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Quarticity Shrinkage: A new Method of Portfolio Allocation

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Abstract

This paper investigates the performance of a new approach to portfolio allocation. Previous literature shows that none of the existing covariance estimators consistently beat the $1/N$ allocation due to estimation uncertainty regarding the covariance matrix. They find that an impractically large estimation window is required to overcome this uncertainty. We propose quarticity shrinkage (QS), a new method that combines the $1/N$ allocation with an existing covariance estimator to beat the $1/N$ strategy using a practical estimation window. We compare the performance of QS in three simulation studies and an empirical analysis with four existing methods. The $1/N$ allocation from DeMiguel & Uppal (2007), linear shrinkage by Ledoit & Wolf (2003a), nonlinear shrinkage from Ledoit & Wolf (2020a) and exponentially weighted moving average by Morgan et al. (1996). We find that QS often outperforms the existing methods on a risk-adjusted basis. The $1/N$ strategy remains unbeaten in terms of mean returns. In our simulation studies, the existing covariance estimators yield a better performance than QS in measures that disregard risk. However, in our empirical analysis, QS is superior to these covariance estimators in most measures, including those that neglect risk.

Contents

1 Introduction

Obtaining an estimate of the optimal portfolio allocation has been a major topic of interest for decades. With an accurate estimated portfolio, one may profit due to (daily) changes in the stock price, a so-called return. Investments in stocks are a vital element of the financial market. In 2020, the Stock Market Capitalization-to-GDP ratio was roughly 150% for the United States (Kenton (2021)). This ratio means that the value of the US stock market is 1.5 times larger than its gross domestic product. The returns on investments depend solely on the performance of assets and thus the portfolio allocation. One of the most commonly used portfolios is the global minimum variance (GMV) portfolio. This portfolio is constructed using only the inverse of the covariance matrix and a vector of ones. Such that ex-ante covariance matrix estimates are of utmost importance.

Markowitz portfolio optimization, introduced by Markowitz (1952), is the foundation for most of the portfolio allocation strategies. In this framework, one assumes that the investor only cares about the mean and variance of the portfolio. The most apparent estimates of these variables are the sample moments. Hence in the Markowitz framework, the sample moments are used to calculate the portfolio allocation. Since the introduction of Markowitz portfolio optimization, many different methods to estimate the optimal allocation were developed. Maybe one of the most obvious and simplistic choices for portfolio allocation would be the equally weighted portfolio from DeMiguel & Uppal (2007). They show that none of the existing covariance estimation methods consistently outperform the $1/N$ strategy. They conclude that a very impractically large estimation window is required for the estimators to outperform this $1/N$ benchmark. When using 25 assets, this window would be around 3000 months. When the number of assets increases, the required window becomes even larger. Thus, making it practically infeasible in the real world with more than 25 assets.

The problem is the uncertainty regarding estimates of each element in the covariance matrix, which leads to non-optimal allocations in practice. To overcome this uncertainty, an impractically large estimation window must be used. This gives a stimulus to try and combine the 1/N strategy with another covariance estimation method, such that the uncertainty can be limited. Therefore, we will introduce a new approach to estimate portfolio allocations by combining the characteristics of the 1/N strategy and an existing covariance estimation method. To our best knowledge, this type of approach to portfolio allocation has not been done before. We call this new method 'quarticity shrinkage' (QS).

QS separates assets based on their uncertainty in the returns. We assess the uncertainty of a stock via its realized quarticity (RQ), from Barndorff-Nielsen & Shephard (2002). This measure is, in essence, the volatility of the volatility. Stocks with a small RQ are deemed certain, and stocks with a large RQ are deemed uncertain. Certain stocks follow the portfolio allocation via the existing covariance estimation method, whereas uncertain stocks follow the 1/N allocation. In the last step, we merge both allocations into and portfolio and normalize it.

To fully exploit the performance of QS, it combines the equally weighted portfolio with linear shrinkage (LS) from Ledoit & Wolf (2003a), nonlinear shrinkage (NLS) by Ledoit & Wolf (2020a) and the exponentially weighted moving average (EWMA) from Morgan et al. (1996).

We examine QS in several simulation studies and an empirical analysis. It will be subject to different estimation windows and a different number of assets. For the data generating process (DGP), we combine the Monte Carlo method with bootstrapping. The underlying structure in the data will be based on a GARCH(1,1) model with Student's t-distributed noise. The procedure of generating returns will follow the principles of Jegadeesh et al. (2019), where we extend the DGP to account for time-varying volatility.

We evaluate the out-of-sample performance with the industry-standard performance measures: the mean return, the Sharpe ratio from Sharpe (1966), the certainty equivalent (CEQ), the turnover and the transaction costs adjusted mean return. These measures are all out-of-sample as the portfolios are created only with historical returns. We evaluate the robustness of the methods via the standard deviation of the performance measures over the replication runs in the simulation studies.

With the introduction of QS, the main question is: *Does quarticity shrinkage beat the* $1/N$ *benchmark using a practical estimation window?*

To support this main question, we propose the following research questions:

- Is there an optimal division between certain and uncertain stocks for quarticity shrinkage?
- Does quarticity shrinkage outperform the existing allocation estimators?
- How does the performance of quarticity shrinkage change when using various existing covariance estimation methods?
- Is quarticity shrinkage more robust than existing methods?

Our main findings are as follows. In the simulation studies, we do not find an optimal partition ratio of certain/uncertain stocks. Further, we find statistically significant differences in the performance of QS against the existing estimators. QS outperforms on a risk-adjusted basis but is beaten in terms of measures that neglect risk. From the empirical analysis, we again find that QS outperforms on a risk-adjusted basis but never attains the largest mean return. We do not find a superior combination for QS. Lastly, we find that QS is more reliable than the existing methods.

The paper proceeds as follows; in Section 2, we provide an overview of the relevant literature. Section 3 describes the data. Section 4 formalizes the methods that we use to estimate the portfolio allocation and describes the performance evaluation of these methods. Section 5 introduces the simulation study with the data generating process and presents the results of said studies. Section 6 shows the results of the estimation methods applied to the empirical stock data. Section 7 discusses the limitations of the paper and gives recommendations for further research. At last, Section 8 concludes the report.

2 Literature Study

One calculates the optimal portfolio in the Markowitz framework with the sample covariance matrix. This matrix is the classic maximum likelihood estimator when the number of observations goes to infinity as derived by Anderson (1973). However, Ledoit & Wolf (2003a) state that nobody should be using the sample covariance matrix. This estimator suffers heavily from estimation uncertainty, as every element is estimated. It is also ill-conditioned when the number of observations is close to the number of assets.

To overcome the issues regarding the sample covariance, numerous methods were developed to estimate the covariance matrix. A widely known and popular approach from statistics is shrinkage; a technique introduced by Stein (1956) to reduce the mean squared error of an estimation. This method shrinks estimates of the mean towards the grand average of all means. Values of the mean greater than the grand average are made smaller, and values smaller than the grand average are made greater. James & Stein (1961) prove that this method dominates the maximum likelihood estimator in terms of total squared error.

Later, Ledoit & Wolf (2003b) apply the principles of shrinkage to covariance estimation: linear shrinkage (LS). Their paper shrinks the sample covariance matrix towards a single factor covariance matrix with a constant shrinkage intensity. LS shrinks the unbiased but variable sample covariance matrix towards the biased but less variable target matrix. This makes sense, as the target matrix adds much structure: the more the sample covariance shrinks towards the target, the fewer parameters have to be estimated. In addition, the resulting estimator is invertible and well-conditioned. In the same year, they propose another shrinkage target: the constant correlation matrix in Ledoit & Wolf (2003a). They find that this target gives a similar performance to the single factor covariance matrix, while it is easier to implement.

Ledoit & Wolf (2012) extend the idea of shrinkage further, which leads to the nonlinear shrinkage (NLS) estimator. The core difference with LS is that NLS uses a varying shrinkage intensity for each element in the covariance matrix, whereas LS uses the same intensity for all elements. Later, Ledoit & Wolf (2020a) propose an analytical nonlinear eigenvalue shrinkage estimator. This method yields the same accuracy and is way faster in estimating the covariance matrix than the previous NLS method. Another benefit of this analytical method is that it uses the eigenvalues of the matrix, such that is does not require any judicious choice for a target matrix.

Stock return data often exhibits volatility clustering, meaning that the volatility changes over time and shows a persistent behaviour. A simplistic yet effective covariance estimator that integrates this dynamic behaviour is the Exponentially Weighted Moving Average (EWMA). Morgan et al. (1996) provide a detailed description of the RiskMetrics methodology that led to the EWMA method.

A widely known method to infer stock return variation and the presence of jumps is the integrated quarticity. Barndorff-Nielsen & Shephard (2002) show that this measure can be estimated consistently via realized quarticity (RQ). RQ is now one of the most used proxies for the uncertainty of intraday stock returns; it is used in Bollerslev et al. (2016), Andersen et al. (2014) and Corsi et al. (2008) among others. In essence, RQ calculates the volatility over the volatility. We estimate RQ in the new method using daily stock returns instead of intraday stock returns. Therefore, we shift from approximating the daily uncertainty of stocks to uncertainty over a longer estimation window.

Within the mean-variance framework, the tangency point is the optimal portfolio of risky assets as it gives the best trade-off between expected return and variance. However, Jagannathan & Ma (2003) find that the tangency portfolio performs worse than the Global Minimum Variance (GMV) portfolio in terms of the out-of-sample Sharpe ratio. This is explained via the noise corresponding to the mean estimates. The GMV portfolio depends solely on the estimation of the covariance matrix, such that it does not rely on mean estimates and can circumvent the corresponding estimation noise.

As mentioned earlier, DeMiguel & Uppal (2007) show that a large estimation window is required to beat the $1/N$ portfolio. Our paper aims to overcome the need for a large estimation window by introducing quarticity shrinkage. Such that it is of interest to analyze the performance using smaller windows. We will be using one year and three years.

Intuitively, one could argue that recent information is more critical than information from a long time ago. The underlying structure of the returns changes over time; what might have been an expansion a year ago or three years could be a recession right now. Therefore, we will also be using three months and six months as estimation windows.

In the simulation study, the data generating process (DGP) must resemble the real world as accurate as possible. However, it is often unclear which distribution to use for the DGP, which leads to a DGP uncertainty. Tu & Zhou (2004) find that the DGP uncertainty is a major issue, as the different distributions lead to substantial differences in the estimates of the parameters.

A commonly used distribution is the normal distribution, even though daily stock returns often contain heavier tails than the normal distribution can account for. Therefore, one should opt for a distribution with fatter tails as suggested by Officer (1972), Hu & Kercheval (2010) and Peiró (1994) among others.

Hu & Kercheval (2010) find that a Student's t-distribution with roughly 6 degrees of freedom (DOF) fits the underlying structure of the true returns quite well. Peiró (1994) also find in their empirical analysis that one should consider a Student's t-distribution with 2.5 - 6 DOF.

3 Data

The dataset consists of daily excess holding-period returns of actively traded stocks on the S&P500 obtained via Wharton Research Data Services[1] from the vendor Center for Research in Security Prices (CRSP). To ensure that enough observations are available and all stocks have the same data range, we remove stocks that do not have the full data throughout 03/01/2007 − 31/12/2020. The reason for missing data of certain companies is simple; new companies are founded and added to the market, whereas companies go bankrupt and are removed from the market. After this filter of the data, we are left with a dataset of $T = 3525$ observations and $N = 327$ stocks. We display the names of the included stocks in a table in the appendix, Section A. We further use the value-weighted market index from the CRSP, and the risk-free rate; both are available at the Kenneth French data library[2].

Table 1 displays the summary statistics of the annualized excess stock return dataset when we use no rolling window. We note that the mean return across the assets and over time is 16.5%, this is way larger than the average annual return of the S&P500 which is 13.6% according to Business Insider[3]. Other sources state that an average annual return of roughly 10% should be expected, which is even lower. We observe that the annual mean return of the excess market index is close to this value but still fairly high. Figure 1 displays the cumulative returns and cumulative market index. We observe an increasing upward trend for both the market and the average stocks. The average stock return has a higher sensitivity to shocks than the market return. In combination with the increasing upward trend, we argue that it makes sense that the average mean return of the stocks is larger than that of the market index. There are a few exceptions to the upward trend, such as the period 2008 − 2009 and 2019 − 2020. Plausible explanations for these downward slopes are the financial banking crisis and the COVID-19 pandemic.

As is quite common for stock returns, we observe that the data exhibits excess kurtosis and a slight skewness. The skewness and kurtosis are different from those expected for a normal distribution, 0 and 3, respectively. Therefore, the returns do not contain an underlying normal distribution but rather a distribution that can capture high peaks and heavier tails. We observe a similar phenomenon for the market index, which contains a typical negative skewness.

Another interesting finding is the volatility clustering for the average stock returns and the market index. Figure 2 displays that the average volatility is time-varying for a

[1]https://wrds-www-wharton-upenn-edu.eur.idm.oclc.org/
[2]https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
[3]https://www.businessinsider.com/personal-finance/average-stock-market-return

rolling window of a year. We observe for both the stock returns and the market index that the highest obtained volatility, around $2008 - 2009$, is roughly three times as large as the lowest one, around $2017 - 2018$.

At last, we find significant autocorrelation within both the stock data and the value-weighted market index. Section B of the appendix contains the autocorrelation function and the partial autocorrelation function for both datasets. The functions display a significant autocorrelation at a level of 5% for all included laggs. We verify these findings with the Ljung-Box test.

	Return	Volatility	Skewness	Kurtosis
Min	0.008	0.179	-1.291	7.279
Mean	0.165	0.351	0.231	17.749
Max	0.492	0.891	4.661	140.040
Mkt	0.109	0.209	-0.343	14.545

Table 1: This table shows the descriptive statistics of the annualized excess holding-period returns. We display the annual minimum, mean, maximum and market (Mkt) return and volatility of the stocks over $03/01/2007 - 31/12/2020$. By annual, we mean that we compute the metric and multiply it by a scale of 252. We multiply the return by 252 and the volatility by $\sqrt{252}$. It further shows the skewness and kurtosis that we calculate over the entire period for each stock.

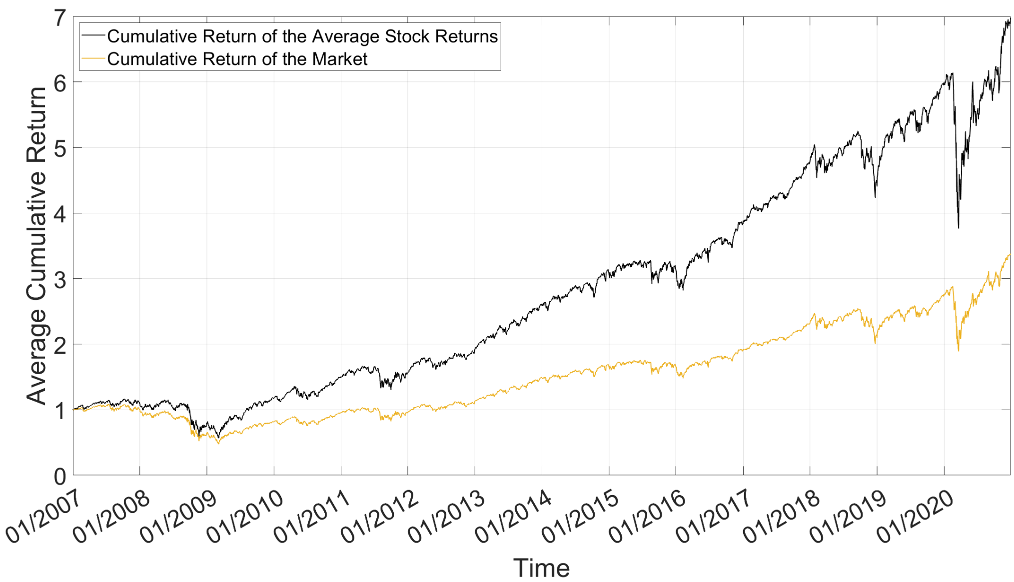

Figure 1: A plot of the cumulative returns of the average stocks and market index with a starting value of 1.

4 Methodology

The methodology will formalize quarticity shrinkage in its approach to estimate the portfolio allocation. It will also cover the $1/N$ strategy from DeMiguel & Uppal (2007), linear

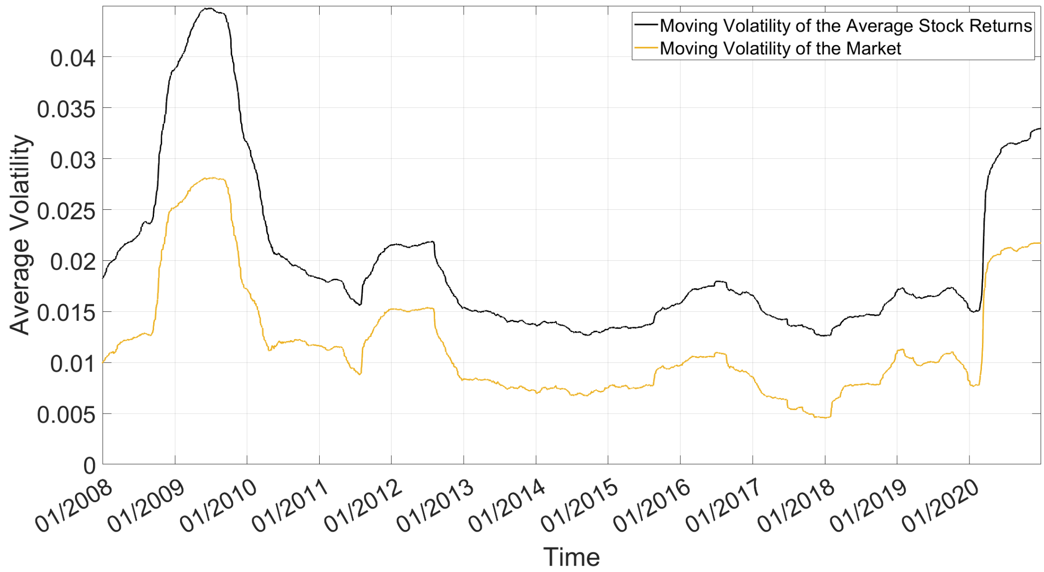

Figure 2: A plot showing the volatility clustering of the average stocks and market index for a one-year (assumption: 252 days) rolling window.

shrinkage from Ledoit & Wolf (2003a), nonlinear shrinkage from Ledoit & Wolf (2020a) and the Exponentially Weighted Moving Average from Morgan et al. (1996).

We also elaborate on the performance evaluation of the methods. We use the following performance measures: the mean return, the Sharpe ratio from Sharpe (1966), the certainty equivalent, the turnover and the transaction costs adjusted mean return. We further introduce testing for significance and portfolio composition.

4.1 Linear Shrinkage

The linear shrinkage method from Ledoit and Wolf is frequently used as it is very efficient in estimating the covariance matrix. LS shrinks the unbiased but variable sample covariance matrix towards a biased but less variable target matrix. One of such targets is the constant correlation matrix from Ledoit & Wolf (2003a). We estimate the covariance matrix as follows:

$$\hat{\mathbf{\Sigma}}_{LS} = (1 - \hat{\delta}^*)\boldsymbol{S} + \hat{\delta}^*\boldsymbol{F}, \tag{1}$$

where $\hat{\delta}^*$ denotes the estimated optimal shrinkage intensity. $\boldsymbol{S}$ is the sample covariance matrix and $\boldsymbol{F}$ denotes the shrinkage target matrix. The optimal shrinkage intensity $\hat{\delta}^*$ is mathematically derived in Ledoit & Wolf (2003a). This paper also provides the code[4] for our implementation of linear shrinkage.

The optimal shrinkage intensity $\hat{\delta}^*$ asymptotically behaves like a constant over the number of observations (T), as proven by Ledoit & Wolf (2003a). We calculate it as

[4]This code is available at: https://www.econ.uzh.ch/en/people/faculty/wolf/publications.html

follows:

$$\begin{aligned}\hat{\delta}^* &= \max\Big\{0, \min\big\{\frac{\hat{\kappa}}{T}, 1\big\}\Big\},\\ \hat{\kappa} &= \frac{\hat{\pi} - \hat{\psi}}{\hat{\gamma}},\end{aligned} \tag{2}$$

where $\frac{\hat{\kappa}}{T}$ denotes the estimated shrinkage intensity in practise. $\hat{\pi}$ is the estimated sum of asymptotic variances of the sample covariance matrix, $\hat{\psi}$ is the estimated sum of asymptotic covariances of the shrinkage target with the covariances of the sample covariance matrix and $\hat{\gamma}$ estimates the offset of the shrinkage target.

We compute these three estimators in the following way:

$$\begin{aligned}\hat{\pi} &= \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\pi}_{ij} \quad \text{where} \quad \hat{\pi}_{ij} = \frac{1}{T}\sum_{t=1}^{T}\Big\{(r_{it} - \bar{r_i})(r_{jt} - \bar{r_j}) - s_{ij}\Big\}^2,\\ \hat{\psi} &= \sum_{i=1}^{N}\hat{\pi}_{ii} + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{\bar{\rho}}{2}\Big(\sqrt{\frac{s_{jj}}{s_{ii}}}\theta_{ii,ij} + \sqrt{\frac{s_{ii}}{s_{jj}}}\theta_{jj,ij}\Big),\\ \theta_{ii,ij} &= \frac{1}{T}\sum_{t=1}^{T}\Big\{(r_{jt} - \bar{r_j})^2 - s_{jj}\Big\}\Big\{(r_{it} - \bar{r_i})(r_{jt} - \bar{r_j}) - s_{ij}\Big\},\\ \hat{\gamma} &= \sum_{i=1}^{N}\sum_{j=1}^{N}(f_{ij} - s_{ij})^2,\end{aligned} \tag{3}$$

note that this is a brief overview of the important equations from the mathematical derivation of the optimal shrinkage intensity. The full derivation can be observed in Ledoit & Wolf (2003a).

The shrinkage target matrix, $\boldsymbol{F}$, consists of diagonal elements equal to the sample variances and off-diagonal elements, which are all equal to the average of the off-diagonal elements of the sample correlation matrix. We estimate this matrix via:

$$\begin{aligned}f_{ii} &= s_{ii} \quad \forall i = 1, \ldots, N,\\ f_{ij} &= \bar{\rho}\sqrt{s_{ii}s_{jj}} \quad \forall i, j = 1, \ldots, N, \quad i \neq j,\end{aligned} \tag{4}$$

where s_{ii}, s_{jj} denotes a diagonal element of the sample covariance matrix. And where in the latter formula $\overline{\rho}$ denotes the average sample correlation, which we calculate as:

$$\begin{aligned}\bar{\rho} &= \frac{2}{(N-1)N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_{ij},\\ \rho_{ij} &= \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}}.\end{aligned} \tag{5}$$

4.2 Nonlinear Shrinkage

Nonlinear shrinkage is an extension to the linear shrinkage methods. The core difference between these methods is that LS uses the same shrinkage intensity for all elements, whereas NLS uses a custom-fit intensity for each element. We use the analytical nonlinear eigenvalue shrinkage from Ledoit & Wolf (2020a).

The first step of this NLS approach is to compute the spectral decomposition of the sample covariance matrix in the following way:

$$\boldsymbol{S} = \sum_{i=1}^{N} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i', \tag{6}$$

where λ_i denotes an eigenvalue of the sample covariance matrix, sorted in increasing order, and where u_i is the corresponding eigenvector.

In the second step, we estimate the sample eigenvalues via the unobservable quantity $\hat{d}_i$ in the following way:

$$\hat{d}_i = \frac{\lambda_i}{[\pi \frac{N}{T} \lambda_i \hat{f}(\lambda_i)]^2 + [1 - \frac{N}{T} - \pi \frac{N}{T} \lambda_i H_{\hat{f}}(\lambda_i)]^2} \quad \forall i = 1, ..., N, \tag{7}$$

where we observe two mathematical tools, $\hat{f}(\lambda_i)$ denotes the Epanechnikov kernel and $H_{\hat{f}}(\lambda_i)$ is its Hilbert transform.

The density of the spectral decomposition is required as $\hat{d}_i$ is unobserved. This unobserved quantity can be estimated with the Epanechnikov kernel by Epanechnikov (1969). We calculate this as follows:

$$\hat{f}(\lambda_i) = \frac{1}{N} \sum_{j=1}^{N} \frac{3}{4\sqrt{5}\lambda_j T^{-1/3}} [1 - \frac{1}{5}(\frac{\lambda_i - \lambda_j}{\lambda_j T^{-1/3}})^2]^+, \tag{8}$$

Next, the other mathematical tool is the Hilbert transform. This tool operates like a local attraction force. Eigenvalues that are further from others are adjusted more than eigenvalues that are close to other eigenvalues. Therefore, local shrinkage intensities are imposed. We apply the Hilbert transform in the following way:

$$H_{\hat{f}}(\lambda_i) = \frac{1}{N} \sum_{j=1}^{N} \Big\{ - \frac{3(\lambda_i - \lambda_j)}{10\pi \lambda_j^2 T^{-2/3}} + \frac{3}{4\sqrt{5}\lambda_j T^{-1/3}} [1 - \frac{1}{5}(\frac{\lambda_i - \lambda_j}{\lambda_j T^{-1/3}})^2] \times log \Big| \frac{\sqrt{5}\lambda_j T^{-1/3} - \lambda_i + \lambda_j}{\sqrt{5}\lambda_j T^{-1/3} + \lambda_i - \lambda_j} \Big| \Big\}. \tag{9}$$

The estimation of the sample eigenvalues might be tough to understand based on equation 7, mainly because this formula also contains the two mathematical tools. In their paper, Ledoit and Wolf provide a more intuitive explanation. When the number of observations goes to infinity, N/T becomes very small. As a result, the denominator in the equation tends to 1. Such that the estimated eigenvalues are very similar to the sample eigenvalues. This corresponds to using the the ML estimator as an estimate of the sample covariance matrix, which is optimal for a large T.

When this ratio is not small, shrinkage should be applied to the eigenvalues. We focus on one eigenvalue λ_i. If its value is smaller than the ones in the neighbourhood, then $H_{\hat{f}}(\lambda_i)$ will be positive, and the eigenvalue will be pushed upwards towards the neighbours. When an eigenvalue is larger than the ones in the neighbourhood, it will be pulled downwards. The last property of this Hilbert transformation is that eigenvalues that are further apart from the group are pushed or pulled more heavily.

In the third and final step, we recompose the covariance matrix estimator:

$$\hat{\boldsymbol{\Sigma}}_{NLS} = \sum_{i=1}^{N} \hat{d}_i \boldsymbol{u}_i \boldsymbol{u}_i'. \tag{10}$$

For our implementation of this analytical nonlinear shrinkage method, we use the code[5] provided by Ledoit & Wolf (2020a).

4.3 Exponentially Weighted Moving Average

The exponentially weighted moving average (EWMA) method is a very simple covariance estimator that incorporates dynamic characteristics in the data. In essence, the method uses all previous estimates of the covariance matrix to estimate the next one at time t. The ones closer to t are valued more heavily, whereas the ones further away from t only give a small influence on tomorrows prediction. We calculate each covariance matrix as follows:

$$\hat{\boldsymbol{\Sigma}}_{EWMA,t} = \phi \hat{\boldsymbol{\Sigma}}_{EWMA,t-1} + (1-\phi)(\boldsymbol{r}_{t-1} - \boldsymbol{\mu}_r)'(\boldsymbol{r}_{t-1} - \boldsymbol{\mu}_r), \tag{11}$$

where $\boldsymbol{r}_{t-1}$ denotes the vector of returns at time $t-1$ and $\boldsymbol{\mu}_r$ is the mean vector of the historical dataset. Further, ϕ denotes the decay factor and will be fixed at 0.94 as Morgan et al. (1996) find that this value is optimal in the RiskMetrics approach. We start this method with the unconditional covariance matrix of the historical dataset. Our implementation of the method will slightly differ from the literature because we use the previous W returns instead of all previous observations, where W corresponds to the size of the moving window. As a result, we obtain estimates of covariance matrices from time $t-W$ to t. We use the estimate of the covariance matrix at time t to compute the portfolio allocation.

To increase the computation time in the simulation and empirical analysis, one could look at the effective number of days used for the EWMA method. Morgan et al. (1996) derive the following formula to determine this number:

[5] This code is available at: https://www.econ.uzh.ch/en/people/faculty/wolf/publications.html

$$K = \frac{ln(\zeta)}{ln(\phi)}, \tag{12}$$

where K is the effective number of days, ζ is the tolerance level which we fix at 0.001% and where ϕ is the decay factor. The resulting number of effective days is 186. Therefore, we will be using the previous 186 days in the moving windows that exceed this number.

For our implementation of the exponentially weighted moving average we use the RiskMetrics function in the MFE toolbox from Kevin Sheppard[6].

4.4 Quarticity Shrinkage

Quarticity shrinkage (QS) combines the $1/N$ allocation with an existing covariance estimator to overcome the uncertainty with regards to estimating the covariance matrix. We obtain estimates of QS via the following three-step approach:

The first step is to assess the uncertainty of the individual stocks via the RQ of the daily excess returns. This computation is done as follows:

$$RQ^i_{t+W} = \frac{W}{3}\sum_{t}^{t+W} r^4_{i,t} \quad \forall i = 1, ..., N, \tag{13}$$

where RQ^i_{t+W} and $r_{i,t}$ denote the Realized Quarticity of stock i over the estimation period and the excess stock return at time t. Further, W is the size of the rolling window. There is no clear value to determine when RQ is considered high or low. So, we will try to find the optimal partition ratio in a simulation study. We elaborate further on this approach in Section 4.4.1. This step leads to two sets of stocks: the first set of certain stocks and the second set of uncertain stocks.

In the second step, we use one of the existing covariance estimators and the global minimum variance portfolio allocation for the certain stocks. This leads to the first set of allocated stocks. We use the equally weighted portfolio for the uncertain stocks to obtain the second set of allocated stocks. When we consider half of the stocks as certain and the other half as uncertain, we obtain the following two portfolios:

$$\begin{aligned} \boldsymbol{\omega}_1 &= \frac{\hat{\boldsymbol{\Sigma}}^{-1}_{1,t}\boldsymbol{\iota}}{\boldsymbol{\iota}'\hat{\boldsymbol{\Sigma}}^{-1}_{1,t}\boldsymbol{\iota}}, \\ \boldsymbol{\omega}_{2,i} &= 1/N \quad \forall i = 1, ..., N/2, \end{aligned} \tag{14}$$

where $\boldsymbol{\omega}_1$ denote the portfolios of certain and $\boldsymbol{\omega}_2$ the portfolio of uncertain stocks. Further, $\hat{\boldsymbol{\Sigma}}^{-1}_{1,t}$ denotes the estimated covariance matrix via the existing estimators using the certain stocks. N and $\boldsymbol{\iota}$ are the number of stocks and a vector of ones, respectively.

In the final step, we simply merge the two sets to obtain one portfolio allocation of

[6] This code is available at: https://www.kevinsheppard.com/code/matlab/mfe-toolbox/

this new method. Which is then normalized such that the weights sum up to one, this is done in the following way:

$$\boldsymbol{w}_{allocation} = \frac{\boldsymbol{w}_{combined}}{\boldsymbol{\iota}'\boldsymbol{w}_{combined}}, \tag{15}$$

where $\boldsymbol{w}_{allocation}$ denotes the portfolio allocation estimated via RQ. $\boldsymbol{w}_{combined}$ is the weight allocation obtained by merging the two sets of allocated stocks.

4.4.1 Optimal Partition

We use our simulation to try and find an optimal structure because there is no explicit value when RQ is considered high or low. In each simulation study, we vary the number of stocks that follow the covariance estimator or $1/N$ allocation by fractions of 0.20. For the simulation with 25 stocks, we have the following partition ratios of certain/uncertain stocks:

- Pf_1 consists of 5 certain and 20 uncertain stocks
- Pf_2 consists of 10 certain and 15 uncertain stocks
- Pf_3 consists of 15 certain and 10 uncertain stocks
- Pf_4 consists of 20 certain and 5 uncertain stocks

where Pf_j denotes the jth portfolio from the new method. We evaluate these portfolios against one other. If a consistent pattern of outperformance is found, then we implement this structure in the new method for the corresponding study. If we find no consistent partition ratio, then we will try to find a middle ground. Say that either the ratio 5/20 or 15/10 yields the best performance in most of the measures, then we will implement the structure of 10/15. As it is likely that this ratio will be the runner up in all of the measures, such that in general it performs well. Now if we also cannot find a middle ground, then we simply use the first half of the stocks as certain, and deem the second half as uncertain.

We note that this approach looks very simplistic, yet it is an effective way. One could argue to program a grid search at each observation and pick the partition that gives the best result on that day. However, it is rather unlikely that a partition will outperform the others in all measures. Further, as no clear ranking exists for the performance measures, it is impossible to use a points system to determine which partition receives the most points.

4.5 Performance Evaluation

We present the performance measures on an annual basis, such that we scale the measures with a factor of the number of days. We assume that one year consists of 252 trading days.

4.5.1 Mean Return

A very simplistic but effective measure is the mean return of the portfolio. From a portfolio optimization perspective, maximizing the mean return is one of the main goals. We calculate it as follows:

$$\mu_p = \frac{\sum_{t=1}^{T} \sum_{i=1}^{N} r_t^i}{TN} \tag{16}$$

where μ_p is the mean return of the portfolio and r_t^i is the return of stock i at day t. To annualize this measure, we multiply it with 252.

4.5.2 Sharpe Ratio

The Sharpe ratio from Sharpe (1966) yields the average return per unit of volatility, which we calculate as follows:

$$Sh = \frac{\mu_p}{\sigma_p}, \tag{17}$$

where Sh is the Sharpe ratio, μ_p is the mean excess return of the portfolio (note that this quantity is the average of the portfolio returns throughout time), and σ_p is the corresponding mean volatility of the portfolio returns. We annualize this measure by multiplying it with the square root of 252.

4.5.3 Certainty Equivalent

The certainty equivalent (CEQ) return can be interpreted as the lowest risk-free rate that an investor is willing to accept instead of the risky return of the portfolio. We calculate this measures via:

$$CEQ = \mu_p - \frac{\gamma}{2}\sigma_p^2, \tag{18}$$

where CEQ denotes the certainty equivalent measure. The parameter γ is the coefficient of risk aversion, which we fix to a value of $\gamma = 1$ and $\gamma = 5$. The first value corresponds to an investor who neglects risk, whereas the latter corresponds to an investor who takes risk into account. We annualize this measure by multiplying it by 252.

4.5.4 Transaction Costs Adjusted Mean Return

From another practical perspective, we evaluate the methods in terms of the transaction costs adjusted mean return. We obtain this measure by making an assumption about the costs and calculating the turnover of the portfolio. The turnover quantifies the fraction of wealth that is re-allocated at each rebalancing moment. We compute is as follows:

$$TO = \frac{1}{T-W}\Sigma_{t=W+1}^{T}\Sigma_{j=1}^{N}(|\omega_{k,j,t+1} - \omega_{k,j,t+}|), \tag{19}$$

where TO denotes the turnover value, $\omega_{k,j,t+1}$ the desired weights and $\omega_{k,j,t+}$ the real weights right before rebalancing in $t+1$. T refers to the total number of observations and W is the size of the rolling window.

We assume that the transaction costs are a fixed fraction of the turnover with a value of either 4 or 8.6 basis points, as done in Marshall et al. (2011). Such that we calculate the transaction costs via:

$$TC = c \cdot \text{TO}, \tag{20}$$

where TC and c denote the transaction costs of the method and the fraction of costs, respectively. Here, c is either 0.00086 or 0.00040. We annualize this measure by multiplying it with 252. Then we simply subtract this number from the mean return to get the transaction costs adjusted mean return (TCAR).

4.5.5 Testing for Significance

We will test whether the performance of the new method differs significantly from the two given inputs. In the sense that for the new method with input LS (QS-LS), we will conduct two tests. One whether the performance differs significantly from the $1/N$ strategy and another independent test for the difference with the LS estimator.

To test whether the mean returns are significantly different from one other, we will be using the t-test. The null hypothesis is a pairwise difference between the means, which is equal to zero. Or in other words, we will conduct the two following tests: $H_0 : \mu_{1/N} = \mu_{QS\text{-}LS}$ and $H_0 : \mu_{LS} = \mu_{QS\text{-}LS}$. Where QS-LS is the new method with LS as input. The data contains autocorrelation and heteroskedasticity such that we adjust the standard errors to follow the Newey-West estimates. It is slightly unconventional to use these estimates on a series of returns instead of a regression. However, we solve this by regressing the series of returns on the constant. The constant depicts the mean, and we can obtain the Newey-West standard errors via the standard procedure.

To evaluate the significance for the Sharpe ratios, we use the Sharpe Ratio test from Ledoit & Wolf (2008)[7]. We use the Quadratic Spectral kernel, and we base the p-values on the pre-Whitened HAC standard errors. We test the null hypothesis in a similar fashion to the tests for the means.

We conduct all tests at significance levels of $\alpha = 1\%$, $\alpha = 5\%$ and $\alpha = 10\%$.

[7]This code is available at: https://www.econ.uzh.ch/en/people/faculty/wolf/publications.html

4.5.6 Portfolio Composition

We evaluate the performance of the existing covariance estimators via the global minimum variance (GMV) portfolio. This portfolio achieves the lowest risk of the portfolios based on the mean-variance framework from Markowitz (1952). We obtain the allocation by solving the following optimization problem:

$$\begin{aligned} \min \quad & \hat{\boldsymbol{\omega}}_{\boldsymbol{t}}{}' \hat{\boldsymbol{\Sigma}}_{\boldsymbol{t}} \hat{\boldsymbol{\omega}}, \\ \text{s.t.} \quad & \hat{\boldsymbol{\omega}}_{\boldsymbol{t}}{}' \boldsymbol{\iota} = 1, \end{aligned} \tag{21}$$

where $\hat{\boldsymbol{\omega}}_{\boldsymbol{t}}$ contains the estimated portfolio weights via the covariance estimator and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{t}}$ is the corresponding covariance matrix. The parameter $\boldsymbol{\iota}$ denotes a vector of ones. The solution to this optimisation problem is the GMV portfolio, which can be estimated as:

$$\boldsymbol{\omega}_{gmv} = \frac{\hat{\boldsymbol{\Sigma}}_{\boldsymbol{t}}^{-1} \boldsymbol{\iota}}{\boldsymbol{\iota}' \hat{\boldsymbol{\Sigma}}_{\boldsymbol{t}}^{-1} \boldsymbol{\iota}}. \tag{22}$$

with the obtained weights we can get the portfolio return as follows:

$$r_{p,t} = \boldsymbol{w}'_{gmv} \boldsymbol{r}_t, \tag{23}$$

where $r_{p,t}$ is the expected portfolio return at time t and $\boldsymbol{r_t}$ is the estimated vector of stock returns at time t.

5 Simulation

Using a simulation enables us to compute standard deviations of the performance measures to evaluate the robustness of the methods. We use the Monte Carlo (MC) method in combination with bootstrapping to simulate 1000 replications of $N = \{25, 100, 250\}$ stock returns. The MC method is straightforward and generates possible returns from an underlying theoretical distribution. We assume noise with an underlying Student's t-distribution with 6 degrees of freedom. The bootstrapping is useful as resampling with replacement maintains the structure of the dataset. We will further apply a burn-in period of one year to try and overcome imperfections at the start of the generated returns. So, we will generate an additional year of daily excess stock returns.

5.1 Data Generating Process

To resemble daily stock returns, we use a data generating process (DGP) similar to the principle used by Jegadeesh et al. (2019). Our paper extends their idea to resemble daily returns by introducing time-varying volatility, whereas their volatility is static. We draw a factor, a corresponding sensitivity and an idiosyncratic return and combine these to obtain the stock return.

We use the daily excess holding-period returns and the excess value-weighted market index from the Fama-French library. We then regress the data without using a rolling window on the factor and a constant to obtain estimates for the alpha, beta and idiosyncratic returns. We conduct this regression according to the single-factor model or CAPM from Sharpe (1964) and Lintner (1965):

$$r_t^i = \alpha^i + \beta_{MKT}^i r_{MKT,t} + \varepsilon_t^i \tag{24}$$

where $\boldsymbol{r}^i$ denotes a vector with excess returns of stock i. The parameters α^i and β_{MKT}^i denote the mispricing and the sensitivity to the market factor, respectively. Further, $\boldsymbol{r}_{MKT,t}$ denotes the market index at time t. And ε_t^i denotes the regression residual of stock i at time t, this value can be interpreted as the idiosyncratic return.

Next, we use the bootstrapping method for each simulation study to draw N stock indices with replacement. We gather the corresponding regression parameters with these obtained indices: α and β_j. We further collect the vectors of idiosyncratic returns that correspond to these stocks.

We fit a GARCH(1,1) model to the idiosyncratic returns and the market, as the real datasets contain significant autocorrelation. Then we generate idiosyncratic returns for stock i according to the fitted model, where we start with the unconditional variance:

$$\begin{aligned} \hat{\varepsilon}_t^i &= \mu^i + \sqrt{h_t^i} z_t, \quad \text{where} \quad z_t \sim t(6), \\ h_t^i &= \omega^i + \alpha^i h_{t-1}^i z_{t-1}^2 + \beta^i h_{t-1}^i, \end{aligned} \tag{25}$$

where μ^i, ω^i, α^i and β^i are parameters from the fitted GARCH(1,1) model for stock i, denoting the offset, constant, ARCH and GARCH parameter, respectively. Further, ε_t^i is the generated daily idiosyncratic return for stock i and h_t^i denotes the conditional variance. We obtain a series of generated idiosyncratic returns by computing these equations recursively. The procedure to generate the market index is very similar, but uses a different fitted GARCH(1,1) model and unconditional variance of the historical market index.

In the last step, we compute the generated daily excess return of stock i as follows:

$$r_t^i = \hat{\alpha}^i + \hat{\beta}_{MKT}^i \hat{r}_{MKT,t} + \hat{\varepsilon}_t^i, \tag{26}$$

where $\hat{\alpha}^i$ and $\hat{\beta}_{MKT}^i$ are the estimated regression parameters for stock i from Equation 24. Further, $\hat{r}_{MKT,t}$ and $\hat{\varepsilon}_t^i$ are the generated market index and idiosyncratic return via Equation 25.

Note that we incorporate the correlation between stocks explicitly via the generated market index.

5.2 Generated Data

Following the procedure for the DGP, we obtain three sets of generated data, each with a different number of stocks. We display the tables with descriptive statistics and plots with autocorrelation functions in the appendix, Section C. We briefly go over the characteristics of the generated dataset and how those are related to the empirical dataset.

We find that the generated datasets' minimum, mean, and maximum return are all way higher than that of the empirical set. This phenomenon makes sense as the estimated α in Equation 24 and market return are often positive, such that the generated return becomes even larger. From Figure 3 we observe that the cumulative generated returns increase more rapidly than the empirical set, while they do not seem to contain any crashes. The cumulative returns further support the large (mean) returns in the generated datasets.

Figure 4 displays that the volatility of the average generated returns seems to fluctuate barely. However, the average volatility of the set with 250 generated stocks is of a larger order than those from the other simulated sets. Therefore, it causes a slight fluctuation in the graph due to the scaling of the vertical axis. Once we look into the average moving volatility of each generated dataset separately, the volatility clustering becomes directly visible. Figure 5 shows the fluctuation of the average moving volatility for the generated dataset with 25 stocks. We display the plots showing the volatility clustering for the other two generated datasets in the appendix, Section C.

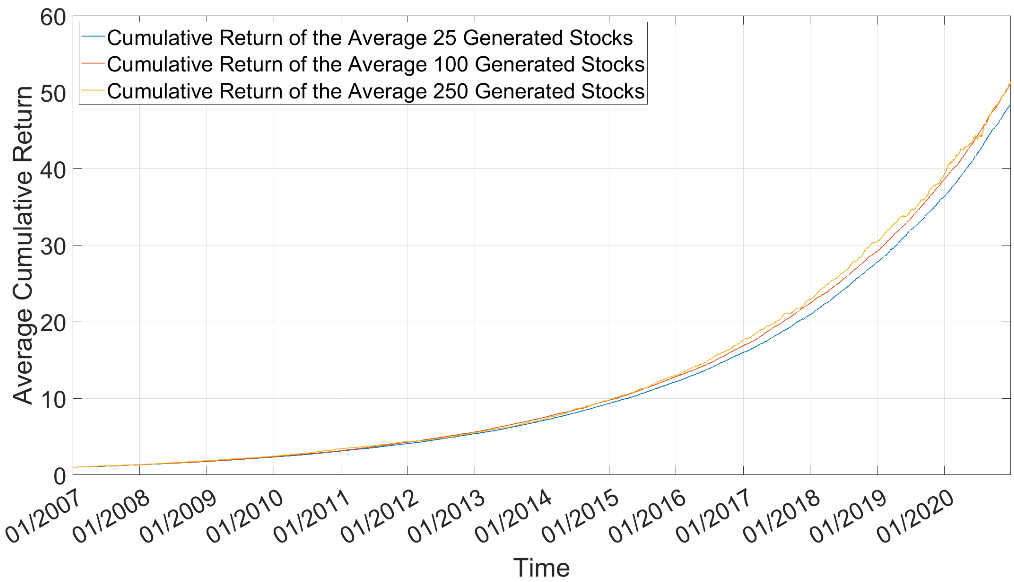

Figure 3: A plot showing the cumulative returns of the average generated datasets with a starting value of 1.

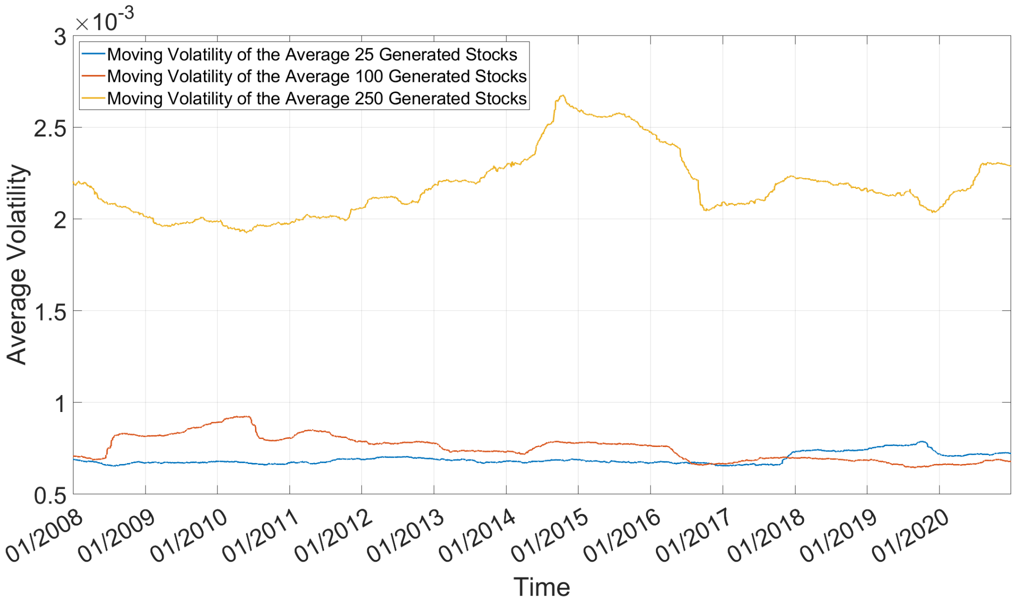

Figure 4: A plot displaying the volatility clustering of the average generated datasets for a one-year (assumption: 252 days) rolling window.

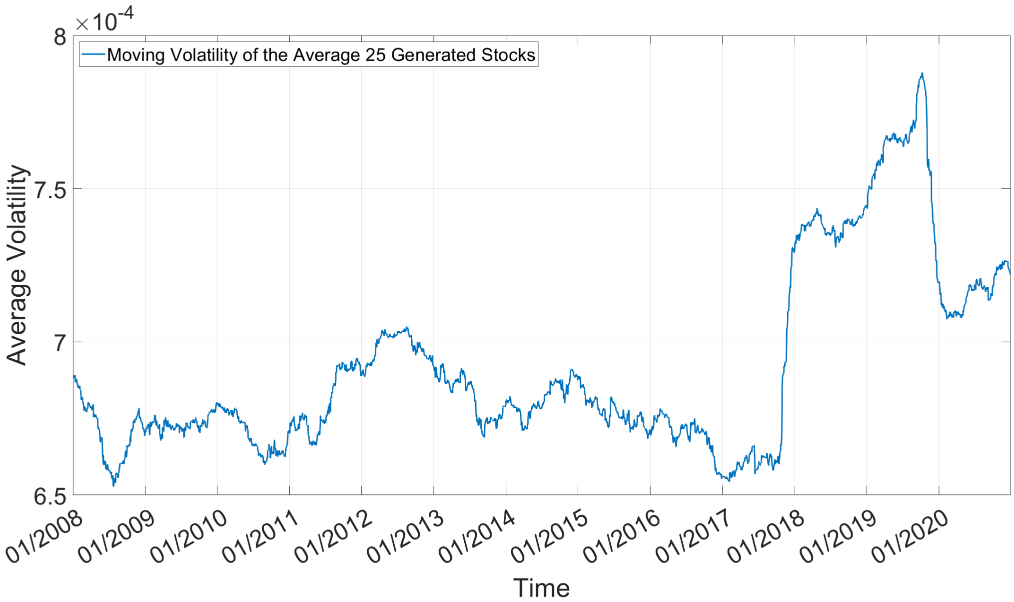

Figure 5: A plot showing the volatility clustering of the average 25 generated returns for a one-year (assumption: 252 days) rolling window.

5.3 Calibrating Quarticity Shrinkage

We start by evaluating the performance of quarticity shrinkage with different ratios of the number of stocks that should follow a certain/uncertain method. We use the generated stock data of $N = \{25, 100, 250\}$ stocks, $T = 3777$ (minus 252 burn-in) days.

We first take a look at the optimal partition of 25 stocks. Table 2 displays the average annualized performance measures for each partition ratio with LS as input, where we average all values over the replications. From this table, it becomes clear that no partition will yield the best performance in all measures for QS regardless of the size of the estimation window or input method. Therefore, in the simulation study with 25 generated stocks, we use a ratio of 12 certain stocks and 13 uncertain stocks. We observe that the two extreme partitions yield the best results. When an investor is purely interested in maximizing the return, the ratio of 5/20 (certain/uncertain) stocks suits best. This ratio yields the largest mean return, certainty equivalent with a small coefficient of risk aversion and largest transaction costs adjusted mean return. When an investor takes risk into account, the ratio of 20/5 (certain/uncertain) stocks fits best. This ratio attains the highest Sharpe ratio and certainty equivalent with a large coefficient of risk aversion.

Surprisingly, the performance in terms of the mean return, Sharpe ratio, and certainty equivalences for both QS-LS and QS-NLS methods worsen once the estimation window increases. Especially for QS-NLS, one would expect that a larger window increases the performance as NLS uses more observations to determine the nonlinear structure in the data. The generated returns change a lot over time. Therefore, only recent returns contain relevant information on tomorrows prediction. So, the estimates will not be accurate when the method uses older observations. As both LS and NLS value past observations equally, they are affected heavily by this phenomenon. For QS-EWMA, the mean return stays relatively flat over the windows. The Sharpe ratio drops for bigger estimation windows. This drop is much smaller than the performance drops in LS and NLS, as the dynamic EWMA method uses an exponential weighting function. In the sense that observations far in the past already have a negligible influence, whereas the most recent observation has the most influence.

The transaction costs adjusted mean returns of all methods improves upon using a larger window. This improvement is directly linked to the turnover, which declines rapidly once the window increases. We explain this decline by the influence of new observations for the estimation of the ex-ante covariance matrix. Using more observations to estimate this matrix implies that the most recent observation has less influence on the estimated covariance matrix. This statement is particularly true for LS and NLS, as each past observation has an equal influence. Again, EWMA is affected less heavily by larger windows, as this method can account for the underlying structure in the generated dataset.

Next, we look into the optimal partition of 100 stocks. In general, we observe a very similar pattern to the partition with 25 stocks. So, we will consider half of the stocks as certain and the other half as uncertain when using QS-LS and QS-NLS. Table 3 displays the average annualized performance measures for each partition ratio with EWMA as input; again, we average the values over the replications. We find that QS-EWMA and the 20/80 certain/uncertain stocks ratio sometimes yields the best performance in terms of the certainty equivalent with a large coefficient of risk aversion. So, an investor who is purely interested in getting a high return will always go for the 20/80 ratio. An investor who takes risk into account might also go for this ratio, especially in the largest window where the difference in Sharpe ratio is small compared to the 80/20 ratio. Consequently, we will use the 20/80 ratio for EWMA in the windows from 126 days and up. For the window of 63 days, we still follow the approach where half of the stocks is considered certain/uncertain.

We explain the outperformance of the 20/80 ratio as follows. We observe that the return stays relatively flat while the Sharpe ratio declines, meaning that a larger volatility comes into place in the windows from 126 days. EWMA with a smaller moving window can adapt quicker to dynamic data than the same method with a longer moving window. Using more past irrelevant observations causes a disturbance in the estimation process. Once the window increases even further to 252 days and 756 days, we note that the volatility increases slightly. This slight increase also makes sense, as the effective number of days is 186. So, the longer windows do add more observations in the estimation, but their influence is negligible.

So, we argue that the volatility of the EWMA method increases with longer estimation windows, whereas the volatility of 1/N stays relatively flat. As the 80/20 ratio has more stocks that follow EWMA, its volatility increases more than the 20/80 ratio. Meanwhile, its mean stays roughly the same. Consequently, the 80/20 ratio loses its advantage, and the certainty equivalent with a small risk aversion coefficient is in favour of the 20/80 ratio for a larger moving window.

Further, we find a peculiar result for QS-NLS, a window of 63 days and a ratio of 60/40 certain/uncertain stocks. Table 4 shows the average annualized performance measures of 100 generated stocks for all ratios with NLS as input. We observe that the turnover suddenly spikes to a large value, and the return is larger than both surrounding ratios. Further, the Sharpe ratio and certainty equivalents attain the lowest values for this ratio. We do not find a similar discrepancy in any of the longer windows or other simulation studies. We find that the weight allocation differs from the surrounding ratios at particular points in time. The minimum (maximum) weights assigned by the 60/40 ratio are roughly three (1.5) times smaller (larger). The minimum (maximum) of the realized weights are four (three) times more negative (more positive). Furthermore, we find that the mean of the weights and realized weights are more moderate than the surrounding ratios. These findings imply that the turnover of 60/40 is way larger than the surrounding ratios in certain replications. The 20 additional stocks greatly influence the weight allocation when the method only uses 63 days. Upon adding another 20 stocks, this strange performance disappears. This sudden outlier is remarkable, but we have seen the major influence of a small number of stocks in the optimal partition with 25 stocks.

Succeeding, we investigate the partition with 250 stocks. We do not observe many differences for QS-LS, such that we use the ratio of 125/125 certain/uncertain stocks. Table 5 shows the average annualized performance measures for each partition ratio of

250 stocks with NLS as input. We observe that the 200/50 ratio seems to outperform in the window with 63 days. This outperformance might not come as a surprise as we know from Ledoit & Wolf (2020a) that the analytical nonlinear shrinkage estimator performs optimally when the number of assets is larger than the number of observations. We note that this outperformance tends to fade away the longer the estimation window, which the increased number of observations might cause. Another explanation for the outperformance might be the underlying dynamic character of the generated data. Longer windows suffer more severely from volatility clustering. The estimates of NLS will be more inaccurate, so the ratio of 200/50 obtains worse estimates as this ratio considers many stocks as certain.

The values for the turnover of QS-NLS are somewhat surprising. We observe that the ratio of 50/200 yields the largest turnover, while 200/50 attains the smallest turnover in the window of 63 days. We expect it the other way around, as the 1/N allocation contains no turnover. We find differences in the minimum and maximum allocated weights over time. The mean allocated weights are similar for both ratios and the average minimum realized weight is slightly smaller for 200/50. However, the average maximum realized weight is almost two times as large for 50/200. We note a huge difference when we look into the minimum and maximum realized weights across all replications. The minimum (maximum) realized weight of ratio of 50/200 is five (six) times more negative (positive) compared to the ratio of 200/50. These findings explain the increased turnover for the ratio of 50/200 in comparison to 200/50. We conclude that 50 stocks cause a large turnover in order to obtain the lowest volatility possible with the GMV allocation. For this method, we use a ratio of 200/50 certain/uncertain stocks for the window of 63 days. The other estimation windows will follow the 125/125 ratio.

Last, we investigate the new method with EWMA in the study with 250 stocks. We find an increasing turnover with an increasing estimation window, which we explain via the rapidly increasing volatility when more stocks follow the EWMA approach. We conclude from the large volatilities that EWMA gives inaccurate estimates of the covariance matrix in this study. We use the 125/125 ratio for QS-EWMA in the moving windows of 63 and 126 days. We use a ratio of 50/200 for the windows of 252 and 756 days as we find that this ratio dominates in the larger moving windows.

	W = 63				W = 126			
	5/20	10/15	15/10	20/5	5/20	10/15	15/10	20/5
Mean Return	**0.257**	0.250	0.242	0.233	**0.256**	0.249	0.240	0.230
Sh	1.347	1.414	1.514	**1.696**	1.324	1.380	1.474	**1.656**
CEQ ($\gamma = 1$)	**0.239**	0.235	0.229	0.224	**0.237**	0.232	0.227	0.220
CEQ ($\gamma = 5$)	0.166	0.172	0.178	**0.186**	0.163	0.167	0.174	**0.182**
TO	**0.053**	0.085	0.116	0.134	**0.028**	0.047	0.065	0.076
TCAR (4 bps)	**0.251**	0.242	0.230	0.220	**0.253**	0.244	0.233	0.222
TCAR (8.6 bps)	**0.245**	0.232	0.217	0.204	**0.250**	0.238	0.226	0.213
	W = 252				**W = 756**			
Mean Return	**0.254**	0.246	0.236	0.226	**0.252**	0.242	0.232	0.221
Sh	1.297	1.337	1.421	**1.600**	1.265	1.285	1.356	**1.521**
CEQ ($\gamma = 1$)	**0.235**	0.229	0.222	0.216	**0.232**	0.224	0.217	0.211
CEQ ($\gamma = 5$)	0.158	0.161	0.167	**0.176**	0.153	0.153	0.159	**0.168**
TO	**0.014**	0.025	0.035	0.043	**0.005**	0.009	0.013	0.016
TCAR (4 bps)	**0.253**	0.243	0.232	0.222	**0.251**	0.241	0.230	0.220
TCAR (8.6 bps)	**0.251**	0.240	0.228	0.217	**0.251**	0.240	0.229	0.218

Table 2: This table shows the performance measures for the optimal partition of QS-LS with 25 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO).

	W = 63				W = 126			
	20/80	40/60	60/40	80/20	20/80	40/60	60/40	80/20
Mean Return	**0.230**	0.218	0.211	0.205	**0.232**	0.220	0.213	0.207
Sh	1.433	1.475	1.610	**1.943**	1.435	1.451	1.489	**1.544**
CEQ ($\gamma = 1$)	**0.217**	0.207	0.203	0.199	**0.219**	0.208	0.203	0.198
CEQ ($\gamma = 5$)	0.166	0.163	0.168	**0.177**	**0.167**	0.162	0.162	0.162
TO	**0.452**	0.772	0.884	0.778	**0.389**	0.711	0.989	1.191
TCAR (4 bps)	**0.185**	0.140	0.122	0.126	**0.193**	0.148	0.113	0.087
TCAR (8.6 bps)	**0.132**	0.051	0.020	0.036	**0.174**	0.066	-0.001	-0.051
	W = 252				**W = 756**			
Mean Return	**0.230**	0.219	0.212	0.206	**0.231**	0.219	0.210	0.205
Sh	1.416	1.425	1.457	**1.509**	1.452	1.441	1.452	**1.495**
CEQ ($\gamma = 1$)	**0.217**	0.207	0.202	0.197	**0.218**	0.207	0.200	0.196
CEQ ($\gamma = 5$)	**0.164**	0.160	0.159	0.160	**0.168**	0.161	0.158	0.158
TO	**0.344**	0.634	0.897	1.123	**0.303**	0.570	0.822	1.056
TCAR (4 bps)	**0.196**	0.155	0.122	0.093	**0.200**	0.161	0.127	0.098
TCAR (8.6 bps)	**0.156**	0.082	0.018	-0.069	**0.165**	0.095	0.032	-0.024

Table 3: This table shows the performance measures for the optimal partition of QS-EWMA with 100 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO).

	W = 63				W = 126			
	20/80	40/60	60/40	80/20	20/80	40/60	60/40	80/20
Mean Return	**0.237**	0.231	0.233	0.226	**0.235**	0.225	0.222	0.219
Sh	1.457	1.539	1.273	**2.017**	1.423	1.534	1.723	**2.077**
CEQ ($\gamma = 1$)	**0.225**	0.220	0.216	0.220	**0.221**	0.215	0.213	0.213
CEQ ($\gamma = 5$)	0.172	0.175	0.149	**0.195**	0.167	0.171	0.180	**0.191**
TO	**0.175**	0.296	1.476	0.367	**0.091**	0.129	0.161	0.207
TCAR (4 bps)	**0.221**	0.201	0.084	0.189	**0.226**	0.212	0.205	0.198
TCAR (8.6 bps)	**0.201**	0.166	-0.087	0.147	**0.215**	0.197	0.187	0.148
	W = 252				**W = 756**			
Mean Return	**0.232**	0.221	0.216	0.213	**0.227**	0.212	0.207	0.202
Sh	1.388	1.493	1.693	**2.164**	1.358	1.443	1.655	**2.114**
CEQ ($\gamma = 1$)	**0.218**	0.210	0.208	0.208	**0.213**	0.201	0.199	0.198
CEQ ($\gamma = 5$)	0.162	0.166	0.175	**0.189**	0.157	0.158	0.168	**0.180**
TO	**0.048**	0.068	0.080	0.089	**0.017**	0.025	0.030	0.034
TCAR (4 bps)	**0.227**	0.214	0.208	0.204	**0.225**	0.210	0.204	0.199
TCAR (8.6 bps)	**0.221**	0.206	0.198	0.194	**0.223**	0.207	0.201	0.195

Table 4: This table shows the performance measures for the optimal partition of QS-NLS with 100 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, we ran the window of 756 days for 500 replications instead of 1000 due to computation issues.

	W = 63				W = 126			
	50/200	100/150	150/100	200/50	50/200	100/150	150/100	200/50
Mean Return	**0.226**	0.221	0.222	0.222	**0.219**	0.213	0.214	0.214
Sh	1.490	1.623	1.890	**2.480**	1.447	1.566	1.790	**2.548**
CEQ ($\gamma = 1$)	0.215	0.212	0.215	**0.218**	0.207	0.203	0.207	**0.211**
CEQ ($\gamma = 5$)	0.169	0.175	0.188	**0.202**	0.162	0.167	0.178	**0.197**
TO	0.413	0.272	0.229	**0.214**	**0.149**	0.282	0.285	0.170
TCAR (4 bps)	0.184	0.194	0.199	**0.201**	**0.204**	0.184	0.186	0.197
TCAR (8.6 bps)	0.137	0.162	0.172	**0.176**	**0.186**	0.151	0.153	0.177
	W = 252				**W = 756**			
Mean Return	**0.215**	0.207	0.207	0.207	**0.210**	0.197	0.196	0.196
Sh	1.409	1.548	1.839	**2.368**	1.340	1.449	1.735	**2.435**
CEQ ($\gamma = 1$)	**0.203**	0.198	0.201	0.203	**0.197**	0.187	0.190	0.193
CEQ ($\gamma = 5$)	0.157	0.162	0.175	**0.188**	0.149	0.151	0.164	**0.180**
TO	**0.077**	0.094	0.118	0.183	**0.028**	0.035	0.040	0.043
TCAR (4 bps)	**0.207**	0.198	0.195	0.189	**0.207**	0.193	0.192	0.192
TCAR (8.6 bps)	**0.198**	0.187	0.181	0.134	**0.204**	0.189	0.187	0.148

Table 5: This table shows the performance measures for the optimal partition of QS-NLS with 250 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, we only ran the simulations with 250 generated stocks for 100 replications due to computation issues.

5.4 Performance of the Methods in the Simulation Studies

Table 6 and Table 7 display the average annualized performance measures of each method for 25 and 250 stocks, respectively. We average the measures over the replications. We show the results for the simulation with 100 stocks in the appendix, Section E.

We start by observing from the tables that no method seems to be superior. In general, quarticity shrinkage yields the highest Sharpe ratio and certainty equivalent with a large coefficient of risk aversion because this method attains the lowest volatility. All differences between the Sharpe ratio of QS with its corresponding counterparts are significant at every level. Further, QS generally yields the lowest mean return and certainty equivalent with a small risk aversion coefficient. This difference is also statistically significant at all levels.

Surprisingly, the mean return and Sharpe ratio obtained by QS do not fall neatly in between the ones attained by the $1/N$ allocation and the existing covariance estimator. In essence, the new method uses some form of shrinkage between the $1/N$ allocation and an existing covariance estimator. This phenomenon is especially remarkable giving the findings in the optimal partition earlier, Section 5.3. For instance, in the optimal partition study with 25 stocks, we do not find that both ratios of 20/5 and 5/20 certain/uncertain stocks give higher mean returns than the ratios in the middle.

This discrepancy gets even more complex once we take a closer look into the GMV allocation. When we use LS, all stocks follow the GMV allocation. When we use QS-LS, only a certain fraction of the stocks follow this allocation. The stocks that follow the GMV allocation for QS-LS are nested in those that use the GMV allocation for LS. Thus, we would expect that the existing method yields the lowest volatility. LS does attain the lowest volatility in a single replication. However, QS-LS yields the lowest volatility once we average over the replications. We further find the highest volatility in a single replication for LS. This finding implies that the estimation of the covariance matrix with LS, NLS and EWMA is not accurate in all replications. Therefore, the GMV portfolio does not give the lowest volatility possible. Instead, it gives large volatilities in these replications. On the contrary, $1/N$ always gives a moderate volatility. We conclude that the new method benefits from both characteristics in all replications, such that it yields the lowest volatility overall.

Next, we evaluate the performance of quarticity shrinkage in the simulation with 25 stocks. We find that the performance of QS differs depending on the input and estimation window. We observe that QS-NLS seems to dominate in the window of 63 days when we neglect transaction costs. QS-LS yields the highest transaction costs adjusted mean return in all windows as this approach yields the lowest turnover. Once the window increases, QS-EWMA starts to dominate in terms of mean return, Sharpe ratio and certainty equivalents. We note that the performance of LS and NLS drops due to the increased volatility, while the performance of EWMA stays relatively flat.

Succeeding, we look at the new method in the simulation with 100 stocks. We generally observe a shift in the dominating method compared to the study with a smaller number of stocks. QS-NLS seems to outperform when taking risk and transaction costs into account. The new method with LS seems to perform subpar, as it no longer gives the largest transaction costs adjusted mean return. Only in the smallest moving window does it give a decent result. QS-EWMA is of interest when risk is not of interest, as it yields the highest mean return and largest certainty equivalent with a small coefficient of risk aversion in the larger moving windows.

Once we look at the simulation with 250 stocks, QS-NLS seems to outperform in all measures across most of the estimation windows. However, we find very peculiar results for the estimation window of 126 days. QS-NLS and a partition ratio of 125/125 certain/uncertain stocks suddenly yields a high mean return at the cost of a large volatility and a huge turnover. We saw a similar phenomenon for the new method with NLS in the optimal partition with 100 stocks. Once again, we investigate the weights of the surrounding partition ratios of 100/150 and 150/100 stocks. We note that the minimum and maximum weights over the replications are similar for the 100/150 and 150/100 ratios. The 125/125 ratio obtains a minimum (maximum) weight of 150 (79) times smaller (larger) than both surrounding ratios. Even when we investigate the ratios of 130/120 and 120/130, we still find that the minimum and maximum weights obtained by 125/125 are much more extreme. So, we again conclude that shifting a small number of stocks between the $1/N$ allocation and an existing covariance estimator causes severe changes in the performance when using the GMV portfolio.

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	**0.278**	**0.278**	**0.278**	0.277	0.247⊗	0.247⊗	0.242⊗
Sh	1.123	0.985	1.022	0.892	1.447⊗	**1.461**⊗	1.451⊗
CEQ ($\gamma = 1$)	**0.247**	0.238	0.241	0.229	0.232	0.233	0.228
CEQ ($\gamma = 5$)	0.125	0.079	0.093	0.036	0.174	**0.176**	0.172
TO	-	0.142	0.201	0.522	**0.099**	0.128	0.296
TCAR (4 bps)	**0.278**	0.264	0.258	0.225	0.237	0.235	0.212
TCAR (8.6 bps)	**0.278**	0.248	0.235	0.164	0.226	0.220	0.178
W = 126							
Mean Return	**0.278**	**0.278**	**0.278**	0.277	0.245⊗	0.245⊗	0.244⊗
Sh	1.124	0.983	1.011	0.896	1.411⊗	1.427⊗	**1.455**⊗
CEQ ($\gamma = 1$)	**0.247**	0.238	0.240	0.229	0.230	0.231	0.230
CEQ ($\gamma = 5$)	0.125	0.078	0.089	0.038	0.170	0.171	**0.174**
TO	-	0.082	0.099	0.528	**0.055**	0.067	0.270
TCAR (4 bps)	**0.278**	0.270	0.268	0.224	0.240	0.239	0.216
TCAR (8.6 bps)	**0.278**	0.260	0.257	0.163	0.233	0.231	0.185
W = 252							
Mean Return	**0.278**	**0.278**	**0.278**	0.277	0.242⊗	0.242⊗	0.243⊗
Sh	1.121	0.981	1.002	0.893	1.363⊗	1.375⊗	**1.448**⊗
CEQ ($\gamma = 1$)	**0.247**	0.238	0.240	0.229	0.226	0.226	0.229
CEQ ($\gamma = 5$)	0.124	0.077	0.085	0.037	0.163	0.165	**0.173**
TO	-	0.046	0.052	0.528	**0.030**	0.035	0.252
TCAR (4 bps)	**0.278**	0.273	0.273	0.224	0.239	0.238	0.218
TCAR (8.6 bps)	**0.278**	0.268	0.267	0.162	0.235	0.234	0.189
W = 756							
Mean Return	**0.277**	**0.277**	**0.277**	0.276	0.238⊗	0.238⊗	0.243⊗
Sh	1.119	0.972	0.983	0.889	1.305⊗	1.312⊗	**1.439**⊗
CEQ ($\gamma = 1$)	**0.246**	0.236	0.237	0.228	0.221	0.222	0.229
CEQ ($\gamma = 5$)	0.124	0.074	0.079	0.035	0.155	0.156	**0.172**
TO	-	0.017	0.018	0.528	**0.011**	0.012	0.237
TCAR (4 bps)	**0.277**	0.275	0.275	0.223	0.237	0.237	0.219
TCAR (8.6 bps)	**0.277**	0.273	0.273	0.162	0.236	0.235	0.192

Table 6: This table displays the performance measures of all methods in the simulation study with 25 generated stocks. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS combined with LS, NLS and EWMA follows the ratio of 12/13 certain/uncertain stocks for all moving windows. Within each row, we display the method that gives the best performance in the respective measure in bold. Note that QS obtains mean returns and Sharpe ratios which are always significantly different from the $1/N$ allocation and the existing method. The differences are significant at all levels following the tests described in Section 4.5.5. We display this phenomenon by ⊗ in the table. We annualize all measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to the number of effective days.

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	0.281	0.281	0.282	**0.283**	0.203⊗	0.222⊗	0.200⊗
Sh	1.115	1.086	1.112	1.090	1.691⊗	**2.480**⊗	1.711⊗
CEQ ($\gamma = 1$)	0.249	0.247	**0.250**	0.249	0.196	0.218	0.193
CEQ ($\gamma = 5$)	0.122	0.114	0.121	0.114	0.167	**0.202**	0.166
TO	-	0.269	**0.190**	0.272	0.308	0.214	0.610
TCAR (4 bps)	**0.281**	0.254	0.263	0.255	0.172	0.201	0.139
TCAR (8.6 bps)	**0.281**	0.223	0.241	0.224	0.136	0.176	0.068
W = 126							
Mean Return	0.280	0.279	**0.281**	0.281	0.197⊗	0.276⊗	0.198⊗
Sh	1.109	1.070	1.104	1.070	**1.661**⊗	0.724⊗	1.436⊗
CEQ ($\gamma = 1$)	0.248	0.245	**0.249**	0.247	0.190	0.203	0.188
CEQ ($\gamma = 5$)	0.121	0.109	0.119	0.108	**0.162**	-0.088	0.150
TO	-	0.241	**0.134**	0.401	0.218	1.104	1.454
TCAR (4 bps)	**0.280**	0.254	0.268	0.241	0.175	0.165	0.051
TCAR (8.6 bps)	**0.280**	0.226	0.252	0.194	0.150	0.037	-0.117
W = 252							
Mean Return	0.280	0.279	0.280	**0.282**	0.193⊗	0.207⊗	0.211⊗
Sh	1.105	1.063	1.104	1.038	1.626⊗	**1.669**⊗	1.352⊗
CEQ ($\gamma = 1$)	**0.248**	0.244	0.248	0.245	0.186	0.199	0.199
CEQ ($\gamma = 5$)	0.119	0.107	0.119	0.097	0.158	**0.168**	0.150
TO	-	0.209	**0.100**	0.694	0.141	0.104	0.758
TCAR (4 bps)	**0.280**	0.258	0.270	0.212	0.179	0.196	0.134
TCAR (8.6 bps)	**0.280**	0.234	0.259	0.132	0.162	0.184	0.047
W = 756							
Mean Return	0.277	**0.278**	**0.278**	0.277	0.188⊗	0.196⊗	0.208⊗
Sh	1.079	1.053	1.071	1.013	1.552⊗	**1.564**⊗	1.306⊗
CEQ ($\gamma = 1$)	**0.244**	0.243	**0.244**	0.240	0.181	0.188	0.195
CEQ ($\gamma = 5$)	0.112	0.104	0.109	0.090	0.151	**0.157**	0.145
TO	-	0.076	0.043	0.694	0.051	**0.037**	0.637
TCAR (4 bps)	**0.277**	0.270	0.274	0.207	0.183	0.192	0.144
TCAR (8.6 bps)	**0.277**	0.261	0.269	0.127	0.177	0.188	0.070

Table 7: This table displays the performance measures of all methods in the simulation study with 250 generated stocks. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS-LS follows the ratio of 125/125 certain/uncertain stocks for all moving windows. QS-NLS follows a ratio of 200/50 for $\mathbf{W} = 63$, and also follows the 125/125 ratio for the other windows. QS-EWMA follows the ratio of 125/125 certain/uncertain stocks for $\mathbf{W} = \{63, 126\}$, and a ratio of 50/200 for $\mathbf{W} = \{252, 756\}$. Within each row, we display the method that gives the best performance in the respective measure in bold. Note that QS obtains mean returns and Sharpe ratios which are always significantly different from the $1/N$ allocation and the existing method. The differences are significant at all levels following the tests described in Section 4.5.5. We display this phenomenon by $\otimes$ in the table. We annualize all measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to the number of effective days.

5.5 Robustness

To assess the robustness of the methods, we look into the annualized standard deviations of the performance measures. We calculate the measures for each replication and take the standard deviation over the replications. We multiply the obtained value by 252 or the square root of 252, which corresponds to the assumption of one year. The turnover is in percentage, such that we do not multiply its standard deviation by any factor. We display the tables with standard deviations of the measures in the appendix, Section E.

We generally find that quarticity shrinkage is the most consistent as it yields the lowest standard deviation in the mean return, certainty equivalents, turnover and transaction costs adjusted mean returns. However, the existing covariance estimators, especially EWMA, give the lowest standard deviations for the Sharpe ratio. The $1/N$ allocation generally does not give the lowest standard deviation in any measure or simulation study.

Earlier in the simulation study, we have seen that the average volatility is way larger for the existing methods. The accuracy of the existing covariance estimators thus varies depending on the characteristics of the dataset. The existing method either perform well, or it does a poor job. This dependency leads to a loss in precision, which we observe in the larger standard deviations. The $1/N$ allocation always yields a moderate performance, it is never very accurate, but it is neither far off. We also note this for $1/N$ that rarely has the largest standard deviations. The new method benefits from both these properties such that it attains the lowest standard deviations in most measures.

Next, we evaluate the robustness of quarticity shrinkage with its different inputs. We observe that QS-LS gives precise estimates in the simulation studies with 100 and 250 stocks. QS-EWMA yields more robust estimates in the simulation study with 25 stocks. Although the new method with NLS gives robust estimates in the windows of 63, it never attains the lowest standard deviation in other studies.

6 Empirical Results

Before we evaluate the empirical backtesting, we briefly address the calibration of quarticity shrinkage with its inputs. We did not find a partition ratio of certain and uncertain stocks that outperforms the others consistently. It depends on the number of stocks, the size of the estimation window and the method used as an input. Consequently, we resort to using half of the stocks as certain and the other half as uncertain.

Looking at the results from the empirical backtesting in Table 8, we first notice that the values all seem to be of a different order compared to the simulation results. The highest mean return in the empirical analysis has a value of 0.181, whereas the largest mean return in the simulation study is 0.283. For the Sharpe ratio we find a maximum value of 1.202 in the empirical results against a top value of 2.480 in the simulation results.

We find a similar discrepancy for both certainty equivalents and the turnover, where the larger values are obtained in the simulation run.

Unsurprisingly, the performance measures in the empirical analysis attain lower values than those in the simulation studies. We found that the underlying structure in the generated data is more moderate in terms of volatility such that we did not observe severe regime switches in the stock returns. These findings led to large returns with low volatilities, such that all methods were able to perform well. The real data contains more severe volatility clustering and has a lower mean return. These characteristics directly lead to worse performances of the estimators.

The empirical results show that the highest mean return is always obtained by $1/N$. However, the difference with the new method is never significant at the respective levels. The highest Sharpe ratio is obtained for EWMA in the smallest window and yielded by QS in longer moving windows. The difference in Sharpe ratios between QS and the $1/N$ allocation is often significant. We further observe the largest certainty equivalent with a small coefficient of risk aversion for either $1/N$ or the QS-NLS. We often find the highest certainty equivalent with a bigger coefficient of risk aversion for QS-NLS. The $1/N$ allocation always yields the largest transaction costs adjusted mean return.

We continue by observing the estimation window of 63 days. It becomes clear that the existing covariance estimators perform subpar, except for the Sharpe ratio of EWMA. An investor who neglects risk will prefer the $1/N$ allocation, due to its high mean return, certainty equivalent with a small coefficient of risk aversion and large transaction costs adjusted mean return. When an investor considers risk, quarticity shrinkage is of interest. QS generally yields a larger Sharpe ratio and certainty equivalent with a large risk aversion coefficient than the $1/N$ allocation and the existing covariance estimators.

The trade-off between a large mean return and a small volatility is optimal for the new method in the window of 126 days. We observe that QS yields the largest Sharpe ratio and certainty equivalent with both coefficients. We find a similar result in the window with 252 days, where QS-NLS outperforms in Sharpe ratio, both certainty equivalents and turnover. This outperformance of QS-NLS might not come as a surprise, given that Ledoit & Wolf (2020b) state that NLS generally outperforms LS. Using 252 observations ensures that NLS uses enough observations to fit the nonlinear function to the data. In the smaller windows QS-NLS did not outperform QS-LS. The existing covariance estimators show a similar pattern, such that they emphasize this finding in the new method.

In the largest moving window, we find that the $1/N$ allocation seems to benefit most from its increased mean return while its volatility decreases. Therefore, this strategy obtains the largest certainty equivalent with a small coefficient of risk aversion again. QS-EWMA gives a poor performance when compared to the EWMA estimator.

The turnovers are generally surprising because the new method does not always yield lower values than the covariance estimators. We expect that the new method gives lower turnovers, considering that it applies an equally weighted approach to half of the stocks. We highlight that the turnover in the smallest window of the new method with EWMA is ten times larger than that of EWMA. Upon analyzing the weights and realized weights, we observe that the new method yields extreme values. The minimum allocated weight by the new method is roughly 1.5 times more negative, while the maximum weight is slightly more positive. We find a considerable difference for the realized weights. The minimum (maximum) allocated realized weight of the new method is roughly 1000 (400) times more negative (positive). The mean allocated weights between the two methods are very similar. We conclude that the turnover for the new method with EWMA is affected severely by noise, such that it gives absurd values.

We find the same phenomenon for the estimation window of 126 days, comparing QS-NLS and NLS. The minimum (maximum) realized weight allocated by QS is roughly ten times (four times) more negative (more positive). The mean of the realized weight is of a similar value. The differences in estimated weights are not as large, where the maximum is 1.3 times larger for the new method with NLS. In this setting, QS-NLS apparently suffers from noise such that the GMV allocation assigns more extreme weights the uncertain stocks.

Next, we evaluate quarticity shrinkage with the different inputs against one other. QS-EWMA seems to perform subpar in most measures, except for the smallest moving window. We observe that the volatility of QS-EWMA increases while the volatility of QS-LS and QS-NLS remains flat. Although the volatility for QS-EWMA between the window of 252 and 756 days stays flat, it is 1.5 times larger than the ones obtained for QS-LS and QS-NLS. We explain this increased volatility by looking at the characteristics of the models. LS and NLS are static models; hence they are less responsive to changes in the stock market. Each observation within the window has the same influence on the estimation of the covariance matrix, such that changes in the matrix occur slowly. EWMA is a dynamic method, which yields more value to recent information. This dynamic characteristic leads to larger and more frequent changes in the covariance matrix, leading to a larger volatility.

We further observe that the performance of QS-LS and QS-NLS seems to be very similar in the windows of 63 and 126 days. In fact, QS-LS outperforms QS-NLS slightly in the smallest moving window. We do not find a similar pattern for LS and NLS. So, we argue that LS with the 163 certain stocks is optimal in the smallest window. Fitting a nonlinear relation to the data while it contains a linear relation can never be as good as fitting a linear relation. This argument is invalid once the estimation window gets longer or the number of stocks gets larger as the outperformance no longer holds.

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	**0.164**	0.096	0.106	0.115	0.139⊕	0.138	0.144
Sh	0.716	0.813	0.890	**1.096**	0.988	0.939*	1.016**
CEQ (gamma = 1)	**0.138**	0.089	0.099	0.109	0.129	0.127	0.134
CEQ (gamma = 5)	0.033	0.061	0.070	0.087	0.090	0.084	**0.094**
TO	-	0.039	0.072	**0.030**	0.050	0.040	0.298
TCAR (4 bps)	**0.164**	-0.894	-1.718	-0.650	-1.141	-0.869	-7.419
TCAR (8.6 bps)	**0.164**	-2.034	-3.814	-1.529	-2.612	-2.026	-16.117
W = 126							
Mean Return	**0.162**	0.106	0.103	0.099	0.148	0.150	0.134
Sh	0.705	0.864	0.872	0.819	**1.036***	1.034**	0.800
CEQ (gamma = 1)	0.136	0.099	0.096	0.092	0.137	**0.139**	0.120
CEQ (gamma = 5)	0.030	0.069	0.068	0.063	**0.097**	**0.097**	0.064
TO	-	**0.029**	0.044	0.052	0.050	0.085	0.069
TCAR (4 bps)	**0.162**	-0.622	-1.024	-1.226	-1.118	-2.004	-1.627
TCAR (8.6 bps)	**0.162**	-1.460	-2.320	-2.750	-2.572	-4.481	-3.652
W = 252							
Mean Return	**0.170**	0.083	0.101	0.118	0.145⊕⊕	0.162⊕	0.162
Sh	0.733	0.658	0.859	0.827	1.008	**1.119** * * *	0.687
CEQ (gamma = 1)	0.143	0.075	0.095	0.107	0.134	**0.151**	0.134
CEQ (gamma = 5)	0.036	0.043	0.067	0.067	0.093	**0.110**	0.023
TO	-	0.056	0.042	0.059	0.119	**0.024**	0.068
TCAR (4 bps)	**0.170**	-1.345	-0.971	-1.392	-2.873	-0.460	-1.553
TCAR (8.6 bps)	**0.170**	-2.987	-2.204	-3.129	-6.342	-1.174	-3.524
W = 756							
Mean Return	**0.181**	0.096	0.126	0.154	0.148⊕	0.157	0.118
Sh	0.958	0.832	1.152	1.153	1.128	**1.202**	0.575⊕
CEQ (gamma = 1)	**0.163**	0.089	0.120	0.145	0.139	0.148	0.097
CEQ (gamma = 5)	0.092	0.063	0.096	0.109	0.105	**0.114**	0.013
TO	-	0.051	0.054	0.060	**0.027**	0.039	0.060
TCAR (4 bps)	**0.181**	-1.202	-1.238	-1.381	-0.535	-0.830	-1.417
TCAR (8.6 bps)	**0.181**	-2.695	-2.806	-3.147	-1.320	-1.966	-3.182

Table 8: This table displays the performance measures of all methods in the empirical analysis with real-world excess holding-period returns. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS always follows the ratio of 163/164 certain/uncertain stocks for all moving windows. Within each row, we display the method that gives the best performance in the respective measure in bold. An asterisk, $*$, indicates that QS is significantly different from the $1/N$ allocation at a level of 10% following the tests described in section 4.5.5. A double asterisk, $**$, indicates that QS is significantly different from the $1/N$ allocation at a level of 5%. And a triple asterisk, $***$, indicates a significant difference at a level of 1%. Further, a $\oplus$ indicates that QS is significantly different from the input method alone at a 10% level. A double $\oplus\oplus$ indicates this for 5% and a triple $\oplus\oplus\oplus$ for 1%. We annualize all measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to be the number of effective days.

7 Discussion

From the results of our simulation studies and empirical analysis, we draw relevant and valuable conclusions. However, we have to acknowledge that our research has its limitations.

We find that a slight change in the number of stocks that follow the certain/uncertain approach in quarticity shrinkage is of extraordinary importance. Although we argue that it might be infeasible to implement a specific grid search, expanding the current calibration might be an option. Another possibility would be to investigate the calibration via a grid search, taking into account the investor's preferences.

For the EWMA approach, we used a constant decay value of 0.94. However, Bollen (2015) found that this particular choice is not always optimal. Their finding imply that the results for the EWMA method might improve when using another value or varying the decay factor depending on the setting. One might find optimal values via cross-validation by assessing the performance of EWMA with different decay factors over time. However, this approach can be very cumbersome, and it is unclear whether the performance of EWMA will improve significantly.

The data generating process produces the returns according to the CAPM model. One can extend this model to try and resemble the real-world data more closely by using, for instance, the Fama-French Three Factor model from Fama & French (1993). Using this model would mean that one fits a multivariate GARCH model to the factors. As a result, one draws the factors while containing autocorrelation and cross-correlation between those factors. Another improvement is to incorporate various states of the world such that the generated returns exhibit crashes, as is the case in the empirical dataset. Including various states might be possible via, for instance, the Markov-Switching GARCH model.

We use the global minimum variance portfolio for the portfolio allocation of the covariance estimators and certain returns. Although this portfolio circumvents the uncertainty in the returns, it might not be the most relevant portfolio for an asset manager. In our results, we find some extreme weights, such that one might consider adding (short-selling) restrictions.

8 Conclusion

This paper investigates the performance of a new approach to portfolio allocation that combines the $1/N$ strategy with linear shrinkage, nonlinear shrinkage and exponentially weighted moving average. DeMiguel & Uppal (2007) show that none of the existing covariance estimators consistently beat the $1/N$ allocation due to uncertainty regarding estimates of the covariance matrix. To overcome this inconsistency, they find that one should use an impractically large estimation window of at least 3000 months. We introduce quarticity shrinkage (QS), a new method to try and beat the $1/N$ allocation consistently without an impractically large estimation window.

Quarticity shrinkage separates stocks into two sets based on their uncertainty, which we proxy with realized quarticity (RQ). Stocks with with a small RQ are regarded certain, and stocks with a large RQ are deemed uncertain. The set of certain stocks follow an existing covariance estimator with the global minimum variance allocation. The set of uncertain stocks follow the $1/N$ allocation. We merge the two sets and normalize the newly obtained portfolio. Further, we investigate the performance of QS with a different number of stocks that follow the certain/uncertain approach.

To assess the performance of QS, we use daily excess holding-period returns of 327 stocks from the S&P500, a value-weighted market index and a risk-free rate. We generate returns via a combination of a Monte Carlo simulation with bootstrapping, where we base the underlying structure of the generated data on a GARCH(1,1) model.

In our simulation studies, we do not find any optimal partition ratio of certain/uncertain stocks. An investor who is only interested in a high return will favour the ratio which uses most stocks as uncertain. A risk averse investor will opt for the ratio that uses most stocks as certain. Further, we find statistically significant differences in the performance between QS and the existing methods. QS generally outperforms on a risk-adjusted basis. An existing estimator will be favoured when the investor neglects risk. Furthermore, we do not find any combination for QS that dominates the other inputs. None of the combinations outperform the others in most measures over all studies. Our last finding from the simulation is that QS yields smaller standard deviations than the existing methods in all measures except the Sharpe ratio. This robustness implies that QS gives the most reliable estimates.

From the empirical analysis, we find that the $1/N$ allocation always attains the largest mean return. However, the difference with quarticity shrinkage is never significant at the respective levels. QS again outperforms on a risk-adjusted basis, where the difference in Sharpe ratio is occasionally significant. QS is superior to the existing covariance estimators in all measures, except the turnover. Furthermore, we find that QS-NLS dominates across the inputs for the new method.

Although the new method seems to outperform the existing covariance estimators, it does not consistently beat the $1/N$ allocation in all measures using a practical estimation window. The performance of quarticity shrinkage varies with the preferences of the investor, input method, number of stocks, size of the estimation window and characteristics in the dataset.

References

Andersen, T., Dobrev, D., & Schaumburg, E. (2014). A robust neighborhood truncation approach to estimation of integrated quarticity. *Econometric Theory*, *30*, 3-59.

Anderson, T. W. (1973). Asymptotically efficient estimation of covariance matrices with linear structure. *The Annals of Statistics*, *1*, 135–141.

Barndorff-Nielsen, O. E., & Shephard, N. (2002). Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, *64*, 253-280.

Bollen, B. (2015). What should the value of lambda be in the exponentially weighted moving average volatility model? *Applied Economics*, *47*, 853-860.

Bollerslev, T., Patton, A. J., & Quaedvlieg, R. (2016). Exploiting the errors: A simple approach for improved volatility forecasting. *Journal of Econometrics*, *192*, 1-18.

Corsi, F., Mittnik, S., Pigorsch, C., & Pigorsch, U. (2008). The volatility of realized volatility. *Econometric Reviews*, *27*, 46-78.

DeMiguel, L., Victor. Garlappi, & Uppal, R. (2007). Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *The Review of Financial Studies*, *22*, 1915-1953.

Epanechnikov, V. A. (1969). Nonparametric estimation of a multidimensional probability density. *Theory Probab. Appl.*, *14*, 153-158.

Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, *33*, 3-56.

Hu, W., & Kercheval, A. (2010). Portfolio optimization for student t and skewed t returns. *Quantitative Finance*, *10*, 91-105.

Jagannathan, R., & Ma, T. (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance*, *58*, 1651-1683.

James, W., & Stein, C. (1961). Estimation with quadratic loss. *Symposium on Mathematical Statistics and Probability*, *1*, 361-379.

Jegadeesh, N., Noh, J., Pukthuanthong, K., Roll, R., & Wang, J. (2019). Empirical tests of asset pricing models with individual assets: Resolving the errors-in-variables bias in risk premium estimation. *Journal of Financial Economics*, *133*, 273-298.

Kenton, W. (2021). *How the stock market capitalization-to-gdp ratio is used.*

Ledoit, O., & Wolf, M. (2003a). Honey, i shrunk the sample covariance matrix. *The Journal of Portfolio Management*, *30*, 110-119.

Ledoit, O., & Wolf, M. (2003b). Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance*, *10*, 603-621.

Ledoit, O., & Wolf, M. (2008). Robust performance hypothesis testing with the sharpe ratio. *Journal of Empirical Finance*, *15*, 850-859.

Ledoit, O., & Wolf, M. (2012). Nonlinear shrinkage estimation of large-dimensional covariance matrices. *Annals of Statistics*, *40*, 1024-1060.

Ledoit, O., & Wolf, M. (2020a). Analytical nonlinear shrinkage of large-dimensional covariance matrices. *Annals of Statistics*, *48*, 3043–3065.

Ledoit, O., & Wolf, M. (2020b). The Power of (Non-)Linear Shrinking: A Review and Guide to Covariance Matrix Estimation. *Journal of Financial Econometrics*.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *The Review of Economics and Statistics*, *47*, 13–37.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, *6*, 77-91.

Marshall, B. R., Nguyen, N. H., & Visaltanachoti, N. (2011). Commodity Liquidity Measurement and Transaction Costs. *The Review of Financial Studies*, *25*, 599-638.

Morgan, J., Longerstaey, J., Limited, R., R. Ltd, M. S., & of New York., M. (1996). Riskmetrics: Technical document. *J.P.Morgan/Reuters*.

Officer, R. (1972). The distribution of stock returns. *Journal of the American Statistical Association*, *67*, 807-812.

Peiró, A. (1994). The distribution of stock returns: international evidence. *Applied Financial Economics*, *4*, 431-439.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk*. *The Journal of Finance*, *19*, 425-442.

Sharpe, W. F. (1966). Mutual fund performance. *Journal of Business*, *39*, 119-138.

Stein, C. (1956). Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. *Berkeley Symposium on Mathematical and Statistical Probability*, *1*, 197 – 206.

Tu, J., & Zhou, G. (2004). Data-generating process uncertainty: What difference does it make in portfolio decisions? *Journal of Financial Economics*, *72*, 385-421.

Appendices

A Names of the Included Stocks

3M CO
A T & T INC
ABBOTT LABORATORIES
ABIOMED INC
ADVANCE AUTO PARTS INC
ADVANCED MICRO DEVICES INC
AFLAC INC
AGILENT TECHNOLOGIES INC
AIR PRODUCTS & CHEMICALS INC
AKAMAI TECHNOLOGIES INC
ALASKA AIRGROUP INC
ALBEMARLE CORP
ALEXANDRIA REAL EST EQUITIES INC
ALEXION PHARMACEUTICALS INC
ALIGN TECHNOLOGY INC
ALLEGHENY TECHNOLOGIES
ALLSTATE CORP
ALTRIA GROUP INC
AMAZON COM INC
AMEREN CORP
AMERICAN ELECTRIC POWER CO INC
AMERICAN EXPRESS CO
AMERICAN INTERNATIONAL GROUP INC
AMERIPRISE FINANCIAL INC
AMERISOURCEBERGEN CORP
AMETEK INC NEW
AMGEN INC
AMPHENOL CORP NEW
ANALOG DEVICES INC
ANSYS INC
APACHE CORP
APPLIED MATERIALS INC
ARCHER DANIELS MIDLAND CO
ASSURANT INC
ATMOS ENERGY CORP
AUTODESK INC
AUTOMATIC DATA PROCESSING INC
AVALONBAY COMMUNITIES INC
AVERY DENNISON CORP
BALL CORP
BANK OF AMERICA CORP
BAXTER INTERNATIONAL INC
BECTON DICKINSON & CO
BERKLEY W R CORP
BEST BUY COMPANY INC
BIO RAD LABORATORIES INC
BLACKROCK INC
BOEING CO
BORGWARNER INC
BOSTON PROPERTIES INC
BOSTON SCIENTIFIC CORP
BRISTOL MYERS SQUIBB CO
C F INDUSTRIES HOLDINGS INC
C M S ENERGY CORP
C S X CORP
CABOT OIL & GAS CORP
CADENCE DESIGN SYSTEMS INC
CAMPBELL SOUP CO
CAPITAL ONE FINANCIAL CORP
CARDINAL HEALTH INC
CARMAX INC
CARNIVAL CORP
CATERPILLAR INC
CENTENE CORP DEL
CENTERPOINT ENERGY INC
CERNER CORP
CH ROBINSON WORLDWIDE INC
CHEVRON CORP NEW
CHIPOTLE MEXICAN GRILL INC
CHURCH & DWIGHT INC
CINCINNATI FINANCIAL CORP
CINTAS CORP
CISCO SYSTEMS INC
CITIGROUP INC
CITRIX SYSTEMS INC
CLOROX CO
COCA COLA CO
COGNIZANT TECHNOLOGY SOLS CORP
COLGATE PALMOLIVE CO
COMCAST CORP NEW
COMERICA INC
CONOCOPHILLIPS
CONSOLIDATED EDISON INC
CONSTELLATION BRANDS INC
COOPER COMPANIES INC
COPART INC
CORE MOLDING TECHNOLOGIES INC
CORNING INC
COSTCO WHOLESALE CORP NEW
CUMMINS INC
D R HORTON INC
D T E ENERGY CO
DANAHER CORP
DARDEN RESTAURANTS INC
DEERE & CO
DEVON ENERGY CORP NEW
DEXCOM INC
DIGITAL REALTY TRUST INC
DISNEY WALT CO
DOMINOS PIZZA INC
DOVER CORP
DUKE ENERGY CORP NEW
DUKE REALTY CORP
EASTMAN CHEMICAL CO
EBAY INC
ECOLAB INC
EDWARDS LIFESCIENCES CORP
ELECTRONIC ARTS INC
EMERSON ELECTRIC CO
ENTERGY CORP NEW
EOG RESOURCES INC
EQUIFAX INC
EQUINIX INC
EQUITY RESIDENTIAL
ESSEX PROPERTY TRUST INC
EVEREST RE GROUP LTD
EXELON CORP
EXPEDITORS INTERNATIONAL WA INC
EXTRA SPACE STORAGE INC
F 5 NETWORKS INC
F M C CORP
FASTENAL COMPANY
FEDERAL REALTY INVESTMENT TRUST
FEDEX CORP
FIDELITY NATIONAL INFO SVCS INC
FIFTH THIRD BANCORP
FIRSTENERGY CORP
FISERV INC
FLIR SYSTEMS INC
FLOWSERVE CORP
FORD MOTOR CO DEL
FRANKLIN RESOURCES INC
GALLAGHER ARTHUR J & CO
GAP INC
GARMIN LTD
GARTNER INC
GENERAL DYNAMICS CORP
GENERAL ELECTRIC CO
GENERAL MILLS INC
GENUINE PARTS CO
GILEAD SCIENCES INC
GLOBAL PAYMENTS INC
GOLDMAN SACHS GROUP INC
GRAINGER W W INC
HALLIBURTON COMPANY
HANESBRANDS INC
HARTFORD FINANCIAL SVCS GRP INC
HASBRO INC
HENRY JACK & ASSOC INC
HERSHEY CO
HESS CORP
HOLOGIC INC
HOME DEPOT INC
HONEYWELL INTERNATIONAL INC
HORMEL FOODS CORP
HOST HOTELS & RESORTS INC
HUMANA INC
HUNT J B TRANSPORT SERVICES INC
HUNTINGTON BANCSHARES INC
I D E X X LABORATORIES INC
I P G PHOTONICS CORP
IDEX CORP
ILLINOIS TOOL WORKS INC
ILLUMINA INC
INCYTE CORP
INTEL CORP
INTERNATIONAL BUSINESS MACHS COR
INTERNATIONAL FLAVORS & FRAG INC
INTERNATIONAL PAPER CO
INTERPUBLIC GROUP COS INC
INTUIT INC
INTUITIVE SURGICAL INC
ITERIS INC
JACOBS ENGINEERING GROUP INC
JOHNSON & JOHNSON
JPMORGAN CHASE & CO
JUNIPER NETWORKS INC
KANSAS CITY SOUTHERN
KELLOGG CO
KEYCORP NEW
KIMBERLY CLARK CORP
KIMCO REALTY CORP
KROGER COMPANY
L K Q CORP
LABORATORY CORP AMERICA HLDGS
LAM RESH CORP
LAS VEGAS SANDS CORP
LAUDER ESTEE COS INC
LEE ENTERPRISES INC
LEGGETT & PLATT INC
LENNAR CORP
LILLY ELI & CO
LINCOLN NATIONAL CORP
LOEWS CORP
M & T BANK CORP
MARATHON OIL CORP
MARKETAXESS HLDGS INC
MARRIOTT INTERNATIONAL INC NEW
MARSH & MCLENNAN COS INC
MARTIN MARIETTA MATERIALS INC
MASCO CORP
MASTERCARD INC
MAXIM INTEGRATED PRODUCTS INC
MCCORMICK & CO INC
MCDONALDS CORP
MCKESSON H B O C INC
METLIFE INC
METTLER TOLEDO INTERNATIONAL INC
MICROCHIP TECHNOLOGY INC
MICRON TECHNOLOGY INC
MICROSOFT CORP
MID AMERICA APT COMMUNITIES INC
MILESTONE SCIENTIFIC INC
MOHAWK INDUSTRIES INC
MONOLITHIC PWR SYS INC
MOODYS CORP
N R G ENERGY INC
N V R INC
NATIONAL OILWELL VARCO INC
NETFLIX INC
NIKE INC
NORFOLK SOUTHERN CORP
NORTHERN TRUST CORP
NORTHROP GRUMMAN CORP
NUCOR CORP
NVIDIA CORP
OCCIDENTAL PETROLEUM CORP
OLD DOMINION FREIGHT LINE INC
OMNICOM GROUP INC
ONEOK INC NEW
ORACLE CORP
P N C FINANCIAL SERVICES GRP INC
P P G INDUSTRIES INC
P P L CORP
PACCAR INC
PACKAGING CORP AMERICA
PARKER HANNIFIN CORP
PAYCHEX INC
PEPSICO INC
PERKINELMER INC
PFIZER INC
PINNACLE WEST CAPITAL CORP
PIONEER NATURAL RESOURCES CO
POOL CORP
PRINCIPAL FINANCIAL GROUP INC
PROCTER & GAMBLE CO

PRUDENTIAL FINANCIAL INC	SMUCKER J M CO	UNITEDHEALTH GROUP INC
PUBLIC SERVICE ENTERPRISE GP INC	SNAP ON INC	UNIVERSAL HEALTH SERVICES INC
QUALCOMM INC	SOUTHERN CO	V F CORP
QUANTA SERVICES INC	SOUTHWEST AIRLINES CO	VALERO ENERGY CORP NEW
QUEST DIAGNOSTICS INC	STARBUCKS CORP	VARIAN MEDICAL SYSTEMS INC
RAYMOND JAMES FINANCIAL INC	STATE STREET CORP	VENTAS INC
REALTY INCOME CORP	SYNOPSYS INC	VERISIGN INC
REGENCY CENTERS CORP	SYSCO CORP	VERIZON COMMUNICATIONS INC
REGENERON PHARMACEUTICALS INC	T J X COMPANIES INC NEW	VERTEX PHARMACEUTICALS INC
REGIONS FINANCIAL CORP NEW	T ROWE PRICE GROUP INC	VORNADO REALTY TRUST
REPUBLIC SERVICES INC	TAKE TWO INTERACTIVE SOFTWR INC	VULCAN MATERIALS CO
RESMED INC	TARGET CORP	WABTEC CORP
ROBERT HALF INTERNATIONAL INC	TELEDYNE TECHNOLOGIES	WASTE MANAGEMENT INC DEL
ROCKWELL AUTOMATION INC	TELEFLEX INC	WATERS CORP
ROLLINS INC	TERADYNE INC	WELLS FARGO & CO NEW
ROSS STORES INC	TEXTRON INC	WEST PHARMACEUTICAL SERVICES INC
ROYAL CARIBBEAN CRUISES LTD	THERMO FISHER SCIENTIFIC INC	WESTAMERICA BANCORPORATION
S L GREEN REALTY CORP	TRACTOR SUPPLY CO NEW	WESTERN DIGITAL CORP
S V B FINANCIAL GROUP	TRANSDIGM GROUP INC	WESTERN UNION CO
SALESFORCE COM INC	TUCOWS INC	WEYERHAEUSER CO
SCHLUMBERGER LTD	TYLER TECHNOLOGIES INC	WHIRLPOOL CORP
SCHWAB CHARLES CORP NEW	TYSON FOODS INC	WILLIAMS COS
SEALED AIR CORP NEW	U S BANCORP DEL	WYNN RESORTS LTD
SEMPRA ENERGY	UNDER ARMOUR INC	X C E L ENERGY INC
SHERWIN WILLIAMS CO	UNION PACIFIC CORP	XILINX INC
SIMON PROPERTY GROUP INC NEW	UNITED PARCEL SERVICE INC	YUM BRANDS INC
SKYWORKS SOLUTIONS INC	UNITED RENTALS INC	ZEBRA TECHNOLOGIES CORP

Table 9: This table shows name of the stocks that we include in our dataset after filtering. We filter the data such that all stocks contains the relevant information over the whole period.

B Autocorrelation of the Empirical Data

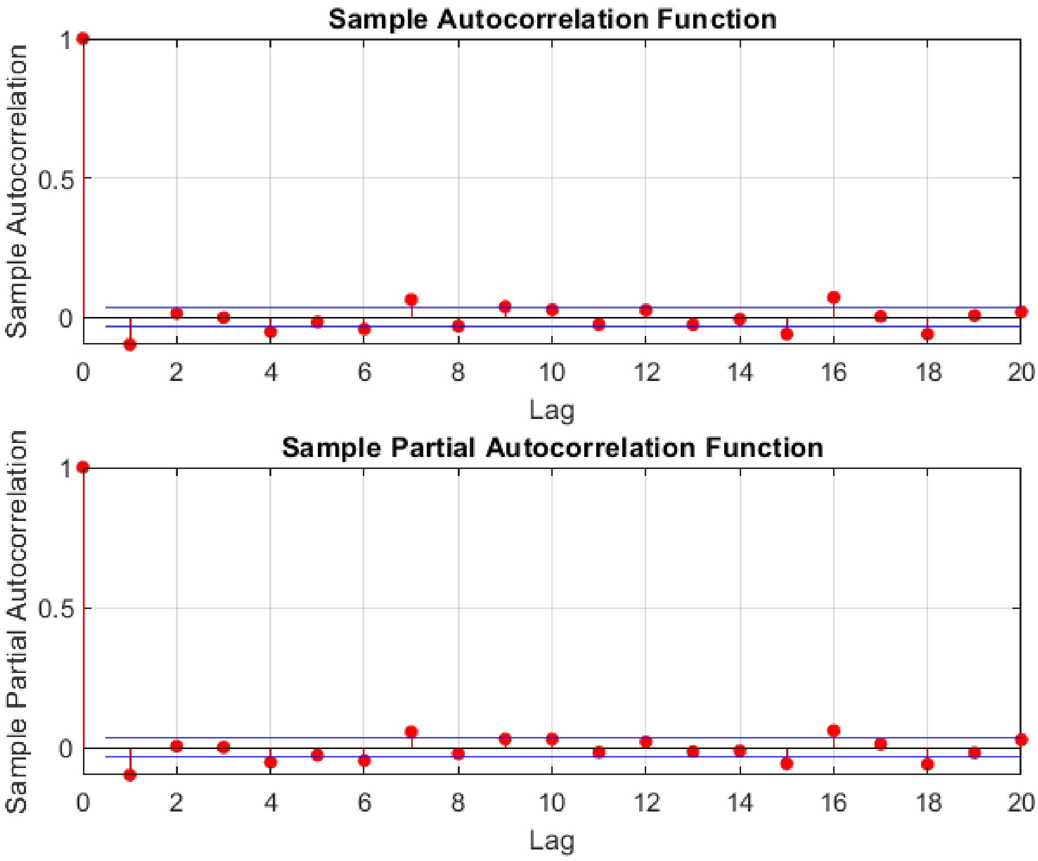

Figure 6: Plots showing the autocorrelation function and partial autocorrelation function of the average excess holding-period returns.

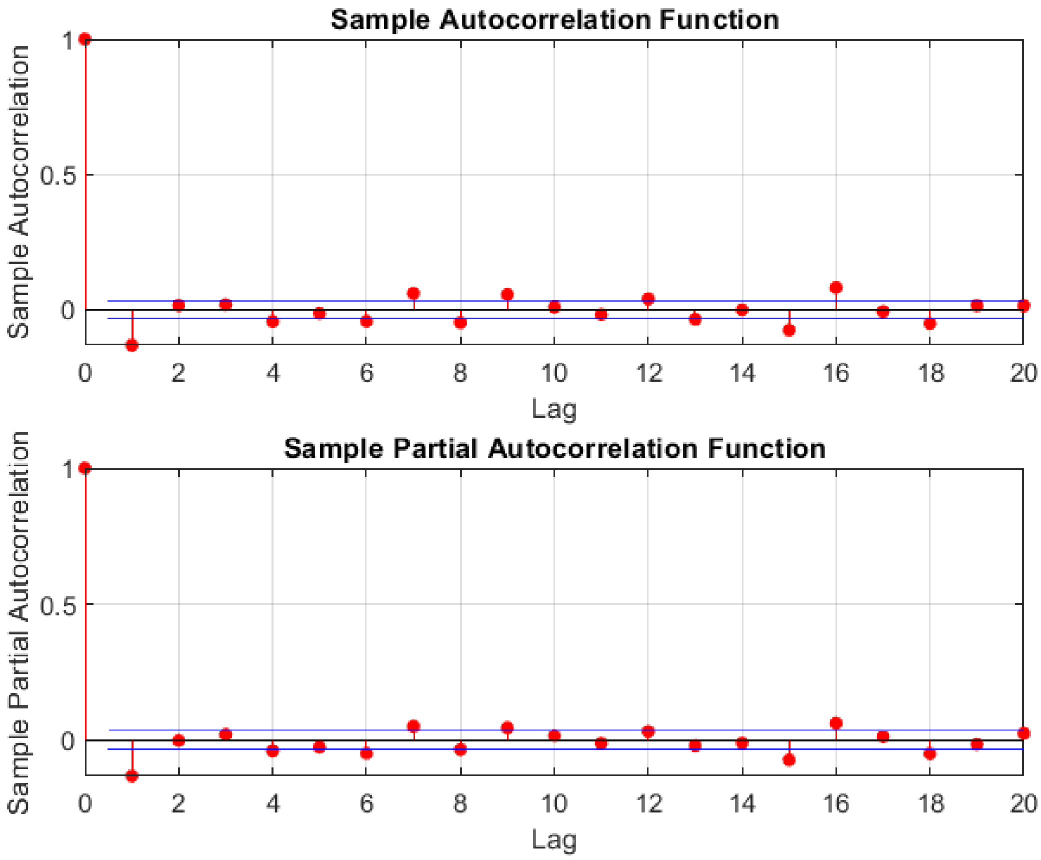

Figure 7: Plots showing the autocorrelation function and partial autocorrelation function of the value-weighted market index.

C Descriptive Statistics of the Generated Data

	Return	Volatility	Skewness	Kurtosis
Min	0.271	0.011	-0.064	3.028
Mean	0.277	0.011	0.005	3.284
Max	0.283	0.011	0.086	4.063

Table 10: This table shows the descriptive statistics of the 25 generated stock returns. We display the annual minimum, mean, maximum and market (Mkt) return and volatility of the stocks over the entire period without the use of any estimation window. By annual, we mean that we compute the metric and multiply it by a scale of 252. We multiply the return by 252 and the volatility by $\sqrt{252}$. It further shows the skewness and kurtosis that we calculate over the entire period for each stock.

	Return	Volatility	Skewness	Kurtosis
Min	0.276	0.011	-0.417	3.129
Mean	0.282	0.012	-0.049	4.288
Max	0.291	0.015	0.364	9.179

Table 11: This table shows the descriptive statistics of the 100 generated stock returns. We display the annual minimum, mean, maximum and market (Mkt) return and volatility of the stocks over the entire period without the use of any estimation window. By annual, we mean that we compute the metric and multiply it by a scale of 252. We multiply the return by 252 and the volatility by $\sqrt{252}$. It further shows the skewness and kurtosis that we calculate over the entire period for each stock.

	Return	Volatility	Skewness	Kurtosis
Min	0.252	0.032	-0.161	3.175
Mean	0.279	0.035	0.028	3.684
Max	0.298	0.039	0.214	5.279

Table 12: This table shows the descriptive statistics of the 250 generated stock returns. We display the annual minimum, mean, maximum and market (Mkt) return and volatility of the stocks over the entire period without the use of any estimation window. By annual, we mean that we compute the metric and multiply it by a scale of 252. We multiply the return by 252 and the volatility by $\sqrt{252}$. It further shows the skewness and kurtosis that we calculate over the entire period for each stock.

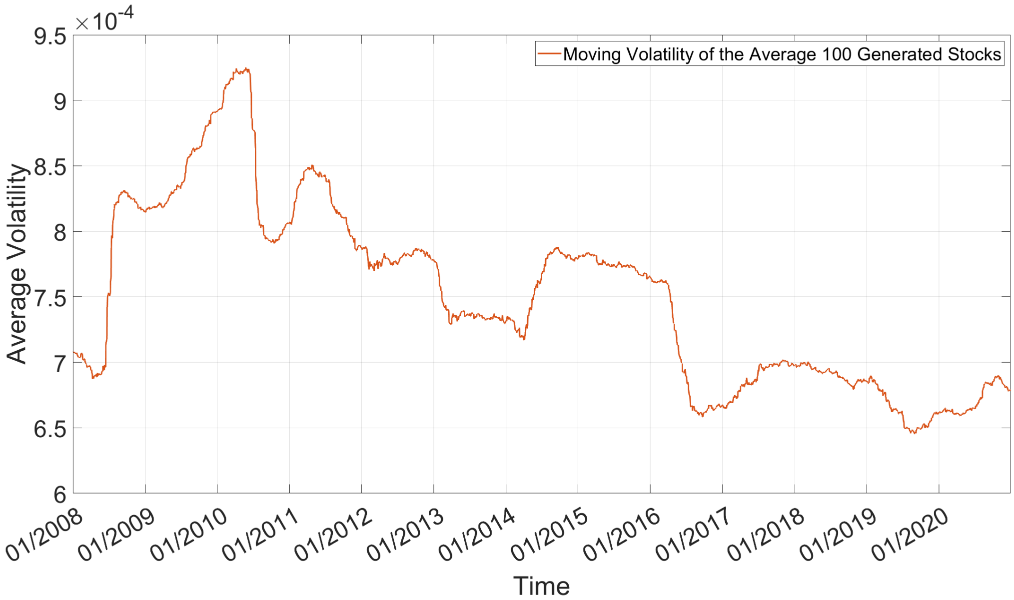

Figure 8: A plot showing the volatility clustering of the average 100 generated returns fora one-year (assumption: 252 days) rolling window.

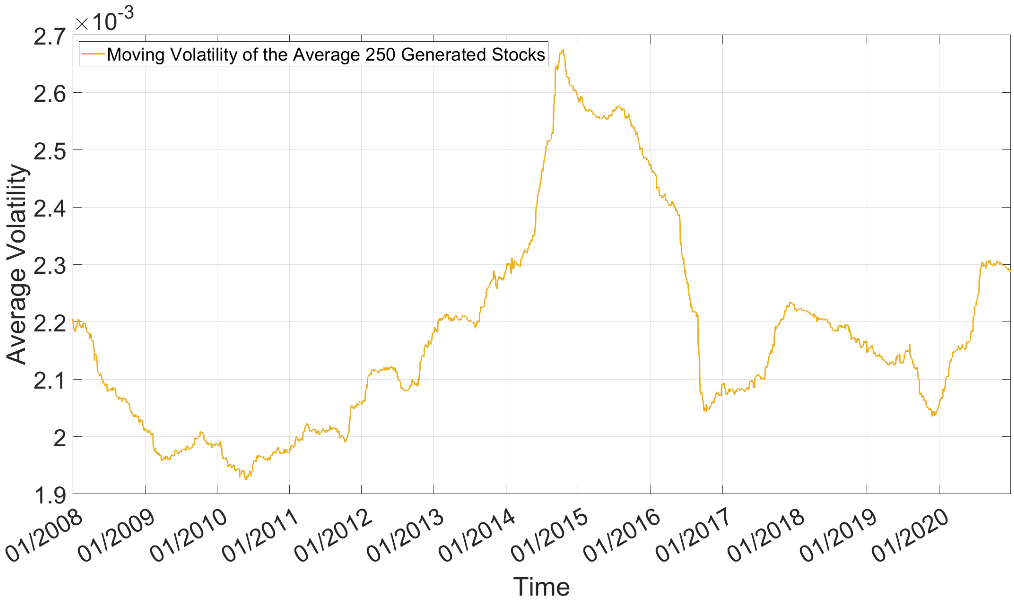

Figure 9: A plot showing the volatility clustering of the average 250 generated returns fora one-year (assumption: 252 days) rolling window.

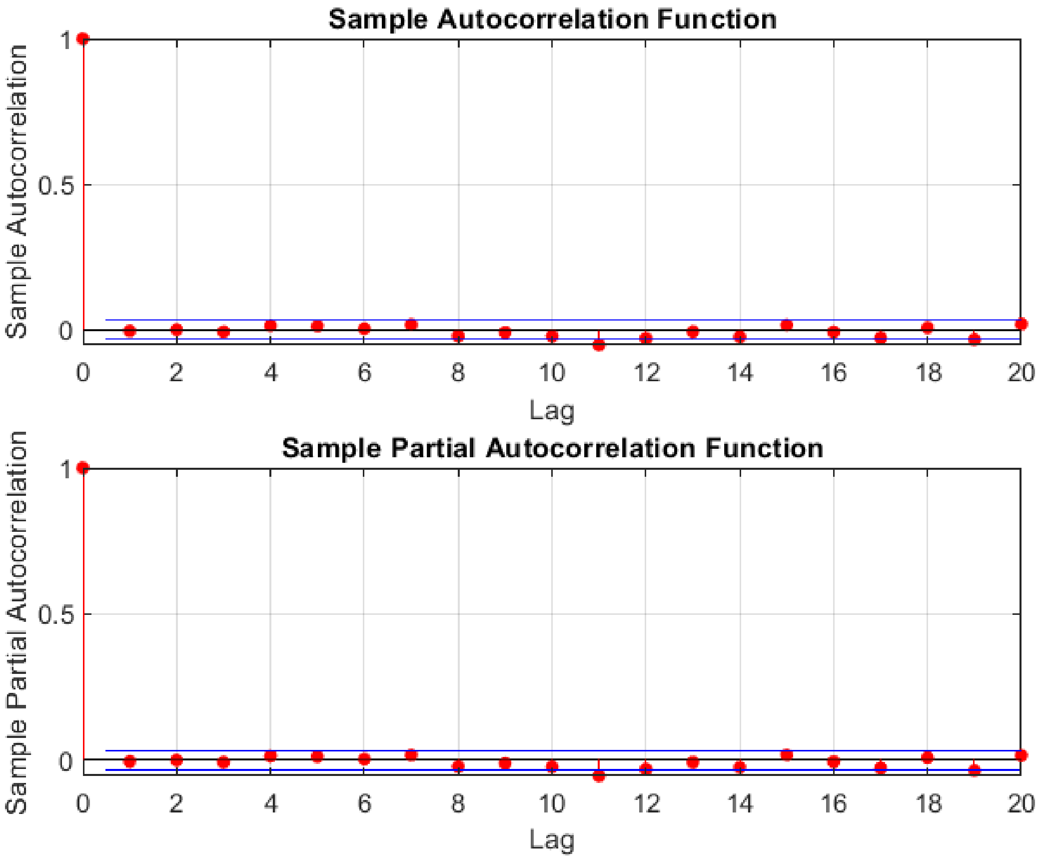

Figure 10: Plots with autocorrelation function and partial autocorrelation function of the 25 generated stocks.

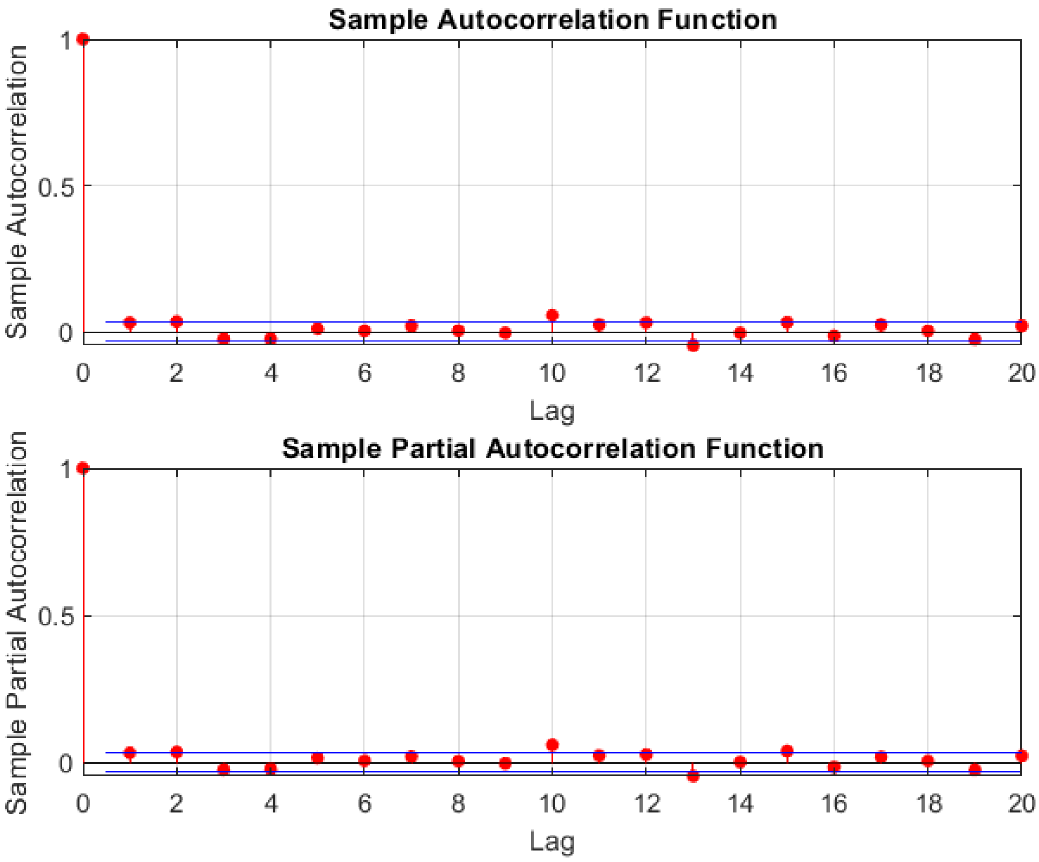

Figure 11: Plots with autocorrelation function and partial autocorrelation function of the 100 generated stocks.

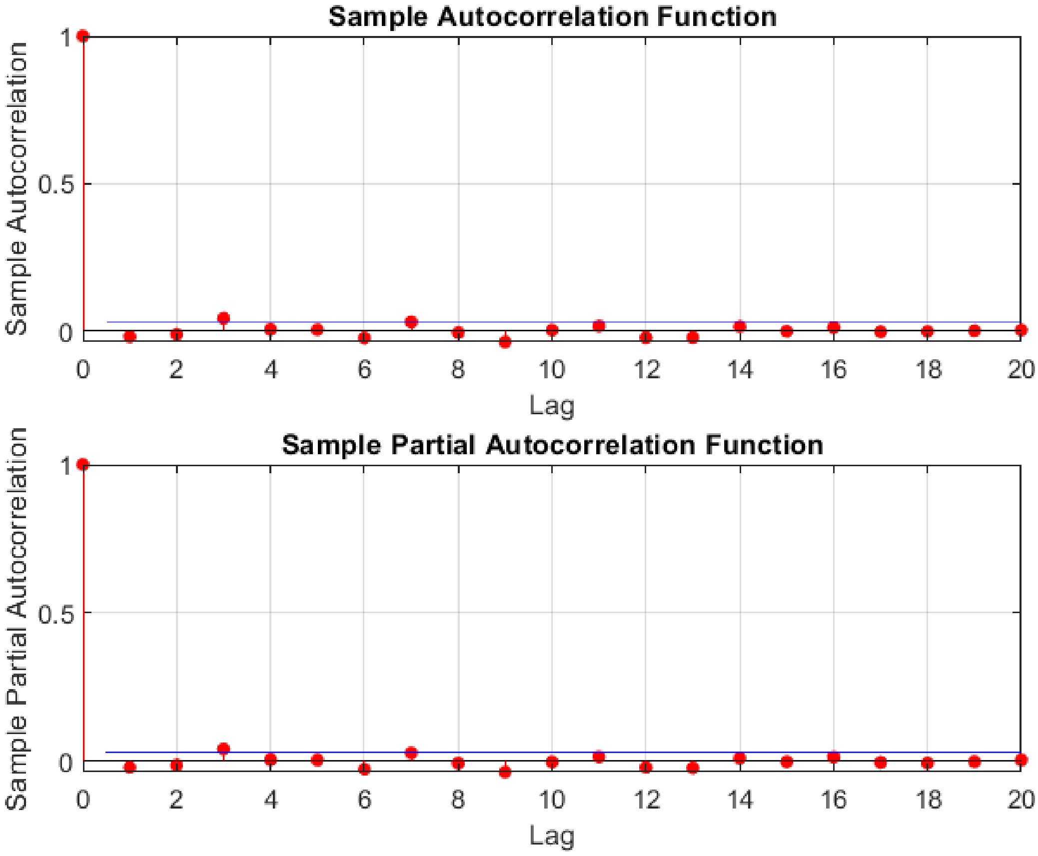

Figure 12: Plots with autocorrelation function and partial autocorrelation function of the 250 generated stocks.

D Results from Calibrating Quarticity Shrinkage

	W = 63				W = 126			
	5/20	10/15	15/10	20/5	5/20	10/15	15/10	20/5
Mean Return	**0.256**	0.250	0.245	0.241	**0.255**	0.248	0.242	0.236
Sh	1.352	1.423	1.536	**1.749**	1.328	1.389	1.498	**1.707**
CEQ ($\gamma = 1$)	**0.238**	0.234	0.232	0.231	**0.236**	0.232	0.229	0.226
CEQ ($\gamma = 5$)	0.166	0.173	0.181	**0.193**	0.163	0.168	0.177	**0.188**
TO	**0.065**	0.111	0.151	0.183	**0.033**	0.058	0.079	0.094
TCAR (4 bps)	**0.249**	0.238	0.230	0.222	**0.251**	0.242	0.234	0.226
TCAR (8.6 bps)	**0.242**	0.225	0.212	0.201	**0.248**	0.235	0.225	0.216
	W = 252				**W = 756**			
Mean Return	**0.253**	0.245	0.238	0.231	**0.251**	0.242	0.233	0.224
Sh	1.301	1.346	1.441	**1.639**	1.267	1.290	1.367	**1.541**
CEQ ($\gamma = 1$)	**0.234**	0.228	0.224	0.221	**0.232**	0.224	0.218	0.214
CEQ ($\gamma = 5$)	0.158	0.162	0.170	**0.181**	0.153	0.154	0.160	**0.171**
TO	**0.017**	0.030	0.041	0.050	**0.005**	0.010	0.014	0.018
TCAR (4 bps)	**0.252**	0.2419	0.2334	0.2257	**0.251**	0.241	0.231	0.222
TCAR (8.6 bps)	**0.250**	0.238	0.229	0.220	**0.250**	0.240	0.230	0.220

Table 13: This table shows the performance measures for the optimal partition of QS-NLS with 25 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO).

	W = 63				W = 126			
	5/20	10/15	15/10	20/5	5/20	10/15	15/10	20/5
Mean Return	**0.254**	0.245	0.237	0.230	**0.254**	0.246	0.239	0.231
Sh	1.364	1.425	1.500	**1.611**	1.365	1.427	1.499	**1.604**
CEQ ($\gamma = 1$)	**0.237**	0.230	0.225	0.220	**0.237**	0.231	0.226	0.221
CEQ ($\gamma = 5$)	0.167	0.171	0.175	**0.179**	0.167	0.172	0.175	**0.179**
TO	**0.131**	0.250	0.362	0.459	**0.116**	0.226	0.335	0.440
TCAR (4 bps)	**0.241**	0.220	0.201	0.184	**0.242**	0.223	0.205	0.187
TCAR (8.6 bps)	**0.226**	0.191	0.159	0.130	**0.229**	0.197	0.166	0.136
	W = 252				**W = 756**			
Mean Return	**0.253**	0.246	0.239	0.231	**0.252**	0.246	0.238	0.230
Sh	1.364	1.423	1.493	**1.594**	1.362	1.419	1.478	**1.570**
CEQ ($\gamma = 1$)	**0.236**	0.231	0.226	0.2205	**0.235**	0.231	0.225	0.2192
CEQ ($\gamma = 5$)	0.167	0.171	0.175	**0.179**	0.166	0.171	0.173	**0.176**
TO	**0.105**	0.210	0.316	0.425	**0.095**	0.195	0.302	0.414
TCAR (4 bps)	**0.243**	0.225	0.207	0.188	**0.243**	0.226	0.208	0.188
TCAR (8.6 bps)	**0.231**	0.200	0.170	0.139	**0.231**	0.204	0.173	0.140

Table 14: This table shows the performance measures for the optimal partition of QS-EWMA with 25 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO).

	W = 63				W = 126			
	20/80	40/60	60/40	80/20	20/80	40/60	60/40	80/20
Mean Return	**0.241**	0.225	0.215	0.207	**0.237**	0.220	0.210	0.201
Sh	1.450	1.551	1.745	**2.142**	1.415	1.512	1.714	**2.130**
CEQ ($\gamma = 1$)	**0.227**	0.214	0.208	0.202	**0.223**	0.209	0.203	0.197
CEQ ($\gamma = 5$)	0.172	0.172	0.177	**0.183**	0.167	0.167	0.173	**0.179**
TO	**0.130**	0.195	0.229	0.239	**0.075**	0.118	0.146	0.160
TCAR (4 bps)	**0.228**	0.205	0.192	0.183	**0.229**	0.208	0.196	0.185
TCAR (8.6 bps)	**0.213**	0.183	0.166	0.155	**0.221**	0.194	0.179	0.167
	W = 252				**W = 756**			
Mean Return	**0.233**	0.216	0.206	0.199	**0.228**	0.209	0.201	0.194
Sh	1.379	1.474	1.664	**2.123**	1.351	1.434	1.644	**2.093**
CEQ ($\gamma = 1$)	**0.219**	0.205	0.198	0.194	**0.213**	0.199	0.193	0.190
CEQ ($\gamma = 5$)	0.162	0.162	0.168	**0.177**	0.157	0.156	0.164	**0.172**
TO	**0.042**	0.068	0.086	0.098	**0.015**	0.025	0.032	0.037
TCAR (4 bps)	**0.229**	0.209	0.197	0.189	**0.226**	0.207	0.198	0.190
TCAR (8.6 bps)	**0.224**	0.201	0.187	0.178	**0.224**	0.204	0.194	0.186

Table 15: This table shows the performance measures for the optimal partition of QS-LS with 100 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, due to computational issues, the window of 756 days has been run for 500 replications instead of 1000.

	W = 63				**W = 126**			
	50/200	100/150	150/100	200/50	50/200	100/150	150/100	200/50
Mean Return	**0.222**	0.206	0.200	0.196	**0.215**	0.201	0.196	0.190
Sh	1.467	1.583	1.842	**2.434**	1.425	1.552	1.824	**2.440**
CEQ ($\gamma = 1$)	**0.210**	0.197	0.195	0.192	**0.204**	0.192	0.190	0.187
CEQ ($\gamma = 5$)	0.165	0.164	0.171	**0.180**	0.158	0.159	0.167	**0.175**
TO	**0.231**	0.297	0.309	0.291	**0.144**	0.202	0.228	0.230
TCAR (4 bps)	**0.198**	0.176	0.169	0.166	**0.201**	0.180	0.173	0.167
TCAR (8.6 bps)	**0.171**	0.142	0.133	0.133	**0.184**	0.157	0.146	0.140
	W = 252				**W = 756**			
Mean Return	**0.212**	0.196	0.191	0.187	**0.208**	0.190	0.187	0.184
Sh	1.387	1.514	1.787	**2.434**	1.324	1.436	1.726	**2.410**
CEQ ($\gamma = 1$)	**0.200**	0.188	0.185	0.184	**0.195**	0.182	0.181	0.182
CEQ ($\gamma = 5$)	0.154	0.154	0.162	**0.172**	0.146	0.146	0.158	**0.170**
TO	**0.084**	0.125	0.154	0.169	**0.030**	0.045	0.058	0.067
TCAR (4 bps)	**0.204**	0.184	0.175	0.170	**0.205**	0.186	0.181	0.178
TCAR (8.6 bps)	**0.194**	0.169	0.157	0.151	**0.201**	0.181	0.174	0.170

Table 16: This table shows the performance measures for the optimal partition of QS-LS with 250 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, we only ran the simulations with 250 generated stocks for 100 replications due to computation issues.

	W = 63				W = 126			
	50/200	100/150	150/100	200/50	50/200	100/150	150/100	200/50
Mean Return	**0.211**	0.201	0.200	0.195	**0.210**	0.201	0.199	0.196
Sh	1.408	1.568	1.901	**2.621**	1.356	1.353	1.655	**2.331**
CEQ ($\gamma = 1$)	**0.199**	0.192	0.194	0.192	**0.198**	0.190	0.192	0.192
CEQ ($\gamma = 5$)	0.155	0.160	0.172	**0.181**	0.150	0.146	0.163	**0.178**
TO	0.928	0.741	0.517	**0.393**	0.899	1.532	1.083	**0.631**
TCAR (4 bps)	0.117	0.126	0.148	**0.155**	0.119	0.047	0.090	**0.132**
TCAR (8.6 bps)	0.009	0.040	0.088	**0.110**	0.015	-0.131	-0.036	**0.059**
	W = 252				**W = 756**			
Mean Return	**0.211**	0.202	0.196	0.197	**0.208**	0.196	0.186	0.191
Sh	1.352	1.320	1.319	**1.384**	**1.306**	1.247	1.219	1.304
CEQ ($\gamma = 1$)	**0.199**	0.190	0.185	0.187	**0.195**	0.184	0.174	0.180
CEQ ($\gamma = 5$)	**0.150**	0.143	0.141	0.146	**0.145**	0.134	0.128	0.137
TO	**0.758**	1.358	1.841	2.163	**0.637**	1.139	1.561	1.913
TCAR (4 bps)	**0.135**	0.065	0.011	-0.021	**0.144**	0.081	0.029	-0.002
TCAR (8.6 bps)	**0.047**	-0.093	-0.203	-0.272	**0.070**	-0.051	-0.153	-0.224

Table 17: This table shows the performance measures for the optimal partition of QS-EWMA with 250 generated stocks. We display the ratio of the number of certain/uncertain stocks under the respective moving window of size **W** days. Within each row, we display the ratio that gives the best performance in the respective measure in bold. Further, note that we annualize all the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, we only ran the simulations with 250 generated stocks for 100 replications due to computation issues.

E Simulation Results & Robustness Results

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	0.073	0.088	0.084	0.097	**0.051**	**0.051**	**0.051**
Sh	0.426	0.396	0.399	**0.366**	0.393	0.395	0.394
CEQ ($\gamma = 1$)	0.114	0.111	0.105	0.116	**0.054**	**0.054**	0.055
CEQ ($\gamma = 5$)	0.411	0.330	0.313	0.324	**0.092**	**0.092**	**0.092**
TO	-	0.020	0.019	0.057	**0.011**	0.016	0.038
TCAR (4 bps)	-	0.088	0.083	0.097	**0.051**	**0.051**	**0.051**
TCAR (8.6 bps)	-	0.088	0.083	0.097	0.051	**0.050**	0.051
W = 126							
Mean Return	0.074	0.090	0.086	0.097	0.054	0.054	**0.052**
Sh	0.430	0.405	0.406	**0.368**	0.401	0.404	0.397
CEQ ($\gamma = 1$)	0.116	0.118	0.110	0.117	0.057	0.057	**0.055**
CEQ ($\gamma = 5$)	0.419	0.367	0.333	0.333	0.096	0.096	**0.092**
TO	-	0.012	0.011	0.057	**0.006**	0.009	0.033
TCAR (4 bps)	-	0.090	0.086	0.097	0.054	0.053	**0.052**
TCAR (8.6 bps)	-	0.090	0.086	0.097	0.054	0.053	**0.051**
W = 252							
Mean Return	0.076	0.094	0.091	0.099	0.057	0.056	**0.052**
Sh	0.437	0.419	0.419	**0.374**	0.415	0.417	0.400
CEQ ($\gamma = 1$)	0.120	0.120	0.116	0.121	0.061	0.060	**0.056**
CEQ ($\gamma = 5$)	0.435	0.366	0.349	0.345	0.103	0.103	**0.095**
TO	-	0.006	0.006	0.057	**0.004**	0.005	0.032
TCAR (4 bps)	-	0.094	0.091	0.099	0.057	0.056	**0.052**
TCAR (8.6 bps)	-	0.094	0.091	0.099	0.057	0.056	**0.052**
W = 756							
Mean Return	0.083	0.104	0.102	0.108	0.065	0.064	**0.057**
Sh	0.463	0.442	0.442	**0.398**	0.463	0.462	0.430
CEQ ($\gamma = 1$)	0.136	0.131	0.127	0.133	0.071	0.071	**0.061**
CEQ ($\gamma = 5$)	0.498	0.389	0.376	0.385	0.131	0.131	**0.108**
TO	-	**0.002**	**0.002**	0.057	**0.002**	**0.002**	0.030
TCAR (4 bps)	-	0.104	0.102	0.108	0.065	0.064	**0.057**
TCAR (8.6 bps)	-	0.104	0.102	0.108	0.065	0.064	**0.056**

Table 19: This table displays the standard deviations of the performance measures from all methods in the simulation with 25 generated stocks. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS combined with LS, NLS and EWMA follows the ratio of 12/13 certain/uncertain stocks for all moving windows. Within each row, we display the method that gives the best performance in the respective measure in bold. We annualize all standard deviations of the measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to the number of effective days.

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	0.095	0.107	0.102	0.109	**0.047**	0.072	0.048
Sh	0.424	0.404	0.419	**0.395**	0.427	0.443	0.405
CEQ ($\gamma = 1$)	0.250	0.292	0.282	0.299	**0.065**	0.212	0.067
CEQ ($\gamma = 5$)	1.020	1.195	1.162	1.208	**0.182**	0.855	0.183
TO	-	0.028	**0.015**	0.039	0.016	0.031	0.073
TCAR (4 bps)	-	0.107	0.102	0.109	**0.047**	0.072	0.049
TCAR (8.6 bps)	-	0.107	0.102	0.110	**0.047**	0.072	0.050
W = 126							
Mean Return	0.096	0.108	0.105	0.112	**0.048**	0.054	0.057
Sh	0.428	0.404	0.421	**0.348**	0.436	0.446	0.418
CEQ ($\gamma = 1$)	0.254	0.297	0.297	0.263	**0.066**	0.107	0.086
CEQ ($\gamma = 5$)	1.038	1.221	1.227	1.035	**0.185**	0.372	0.264
TO	-	0.026	0.017	0.080	**0.010**	0.012	0.032
TCAR (4 bps)	-	0.108	0.105	0.113	**0.048**	0.054	0.057
TCAR (8.6 bps)	-	0.108	0.105	0.114	**0.048**	0.054	0.057
W = 252							
Mean Return	0.100	0.112	0.109	0.116	**0.050**	**0.050**	0.058
Sh	0.433	0.416	0.428	**0.353**	0.451	0.459	0.426
CEQ ($\gamma = 1$)	0.264	0.312	0.307	0.269	**0.069**	0.070	0.089
CEQ ($\gamma = 5$)	1.077	1.276	1.264	1.052	**0.192**	0.195	0.273
TO	-	0.013	**0.006**	0.081	0.007	0.007	0.029
TCAR (4 bps)	-	0.112	0.109	0.117	**0.050**	**0.050**	0.059
TCAR (8.6 bps)	-	0.112	0.109	0.118	**0.050**	**0.050**	0.059
W = 756							
Mean Return	0.081	0.090	0.088	0.101	**0.047**	**0.047**	0.051
Sh	0.465	0.453	0.459	**0.376**	0.498	0.503	0.446
CEQ ($\gamma = 1$)	0.121	0.114	0.115	0.122	**0.049**	0.050	0.055
CEQ ($\gamma = 5$)	0.491	0.384	0.404	0.374	**0.092**	0.095	0.129
TO	-	0.005	**0.003**	0.083	**0.003**	**0.003**	0.027
TCAR (4 bps)	-	0.090	0.088	0.102	**0.047**	**0.047**	0.051
TCAR (8.6 bps)	-	0.090	0.088	0.104	**0.047**	**0.0477**	0.051

Table 20: This table displays the standard deviations of the performance measures from all methods in the simulation with 100 generated stocks. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS-LS and QS-NLS follow the ratio of 50/50 certain/uncertain stocks for all moving windows. QS-EWMA follows the ratio of 50/50 certain/uncertain stocks for $\mathbf{W} = 63$, and a ratio of 20/80 for $\mathbf{W} = 126$ and up. Within each row, we display the method that gives the best performance in the respective measure in bold. We annualize the standard deviations of all measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to the number of effective days.

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	0.062	0.067	0.063	0.068	0.031	**0.024**	0.032
Sh	**0.372**	0.429	0.455	0.463	0.427	0.451	0.474
CEQ ($\gamma = 1$)	0.065	0.071	0.068	0.073	0.031	**0.024**	0.032
CEQ ($\gamma = 5$)	0.170	0.178	0.175	0.179	0.039	**0.025**	0.042
TO	-	0.016	**0.012**	0.014	0.018	0.013	0.038
TCAR (4 bps)	-	0.067	0.063	0.068	0.031	**0.024**	0.032
TCAR (8.6 bps)	-	0.067	0.063	0.068	0.030	**0.023**	0.033
W = 126							
Mean Return	0.062	0.069	0.064	0.070	**0.032**	0.206	0.037
Sh	0.379	0.429	0.480	0.430	0.456	0.431	**0.379**
CEQ ($\gamma = 1$)	0.066	0.074	0.070	0.076	**0.032**	2.008	0.037
CEQ ($\gamma = 5$)	0.174	0.180	0.178	0.178	**0.041**	9.442	0.046
TO	-	0.020	**0.009**	0.022	0.015	2.010	0.093
TCAR (4 bps)	-	0.068	0.064	0.070	**0.032**	0.373	0.037
TCAR (8.6 bps)	-	0.068	0.064	0.070	**0.031**	0.594	0.040
W = 252							
Mean Return	0.065	0.071	0.064	0.072	**0.033**	**0.033**	0.041
Sh	0.380	0.411	0.380	**0.366**	0.458	0.467	0.377
CEQ ($\gamma = 1$)	0.069	0.077	0.069	0.079	**0.033**	**0.033**	0.041
CEQ ($\gamma = 5$)	0.181	0.178	0.179	0.177	**0.042**	0.045	0.057
TO	-	0.024	0.239	0.039	0.012	**0.008**	0.050
TCAR (4 bps)	-	0.070	0.070	0.072	**0.032**	0.033	0.041
TCAR (8.6 bps)	-	0.070	0.085	0.072	**0.032**	0.033	0.042
W = 756							
Mean Return	0.079	0.083	0.080	0.083	**0.040**	**0.040**	0.047
Sh	0.423	0.494	0.527	**0.396**	0.502	0.511	0.403
CEQ ($\gamma = 1$)	0.084	0.091	0.088	0.092	**0.040**	0.041	0.047
CEQ ($\gamma = 5$)	0.215	0.226	0.220	0.207	**0.052**	0.056	0.064
TO	-	2.400	**0.760**	10.149	1.340	0.913	11.306
TCAR (4 bps)	-	0.083	0.080	0.083	**0.040**	**0.040**	0.048
TCAR (8.6 bps)	-	0.083	0.080	0.083	**0.040**	**0.040**	0.048

Table 21: This table displays the standard deviations of the performance measures from all methods in the simulation with 250 generated stocks. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS-LS follows the ratio of 125/125 certain/uncertain stocks for all moving windows. QS-NLS follows a ratio of 200/50 for $\mathbf{W} = 63$, and also follows the 125/125 ratio for the other windows. QS-EWMA follows the ratio of 125/125 certain/uncertain stocks for $\mathbf{W} = \{63, 126\}$, and a ratio of 50/200 for $\mathbf{W} = \{252, 756\}$. Within each row, we display the method that gives the best performance in the respective measure in bold. We annualize the standard deviations of all measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to the number of effective days.

F Significance Testing Empirical Analysis

	QS-LS		QS-NLS		QS-EWMA	
	1/N	LS	1/N	NLS	1/N	EWMA
W = 63						
p-value	0.520	0.065	0.374	0.247	0.556	0.347
W = 126						
p-value	0.700	0.131	0.684	0.117	0.488	0.385
W = 252						
p-value	0.495	0.049	0.808	0.058	0.898	0.484
W = 756						
p-value	0.233	0.057	0.371	0.216	0.250	0.526

Table 22: This table displays the p-values from the t-test in the empirical analysis. We display the existing method used as input for the new method above the separation line. Underneath this line is the name of the estimator with which we compare the mean return. The tests are independent from one other. The null hypothesis is that of an equal mean return.

	NM with LS		NM with NLS		NM with EWMA	
	1/N	LS	1/N	NLS	1/N	EWMA
W = 63						
p-value	0.117	0.326	0.074	0.814	0.035	0.743
W = 126						
p-value	0.065	0.413	0.016	0.479	0.634	0.941
W = 252						
p-value	0.109	0.140	0.006	0.276	0.857	0.639
W = 756						
p-value	0.285	0.175	0.107	0.805	0.177	0.085

Table 23: This table displays the p-values from the Sh-test in the empirical analysis. We display the existing method used as input for the new method above the separation line. Underneath this line is the name of the estimator with which we compare the Sharpe ratio. The tests are independent from one other. The null hypothesis is that of an equal Sharpe ratio.

	Existing Methods				Quarticity Shrinkage		
	1/N	LS	NLS	EWMA	LS	NLS	EWMA
W = 63							
Mean Return	**0.281**	**0.281**	**0.281**	**0.281**	0.220⊗	0.229⊗	0.214⊗
Sh	1.126	1.051	1.093	1.026	**1.633**⊗	1.576⊗	1.522⊗
CEQ ($\gamma = 1$)	**0.250**	0.245	0.248	0.244	0.211	0.218	0.204
CEQ ($\gamma = 5$)	0.125	0.102	0.116	0.093	0.175	**0.176**	0.165
TO	-	0.263	0.252	0.585	**0.216**	0.433	0.862
TCAR (4 bps)	**0.281**	0.254	0.255	0.222	0.198	0.185	0.127
TCAR (8.6 bps)	**0.281**	0.224	0.226	0.154	0.173	0.135	0.027
W = 126							
Mean Return	0.281	0.281	0.281	**0.283**	0.215⊗	0.223⊗	0.232⊗
Sh	1.125	1.044	1.085	0.919	1.596⊗	**1.617**⊗	1.435⊗
CEQ ($\gamma = 1$)	**0.250**	0.245	0.247	0.236	0.206	0.214	0.219
CEQ ($\gamma = 5$)	0.125	0.100	0.113	0.046	0.169	**0.176**	0.167
TO	-	0.193	0.301	30.029	**0.133**	0.144	0.389
TCAR (4 bps)	**0.281**	0.262	0.250	0.160	0.201	0.209	0.193
TCAR (8.6 bps)	**0.281**	0.239	0.215	0.019	0.186	0.192	0.148
W = 252							
Mean Return	0.281	0.282	0.281	**0.283**	0.211⊗	0.218⊗	0.232⊗
Sh	1.126	1.057	1.085	0.911	1.560⊗	**1.584**⊗	1.434⊗
CEQ ($\gamma = 1$)	**0.250**	0.246	0.248	0.235	0.202	0.209	0.219
CEQ ($\gamma = 5$)	0.125	0.104	0.113	0.042	0.165	**0.171**	0.166
TO	-	0.115	0.090	1.245	0.078	**0.074**	0.342
TCAR (4 bps)	**0.281**	0.270	0.272	0.157	0.203	0.211	0.197
TCAR (8.6 bps)	**0.281**	0.257	0.261	0.013	0.194	0.202	0.157
W = 756							
Mean Return	**0.278**	**0.278**	**0.278**	**0.278**	0.205⊗	0.210⊗	0.231⊗
Sh	1.151	1.087	1.100	0.917	1.523⊗	**1.533**⊗	1.452⊗
CEQ ($\gamma = 1$)	**0.249**	0.246	0.246	0.232	0.196	0.200	0.218
CEQ ($\gamma = 5$)	0.132	0.115	0.118	0.048	0.160	0.163	**0.168**
TO	-	0.040	0.034	1.245	0.028	**0.027**	0.303
TCAR (4 bps)	**0.278**	0.274	0.275	0.152	0.202	0.207	0.200
TCAR (8.6 bps)	**0.278**	0.270	0.271	0.008	0.199	0.204	0.165

Table 18: This table displays the performance measures of all methods in the simulation study with 100 generated stocks. We display the existing method that we use as input for quarticity shrinkage underneath 'Quarticity Shrinkage'. QS-LS and QS-NLS follow the ratio of 50/50 certain/uncertain stocks for all moving windows. QS-EWMA follows the ratio of 50/50 certain/uncertain stocks for $\mathbf{W} = 63$, and a ratio of 20/80 for $\mathbf{W} = 126$ and up. Within each row, we display the method that gives the best performance in the respective measure in bold. Note that QS obtains mean returns and Sharpe ratios which are always significantly different from the $1/N$ allocation and the existing method. The differences are significant at all levels following the tests described in Section 4.5.5. We display this phenomenon by ⊗ in the table. We annualize all measures by means of a scaling factor of 252 or its square root. We do not annualize the turnover (TO). Furthermore, for both EWMA and QS-EWMA we use the previous 186 days. This value corresponds to the number of effective days.