioned in Section 3.6.1, the p-value is calculated using the following limiting covariance matrix in Equation 5:

$$\Psi = \lim_{N\to\infty} \frac{1}{N} \sum_{s=1}^{N} \sum_{t=1}^{N} \mathbb{E}\left(y_s y_t'\right),$$

where $y_t' = \left(r_{t,i} - \mu_i, r_{t,j} - \mu_j, \sigma_i^2, \sigma_j^2\right)$.
By change of variables, the limit can be alternatively expressed as:

$$\Psi = \lim_{N\to\infty} \Psi_N, \text{ with } \Psi_N = \sum_{j=-N+1}^{N-1} \Gamma_N(j), \text{ where}$$

$$\Gamma_N(j) = \begin{cases} \frac{1}{N}\sum_{t=j+1}^{N} \mathbb{E}\left(y_t y_{t-j}'\right) & \text{for } j \geq 0 \\ \frac{1}{N}\sum_{t=-j+1}^{N} \mathbb{E}\left(y_{t+j} y_t'\right) & \text{for } j < 0 \end{cases}$$

A HAC robust kernel k is used to come up with a consistent estimator $\widehat{\Psi}_N$. The kernel estimate for Ψ is given by:

$$\widehat{\Psi} = \widehat{\Psi}_N = \frac{T}{T-4} \sum_{j=-N+1}^{N-1} k\left(\frac{j}{S_N}\right) \widehat{\Gamma}_N(j), \text{ where}$$

$$\widehat{\Gamma}_N(j) = \begin{cases} \frac{1}{N}\sum_{t=j+1}^{N} \widehat{y}_t \widehat{y}_{t-j}' & \text{for } j \geq 0 \\ \frac{1}{N}\sum_{t=-j+1}^{N} \widehat{y}_{t+j} \widehat{y}_t' & \text{for } j < 0, \end{cases}$$

where $\widehat{y}_t' = \left(r_{t,i} - \mathbb{E}\left(r_{t,i}\right), r_{t,j} - \mathbb{E}\left(r_{t,j}\right), r_{t,i}^2 - \mathbb{E}\left(r_{t,i}^2\right), r_{t,j}^2 - \mathbb{E}\left(r_{t,j}^2\right)\right)$ and S_N is the bandwidth. Most commonly used kernel is the Parzen-Gallant kernel, which will be used in this thesis. There are several automatic methods to determine the asymptotic optimal bandwidth (Andrews and Monahan, 1992; Newey and West, 1994). In this thesis, the automatic bandwidth selection of Andrews and Monahan (1992) is used.

D Return distributions

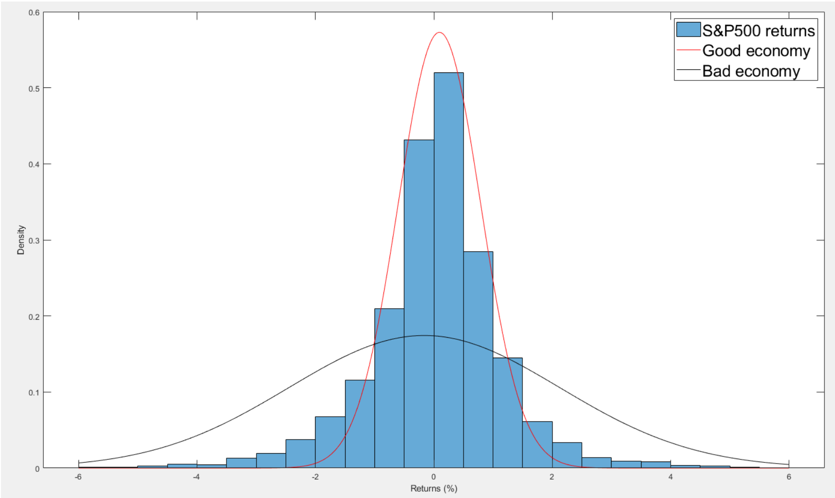

Figure 6: Histogram of daily S&P 500 index returns and the two density curves of the normal mixture model.

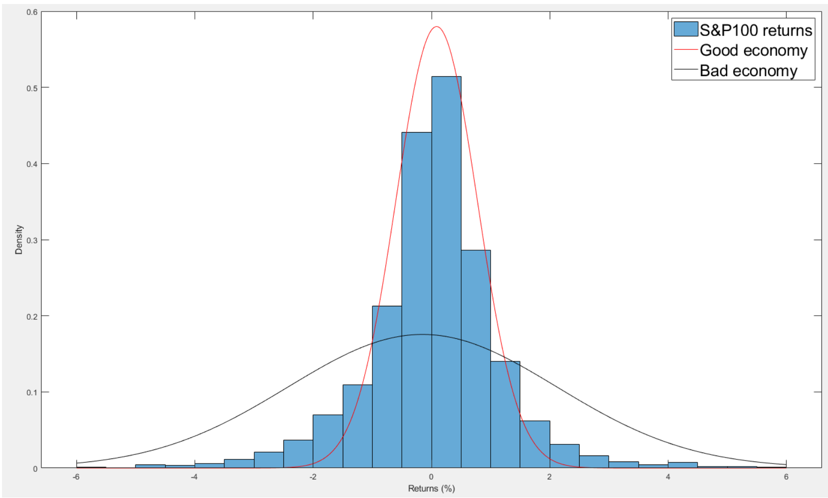

Figure 7: Histogram of daily S&P 100 index returns and the two density curves of the normal mixture model.

E Cumulative returns

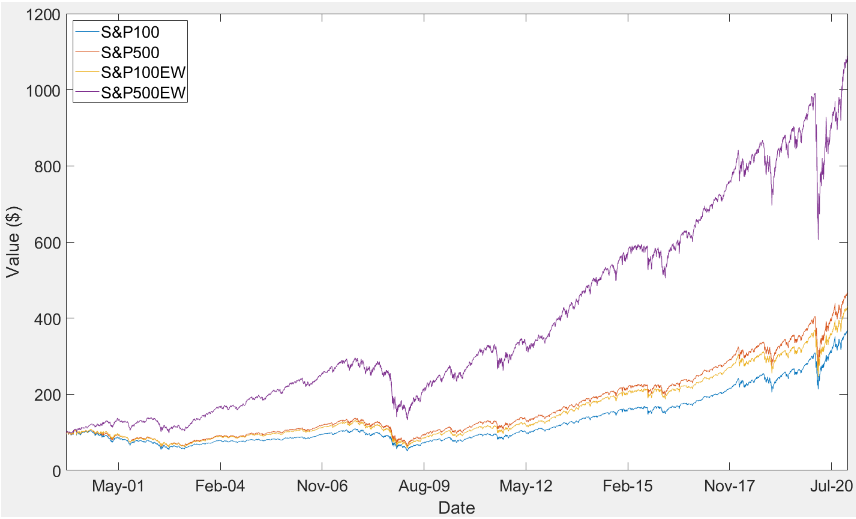

Figure 8: Value over time for the four different indices.

F Value of indices during recessions

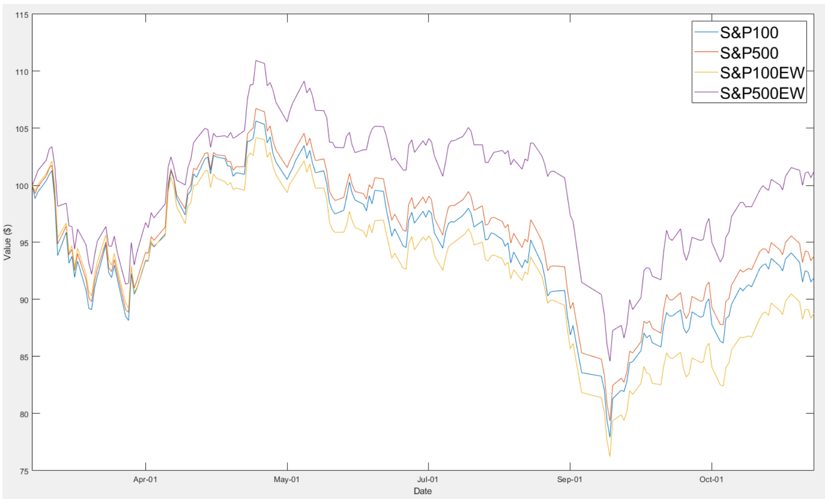

Figure 9: Returns of S&P indices during the early 2000 recession.

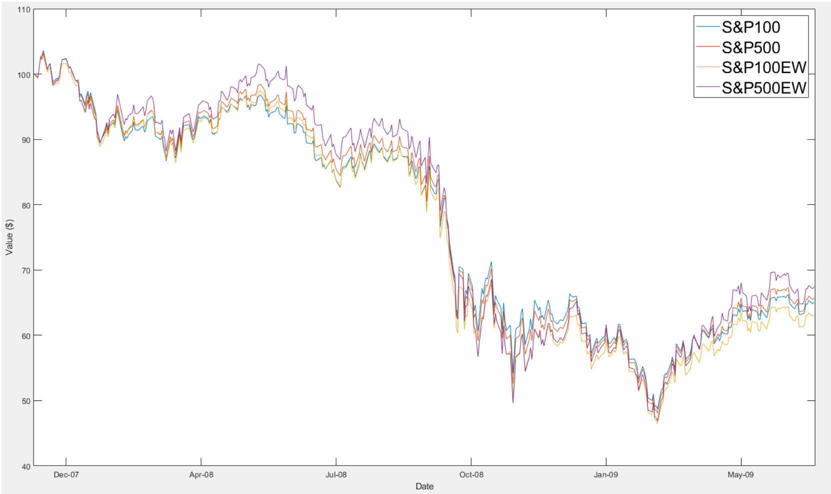

Figure 10: Returns of S&P indices during the global financial crisis.

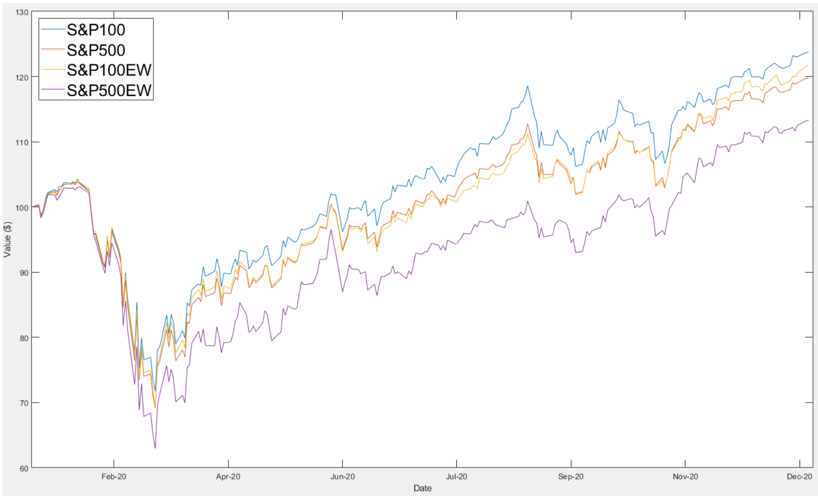

Figure 11: Returns of S&P indices during the COVID-19 pandemic.

G Dynamic constraint

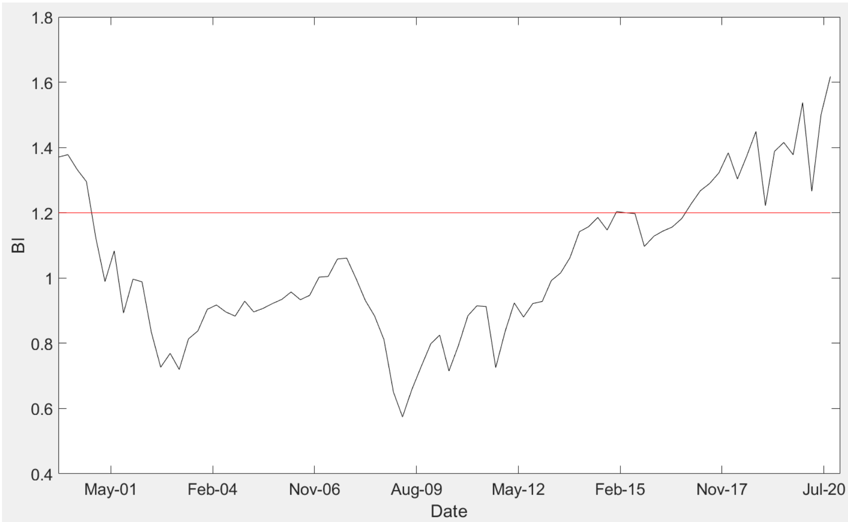

Figure 12: Graph of the Buffett Indicator with a value of γ (1.2) displayed.

H Stock fundamentals

Table 21: Explanation of stock fundamentals*.

Value	
Book-to-Market ratio	This ratio compares a firm's book value to its market value. A company's book value is calculated by looking at the company's historical cost, or accounting value. The firm's market value is equal to the market capitalization of the company. A Book-to-Market (B/M) ratio under 1 could indicate that a company is undervalued, since the stock price of a company is trading for less than the worth of its assets. The opposite could be true for a B/M ratio above 1.
Price-to-Book ratio	This ratio compares a firm's market value to its book value. It's calculated by dividing the stock price per share by its book value per share. Therefore, this ratio takes into account possible stock splits. High-growth companies often show Price-to-Book (P/B) ratios well above 1, whereas companies facing severe distress will show ratios below 1.
Price-to-Cash Flow ratio	The Price-to-Cash Flow (P/CF) ratio measures the amount of cash generated by a company relative to its stock price. Cash flows can not be manipulated as easily as earnings, making it a robust measure. A low P/CF ratio may imply undervaluation of a stock.

Price-to-Earnings ratio	This ratio compares a company's market value to its per-share earnings. This metric can be used to determine the relative value of a company's shares. Again, this metric takes into account possible stock splits by taking the stock price per share instead of the stock price itself. A high Price-to-Earnings (P/E) ratio could mean that a company's stock is overvalued or that investors expect high future growth rates. Companies that have no earnings or that are losing money do not have a P/E ratio.
Price-to-Earnings-to-Growth ratio	This ratio is calculated by taking the P/E ratio and divide it by the growth rate of the company's earnings. It is used to determine a company's stock value while factoring in its historical earnings growth. Therefore it is thought to provide a more complete picture than the simple P/E ratio. CRSP uses a historical 3 year EPS growth rate, A high Price-to-Earnings-to-Growth (PEG) ratio could mean that a company's stock is overvalued. Companies that have no earnings or that are losing money do not have a PEG ratio.
Price-to-Sales ratio	This ratio compares a company's stock price to its revenues. It shows how much investors are willing to pay per dollar of sales for a stock. One downside of the Price-to-Sales (P/S) ratio is that it does not take into account if the company makes any earnings. A low P/S ratio could mean that a company's stock is undervalued, while the opposite hold for a high P/S ratio.
Profitability	
Debt-to-Equity ratio	The Debt-to-Equity (D/E) ratio is used to evaluate a company's financial leverage and is calculated by dividing a company's total liabilities by its shareholder equity. It measures if a company finances its operations through debt versus owned funds. A higher D/E ratio often indicates a riskier stock.
Operating Profit Margin	The Operating Profit Margin (OPM) measures how much profit a company makes on one dollar of sales after paying wages and production costs, but before paying interest or taxes. Higher OPM values illustrate more efficient operations. This metric is hard to compare across different sectors, but can be used for competitors.
Return On Equity	Return On Equity (ROE) is a measure of financial performance calculated by dividing net income by a company's asset minus its debt. A higher ROE reflects higher profitability.

*Source: https://www.investopedia.com/financial-term-dictionary-4769738

Table 22: Mean index fundamentals.

	B/M	P/B	P/CF	P/E	PEG	P/S	D/E	OPM($)	ROE(%)
S&P 100	0.36	5.20	14.4	26.5	2.03	3.59	2.17	0.23	0.23
S&P 500	0.38	4.85	14.2	24.4	1.87	3.37	2.41	0.19	0.21
S&P 100 EW	0.37	4.98	14.7	21.4	1.79	3.60	2.20	0.21	0.22
S&P 500 EW	0.41	4.01	13.0	18.6	1.49	2.81	2.19	0.10	0.17

I Sector allocation

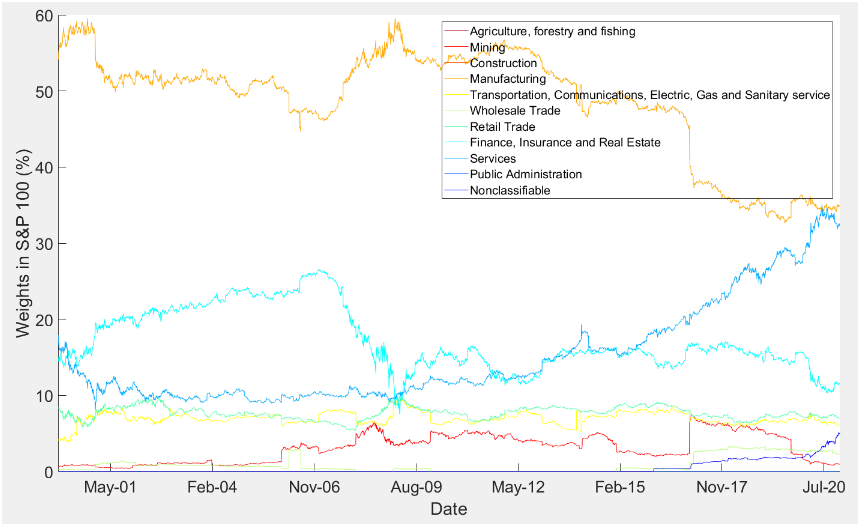

Figure 13: SIC sector weights over time for the S&P 100.

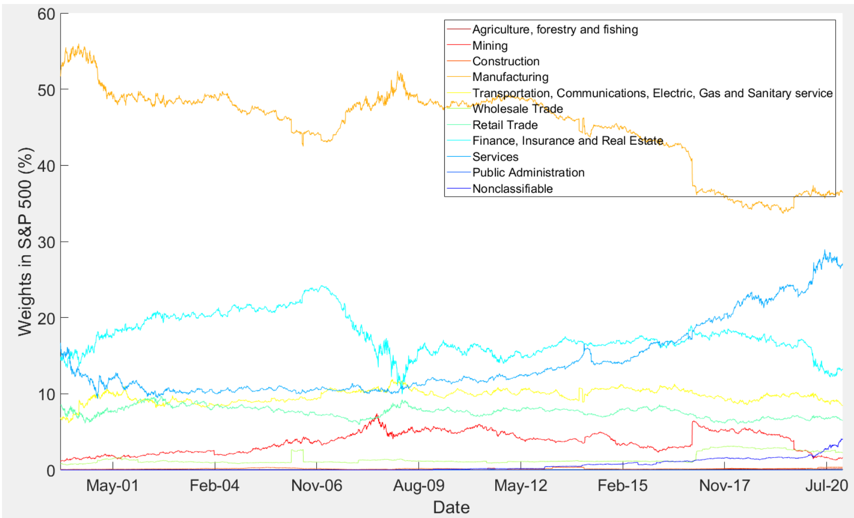

Figure 14: SIC sector weights over time for the S&P 500.

J Static constrained indices

J.1 Constant constraints

Figure 15: Value over time for different constant constraints for the S&P 500.

Table 23: Annualized summary statistics of quarterly rebalanced constrained S&P 500 using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.62	19.80	0.00	0.00	0.39	0.77	0.77	3.23	0.16	108	3.00	3.20
α of 4.5%	7.63	19.80	0.00	0.00	0.39	0.69	0.71	3.31	0.22	108	3.00	3.20
α of 4.0%	7.65	19.79	0.00	0.00	0.39	0.48	0.50	3.43	0.30	110	3.00	3.20
α of 3.5%	7.68	19.76	0.00	0.00	0.39	0.34	0.35	3.68	0.41	113	2.99	3.19
α of 3.0%	7.70	19.74	0.00	0.00	0.40	0.38	0.38	3.93	0.55	118	2.99	3.19
α of 2.5%	7.69	19.73	0.00	0.00	0.40	0.52	0.52	4.03	0.70	125	2.99	3.19
α of 2.0%	7.73	19.72	0.00	0.00	0.40	0.48	0.48	4.25	0.90	134	2.99	3.18
α of 1.5%	7.87	19.74	0.03	0.03	0.40	0.29	0.29	4.86	1.15	149	2.99	3.19
α of 1.0%	8.15	19.81	0.83	0.82	0.42	0.12	0.12	5.65	1.49	179	3.00	3.19
α of 0.5%	8.67	19.98	0.02	0.02	0.44	0.05	0.05	6.62	2.14	248	3.02	3.21

Table 24: Annualized summary statistics of quarterly rebalanced constrained S&P 500 using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.61	19.81	0.00	0.00	0.39	0.96	0.96	3.28	0.13	107	3.00	3.20
α of 4.5%	7.62	19.80	0.00	0.00	0.39	0.87	0.87	3.36	0.18	108	3.00	3.20
α of 4.0%	7.64	19.79	0.00	0.00	0.39	0.57	0.58	3.50	0.25	110	3.00	3.20
α of 3.5%	7.66	19.77	0.00	0.00	0.39	0.39	0.39	3.78	0.34	112	3.00	3.19
α of 3.0%	7.68	19.75	0.00	0.00	0.39	0.42	0.41	4.04	0.45	116	2.99	3.19
α of 2.5%	7.67	19.74	0.00	0.00	0.39	0.59	0.59	4.18	0.59	123	2.99	3.19
α of 2.0%	7.70	19.73	0.00	0.00	0.40	0.52	0.52	4.42	0.76	132	2.99	3.18
α of 1.5%	7.84	19.73	0.00	0.00	0.40	0.28	0.28	5.08	0.99	145	2.99	3.18
α of 1.0%	8.08	19.78	0.21	0.20	0.41	0.12	0.12	5.99	1.30	173	2.99	3.19
α of 0.5%	8.58	19.94	0.02	0.01	0.43	0.05	0.04	7.09	1.92	238	3.01	3.20

Table 25: Annualized summary statistics of constrained S&P 100 using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12.00	2.13	97	2.97	3.18
α of 5.0%	6.46	19.64	0.00	0.00	0.34	0.27	0.27	4.91	0.37	53	2.97	3.19
α of 4.5%	6.43	19.62	0.00	0.00	0.34	0.55	0.54	5.00	0.46	54	2.97	3.19
α of 4.0%	6.41	19.59	0.00	0.00	0.34	0.76	0.77	5.08	0.57	56	2.96	3.19
α of 3.5%	6.38	19.56	0.00	0.00	0.33	0.92	0.93	5.18	0.71	58	2.96	3.18
α of 3.0%	6.37	19.53	0.00	0.00	0.33	0.96	0.96	5.33	0.87	60	2.95	3.17
α of 2.5%	6.41	19.49	0.00	0.00	0.34	0.81	0.82	5.58	1.06	63	2.95	3.17
α of 2.0%	6.48	19.46	0.00	0.00	0.34	0.66	0.66	5.99	1.29	67	2.94	3.16
α of 1.5%	6.60	19.45	0.00	0.00	0.35	0.48	0.49	6.99	1.54	73	2.94	3.15
α of 1.0%	6.75	19.49	0.00	0.00	0.35	0.37	0.36	7.54	1.87	82	2.94	3.16
α of 0.5%	6.95	19.65	0.23	0.21	0.36	0.34	0.35	10.00	2.44	91	2.97	3.18

Table 26: Annualized summary statistics of quarterly rebalanced constrained S&P 100 using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	6.43	19.65	0.00	0.00	0.34	0.59	0.60	6.14	0.49	54	2.97	3.18
α of 4.5%	6.44	19.62	0.00	0.00	0.34	0.63	0.64	6.41	0.61	56	2.97	3.17
α of 4.0%	6.42	19.59	0.00	0.00	0.34	0.77	0.78	6.58	0.75	57	2.97	3.17
α of 3.5%	6.38	19.58	0.00	0.00	0.33	0.94	0.94	6.73	0.91	59	2.96	3.17
α of 3.0%	6.36	19.57	0.00	0.00	0.33	0.98	0.98	6.97	1.08	62	2.96	3.17
α of 2.5%	6.42	19.57	0.00	0.00	0.34	0.84	0.84	7.56	1.30	65	2.96	3.16
α of 2.0%	6.53	19.58	0.01	0.01	0.34	0.63	0.63	8.48	1.56	70	2.97	3.17
α of 1.5%	6.69	19.63	0.10	0.09	0.35	0.44	0.45	9.67	1.84	76	2.97	3.17
α of 1.0%	6.88	19.73	0.92	0.92	0.36	0.33	0.33	11.55	2.23	86	2.99	3.19
α of 0.5%	7.11	19.92	0.05	0.05	0.37	0.31	0.31	15.92	2.92	97	3.01	3.21

Table 27: Annualized summary statistics of quarterly rebalanced constrained S&P 100 using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	6.42	19.65	0.00	0.00	0.34	0.62	0.63	6.38	0.41	54	2.97	3.18
α of 4.5%	6.42	19.62	0.00	0.00	0.34	0.65	0.65	6.65	0.51	55	2.97	3.17
α of 4.0%	6.40	19.59	0.00	0.00	0.34	0.80	0.80	6.82	0.63	57	2.97	3.17
α of 3.5%	6.37	19.57	0.00	0.00	0.33	0.99	0.99	7.01	0.77	59	2.96	3.17
α of 3.0%	6.35	19.55	0.00	0.00	0.33	0.95	0.96	7.23	0.92	61	2.96	3.16
α of 2.5%	6.42	19.54	0.00	0.00	0.34	0.82	0.82	7.84	1.11	64	2.96	3.16
α of 2.0%	6.51	19.53	0.00	0.00	0.34	0.60	0.60	8.80	1.33	68	2.95	3.16
α of 1.5%	6.63	19.54	0.00	0.00	0.35	0.45	0.46	9.98	1.60	74	2.96	3.16
α of 1.0%	6.81	19.61	0.03	0.02	0.35	0.33	0.33	11.85	1.97	83	2.96	3.17
α of 0.5%	7.07	19.79	0.41	0.41	0.37	0.28	0.28	15.97	2.60	92	2.99	3.19

J.2 Stock fundamentals constraints

Constraints based on the different stock fundamentals mentioned in Appendix 21 have been implemented. However, this Appendix only shows the two constrained indices that show the best results based on downside risk reduction. This was decided to keep the research compact and clear. If you are interested in the other results, please contact 543857mk@eur.nl.

J.2.1 Price-to-Earnings ratio

Table 28: Annualized summary statistics of constrained S&P 500 using price-to-earnings ratio and the general normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.70	19.74	0.00	0.00	0.40	0.07	0.07	2.50	0.25	106	2.99	3.21
α of 4.5%	7.70	19.74	0.00	0.00	0.40	0.07	0.08	2.50	0.25	106	2.99	3.21
α of 4.0%	7.70	19.74	0.00	0.00	0.40	0.07	0.07	2.50	0.25	106	2.99	3.21
α of 3.5%	7.70	19.74	0.00	0.00	0.40	0.07	0.07	2.50	0.25	106	2.99	3.21
α of 3.0%	7.70	19.74	0.00	0.00	0.40	0.06	0.07	2.52	0.25	106	2.99	3.21
α of 2.5%	7.71	19.74	0.00	0.00	0.40	0.05	0.05	2.55	0.26	106	2.99	3.21
α of 2.0%	7.73	19.72	0.00	0.00	0.40	0.03	0.03	2.61	0.28	107	2.99	3.20
α of 1.5%	7.76	19.70	0.00	0.00	0.40	0.02	0.02	2.72	0.34	108	2.98	3.20
α of 1.0%	7.86	19.68	0.00	0.00	0.40	0.00	0.01	2.98	0.44	108	2.98	3.19
α of 0.5%	8.01	19.64	0.00	0.00	0.41	0.00	0.00	3.54	0.60	108	2.97	3.18

Table 29: Annualized summary statistics of constrained S&P 500 using price-to-earnings ratio and the sector neutral normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.68	19.77	0.00	0.00	0.39	0.10	0.11	2.48	0.22	106	3.00	3.21
α of 4.5%	7.68	19.77	0.00	0.00	0.39	0.10	0.11	2.48	0.22	106	3.00	3.21
α of 4.0%	7.68	19.77	0.00	0.00	0.39	0.10	0.11	2.48	0.22	106	3.00	3.21
α of 3.5%	7.69	19.77	0.00	0.00	0.39	0.10	0.10	2.48	0.22	106	3.00	3.21
α of 3.0%	7.69	19.77	0.00	0.00	0.39	0.08	0.09	2.50	0.22	106	3.00	3.21
α of 2.5%	7.70	19.77	0.00	0.00	0.40	0.06	0.07	2.53	0.22	106	3.00	3.21
α of 2.0%	7.72	19.76	0.00	0.00	0.40	0.02	0.02	2.60	0.25	107	2.99	3.21
α of 1.5%	7.77	19.74	0.00	0.00	0.40	0.01	0.01	2.74	0.30	108	2.99	3.21
α of 1.0%	7.85	19.71	0.00	0.00	0.40	0.00	0.00	3.05	0.39	109	2.99	3.20
α of 0.5%	8.01	19.68	0.00	0.00	0.41	0.00	0.00	3.71	0.54	108	2.98	3.19

Table 30: Annualized summary statistics of constrained S&P 100 using price-to-earnings ratio and the general normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.48	19.64	0.00	0.00	0.34	0.07	0.08	4.55	0.30	51	2.97	3.20
α of 4.5%	6.48	19.64	0.00	0.00	0.34	0.07	0.08	4.55	0.30	51	2.97	3.19
α of 4.0%	6.48	19.64	0.00	0.00	0.34	0.07	0.08	4.55	0.30	51	2.97	3.20
α of 3.5%	6.48	19.64	0.00	0.00	0.34	0.09	0.09	4.58	0.30	51	2.97	3.19
α of 3.0%	6.48	19.63	0.00	0.00	0.34	0.09	0.11	4.64	0.32	51	2.97	3.19
α of 2.5%	6.50	19.61	0.00	0.00	0.34	0.07	0.07	4.75	0.35	52	2.96	3.19
α of 2.0%	6.53	19.59	0.00	0.00	0.34	0.04	0.05	4.92	0.41	52	2.96	3.18
α of 1.5%	6.61	19.57	0.00	0.00	0.35	0.01	0.01	5.21	0.50	52	2.96	3.18
α of 1.0%	6.74	19.53	0.00	0.00	0.35	0.00	0.01	5.68	0.63	51	2.95	3.17
α of 0.5%	6.84	19.50	0.00	0.00	0.36	0.00	0.00	6.66	0.80	50	2.94	3.16

Table 31: Annualized summary statistics of constrained S&P 100 using price-to-earnings ratio and the sector neutral normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.46	19.68	0.00	0.00	0.34	0.11	0.14	4.64	0.24	51	2.98	3.20
α of 4.5%	6.46	19.68	0.00	0.00	0.34	0.11	0.14	4.64	0.24	51	2.98	3.20
α of 4.0%	6.46	19.68	0.00	0.00	0.34	0.11	0.14	4.64	0.24	51	2.98	3.20
α of 3.5%	6.46	19.68	0.00	0.00	0.34	0.11	0.14	4.66	0.25	51	2.98	3.20
α of 3.0%	6.47	19.67	0.00	0.00	0.34	0.10	0.12	4.72	0.26	51	2.97	3.20
α of 2.5%	6.49	19.66	0.00	0.00	0.34	0.06	0.06	4.84	0.29	52	2.97	3.20
α of 2.0%	6.52	19.64	0.00	0.00	0.34	0.03	0.04	5.05	0.33	52	2.97	3.19
α of 1.5%	6.59	19.62	0.00	0.00	0.34	0.01	0.02	5.41	0.41	52	2.97	3.19
α of 1.0%	6.69	19.60	0.00	0.00	0.35	0.00	0.01	6.07	0.52	52	2.96	3.18
α of 0.5%	6.79	19.57	0.00	0.00	0.35	0.01	0.01	7.27	0.70	51	2.96	3.17

J.2.2 Price-to-Earnings-to-Growth ratio

Table 32: Annualized summary statistics of constrained S&P 500 using price-to-earnings-to-growth ratio and the general normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.70	19.74	0.00	0.00	0.40	0.07	0.07	2.50	0.25	106	2.99	3.21
α of 4.5%	7.70	19.74	0.00	0.00	0.40	0.07	0.08	2.50	0.25	106	2.99	3.21
α of 4.0%	7.70	19.74	0.00	0.00	0.40	0.07	0.07	2.50	0.25	106	2.99	3.21
α of 3.5%	7.70	19.74	0.00	0.00	0.40	0.06	0.06	2.52	0.25	106	2.99	3.21
α of 3.0%	7.71	19.74	0.00	0.00	0.40	0.04	0.05	2.56	0.26	106	2.99	3.21
α of 2.5%	7.72	19.74	0.00	0.00	0.40	0.03	0.03	2.63	0.26	107	2.99	3.21
α of 2.0%	7.74	19.74	0.00	0.00	0.40	0.02	0.02	2.78	0.27	107	2.99	3.21
α of 1.5%	7.77	19.74	0.00	0.00	0.40	0.01	0.01	3.01	0.30	108	2.99	3.21
α of 1.0%	7.84	19.74	0.00	0.00	0.40	0.00	0.00	3.46	0.35	109	2.99	3.21
α of 0.5%	7.94	19.75	0.00	0.00	0.41	0.00	0.00	4.27	0.46	108	2.99	3.21

Table 33: Annualized summary statistics of constrained S&P 500 using price-to-earnings-to-growth ratio and the sector neutral normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.68	19.77	0.00	0.00	0.39	0.10	0.11	2.48	0.22	106	3.00	3.21
α of 4.5%	7.68	19.77	0.00	0.00	0.39	0.10	0.11	2.48	0.22	106	3.00	3.21
α of 4.0%	7.68	19.77	0.00	0.00	0.39	0.10	0.11	2.48	0.22	106	3.00	3.21
α of 3.5%	7.69	19.77	0.00	0.00	0.39	0.08	0.09	2.49	0.22	106	3.00	3.21
α of 3.0%	7.70	19.77	0.00	0.00	0.40	0.06	0.06	2.54	0.22	106	3.00	3.21
α of 2.5%	7.71	19.77	0.00	0.00	0.40	0.04	0.04	2.62	0.22	107	3.00	3.21
α of 2.0%	7.73	19.77	0.00	0.00	0.40	0.02	0.03	2.77	0.24	108	3.00	3.21
α of 1.5%	7.76	19.77	0.00	0.00	0.40	0.01	0.01	3.01	0.27	108	3.00	3.21
α of 1.0%	7.82	19.78	0.00	0.00	0.40	0.00	0.00	3.52	0.32	109	3.00	3.21
α of 0.5%	7.92	19.79	0.00	0.00	0.41	0.00	0.00	4.49	0.42	109	3.00	3.22

Table 34: Annualized summary statistics of constrained S&P 100 using price-to-earnings-to-growth ratio and the general normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.49	19.64	0.00	0.00	0.34	0.06	0.07	4.57	0.30	51	2.97	3.20
α of 4.5%	6.49	19.64	0.00	0.00	0.34	0.05	0.05	4.60	0.30	51	2.97	3.19
α of 4.0%	6.50	19.64	0.00	0.00	0.34	0.04	0.05	4.63	0.31	52	2.97	3.19
α of 3.5%	6.51	19.64	0.00	0.00	0.34	0.04	0.04	4.69	0.32	52	2.97	3.19
α of 3.0%	6.52	19.64	0.00	0.00	0.34	0.03	0.03	4.77	0.33	52	2.97	3.19
α of 2.5%	6.53	19.64	0.00	0.00	0.34	0.03	0.03	4.91	0.35	52	2.97	3.20
α of 2.0%	6.55	19.64	0.00	0.00	0.34	0.03	0.03	5.12	0.38	52	2.97	3.19
α of 1.5%	6.59	19.64	0.00	0.00	0.34	0.02	0.02	5.48	0.42	52	2.97	3.20
α of 1.0%	6.65	19.65	0.00	0.00	0.35	0.01	0.02	6.06	0.48	52	2.97	3.20
α of 0.5%	6.70	19.66	0.00	0.00	0.35	0.02	0.02	7.12	0.58	51	2.97	3.20

Table 35: Annualized summary statistics of constrained S&P 100 using price-to-earnings-to-growth ratio and the sector neutral normalization method, where a constraint of 5.0% corresponds to α of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.46	19.68	0.00	0.00	0.34	0.10	0.12	4.64	0.25	51	2.98	3.20
α of 4.5%	6.47	19.68	0.00	0.00	0.34	0.07	0.10	4.68	0.25	51	2.98	3.20
α of 4.0%	6.48	19.68	0.00	0.00	0.34	0.05	0.07	4.73	0.25	52	2.98	3.20
α of 3.5%	6.49	19.68	0.00	0.00	0.34	0.04	0.05	4.79	0.26	52	2.98	3.20
α of 3.0%	6.51	19.68	0.00	0.00	0.34	0.03	0.03	4.89	0.27	52	2.98	3.20
α of 2.5%	6.53	19.68	0.00	0.00	0.34	0.02	0.02	5.05	0.29	52	2.98	3.20
α of 2.0%	6.56	19.68	0.00	0.00	0.34	0.01	0.02	5.29	0.31	52	2.98	3.20
α of 1.5%	6.60	19.68	0.00	0.00	0.34	0.01	0.01	5.70	0.36	52	2.98	3.20
α of 1.0%	6.67	19.69	0.00	0.00	0.35	0.00	0.01	6.36	0.42	52	2.98	3.21
α of 0.5%	6.73	19.72	0.43	0.43	0.35	0.01	0.01	7.62	0.54	51	2.98	3.21

J.3 Sector constraints

Table 36: Annualized summary statistics of constrained S&P 100 using a sector constraint, where a constraint of 85% corresponds to a β of 85% at the moment of index creation.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
β of 85%	6.48	19.64	0.00	0.00	0.34	0.09	0.09	4.55	0.30	51	2.97	3.20
β of 84%	6.48	19.64	0.00	0.00	0.34	0.09	0.11	4.55	0.30	51	2.97	3.19
β of 83%	6.48	19.65	0.00	0.00	0.34	0.10	0.11	4.55	0.31	51	2.97	3.20
β of 82%	6.47	19.65	0.00	0.00	0.34	0.13	0.14	4.56	0.31	51	2.97	3.20
β of 81%	6.46	19.65	0.00	0.00	0.34	0.17	0.18	4.56	0.31	51	2.97	3.20
β of 80%	6.46	19.66	0.00	0.00	0.34	0.22	0.23	4.57	0.31	51	2.97	3.20
β of 79%	6.45	19.67	0.00	0.00	0.34	0.28	0.28	4.58	0.31	51	2.97	3.20
β of 78%	6.44	19.67	0.00	0.00	0.34	0.34	0.34	4.60	0.32	52	2.97	3.20
β of 77%	6.44	19.68	0.00	0.00	0.34	0.41	0.41	4.63	0.32	52	2.98	3.20
β of 76%	6.44	19.69	0.00	0.00	0.34	0.48	0.48	4.65	0.33	52	2.98	3.20

K Dynamic constrained indices

Indices based on dynamic constraints and different values for δ are calculated. However, this Appendix only shows the two dynamic constrained indices that show the best results based on downside risk reduction. This was decided to keep the research compact and clear. If you are interested in the other results, please contact 543857mk@eur.nl.

K.1 VIX

Table 37: Annualized summary statistics of dynamic constrained S&P 500 for $\gamma = 40$ using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.69	19.79	0.00	0.00	0.39	0.09	0.13	5.90	0.20	106	3.00	3.20
α of 4.5%	7.68	19.79	0.00	0.00	0.39	0.10	0.12	6.41	0.20	106	3.00	3.20
α of 4.0%	7.68	19.78	0.00	0.00	0.39	0.13	0.14	7.07	0.20	106	3.00	3.20
α of 3.5%	7.66	19.78	0.00	0.00	0.39	0.23	0.27	8.03	0.21	106	3.00	3.20
α of 3.0%	7.65	19.78	0.00	0.00	0.39	0.41	0.41	9.68	0.23	107	3.00	3.19
α of 2.5%	7.63	19.77	0.00	0.00	0.39	0.65	0.67	11.95	0.25	107	3.00	3.19
α of 2.0%	7.61	19.78	0.00	0.00	0.39	0.88	0.89	15.09	0.28	107	3.00	3.20
α of 1.5%	7.60	19.79	0.01	0.02	0.39	0.96	0.96	20.26	0.33	107	3.00	3.20
α of 1.0%	7.60	19.80	0.31	0.34	0.39	0.88	0.88	30.01	0.42	108	3.00	3.20
α of 0.5%	7.56	19.86	0.18	0.20	0.39	0.66	0.67	50.13	0.63	109	3.01	3.21

Table 38: Annualized summary statistics of dynamic VIX constrained S&P 500 for $\gamma = 40$ using a sector constraint, where a constraint of 85% corresponds to a β of 85%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
β of 85%	7.69	19.78	0.00	0.00	0.39	0.09	0.13	2.73	0.22	106	3.00	3.19
β of 84%	7.69	19.78	0.00	0.00	0.39	0.09	0.13	2.73	0.22	106	3.00	3.19
β of 83%	7.69	19.78	0.00	0.00	0.39	0.09	0.14	2.73	0.22	106	3.00	3.19
β of 82%	7.69	19.78	0.00	0.00	0.39	0.09	0.14	2.73	0.22	106	3.00	3.19
β of 81%	7.69	19.78	0.00	0.00	0.39	0.09	0.13	2.73	0.22	106	3.00	3.19
β of 80%	7.69	19.78	0.00	0.00	0.39	0.09	0.12	2.73	0.22	106	3.00	3.20
β of 79%	7.69	19.78	0.00	0.00	0.39	0.09	0.14	2.74	0.22	106	3.00	3.19
β of 78%	7.69	19.78	0.00	0.00	0.39	0.09	0.14	2.80	0.22	106	3.00	3.19
β of 77%	7.69	19.78	0.00	0.00	0.39	0.09	0.14	2.95	0.22	106	3.00	3.19
β of 76%	7.69	19.78	0.00	0.00	0.39	0.09	0.15	3.19	0.22	106	3.00	3.19

Table 39: Annualized summary statistics of dynamic constrained S&P 100 for $\gamma = 40$ using a constant and the general normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.44	19.67	0.00	0.00	0.34	0.32	0.33	8.69	0.29	52	2.98	3.18
α of 4.5%	6.43	19.67	0.00	0.00	0.34	0.50	0.52	10.36	0.31	52	2.98	3.18
α of 4.0%	6.41	19.67	0.00	0.00	0.34	0.71	0.72	12.41	0.33	52	2.98	3.18
α of 3.5%	6.39	19.67	0.00	0.01	0.33	0.93	0.93	14.79	0.37	52	2.98	3.18
α of 3.0%	6.37	19.68	0.01	0.04	0.33	0.87	0.86	17.77	0.41	52	2.98	3.19
α of 2.5%	6.36	19.70	0.16	0.22	0.33	0.76	0.76	21.92	0.46	52	2.98	3.19
α of 2.0%	6.34	19.71	0.59	0.63	0.33	0.71	0.70	28.10	0.53	52	2.98	3.19
α of 1.5%	6.35	19.74	0.80	0.81	0.33	0.74	0.75	36.71	0.61	52	2.99	3.20
α of 1.0%	6.34	19.78	0.17	0.19	0.33	0.75	0.75	49.12	0.74	52	3.00	3.20
α of 0.5%	6.33	19.92	0.00	0.03	0.33	0.69	0.70	67.71	1.04	52	3.02	3.23

Table 40: Annualized summary statistics of dynamic constrained S&P 100 for $\gamma = 40$ using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.44	19.67	0.00	0.00	0.34	0.24	0.24	11.87	0.25	52	2.98	3.18
α of 4.5%	6.43	19.67	0.00	0.00	0.34	0.39	0.39	13.41	0.26	52	2.98	3.18
α of 4.0%	6.41	19.67	0.00	0.00	0.34	0.57	0.58	15.32	0.29	52	2.98	3.18
α of 3.5%	6.40	19.66	0.00	0.00	0.33	0.78	0.79	17.56	0.31	52	2.98	3.18
α of 3.0%	6.39	19.66	0.00	0.00	0.33	0.93	0.94	20.48	0.34	52	2.98	3.18
α of 2.5%	6.38	19.67	0.00	0.00	0.33	0.98	0.98	24.61	0.38	52	2.98	3.18
α of 2.0%	6.38	19.67	0.00	0.01	0.33	0.95	0.96	30.44	0.44	52	2.98	3.18
α of 1.5%	6.38	19.67	0.03	0.04	0.33	0.97	0.97	38.96	0.51	52	2.98	3.18
α of 1.0%	6.37	19.69	0.25	0.28	0.33	0.91	0.91	51.40	0.62	52	2.98	3.19
α of 0.5%	6.35	19.78	0.19	0.23	0.33	0.83	0.83	70.11	0.86	52	2.99	3.20

Table 41: Annualized summary statistics of dynamic constrained S&P 100 for $\gamma = 40$ using a sector constraint, where a constraint of 85% corresponds to a β of 85%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
β of 85%	6.48	19.68	0.00	0.00	0.34	0.06	0.09	4.83	0.25	51	2.98	3.18
β of 84%	6.48	19.68	0.00	0.00	0.34	0.06	0.09	4.83	0.25	51	2.98	3.18
β of 83%	6.48	19.68	0.00	0.00	0.34	0.06	0.10	4.83	0.25	51	2.98	3.18
β of 82%	6.48	19.68	0.00	0.00	0.34	0.06	0.09	4.87	0.25	51	2.98	3.18
β of 81%	6.48	19.68	0.00	0.00	0.34	0.07	0.10	5.03	0.25	51	2.98	3.18
β of 80%	6.48	19.68	0.00	0.00	0.34	0.07	0.11	5.35	0.25	51	2.98	3.18
β of 79%	6.48	19.68	0.00	0.00	0.34	0.08	0.12	5.75	0.25	51	2.98	3.18
β of 78%	6.48	19.68	0.00	0.00	0.34	0.08	0.12	6.26	0.25	51	2.98	3.18
β of 77%	6.48	19.69	0.00	0.00	0.34	0.09	0.12	6.82	0.26	51	2.98	3.18
β of 76%	6.48	19.69	0.00	0.00	0.34	0.09	0.12	7.44	0.26	51	2.98	3.18

K.2 Buffett Indicator

Table 42: Annualized summary statistics of dynamic constrained S&P 500 for $\gamma = 1.2$ using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
α of 5.0%	7.69	19.78	0.00	0.00	0.39	0.08	0.09	5.40	0.21	106	3.00	3.19
α of 4.5%	7.69	19.78	0.00	0.00	0.39	0.11	0.13	5.47	0.22	107	3.00	3.19
α of 4.0%	7.68	19.77	0.00	0.00	0.39	0.18	0.21	5.60	0.25	107	3.00	3.19
α of 3.5%	7.68	19.76	0.00	0.00	0.39	0.25	0.25	5.78	0.30	109	2.99	3.19
α of 3.0%	7.67	19.75	0.00	0.00	0.39	0.40	0.41	6.06	0.38	110	2.99	3.19
α of 2.5%	7.67	19.74	0.00	0.00	0.39	0.55	0.56	6.43	0.49	112	2.99	3.19
α of 2.0%	7.65	19.73	0.00	0.00	0.39	0.72	0.72	6.93	0.62	114	2.99	3.18
α of 1.5%	7.63	19.72	0.00	0.00	0.39	0.83	0.84	7.62	0.78	117	2.99	3.18
α of 1.0%	7.63	19.72	0.00	0.00	0.39	0.88	0.88	8.74	0.95	121	2.98	3.18
α of 0.5%	7.71	19.73	0.01	0.01	0.40	0.69	0.70	11.28	1.25	126	2.98	3.18

Table 43: Annualized summary statistics of dynamic constrained S&P 500 for $\gamma = 1.2$ using a sector constraint, where a constraint of 85% corresponds to a β of 85%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 500	7.61	19.82	-	-	0.39	-	-	0.61	-	107	3.00	3.22
S&P 500 EW	12.04	20.37	0.00	0.00	0.58	0.00	0.00	9.66	4.40	477	3.07	3.25
β of 85%	7.69	19.75	0.00	0.00	0.39	0.09	0.10	2.53	0.24	106	2.99	3.19
β of 84%	7.69	19.75	0.00	0.00	0.39	0.09	0.09	2.53	0.24	106	2.99	3.19
β of 83%	7.69	19.75	0.00	0.00	0.39	0.09	0.10	2.53	0.24	106	2.99	3.19
β of 82%	7.69	19.75	0.00	0.00	0.39	0.09	0.10	2.53	0.24	106	2.99	3.19
β of 81%	7.69	19.75	0.00	0.00	0.39	0.10	0.10	2.53	0.24	106	2.99	3.19
β of 80%	7.68	19.75	0.00	0.00	0.39	0.11	0.12	2.53	0.24	106	2.99	3.19
β of 79%	7.68	19.75	0.00	0.00	0.39	0.11	0.12	2.53	0.25	106	2.99	3.19
β of 78%	7.68	19.75	0.00	0.00	0.39	0.12	0.13	2.54	0.25	106	2.99	3.19
β of 77%	7.68	19.75	0.00	0.00	0.39	0.13	0.14	2.55	0.26	106	2.99	3.19
β of 76%	7.68	19.75	0.00	0.00	0.39	0.14	0.15	2.57	0.26	106	2.99	3.19

Table 44: Annualized summary statistics of dynamic constrained S&P 100 for $\gamma = 1.2$ using a constant and the sector neutral normalization method, where a constraint of 5.0% corresponds to a max index weight of 5.0%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
α of 5.0%	6.42	19.66	0.00	0.00	0.34	0.51	0.51	8.79	0.34	52	2.97	3.18
α of 4.5%	6.40	19.66	0.00	0.00	0.33	0.77	0.78	9.05	0.40	53	2.97	3.18
α of 4.0%	6.39	19.64	0.00	0.00	0.33	0.94	0.94	9.36	0.49	53	2.97	3.18
α of 3.5%	6.37	19.63	0.00	0.00	0.33	0.96	0.96	9.73	0.59	54	2.97	3.17
α of 3.0%	6.34	19.61	0.00	0.00	0.33	0.83	0.83	10.17	0.71	55	2.96	3.17
α of 2.5%	6.33	19.60	0.00	0.00	0.33	0.80	0.80	10.74	0.86	55	2.96	3.17
α of 2.0%	6.30	19.58	0.00	0.00	0.33	0.76	0.75	11.56	1.01	56	2.96	3.16
α of 1.5%	6.26	19.59	0.00	0.00	0.33	0.66	0.66	12.62	1.16	57	2.96	3.16
α of 1.0%	6.29	19.59	0.00	0.00	0.33	0.79	0.79	14.37	1.32	58	2.96	3.16
α of 0.5%	6.36	19.61	0.02	0.02	0.33	0.97	0.97	17.42	1.59	59	2.96	3.17

Table 45: Annualized summary statistics of dynamic constrained S&P 100 for $\gamma = 1.2$ using a sector constraint, where a constraint of 85% corresponds to a β of 85%.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TO	TE	AS	ES_{HS}	ES_{NM}
S&P 100	6.39	19.73	-	-	0.33	-	-	2.23	-	51	2.98	3.21
S&P 100 EW	7.21	19.68	0.61	0.59	0.37	0.24	0.23	12	2.13	97	2.97	3.18
β of 85%	6.47	19.65	0.00	0.00	0.34	0.09	0.09	4.59	0.28	51	2.97	3.18
β of 84%	6.47	19.65	0.00	0.00	0.34	0.10	0.10	4.59	0.28	51	2.97	3.18
β of 83%	6.47	19.65	0.00	0.00	0.34	0.11	0.12	4.59	0.28	51	2.97	3.18
β of 82%	6.47	19.66	0.00	0.00	0.34	0.12	0.13	4.59	0.29	51	2.97	3.18
β of 81%	6.46	19.66	0.00	0.00	0.34	0.14	0.15	4.59	0.29	51	2.97	3.18
β of 80%	6.46	19.66	0.00	0.00	0.34	0.15	0.16	4.61	0.29	51	2.97	3.18
β of 79%	6.46	19.66	0.00	0.00	0.34	0.17	0.18	4.64	0.30	51	2.97	3.18
β of 78%	6.46	19.66	0.00	0.00	0.34	0.19	0.20	4.68	0.30	51	2.97	3.18
β of 77%	6.46	19.66	0.00	0.00	0.34	0.21	0.21	4.72	0.31	51	2.97	3.18
β of 76%	6.46	19.66	0.00	0.00	0.34	0.23	0.23	4.77	0.31	51	2.97	3.18

L Index combinations

Table 46: Annualized summary statistics of the combined S&P 100 index where the combined index will exist out of 1. market cap index, 2. dynamic constant constrained index, 3. constant constrained index and 4. sector constrained index. A training time of 1 year is applied and the weight calculation is based on volatility.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TE	AS	ES_{HS}	ES_{NM}
S&P 100	7.43	19.47	-	-	0.40	-	-	-	52	2.96	3.17
Comb 4 indices	7.32	19.35	0.00	0.00	0.39	0.55	0.55	0.57	61	2.95	3.15
Comb 3 indices	7.27	19.34	0.00	0.00	0.39	0.48	0.49	0.83	63	2.95	3.15
Comb 2 indices	7.22	19.38	0.00	0.00	0.39	0.13	0.13	0.57	55	2.95	3.16

Table 47: Annualized summary statistics of the combined S&P 100 index where the combined index will exist out of 1. market cap index, 2. dynamic constant constrained index, 3. constant constrained index and 4. sector constrained index. A training time of 1 year is applied and the weight calculation is based on expected shortfall.

	r_p	σ_p	$p_{H,\sigma}$	$p_{B,\sigma}$	SR	$p_{H,SR}$	$p_{B,SR}$	TE	AS	ES_{HS}	ES_{NM}
S&P 100	7.43	19.47	-	-	0.40	-	-	-	52	2.96	3.17
Comb 4 indices	7.32	19.35	0.00	0.00	0.39	0.55	0.55	0.55	61	2.95	3.15
Comb 3 indices	7.27	19.34	0.00	0.00	0.39	0.48	0.48	0.83	63	2.95	3.15
Comb 2 indices	7.22	19.38	0.00	0.00	0.39	0.13	0.12	0.57	55	2.95	3.16