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MSc Thesis Quantitative Finance

Economic Forecast Evaluation of the Nelson-Siegel Class of Yield Curve Models and Estimation by using the EM Algorithm

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Abstract

This paper examines the out-of-sample forecasting performance of several nested Nelson-Siegel yield curve model specifications with statistical and economic measures. I introduce the application of a constrained EM algorithm for estimating the Nelson-Siegel class of models. Furthermore, I implement an economic forecast evaluation based on mean-variance optimal bond portfolios and the comparison of their Sharpe Ratios. Next, I examine the relation between the statistical and economic measures. The empirical application of this research is based on the US Treasury yield dataset by Liu and Wu (2019). The results indicate that the EM approach show quite promising results for 6 and 12-step ahead forecasting horizons. Furthermore, the outcome of the statistical and economic evaluation can deviate, and the reported strong negative relation for specific risk aversion choices and maturities can help on the decision-making of which model specification is preferable.

Keywords: Term Structure of Interest Rates; Nelson-Siegel model; State-Space model; Forecasting; Mean-Variance Portfolio Optimization; EM Algorithm; Kalman Filter

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1 Introduction

The term structure of interest rates is of crucial importance for a variety of economic agents such as central banks, treasuries, financial institutions and portfolio managers. Modelling and forecasting the term structure of interest rates remain a very challenging task. A class of yield curve models which is widely used in several applications is the Nelson-Siegel class.

This research is focused on the evaluation of the out-of-sample forecasting performance of several yield curve model specifications within the Nelson-Siegel class of models. This class was initially introduced by Nelson and Siegel (1987) to fit the term structure and later extended by Diebold and Li (2006) by combining the factor representation of the yields with autoregressive specifications for the dynamics of the three factors. They showed that the dynamic version of this model can produce quite accurate forecasts as well. Following De Pooter (2007), I examine several Dynamic Nelson-Siegel (DNS) specifications with different number of factors and decay parameters (Diebold and Li (2006); Diebold et al. (2005); Björk and Christensen (1999); Bliss (1997); Svensson (1994)).

Since the aforementioned yield curve models are nested, I capture their dynamics in a state-space representation and estimate them under two different approaches, a Maximum Likelihood (ML) approach and a new for this class of models, constrained Expectation Maximization (EM) approach, which is a contribution of this research. As Diebold et al. (2006) have shown, these models can be implemented under a state-space modelling framework that lets us simultaneously fit the yield curve at each point in time and estimate the underlying dynamics of the factors. De Pooter (2007) performed two different estimation methods, a two-step procedure of a cross-sectional model and a time series model which are estimated separately, and furthermore, a one-step procedure under the state-space framework, which estimates the parameters with the maximum likelihood and applies the Kalman filter to obtain optimal factor estimates. As the chosen framework of this research is the state-space, I only implement the one-step procedure, named as ML approach.

The constrained version of the Expectation-Maximisation algorithm that this research implements, is known as more robust and attractive estimation method when there are many parameters. Holmes (2013) present derivations of the EM algorithm for general cases of unconstrained and constrained multivariate autoregressive Gaussian state-space models. By following their paper, I derive the analytical expressions for the model parameters of this research. The EM algorithm was originally introduced by Dempster et al. (1977) and firstly applied in the state-space framework by Shumway and Stoffer (1982). Creal and Wu (2015) apply the classical EM algorithm to affine term structure models and Watson and Engle (1983) implement the EM in dynamic factor models. However, this method has not yet been applied in the Nelson-Siegel class of models and it is of interest to compare the results of this implementation with the ML approach.

Since the forecasting ability of individual models usually varies over time considerably and often they have similar predictive accuracy, this thesis examines the performance of a forecast combination of the their outcome. The motivation to combine forecasts comes from an important result from the methodological literature on forecasting, which shows that a linear combination of two or more forecasts may yield more accurate predictions than using only a single forecast (Granger (1989); Newbold and Harvey (2002); Aiolfi and Timmermann (2006)). Taking averages of such forecasts is likely to improve predictive accuracy. According to Clark and McCracken (2009), a forecast combination of nested models, as the ones that are used in this research, can improve the forecasting accuracy comparing to the performance of the individual models. Furthermore, their forecasting performance is compared with competitive statistical models such as the random walk, the AR(1), the VAR(1).

Furthermore, another point of contribution of this research is the extension of the classic forecasting evaluation of the nested DNS models based on statistical measures, by implementing an economic evaluation of their forecasts. The common way to evaluate the forecasting performance of yield curve models in the existing literature is by computing statistical measures such as the Root Mean Squared Forecast Error (RMSFE) and the Trace Root Mean Squared Forecast Error (TRMSFE). Since the statistical differences between the accuracy of different forecasts are usually not significant, economic criteria can be taken into account in the economic and financial decision making process. Leitch and Tanner (1991) argue that these statistical measures may not be closely related to a forecast's profits. Among several attempts in the literature to approach this puzzle, Caldeira et al. (2016b) propose an economic evaluation of the forecasts based on mean–variance optimal bond portfolios as introduced by Markowitz (1952), and then a comparison based on their Sharpe ratios. This thesis follows this approach to economically evaluate the performance of the nested Nelson-Siegel models and deviates from Caldeira et al. (2016b) by not restricting the short-selling.

Another point of this research that comes naturally, is the examination of the relation between the results of the statistical and economic evaluation. Caldeira et al. (2016b) discuss the relation between the statistical and the economic performance measures, and according to their findings a potential negative relation between the performance measures is observed. Therefore, a closer look on this relation can help by indicating the cases where the statistical and economic criteria can be used together for the desicion-making of which nested model is preferable. This thesis deviates from Caldeira et al. (2016b) by examining the correlation between the TRMSFE and the Sharpe Ratios for each risk aversion choice, and extends this analysis by also taking into account the correlation between the RMSFE per maturity and the Sharpe Ratios for each risk aversion choice.

Finally, this research uses the new dataset by Liu and Wu (2019). Their method generates a smoothed yield curve and preserves information in the raw data and has much smaller pricing errors than another popular method, Gürkaynak et al. (2007). It is very interesting that Liu and Wu (2019) repeat two significant studies of Cochrane and Piazzesi (2005) and Giglio and Kelly (2018) and draw different conclusions. Hence, the use of this dataset in my research can also result in interesting outcomes. I therefore repeat the one-step approach with AR(1) dynamics of De Pooter (2007) with this dataset, and directly compare with the De Pooter (2007) results, that use the Fama and Bliss (1987) dataset, for the sample period Jan 1984 to Dec 2003 and the presented maturities 1, 3, 6, 12, 24, 60, 84 and 120 months.

The main findings of this paper are that the DNS models under the EM approach are quite competitive for 6 and 12-step ahead forecasting horizons, according to the statistical evaluation. Also, the Forecast Combination of the nested models performs better than most of the individual models. Furthermore, the direct comparison with the De Pooter (2007) results, shows that indeed, the conclusions differ significantly when one uses the Liu and Wu (2019) dataset, as the performance of DNS models seems to be in general lower. Regarding the economic evaluation, it seems that the results can sometimes deviate from what one concludes by using statistical criteria, and furthermore, the choice of risk aversion plays major role on the choice of which DNS variant produces more economically meaningful forecasts. Finally, by analysing the correlation coefficient per risk aversion value, better insights regarding the relation of the statistical measures and the Sharpe Ratios can be obtained. Strong negative relation is reported in several cases, in an aggregate level and per maturity as well.

The remainder of the paper is organised as follows: Section 2 describes the models and methodology, Section 3 describes the data, Section 4 presents the statistical and economic evaluation of the forecasting results, with Section 5 concluding the thesis with suggested extensions to the future scope of work.

2 Models and Methodology

In this section a general introduction of the Term Structure of Interest Rates (Section 2.1), the nested Nelson-Siegel models (Section 2.2) and the State-Space framework that captures them (Section 2.3) are introduced. Furthermore, the two estimation methods (Section 2.4) are outlined, as well as the Forecasting procedure (Section 2.5) , the combination scheme and the two difference forecast evaluation methods.

2.1 Term Structure of Interest Rates

Let $P_t(\tau)$ denote the price of a zero-coupon bond at time t with time to maturity $\tau = T - t$, where T is the maturity date. In other words, if the zero-coupon bond pays back 1$ at maturity date, then $P_t(\tau)$ is the today's value of receiving 1$ in $t = T - \tau$ years. Furthermore, let $y_t(\tau)$ denote the continuously compounded nominal yield to maturity of the zero-coupon bond. The collection of bond prices for all different maturities τ is the discount curve, and can be obtained from the yield curve: $P_t(\tau) = e^{-\tau y_t(\tau)}$. Moreover, from the discount curve we can obtain the instantaneous forward rate curve $f_t(\tau) = -\frac{1}{P_t(\tau)}\frac{\mathrm{d}P_t(\tau)}{\mathrm{d}\tau}$.

The term structure of interest rates can be characterized by any of the previously defined curves. In practice, they cannot be observed for all maturities, however they can be estimated from the observed bond prices. Several methods have been proposed for the estimation of the term structure, with the Fama and Bliss (1987) and Gürkaynak et al. (2007) being among the most popular approaches. The chosen dataset for this research Liu and Wu (2019) overcomes limitations of the aforementioned popular approaches. Modeling and forecasting the development of the term structure of interest rates over time comes naturally as the next step. The Nelson-Siegel class of models is the main focus of this research.

2.2 Nelson-Siegel class of models

Nelson and Siegel (1987) proposed as an approximation of the forward rate curve the following factor-formed function,

$$f(\tau) = \beta_1 + \beta_2 e^{-\lambda\tau} + \beta_3 \lambda e^{-\lambda\tau}, \tag{1}$$

which is given by the solution of a second-order differential equation with real and equal roots. The term $f(\tau)$ represents the instantaneous forward rate with time to maturity τ, the β_1, β_2 and β_3 are static parameters and λ is a decay parameter. The aforementioned function can be transformed to the yield curve below,

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} \right) + \beta_3 \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} - e^{\lambda\tau} \right). \tag{2}$$

The term $y(\tau)$ represents the bond yield with time to maturity τ, the β_1, β_2 and β_3 are static parameters and λ is a decay parameter. Due to the parsimonious structure of the Nelson-Siegel model, this method is widely used from many central banks. The characteristic of calculating easily the instantaneous short rate value $\lim_{\tau \to 0} y(\tau) = \beta_1 + \beta_2$, and the value of $y(\tau)$ when the maturity tends to infinity, $\lim_{\tau \to +\infty} y(\tau) = \beta_1$, is also important. The main focus of this research is the evaluation of some dynamic versions of the Nelson-Siegel model. Similarly with De Pooter (2007), I evaluate different approaches with two, three and four dynamic factors.

2.2.1 Diebold and Li (2006)

The following is a dynamic version of the Nelson-Siegel model, introduced by Diebold and Li (2006) (**DNS 1**). The three factors $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ are allowed to vary over time and can be interpreted as latent dynamic factors that captures the level, slope and curvature of the yield curve. The DNS 1 specification is given below:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t} \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} \right) + \beta_{3,t} \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} - e^{\lambda\tau} \right), \tag{3}$$

where $y_t(\tau)$ represents the bond yield at time t with time to maturity τ. Furthermore, λ is a decay parameter and is assumed to be constant over time, as Diebold and Li (2006) suggest, in order to simplify the computations while maintaining the consequential loss in generality at a minimal level. However, the decay parameter λ_t can be also treated as a time-varying parameter. For instance, Koopman et al. (2010) propose to treat λ_t as a fourth latent factor, by also taking account of its dynamics by including it in a Vector Auto-Regressive specification of all four latent factors.

2.2.2 Diebold, Piazzesi, and Rudebusch (2005)

The second specification is a two-factor Nelson-Siegel model, as it contains only the level and slope factors, $\beta_{1,t}$ and $\beta_{2,t}$. Diebold et al. (2005) examine this specification (**DNS 2**), due to the fact that the first two principal components explain the greater amount of variation in interest rates could result accurate forecasts. The DNS 2 specification is given by:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t} \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} \right), \tag{4}$$

where $y_t(\tau)$ represents the bond yield at time t with time to maturity τ and λ is a decay parameter.

2.2.3 Björk and Christensen (1999)

Björk and Christensen (1999) (**DNS 3**) suggest the addition of a fourth factor $\beta_{4,t}$, which can be interpreted as a second slope factor. The DNS 3 specification is the following:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t} \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} \right) + \beta_{3,t} \left(\frac{1 - e^{\lambda\tau}}{\lambda\tau} - e^{\lambda\tau} \right) + \beta_{4,t} \left(\frac{1 - e^{2\lambda\tau}}{2\lambda\tau} \right), \tag{5}$$

where $y_t(\tau)$ represents the bond yield at time t with time to maturity τ, λ is a decay parameter and the $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ can be interpreted as the level factor, the first slope factor and the curvature factor of the yield curve.

2.2.4 Bliss (1997)

The next specification is introduced by Bliss (1997) (**DNS 4**). The difference with the three-factor Nelson-Siegel model is that the slope and curvature factors are governed by two different decay parameters λ_1 and λ_2. The DNS 4 specification is the following:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t} \left(\frac{1 - e^{\lambda_1\tau}}{\lambda_1\tau} \right) + \beta_{3,t} \left(\frac{1 - e^{\lambda_2\tau}}{\lambda_2\tau} - e^{\lambda_2\tau} \right), \tag{6}$$

where $y_t(\tau)$ represents the bond yield at time t with time to maturity τ and the $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ are dynamic latent factors.

2.2.5 Svensson (1994)

Svensson (1994) proposes a four-factor Nelson-Siegel model (**DNS 5**). The purpose of the fourth factor, $\beta_{4,t}$, is to add extra flexibility and fit to the model. This factor has its own decay parameter and can be interpreted as a second curvature. The DNS 5 specification is given by:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t} \left(\frac{1 - e^{\lambda_1\tau}}{\lambda_1\tau} \right) + \beta_{3,t} \left(\frac{1 - e^{\lambda_1\tau}}{\lambda_1\tau} - e^{\lambda_1\tau} \right) + \beta_{4,t} \left(\frac{1 - e^{\lambda_2\tau}}{\lambda_2\tau} - e^{\lambda_2\tau} \right), \tag{7}$$

where $y_t(\tau)$ represents the bond yield at time t with time to maturity τ, λ_1 and λ_2 are decay parameters and the $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ can be interpreted as the level, slope and the first curvature factor of the yield curve.

2.2.6 Adjusted Svensson (1994)

The final specification of this research is an adjusted Svensson model, introduced by De Pooter (2007) (**DNS 6**). In order to address a potential multicolinearity issue of the Svensson model, De Pooter (2007) adjusts the second curvature component. The DNS 6 specification is the following:

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{\lambda_1\tau}}{\lambda_1\tau}\right) + \beta_{3,t}\left(\frac{1-e^{\lambda_1\tau}}{\lambda_1\tau} - e^{\lambda_1\tau}\right) + \beta_{4,t}\left(\frac{1-e^{\lambda_2\tau}}{\lambda_2\tau} - e^{2\lambda_2\tau}\right), \quad (8)$$

where $y_t(\tau)$ represents the bond yield at time t with time to maturity τ, λ_1 and λ_2 are decay parameters and the $\beta_{1,t}$, $\beta_{2,t}$, $\beta_{3,t}$ and $\beta_{4,t}$ can be interpreted as the level, slope, first and second curvature factor of the yield curve.

2.3 State Space framework

Due to the fact that Diebold and Li (2006) show that the dynamics of the latent factors can be captured by autoregressive processes, Diebold et al. (2006) identify that the Diebold and Li (2006) model can be recognized as a state-space form. Therefore, the nested Nelson-Siegel specifications of this research can be captured in one general state-space representation, which is given below. The observation equation regards the cross-section of the yields at any point in time with the latent factors that represent the level, slope and curvature of the yield curve and their number differ, depending the specification of this research. The state equation captures the dynamics of the latent factors by a first-order autoregressive specification, which is preferred in this research.

$$\begin{aligned} \mathrm{Y_t} &= \boldsymbol{\Lambda}\boldsymbol{\beta}_t + \boldsymbol{\varepsilon}_t \\ \boldsymbol{\beta}_t &= \boldsymbol{\mu} + \mathbf{A}\boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \end{aligned} \quad (9)$$

where $\boldsymbol{\beta}_t$ is the vector of the latent factors, $\boldsymbol{\Lambda}$ the matrix of factor loadings, $\mathrm{Y_t} = [y_t(\tau_1)\cdots y_t(\tau_N)]'$ the vector of yields on N different maturities, $\boldsymbol{\mu}$ a vector of intercepts, $\mathbf{A}$ the matrix of autoregressive parameters, and the vectors of error terms are assumed to follow normal distribution:

$$\begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\eta}_t \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mathbf{0}_{N\times 1} \\ \mathbf{0}_{K\times 1} \end{pmatrix}, \begin{pmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{pmatrix}\right]. \tag{10}$$

Each of the above Nelson-Siegel models is a special case of this framework with a different number of factors and/or a different specification for the factor loadings. The covariance matrix of the observation equation $\mathbf{H}$, is assumed to be diagonal, which means that there is no correlation between the deviations of yields from the yield curve for different maturities. Due to the AR(1) specification of the state equation that is preferred in this research based on the "shrinkage principle" of Diebold and Li (2006), the covariance matrix $\mathbf{Q}$ is also diagonal. Hence the potential correlation between the latent factors is not taken into consideration.

2.4 Estimation

In order to estimate the latent factors and parameters I follow two different approaches. Both are based on simultaneous estimation of the parameters of the state-space model. In the first approach, I perform the one-step state-space estimation in a similar way with De Pooter (2007). The parameters are estimated concurrently by maximizing the likelihood function, and the latent factors by using the Kalman filter. This approach for the Nelson-Siegel model was initially introduced by Diebold et al. (2006). The second approach involves a simultaneous estimation of the parameters by using a constrained version of the EM algorithm and the Kalman filter for filtering and smoothing the latent factors. The ability of the EM algorithm to manage in a better way the estimation of large systems of parameters, motivates the implementation of this approach for the estimation of the Nelson-Siegel class of models.

2.4.1 Maximum Likelihood (ML) approach

The main goal of this approach is to estimate the unknown parameters and the latent factors on the basis of the observations of yields. Under the normality assumption for the conditional distribution of Y_t, the estimation of the parameters is done by finding the values for the parameters that maximize the prediction error decomposition of the log-likelihood function (see Harvey (1990)),

$$\begin{aligned} \mathbf{L} &= \sum_{t=1}^{T} \log f(\mathrm{Y_t}|\mathcal{I}_{t-1};\boldsymbol{\theta}) \\ &= -\frac{T17}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log|\Sigma_t(\boldsymbol{\theta})| - \frac{1}{2}\sum_{t=1}^{T}\left(\mathrm{Y_t}-\mu_t(\boldsymbol{\theta})\right)'\left(\Sigma_t(\boldsymbol{\theta})\right)^{-1}\left(\mathrm{Y_t}-\mu_t(\boldsymbol{\theta})\right), \end{aligned} \tag{11}$$

where $\mu_t(\boldsymbol{\theta}) = \mathbb{E}(\mathrm{Y_t}|\mathcal{I}_{t-1};\boldsymbol{\theta}) = \boldsymbol{\Lambda}\hat{\boldsymbol{\beta}}_{t|t-1}$ and $\Sigma_t(\boldsymbol{\theta}) = \mathrm{Var}(\mathrm{Y_t}|\mathcal{I}_{t-1};\boldsymbol{\theta}) = \boldsymbol{\Lambda}\mathbf{P}_{t|t-1}\boldsymbol{\Lambda}' + \mathbf{H}$, and $\boldsymbol{\theta}$ is the set of parameters. The estimates for the latent factors $\left[\hat{\boldsymbol{\beta}}_{t|t-1}\right]_{t=1}^{T}$ and their conditional variance $\left[\mathbf{P}_{t|t-1}\right]_{t=1}^{T}$ are obtained by using the Kalman Filter[1].

In order to maximize the expression (11) above, a numerical optimization method in Matlab is used, the $fminunc$[2]. This nonlinear programming method starts by using an initialization of the unknown parameters, it attempts by making better guesses to find the minimum of the relevant function. The maximization is achieved by optimizing the negative log-likelihood function. I impose constraints such that the estimates λ and the variances avoid negative values, and $\alpha_{i,i} \in [-1, 1]$ by estimating the log parameters $\log(\lambda)$, $\log(h_{i,i})$, $\log(q_{i,i})$ and $\log(\frac{1+\alpha_{i,i}}{1-\alpha_{i,i}})$.

Initialization

The initialization of the parameters could play significant role on the accuracy of the estimates. I follow the two-step procedure of Diebold and Li (2006). The decay parameter(s) of the models are set to 0.0609 as Diebold and Li (2006) and afterwards, the first step is to estimate the values of the latent factors $\forall t \in \{1, \ldots, T\}$ by using cross-sectional OLS. In the second step, I estimate the parameters of an AR(1) specification for each one of the estimated latent factors from the first step. Hence, the initial estimates for $\boldsymbol{\mu}$, $\mathbf{A}$, $\mathbf{H}$ and $\mathbf{Q}$ are obtained.

Regarding the initialization of the Kalman Filter, I follow Kim et al. (1999) and use the unconditional mean and variance matrix of $\boldsymbol{\beta_t}$ as values for $\hat{\boldsymbol{\beta}}_{0|0}$ and $\mathbf{P}_{0|0}$. Their derived expressions are the following:

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{0|0} &= \left(\mathbf{I}_m - \mathbf{A}\right)^{-1}\boldsymbol{\mu} \\ \mathrm{vec}(\mathbf{P}_{0|0}) &= \left(\mathbf{I}_{m^2} - \mathbf{A}\otimes\mathbf{A}\right)^{-1}\mathrm{vec}(\mathbf{Q}), \end{aligned} \tag{12}$$

where m is the number of the latent factors of the estimated model.

[1] A brief explanation of the Kalman Filter and the derived expressions of the Prediction and Updating steps can be found in the Appendix A

[2] `https://nl.mathworks.com/help/optim/ug/fminunc.html`

Algorithm

The ML-Kalman Filter estimation approach can be summarized in the following steps.

1. Initialization of the parameters $\boldsymbol{\theta}^{(0)}$;
2. Use Kalman Filter to obtain estimates for the latent factors $\left[\hat{\boldsymbol{\beta}}_{t|t-1}(\boldsymbol{\theta}^{(0)})\right]_{t=1}^{T}$ and their conditional variance $\left[\mathbf{P}_{t|t-1}(\boldsymbol{\theta}^{(0)})\right]_{t=1}^{T}$;
3. Use $\left[\hat{\boldsymbol{\beta}}_{t|t-1}(\boldsymbol{\theta}^{(0)})\right]_{t=1}^{T}$ and $\left[\mathbf{P}_{t|t-1}(\boldsymbol{\theta}^{(0)})\right]_{t=1}^{T}$ in the log-likelihood to obtain new estimates for the parameters, $\boldsymbol{\theta}^{(1)}$;
   - The quasi-newton approach of the *fminunc* algorithm attempts a new step, in order to optimize its criteria. The new set of parameters $\boldsymbol{\theta}^{(1)}$ will be used as input of the Kalman Filter in the next iteration.
4. Iterate steps 2. and 3. until the finding of the optimal parameter values $\boldsymbol{\theta}^{ML}$ that maximize the negative log-likelihood;

2.4.2 Expectation-Maximization (EM) approach

The EM Algorithm is an iterative estimation method which provides a maximum of the likelihood function using a procedure of two steps; the Expectation-step and the Maximization-step. The M-step calculates the parameter estimates given the full data set. The E-step estimates the unobserved latent factors based on the observed data and the given parameter values. Since the parameter matrices **A**, **Q** and **H** in the models of this research are diagonal, this case is considered as a constrained EM algorithm. This means that the analytical derivations of the parameter estimates under the simple case of using the EM algorithm for a state space model do not apply. Hence, by following Holmes (2013), the analytical solutions for the parameters of this research are derived.

The joint log-likelihood of the complete data is the following:

$$\begin{aligned}\mathbf{L} = &-\frac{1}{2}\sum_{t=1}^{T}\left(\mathrm{Y}_t - \boldsymbol{\Lambda}\boldsymbol{\beta}_t\right)'\mathbf{H}^{-1}\left(\mathrm{Y}_t - \boldsymbol{\Lambda}\boldsymbol{\beta}_t\right) - \frac{T}{2}\log|\mathbf{H}| \\ &-\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{\beta}_t - \boldsymbol{\mu} - \mathbf{A}\boldsymbol{\beta}_{t-1}\right)'\mathbf{Q}^{-1}\left(\boldsymbol{\beta}_t - \boldsymbol{\mu} - \mathbf{A}\boldsymbol{\beta}_{t-1}\right) - \frac{T}{2}\log|\mathbf{Q}|\,.\end{aligned} \tag{13}$$

By implementing the vec-rule, $\text{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\text{vec}(\mathbf{B})$ the joint log-likelihood can be rewritten as:

$$\begin{aligned}
\mathbf{L} =& -\frac{1}{2}\sum_{t=1}^{T}\left(\mathrm{Y}_t - (\boldsymbol{\beta}_t' \otimes \mathbf{I}_n)\text{vec}(\boldsymbol{\Lambda})\right)' \mathbf{H}^{-1}\left(\mathrm{Y}_t - (\boldsymbol{\beta}_t' \otimes \mathbf{I}_n)\text{vec}(\boldsymbol{\Lambda})\right) - \frac{T}{2}\log|\mathbf{H}| \\
& -\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{\beta}_t - \text{vec}(\boldsymbol{\mu}) - (\boldsymbol{\beta}_{t-1}' \otimes \mathbf{I}_m)\text{vec}(\mathbf{A})\right)' \mathbf{Q}^{-1}\left(\boldsymbol{\beta}_t - \text{vec}(\boldsymbol{\mu}) - (\boldsymbol{\beta}_{t-1}' \otimes \mathbf{I}_m)\text{vec}(\mathbf{A})\right) \\
& -\frac{T}{2}\log|\mathbf{Q}|\,.
\end{aligned} \tag{14}$$

As the above function depends on the unobserved factors, one can take the expectation of the joint log-likelihood conditional on the yield curve data. Afterwards, by taking the first-order conditions the following analytical expressions can be derived, that maximize the joint log-likelihood:

$$\begin{aligned}
\boldsymbol{\mu} =& \frac{1}{T}\sum_{t=1}^{T}\left(\hat{\boldsymbol{\beta}}_{t|T} - \left(\hat{\boldsymbol{\beta}}_{t-1|T}' \otimes \mathbf{I}_m\right)\text{vec}(\mathbf{A})\right), \\
\boldsymbol{\alpha} =& \left(\mathbf{D}_{\boldsymbol{\alpha}}'\left(\sum_{t=1}^{T}\left(\mathbf{P}_{t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}_{t-1|T}'\right) \otimes \mathbf{Q}^{-1}\right)\mathbf{D}_{\boldsymbol{\alpha}}\right)^{-1} \cdot \\
& \cdot \mathbf{D}_{\boldsymbol{\alpha}}'\left(\sum_{t=1}^{T}\text{vec}\left(\mathbf{Q}^{-1}\left(\mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t-1|T}'\right)\right) - \sum_{t=1}^{T}\text{vec}\left(\mathbf{Q}^{-1}\boldsymbol{\mu}\hat{\boldsymbol{\beta}}_{t-1|T}'\right)\right), \\
\boldsymbol{q} =& \frac{1}{T}\mathbf{D}_{\boldsymbol{q}}'\sum_{t=1}^{T}\text{vec}[\mathbf{P}_{t|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t|T}' - \left(\mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t-1|T}'\right)\mathbf{A}' - \mathbf{A}\left(\mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}_{t|T}'\right) \\
& + \mathbf{A}\left(\mathbf{P}_{t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}_{t-1|T}'\right)\mathbf{A}' + \hat{\boldsymbol{\beta}}_{t|T}\boldsymbol{\mu}' - \boldsymbol{\mu}\hat{\boldsymbol{\beta}}_{t|T}' + \mathbf{A}\hat{\boldsymbol{\beta}}_{t-1|T}\boldsymbol{\mu}' + \boldsymbol{\mu}\hat{\boldsymbol{\beta}}_{t-1|T}'\mathbf{A}' + \boldsymbol{\mu}\boldsymbol{\mu}'], \\
\boldsymbol{h} =& \frac{1}{T}\mathbf{D}_{\boldsymbol{h}}'\sum_{t=1}^{T}\text{vec}\left[\mathrm{Y}_t\mathrm{Y}_t' - \mathrm{Y}_t\hat{\boldsymbol{\beta}}_{t|T}'\boldsymbol{\Lambda}' - \boldsymbol{\Lambda}\hat{\boldsymbol{\beta}}_{t|T}\mathrm{Y}_t' + \boldsymbol{\Lambda}\left(\mathbf{P}_{t|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t|T}'\right)\boldsymbol{\Lambda}'\right],
\end{aligned} \tag{15}$$

where $\text{vec}(\mathbf{A}) = \mathbf{D}_{\boldsymbol{\alpha}}\boldsymbol{\alpha}$, $\text{vec}(\mathbf{Q}) = \mathbf{D}_{\boldsymbol{q}}\boldsymbol{q}$ and $\text{vec}(\mathbf{H}) = \mathbf{D}_{\boldsymbol{h}}\boldsymbol{h}$. The $\boldsymbol{\alpha}$, $\boldsymbol{q}$ and $\boldsymbol{h}$ are vectors that consist of the diagonal elements of the parameter matrices $\mathbf{A}$, $\mathbf{Q}$ and $\mathbf{H}$. The exact expressions of the constrained matrices $\mathbf{D}_{\boldsymbol{\alpha}}$, $\mathbf{D}_{\boldsymbol{q}}$ and $\mathbf{D}_{\boldsymbol{h}}$ and the derivations of the expressions above can be found in the Appendix C. The smoothed estimates of the latent factors $\left[\hat{\boldsymbol{\beta}}_{t|T}(\boldsymbol{\theta}^{(0)})\right]_{t=0}^{T}$, their conditional variance $\left[\mathbf{P}_{t|T}(\boldsymbol{\theta}^{(0)})\right]_{t=0}^{T}$ and their cross covariance $\left[\mathbf{P}_{t,t-1|T}(\boldsymbol{\theta}^{(0)})\right]_{t=1}^{T}$ can be obtained by running the Kalman Smoother[3]. Due to insurmentable difficulties on the attempt to derive an

[3] A brief explanation of the Kalman Smoother and the derived expressions of the Smoothing step can be found in the Appendix B

analytical expression for the decay parameter(s) λ, it is considered as fixed. The fixed values of the decay parameters per model are equal to the in-sample estimates[4] of the ML approach.

Initialization

The model parameters are initialized by following the two-step procedure, in the exact way as it is described for the ML approach, section 2.4.1. Furthermore, the initialization of the Kalman Smoother is based on the equations (12):

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{0|T} &= \hat{\boldsymbol{\beta}}_{0|0} + \mathbf{J}_t(\hat{\boldsymbol{\beta}}_{1|T} - \hat{\boldsymbol{\beta}}_{1|0}) \\ \mathbf{P}_{0|T} &= \mathbf{P}_{0|0} + \mathbf{J}_0(\mathbf{P}_{1|T} - \mathbf{P}_{1|t})\mathbf{J}_0'. \end{aligned} \tag{16}$$

This specification is necessary due to the existence of the cross covariance matrix $\mathbf{P}_{1,0|T}$ in the derived analytical equations (15) of the EM Algorithm.

Algorithm

The EM Algorithm approach can be summarized in the following steps.

1. Initialization of the parameters $\boldsymbol{\theta}^{(0)}$;
2. E-Step: Run Kalman Smoother to obtain the smoothed estimates of the latent factors $\left[\hat{\boldsymbol{\beta}}_{t|T}(\boldsymbol{\theta}^{(0)})\right]_{t=0}^{T}$, their conditional variance $\left[\mathbf{P}_{t|T}(\boldsymbol{\theta}^{(0)})\right]_{t=0}^{T}$ and their cross covariance $\left[\mathbf{P}_{t,t-1|T}(\boldsymbol{\theta}^{(0)})\right]_{t=1}^{T}$;
3. M-Step: Use the output of the previous step to calculate the new set of parameters $\boldsymbol{\theta}^{(1)}$ by using the equations 15 in the following order:
   (a) Use $\mathbf{Q}^{(0)}$ and $\boldsymbol{\mu}^{(0)}$ to obtain $\mathbf{A}^{(1)}$;
   (b) Use $\mathbf{A}^{(1)}$ to obtain $\boldsymbol{\mu}^{(1)}$;
   (c) Use $\mathbf{A}^{(1)}$ and $\boldsymbol{\mu}^{(1)}$ to obtain $\mathbf{Q}^{(1)}$;
4. Use the new set $\boldsymbol{\theta}^{(1)}$ as input of the Kalman Smoother;
5. Iterate steps 2. to 4. until convergence of the parameters;

In order to assure that the estimated parameters will converge, the number of iterations is set to I = 1000.

[4]The in-sample estimates be found in Table 30, Appendix.

2.5 Forecasting

In order to compare the out-of-sample forecasting power of the six variants from the Nelson-Siegel class of models, I produce forecasts from December 1981 until December 2019. Furthermore, I use moving windows of 120 observations to estimate the models. The initial window is from December 1971 until November 1981. Forecasting horizons of 1, 3, 6 and 12 step-ahead are considered. The forecasts for the latent factors and the yields are given below:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{T+h|T} &= \left(I_m - \hat{\mathbf{A}}^h\right)\left(I_m - \hat{\mathbf{A}}\right)^{-1}\hat{\boldsymbol{\mu}} + \hat{\mathbf{A}}^h\hat{\boldsymbol{\beta}}_{T|T} \\ \hat{\mathbf{Y}}_{T+h|T} &= \hat{\mathbf{\Lambda}}\hat{\boldsymbol{\beta}}_{T+h|T} = \hat{\mathbf{\Lambda}}\left(I_m - \hat{\mathbf{A}}^h\right)\left(I_m - \hat{\mathbf{A}}\right)^{-1}\hat{\boldsymbol{\mu}} + \hat{\mathbf{\Lambda}}\hat{\mathbf{A}}^h\hat{\boldsymbol{\beta}}_{T|T},\end{aligned} \tag{17}$$

where m is the number of latent factors, $\hat{\mathbf{A}}$, $\hat{\boldsymbol{\mu}}$ and $\hat{\mathbf{\Lambda}}$ are the parameter estimates, and $\hat{\boldsymbol{\beta}}_{T|T}$ is the last updated estimates of the factors.

2.5.1 Competitor models

Random Walk

The Random Walk (RW) is a simple model that can be a good benchmark for evaluating the forecasting abilities of other models. In many cases in the literature, regarding interest rate forecasting is clear this benchmark model is very difficult to be consistently outperformed (Duffee (2002); Ang and Piazzesi (2003); Diebold and Li (2006)).

$$y_t(\tau_i) = y_{t-1}(\tau_i) + \varepsilon_t(\tau_i) \quad , \quad \varepsilon_t(\tau_i) \sim \mathcal{N}\left(0, \sigma^2(\tau_i)\right), \tag{18}$$

where $y_t(\tau_i)$ represents the bond yield at time t with time to maturity τ_i, $i \in \{1, \dots N\}$, the $\varepsilon_t(\tau_i)$ is the error term and $\sigma^2(\tau_i)$ its variance. The h-step ahead forecast is equal to:

$$\hat{y}_{T+h|T}(\tau_i) = y_T(\tau_i) \ , \forall h \in \{1, 3, 6, 12\}. \tag{19}$$

AR(1)

The first-order univariate autoregressive model is also a proper choice due to its property of mean-reversion.

$$y_t(\tau_i) = \mu(\tau_i) + \phi(\tau_i) y_{t-1}(\tau_i) + \varepsilon_t(\tau_i) \quad , \quad \varepsilon_t(\tau_i) \sim \mathcal{N}\left(0, \sigma^2(\tau_i)\right), \tag{20}$$

where $y_t(\tau_i)$ represents the bond yield at time t with time to maturity τ_i, $i \in \{1, \dots N\}$, the $\mu(\tau_i)$ and $\phi(\tau_i)$ are the unknown parameters, the $\varepsilon_t(\tau_i)$ is the error term and $\sigma^2(\tau_i)$ its variance. The h-step ahead forecast is obtained by:

$$\begin{aligned} \hat{y}_{T+1|T}(\tau_i) =& \hat{\mu}(\tau_i) + \hat{\phi}(\tau_i) y_T(\tau_i) \text{ , for } h = 1 \\ \hat{y}_{T+1|T}(\tau_i) =& \hat{\mu}\left(1 + \hat{\phi}(\tau_i) + \cdots + \hat{\phi}^{h-1}(\tau_i)\right) + \hat{\phi}^h(\tau_i) y_T(\tau_i) \text{ , } h \in \{3, 6, 12\}. \end{aligned} \tag{21}$$

VAR(1)-PCA

A first-order unrestricted vector autoregressive model for yield levels exploits the cross-section information of different maturities, which is not the case for the random walk and the AR(1). As in De Pooter (2007), the first three principal components of the lagged yields replace them, which explain over 99 percent of the total variation. The purpose of this replacement is to overcome the drawback of using an unrestricted VAR(1) model, which implies the necessity of estimating a significant number of parameters.

$$\mathrm{Y}_t = \boldsymbol{\mu} + \mathbf{F}_{t-1}\Phi + \boldsymbol{\varepsilon}_t \quad , \quad \boldsymbol{\varepsilon}_t \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}\right), \tag{22}$$

where $\mathrm{Y}_t = [y_t(\tau_1) \cdots y_t(\tau_N)]'$ is the vector of bond yields at time t for all maturities, the vector $\boldsymbol{\mu}$ and the matrix $\Phi(\tau_i)$ consists of the unknown parameters, the $\mathbf{F}_{t-1}$ is the vector of the first three principal components of the lagged yields, the $\boldsymbol{\varepsilon}_t$ is the vector of error terms and Σ its variance-covariance matrix. The h-step ahead forecast is obtained by:

$$\begin{aligned} \hat{\mathrm{Y}}_{T+1|T} =& \hat{\boldsymbol{\mu}} + \mathbf{F}_T \hat{\Phi} = \hat{\boldsymbol{\mu}} + \left(\mathrm{Y}_T - \bar{\mathrm{Y}}\right) \mathbf{V}\Phi \text{ , for } h = 1 \\ \hat{\mathrm{Y}}_{T+h|T} =& \hat{\boldsymbol{\mu}}\left(\mathbf{I} + \mathbf{V}\Phi + \cdots + (\mathbf{V}\Phi)^{h-1}\right) + \mathrm{Y}_T(\mathbf{V}\Phi)^h - \bar{\mathrm{Y}}\left(\mathbf{V}\Phi + \cdots + (\mathbf{V}\Phi)^h\right) \text{ , } h \in \{3, 6, 12\}, \end{aligned} \tag{23}$$

where $\bar{\mathrm{Y}}$ is the vector of yields' mean per maturity, and the matrix $\mathbf{V}$ consists of the right eigenvectors.

2.5.2 Forecast Combination Scheme

A combined forecast for a h-month horizon for the yield with maturity τ_i is given by:

$$\hat{y}_{T+h|T}(\tau_i) = \sum_{m=1}^{N} w_{T+h|T,m}(\tau_i) \hat{y}_{T+h|T,m}(\tau_i). \tag{24}$$

According to Clark and McCracken (2009), it is hard to beat the scheme of equally weighted forecasts of nested models. This scheme, which is the one that is used in this research, is just the assignment of equal weights to the forecasts from all individual models, $w_{T+h|T,m}(\tau_i) = 1/N$, where N is the number of models.

2.5.3 Forecast Evaluation

Statistical Evaluation

In order to evaluate statistically the out-of-sample forecasts of each model, I calculate two popular measures, the Root Mean Squared Forecast Error for each maturity, and Trace Root Mean Squared Forecast Error which measures the forecasting error for all maturities together. The RMSFE for a h-step ahead forecast horizon, regarding the model m that takes into account T out-of-sample point forecasts of the τ_i yield maturity is the following:

$$\text{RMSFE}_{m,h}(\tau_i) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{y}_{t+h|t}(\tau_i) - y_{t+h}(\tau_i)\right)^2}. \tag{25}$$

Moreover, the TRMSFE for a h-step ahead forecast horizon, regarding the model m that takes into account T out-of-sample point forecasts for all maturities is given below:

$$\text{TRMSFE}_{m,h} = \sqrt{\frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\hat{y}_{t+h|t}(\tau_i) - y_{t+h}(\tau_i)\right)^2}. \tag{26}$$

Economic Evaluation

In parallel with the statistical forecast evaluation, this research attempts to see if the yield forecasts from the relevant nested models are economically meaningful. Due to the anticipated minor differences between the statistical performance of the nested models, an economic approach for evaluating them can support the statistical results and indicate the favourite ones. The economic measures that this research uses to evaluate the models are the Sharpe Ratio and the Turnover of the mean-variance optimized portfolios, a framework that was initially introduced by Markowitz (1952). In the mean-variance portfolio construction, individuals choose their allocations in risky assets based on the trade-off between expected returns and risk. I consider the case of an investor who has a h-month investment horizon and re-balances his/her portfolio also on a h-months period basis, $h \in \{1, 3, 6, 12\}$.

The formulation of the mean–variance optimization problem is given by

$$\begin{aligned} \min_{w_t} w_t' \Sigma w_t - \frac{1}{\delta} w_t' \mu_{r_{t|t-h}} \\ \text{s.t. } w_t' \iota = 1, \end{aligned} \tag{27}$$

where $\mu_{r_{t|t-h}}$ is the h-month period-ahead vector of expected bond returns, τ is the vector of observed maturities, Σ is the variance–covariance matrix of bond returns, w_t is the vector of portfolio weights at time t chosen at time $t-h$, ι is an $N \times 1$ vector of ones, and δ is the coefficient of risk aversion, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$.

Following the Caldeira et al. (2016a) and Caldeira et al. (2016b), the inputs for the aforementioned optimization problem can be obtained by using the forecasts for each yield maturity to derive estimates of expected bond returns, by holding each bond from $t-h$ to t, when at the same time the maturity of the bonds decreases from τ to $\tau - h$:

$$\mu_{r_{t|t-h}} = -(\tau - h) \odot \mathrm{Y}_{t|t-h}^{(\tau-h)} + \tau \odot \mathrm{Y}_{t-h}^{(\tau)}, \tag{28}$$

where $\tau - h \in \{3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120\}$ for each value[5] of h, and for dynamic factor models as the Nelson-Siegel models, according to Caldeira et al. (2016a) one can obtain a closed form expression for the covariance matrix of the bond returns:

$$\Sigma_{r_{t|t-h}} = (\tau - h)'(\tau - h) \odot \left(\mathbf{\Lambda} \mathbf{P}_{t|t-h} \mathbf{\Lambda} + \mathbf{H}\right), \tag{29}$$

where $\mathbf{P}_{t|t-h}$, $h \in \{1, 3, 6, 12\}$ can be obtain while using the Kalman Filter.

Caldeira et al. (2016b) and Caldeira et al. (2016a) add a constraint to the portfolio weights, to exclude the the short-selling and use numerical methods to solve the constrained optimization problem. The main motive of their choice is that previous studies (see Jagannathan and Ma (2003)) have shown that a restriction like that can reduce the turnover of the portfolio. In opposition to their approach, this research does not restrict the portfolio weight. The reason of this divergence to their choice is that experiments which are not included in this paper, showed that due to the highly correlated nature of the yields for different maturities, more than 99% of the portfolio weight is accumulated to one maturity, when using numerical methods to solve the

[5] In order to exploit the 1, 3, 6 and 12-step ahead yield forecasts for the maturities that are mentioned in section 3, at time $t-h$ the bonds that are taken into account have maturities $\{3+h, 6+h, 9+h, 12+h, 15+h, 18+h, 21+h, 24+h, 30+h, 36+h, 48+h, 60+h, 72+h, 84+h, 96+h, 108+h, 120+h\}$, and at time t their maturities will be decreased to $\{3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120\}$.

problem.

Thus, the analytical solution of the mean-variance optimization problem is:

$$w_t = \frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota} + \frac{1}{2\delta}\left(\Sigma^{-1} - \frac{\Sigma^{-1}\iota\iota'\Sigma^{-1}}{\iota'\Sigma^{-1}\iota}\right)\mu. \tag{30}$$

The performance of optimal mean–variance portfolios is evaluated in terms of Sharpe ratio, which is the ratio of the realized portfolio returns over its standard deviation. It can be computed by using the excess returns of the bond portfolios to the risk free rate (see Section 3):

$$\text{SR} = \frac{\mu_p - R_f}{\sigma_p}. \tag{31}$$

Furthermore, the choice of not restricting the portfolio weights, naturally demands the computation of portfolios' Turnover as a supporting measure. As DeMiguel et al. (2009) states, the turnover can provide insights on the trading volume that a portfolio strategy needs to be implemented. The Turnover is given by:

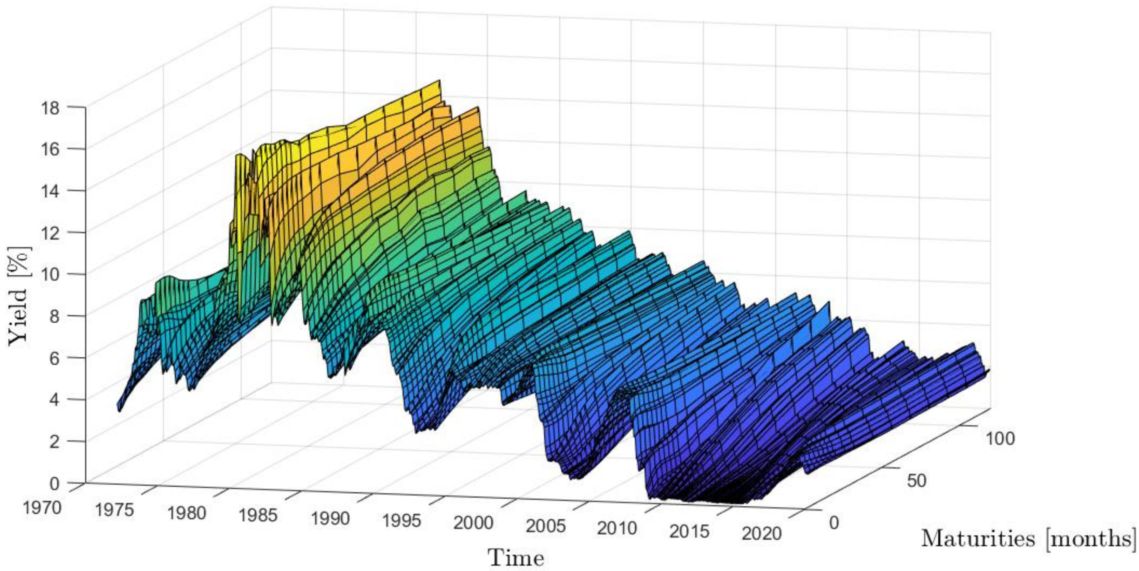
$$\text{Turnover} = \frac{1}{T-1}\sum_{t=1}^{T-1}\sum_{j=1}^{N}\left(|w_{j,t+1} - w_{j,t^+}|\right), \tag{32}$$

where $w_{j,t+1}$ is the portfolio weight of the bond j at time $t+1$, and w_{j,t^+} is the portfolio weight of the bond j before rebalancing at time $t+1$.

3 Data

The dataset that is used in this research, developed by Liu and Wu (2019), is preferred against the Fama and Bliss (1987) and Gürkaynak et al. (2007), as according to their paper, their method overcomes the limitations of the aforementioned datasets and provides better estimate of the US yield curve raw data. I use monthly yield data from December of 1971 until December of 2019, a time series of 577 observations, and 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$. Figure 1 below illustrates the yield curve over time and different maturities. It is clear that the level yields vary over time, but also the their slope and curvature. As can be seen in the surface plot, the yield curve evolves by taking a variety of different shapes across the time horizon. Overall, we can also see how the yields across all maturities tend to decrease with time.

Figure 1: Yield Curve



Note: This figure represents a surface plot of the US Treasury Yields, spanning the data from December of 1971 until December of 2019, and at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$

The main reason of choosing the specific time period is that it includes the Volcker period (December 1971 - December 1983), a period that is stigmatized by the Great Inflation and the way that Paul Volcker[6] tackled it, as the 12^{th} Chairman of the Federal Reserve of US, by raising the federal funds rates in enormous levels, until the most recent period, after the global economic crisis of 2008, that is characterized by the extreme low interest rates environment. Therefore, one can obtain a clear idea about the overall performance of the DNS models under all kind of conditions. Moreover, this thesis also examines a sub-sample, which chronologically starts after the Volcker period and is aligned with De Pooter (2007) sample period (January 1984 - December 2003, $T = 240$ observations). The results of a sub-sample can provide further insights regarding the performance of the models and methods and furthermore, this particular choice lets this research to fairly compare its results with De Pooter (2007), and have a clearer idea about the benefits of using the new dataset of Liu and Wu (2019).

[6]`https://en.wikipedia.org/wiki/Paul_Volcker`

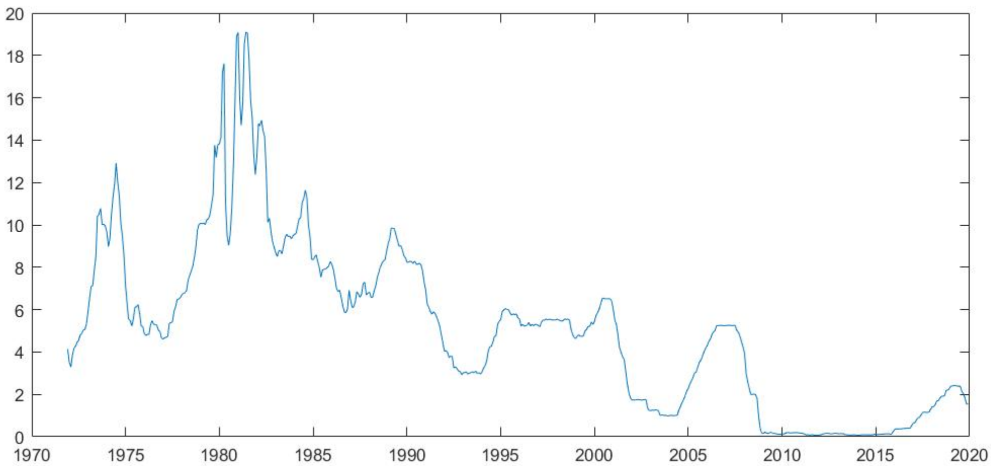
Table 1: Summary statistics

Maturity	Mean	Std Dev	Min	Max	$\hat{\rho}_1$	$\hat{\rho}_{12}$	$\hat{\rho}_{24}$
3-month	4.720	3.516	0.023	16.171	0.988	0.864	0.712
6-month	4.872	3.555	0.043	16.214	0.990	0.874	0.729
1-year	5.070	3.552	0.105	16.026	0.990	0.886	0.756
2-year	5.320	3.495	0.201	15.664	0.991	0.900	0.797
3-year	5.517	3.405	0.321	15.551	0.992	0.908	0.818
4-year	5.698	3.323	0.469	15.417	0.992	0.910	0.829
5-year	5.833	3.231	0.644	15.009	0.992	0.912	0.837
6-year	5.970	3.178	0.824	14.994	0.992	0.913	0.841
7-year	6.070	3.111	1.002	14.964	0.992	0.911	0.840
8-year	6.156	3.056	1.210	14.895	0.993	0.913	0.845
9-year	6.228	3.003	1.414	14.812	0.993	0.913	0.846
10-year	6.285	2.927	1.497	14.779	0.992	0.908	0.843

Note: Descriptive statistics of the US Treasury Yields. The values for mean, standard deviation, maximum and minimum are expressed in percentage.

Table 1 shows summary statistics of the US Treasury yields for several maturities. One can see that the short end of the yield curve is more volatile than the long end, by steadily decreasing from the one end to the other. This confirms that the short end is more sensitive to policy changes. Furthermore, the the average yield curve increases over time, not concave though due to the shape of the average short end. In terms of autocorrelation, one can see that the yield dynamics are quite persistent, especially for long maturities. Correlations between different maturities are quite high - at least 92%. Clearly, the majority of the stylized facts is present.

Figure 2: Federal Funds Rate



Note: This figure shows the time series evolution of the Federal Funds Rate, spanning the period from December of 1971 until December of 2019

Regarding the economic evaluation on this research, the Effective Federal Funds Rate (FFR) is considered as risk-free rate, which is obtained from the Federal Reserve Bank of St. Louis[7]. Figure 2 shows the time series of the FFR from December of 1971 until December of 2019. Furthermore, in order to take into account the maturity decrease of the holding bonds for the estimates of the expected bond returns, I exploit further the Liu and Wu (2019) dataset and also use the monthly yield data from December of 1971 until December of 2019, a time series of 577 observations, for the maturities, $\tau \in \{3+h, 6+h, 9+h, 12+h, 15+h, 18+h, 21+h, 24+h, 30+h, 36+h, 48+h, 60+h, 72+h, 84+h, 96+h, 108+h, 120+h\}$, where $h \in \{1, 3, 6, 12\}$.

4 Out-of-sample Forecasting Results

This section discusses the forecasting results for the full sample (Section 4.1) and the De Pooter (2007) Sub-sample (Section 4.2). More specifically, the statistical and economic evaluation of the DNS models are presented, as well as discussion of the relation between the statistical and economic measures. Lastly, a direct comparison to De Pooter (2007) results (Section 4.3) to analyse the performance of the models for the different datasets is introduced.

4.1 Full Sample

4.1.1 Statistical Evaluation

Table 2 shows the TRMSFE[8] of the benchmark model and the ratios of the TRMSFE of the models relative to the benchmark and furthermore and Tables 3-6 shows the ratios of the RMSFE of the models relative to the benchmark for all maturities. Starting with the comparison of the predictive ability of the models for 1-step ahead horizon, in an aggregate level, TRMSFEs in Table 2 indicate that the DNS models perform worse than the RW. Furthermore, the DNS-ML models produce better forecasts than the DNS-EM model. DNS 3 has the best performance within the DNS-ML and the DNS-EM models, with a 2% and 12% lower performance than the RW, respectively. The worst variant of the DNS nested models is the DNS 2, which is a two-factor model.

[7] `https://fred.stlouisfed.org`

[8] In the Appendix one can also find the TRMSFE Rolling Forecasting Performance of the six DNS specification and the Forecast Combination, under both estimation approaches, comparing to the RW as well (See Figures 3-9). This could provide useful insights regarding the out-of-sample forecasting performance of the models throughout the whole time period that is taken into account in this research, and for all step-ahead horizons.

Table 2: TRMSFE - Full Sample

Forecasting Horizon	**1-step**	**3-step**	**6-step**	**12-step**
Random Walk	0.31	0.59	0.89	1.30
AR(1)	1.02	1.03	1.03	1.05
VAR(1) PCA	1.05	1.08	1.12	1.17
DNS - ML approach				
DNS 1	1.09	1.12	1.15	1.21
DNS 2	1.13	1.13	1.17	1.24
DNS 3	1.02	1.01	1.02	1.06
DNS 4	1.09	1.11	1.15	1.21
DNS 5	1.07	1.12	1.14	1.17
DNS 6	1.08	1.14	1.16	1.20
Forecast Combination	1.04	1.06	1.09	1.13
DNS - EM approach				
DNS 1	1.14	1.17	1.18	1.21
DNS 2	1.26	1.22	1.23	1.25
DNS 3	1.12	1.19	1.19	1.19
DNS 4	1.14	1.18	1.20	1.23
DNS 5	1.16	1.24	1.26	1.26
DNS 6	1.15	1.23	1.25	1.23
Forecast Combination	1.11	1.17	1.19	1.20

Note: This table reports the TRMSFE of the RW, for the 1, 3, 6 and 12-step ahead forecasts for the full sample, and also the relative TRMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW.

Regarding the Forecast Combination, it seems that for the ML approach outperforms all the individual models, but the DNS 3, and for the EM approach outperforms all the individual variations. In a maturity-specific level, Table 3 shows that the DNS-ML and DNS-EM models are significantly outperformed by RW for the short maturities, 3 and 6 months. However, their forecasting ability is better for the medium and long-term maturities, especially for DNS 3 under the ML approach, which performs slightly better by 1% or equally well to the benchmark for the 9, 12, 15, 18, 30, 36, 48, 72 and 96-month maturities. The rest of the competitor models outperforms the aforementioned model only for the short-term maturities, 3 and 6 months, and the long-term 120 month maturity. Regarding the comparison between the DNS-EM and DNS-ML models, the results in Table 31 in Appendix, are mostly in favor of the ML approach, except that for the 120-month maturity, the DNS 1 and 4 perform better under the EM approach by 4% and 2%, respectively. Moreover, the DNS 1 has equal predictive abilities under the EM approach for the 15, 18 and 108-months maturities, and the DNS 2 performs equally or slightly better by 1% under the EM approach for the 21, 24 and 30-months maturities.

Table 3: RMSFE 1-step ahead - Full Sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	0.33	0.31	0.31	0.31	0.31	0.31	0.31	0.32	0.32	0.32	0.32	0.32	0.32	0.31	0.31	0.30	0.30
AR(1)	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.01	1.02	1.02
VAR(1) PCA	1.03	1.03	1.03	1.05	1.06	1.07	1.07	1.06	1.05	1.04	1.04	1.05	1.03	1.04	1.04	1.04	1.06
DNS - ML approach																	
DNS 1	1.19	1.11	1.09	1.09	1.09	1.09	1.09	1.09	1.08	1.08	1.07	1.09	1.04	1.05	1.03	1.05	1.14
DNS 2	1.47	1.24	1.11	1.06	1.06	1.07	1.08	1.10	1.11	1.11	1.10	1.10	1.04	1.05	1.07	1.09	1.20
DNS 3	1.12	1.04	**1.00**	**0.99**	**1.00**	**1.00**	1.01	1.01	**1.00**	**0.99**	**1.00**	1.02	**0.99**	1.01	**1.00**	1.01	1.10
DNS 4	1.20	1.12	1.09	1.09	1.09	1.09	1.08	1.08	1.08	1.07	1.07	1.08	1.04	1.05	1.03	1.05	1.14
DNS 5	1.19	1.12	1.08	1.08	1.07	1.07	1.07	1.07	1.06	1.04	1.03	1.06	1.02	1.07	1.06	1.04	1.13
DNS 6	1.16	1.12	1.10	1.09	1.09	1.08	1.09	1.09	1.07	1.06	1.05	1.08	1.04	1.07	1.06	1.04	1.13
Forecast Combination	1.11	1.06	1.04	1.04	1.04	1.04	1.05	1.05	1.04	1.03	1.03	1.05	1.01	1.03	1.02	1.01	1.10
DNS - EM approach																	
DNS 1	1.47	1.27	1.15	1.10	1.09	1.09	1.09	1.10	1.10	1.10	1.10	1.13	1.09	1.10	1.07	1.05	1.09
DNS 2	1.97	1.67	1.42	1.24	1.14	1.09	1.08	1.09	1.12	1.13	1.14	1.16	1.10	1.10	1.10	1.11	1.21
DNS 3	1.18	1.15	1.14	1.13	1.13	1.13	1.13	1.13	1.12	1.10	1.09	1.11	1.07	1.10	1.07	1.06	1.13
DNS 4	1.46	1.26	1.15	1.11	1.10	1.10	1.10	1.11	1.11	1.11	1.11	1.14	1.09	1.10	1.07	1.06	1.11
DNS 5	1.22	1.20	1.19	1.19	1.18	1.17	1.17	1.17	1.16	1.14	1.12	1.14	1.10	1.12	1.10	1.09	1.16
DNS 6	1.26	1.23	1.20	1.19	1.18	1.17	1.17	1.17	1.15	1.13	1.11	1.13	1.09	1.11	1.09	1.09	1.15
Forecast Combination	1.23	1.16	1.12	1.10	1.10	1.09	1.10	1.11	1.11	1.10	1.09	1.11	1.07	1.09	1.07	1.06	1.12

Note: This table reports the RMSFE of the RW, for the 1-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

For the 3-step ahead horizon, Table 2 shows that overall RW has better predictive abilities than the DNS family of models. Worth mentioning is that comparing to the DNS 3 under the ML approach, the benchmark performs better just by 1%. The DNS models under the ML approach seems to be more accurate than the ones under the EM approach. Among the DNS-ML models, DNS 3 performs significantly better and among the DNS-EM models DNS 1 shows slightly better performance. As regards the Forecast Combination, under the ML approach beats all the individuals but the DNS 3, and under the EM approach performs equally with the DNS 1 model. Furthermore, Table 4 shows the 3-step ahead RMSFE results. One can observe that the DNS 3 under the ML approach, is only one that performs equally or better by 1-2% than the RW, for the maturities, 12, 15, 18, 21, 24, 30, 36, 48, 72, 84 and 108-months. The DNS models shows low predictive abilities especially for the short-term maturities. Furthermore, the AR(1) and the VAR(1)-PCA competitors are outperformed by the DNS 3 under the ML approach throughout the different maturities. Regarding the performance of the DNS-EM against the DNS-ML models, Table 32 in Appendix, reports that under the ML approach the models produce more accurate forecasts. However, for the maturities 15, 18 and 120-months the DNS 1 under the EM approach is worse only by 1%.

Table 4: RMSFE 3-step ahead - Full Sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	0.62	0.60	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.60	0.59	0.56	0.56	0.54	0.53	0.53
AR(1)	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.03	1.03	1.03	1.04	1.04	1.05
VAR(1) PCA	1.07	1.08	1.08	1.08	1.09	1.10	1.10	1.09	1.08	1.07	1.06	1.07	1.07	1.08	1.09	1.08	1.08
DNS - ML approach																	
DNS 1	1.17	1.15	1.15	1.15	1.14	1.13	1.13	1.13	1.12	1.11	1.09	1.10	1.07	1.08	1.07	1.07	1.10
DNS 2	1.24	1.19	1.15	1.14	1.13	1.12	1.12	1.12	1.11	1.10	1.09	1.10	1.07	1.10	1.11	1.11	1.16
DNS 3	1.05	1.02	1.01	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**0.99**	**0.99**	1.01	**0.98**	**1.00**	1.01	**1.00**	1.05
DNS 4	1.17	1.16	1.15	1.14	1.14	1.13	1.12	1.12	1.11	1.10	1.08	1.09	1.06	1.07	1.07	1.06	1.10
DNS 5	1.20	1.18	1.15	1.14	1.13	1.12	1.11	1.11	1.10	1.09	1.07	1.09	1.07	1.10	1.11	1.09	1.13
DNS 6	1.20	1.20	1.18	1.17	1.16	1.15	1.14	1.14	1.12	1.11	1.09	1.11	1.08	1.11	1.11	1.08	1.13
Forecast Combination	1.11	1.10	1.09	1.08	1.08	1.07	1.07	1.07	1.06	1.05	1.03	1.05	1.02	1.04	1.04	1.02	1.07
DNS - EM approach																	
DNS 1	1.32	1.25	1.20	1.17	1.16	1.15	1.15	1.15	1.15	1.15	1.14	1.16	1.14	1.15	1.14	1.11	1.11
DNS 2	1.51	1.39	1.30	1.23	1.19	1.16	1.16	1.16	1.16	1.16	1.16	1.18	1.16	1.17	1.18	1.17	1.20
DNS 3	1.22	1.22	1.22	1.21	1.21	1.20	1.20	1.20	1.19	1.18	1.16	1.17	1.15	1.16	1.16	1.13	1.16
DNS 4	1.32	1.25	1.21	1.19	1.17	1.17	1.17	1.17	1.17	1.17	1.16	1.17	1.15	1.16	1.15	1.12	1.13
DNS 5	1.30	1.30	1.29	1.28	1.27	1.26	1.25	1.25	1.24	1.22	1.19	1.20	1.18	1.19	1.18	1.17	1.19
DNS 6	1.27	1.28	1.27	1.26	1.25	1.25	1.24	1.24	1.23	1.22	1.19	1.20	1.19	1.20	1.20	1.18	1.20
Forecast Combination	1.23	1.21	1.20	1.18	1.17	1.17	1.17	1.17	1.16	1.16	1.14	1.16	1.14	1.15	1.15	1.13	1.15

Note: This table reports the RMSFE of the RW, for the 3-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

Similarly, for the 6-step ahead Table 2 indicates that the aggregated performance of the RW is better than the DNS family's. The DNS 3 under the ML approach is again the most accurate among the DNS family, and is outperformed by 2% by the benchmark, while the DNS 1 has the better predictive ability among the ones under the EM approach, although it is outperformed by 18% by the benchmark. Moreover, the DNS models under the EM approach are less accurate than under the ML approach, and the Forecast combination follows the same pattern as for the 3-step ahead. In a more granular level, Table 5 shows that for the 6-step ahead horizon, the DNS models are consistently beaten by the benchmark for all maturities. The only exception is the DNS 3 under the ML approach which for the maturities 36, 48 and 72-months performs better by 1% or equally well with the RW. Furthermore, Table 33 in Appendix, indicates that the models under the ML approach perform better that under the EM approach. Although, for the maturities 12, 15, 18, 21, 24 and 120-months the DNS 1 under the EM approach has the same forecast abilities or worse by 1% than under the ML approach.

Table 5: RMSFE 6-step ahead - Full Sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	0.96	0.96	0.96	0.96	0.96	0.95	0.94	0.93	0.92	0.90	0.87	0.85	0.82	0.81	0.78	0.77	0.77
AR(1)	1.04	1.04	1.03	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.03	1.03	1.04	1.05	1.06	1.06	1.07
VAR(1) PCA	1.11	1.12	1.12	1.12	1.12	1.13	1.13	1.13	1.12	1.12	1.11	1.11	1.11	1.12	1.14	1.13	1.12
DNS - ML approach																	
DNS 1	1.20	1.19	1.18	1.18	1.17	1.16	1.16	1.16	1.15	1.15	1.13	1.13	1.10	1.11	1.12	1.11	1.13
DNS 2	1.24	1.21	1.19	1.18	1.17	1.16	1.16	1.16	1.15	1.15	1.14	1.14	1.13	1.14	1.16	1.16	1.19
DNS 3	1.05	1.03	1.02	1.02	1.01	1.01	1.01	1.01	1.01	**1.00**	**1.00**	1.01	**0.99**	1.01	1.02	1.02	1.05
DNS 4	1.20	1.19	1.18	1.17	1.17	1.16	1.15	1.15	1.15	1.14	1.12	1.12	1.09	1.10	1.11	1.10	1.12
DNS 5	1.23	1.20	1.18	1.17	1.15	1.14	1.14	1.14	1.13	1.12	1.10	1.11	1.09	1.11	1.12	1.11	1.14
DNS 6	1.24	1.22	1.21	1.20	1.18	1.17	1.17	1.16	1.15	1.14	1.11	1.12	1.10	1.11	1.12	1.11	1.14
Forecast Combination	1.14	1.13	1.12	1.11	1.10	1.09	1.09	1.09	1.08	1.07	1.06	1.06	1.04	1.05	1.06	1.05	1.08
DNS - EM approach																	
DNS 1	1.29	1.24	1.21	1.19	1.18	1.17	1.17	1.17	1.17	1.17	1.16	1.17	1.15	1.16	1.16	1.14	1.13
DNS 2	1.39	1.32	1.27	1.23	1.21	1.19	1.19	1.19	1.19	1.19	1.18	1.20	1.18	1.20	1.21	1.21	1.23
DNS 3	1.21	1.21	1.20	1.20	1.20	1.19	1.19	1.19	1.19	1.18	1.16	1.17	1.15	1.17	1.17	1.16	1.17
DNS 4	1.30	1.25	1.22	1.21	1.20	1.19	1.19	1.19	1.19	1.19	1.17	1.18	1.16	1.17	1.17	1.15	1.15
DNS 5	1.32	1.31	1.30	1.29	1.28	1.27	1.27	1.27	1.26	1.25	1.22	1.22	1.20	1.20	1.21	1.20	1.20
DNS 6	1.28	1.28	1.28	1.27	1.27	1.26	1.26	1.26	1.25	1.24	1.22	1.22	1.21	1.21	1.22	1.21	1.21
Forecast Combination	1.24	1.22	1.21	1.20	1.19	1.18	1.18	1.18	1.18	1.18	1.16	1.17	1.16	1.16	1.17	1.16	1.16

Note: This table reports the RMSFE of the RW, for the 6-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

Finally, for the 12-step ahead, one can see in Table 2 that, in an aggregate level, the DNS 3 under the ML is the most accurate one among the DNS models, but still worse than the benchmark by 6%. It is worth mentioning that the DNS 1 under both approaches has equal performance, and the DNS 2 under the ML approach beats the one under the EM just by 1%. The Forecast Combination in both approaches produces better predictions than the individual models, except for the DNS 3. In a maturity-specific perspective, the reported RMSFEs in Table 6 indicate that the RW has the best forecast abilities. Also, the AR(1) outperforms the best of the DNS models, the DNS 3 under the ML approach, for short and medium-term maturities. Regarding the comparison of the two estimation methods, Table 34 in Appendix, shows the DNS 1 under the EM approach seems to produce better 12-step ahead forecasts for the maturities 12, 15, 18, 21, 24, 30 and 120-months by 1-2%. The same applies for the DNS 2 for the maturities 18 and 21-months, where one can also see that this model under the two approaches has the same performance for the maturities 12, 15, 24, 30, 36, 48, 108 and 120-months. Finally, the DNS 6 has slightly better performance for short-term maturities and the EM approach.

Table 6: RMSFE 12-step ahead - Full Sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	1.44	1.46	1.46	1.46	1.44	1.42	1.40	1.38	1.33	1.29	1.23	1.18	1.15	1.14	1.09	1.07	1.07
AR(1)	1.05	1.04	1.03	1.03	1.02	1.02	1.02	1.03	1.03	1.03	1.04	1.06	1.07	1.08	1.10	1.12	1.13
VAR(1) PCA	1.14	1.15	1.16	1.16	1.16	1.17	1.18	1.18	1.18	1.19	1.19	1.19	1.18	1.17	1.20	1.19	1.17
DNS - ML approach																	
DNS 1	1.24	1.23	1.22	1.22	1.21	1.21	1.21	1.22	1.22	1.22	1.21	1.21	1.19	1.18	1.20	1.19	1.19
DNS 2	1.29	1.27	1.25	1.24	1.24	1.23	1.23	1.23	1.24	1.24	1.23	1.24	1.22	1.22	1.25	1.25	1.26
DNS 3	1.07	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.05	1.05	1.04	1.05	1.04	1.05	1.07	1.07	1.09
DNS 4	1.25	1.23	1.22	1.22	1.21	1.21	1.21	1.21	1.22	1.21	1.20	1.20	1.18	1.17	1.19	1.19	1.19
DNS 5	1.24	1.21	1.19	1.18	1.17	1.16	1.16	1.16	1.16	1.16	1.14	1.15	1.14	1.14	1.15	1.15	1.16
DNS 6	1.26	1.24	1.23	1.22	1.21	1.20	1.20	1.20	1.20	1.19	1.17	1.18	1.16	1.15	1.17	1.16	1.17
Forecast Combination	1.18	1.16	1.15	1.15	1.14	1.13	1.13	1.13	1.13	1.13	1.12	1.12	1.10	1.10	1.11	1.11	1.12
DNS - EM approach																	
DNS 1	1.28	1.24	1.22	1.20	1.20	1.19	1.19	1.20	1.21	1.22	1.21	1.22	1.20	1.19	1.20	1.19	1.18
DNS 2	1.36	1.31	1.27	1.25	1.23	1.22	1.22	1.23	1.23	1.24	1.23	1.25	1.24	1.23	1.26	1.26	1.26
DNS 3	1.19	1.19	1.19	1.18	1.18	1.18	1.18	1.19	1.20	1.20	1.19	1.20	1.19	1.19	1.21	1.21	1.21
DNS 4	1.29	1.26	1.24	1.22	1.22	1.21	1.21	1.22	1.23	1.23	1.22	1.23	1.21	1.20	1.21	1.20	1.19
DNS 5	1.29	1.28	1.27	1.27	1.26	1.26	1.26	1.27	1.27	1.27	1.26	1.26	1.24	1.23	1.25	1.25	1.24
DNS 6	1.25	1.24	1.23	1.23	1.22	1.22	1.22	1.23	1.24	1.23	1.22	1.22	1.21	1.20	1.22	1.22	1.21
Forecast Combination	1.24	1.22	1.21	1.20	1.19	1.19	1.19	1.20	1.21	1.21	1.20	1.21	1.20	1.19	1.21	1.20	1.20

Note: This table reports the RMSFE of the RW, for the 12-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

To summarize the full-sample results, RW proves that is a hard-to-beat benchmark model. The DNS 3 under the ML approach has the best predictive performance among the DNS family for both approaches and for all the forecasting horizons. Furthermore, the DNS 3 under the ML approach is the only that challenges the dominance of the RW model. Overall, the rest of the DNS models per approach have similar predictive abilities. Furthermore, the Forecast Combination seems to have improved forecast abilities than the most of the individual models. Also, for the full-sample which includes several yield environments, as described in Section 3, the ML approach seems preferable in front of the EM approach for the 1, 3 and 6-step ahead horizons, however for the 12-step ahead horizon several DNS-EM specifications perform equally or even better the the DNS-ML for specific parts of the yield curve.

4.1.2 Economic Evaluation

For the 1-month rebalancing portfolios, in Table 35 in Appendix, one can observe that the DNS nested models for both approaches have significantly lower Turnover than the competitor models, for all the difference risk aversion coefficients. Furthermore, the combination scheme seems to result higher Turnover as well. Regarding the Sharpe Ratios that are achieved by the mean-variance portfolios obtained with the DNS models under the ML approach, Table 7 shows that for different δ there is a similar pattern with slight or sometimes more clear differences. More specifically, for δ equal to 0.1, 0.25, 0.5 and 1, the DNS 1 model delivers higher Sharpe Ratios where is more evident for the latter three, whereas for higher δ shows equal or slightly lower performance than the DNS 5 and 6. Worth mentioning is that the two-factor model DNS 2 has the worst performance for all values of risk aversion. For the DNS - EM models, even though the differences in magnitude per risk aversion are small, one can see that DNS 3 is slightly better for $\delta = 0.1$, DNS 1 and 4 for δ equal to 0.25, 0.5 and 1, whereas DNS seems to perform slightly better for higher values of the risk aversion. Similarly with the ML approach, DNS 2 shows poor performance comparing to the rest DNS variants. Between the two estimation approaches, there is a consistency for all δ when one compares the nested models. Under the ML approach the models DNS 1, 3 and 6 delivers higher Sharpe Ratios, whereas DNS 2 and 4 perform better under the EM approach. In addition, the combination scheme seems to be outperformed in both approaches by the majority of the individual models, and also the competitor models shows lower performance[9].

Table 7: Sharpe Ratio 1-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	0.40	0.56	0.79	1.07	1.24	1.27	1.28
AR(1)	0.44	0.61	0.85	1.12	1.26	1.28	1.28
VAR(1) PCA	0.29	0.33	0.40	0.54	0.78	0.95	1.15
DNS - ML approach							
DNS 1	0.57	0.80	1.05	1.23	1.28	1.27	1.26
DNS 2	0.35	0.54	0.77	0.97	1.07	1.10	1.11
DNS 3	0.56	0.72	0.94	1.16	1.27	1.27	1.26
DNS 4	0.51	0.70	0.94	1.17	1.26	1.26	1.26
DNS 5	0.55	0.70	0.92	1.16	1.28	1.29	1.26
DNS 6	0.55	0.71	0.92	1.17	1.28	1.29	1.26
Forecast Combination	0.43	0.49	0.59	0.76	1.00	1.13	1.22
DNS - EM approach							
DNS 1	0.52	0.72	0.96	1.18	1.26	1.26	1.24
DNS 2	0.36	0.57	0.81	1.00	1.08	1.10	1.10
DNS 3	0.55	0.71	0.92	1.15	1.25	1.25	1.23
DNS 4	0.52	0.72	0.96	1.19	1.26	1.26	1.24
DNS 5	0.54	0.70	0.91	1.16	1.28	1.28	1.26
DNS 6	0.53	0.69	0.90	1.13	1.25	1.26	1.24
Forecast Combination	0.38	0.43	0.51	0.67	0.90	1.05	1.18

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 1-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Full sample.

[9]Since the focus of the economic evaluation in this research is on the nested models, the comparison with the competitor models is not taken place thoroughly.

For the 3-months rebalancing portfolios, the resulting Turnovers in Table Table 36 in Appendix, indicates that the DNS models perform substantially better than the competitor models in that perspective. The Forecast Combination does not outperform the majority of the models for both estimation approaches as well. Next, Table 8 shows the Sharpe Ratios that are delivered from the DNS-ML models have significant differences for δ equal to 0.1, 0.25 and 0.5, where DNS 1 and 4 equally dominate. For $\delta = 1$, DNS 6 performs slightly better than the aforementioned ones, whereas for higher values of risk aversion DNS 3 and 5 seems to achieve higher Sharpe Ratios. The worst performing model for lower δ is the DNS 5, whereas for higher δ is the DNS 2 variant. As regards the DNS-EM models, the variants that constantly deliver high Sharpe Ratios regardless the choice of risk aversion coefficient are the DNS 1 and 4. However, for $\delta = 0.25$, DNS 2 has slightly better performance, and for $\delta = 3$, DNS 5 performs equally well with the DNS 1 and 4. The worst performing model for the EM approach for the lower values of risk aversion is the DNS 6, while similarly to the ML approach, for higher δ is the DNS 2. Regarding the comparison between the two estimation methods, DNS 2, 3 and 6 achieves higher Sharpe Ratios for all δ under the ML approach, as well as DNS 1 and 4 for the lower values of δ and DNS 5 for higher values of δ. Moreover, the DNS 5 performs better under the EM approach for δ equal to 0.1, 0.25 and 0.5.

Table 8: Sharpe Ratio 3-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	0.36	0.51	0.72	1.00	1.23	1.30	1.33
AR(1)	0.32	0.45	0.65	0.93	1.18	1.26	1.31
VAR(1) PCA	0.48	0.60	0.79	1.07	1.32	1.38	1.39
DNS - ML approach							
DNS 1	0.82	1.11	1.29	1.36	1.35	1.34	1.33
DNS 2	0.76	1.06	1.21	1.25	1.25	1.24	1.23
DNS 3	0.68	0.90	1.15	1.33	1.37	1.36	1.34
DNS 4	0.82	1.10	1.29	1.36	1.35	1.34	1.33
DNS 5	0.63	0.86	1.13	1.33	1.37	1.36	1.33
DNS 6	0.71	0.97	1.23	1.37	1.37	1.34	1.31
Forecast Combination	0.58	0.69	0.85	1.08	1.27	1.32	1.34
DNS - EM approach							
DNS 1	0.72	1.01	1.25	1.36	1.35	1.34	1.32
DNS 2	0.69	1.02	1.19	1.23	1.22	1.22	1.21
DNS 3	0.66	0.88	1.13	1.30	1.34	1.33	1.31
DNS 4	0.72	1.00	1.25	1.36	1.36	1.34	1.33
DNS 5	0.66	0.88	1.14	1.32	1.35	1.34	1.31
DNS 6	0.61	0.83	1.07	1.27	1.32	1.32	1.31
Forecast Combination	0.46	0.54	0.68	0.90	1.15	1.26	1.32

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 3-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full sample.

Finally, the Forecast Combination is outperformed for most of the δ, except for $\delta = 5$, where this scheme competes the best models from both approaches, and additionally, the RW and AR(1) have lower performance than the most of the DNS models for all δ. The VAR(1)-PCA is the only model from the competitor ones, which achieves higher Sharpe Ratios for δ equal to 3 and 5.

Similarly, for the 6-month rebalancing portfolios Table 37 in Appendix, shows that the DNS models produce portfolios with dramatically lower Turnover than the competitors for all δ, and that the combination scheme performs results to a higher Turnover than the most of the individual models, for all the risk aversion levels. As regards the Sharpe Ratios, according to the Table 9, under the ML approach, the four-factor models achieves higher Sharpe Ratios for all δ. Specifically, DNS 6 performs better than the other variants for δ equal to 0.1 and 0.25, DNS 5 for δ equal to 0.5 and 1, and DNS 3 for higher risk aversion values. Worth mentioning is that DNS 2 for δ equal to 0.5, 1, 2, 3 and 5 has the worst performance.

Table 9: Sharpe Ratio 6-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	0.35	0.45	0.59	0.82	1.08	1.19	1.25
AR(1)	0.35	0.44	0.58	0.81	1.07	1.18	1.25
VAR(1) PCA	0.33	0.42	0.56	0.79	1.06	1.17	1.25
DNS - ML approach							
DNS 1	0.97	1.23	1.36	1.39	1.38	1.38	1.37
DNS 2	0.98	1.21	1.28	1.30	1.30	1.29	1.29
DNS 3	0.97	1.24	1.42	1.46	1.43	1.41	1.38
DNS 4	0.98	1.24	1.36	1.39	1.38	1.38	1.37
DNS 5	0.98	1.29	1.46	1.47	1.42	1.39	1.36
DNS 6	1.00	1.30	1.45	1.45	1.39	1.37	1.34
Forecast Combination	0.75	0.90	1.08	1.26	1.36	1.37	1.37
DNS - EM approach							
DNS 1	0.87	1.18	1.37	1.43	1.41	1.40	1.38
DNS 2	0.97	1.23	1.29	1.29	1.28	1.27	1.27
DNS 3	0.89	1.16	1.34	1.39	1.38	1.36	1.34
DNS 4	0.87	1.18	1.37	1.43	1.42	1.40	1.39
DNS 5	0.83	1.09	1.28	1.36	1.35	1.34	1.32
DNS 6	0.79	1.05	1.26	1.36	1.37	1.36	1.34
Forecast Combination	0.60	0.73	0.92	1.16	1.34	1.38	1.39

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 6-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full sample.

Under the EM approach, the two and three-factor models dominate, as the DNS 2 achieves the highest Sharpe Ratio for δ equal to 0.1 and 0.25, and additionally, DNS 1 and 4 have equally the best performance for all the higher values of risk aversion coefficient. However, DNS 2 has the worst results for δ equal to 1, 2, 3 and 5. When comparing the two approaches, one see that under the EM approach, only DNS 1 and 4 achieves higher Sharpe Ratios, for high values of δ and DNS 2 for δ equal to 0.25 and 0.5. Furthermore, the Forecast Combination delivers competitive results for high values of δ under both estimation approaches, and also, it is clear that the DNS models outperforms the competitor models regardless the value of the risk aversion coefficient.

Finally, for the 12-months rebalancing portfolios, Table 38 in Appendix, shows that the pattern for the Turnover holds, and the DNS models clearly produce more reliable portfolios, regarding the trading volume. It is also evident that the combination scheme increase the Turnover. Furthermore, in Table 10 the Sharpe Ratios for the 12-months rebalancing portfolios produced from the models of this research are presented. One can see that under the ML approach, the DNS 2 model achieves the highest Sharpe Ratios comparing to the other nested models for δ equal to 0.1, 0.25 and 0.5, but on the contrary the worst for δ equal to 1, 2, 3 and 5, while the four-factor models dominate for these δ. More specifically, the DNS 3, 5 and 6 has the best performance for $\delta = 1$, and the DNS 3 for the higher risk aversion coefficients as well.

Table 10: Sharpe Ratio 12-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	0.31	0.38	0.50	0.73	1.09	1.28	1.34
AR(1)	0.31	0.38	0.49	0.72	1.09	1.28	1.34
VAR(1) PCA	0.29	0.35	0.47	0.70	1.09	1.29	1.36
DNS - ML approach							
DNS 1	0.84	1.15	1.33	1.40	1.41	1.40	1.40
DNS 2	1.16	1.36	1.39	1.37	1.36	1.35	1.35
DNS 3	0.73	1.00	1.26	1.42	1.45	1.44	1.42
DNS 4	0.87	1.17	1.35	1.41	1.42	1.41	1.40
DNS 5	0.80	1.09	1.32	1.42	1.42	1.41	1.40
DNS 6	0.82	1.11	1.33	1.42	1.41	1.40	1.38
Forecast Combination	0.61	0.75	0.93	1.16	1.33	1.38	1.40
DNS - EM approach							
DNS 1	0.70	1.02	1.29	1.44	1.45	1.44	1.42
DNS 2	0.98	1.27	1.35	1.35	1.34	1.33	1.33
DNS 3	0.66	0.91	1.17	1.35	1.40	1.40	1.39
DNS 4	0.71	1.02	1.30	1.44	1.46	1.44	1.43
DNS 5	0.60	0.85	1.12	1.33	1.40	1.40	1.39
DNS 6	0.57	0.80	1.06	1.29	1.39	1.40	1.39
Forecast Combination	0.47	0.58	0.74	0.99	1.26	1.36	1.41

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 12-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full sample.

Under the EM approach, similarly to the ML approach, the DNS 2 achieves the highest Sharpe Ratios for the lower values of δ, and has the worst performance for δ equal to 2, 3 and 5. Furthermore, as for the 6-month rebalancing portfolios, the three-factor models DNS 1 and 4, dominate for δ equal to 1, 2, 3 and 5. As regards the comparison between the two approaches, overall DNS models deliver larger Sharpe Ratios under the ML approach, except for δ equal to 1, 2, 3 and 5, where DNS 1 and 4 performs better under the EM approach. Moreover, the combination scheme does not improve the Sharpe Ratios for the majority of the models for both approaches. Lastly, the competitor models are outperformed for all the different risk aversion coefficients.

To summarize, the results for all rebalancing periods, regarding the Turnover of the produced bond portfolios based on the forecasts of the nested DNS models, support the approach of this research of not restricting the the portfolio weights. Since the nested models are the epicentre of the economic evaluation, the large Turnover of the competitors is not taken into further consideration. The comparison of the Sharpe Ratios from the models of each estimation approach shows that, the choice under this economic criterion is indissolubly related with the risk aversion coefficient, and that most of the cases the differences are small in magnitude. Overall, the achieved Sharpe Ratios indicate that for all the rebalancing periods and in most of the risk aversion choices, the four-factors models DNS 3, 5, and 6 perform better under the ML approach, whereas for 6 and 12-months rebalancing and high δ, the three-factors models DNS 1 and 4 show higher performance under the EM. Lastly, the Forecast Combination of the nested models does not seem to add economic value, since this scheme does not reduce the Turnover and at the same time is outperformed by the individual models when on compares the delivered Sharpe Ratios for the most of the values of risk aversion.

4.1.3 Relation between Statistical and Economic measures

In order to exploit further the results, following the initial idea of Caldeira et al. (2016b), this thesis examines how the statistical measures and the Sharpe Ratios are correlated. The focus of this analysis is on the nested DNS models, due to the small differences that are observed in the two evaluation methods. Therefore, the decision of which of the nested models is the best could be based on statistical criteria, economic criteria or ideally in the combination of both. In the Table 11 below, one can see the correlation coeficient between the TRMSFE and Sharpe Ratios for the different values of risk aversion. One can see that the correlation between the two measures is negative for 1, 3 and 6-step ahead horizon, for all δ. More specifically, For the 1 and 3-step ahead horizons the two measures are highly negatively correlated from -0.47 to -0.66, whereas for 6-step ahead horizon the negative correlation of the TRMSFE and Sharpe

Ratios is lower, from -0.22 to -0.37, but not negligible. As regards the 12-step ahead horizon, the correlation coefficient for δ equal to 0.1 and 0.25 is positive, for δ equal to 0.5 is slightly negative, -0.05,and for higher δ is significantly negative, from -0.43 to -0.55.

Table 11: Correlation between TRMSFE and Sharpe Ratio - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
1-step	-0.63	-0.56	-0.47	-0.54	-0.60	-0.62	-0.64
3-step	-0.36	-0.22	-0.27	-0.37	-0.32	-0.33	-0.35
6-step	-0.61	-0.61	-0.66	-0.57	-0.51	-0.50	-0.48
12-step	0.18	0.14	-0.05	-0.43	-0.55	-0.52	-0.48

Note: This table reports the correlation between the TRMSFE of the DNS-ML and DNS-EM models and the Sharpe Ratios of the mean-variance bond portfolios with 1, 3, 6 and 12-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full Sample.

These results are generally aligned with the previous statistical and economic evaluation of the full sample, as even if the final ranking of the DNS nested models has diffences under the two evaluation methods. For instance, for 3 and 6-step ahead the DNS 1 under the ML approach is the best among this approach according to the TRMSFE, Table 2 and to the Sharpe Ratios, Table 8 and 9, for high values of δ. For 6 and 12-step ahead horizon, among the DNS-ML models, the DNS 3 variant has the lowest TRMSFE and the highest Sharpe Ratios for δ equal to 2, 3 and 5. Lastly, for 1-step ahead horizon, DNS 1 under both estimation methods seems to be the worst performing variant according to both statistical and economic measures.

In a more granular level, in Tables 12-15, one can see what is the correlation of the RMSFEs per maturity with the Sharpe Ratios for the different values of risk aversion. In this way the strongest negative correlations per maturity can be highlighted, and make clearer in which cases the economic evaluation can be used as a supplementary method to indicate the best performing nested model per maturity. For the 1-step ahead horizon, significant levels of negative correlation, that in some cases exceeds the 80%, are indicated for the short and long-term maturities 3 to 9 months and 108 to 120 months, for all values of δ. For the 3-step ahead horizon, the RMSFEs of the short and long-term maturities, 3-6 and 96-120 months, exceeds the 60% of negative correlation with the Sharpe ratios for higher values of risk aversion. Similarly, for the 6-step ahead horizon, the maturities that there is high negative correlation are the short-term 3-6 months and the long-term 108-120 months for higher δ, as well as the medium-term maturities for lower δ. Finally, for the 12-month ahead horizon, the highlighted negative correlations can be seen for the long-term maturities 108 to 120 months, where for high values of delta the negative relation between the RMSFE and Sharpe Ratios in some cases surpasses the 70%.

Table 12: Correlation between RMSFE and Sharpe Ratio 1-step - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	-0.81	-0.67	-0.54	-0.65	-0.77	-0.80	-0.82
6-m	-0.74	-0.64	-0.54	-0.63	-0.72	-0.74	-0.76
9-m	-0.58	-0.52	-0.45	-0.51	-0.56	-0.58	-0.61
12-m	-0.31	-0.30	-0.27	-0.28	-0.29	-0.30	-0.33
15-m	-0.02	-0.05	-0.07	-0.03	0.00	0.00	-0.02
18-m	0.11	0.05	0.01	0.08	0.13	0.14	0.12
21-m	0.08	0.00	-0.04	0.03	0.10	0.10	0.09
24-m	-0.03	-0.10	-0.12	-0.06	-0.01	-0.01	-0.02
30-m	-0.27	-0.29	-0.27	-0.26	-0.25	-0.25	-0.27
36-m	-0.41	-0.37	-0.32	-0.35	-0.38	-0.39	-0.41
48-m	-0.53	-0.43	-0.34	-0.41	-0.50	-0.52	-0.53
60-m	-0.41	-0.33	-0.24	-0.29	-0.36	-0.39	-0.41
72-m	-0.26	-0.22	-0.16	-0.17	-0.23	-0.25	-0.28
84-m	-0.10	-0.16	-0.16	-0.10	-0.07	-0.08	-0.12
96-m	-0.39	-0.47	-0.48	-0.42	-0.36	-0.36	-0.39
108-m	-0.63	-0.61	-0.57	-0.61	-0.62	-0.62	-0.63
120-m	-0.80	-0.77	-0.74	-0.81	-0.80	-0.79	-0.78

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 1-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 1-month rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full Sample.

Table 13: Correlation between RMSFE and Sharpe Ratio 3-step - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	-0.20	0.08	-0.02	-0.49	-0.62	-0.64	-0.66
6-m	-0.29	-0.07	-0.15	-0.46	-0.51	-0.53	-0.56
9-m	-0.33	-0.17	-0.24	-0.39	-0.37	-0.39	-0.41
12-m	-0.33	-0.24	-0.28	-0.31	-0.25	-0.25	-0.28
15-m	-0.33	-0.27	-0.29	-0.25	-0.16	-0.16	-0.19
18-m	-0.33	-0.29	-0.30	-0.22	-0.12	-0.12	-0.14
21-m	-0.34	-0.30	-0.31	-0.22	-0.11	-0.11	-0.13
24-m	-0.35	-0.30	-0.31	-0.22	-0.12	-0.11	-0.13
30-m	-0.35	-0.28	-0.29	-0.24	-0.15	-0.14	-0.16
36-m	-0.34	-0.25	-0.27	-0.26	-0.18	-0.18	-0.19
48-m	-0.32	-0.21	-0.24	-0.30	-0.24	-0.24	-0.25
60-m	-0.37	-0.24	-0.26	-0.32	-0.26	-0.26	-0.27
72-m	-0.43	-0.29	-0.31	-0.33	-0.24	-0.24	-0.26
84-m	-0.49	-0.35	-0.38	-0.37	-0.26	-0.27	-0.29
96-m	-0.50	-0.35	-0.41	-0.46	-0.35	-0.35	-0.39
108-m	-0.46	-0.30	-0.40	-0.53	-0.43	-0.43	-0.45
120-m	-0.44	-0.27	-0.44	-0.68	-0.59	-0.59	-0.63

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 3-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 3-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full Sample.

Table 14: Correlation between RMSFE and Sharpe Ratio 6-step - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	-0.36	-0.32	-0.49	-0.55	-0.56	-0.56	-0.54
6-m	-0.46	-0.45	-0.56	-0.56	-0.55	-0.55	-0.54
9-m	-0.53	-0.53	-0.59	-0.54	-0.52	-0.51	-0.50
12-m	-0.57	-0.58	-0.61	-0.52	-0.48	-0.47	-0.46
15-m	-0.60	-0.62	-0.62	-0.50	-0.45	-0.43	-0.43
18-m	-0.63	-0.65	-0.63	-0.49	-0.43	-0.41	-0.40
21-m	-0.65	-0.67	-0.63	-0.49	-0.41	-0.40	-0.38
24-m	-0.67	-0.68	-0.64	-0.49	-0.41	-0.39	-0.38
30-m	-0.69	-0.70	-0.66	-0.50	-0.41	-0.39	-0.37
36-m	-0.69	-0.70	-0.68	-0.52	-0.43	-0.40	-0.37
48-m	-0.69	-0.70	-0.71	-0.56	-0.46	-0.43	-0.40
60-m	-0.68	-0.69	-0.70	-0.56	-0.48	-0.46	-0.43
72-m	-0.69	-0.69	-0.70	-0.57	-0.49	-0.47	-0.44
84-m	-0.65	-0.66	-0.70	-0.59	-0.53	-0.52	-0.51
96-m	-0.61	-0.65	-0.74	-0.66	-0.61	-0.60	-0.58
108-m	-0.57	-0.63	-0.76	-0.70	-0.66	-0.65	-0.64
120-m	-0.41	-0.52	-0.75	-0.77	-0.78	-0.78	-0.77

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 6-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 6-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full Sample.

Table 15: Correlation between RMSFE and Sharpe Ratio 12-step - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	0.31	0.33	0.19	-0.21	-0.49	-0.52	-0.51
6-m	0.27	0.26	0.10	-0.31	-0.53	-0.53	-0.52
9-m	0.22	0.20	0.02	-0.37	-0.54	-0.53	-0.51
12-m	0.19	0.15	-0.03	-0.41	-0.54	-0.51	-0.48
15-m	0.15	0.12	-0.07	-0.43	-0.52	-0.49	-0.45
18-m	0.13	0.09	-0.09	-0.44	-0.51	-0.47	-0.43
21-m	0.10	0.06	-0.12	-0.44	-0.50	-0.45	-0.41
24-m	0.08	0.05	-0.13	-0.45	-0.48	-0.44	-0.39
30-m	0.07	0.03	-0.14	-0.44	-0.47	-0.42	-0.38
36-m	0.07	0.04	-0.14	-0.44	-0.47	-0.42	-0.37
48-m	0.10	0.07	-0.11	-0.42	-0.48	-0.43	-0.38
60-m	0.11	0.07	-0.11	-0.44	-0.50	-0.46	-0.41
72-m	0.11	0.07	-0.13	-0.47	-0.53	-0.49	-0.45
84-m	0.16	0.11	-0.12	-0.51	-0.60	-0.56	-0.52
96-m	0.21	0.14	-0.10	-0.54	-0.65	-0.61	-0.57
108-m	0.26	0.19	-0.08	-0.57	-0.71	-0.68	-0.64
120-m	0.37	0.29	-0.01	-0.58	-0.79	-0.77	-0.73

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 12-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 12-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full Sample.

4.2 De Pooter (2007) Sub-sample

4.2.1 Statistical Evaluation

Table 16 shows the TRMSFE of the benchmark model and the ratios of the TRMSFE of the models relative to the benchmark and furthermore, Tables 17-20 shows the ratios of the RMSFE of the models relative to the benchmark for all maturities.

Table 16: TRMSFE - Sub-sample

Forecasting Horizon	**1-step**	**3-step**	**6-step**	**12-step**
Random Walk	0.27	0.54	0.85	1.29
AR(1)	1.01	1.01	1.01	**1.00**
VAR(1) PCA	1.05	1.11	1.16	1.21
DNS - ML approach				
DNS 1	1.04	1.06	1.07	1.08
DNS 2	1.09	1.03	1.05	1.11
DNS 3	1.02	1.02	1.02	1.04
DNS 4	1.07	1.09	1.09	1.09
DNS 5	1.05	1.05	1.04	1.02
DNS 6	1.09	1.12	1.13	1.15
Forecast Combination	1.03	1.04	1.04	1.05
DNS - EM approach				
DNS 1	1.13	1.16	1.13	1.09
DNS 2	1.21	1.18	1.15	1.11
DNS 3	1.04	1.08	1.02	**0.95**
DNS 4	1.16	1.21	1.18	1.12
DNS 5	1.11	1.14	1.11	1.04
DNS 6	1.17	1.14	1.12	1.04
Forecast Combination	1.07	1.12	1.09	1.04

Note: This table reports the TRMSFE of the RW, for the 1, 3, 6 and 12-step ahead forecasts for the Sub-sample, and also the relative TRMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

For 1-step ahead horizon, Table 16 shows that according to the TRMSFEs, RW outperforms the DNS models. One can see that the DNS 3 is the best variant under both approaches and also, that the DNS models have better performance under the ML approach. Furthermore, the combination scheme seems slightly outperform the most of the DNS nested models. It is worth mentioning that under the EM approach, one can distinguish the best variant, DNS 3, and the worst, DNS 2. In a maturity specific level, Table 17 indicates the performance of the models relative to the benchmark. For short-term maturities, RW clearly outperforms the DNS models under both approaches, whereas for longer maturities the DNS models under the ML approach increase their forecast ability, and most of the variants show equal or slightly worse performance than the benchmark model. Especially the DNS 3, outperforms the benchmark by 1% for the maturities of 36 and 48-months. Worth mentioning is that the Forecast Combination produces significantly better forecasts than all the DNS nested models, under both estimation approaches for the short-term maturities of 3 and 6-months, and also that for these maturities VAR(1)-PCA outperforms all the models of this research by at least 14% and 5% respectively. As regards, the comparison between the two approaches, one can see in Table 39 in Appendix, that generally the DNS-ML models performs better. However, for the 3-month maturity the DNS 3 and 5 have higher forecasting abilities under the EM approach, by 4% and 3% respectively, and for the 120-month maturity each of the DNS 1 and 3 models perform equally under both approaches.

Table 17: RMSFE 1-step ahead - Sub-sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	0.22	0.21	0.23	0.24	0.26	0.27	0.28	0.28	0.29	0.30	0.30	0.30	0.29	0.28	0.27	0.26	0.26
AR(1)	**1.00**	**1.00**	**1.00**	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02
VAR(1) PCA	**0.86**	**0.95**	1.03	1.07	1.08	1.08	1.08	1.07	1.05	1.04	1.03	1.06	1.05	1.08	1.08	1.10	1.10
DNS - ML approach																	
DNS 1	1.12	1.09	1.09	1.10	1.09	1.08	1.06	1.04	1.01	**1.00**	**1.00**	1.02	1.01	1.01	1.01	1.03	1.06
DNS 2	1.70	1.30	1.05	**1.00**	1.02	1.06	1.09	1.10	1.09	1.07	1.04	1.03	1.01	1.01	1.02	1.04	1.08
DNS 3	1.13	1.06	1.03	1.02	1.02	1.02	1.02	1.01	**1.00**	**0.99**	**0.99**	**1.00**	**1.00**	**1.00**	1.01	1.02	1.05
DNS 4	1.24	1.17	1.14	1.13	1.13	1.11	1.09	1.08	1.05	1.02	1.01	1.03	1.02	1.02	1.02	1.03	1.04
DNS 5	1.35	1.16	1.09	1.07	1.07	1.07	1.06	1.05	1.03	1.01	1.01	1.02	1.02	1.02	1.01	1.01	1.01
DNS 6	1.25	1.19	1.15	1.14	1.14	1.13	1.13	1.11	1.08	1.06	1.04	1.05	1.04	1.04	1.03	1.03	1.02
Forecast Combination	1.01	1.01	1.03	1.05	1.07	1.07	1.07	1.06	1.03	1.01	**1.00**	1.02	1.01	1.01	1.01	1.01	1.02
DNS - EM approach																	
DNS 1	1.44	1.26	1.15	1.14	1.14	1.14	1.14	1.13	1.10	1.07	1.07	1.10	1.11	1.10	1.08	1.07	1.06
DNS 2	2.11	1.74	1.38	1.20	1.13	1.11	1.11	1.12	1.11	1.09	1.08	1.10	1.10	1.11	1.10	1.11	1.13
DNS 3	1.09	1.07	1.05	1.05	1.05	1.05	1.05	1.05	1.03	1.02	1.01	1.04	1.05	1.05	1.04	1.04	1.04
DNS 4	1.47	1.31	1.20	1.18	1.18	1.18	1.17	1.16	1.13	1.11	1.09	1.13	1.13	1.12	1.10	1.09	1.08
DNS 5	1.30	1.22	1.17	1.15	1.15	1.14	1.13	1.12	1.09	1.07	1.06	1.09	1.09	1.09	1.07	1.07	1.06
DNS 6	1.51	1.44	1.33	1.26	1.22	1.18	1.16	1.14	1.11	1.09	1.07	1.10	1.11	1.11	1.10	1.09	1.09
Forecast Combination	1.02	1.05	1.06	1.08	1.09	1.10	1.10	1.09	1.07	1.06	1.05	1.08	1.09	1.09	1.07	1.07	1.06

Note: This table reports the RMSFE of the RW, for the 1-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the Sub-sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

For the 3-step ahead horizon, Table 16 indicates that in an aggregate level, the DNS 2 an 3 under the ML approach produce the best forecasts among the DNS-ML models, and the DNS 3 among the DNS-EM models. Furthermore, the DNS models seems to have better performarce under the ML approach, except the DNS 6, which performs the same under both approaches, the DNS 4 delivers the worst results under both approaches, and the combination scheme beats the most of the individual models. Moreover, in Table 18 one can see the RMSFEs of the models relative to the RW. For the maturities of 3, 6 and 9-months, the DNS 2 under the ML approach performs equally or better by 2%, and for the maturities 30, 36, 48, 60, 72, 84 and 96-months, DNS 3 under the ML approach has the same forecasting abilities than RW of better by 1%, which is also the case for the DNS 5 for 96, 108 and 120-months maturities. The Forecast Combination seems to outperform the most of the models and being a particularly competitive scheme for long-term maturities. Another interesting point regarding the VAR(1)-PCA, is that for the 3-months maturity outperforms all the models by at least 10%. Finally, one can observe in Table 40 in Appendix, that for the 3-months maturity, the DNS 3 model shows higher performance under the EM approach, by 2% and also, for the maturities of 3, 6, 9, 12, 15, 18 and 21-months, DNS 6 performs better under the EM approach by 1% to 8%.

Table 18: RMSFE 3-step ahead - Sub-sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	0.48	0.50	0.52	0.54	0.56	0.57	0.58	0.58	0.59	0.59	0.57	0.56	0.53	0.52	0.50	0.48	0.47
AR(1)	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02
VAR(1) PCA	**0.88**	**0.99**	1.06	1.10	1.12	1.13	1.13	1.13	1.11	1.11	1.12	1.14	1.14	1.15	1.16	1.17	1.17
DNS - ML approach																	
DNS 1	1.09	1.11	1.11	1.12	1.11	1.10	1.09	1.07	1.04	1.03	1.02	1.03	1.02	1.02	1.02	1.03	1.04
DNS 2	**0.98**	**0.98**	**1.00**	1.03	1.05	1.06	1.06	1.05	1.03	1.02	1.01	1.02	1.02	1.03	1.04	1.06	1.08
DNS 3	1.07	1.05	1.04	1.03	1.03	1.02	1.02	1.01	**1.00**	**0.99**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	1.01	1.02
DNS 4	1.17	1.17	1.17	1.17	1.16	1.14	1.13	1.11	1.07	1.05	1.03	1.03	1.02	1.02	1.02	1.02	1.03
DNS 5	1.21	1.13	1.10	1.09	1.08	1.07	1.06	1.05	1.03	1.01	1.01	1.01	1.01	1.01	**1.00**	**1.00**	**0.99**
DNS 6	1.24	1.22	1.21	1.20	1.19	1.18	1.16	1.15	1.11	1.09	1.07	1.06	1.05	1.04	1.03	1.03	1.01
Forecast Combination	1.04	1.07	1.08	1.08	1.09	1.08	1.07	1.06	1.03	1.02	1.01	1.01	1.01	**1.00**	**1.00**	1.01	1.01
DNS - EM approach																	
DNS 1	1.20	1.20	1.19	1.19	1.19	1.18	1.18	1.17	1.14	1.12	1.12	1.14	1.15	1.14	1.13	1.12	1.10
DNS 2	1.25	1.21	1.18	1.17	1.17	1.17	1.17	1.17	1.15	1.14	1.14	1.16	1.18	1.18	1.19	1.20	1.21
DNS 3	1.04	1.07	1.08	1.09	1.09	1.09	1.09	1.08	1.07	1.06	1.06	1.07	1.09	1.09	1.08	1.08	1.08
DNS 4	1.24	1.25	1.24	1.24	1.24	1.23	1.23	1.22	1.19	1.18	1.17	1.19	1.19	1.19	1.17	1.17	1.15
DNS 5	1.26	1.22	1.19	1.18	1.17	1.16	1.15	1.14	1.11	1.10	1.10	1.12	1.13	1.12	1.12	1.11	1.10
DNS 6	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.14	1.12	1.11	1.11	1.14	1.15	1.15	1.15	1.15	1.14
Forecast Combination	1.04	1.09	1.12	1.13	1.14	1.14	1.13	1.13	1.11	1.10	1.10	1.12	1.13	1.13	1.13	1.13	1.12

Note: This table reports the RMSFE of the RW, for the 3-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the Sub-sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

For the 6-step ahead horizon, TRMSFEs in Table 16 show that RW outperforms all the DNS models. The DNS 3 under both approaches is the only competitive variant, as it is slightly worse than the benchmark, and at the same time has significantly better forecasting abilities than the rest of the nested models, especially under the EM approach. Furthermore, the combination scheme outperforms the most of the individuals under both approaches, except the DNS 3.

Regarding the performance of the models per maturity, Table 19 shows that for the maturities 3 to 24-months RW generally outperforms the DNS models, except for 3-months maturity, where the DNS 2 under the ML approach shows better results by 3% and the DNS 3 under the EM approach performs equally well with the RW. Furthermore, for the maturities 30 to 96 months, DNS 3 under the ML approach produce equal or better forecasts than the benchmark, while DNS 5 under the ML approach constantly beats the benchmark for maturities of 48 to 120 months. Worth mentioning is that for the 3-months maturity the rest of the competitors shows greater performance than the RW, and that the combination scheme for the DNS-ML models, for long-term maturities shows similar predictive abilities as the best ones of this approach. In addition, the comparison of the two estimation approaches follows the same pattern as for the 3-step ahead horizon. This can be seen in Table 41 in Appendix, where the DNS 3 seems to perform better under the EM approach for the maturities 3, 6, 9 and 12-months by 1% to 6%, and similarly the DNS 6 for the maturities 3 to 36 months by 1% to 9%.

Table 19: RMSFE 6-step ahead - Sub-sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	0.83	0.84	0.87	0.88	0.90	0.90	0.91	0.91	0.91	0.90	0.87	0.85	0.81	0.78	0.75	0.72	0.70
AR(1)	**0.98**	**0.99**	**0.99**	**0.99**	**0.99**	**1.00**	**1.00**	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.03
VAR(1) PCA	**0.96**	1.05	1.11	1.14	1.17	1.18	1.18	1.18	1.17	1.17	1.18	1.20	1.21	1.21	1.22	1.24	1.24
DNS - ML approach																	
DNS 1	1.11	1.12	1.13	1.13	1.12	1.11	1.10	1.08	1.06	1.04	1.03	1.03	1.02	1.02	1.02	1.03	1.04
DNS 2	**0.97**	1.02	1.05	1.06	1.07	1.07	1.07	1.06	1.04	1.03	1.03	1.03	1.04	1.05	1.07	1.09	1.12
DNS 3	1.06	1.05	1.04	1.03	1.03	1.02	1.02	1.01	**0.99**	**0.99**	**0.99**	**1.00**	**0.99**	**1.00**	**1.00**	1.01	1.03
DNS 4	1.15	1.17	1.17	1.17	1.16	1.14	1.12	1.10	1.07	1.05	1.02	1.01	1.01	**1.00**	1.01	1.02	1.03
DNS 5	1.16	1.12	1.10	1.08	1.07	1.06	1.05	1.04	1.01	**1.00**	**0.99**	**0.99**	**0.99**	**0.98**	**0.98**	**0.98**	**0.97**
DNS 6	1.24	1.24	1.23	1.22	1.21	1.19	1.18	1.16	1.12	1.09	1.07	1.05	1.03	1.02	1.02	1.01	**1.00**
Forecast Combination	1.07	1.09	1.09	1.09	1.09	1.08	1.07	1.05	1.03	1.01	**1.00**	**1.00**	**0.99**	**0.99**	**0.99**	**1.00**	**1.00**
DNS - EM approach																	
DNS 1	1.18	1.18	1.18	1.18	1.17	1.16	1.15	1.13	1.11	1.10	1.09	1.10	1.10	1.09	1.09	1.09	1.08
DNS 2	1.15	1.16	1.16	1.17	1.16	1.16	1.15	1.15	1.13	1.12	1.12	1.14	1.15	1.16	1.17	1.19	1.20
DNS 3	**1.00**	1.01	1.02	1.02	1.03	1.02	1.02	1.02	1.01	**1.00**	**1.00**	1.02	1.03	1.04	1.04	1.05	1.06
DNS 4	1.22	1.23	1.23	1.23	1.22	1.21	1.20	1.19	1.16	1.14	1.14	1.14	1.15	1.14	1.13	1.13	1.13
DNS 5	1.20	1.18	1.16	1.14	1.13	1.12	1.11	1.10	1.07	1.06	1.06	1.06	1.07	1.07	1.07	1.08	1.07
DNS 6	1.14	1.15	1.16	1.16	1.15	1.15	1.13	1.12	1.10	1.08	1.07	1.08	1.09	1.09	1.09	1.10	1.10
Forecast Combination	1.08	1.11	1.12	1.12	1.12	1.11	1.11	1.10	1.08	1.07	1.06	1.08	1.09	1.09	1.09	1.09	1.10

Note: This table reports the RMSFE of the RW, for the 6-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the Sub-sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

Finally, for the 12-step ahead forecasting horizon, in Table 16 one can see that in an aggregate level, the DNS-ML models are outperformed by the benchmark, as well as most of the DNS-EM models, with the exception that the DNS 3 significantly outperform the RW, by 5%. Additionally, the best performing model under the ML approach is the DNS 5. The Forecast Combination, does not show a substantial improvement comparing to the individual models. Another interesting point is that the VAR-PCA(1) models has quite poor performance. In a maturity level of scope, the Table 20 shows that for short and medium-term maturities, 3 to 30-months, RW is outperformed only by the DNS 3 under the EM approach, by 5% to 8%. For longer maturities, one can observe that the DNS 3 and 5 under the ML approach, have equal or slightly better performance than the RW. Moreover, DNS 3 under the EM approach continues showing substantial performance for the maturities of 36 to 72-months, where outperforms the benchmark by 2% to 8%. In addition, AR(1) beats the RW for short and medium-term maturities, but still is outperformed by the DNS 3 under the EM, and the Forecast Combination of the DNS-ML models shows an equal performance to the benchmark for long-term maturities. Last, the Table 42 in Appendix, indicates that under the EM estimation approach, DNS 1 and 2 performs slightly better for medium-term maturities, as well as DNS 5 for short-maturities, and DNS 3 and 6 performs significantly better for the maturities of 3 to 60-months, by 1% to 15%.

Table 20: RMSFE 12-step ahead - Sub-sample

Maturity	**3-m**	**6-m**	**9-m**	**12-m**	**15-m**	**18-m**	**21-m**	**24-m**	**30-m**	**36-m**	**48-m**	**60-m**	**72-m**	**84-m**	**96-m**	**108-m**	**120-m**
Random Walk	1.40	1.43	1.45	1.46	1.45	1.44	1.42	1.40	1.37	1.32	1.25	1.19	1.13	1.08	1.03	0.99	0.96
AR(1)	**0.96**	**0.97**	**0.97**	**0.97**	**0.98**	**0.98**	**0.99**	**1.00**	1.01	1.02	1.04	1.04	1.05	1.05	1.05	1.05	1.05
VAR(1) PCA	1.07	1.11	1.14	1.17	1.19	1.20	1.21	1.22	1.22	1.23	1.25	1.26	1.28	1.29	1.31	1.33	1.33
DNS - ML approach																	
DNS 1	1.12	1.12	1.11	1.11	1.11	1.10	1.10	1.09	1.07	1.06	1.05	1.04	1.03	1.03	1.04	1.04	1.05
DNS 2	1.11	1.11	1.12	1.12	1.12	1.12	1.12	1.11	1.10	1.09	1.09	1.09	1.11	1.12	1.14	1.16	1.18
DNS 3	1.10	1.08	1.07	1.06	1.05	1.05	1.04	1.03	1.01	**1.00**	**0.99**	**0.99**	**0.99**	**0.99**	**1.00**	1.01	1.02
DNS 4	1.13	1.13	1.13	1.13	1.12	1.11	1.10	1.09	1.07	1.05	1.04	1.03	1.02	1.02	1.04	1.05	1.06
DNS 5	1.11	1.08	1.06	1.05	1.04	1.03	1.02	1.01	**1.00**	**0.99**	**0.98**	**0.98**	**0.99**	**0.99**	**0.99**	**0.99**	**0.99**
DNS 6	1.23	1.22	1.21	1.20	1.20	1.19	1.17	1.16	1.13	1.11	1.09	1.06	1.05	1.04	1.04	1.03	1.02
Forecast Combination	1.10	1.10	1.09	1.09	1.08	1.07	1.06	1.05	1.03	1.02	1.01	**1.00**	**1.00**	**1.00**	**1.00**	1.01	1.01
DNS - EM approach																	
DNS 1	1.15	1.13	1.11	1.10	1.09	1.09	1.08	1.07	1.06	1.05	1.05	1.06	1.08	1.08	1.09	1.09	1.09
DNS 2	1.14	1.13	1.12	1.11	1.10	1.09	1.09	1.09	1.07	1.07	1.08	1.09	1.12	1.14	1.16	1.18	1.20
DNS 3	**0.95**	**0.95**	**0.95**	**0.94**	**0.94**	**0.94**	**0.93**	**0.93**	**0.92**	**0.92**	**0.93**	**0.96**	**0.98**	**1.00**	1.02	1.04	1.05
DNS 4	1.17	1.16	1.15	1.14	1.13	1.12	1.12	1.11	1.10	1.09	1.09	1.10	1.11	1.12	1.12	1.13	1.13
DNS 5	1.09	1.06	1.05	1.04	1.03	1.03	1.02	1.02	1.01	**1.00**	1.01	1.02	1.04	1.05	1.07	1.08	1.08
DNS 6	1.04	1.05	1.05	1.05	1.04	1.04	1.04	1.03	1.02	1.01	1.02	1.03	1.05	1.06	1.08	1.09	1.10
Forecast Combination	1.06	1.05	1.05	1.04	1.04	1.03	1.03	1.02	1.01	1.01	1.01	1.03	1.05	1.06	1.08	1.09	1.10

Note: This table reports the RMSFE of the RW, for the 12-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the Sub-sample. Furthermore, the table reports the relative RMSFE to the RW of the AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination. A value smaller than one indicates that the model outperform the benchmark model, RW. The highlighted values indicate that the respective model performs equally or better than the RW.

To summarize, the strong performance of the benchmark model is being challenged sometimes from the DNS models, especially for long forecasting horizons. Worth mentioning is that the DNS 3 under the EM for 12-step horizon, outperforms the benchmark for most of the maturities. In general, the DNS 3 shows the best results among the DNS models under both approaches. Furthermore, for 1 and 3-step ahead horizons it seems that the DNS-ML models are performs better than the DNS-EM, whereas for the 6-step and, especially for the 12-step ahead, DNS-EM are quite competitive and in many cases better that the DNS-ML. More specifically, DNS 3 and 6 variants show better short and medium-term results, under the EM approach, for the 3, 6 and 12-step ahead horizon, and also the DNS-ML models outperform the DNS-EM for long-term maturities for all forecasting horizons. Finally, the Forecast Combination performs better than most of the individuals, with highlighted performance, the short-term forecasts for 1 and 3-step ahead horizon, and the long-term forecasts for the 6 and 12-step ahead horizon.

4.2.2 Economic Evaluation

For the 1-month rebalancing portfolios, Table 43 in Appendix, indicated that the DNS models produce portfolios with lower Turnover than the competitor models, especially for higher risk aversion coefficients. The combination scheme results to higher Turnovers, under both estimation approaches. As regards the Sharpe Ratios, in Table 21 one can see that for the DNS-ML models, DNS 3 achieves higher Sharpe ratios for most of the δ, except for δ equal to 0.1 and 5, where DNS 1 and 5 respectively, slightly dominate. For the DNS-EM models, DNS 5 performs better for δ equal to 0.1, 2, 3 and 5, and furthermore, DNS 1 and 4 present similarly better results than the rest of the DNS-EM models for δ equal to 0.25, 0.5 and 1. In addition, the DNS 2 is the worst performing model under both estimation approaches, for most of the values of the risk aversion. When comparing the the performance of the DNS models under the two difference approaches, one can see that under the EM approach, DNS 2 and 5 perform better for all δ, DNS 4 and 6 for δ equal to 0.1, 0.25, 0.5, 1 and 2, whereas the DNS 1 shows higher Sharpe Ratios for all δ under the ML approach, as well as DNS 3 for higher δ. Also, Forecast combination does not outperform the individual models, under both approaches, and the competitor models significantly outperform the DNS models for lower δ.

Table 21: Sharpe Ratio 1-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	1.82	2.78	2.84	2.68	2.57	2.52	2.49
AR(1)	1.90	2.73	2.76	2.64	2.54	2.51	2.48
VAR(1) PCA	0.85	1.36	2.00	2.53	2.64	2.61	2.55
DNS - ML approach							
DNS 1	0.64	1.01	1.56	2.24	2.56	2.58	2.54
DNS 2	0.30	0.53	0.85	1.25	1.55	1.65	1.72
DNS 3	0.66	0.88	1.21	1.72	2.19	2.32	2.35
DNS 4	0.43	0.62	0.92	1.43	2.01	2.24	2.39
DNS 5	0.62	0.83	1.15	1.68	2.26	2.47	2.58
DNS 6	0.57	0.75	1.05	1.56	2.17	2.41	2.53
Forecast Combination	0.46	0.54	0.68	0.95	1.41	1.75	2.13
DNS - EM approach							
DNS 1	0.63	0.94	1.38	1.91	2.16	2.17	2.14
DNS 2	0.47	0.78	1.20	1.61	1.77	1.78	1.76
DNS 3	0.65	0.87	1.21	1.75	2.24	2.36	2.38
DNS 4	0.65	0.95	1.39	1.91	2.19	2.21	2.18
DNS 5	0.67	0.90	1.26	1.83	2.39	2.56	2.62
DNS 6	0.58	0.79	1.12	1.65	2.22	2.42	2.52
Forecast Combination	0.50	0.58	0.71	0.97	1.40	1.72	2.07

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 1-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

For the 3-month rebalancing horizon, Table 44 in Appendix, shows that the DNS models achieve significantly lower Turnover than the competitor models. Furthermore, the combination scheme does not increase the performance of the individual nested models on that perspective. Regarding the comparison of the Sharpe Ratios, Table 22 indicates that, under the ML approach, DNS 5 dominates for the most of the risk aversion coefficients, except for δ equal to 0.5, where DNS 1 achieves higher Sharpe Ratio, and for δ equal to 1, where DNS 1 and 5 have equally higher performance than the rest of the variants. Under the ML approach, DNS 5 delivers clearly higher Sharpe Ratios than the rest of the DNS models, for all values of the risk aversion coefficient. Furthermore, the worst performing variant under both approaches is the DNS 2, for all δ. Regarding the comparison between the DNS-ML and DNS-EM models, one can see the under the EM approach, the models DNS 2, 3 and 6 perform better for all δ, as well as the the DNS 4 and 5 for most of the δ, except for the highest ones. In addition, the combination scheme shows competitive results, although not better than the best performing DNS models, for higher δ. Last, the competitor models, especially the RW and AR(1), outperform the DNS models for all δ.

Table 22: Sharpe Ratio 3-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	1.48	2.33	3.77	5.20	4.89	4.51	4.15
AR(1)	1.45	2.25	3.56	4.95	4.80	4.46	4.13
VAR(1) PCA	1.11	1.60	2.43	3.73	4.46	4.39	4.16
DNS - ML approach							
DNS 1	1.10	1.98	2.75	3.03	2.98	2.93	2.88
DNS 2	0.63	1.14	1.73	2.15	2.27	2.27	2.26
DNS 3	1.05	1.56	2.22	2.81	2.97	2.95	2.89
DNS 4	0.72	1.23	1.88	2.50	2.78	2.83	2.83
DNS 5	1.19	1.78	2.47	3.03	3.18	3.17	3.13
DNS 6	1.16	1.74	2.45	3.01	3.12	3.07	3.00
Forecast Combination	0.83	1.10	1.54	2.25	2.90	3.03	3.01
DNS - EM approach							
DNS 1	1.04	1.90	2.79	3.01	2.80	2.70	2.60
DNS 2	0.83	1.58	2.33	2.55	2.43	2.36	2.29
DNS 3	1.15	1.79	2.60	3.17	3.18	3.09	2.98
DNS 4	1.14	2.00	2.81	2.99	2.82	2.73	2.64
DNS 5	1.42	2.17	2.95	3.32	3.26	3.17	3.09
DNS 6	1.28	2.00	2.77	3.18	3.16	3.10	3.03
Forecast Combination	0.80	1.06	1.49	2.24	2.96	3.09	3.03

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 3-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

Next, for the 6-month rebalancing horizon, Table 45 in Appendix, shows that as for the previous rebalancing horizons, the DNS models produce mean-variance portfolios with much lower Turnover than the competitor models, and furthermore, the Forecast Combination does not improve the performance of the individuals on that perspective. Regarding the Sharpe Ratios of this rebalancing horizon, one can observe in Table 23 that, under the ML approach, for δ equal to 0.1, 0.25 and 0.5, DNS 1 is the best performing model, whereas for higher delta DNS 5 and 6 clearly dominate. The worst performing model is the DNS 2 for all δ. Under the EM approach, similarly to the 3-month horizon results, DNS 5 achieves the highest Sharpe Ratios for all values of the risk aversion coefficient. Furthermore, the worst performing model for lower δ is the DNS 3, whereas for higher δ is the DNS 2. Regarding the comparison of the two estimation approaches, it seems that under the EM the DNS 4, 5 and 6 deliver higher Sharpe Ratios for lower δ, DNS 3 under higher δ, and DNS 2 for all values of the risk aversion coefficient. However DNS 1 performs better under the ML approach for all the δ, except for δ =0.1. Worth mentioning is that is that the Forecast Combination of the DNS-ML models significantly outperforms the individual nested models for δ equal to 1, 2 and 3, and the Forecast Combination of the DNS-EM for δ equal to 2 and 3. Last, the best performing models under both approaches deliver higher results than the competitor models for most of the δ.

Table 23: Sharpe Ratio 6-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	1.56	2.33	3.24	3.81	3.77	3.67	3.55
AR(1)	1.78	2.65	3.58	3.99	3.83	3.70	3.57
VAR(1) PCA	1.34	2.03	2.90	3.58	3.69	3.62	3.53
DNS - ML approach							
DNS 1	2.91	4.26	3.98	3.57	3.34	3.26	3.19
DNS 2	1.24	2.21	2.77	2.80	2.69	2.64	2.59
DNS 3	1.60	2.58	3.61	3.89	3.62	3.47	3.33
DNS 4	1.41	2.42	3.10	3.29	3.25	3.22	3.18
DNS 5	1.91	3.07	3.94	4.05	3.85	3.74	3.64
DNS 6	1.48	2.54	3.76	4.15	3.85	3.67	3.52
Forecast Combination	1.33	1.95	3.00	4.27	4.22	3.91	3.58
DNS - EM approach							
DNS 1	1.50	2.92	3.94	3.61	3.17	3.01	2.88
DNS 2	1.59	3.11	3.46	3.09	2.81	2.72	2.64
DNS 3	0.88	1.48	2.46	3.59	3.72	3.56	3.36
DNS 4	1.71	3.23	3.93	3.53	3.14	3.01	2.90
DNS 5	2.25	3.84	4.61	4.25	3.84	3.68	3.56
DNS 6	1.80	3.16	4.15	4.07	3.74	3.59	3.47
Forecast Combination	1.08	1.57	2.39	3.55	3.92	3.76	3.52

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 6-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Sub-sample.

Finally, for the 12-month rebalancing horizon, Table 46 in Appendix, indicates that, the Turnover of the portfolios that are produced by the DNS models is substantially lower than the ones produces by the competitor models. Regarding the Sharpe Ratio comparison, Table 24 shows that, under the ML approach, DNS 2 performs better for lower values of δ, whereas DNS 5 is the best performing model for higher δ. Under the EM approach, for δ equal to 0.1 and 0.25, DNS 2 has the higher results, for δ equal to 0.5, DNS 4 seems to outperform the rest DNS-EM models, whereas for δ equal to 1, 2 and 3, DNS 5 is the dominant one, and lastly for δ equal to 5, DNS 3 is the best variant. When comparing the to approaches, one can see mixed results, since the ML approach is preferable for the DNS 2, 3 and 6 variants for most of the risk aversion values, whereas the DNS 4 and 5 variants produce better results under the EM approach for higher and lower δ, respectively. In addition, the combination scheme seems to improve the performance of the majority of the individual models, under both estimation approaches, for high δ values. Comparing to the competitor models, the best DNS variants deliver higher Sharpe Ratios for all δ.

Table 24: Sharpe Ratio 12-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	1.38	1.93	2.42	2.68	2.69	2.66	2.63
AR(1)	1.67	2.30	2.77	2.89	2.79	2.73	2.66
VAR(1) PCA	1.48	2.08	2.55	2.73	2.70	2.66	2.63
DNS - ML approach							
DNS 1	2.58	4.21	4.43	4.12	3.88	3.79	3.72
DNS 2	3.32	5.20	4.67	4.04	3.71	3.60	3.51
DNS 3	1.58	2.69	3.90	4.35	4.15	4.00	3.87
DNS 4	2.71	4.03	4.24	4.03	3.84	3.77	3.70
DNS 5	1.94	3.34	4.40	4.51	4.25	4.12	4.00
DNS 6	1.87	3.16	4.12	4.28	4.09	3.98	3.89
Forecast Combination	1.33	1.97	2.97	4.13	4.39	4.24	4.03
DNS - EM approach							
DNS 1	2.12	4.23	5.03	4.38	3.88	3.71	3.57
DNS 2	3.55	4.50	4.15	3.80	3.60	3.54	3.48
DNS 3	0.42	0.51	0.66	1.01	1.93	3.15	5.01
DNS 4	2.23	4.46	5.15	4.41	3.89	3.72	3.58
DNS 5	2.16	3.64	4.49	4.44	4.18	4.06	3.96
DNS 6	1.87	3.14	3.99	4.13	4.00	3.92	3.85
Forecast Combination	1.27	1.91	2.94	4.25	4.51	4.31	4.05

Note: This table reports the Sharpe Ratio of the mean-variance bond portfolios with 12-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Sub-sample.

To summarize, on the Turnover perspective it is clear that the DNS models perform quite better than the competitors and that the combination scheme does not reduce this measure, for all rebalancing horizons. One can see that the differences on the Sharpe Ratios are more profound than for the full sample, which helps the decision-making for this chronological period. Worth mentioning is that for higher values of risk aversion, the DNS 5 variant shows generally greater performance for all forecasting horizons, under both forecasting approaches. Furthermore, DNS 1 under the ML approach, is quite competitive for the 1, 3 and 6-months rebalancing horizon, whereas the aforementioned variant, DNS 5 is the best among the DNS-EM models for the 3 and 6-month rebalancing period for lower δ as well. As regards the comparison of the two estimation approaches, one can see that for the 1, 3 and 6-month rebalancing horizon, under the EM approach the DNS 2 variant performs better for all δ, the DNS 4, 5 and 6 for lower values of risk aversion achieve higher Sharpe Ratios, and the DNS 3 for higher δ, whereas the DNS 1 delivers better results under the ML approach. However, for the 12-month rebalancing horizon, the DNS-ML models shows in general better performance comparing to the lower rebalancing periods. Finally, regarding the Forecast Combination, it can be seen that for 1-month rebalancing period does not improve the performance of the individual models, for 3-month horizon this scheme is competitive for high δ, whereas for 6 and 12-month horizon it outperforms the individual models for high values of risk aversion.

4.2.3 Relation between Statistical and Economic measures

In order to exploit the results of the economic evaluation I examine if the statistical measures and the Sharpe Ratios of the DNS nested models are correlated. Table 25 below indicates the correlation coefficients between the TRMSFE and the Sharpe Ratios for different δ. For the 1 and 6-step ahead horizon the two measures show a significantly negative correlation for high values of risk aversion, from 28% to 44%, whereas for lower δ one can observe that the correlation coefficient is slightly negative or positive, except for δ equal to 0.1 where the 1-step ahead results indicate a negative correlation of 19%. For the 3-step ahead horizon, the two measures show for δ equal to 3 and 5 a negative correlation of 11% to 17%, whereas for lower values show a strong positive correlation. Similarly, for the 12-step ahead horizon a strong positive correlation is observed, which declines throughout the increase of the risk aversion value. However, for δ equal to 5, a substantially negative correlation of 82% is indicated.

Table 25: Correlation between TRMSFE and Sharpe Ratio - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
1-step	-0.19	-0.04	0.03	-0.08	-0.34	-0.42	-0.44
3-step	0.27	0.50	0.55	0.31	-0.01	-0.11	-0.17
6-step	0.13	0.39	0.44	-0.02	-0.28	-0.32	-0.32
12-step	0.71	0.79	0.72	0.61	0.51	0.19	-0.82

Note: This table reports the correlation between the TRMSFE of the DNS-ML and DNS-EM models and the Sharpe Ratios of the mean-variance bond portfolios with 1, 3, 6 and 12-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

Furthermore, in a granular level, Tables 26-29, show the correlation of the RMSFEs per maturity with the Sharpe Ratios for the different values of risk aversion. For the 1-step ahead horizon, there is a worth mentioning level of negative correlation for the short term maturities 3 and 6-months, which on a few cases exceeds the 50%, and for the 120-months maturity and for δ equal to 0.1, 2, 3 and 5 the negative correlation ranges from 33% to 67%. Moreover, for the 3-step ahead horizon, the RMSFE of the long-term maturities, 96-120 months, is negatively correlated with the Sharpe ratios for higher values of risk aversion, from 17% to 53%. For the 6-step ahead horizon, worth mentioning is that for the maturities of 18 to 72-months, the level of the negative correlation between the two measures for high values of δ ranges from 28% to 55%, and additionally, for longer maturities one can see even higher negative correlation from 51% to 76%. Finally, for the 12-step ahead, all maturities for δ equal to 5, show strong levels of negative correlation, as it is also mentioned above in an aggregate level, and furthermore, the two measures for the maturities of 72 to 120-months and for δ equal to 3, are negatively correlated from 18% to 44%.

Table 26: Correlation between RMSFE and Sharpe Ratio 1-step - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	-0.55	-0.35	-0.22	-0.37	-0.71	-0.77	-0.77
6-m	-0.39	-0.22	-0.12	-0.25	-0.53	-0.59	-0.59
9-m	-0.09	0.01	0.06	-0.02	-0.16	-0.19	-0.20
12-m	0.16	0.20	0.20	0.18	0.17	0.16	0.15
15-m	0.20	0.22	0.21	0.22	0.24	0.23	0.22
18-m	0.11	0.14	0.15	0.14	0.12	0.11	0.11
21-m	-0.01	0.04	0.06	0.02	-0.06	-0.08	-0.08
24-m	-0.12	-0.05	-0.01	-0.08	-0.21	-0.24	-0.24
30-m	-0.23	-0.14	-0.08	-0.18	-0.38	-0.42	-0.42
36-m	-0.23	-0.12	-0.06	-0.17	-0.40	-0.46	-0.47
48-m	-0.07	0.04	0.08	-0.03	-0.26	-0.34	-0.37
60-m	0.13	0.23	0.24	0.15	-0.07	-0.17	-0.23
72-m	0.27	0.35	0.35	0.26	0.05	-0.06	-0.13
84-m	0.27	0.35	0.34	0.25	0.04	-0.07	-0.14
96-m	0.18	0.29	0.30	0.20	-0.03	-0.15	-0.22
108-m	-0.01	0.17	0.25	0.13	-0.20	-0.34	-0.41
120-m	-0.33	-0.04	0.13	0.00	-0.42	-0.59	-0.67

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 1-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 1-month rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Sub-sample.

Table 27: Correlation between RMSFE and Sharpe Ratio 3-step - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	0.41	0.48	0.48	0.40	0.27	0.22	0.20
6-m	0.39	0.52	0.54	0.42	0.24	0.18	0.15
9-m	0.34	0.50	0.54	0.40	0.19	0.13	0.09
12-m	0.29	0.48	0.52	0.35	0.13	0.07	0.02
15-m	0.25	0.46	0.51	0.31	0.07	0.00	-0.04
18-m	0.22	0.44	0.49	0.28	0.01	-0.06	-0.11
21-m	0.20	0.43	0.49	0.26	-0.03	-0.11	-0.16
24-m	0.20	0.43	0.49	0.25	-0.05	-0.14	-0.20
30-m	0.21	0.45	0.51	0.25	-0.07	-0.17	-0.23
36-m	0.22	0.46	0.53	0.26	-0.08	-0.18	-0.25
48-m	0.25	0.49	0.56	0.29	-0.07	-0.18	-0.26
60-m	0.27	0.52	0.58	0.31	-0.06	-0.18	-0.25
72-m	0.27	0.52	0.58	0.31	-0.06	-0.18	-0.26
84-m	0.24	0.48	0.55	0.27	-0.10	-0.22	-0.30
96-m	0.18	0.43	0.49	0.20	-0.17	-0.28	-0.36
108-m	0.09	0.35	0.41	0.11	-0.25	-0.36	-0.43
120-m	-0.03	0.23	0.29	-0.02	-0.37	-0.47	-0.53

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 3-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 3-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

Table 28: Correlation between RMSFE and Sharpe Ratio 6-step - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	0.35	0.51	0.73	0.50	0.26	0.25	0.26
6-m	0.29	0.48	0.64	0.33	0.09	0.07	0.08
9-m	0.25	0.44	0.56	0.21	-0.04	-0.05	-0.04
12-m	0.22	0.41	0.50	0.12	-0.12	-0.13	-0.12
15-m	0.19	0.39	0.46	0.06	-0.17	-0.19	-0.18
18-m	0.16	0.37	0.43	0.02	-0.22	-0.24	-0.23
21-m	0.14	0.36	0.41	-0.01	-0.25	-0.28	-0.27
24-m	0.12	0.35	0.40	-0.03	-0.28	-0.30	-0.31
30-m	0.09	0.34	0.39	-0.06	-0.31	-0.35	-0.36
36-m	0.08	0.34	0.38	-0.09	-0.35	-0.39	-0.40
48-m	0.05	0.32	0.34	-0.15	-0.41	-0.46	-0.48
60-m	0.00	0.28	0.29	-0.19	-0.43	-0.49	-0.52
72-m	-0.04	0.24	0.23	-0.23	-0.46	-0.51	-0.55
84-m	-0.08	0.20	0.16	-0.30	-0.51	-0.56	-0.60
96-m	-0.11	0.16	0.08	-0.39	-0.58	-0.63	-0.66
108-m	-0.13	0.12	0.00	-0.48	-0.64	-0.68	-0.71
120-m	-0.15	0.08	-0.10	-0.57	-0.70	-0.74	-0.76

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 6-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 6-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

Table 29: Correlation between RMSFE and Sharpe Ratio 12-step - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
3-m	0.52	0.65	0.75	0.72	0.67	0.46	-0.73
6-m	0.56	0.67	0.73	0.69	0.62	0.39	-0.75
9-m	0.58	0.68	0.71	0.65	0.59	0.34	-0.75
12-m	0.60	0.69	0.69	0.63	0.57	0.31	-0.75
15-m	0.61	0.70	0.68	0.62	0.55	0.29	-0.75
18-m	0.63	0.71	0.68	0.61	0.54	0.28	-0.75
21-m	0.64	0.72	0.69	0.61	0.54	0.26	-0.76
24-m	0.66	0.74	0.70	0.61	0.53	0.24	-0.78
30-m	0.69	0.77	0.71	0.61	0.52	0.22	-0.80
36-m	0.72	0.80	0.72	0.60	0.51	0.19	-0.82
48-m	0.76	0.83	0.71	0.56	0.44	0.09	-0.83
60-m	0.76	0.82	0.65	0.45	0.32	-0.05	-0.79
72-m	0.71	0.74	0.55	0.33	0.19	-0.18	-0.70
84-m	0.68	0.68	0.45	0.22	0.08	-0.28	-0.62
96-m	0.68	0.64	0.38	0.15	0.01	-0.34	-0.58
108-m	0.66	0.60	0.31	0.08	-0.05	-0.40	-0.53
120-m	0.67	0.58	0.26	0.03	-0.10	-0.44	-0.49

Note: This table reports the correlation between the RMSFE per maturity of the DNS-ML and DNS-EM models for 12-step forecasting horizon and the Sharpe Ratios of the mean-variance bond portfolios, with 12-months rebalancing horizon that are produced by the DNS-ML and DNS-EM models, for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Sub-sample.

4.3 Data comparison

In order to directly observe the effect of the new Liu and Wu (2019) dataset, and how the DNS models perform relatively to the benchmark RW, I compare the RMSFE of the DNS-ML models for the maturities of 3, 6, 12, 24, 60, 84 and 120 to the results that De Pooter (2007) reports. Due to the fact that this research uses moving windows for estimating the parameters and the factors, where De Pooter (2007) uses expanding windows, for this specific comparison I produce new parallel results, by using expanding windows. Therefore, the RMSFEs of the DNS-ML models relative to the RW for the Liu and Wu (2019) dataset, by using expanding windows for estimating the parameters of the models, can be found in Table 47, in Appendix.

When one compares the results in Table 47 in Appendix, with De Pooter (2007) can see that for the 1-step ahead horizon,the DNS 1 and 3 have similar or worse performance for all reported maturities by using the new dataset, whereas DNS 4 and 6 show reduced performance for all reported maturities but the 120-months maturity, where they perform better. Furthermore, the two factor DNS 2 model has similar or improved performance relative to the benchmark, for the new yield dataset. Last, the DNS 5 variant, for the 3, 6 and 12-month maturities performs worst for the Liu and Wu (2019) dataset, while for the longer maturities shows equal or better results. Worth mentioning is that for the 1-step ahead forecast horizon, the DNS models have different ranking, especially for longer maturities. Although, for the maturities of 3 to 84 months DNS 3 is the best performing variant for both datasets.

For the 3-step ahead horizon, the use of the Liu and Wu (2019) dataset results to a reduced performance for the DNS 1, 4 and 6 for all reported maturities, except for the 120-months where the their results are identical to De Pooter (2007). As regards the DNS 5 variant, the new results indicate that performs similarly or worse for all maturities, expect for the 120-months maturity. Moreover, the DNS 2 performs similarly or better for all maturities, expect for the 60 and 84-months maturities. Last, the DNS 3 variant shows reduced performance for all maturities. Worth mentioning is that differences on the ranking of the models are also observed for all maturities.

Next, for the 6-step ahead horizon, the results of the DNS 1, 3, 4, 5 and 6 variants indicates that their relative performance to the RW is reduced by using the Liu and Wu (2019) dataset. The same stands for the DNS 2 variant, for the maturities 24, 60 and 84 months, whereas its performance for the maturities 3, 6 and 12-months is similar and for 120-months is improved. Regarding the ranking of the models, there are differences for all the maturities, especially for the short-term maturities where the ranking is almost reversed. Finally, for the 12-step ahead, the results are more profound, as all the DNS variants show lower performance by using the new dataset, with differences on their ranking for all maturities as well.

5 Conclusion

This research examines the out-of-sample forecasting performance of several nested DNS model specifications. I use the same DNS specifications as De Pooter (2007) under a state-space framework with first order autoregressive factor dynamics. I therefore follow the one-step estimation approach of De Pooter (2007), which combines the Maximum Likelihood estimation of parameters and the Kalman Filter to estimate the latent factors, and extend De Pooter (2007) by introducing the application of a constrained Expectation Maximization algorithm on the Nelson-Siegel class of models. By following Holmes (2013), I derive analytical expressions for the parameters of the DNS models. Furthermore, a comparison of the forecasting performance between the two approaches is taken place, and also to three competitor models, the RW, AR(1) and the VAR(1)-PCA, and a Forecast Combination scheme of the nested models. I find that for both full sample and sub-sample of this research, DNS 3 shows the best performance among the DNS models, under both estimation approaches, and in several cases has similar of greater results than the benchmark model, RW. The differences on the performance of the rest specifications are small, especially for the full sample. Moreover, for the 1 and 3-step ahead horizons the DNS models perform better under the ML approach, whereas for 6 and 12-step horizons the DNS-EM models are more competitive and in many cases better than the DNS-ML models, especially when examining them for the sub-sample. In addition, the Forecast Combination outperforms most of the individual models.

Furthermore, this research uses the Liu and Wu (2019) yield dataset, which has much lower pricing error than other popular datasets. The Liu and Wu (2019) dataset provides the opportunity to compare the DNS nested models in a quite longer time-horizon than De Pooter (2007), which includes many different economic situation, from a high inflation period with tremendous levels of interest rates until a period stigmatized by a global crisis and extremely low interest rates. Due to the fact that in their research find that other significant studies have different conclusions if one uses their dataset, this thesis applies a direct comparison to the De Pooter (2007) RMSFEs of the DNS specifications relative to the benchmark, RW, for the one-step approach with AR(1) factor dynamics. The outcome of this comparison shows that the use of this novel dataset produces different results. One can see that overall, most of the DNS specifications have lower performance relative to the benchmark, when using the Liu and Wu (2019) dataset, and also many differences on the ranking of the models are observed for all forecasting horizons.

Next, I examine the economic forecasting performance of several nested DNS model specifications, the same ones as De Pooter (2007), and extend their evaluation from an economic point of view. This research follows Caldeira et al. (2016b) and Caldeira et al. (2016a) by producing mean-variance optimal portfolios by using the forecasted yields for 17 difference maturities from the six different DNS nested models, and compares the performance of the portfolios based on their Sharpe Ratios. I deviate from Caldeira et al. (2016b) by not restricting the negative positions on the portfolio weights, and I support this choice by calculating the Turnover of the portfolios. The results indicate that the different values of risk aversion have significant role on the choice of which DNS variant produces more economically meaningful forecasts. It is clear that for some cases the best model according to the statistical and economic measures can deviate. Furthermore, similarly to the statistical evaluation, the differences on the Sharpe Ratios are smaller for the full sample, whereas for the sub-sample are more profound.

Finally, I implement and extend the idea of Caldeira et al. (2016b) to examine the relation between the statistical measures and the Sharpe Ratios. This research specifically examines the aforementioned relation for nested models of the Nelson-Siegel family, in order to improve the decision-making of which specification should be chosen. Furthermore, instead of reporting the correlation coefficients of the average value across all maturities of the RMSFE and the Sharpe Ratios for all risk aversion values, I examine the correlation between the TRMSFE and the Sharpe Ratios for each risk aversion choice, and extend this idea in a more granular level, by examining the correlation between the RMSFE per maturity and the Sharpe Ratios for each risk aversion choice. I find that taking into account the correlation coefficient for each value of risk aversion provides a better understanding about the relation of the statistical measures and the Sharpe Ratios. The results indicate in a clear way in which cases there is a strong negative relation between the measures, in an aggregate level and in a maturity level, and therefore when the combination of the statistical and economic criteria can benefit the decision of which of the nested DNS models is the best for predicting the yields of the bonds.

This research can be extended in several ways. A choice of a VAR(1) specification for the factor dynamics of the state-space model would be worthwhile, since the number of estimated coefficients would be significantly increased, something that could emphasize even further the value of the EM approach. Furthermore, a potential analytical derivation for the decay parameter λ could also increase the value of the EM approach. Also, the assessment of the statistical significance of the difference between the Sharpe Ratios by following Ledoit and Wolf (2008) could help even further on the decision making of which nested model performs better. Lastly, the use of the daily yield data that Liu and Wu (2019) provide, can be beneficial for comparing the models to even more specific sub-samples.

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Appendix

A Kalman Filter

The Kalman filter is an iterative updating process based on the assumption that the underlying and observed states are jointly Gaussian (Hamilton (1994)). This process provides us with forecasts of the the latent factors $\hat{\boldsymbol{\beta}}_{t|t-1}$, where $t = 1, 2, \ldots, T$ given the information we have until time $t-1$, and assuming that the parameters of the model are given. These forecasts are generating recursively; $\hat{\boldsymbol{\beta}}_{1|0}$, $\hat{\boldsymbol{\beta}}_{2|1}$, $\ldots$, $\hat{\boldsymbol{\beta}}_{T|T-1}$. In parallel, the mean squared error matrices of these forecasts, $\mathbf{P}_{t|t-1}$ are being estimated.
In the meanwhile, after each forecast $\hat{\boldsymbol{\beta}}_{t|t-1}$ and estimation of its conditional variance $\mathbf{P}_{t|t-1}$, the yield observations at time t become available. At this point, the construction of their updates, $\hat{\boldsymbol{\beta}}_{t|t}$ and $\mathbf{P}_{t|t}$ is possible. The derived formulas of the aforementioned quantities are given below:
Prediction step:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{t|t-1} &= \boldsymbol{\mu} + \mathbf{A}\hat{\boldsymbol{\beta}}_{t-1|t-1} \\ \mathbf{P}_{t|t-1} &= \mathbf{A}\mathbf{P}_{t-1|t-1}\mathbf{A}' + \mathbf{Q}\end{aligned} \tag{33}$$

Updating step:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{t|t} &= \hat{\boldsymbol{\beta}}_{t|t-1} + \mathbf{P}_{t|t-1}\boldsymbol{\Lambda}'(\boldsymbol{\Lambda}\mathbf{P}_{t|t-1}\boldsymbol{\Lambda}' + \mathbf{H})^{-1}(\mathrm{Y}_t - \boldsymbol{\Lambda}\hat{\boldsymbol{\beta}}_{t|t-1}) \\ \mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}\boldsymbol{\Lambda}'(\boldsymbol{\Lambda}\mathbf{P}_{t|t-1}\boldsymbol{\Lambda}' + \mathbf{H})^{-1}\boldsymbol{\Lambda}\mathbf{P}_{t|t-1}\end{aligned} \tag{34}$$

B Kalman Smoother

In addition to the estimation through Kalman Filter, Hamilton (1994) presented an improved way to estimate the latent factors, which uses information until time T. This method is known as the Kalman Smoother and was initially introduced by Rauch et al. (1965). The smoothed estimates are produced through a backwards iterative procedure; $\hat{\boldsymbol{\beta}}_{T|T}$, $\hat{\boldsymbol{\beta}}_{T-1|T}$, $\ldots$, $\hat{\boldsymbol{\beta}}_{1|T}$, as well as their respective covariance matrices $\mathbf{P}_{T|T}, \mathbf{P}_{T-1|T}, \cdots, \mathbf{P}_{1|T}$.

The first step of this procedure is, given the parameters of the model, the use of the Kalman Filter (see A) to obtain the estimates of the prediction step, $\left[\hat{\boldsymbol{\beta}}_{t|t-1}\right]_{t=1}^{T}$, $\left[\mathbf{P}_{t|t-1}\right]_{t=1}^{T}$ and of the updating step, $\left[\hat{\boldsymbol{\beta}}_{t|t}\right]_{t=1}^{T}$, $\left[\mathbf{P}_{t|t}\right]_{t=1}^{T}$. The last estimate of the updating step, $\hat{\boldsymbol{\beta}}_{T|T}$ and $\mathbf{P}_{T|T}$ is the point that Kalman Smoother starts. Afterwards, the smoothed estimates can be obtained from the following derived expressions:

Smoothing step:

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_{t|T} &= \hat{\boldsymbol{\beta}}_{t|t} + \mathbf{J}_t(\hat{\boldsymbol{\beta}}_{t+1|T} - \hat{\boldsymbol{\beta}}_{t+1|t}) \ , \ t = T-1, T-2, \ldots \\
\mathbf{P}_{t|T} &= \mathbf{P}_{t|t} + \mathbf{J}_t(\mathbf{P}_{t+1|T} - \mathbf{P}_{t+1|t})\mathbf{J}_t' \ , \ t = T-1, T-2, \ldots \\
\mathbf{J}_t &= \mathbf{P}_{t|t}\mathbf{A}'\mathbf{P}_{t+1|t}^{-1} \\
\mathbf{P}_{t,t-1|T} &= \mathbf{P}_{t|T}\mathbf{P}_{t|t-1}^{-1}\mathbf{A}\mathbf{P}_{t-1|t-1} \ , \ t = T, T-1, \ldots
\end{aligned} \tag{35}$$

Furthermore, the EM Algorithm requires the estimation of the cross covariance between the $\hat{\boldsymbol{\beta}}_t$ and $\hat{\boldsymbol{\beta}}_{t-1}$. This expression can be obtained by their joint distribution. The derivation of the $\mathbf{P}_{t,t-1|T}$ above, can be found in Särkkä (2013).

C EM Algorithm - Derivations and Additional Information

The following derivations of the constrained EM algorithm that is used for estimating the Nelson-Siegel class of models under a state-space framework in this research, are based on Holmes (2013).

E-step

First, we take the expectation of the joint log-likelihood function of the complete data (14) over $\boldsymbol{\beta}$ and Y, conditional on $\mathcal{I}_T$ and the set of parameters for the iteration i, $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}]$. The goal is to derive analytical expressions of the new set of parameters $\boldsymbol{\theta}^{(i+1)}$.

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}] = &-\frac{1}{2}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\sum_{t=1}^{T}\left(\mathrm{Y}_t - (\boldsymbol{\beta}_t' \otimes \mathbf{I}_n)\mathrm{vec}(\boldsymbol{\Lambda})\right)' \mathbf{H}^{-1}\left(\mathrm{Y}_t - (\boldsymbol{\beta}_t' \otimes \mathbf{I}_n)\mathrm{vec}(\boldsymbol{\Lambda})\right)\right] \\
&- \frac{T}{2}\log|\mathbf{H}| - \frac{1}{2}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\Bigg[\sum_{t=1}^{T}\left(\boldsymbol{\beta}_t - \mathrm{vec}(\boldsymbol{\mu}) - (\boldsymbol{\beta}_{t-1}' \otimes \mathbf{I}_m)\mathrm{vec}(\mathbf{A})\right)' \mathbf{Q}^{-1} \\
&\left(\boldsymbol{\beta}_t - \mathrm{vec}(\boldsymbol{\mu}) - (\boldsymbol{\beta}_{t-1}' \otimes \mathbf{I}_m)\mathrm{vec}(\mathbf{A})\right)\Bigg] - \frac{T}{2}\log|\mathbf{Q}| \,.
\end{aligned}$$

M-step

Next, we maximize the above expression by taking the first-order condition with respect to the parameters and setting to zero.

Derivation of $\boldsymbol{\mu}$

Starting with $\boldsymbol{\mu}$:

$$\begin{aligned}\frac{\partial \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}]}{\partial \boldsymbol{\mu}} =& -\frac{1}{2}\sum_{t=1}^{T}\left(-\frac{\partial}{\partial \boldsymbol{\mu}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\mathbf{Q}^{-1}\boldsymbol{\mu}\right] - \frac{\partial}{\partial \boldsymbol{\mu}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{\partial}{\partial \boldsymbol{\mu}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\left((\boldsymbol{\beta}_{t-1}'\otimes\mathbf{I}_m)\mathrm{vec}(\mathbf{A})\right)'\mathbf{Q}^{-1}\boldsymbol{\mu}\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{\partial}{\partial \boldsymbol{\mu}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes\mathbf{I}_m)\mathrm{vec}(\mathbf{A})\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{\partial}{\partial \boldsymbol{\mu}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}\boldsymbol{\mu}\right]\right)=0,\end{aligned}$$

$\Rightarrow$

$$-\frac{1}{2}\sum_{t=1}^{T}\left(-2\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\mathbf{Q}^{-1}\right]+2\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\left((\boldsymbol{\beta}_{t-1}'\otimes\mathbf{I}_m)\mathrm{vec}(\mathbf{A})\right)'\mathbf{Q}^{-1}\right]+2\boldsymbol{\mu}'\mathbf{Q}^{-1}\right)=0,$$

$\Rightarrow$

$$\sum_{t=1}^{T}\left(\mathbf{Q}^{-1}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right]-\mathbf{Q}^{-1}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\right]'\otimes\mathbf{I}_m\right)\mathrm{vec}(\mathbf{A})-\mathbf{Q}^{-1}\boldsymbol{\mu}\right)=0,$$

$\Rightarrow$

$$\boldsymbol{\mu}=\frac{1}{T}\sum_{t=1}^{T}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right]-\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\right]'\otimes\mathbf{I}_m\right)\mathrm{vec}(\mathbf{A})\right),$$

where $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right]=\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}\right]=\hat{\boldsymbol{\beta}}_{t|T}$. Thus, the analytical solution for $\boldsymbol{\mu}$ is the following:

$$\boldsymbol{\mu}=\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\boldsymbol{\beta}}_{t|T}-\left(\hat{\boldsymbol{\beta}}_{t-1|T}'\otimes\mathbf{I}_m\right)\mathrm{vec}(\mathbf{A})\right).$$

Derivation of A

Since the parameter matrix $\mathbf{A}$ is diagonal in this research, I define a constrained matrix $\mathbf{D}_{\boldsymbol{\alpha}}$, such that $\mathrm{vec}(\mathbf{A})=\mathbf{D}_{\alpha}\boldsymbol{\alpha}$. The $\mathbf{D}_{\boldsymbol{\alpha}}$ for the two, three and four-factor models can be found at the end of this section. Therefore, the $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}]$ is maximized with respect to $\boldsymbol{\alpha}$:

$$\begin{aligned}
\frac{\partial \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}]}{\partial \boldsymbol{\alpha}} =& -\frac{1}{2}\sum_{t=1}^{T}\left(-\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\boldsymbol{\alpha}\right]\right)\\
&-\frac{1}{2}\sum_{t=1}^{T}\left(-\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[((\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\boldsymbol{\alpha})'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right]\right)\\
&-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\left((\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\boldsymbol{\alpha}\right)'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\boldsymbol{\alpha}\right]\right)\\
&-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\boldsymbol{\alpha}\right]\right)\\
&-\frac{1}{2}\sum_{t=1}^{T}\left(\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[((\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\boldsymbol{\alpha})'\mathbf{Q}^{-1}\boldsymbol{\mu}\right]\right) = 0,
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
0 =& -\frac{1}{2}\sum_{t=1}^{T}\left(-2\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\right] + 2\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\alpha}'\mathbf{D}_\alpha'(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\right]\right)\\
&-\frac{1}{2}\sum_{t=1}^{T}\left(2\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\mathbf{D}_\alpha\right]\right),
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
0 =& \left(\mathbf{D}_\alpha'\sum_{t=1}^{T}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\right]\mathbf{D}_\alpha\right)\boldsymbol{\alpha}\\
&-\mathbf{D}_\alpha'\left(\sum_{t=1}^{T}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right] - \sum_{t=1}^{T}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}\boldsymbol{\mu}\right]\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)\right] =& \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_{t-1}'\right]\otimes \mathbf{Q}^{-1},\\
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right] =& \mathrm{vec}\left(\mathbf{Q}^{-1}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_{t-1}'\right]\right),\\
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\otimes \mathbf{I}_m)'\mathbf{Q}^{-1}\boldsymbol{\mu}\right] =& \mathrm{vec}\left(\mathbf{Q}^{-1}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\right]\right).
\end{aligned}$$

Thus,

$$\boldsymbol{\alpha} = \left(\mathbf{D}'_{\boldsymbol{\alpha}} \left(\sum_{t=1}^{T} \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}'_{t-1}\right] \otimes \mathbf{Q}^{-1}\right) \mathbf{D}_{\boldsymbol{\alpha}}\right)^{-1} \cdot$$
$$\cdot \mathbf{D}'_{\boldsymbol{\alpha}} \left(\sum_{t=1}^{T} \mathrm{vec}\left(\mathbf{Q}^{-1}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t}\boldsymbol{\beta}'_{t-1}\right]\right) - \sum_{t=1}^{T} \mathrm{vec}\left(\mathbf{Q}^{-1}\boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}'_{t-1}\right]\right)\right).$$

where

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}'_{t-1}\right] =& \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}'_{t-1}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}\right] = \mathbf{P}_{t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}'_{t-1|T},\\
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t}\boldsymbol{\beta}'_{t-1}\right] =& \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t}\boldsymbol{\beta}'_{t-1}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}\right] = \mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}'_{t-1|T},\\
\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}'_{t-1}\right] =& \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}'_{t-1}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}\right] = \hat{\boldsymbol{\beta}}'_{t-1|T}.
\end{aligned}$$

Thus, the analytical solution for $\boldsymbol{\alpha}$ is the following:

$$\boldsymbol{\alpha} = \left(\mathbf{D}'_{\boldsymbol{\alpha}} \left(\sum_{t=1}^{T} \left(\mathbf{P}_{t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}'_{t-1|T}\right) \otimes \mathbf{Q}^{-1}\right) \mathbf{D}_{\boldsymbol{\alpha}}\right)^{-1} \cdot$$
$$\cdot \mathbf{D}'_{\boldsymbol{\alpha}} \left(\sum_{t=1}^{T} \mathrm{vec}\left(\mathbf{Q}^{-1}\left(\mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}'_{t-1|T}\right)\right) - \sum_{t=1}^{T} \mathrm{vec}\left(\mathbf{Q}^{-1}\boldsymbol{\mu}\hat{\boldsymbol{\beta}}'_{t-1|T}\right)\right)$$

Derivation of Q

Since the parameter matrix $\mathbf{Q}$ is diagonal in this research, I define a constrained matrix $\mathbf{D}_{\boldsymbol{q}}$, such that $\mathrm{vec}(\mathbf{Q}) = \mathbf{D}_{\boldsymbol{q}}\boldsymbol{q}$. The $\mathbf{D}_{\boldsymbol{q}}$ for the two, three and four-factor models can be found at the end of this section. Due to the fact that the joint log-likelihood of the complete data (13) has the term $\mathbf{Q}^{-1}$, we can maximize $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}]$ with respect to $\boldsymbol{q}_{inv}$, which is a vector of the reciprocal diagonal elements of $\mathbf{Q}$, $q_{inv,1} = \frac{1}{q_1}, \ldots, q_{inv,m} = \frac{1}{q_m}$, where $m \in \{2, 3, 4\}$. Since $\mathbf{Q}$ is diagonal, the following applies: $\mathrm{vec}(\mathbf{Q}^{-1}) = \mathbf{D}_{\boldsymbol{q}}\boldsymbol{q}_{inv}$. Therefore, the $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}]$ is maximized with respect to $\boldsymbol{q}_{inv}$:

$$\begin{aligned}\frac{\partial\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}]}{\partial\boldsymbol{q}_{inv}} =& -\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\mathbf{Q}^{-1}\mathbf{A}\boldsymbol{\beta}_{t-1}\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\mathbf{A}'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right]+\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\mathbf{A}'\mathbf{Q}^{-1}\mathbf{A}\boldsymbol{\beta}_{t-1}\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\mathbf{Q}^{-1}\boldsymbol{\mu}\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}\boldsymbol{\beta}_t\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\mathbf{A}'\mathbf{Q}^{-1}\boldsymbol{\mu}\right]+\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}\mathbf{A}\boldsymbol{\beta}_{t-1}\right]\right)\\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\mathbf{Q}^{-1}\boldsymbol{\mu}\right]\right)-\frac{T}{2}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\log|\mathbf{Q}|=0.\end{aligned}$$

By using the relation $\boldsymbol{a}'\mathbf{Q}^{-1}\boldsymbol{b}=(\boldsymbol{b}'\otimes\boldsymbol{a}')\mathrm{vec}(\mathbf{Q}^{-1})$, the above can be written as:

$$\begin{aligned}0=&-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\otimes\boldsymbol{\beta}_t'\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\mathbf{A}')\otimes\boldsymbol{\beta}_t'\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\otimes(\boldsymbol{\beta}_{t-1}'\mathbf{A}')\right]\right)\mathrm{vec}(\mathbf{Q}^{-1})\\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\mathbf{A}')\otimes(\boldsymbol{\beta}_{t-1}'\mathbf{A}')\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\otimes\boldsymbol{\beta}_t'\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\otimes\boldsymbol{\mu}'\right]\right)\mathrm{vec}(\mathbf{Q}^{-1})\\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\mu}'\otimes(\boldsymbol{\beta}_{t-1}'\mathbf{A}')\right]+\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_{t-1}'\mathbf{A}')\otimes\boldsymbol{\mu}'\right]+\boldsymbol{\mu}'\otimes\boldsymbol{\mu}'\right)\mathrm{vec}(\mathbf{Q}^{-1})\\ &-\frac{T}{2}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\log|\mathbf{Q}|.\end{aligned}$$

Since $\boldsymbol{b}'\otimes\boldsymbol{a}'=(\mathrm{vec}(\boldsymbol{a}\boldsymbol{b}'))'$ and $\log|\mathbf{Q}|=-\log\left|\mathbf{Q}^{-\mathbf{1}}\right|$, we can write:

$$\begin{aligned}0=&-\frac{1}{2}\sum_{t=1}^{T}(\mathrm{vec}(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'-\mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_t'\right]+\mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'\\ &-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right]\boldsymbol{\mu}'-\boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\right]+\mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\right]\boldsymbol{\mu}'+\boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'+\boldsymbol{\mu}\boldsymbol{\mu}'))'\frac{\partial}{\partial\boldsymbol{q}_{inv}}\mathrm{vec}(\mathbf{Q}^{-1})\\ &+\frac{T}{2}\frac{\partial}{\partial\boldsymbol{q}_{inv}}\log\left|\mathbf{Q}^{-1}\right|,\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}0=&\frac{1}{2}\mathbf{D}_{\boldsymbol{q}}'\sum_{t=1}^{T}\mathrm{vec}(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right]-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'-\mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_t'\right]+\mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'\\ &-\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right]\boldsymbol{\mu}'-\boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\right]+\mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\right]\boldsymbol{\mu}'+\boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'\\ &+\boldsymbol{\mu}\boldsymbol{\mu}')-\frac{T}{2}\boldsymbol{q},\end{aligned}$$

$\Rightarrow$

$$\boldsymbol{q} = \frac{1}{T}\mathbf{D}_{\boldsymbol{q}}' \sum_{t=1}^{T} \text{vec}(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}' - \mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_t'\right] + \mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}'$$
$$- \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right]\boldsymbol{\mu}' - \boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\right] + \mathbf{A}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}\right]\boldsymbol{\mu}' + \boldsymbol{\mu}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_{t-1}'\right]\mathbf{A}' + \boldsymbol{\mu}\boldsymbol{\mu}'),$$

where

$$\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right] = \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}\right] = \mathbf{P}_{t|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t|T}',$$
$$\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_{t-1}'\right] = \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_{t-1}'|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}\right] = \mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t-1|T}',$$
$$\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\right] = \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}\right] = \hat{\boldsymbol{\beta}}_{t|T}.$$

Hence, the analytical solution for $\boldsymbol{q}$ is the following:

$$\boldsymbol{q} = \frac{1}{T}\mathbf{D}_{\boldsymbol{q}}' \sum_{t=1}^{T} \text{vec}[\mathbf{P}_{t|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t|T}' - \left(\mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t-1|T}'\right)\mathbf{A}' - \mathbf{A}\left(\mathbf{P}_{t,t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}_{t|T}'\right)$$
$$+ \mathbf{A}\left(\mathbf{P}_{t-1|T} + \hat{\boldsymbol{\beta}}_{t-1|T}\hat{\boldsymbol{\beta}}_{t-1|T}'\right)\mathbf{A}' + \hat{\boldsymbol{\beta}}_{t|T}\boldsymbol{\mu}' - \boldsymbol{\mu}\hat{\boldsymbol{\beta}}_{t|T}' + \mathbf{A}\hat{\boldsymbol{\beta}}_{t-1|T}\boldsymbol{\mu}' + \boldsymbol{\mu}\hat{\boldsymbol{\beta}}_{t-1|T}'\mathbf{A}' + \boldsymbol{\mu}\boldsymbol{\mu}'].$$

Derivation of H

Since the parameter matrix $\mathbf{H}$ is diagonal in this research, I define a constrained matrix $\mathbf{D}_{\boldsymbol{h}}$, such that $\text{vec}(\mathbf{H}) = \mathbf{D}_{\boldsymbol{h}}\boldsymbol{h}$. The $\mathbf{D}_{\boldsymbol{h}}$ can be found at the end of this section. Due to the fact that the joint log-likelihood of the complete data (13) has the term $\mathbf{H}^{-1}$, we can maximize $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}]$ with respect to $\boldsymbol{h}_{inv}$, which is a vector of the reciprocal diagonal elements of $\mathbf{H}$, $q_{inv,1} = \frac{1}{q_1}, \ldots, q_{inv,n} = \frac{1}{h_n}$, where $n = 17$. Since $\mathbf{H}$ is diagonal, the following applies: $\text{vec}(\mathbf{H}^{-1}) = \mathbf{D}_{\boldsymbol{h}}\boldsymbol{h}_{inv}$. Therefore, the $\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}]$ is maximized with respect to $\boldsymbol{h}_{inv}$:

$$\frac{\partial\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}[\mathbf{L}|\mathcal{I}_T,\boldsymbol{\theta}^{(i)}]}{\partial\boldsymbol{h}_{inv}} = -\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{h}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t'\mathbf{H}^{-1}\mathrm{Y}_t\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t'\mathbf{H}^{-1}\boldsymbol{\Lambda}\boldsymbol{\beta}_t\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\boldsymbol{\Lambda}'\mathbf{H}^{-1}\mathrm{Y}_t\right]\right)$$
$$-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{h}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t'\boldsymbol{\Lambda}'\mathbf{H}^{-1}\boldsymbol{\Lambda}\boldsymbol{\beta}_t\right]\right) - \frac{T}{2}\frac{\partial}{\partial\boldsymbol{h}_{inv}}\log|\mathbf{H}| = 0.$$

By using the relation $\boldsymbol{a}'\mathbf{H}^{-1}\boldsymbol{b} = (\boldsymbol{b}' \otimes \boldsymbol{a}')\text{vec}(\mathbf{H}^{-1})$, the above can be written as:

$$\begin{aligned} 0 = &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial \boldsymbol{h}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t' \otimes \mathrm{Y}_t'\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_t'\boldsymbol{\Lambda}') \otimes \mathrm{Y}_t'\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t' \otimes (\boldsymbol{\beta}_t'\boldsymbol{\Lambda}')\right]\right)\text{vec}(\mathbf{H}^{-1}) \\ &-\frac{1}{2}\sum_{t=1}^{T}\frac{\partial}{\partial \boldsymbol{h}_{inv}}\left(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[(\boldsymbol{\beta}_t'\boldsymbol{\Lambda}') \otimes (\boldsymbol{\beta}_t'\boldsymbol{\Lambda}')\right]\right)\text{vec}(\mathbf{H}^{-1}) - \frac{T}{2}\frac{\partial}{\partial \boldsymbol{h}_{inv}}\log|\mathbf{H}|. \end{aligned}$$

Since $\boldsymbol{b}' \otimes \boldsymbol{a}' = (\text{vec}(\boldsymbol{a}\boldsymbol{b}'))'$ and $\log|\mathbf{H}| = -\log\left|\mathbf{H^{-1}}\right|$, we can write:

$$\begin{aligned} 0 = &-\frac{1}{2}\sum_{t=1}^{T}\text{vec}(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\mathrm{Y}_t'\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\boldsymbol{\beta}_t'\right]\boldsymbol{\Lambda}' - \boldsymbol{\Lambda}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\mathrm{Y}_t'\right] \\ &+ \boldsymbol{\Lambda}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right]\boldsymbol{\Lambda}')'\frac{\partial}{\partial \boldsymbol{h}_{inv}}\text{vec}(\mathbf{H}^{-1}) + \frac{T}{2}\frac{\partial}{\partial \boldsymbol{h}_{inv}}\log\left|\mathbf{H}^{-1}\right|, \end{aligned}$$

$\Rightarrow$

$$0 = \frac{1}{2}\mathbf{D}_{\boldsymbol{h}}'\sum_{t=1}^{T}\text{vec}(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\mathrm{Y}_t'\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\boldsymbol{\beta}_t'\right]\boldsymbol{\Lambda}' - \boldsymbol{\Lambda}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\mathrm{Y}_t'\right] + \boldsymbol{\Lambda}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right]\boldsymbol{\Lambda}') - \frac{T}{2}\boldsymbol{h},$$

$\Rightarrow$

$$\boldsymbol{h} = \frac{1}{T}\mathbf{D}_{\boldsymbol{h}}'\sum_{t=1}^{T}\text{vec}(\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\mathrm{Y}_t'\right] - \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\boldsymbol{\beta}_t'\right]\boldsymbol{\Lambda}' - \boldsymbol{\Lambda}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\mathrm{Y}_t'\right] + \boldsymbol{\Lambda}\mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right]\boldsymbol{\Lambda}'),$$

where

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\mathrm{Y}_t'\right] &= \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\mathrm{Y}_t'|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}\right] = \mathrm{Y}_t\mathrm{Y}_t', \\ \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\boldsymbol{\beta}_t'\right] &= \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\mathrm{Y}_t\boldsymbol{\beta}_t'|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}\right] = \mathrm{Y}_t\hat{\boldsymbol{\beta}}_{t|T}', \\ \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'\right] &= \mathbb{E}_{\boldsymbol{\beta}\mathrm{Y}}\left[\boldsymbol{\beta}_t\boldsymbol{\beta}_t'|\mathcal{I}_T, \boldsymbol{\theta}^{(i)}\right] = \mathbf{P}_{t|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t|T}'. \end{aligned}$$

Therefore, the analytical solution for $\boldsymbol{h}$ is:

$$\boldsymbol{h} = \frac{1}{T}\mathbf{D}_{\boldsymbol{h}}'\sum_{t=1}^{T}\text{vec}\left[\mathrm{Y}_t\mathrm{Y}_t' - \mathrm{Y}_t\hat{\boldsymbol{\beta}}_{t|T}'\boldsymbol{\Lambda}' - \boldsymbol{\Lambda}\hat{\boldsymbol{\beta}}_{t|T}\mathrm{Y}_t' + \boldsymbol{\Lambda}\left(\mathbf{P}_{t|T} + \hat{\boldsymbol{\beta}}_{t|T}\hat{\boldsymbol{\beta}}_{t|T}'\right)\boldsymbol{\Lambda}'\right].$$

Constrained Matrices

For the two-factor models:

$$\mathbf{D}_{\boldsymbol{a}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{D}_{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the three-factor models:

$$\mathbf{D}_{\boldsymbol{a}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{D}_{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the four-factor models:

$$\mathbf{D}_{\boldsymbol{a}} = \begin{pmatrix} D_{\boldsymbol{a}1} \\ D_{\boldsymbol{a}2} \\ D_{\boldsymbol{a}3} \\ D_{\boldsymbol{a}4} \end{pmatrix}, \mathbf{D}_{\boldsymbol{q}} = \begin{pmatrix} D_{\boldsymbol{q}1} \\ D_{\boldsymbol{q}2} \\ D_{\boldsymbol{q}3} \\ D_{\boldsymbol{q}4} \end{pmatrix},$$

where the $D_{\boldsymbol{a}j}$ and $D_{\boldsymbol{q}j}$, $j = 1, 2, 3, 4$ are 4×4 matrices with all elements equal to zero, except the $D_{\boldsymbol{a}j}(j, j) = 1$ and $D_{\boldsymbol{q}j}(j, j) = 1$. For all cases:

$$\mathbf{D}_{\boldsymbol{h}} = \begin{pmatrix} D_{\boldsymbol{h}1} \\ D_{\boldsymbol{h}2} \\ \vdots \\ D_{\boldsymbol{h}17} \end{pmatrix},$$

where the $D_{\boldsymbol{h}j}$, $j = 1, \ldots 17$ is a 17×17 matrix with all elements equal to zero, except the $D_{\boldsymbol{h}j}(j, j) = 1$.

DNS-EM λ: ML in-sample λ estimates

Table 30: DNS-EM: ML in-sample λ values

Models	λ	λ_1	λ_2
DNS 1	0.0461	-	-
DNS 2	0.0271	-	-
DNS 3	0.0565	-	-
DNS 4	-	0.0302	0.0501
DNS 5	-	0.1264	0.0524
DNS 6	-	0.0429	0.0540

Note: This table reports the in-sample estimated values of the decay parameters per DNS model of the ML approach, that are taken as fixed for the EM approach.

D Full Sample - Additional Tables and Figures

Table 31: RMSFE 1-step ahead ML vs EM - Full Sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.24	1.34	1.06	1.22	1.02	1.09	1.10
6-m	1.14	1.35	1.11	1.13	1.07	1.10	1.09
9-m	1.05	1.28	1.14	1.05	1.10	1.09	1.08
12-m	1.01	1.17	1.14	1.02	1.10	1.08	1.06
15-m	**1.00**	1.08	1.13	1.01	1.10	1.08	1.05
18-m	**1.00**	1.02	1.12	1.01	1.10	1.08	1.05
21-m	1.01	**1.00**	1.12	1.02	1.10	1.07	1.05
24-m	1.01	**0.99**	1.12	1.02	1.10	1.07	1.05
30-m	1.02	**1.00**	1.12	1.03	1.10	1.07	1.06
36-m	1.02	1.02	1.11	1.04	1.10	1.07	1.06
48-m	1.03	1.04	1.09	1.04	1.09	1.06	1.06
60-m	1.04	1.05	1.09	1.05	1.08	1.05	1.06
72-m	1.05	1.06	1.08	1.05	1.08	1.05	1.07
84-m	1.05	1.05	1.08	1.05	1.05	1.04	1.06
96-m	1.03	1.03	1.07	1.04	1.03	1.03	1.05
108-m	**1.00**	1.02	1.05	1.01	1.06	1.05	1.05
120-m	**0.96**	1.01	1.02	**0.98**	1.03	1.02	1.02

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 1-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 32: RMSFE 3-step ahead ML vs EM - Full Sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.13	1.21	1.16	1.12	1.08	1.06	1.10
6-m	1.08	1.17	1.19	1.08	1.11	1.07	1.10
9-m	1.04	1.12	1.21	1.05	1.12	1.07	1.10
12-m	1.02	1.08	1.21	1.04	1.13	1.08	1.09
15-m	1.01	1.06	1.21	1.03	1.13	1.08	1.09
18-m	1.01	1.04	1.20	1.03	1.13	1.09	1.09
21-m	1.02	1.04	1.20	1.04	1.13	1.09	1.09
24-m	1.02	1.04	1.20	1.05	1.12	1.09	1.09
30-m	1.03	1.05	1.19	1.06	1.12	1.09	1.10
36-m	1.04	1.05	1.19	1.06	1.12	1.10	1.10
48-m	1.05	1.06	1.17	1.07	1.11	1.09	1.10
60-m	1.06	1.07	1.16	1.08	1.10	1.09	1.11
72-m	1.07	1.08	1.17	1.09	1.10	1.10	1.12
84-m	1.07	1.07	1.16	1.08	1.08	1.09	1.11
96-m	1.06	1.06	1.15	1.07	1.07	1.08	1.10
108-m	1.04	1.06	1.13	1.06	1.07	1.09	1.10
120-m	1.01	1.04	1.10	1.03	1.05	1.06	1.07

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 3-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 33: RMSFE 6-step ahead ML vs EM - Full Sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.08	1.12	1.16	1.08	1.07	1.03	1.09
6-m	1.05	1.09	1.17	1.05	1.09	1.05	1.09
9-m	1.02	1.06	1.18	1.04	1.10	1.06	1.08
12-m	1.01	1.04	1.18	1.03	1.11	1.06	1.08
15-m	1.01	1.03	1.18	1.02	1.11	1.07	1.08
18-m	**1.00**	1.03	1.18	1.03	1.12	1.07	1.08
21-m	1.01	1.02	1.18	1.03	1.12	1.08	1.09
24-m	1.01	1.03	1.18	1.03	1.12	1.08	1.09
30-m	1.02	1.03	1.18	1.04	1.12	1.09	1.09
36-m	1.02	1.03	1.18	1.05	1.12	1.09	1.10
48-m	1.03	1.04	1.17	1.05	1.11	1.09	1.10
60-m	1.04	1.05	1.16	1.06	1.10	1.09	1.10
72-m	1.05	1.05	1.16	1.06	1.10	1.10	1.11
84-m	1.04	1.05	1.15	1.06	1.08	1.09	1.11
96-m	1.04	1.04	1.14	1.05	1.08	1.09	1.10
108-m	1.03	1.04	1.14	1.04	1.07	1.09	1.10
120-m	1.01	1.03	1.11	1.03	1.05	1.07	1.08

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 6-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 34: RMSFE 12-step ahead ML vs EM - Full Sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.03	1.05	1.12	1.04	1.04	**0.99**	1.05
6-m	1.01	1.03	1.12	1.02	1.06	**1.00**	1.05
9-m	**1.00**	1.01	1.12	1.01	1.07	1.01	1.05
12-m	**0.99**	**1.00**	1.12	**1.00**	1.07	1.01	1.05
15-m	**0.98**	**1.00**	1.12	**1.00**	1.08	1.01	1.05
18-m	**0.98**	**0.99**	1.13	**1.00**	1.08	1.02	1.05
21-m	**0.99**	**0.99**	1.13	**1.00**	1.09	1.02	1.05
24-m	**0.99**	**1.00**	1.14	1.01	1.09	1.03	1.06
30-m	**0.99**	**1.00**	1.14	1.01	1.09	1.03	1.06
36-m	**1.00**	**1.00**	1.14	1.02	1.10	1.04	1.07
48-m	**1.00**	**1.00**	1.14	1.02	1.10	1.04	1.07
60-m	1.01	1.01	1.14	1.02	1.09	1.04	1.08
72-m	1.01	1.01	1.14	1.03	1.09	1.05	1.09
84-m	1.01	1.01	1.14	1.02	1.09	1.04	1.09
96-m	1.01	1.01	1.13	1.02	1.09	1.04	1.09
108-m	**1.00**	**1.00**	1.13	1.01	1.08	1.04	1.08
120-m	**0.99**	**1.00**	1.11	**1.00**	1.07	1.03	1.07

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 12-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the full sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 35: Turnover 1-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	5690.97	2342.35	1287.23	823.09	642.80	599.98	576.22
AR(1)	5757.57	2378.97	1312.02	836.91	648.33	602.51	576.56
VAR(1) PCA	8940.82	3618.43	1885.70	1077.58	742.75	658.60	606.02
DNS - ML approach							
DNS 1	498.65	202.70	105.02	57.55	35.46	29.04	24.85
DNS 2	257.81	104.29	54.63	31.58	21.75	19.03	17.17
DNS 3	698.52	282.96	145.69	78.42	46.48	36.66	29.74
DNS 4	539.51	218.67	112.79	61.30	37.07	29.85	25.02
DNS 5	818.49	332.62	171.88	92.82	54.79	43.00	34.49
DNS 6	754.64	304.74	155.99	82.93	47.60	36.48	28.25
Forecast Combination	1439.50	576.39	289.29	146.40	75.74	52.68	34.94
DNS - EM approach							
DNS 1	473.68	190.18	97.37	53.15	33.54	28.43	25.63
DNS 2	220.91	90.80	49.21	30.33	22.37	20.24	18.94
DNS 3	660.45	268.16	138.78	75.47	45.61	36.53	30.25
DNS 4	496.50	199.41	101.98	55.26	34.37	28.74	25.60
DNS 5	746.14	302.47	156.45	85.02	51.12	40.93	33.78
DNS 6	683.83	277.35	143.20	77.73	46.63	37.20	30.60
Forecast Combination	1425.80	570.96	286.62	145.25	75.74	53.29	36.21

Note: This table reports the Turnover of the mean-variance bond portfolios with 1-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Full sample.

Table 36: Turnover 3-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	6061.64	2605.83	1622.94	1236.61	1125.52	1111.93	1105.86
AR(1)	6138.38	2654.92	1648.30	1242.93	1125.29	1110.61	1105.21
VAR(1) PCA	5150.13	2310.01	1504.53	1212.02	1128.47	1114.89	1107.42
DNS - ML approach							
DNS 1	452.20	185.52	97.61	54.74	35.15	29.58	26.25
DNS 2	198.99	80.37	42.61	25.72	18.96	17.06	15.92
DNS 3	634.24	257.02	132.27	71.50	42.74	34.13	28.33
DNS 4	467.64	191.13	100.39	56.38	36.09	30.42	27.06
DNS 5	759.77	310.47	162.36	89.93	55.88	45.67	38.57
DNS 6	645.30	261.53	135.06	73.58	44.98	36.33	30.25
Forecast Combination	1183.16	474.06	238.33	121.42	64.10	45.62	31.60
DNS - EM approach							
DNS 1	377.28	151.66	78.22	43.70	29.21	25.79	24.01
DNS 2	161.45	68.11	38.44	25.01	19.32	17.81	16.96
DNS 3	584.92	237.77	123.49	68.23	42.34	34.61	29.44
DNS 4	397.13	159.75	82.46	46.10	30.57	26.69	24.80
DNS 5	667.37	273.92	144.29	81.05	51.55	42.80	36.51
DNS 6	587.78	239.90	125.66	70.45	44.99	37.43	32.17
Forecast Combination	1075.19	430.41	216.49	110.41	58.57	42.06	29.82

Note: This table reports the Turnover of the mean-variance bond portfolios with 3-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Full sample.

Table 37: Turnover 6-month rebalancing - Full Sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	5677.53	2448.15	1465.14	1045.86	871.59	829.14	803.71
AR(1)	5973.94	2557.64	1512.88	1067.35	882.21	834.58	806.67
VAR(1) PCA	5057.22	2277.05	1412.57	1031.83	870.49	831.11	806.11
DNS - ML approach							
DNS 1	416.33	168.26	87.70	49.82	32.90	28.52	26.25
DNS 2	187.01	76.51	41.72	26.00	19.07	17.17	16.06
DNS 3	672.67	268.84	135.62	71.77	42.09	33.64	28.27
DNS 4	428.50	172.75	89.71	50.99	33.83	29.49	27.31
DNS 5	727.95	292.57	151.19	83.53	52.41	43.81	38.54
DNS 6	662.27	265.50	135.25	73.12	44.59	36.33	31.00
Forecast Combination	1163.20	463.50	231.06	116.01	60.05	42.33	29.39
DNS - EM approach							
DNS 1	355.47	142.43	73.63	41.84	29.12	26.10	24.39
DNS 2	142.07	60.91	35.03	23.04	18.05	16.88	16.33
DNS 3	569.81	227.53	115.37	62.65	38.81	32.38	28.67
DNS 4	370.39	147.76	75.73	42.84	29.82	26.83	25.18
DNS 5	690.01	280.67	145.69	80.68	51.09	42.33	36.56
DNS 6	564.96	227.41	117.23	64.96	41.45	35.19	31.31
Forecast Combination	996.56	397.48	199.41	101.75	54.33	39.40	28.88

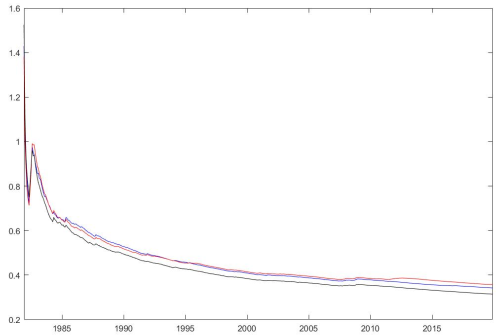
Note: This table reports the Turnover of the mean-variance bond portfolios with 6-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Full sample.

Table 38: Turnover 12-month rebalancing - Full Sample

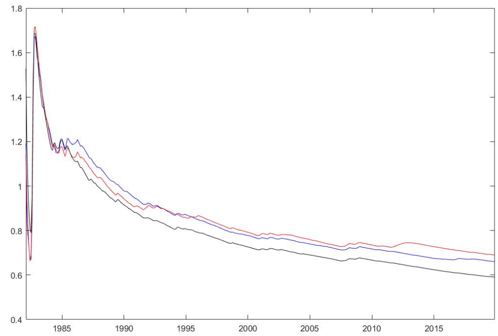
Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	6505.20	3484.60	2634.18	2325.33	2213.51	2181.42	2156.81
AR(1)	6971.74	3671.73	2725.14	2367.06	2232.14	2193.15	2163.48
VAR(1) PCA	5655.78	3308.70	2653.94	2366.78	2237.26	2197.07	2166.13
DNS - ML approach							
DNS 1	385.93	156.36	82.91	49.50	35.97	32.09	29.34
DNS 2	164.50	69.02	38.75	24.83	18.79	17.20	16.20
DNS 3	674.33	267.93	134.58	71.16	43.41	35.59	30.42
DNS 4	420.76	167.72	87.02	51.03	36.87	33.09	30.56
DNS 5	760.38	297.66	147.51	77.61	49.71	43.08	39.35
DNS 6	728.56	288.02	143.69	75.97	46.05	37.96	32.90
Forecast Combination	1212.37	482.94	241.18	122.94	66.21	48.21	35.39
DNS - EM approach							
DNS 1	408.38	166.64	88.43	51.20	34.44	29.75	26.45
DNS 2	173.86	72.90	40.26	25.04	18.43	16.84	16.08
DNS 3	607.22	239.83	120.45	64.57	41.22	35.12	31.57
DNS 4	430.81	176.01	93.37	53.94	36.33	31.39	28.05
DNS 5	705.85	281.15	142.67	76.45	47.77	40.57	35.96
DNS 6	585.24	230.91	115.61	63.30	41.29	35.58	32.16
Forecast Combination	1188.68	475.50	238.75	123.11	66.92	49.42	36.31

Note: This table reports the Turnover of the mean-variance bond portfolios with 12-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Full sample.

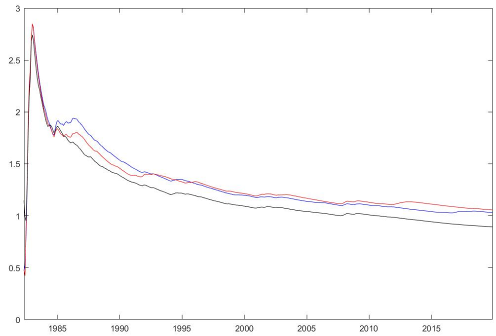
Figure 3: DNS 1: Rolling Forecasting Performance of the TRMSFE - Full Sample



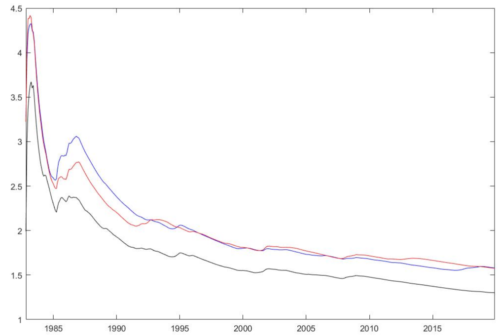
(a) 1-step ahead



(b) 3-step ahead



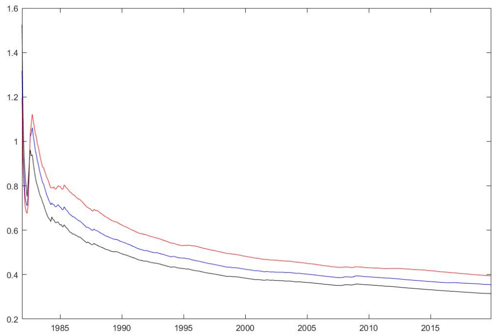
(c) 6-step ahead



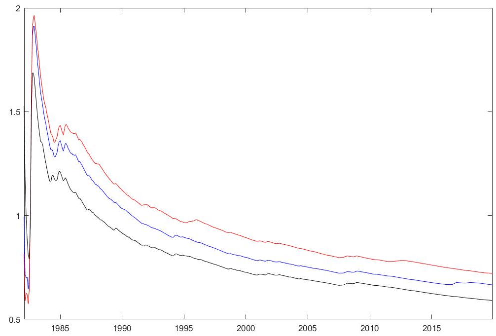
(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the DNS 1 under the ML approach which is indicated by the blue line and the DNS 1 under the EM approach which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

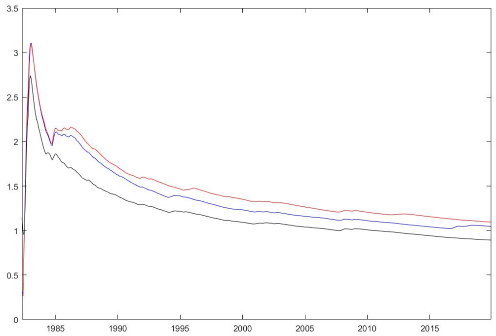
Figure 4: DNS 2: Rolling Forecasting Performance of the TRMSFE - Full Sample



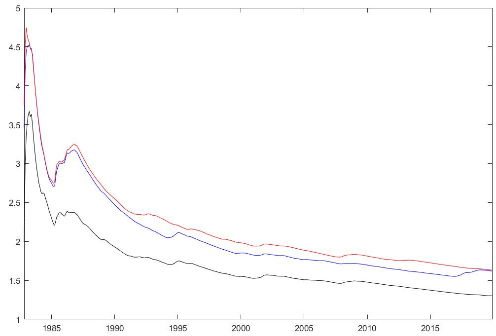
(a) 1-step ahead



(b) 3-step ahead



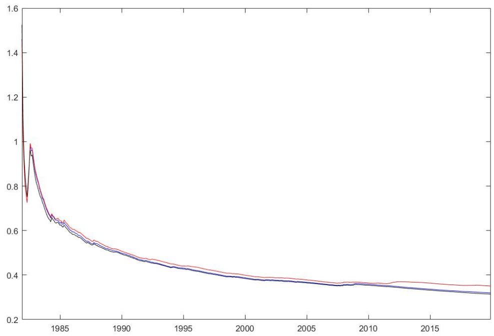
(c) 6-step ahead



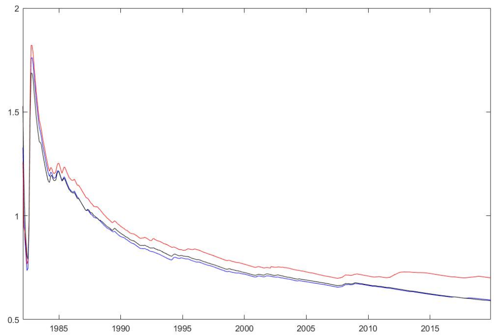
(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the DNS 2 under the ML approach which is indicated by the blue line and the DNS 2 under the EM approach which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

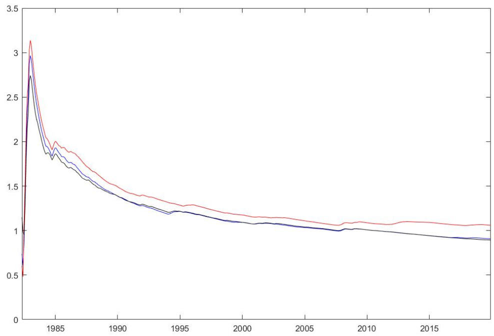
Figure 5: DNS 3: Rolling Forecasting Performance of the TRMSFE - Full Sample



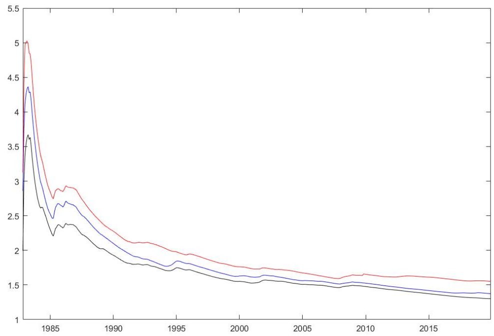
(a) 1-step ahead



(b) 3-step ahead



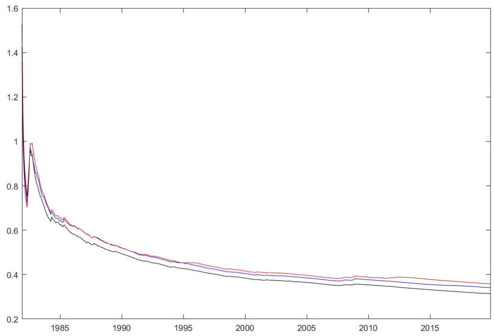
(c) 6-step ahead



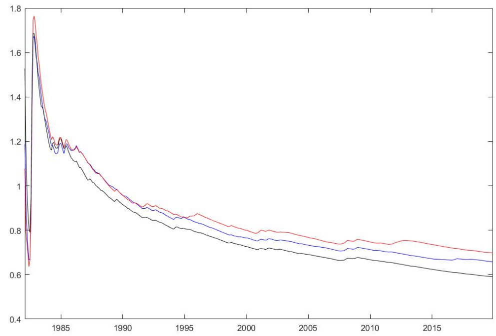
(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the DNS 3 under the ML approach which is indicated by the blue line and the DNS 3 under the EM approach which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

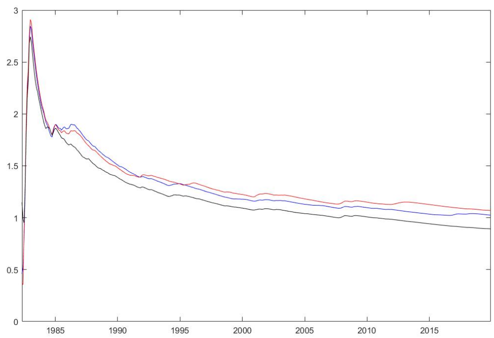
Figure 6: DNS 4: Rolling Forecasting Performance of the TRMSFE - Full Sample



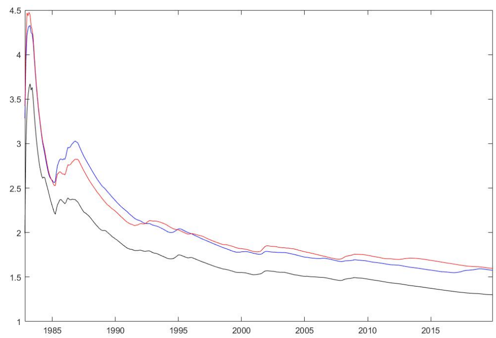
(a) 1-step ahead



(b) 3-step ahead



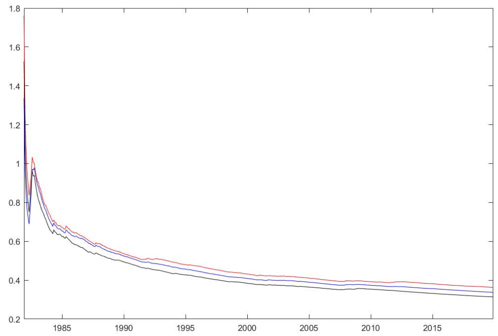
(c) 6-step ahead



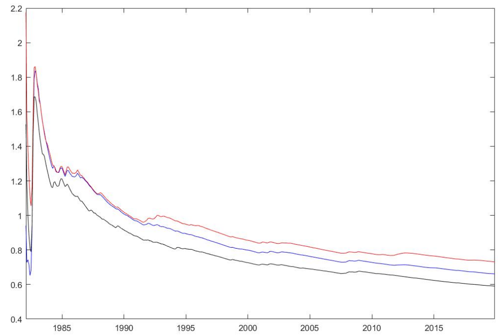
(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the DNS 4 under the ML approach which is indicated by the blue line and the DNS 4 under the EM approach which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

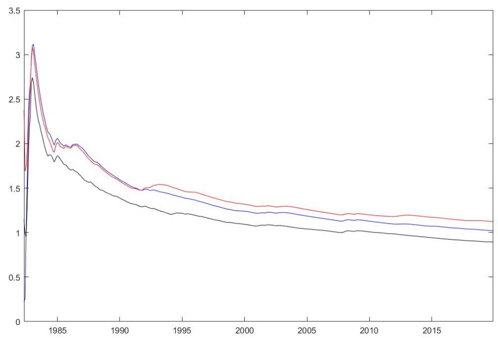
Figure 7: DNS 5: Rolling Forecasting Performance of the TRMSFE - Full Sample



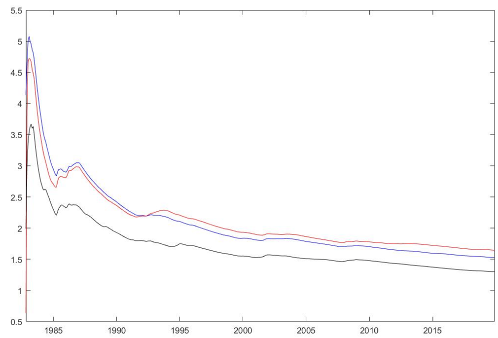
(a) 1-step ahead



(b) 3-step ahead



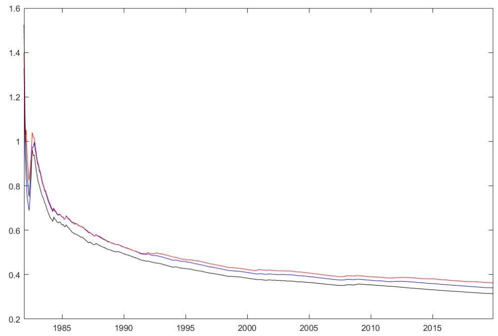
(c) 6-step ahead



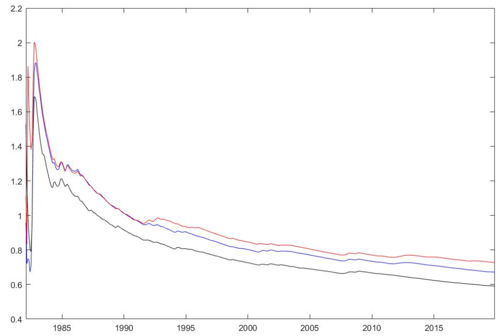
(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the DNS 5 under the ML approach which is indicated by the blue line and the DNS 5 under the EM approach which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

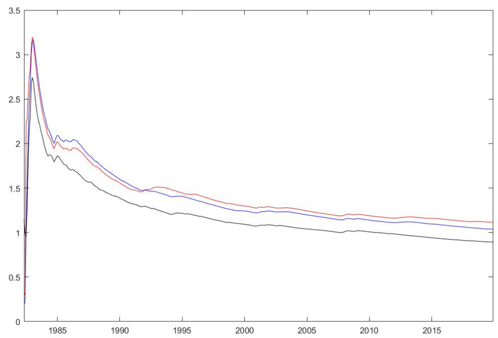
Figure 8: DNS 6: Rolling Forecasting Performance of the TRMSFE - Full Sample



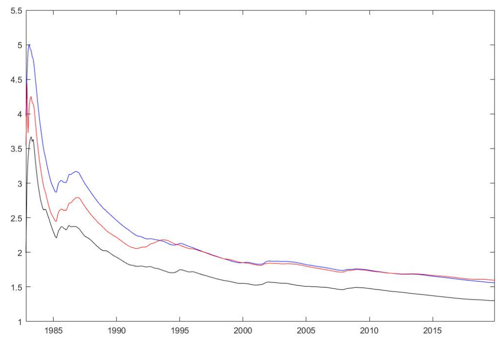
(a) 1-step ahead



(b) 3-step ahead



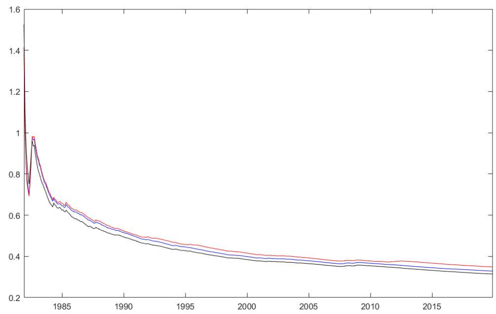
(c) 6-step ahead



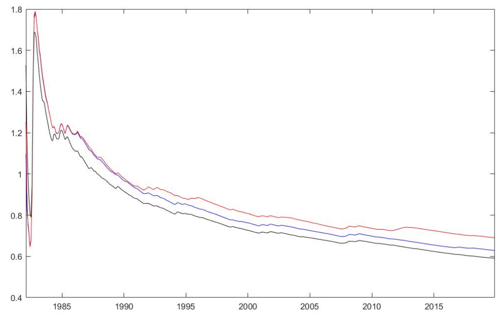
(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the DNS 6 under the ML approach which is indicated by the blue line and the DNS 6 under the EM approach which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

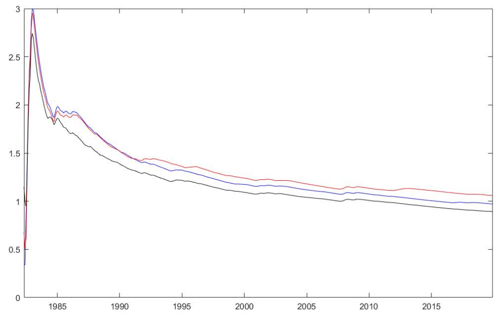
Figure 9: Forecast Combination: Rolling Forecasting Performance of the TRMSFE - Full sample



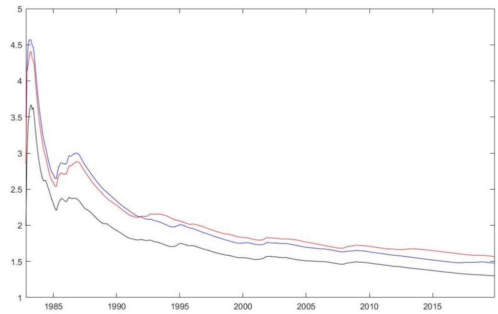
(a) 1-step ahead



(b) 3-step ahead



(c) 6-step ahead



(d) 12-step ahead

Note: This figure presents the rolling forecasting performance of the TRMSFE of the RW which is indicated by the black line, the Forecast Combination of the DNS-ML models which is indicated by the blue line and the Forecast Combination of the DNS-EM models which is indicated by the red line, for the 1, 3, 6 and 12-step ahead forecasting horizon.

E De Pooter (2007) Sub-sample - Tables

Table 39: RMSFE 1-step ahead ML vs EM - Sub-sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.28	1.24	**0.96**	1.18	**0.97**	1.20	1.02
6-m	1.16	1.34	**1.00**	1.12	1.05	1.21	1.04
9-m	1.06	1.32	1.02	1.06	1.07	1.16	1.02
12-m	1.04	1.21	1.03	1.04	1.07	1.10	1.02
15-m	1.05	1.11	1.03	1.05	1.07	1.07	1.02
18-m	1.06	1.05	1.03	1.06	1.07	1.04	1.02
21-m	1.07	1.02	1.03	1.07	1.06	1.03	1.03
24-m	1.08	1.02	1.03	1.08	1.06	1.03	1.04
30-m	1.08	1.02	1.03	1.08	1.06	1.03	1.04
36-m	1.08	1.02	1.03	1.08	1.05	1.03	1.04
48-m	1.07	1.04	1.02	1.08	1.05	1.03	1.05
60-m	1.08	1.07	1.04	1.10	1.06	1.05	1.06
72-m	1.09	1.09	1.05	1.11	1.07	1.06	1.08
84-m	1.09	1.10	1.05	1.10	1.07	1.07	1.08
96-m	1.07	1.08	1.03	1.08	1.06	1.06	1.06
108-m	1.04	1.07	1.02	1.06	1.06	1.07	1.05
120-m	**1.00**	1.04	**1.00**	1.03	1.05	1.07	1.03

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 1-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the sub-sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 40: RMSFE 3-step ahead ML vs EM - Sub-sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.09	1.28	**0.98**	1.06	1.04	**0.92**	**1.00**
6-m	1.08	1.24	1.02	1.06	1.07	**0.94**	1.03
9-m	1.07	1.18	1.04	1.06	1.08	**0.95**	1.04
12-m	1.07	1.14	1.06	1.06	1.08	**0.96**	1.04
15-m	1.07	1.12	1.06	1.07	1.08	**0.97**	1.05
18-m	1.08	1.11	1.07	1.08	1.08	**0.98**	1.05
21-m	1.08	1.11	1.07	1.09	1.08	**0.99**	1.06
24-m	1.09	1.11	1.07	1.10	1.08	**1.00**	1.07
30-m	1.10	1.11	1.07	1.11	1.09	1.01	1.08
36-m	1.10	1.12	1.07	1.12	1.09	1.02	1.08
48-m	1.10	1.12	1.06	1.13	1.09	1.04	1.09
60-m	1.11	1.14	1.07	1.15	1.10	1.07	1.11
72-m	1.12	1.16	1.09	1.17	1.11	1.09	1.12
84-m	1.12	1.15	1.09	1.17	1.12	1.11	1.13
96-m	1.11	1.14	1.08	1.15	1.12	1.11	1.12
108-m	1.09	1.13	1.07	1.14	1.12	1.12	1.12
120-m	1.06	1.12	1.06	1.12	1.12	1.13	1.11

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 3-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the sub-sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 41: RMSFE 6-step ahead ML vs EM - Sub-sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.07	1.18	**0.94**	1.06	1.04	**0.91**	1.01
6-m	1.06	1.14	**0.97**	1.06	1.05	**0.93**	1.02
9-m	1.05	1.11	**0.98**	1.05	1.05	**0.94**	1.02
12-m	1.04	1.09	**0.99**	1.05	1.05	**0.95**	1.02
15-m	1.04	1.09	**1.00**	1.05	1.05	**0.95**	1.03
18-m	1.04	1.08	**1.00**	1.06	1.06	**0.96**	1.03
21-m	1.04	1.08	1.01	1.07	1.06	**0.97**	1.03
24-m	1.05	1.08	1.01	1.07	1.06	**0.97**	1.04
30-m	1.05	1.09	1.01	1.09	1.06	**0.98**	1.05
36-m	1.05	1.09	1.01	1.10	1.06	**0.99**	1.05
48-m	1.05	1.09	1.01	1.11	1.07	1.01	1.06
60-m	1.07	1.10	1.02	1.12	1.08	1.03	1.08
72-m	1.08	1.11	1.04	1.14	1.09	1.05	1.09
84-m	1.08	1.11	1.04	1.13	1.09	1.07	1.10
96-m	1.06	1.10	1.04	1.12	1.10	1.07	1.10
108-m	1.05	1.09	1.04	1.11	1.10	1.08	1.10
120-m	1.04	1.08	1.03	1.09	1.10	1.09	1.09

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 6-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the sub-sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 42: RMSFE 12-step ahead ML vs EM - Sub-sample

Models	DNS 1	DNS 2	DNS 3	DNS 4	DNS 5	DNS 6	FC
3-m	1.02	1.03	**0.87**	1.03	**0.98**	**0.85**	**0.96**
6-m	1.01	1.01	**0.88**	1.02	**0.99**	**0.86**	**0.96**
9-m	**1.00**	**1.00**	**0.88**	1.01	**0.99**	**0.86**	**0.96**
12-m	**0.99**	**0.99**	**0.89**	1.01	**0.99**	**0.87**	**0.96**
15-m	**0.99**	**0.98**	**0.89**	1.01	**0.99**	**0.87**	**0.96**
18-m	**0.98**	**0.98**	**0.90**	1.01	**1.00**	**0.88**	**0.96**
21-m	**0.98**	**0.98**	**0.90**	1.01	**1.00**	**0.88**	**0.96**
24-m	**0.98**	**0.98**	**0.91**	1.02	**1.00**	**0.89**	**0.97**
30-m	**0.99**	**0.98**	**0.91**	1.02	1.01	**0.90**	**0.98**
36-m	**0.99**	**0.98**	**0.92**	1.03	1.02	**0.91**	**0.99**
48-m	**1.00**	**0.99**	**0.94**	1.05	1.03	**0.94**	**1.00**
60-m	1.02	**1.00**	**0.97**	1.07	1.04	**0.97**	1.03
72-m	1.04	1.01	**0.99**	1.08	1.06	**1.00**	1.05
84-m	1.05	1.02	1.01	1.09	1.07	1.02	1.07
96-m	1.05	1.02	1.02	1.08	1.08	1.04	1.08
108-m	1.05	1.02	1.03	1.08	1.08	1.06	1.09
120-m	1.04	1.01	1.03	1.07	1.09	1.07	1.09

Note: This table reports the RMSFE of the DNS-EM models relative to the respective DNS-ML specifications, for the 12-step ahead forecasts, at 17 different maturities, $\tau \in \{3,6,9,12,15,18,21,24,30,36,48,60,72,84,96,108,120\}$, for the sub-sample. A value smaller than one indicates that the DNS-EM specification outperform the respective DNS-ML specification. The highlighted values indicate that the DNS-EM specification performs equally or better than the respective DNS-ML specification.

Table 43: Turnover 1-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	1493.34	850.92	691.69	639.25	623.62	619.06	615.58
AR(1)	1490.61	852.62	693.42	640.43	624.34	619.53	615.86
VAR(1) PCA	2016.91	1009.49	736.21	645.28	622.39	617.20	614.13
DNS - ML approach							
DNS 1	741.91	298.01	150.64	77.94	43.05	32.26	24.59
DNS 2	322.55	129.10	65.44	35.60	23.08	19.72	17.56
DNS 3	1127.43	451.67	227.59	116.93	63.48	46.72	34.35
DNS 4	890.39	356.11	179.16	92.07	50.00	36.83	27.36
DNS 5	1296.86	519.21	261.45	133.99	72.29	53.03	38.79
DNS 6	1178.25	471.96	237.33	121.67	65.19	46.98	33.21
Forecast Combination	1958.71	783.73	392.51	197.58	100.84	69.02	44.05
DNS - EM approach							
DNS 1	685.44	274.83	139.02	72.77	42.05	33.31	27.89
DNS 2	301.45	121.27	62.74	35.86	24.67	21.77	20.14
DNS 3	1034.18	415.65	210.91	110.22	61.90	46.90	36.12
DNS 4	714.52	286.04	144.39	75.10	42.58	33.13	27.21
DNS 5	1104.84	444.17	225.78	119.31	67.92	52.11	40.87
DNS 6	1006.00	404.22	205.05	107.78	60.83	46.01	35.37
Forecast Combination	2067.58	827.95	415.29	209.84	108.30	75.09	49.37

Note: This table reports the Turnover of the mean-variance bond portfolios with 1-month rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Sub-sample.

Table 44: Turnover 3-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	7835.83	3745.07	2576.83	2167.45	2050.43	2034.67	2024.75
AR(1)	7963.50	3849.22	2671.47	2241.91	2094.88	2064.92	2042.99
VAR(1) PCA	6821.61	3427.12	2472.86	2147.73	2054.05	2036.18	2024.90
DNS - ML approach							
DNS 1	566.97	228.98	116.97	62.39	37.01	29.64	24.81
DNS 2	218.53	87.33	45.57	27.27	19.81	17.66	16.27
DNS 3	965.50	385.08	193.18	99.52	54.87	41.19	31.49
DNS 4	679.85	272.34	137.67	72.38	42.40	33.46	28.00
DNS 5	1039.40	416.51	211.68	111.75	64.88	50.75	40.98
DNS 6	967.77	387.60	195.54	101.79	57.21	43.27	33.52
Forecast Combination	1426.16	571.42	287.07	145.47	75.91	53.14	35.39
DNS - EM approach							
DNS 1	499.17	200.95	102.81	56.13	35.21	29.49	25.99
DNS 2	200.57	81.97	44.48	27.75	20.92	19.22	18.27
DNS 3	827.62	332.19	169.16	89.56	52.33	41.32	33.78
DNS 4	501.11	201.21	102.84	55.67	34.64	28.83	25.67
DNS 5	914.63	369.90	190.38	103.62	62.62	50.29	41.66
DNS 6	828.55	333.57	170.86	91.49	54.01	42.69	35.01
Forecast Combination	1348.87	541.56	273.36	140.00	74.63	53.63	37.88

Note: This table reports the Turnover of the mean-variance bond portfolios with 3-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1, 0.25, 0.5, 1, 2, 3, 5\}$. The results refer to the Sub-sample.

Table 45: Turnover 6-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	9226.74	4008.03	2418.15	1708.04	1428.26	1363.16	1320.18
AR(1)	9140.41	3912.54	2344.78	1676.41	1409.45	1351.81	1313.25
VAR(1) PCA	7844.77	3521.44	2178.35	1597.13	1389.01	1341.55	1309.35
DNS - ML approach							
DNS 1	449.46	180.90	93.11	51.37	32.93	28.04	25.32
DNS 2	167.25	68.53	37.61	24.32	18.65	17.03	16.01
DNS 3	1008.29	399.53	198.68	100.72	55.68	42.61	33.11
DNS 4	555.70	221.99	111.95	60.26	37.47	31.92	29.51
DNS 5	938.22	372.10	186.43	96.73	55.86	46.16	40.13
DNS 6	928.98	369.74	184.92	95.64	53.48	41.70	34.44
Forecast Combination	1371.81	547.36	273.21	137.55	71.82	50.83	34.97
DNS - EM approach							
DNS 1	454.88	185.85	98.37	56.12	36.90	31.50	27.77
DNS 2	176.75	72.69	39.50	24.84	19.77	18.45	17.86
DNS 3	839.85	332.29	165.77	86.53	50.54	40.78	35.12
DNS 4	472.52	191.00	98.76	55.76	36.92	31.89	28.29
DNS 5	877.97	354.75	182.12	98.04	58.59	47.54	40.52
DNS 6	855.79	341.22	172.50	90.78	52.66	41.75	34.86
Forecast Combination	1364.23	547.06	276.04	142.19	76.85	55.87	40.24

Note: This table reports the Turnover of the mean-variance bond portfolios with 6-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

Table 46: Turnover 12-month rebalancing - Sub-sample

Risk Aversion	$\delta = 0.1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 5$
Random Walk	6230.74	5369.03	5269.92	5272.29	5282.25	5286.71	5291.30
AR(1)	6034.56	5280.59	5224.13	5251.16	5271.57	5279.70	5286.99
VAR(1) PCA	5253.16	4959.30	5076.95	5183.38	5237.79	5257.78	5273.88
DNS - ML approach							
DNS 1	361.13	149.81	80.81	47.54	33.68	29.37	26.13
DNS 2	111.61	46.19	25.99	18.82	15.85	15.03	14.56
DNS 3	878.37	348.23	172.20	87.93	48.58	37.51	31.00
DNS 4	394.63	157.71	82.43	49.22	36.51	32.58	30.15
DNS 5	812.18	315.08	153.66	78.63	49.26	42.06	38.63
DNS 6	805.37	313.62	150.19	75.49	42.53	34.46	30.45
Forecast Combination	1120.74	448.20	224.23	115.41	62.56	45.58	33.72
DNS - EM approach							
DNS 1	410.84	168.89	89.38	51.99	34.73	29.46	25.49
DNS 2	141.34	57.19	31.15	20.44	17.12	16.71	16.74
DNS 3	692.23	269.79	133.05	68.62	43.21	36.70	33.22
DNS 4	431.01	177.18	93.43	53.08	35.59	30.19	26.41
DNS 5	820.62	324.22	162.30	86.13	52.19	43.81	38.54
DNS 6	774.20	304.22	149.67	77.69	45.04	35.83	31.48
Forecast Combination	1046.52	418.89	213.32	111.76	62.14	46.55	34.50

Note: This table reports the Turnover of the mean-variance bond portfolios with 12-months rebalancing horizon, for the RW, AR(1), VAR(1) PCA, DNS-ML and DNS-EM models, as well as of the Forecast Combination , for different values of risk aversion coefficient, $\delta \in \{0.1,0.25,0.5,1,2,3,5\}$. The results refer to the Sub-sample.

Table 47: DNS-ML: RMSFE relative to the benchmark, expanding window - Sub-sample

Maturity	**3-m**	**6-m**	**12-m**	**24-m**	**60-m**	**84-m**	**120-m**
1-step ahead							
DNS 1	1.31	1.18	1.09	1.07	1.05	1.05	1.01
DNS 2	2.11	1.67	1.09	1.05	1.05	1.05	1.09
DNS 3	1.09	1.07	1.04	1.03	1.02	1.03	1.01
DNS 4	1.28	1.22	1.16	1.11	1.08	1.07	1.02
DNS 5	1.16	1.08	1.06	1.05	1.02	1.03	1.00
DNS 6	1.18	1.15	1.13	1.10	1.06	1.05	1.01
3-step ahead							
DNS 1	1.13	1.13	1.12	1.08	1.06	1.05	1.00
DNS 2	1.13	1.07	1.05	1.06	1.06	1.07	1.07
DNS 3	1.08	1.08	1.05	1.03	1.02	1.03	1.01
DNS 4	1.21	1.22	1.20	1.15	1.09	1.08	1.03
DNS 5	1.12	1.09	1.07	1.05	1.02	1.02	0.99
DNS 6	1.16	1.17	1.16	1.12	1.06	1.05	1.01
6-step ahead							
DNS 1	1.12	1.13	1.13	1.10	1.06	1.06	1.02
DNS 2	1.05	1.07	1.09	1.09	1.08	1.09	1.10
DNS 3	1.09	1.09	1.07	1.04	1.02	1.03	1.04
DNS 4	1.19	1.21	1.22	1.17	1.10	1.09	1.06
DNS 5	1.10	1.09	1.08	1.06	1.02	1.02	1.01
DNS 6	1.14	1.16	1.17	1.13	1.06	1.04	1.02
12-step ahead							
DNS 1	1.12	1.12	1.11	1.11	1.11	1.12	1.11
DNS 2	1.13	1.13	1.13	1.13	1.15	1.17	1.20
DNS 3	1.11	1.10	1.08	1.06	1.05	1.09	1.12
DNS 4	1.14	1.16	1.17	1.18	1.15	1.16	1.16
DNS 5	1.05	1.05	1.05	1.06	1.06	1.08	1.09
DNS 6	1.08	1.10	1.12	1.12	1.08	1.09	1.09

Note: This table reports the RMSFE of the RW, for the 1, 3, 6 and 12-step ahead forecasts, at 7 different maturities, $\tau \in \{3,6,12,24,60,84,120\}$, for the Sub-sample. Furthermore, the table reports the relative RMSFE to the RW of the DNS-ML models. A value smaller than one indicates that the model outperform the benchmark model, RW. Expanding windows have been used for the estimation of the models.