Erasmus University Rotterdam
Erasmus School of Economics

Master Thesis Quantitative Finance

Optimal Rebalancing with Costs: Theory and Application for Proportional and Market-Impact Costs

by

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Submission Date:	April 26, 2021	In Cooperation With:	APG Asset Management
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Abstract

We consider a framework for large institutional investors, who deal with proportional transactions costs as well as market-impact transaction costs. We consider a stock price model based on Geometric Brownian Motion and derive the Hamilton-Jacobi-Bellman (HJB) equation and an optimal turnover rate governed by this equation. Our model is a generalization of previously considered models as we allow for multiple assets and a combination of proportional costs and market-impact costs. To solve the HJB equation and the optimal turnover rate, we propose several approximation methods. In a simulation study these approximation methods are compared to each other and existing methods found in practice and literature. We find that an optimal policy is described by a no-trade zone around the Merton point and (depending on costs structure) linear trading towards this optimum outside the no-trade region. We also find that this policy helps with decreasing downside risk and is quite robust to misspecification.

Contents

1 Introduction

In the world of modern portfolio theory stylized approaches give well-understood and easily implementable results, see for example Merton (1969). However, in practice these results are too good to be true. An assumption that directly forfeits the results from a stylized setting is the assumption of a perfect market. This assumption takes the form of information equilibria, infinite market dept, and most importantly no transaction costs. For a small investor transaction costs consist of a percentage of the traded amount or a fixed cost per trade. For larger investors we see transaction cost in the form of market impact costs. The optimal allocation problem with fixed or proportional transaction costs has been reviewed many times before, see Cadenillas (2000) for a review.

We consider specifically how a large institutional investor, for example a pension fund, should approach transaction costs. Two methods currently used in practice are a method where trading is done to remain in a certain bandwidth around the optimum, and a method where trading is done towards a target with trading speed proportional towards the target (see Gârleanu and Pedersen (2013)). It is however unclear how the approaches differ in performance compared to each other, as most literature is more focused towards solving the mathematical problem in different settings, than to discuss the implications this has for practice.

The aim of this thesis is to make clear what the implications are for large (institutional) investors. We furthermore propose a new specification of transactions costs combining approaches from previous literature. We set up a mathematical model for the transaction costs, which generalizes previous literature and we also extend this model to the setting for multiple risky assets and cash. Our transaction costs structure consists of both proportional costs and market-impact (power) costs. We consider the isoelastic utility in our optimization problem. By dynamic programming and the Bellman principle, we find the non-linear dynamical system the value function for this problem should adhere to. On top of this, we find an intuitive solution for the optimal trading strategy depending on the value function. We use an asymptotic approximation also used in literature and a Taylor approximation to circumvent solving the non-linear system.

We apply these trading strategies in a simulation setting and measure their performance against other strategies found in literature and practice. We also simulate certain settings to check how robust the strategies are for misspecification. In the simulation setting we look at the mean performance and the performance in a tail-event. We find that our proposed trading strategies perform well and outperform other methods. We also find that even if the costs are actually zero, our strategies still perform near the optimal Merton strategy.

In Section 2, we provide a more extensive overview of past literature in the field of portfolio theory with transactions costs. Building on these past approaches, we formulate the problem mathematically, derive the dynamical system and discuss how to solve this equation numerically for the single-asset and multi-asset formulation in Section 3. We also show some results for the value function and optimal trading strategy in this section. In Section 4, we discuss the methods used in our simulation setting and also discuss their results. Concluding and summarizing in Section 5, we discuss the results and propose possible further applications and advances that can be made using this work for theorists and practitioners.

2 Literature

The foundations of modern portfolio theory in a long-term setting date back to the very influential work done by Merton (1969), Samuelson (1975) and Merton (1975). Although a stylized setting is used, the conclusions that an agent should hold a single optimal portfolio is found over and over again. This stylized setting mainly consist of a certain process for stock prices, assumptions on utility preferences and most importantly assumptions that market-friction and transaction costs are negligible. For an overview of previous literature on long-run portfolio theory without transaction cost, see Campbell and Viceira (2002).

However, when transaction costs are incorporated, the nature of the problem changes tremendously, as there now exists a trade-off between the rebalancing to the optimal allocation and the transaction costs this trade has. A model including proportional transaction costs was first posed by Magill and Constantinides (1976) and sequentially also considered by Constantinides (1986), Taksar et al. (1988), Davis and Norman (1990) and Shreve and Soner (1994). Cadenillas (2000) gives an overview of the accomplishments from this string of literature, Davis and Norman (1990) is generally considered to be the first solution for proportional transaction costs. In Davis and Norman (1990) a geometric Brownian motion is considered for stock prices in combination with a power utility. Davis and Norman (1990) and the other literature above almost all form a free-boundary problem based on (quasi-)variational inequalities and solve this for the optimal solution. Gerhold et al. (2013) takes a different approach and solves the problem through a dual optimizer. Even-though, the papers above consider different optimally criteria, it is concluded that rebalancing should only happen if the portfolio allocation falls outside a certain bandwidth around the Merton-portfolio, the so-called no-trade (NT) zone. The models above solve the stochastic control problem with proportional transaction costs, however the results are still not practically viable, as rebalancing at the boundary still happens continuously to stay inside the boundary. To implicitly solve this issue, one could consider fixed transaction cost in addition to proportional costs. Pliska and Suzuki (2004) solve a settings with both fixed and proportional costs and find a NT zone again, however if allocations exceed this zone rebalancing is done to some optimal portfolio inside the NT zone. Results with fixed and proportional transaction costs are also derived in Korn (1998) and Buckley and Korn (1998). As mentioned before from the above literature, it is often found that the problem translates to a free boundary problem, numerical schemes based on finite element methods are used in Muthuraman and Kumar (2006) to solve the problem numerically.

In a finite horizon setting, Liu and Loewenstein (2002) finds that the optimal trading strategy is based on the horizon. They also consider a model with a random end-date and solve this. Even with all the above results, applicability of the theorems is hard when the number of assets increase. Therefore, Brown and Smith (2011) come up with heuristics and finds that these perform near-optimal when used in simulations.

Despite the fact that the problem with proportional transaction costs could be declared solved as of Davis and Norman (1990) and the problem with fixed transaction costs proved to be solved quite fast after that, other costs function have hardly been considered. It is however important to also consider for example quadratic costs, because proportional costs imply costs caused by the bid-ask spread, whereas quadratic cost imply cost for market impact. These market-impact costs are of interest for larger investors. Gârleanu and Pedersen (2013) consider a discrete time setting in which quadratic costs are implemented. They find that the optimal portfolio consists of a weighted average of the current portfolio and a target portfolio, such that each time step you move closer to the optimal portfolio. This is intuitive as it becomes more costly to trade if you are far away from the optimal portfolio and thus we want to stay close, however trading each time step directly to the optimal portfolio at this time will cause the portfolio at the next time step to be less optimal, thus we take a weighted average of future optimal portfolios. Where Gârleanu and Pedersen (2013) use a setting with prices driven by an arithmetic Brownian motion and direct mean-variance preferences, Guasoni and Weber (2017) consider the problem in the usual portfolio theoretical setting, namely with geometric Brownian motion and a terminal wealth problem with isoelastic utility. Similar to Guasoni and Weber (2018), it is found that optimal trading takes place towards a target portfolio. The latter 2018 paper even extends the result in a multi-asset market, where market-impact also affects prices of other assets. The results from Gârleanu and Pedersen (2013), Gârleanu and Pedersen (2016), Guasoni and Weber (2017) and Guasoni and Weber (2018) seem to be along the same lines, this is confirmed in Moreau et al. (2017) where a general model is presented, where the trading rate is dependent on the investors preferences, market volatility, transaction costs and most importantly is proportional to the distance from frictionless target. They also show that the case of fixed and proportional transaction costs has certain properties found with quadratic transaction costs.

Extending on previous results with proportional transaction costs Rej et al. (2015) combines proportional and (small) quadratic transaction costs and find that the solution consist of a mix of the above mentioned solutions. That is, the optimal policy consist of some bandwidth and a target portfolio outside this bandwidth. The bandwidth found is smaller than the bandwidth that would be found in for example a Davis and Norman (1990) setting. The combination of these two makes sense as we would like to model the proportional costs from for example the bid-ask spread and the quadratic costs from market-impact. Liu et al. (2017) confirms this result and incorporates proportional costs in the model of Guasoni and Weber (2017). The results is again a no-trade zone and targeted trading outside this region, however it should be noted that both the no-trade zone as well as the trading rate are both smaller than in models with solely proportional or solely quadratic costs.

The above results are all found when using quadratic transaction costs as a proxy for market-impact. The literature in market-impact started with Kyle (1985), who introduces a model to measure market depth and liquidity. Guasoni and Weber (2020) derive asymptotic behavior which show that other specifications of market-impact costs lead to the same result. This result is also fortified by the work by Gonon et al. (2021), who consider a specification for equilibrium asset prices. They find that the use of quadratic costs is justified as the effects of other specifications are rather similar. In all the above literature including quadratic costs (especially Guasoni and Weber (2017), Liu et al. (2017) and Guasoni and Weber (2020)) a similar non-linear ordinary differential equation (ODE) is found, which unfortunately has no closed-form solutions. We will continue on these papers by extending the results with general power transaction costs from Guasoni and Weber (2020) with proportional transaction costs in the same manner as in Liu et al. (2017). We will also extend the equations from these papers with a risk-free rate process and in a multi-asset settings as in Guasoni and Weber (2018). To summarize the significance of this thesis we look at the following table.

Table 1: Important models in literature and their properties

Literature	**Proportional Costs**	**Market-impact Costs**	**Single-asset**	**Multi-asset**
Davis and Norman (1990)	✓		✓	
Guasoni and Weber (2017)		✓	✓	
Liu et al. (2017)	✓	✓	✓	
Moreau et al. (2017)		✓	✓	
Guasoni and Weber (2018)		✓		✓
Guasoni and Weber (2020)		✓	✓	
This thesis	✓	✓	✓	✓

The literature in this table describes models, we combine to make a new model that incorporates both proportional costs and market-impact costs. Furthermore, we describe our model in a single-asset setting and a multi-asset setting.

3 Optimal trading

In this section, we will discuss how we attack the problem from different angles. We will start by discussing the theoretically optimal solution in both a single-asset setting and a multi-asset setting with asymptotic approximations that can made to ease calculation.

3.1 Single-asset Model

We model a financial market with two assets. One cash asset which is risk-free for which we normalize the risk-free rate to zero. The second asset, a risky asset, follows a geometric Brownian motion, such that

$$\frac{\mathrm{d}S_t}{St} = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t, \tag{1}$$

where W_t is a standard Brownian motion. $\mu > 0$ is the expected excess return and $\sigma > 0$ denotes the volatility. We assume that trading does not occur at the exogenous price S_t, but at $\tilde{S}_t$ which is given by

$$\tilde{S}_t = S_t \left(1 + \varepsilon \text{ sign}(\Delta\theta) + \lambda \text{ sign } (\Delta\theta) \left|\frac{S_t \Delta\theta}{X_t \Delta t}\right|^{\alpha}\right), \tag{2}$$

where $\Delta\theta$ is the number of shares traded and Δt is the execution time. We denote wealth by X_t and use this as a proxy for the market capitalization. Here λ and ε (non-negative real numbers) denote the costs parameters. We denote $\alpha \in (0, 1]$ as the elasticity of the price impact to the order flow. If $\varepsilon = 0$ we find the model of Guasoni and Weber (2020). Likewise, if $\lambda = 0$, we find the model with proportional costs comparable to the model in Davis and Norman (1990). If we set $\alpha = 1$, we find a model with proportional costs and linear price impact similar to Liu et al. (2017). To make the model more tractable, we change to the continuous equivalent. Thus we define $\dot{\theta}$ as the time derivative for θ_t, such that we can replace $\frac{\Delta\theta_t}{\Delta t}$. From the previous equation, we find that the cash position of the investor evolves as

$$\mathrm{d}C_t = -S_t \left(1 + \varepsilon \text{sign}(\dot{\theta}_t) + \lambda \left|\frac{\dot{\theta}_t S_t}{X_t}\right|\right) \mathrm{d}\theta_t. \tag{3}$$

Following Guasoni and Weber (2020), we define the wealth turnover as $u_t := \frac{\dot{\theta} S_t}{X_t}$ and the risky portfolio weight $Y_t := \frac{\theta_t S_t}{X_t}$. We can then use Ito's rule to find

$$\frac{\mathrm{d}X_t}{X_t} = Y_t(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t) - \varepsilon|u_t| \,\mathrm{d}t - \lambda|u_t|^{\alpha+1} \,\mathrm{d}t \tag{4}$$

$$\mathrm{d}Y_t = \left(Y_t(1 - Y_t)(\mu - Y_t\sigma^2) + u_t + \varepsilon|u_t|Y_t + \lambda Y_t|u_t|^{\alpha+1}\right) \mathrm{d}t + Y_t(1 - Y_t)\sigma \,\mathrm{d}W_t \tag{5}$$

Due to the non-linear costs term the risky asset weight is no longer a control variable, but becomes a state variable. We can influence the drift term by variable u_t, which is now the control variable. To be precise, we set up a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with Brownian motion $(W_t)_{t\geq 0}$ and

$(\mathcal{F}_t)$ the augmentation of the filtration generated by $(W_t)_{t\geq 0}$ as defined in Liu et al. (2017). We define admissible strategies $(u_t)_{t\geq 0}$, which we require to be square integrable, such that (4) has a unique solution for $t \in [0, \infty)$ and $Y_0 \in [0, 1]$. We find

$$X_t^u = X_0 \exp\left(\int_0^T \mu Y_t - \frac{\sigma^2}{2} Y_t^2 - \varepsilon|u_t| - \lambda|u_t|^{\alpha+1} \,\mathrm{d}t + \int_0^T \sigma Y_t \,\mathrm{d}W_t\right) \tag{6}$$

And find optimal admissible strategy $(u_t)_{t\geq 0}$ if they maximize the equivalent safe rate.

Definition 3.1 *The equivalent safe rate (ESR) is given by*

$$ESR_\gamma(u) := \liminf_{T\to\infty} \frac{1}{T} \log \mathbb{E}\left[(X_T^u)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, \tag{7}$$

where $\gamma > 0, \gamma \neq 1$ *is the relative risk aversion parameter. The ESR denotes the risk-free saving rate needed to achieve a similar utility as the portfolio.*

The equivalent safe rate as specified above implies a isoelastic utility function. We find that value function for this problem is dependent on Y_t and X_t and therefore should have the following form by Îto's lemma

$$\begin{aligned} \mathrm{d}V(t,x,y) = V_t \,\mathrm{d}t + V_x \,\mathrm{d}X_t + V_y \,\mathrm{d}Y_t + \frac{V_{xx}}{2} \,\mathrm{d}\langle X_t, X_t\rangle \\ + \frac{V_{yy}}{2} \,\mathrm{d}\langle Y_t, Y_t\rangle + V_{xy} \,\mathrm{d}\langle X_t, Y_t\rangle. \end{aligned} \tag{8}$$

By Bellman's principle of optimality, we must have that the value function is a super-martingale and furthermore for the optimal control it should be a martingale. Therefore, by filling in equation (4) and equation (5), we find the following differential equation for the value function

$$\begin{aligned} 0 = {} & V_t + V_x \mu X_t Y_t + V_y Y_t(1-Y_t)(\mu - \sigma^2 Y_t) \\ & + \sigma^2 \left(\frac{V_{xx}}{2} X_t^2 Y_t^2 + \frac{V_{yy}}{2} Y_t^2 (1-Y_t)^2 + V_{xy} X_t Y_t^2 (1-Y_t)\right) \\ & + \max_{u_t} \left\{X_t V_x(-\varepsilon|u_t| - \lambda|u_t|^{\alpha+1}) + V_y(u_t + \varepsilon|u_t|Y_t + \lambda|u_t|^{\alpha+1} Y_t)\right\}, \end{aligned} \tag{9}$$

where we have to maximize over control variable u_t to find the optimal control. When we use the following ansatz for the solution we can simplify this differential equation.

Ansatz 3.2 *As we make assumptions that the utility function is homogeneous of the first degree and the limited influence time should have on the problem, we find that the value function should have the following form*

$$v(t, x, \boldsymbol{y}) = U(x) \exp\left\{(1-\gamma)\left(\beta(T-t) + \int_{y_0}^{y} q(\hat{y}) \,\mathrm{d}\hat{y}\right)\right\}. \tag{10}$$

In this ansatz $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ *denotes the isoelastic utility function,* $\beta \in (0, \frac{\mu^2}{2\gamma\sigma^2})$ *denotes the ESR* *and the function $q(y) : [0,1] \to \mathbb{R}$ is unknown. The integration lower-bound y_0 is unknown, but irrelevant for our applications.*

We find β, function $q(y)$ exist that solve the ordinary differential equation

$$\begin{aligned}0 = &-\beta + \mu y - \frac{\gamma\sigma^2}{2}y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q \\ &+ \frac{\sigma^2}{2}y^2(1-y)^2(q' + (1-\gamma)q^2) \\ &+ \max_u \left\{-\varepsilon|u| - \lambda|u|^{\alpha+1} + (u + \varepsilon|u|y + \lambda|u|^{\alpha+1}y)q\right\}.\end{aligned} \tag{11}$$

simplifying this and filling in the optimal control which we solve analytically, we find

Proposition 3.3 *There exists a function $q : [0,1] \to \mathbb{R}$ and constants $\beta \in (0, \frac{\mu^2}{2\gamma\sigma^2})$ and $0 \le y^- \le y^+ \le 1$ which solve the equation*

$$\begin{aligned}0 = &-\beta + \mu y - \frac{\gamma\sigma^2}{2}y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q \\ &+ \frac{\sigma^2}{2}y^2(1-y)^2(q' + (1-\gamma)q^2) \\ &+ \begin{cases}\alpha\lambda^{-1/\alpha}(\alpha+1)^{-\frac{\alpha+1}{\alpha}}\frac{(q-\varepsilon(1-yq))^{\frac{\alpha+1}{\alpha}}}{(1-yq)^{1/\alpha}}, & \textit{if} \quad y \in [0, y_-) \\ 0, & \textit{if} \quad y \in [y_-, y_+] \\ \alpha\lambda^{-1/\alpha}(\alpha+1)^{-\frac{\alpha+1}{\alpha}}\frac{(-q-\varepsilon(1-yq))^{\frac{\alpha+1}{\alpha}}}{(1-yq)^{1/\alpha}}, & \textit{if} \quad y \in (y_+, 1]\end{cases}.\end{aligned} \tag{12}$$

Where $q' = \mathrm{d}q/\,\mathrm{d}y$ and $0 \le y_- \le y_+ \le 1$ define the no-trade region. On the boundaries the function q and parameters β, y^- and y^+ adhere to the following

$$q(0^+) = \varepsilon + (\alpha+1)\left(\lambda\beta^\alpha\alpha^{-\alpha}\right)^{\frac{1}{\alpha+1}}, \tag{13}$$

$$q(y_-) = \frac{\varepsilon}{1 + y_-\varepsilon}, \tag{14}$$

$$q(y_+) = \frac{-\varepsilon}{1 - y_+\varepsilon}, \tag{15}$$

$$q(1^+) = \frac{d - \varepsilon}{d - \varepsilon + 1} \quad \textit{with} \quad d = \lambda(\alpha+1)\left(\beta - \mu + \frac{\gamma\sigma^2}{2}\right)^\alpha, \tag{16}$$

where $q(0^+)$ and $q(1^-)$ denote $\lim_{y\downarrow 0} q(y)$ and $\lim_{y\uparrow 1} q(y)$, respectively. The optimal policy found in this setting is given by

$$u_\pm = \max\left\{0, \left[\frac{1}{\lambda(\alpha+1)}\left(\pm\frac{q(y)}{1 - yq(y)} - \varepsilon\right)\right]^{1/\alpha}\right\}, \tag{17}$$

where u^+ (u^-) is the optimal buy (sell) policy.

We see that the ODE is equal to equation (12) in Guasoni and Weber (2020) if $\varepsilon = 0$ and $y_- = y_+$. If $\alpha = 1$ the equation is similar to equation (3.1) of Liu et al. (2017). We also see that the optimal policy implies a no-trade zone around the optimum of the value function and weighs the trading outside this no-trade zone dependent on market-impact costs.

3.2 Multi-asset model

We can also model a financial world with two risky assets and a riskless cash asset. The model we describe is general enough to describe a model with cash and any number of assets, however as we will use a two asset setting later, we will describe the model as such. We model these risky assets by a geometric Brownian motion, such that

$$\frac{\mathrm{d}S_t^i}{S_t^i} = (r_t + \boldsymbol{\lambda}'\boldsymbol{\sigma_i})dt + \boldsymbol{\sigma}_i' \,\mathrm{d}\boldsymbol{W}_t, \tag{18}$$

where r_t is the risk-free interest rate, $\boldsymbol{\lambda}$ is the market price of risk and $\boldsymbol{\sigma}_i$ denotes a quantity that has properties similar to volatility, thus we will refer to it as volatility. $\mathrm{d}\boldsymbol{W}_t$ are the increments in the uncorrelated multivariate Brownian motion $\boldsymbol{W}_t$. As we have two assets we also use two Brownian motions, such that all vectors are of length 2. We also write $\boldsymbol{\sigma}$ to denote the 2×2 matrix of the stacked vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. We define $\boldsymbol{S}_t = (S_t^1, S_t^2)'$. However, we assume that instead of $\boldsymbol{S}_t$ we pay $\hat{\boldsymbol{S}}_t$, which is given by

$$\hat{\boldsymbol{S}}_t = \boldsymbol{S}_t \left(1 + \boldsymbol{\varepsilon}_1 \,\mathrm{sgn}(\Delta\boldsymbol{\theta}_t) + \boldsymbol{\varepsilon}_2 \left|\frac{\boldsymbol{S}_t \Delta\boldsymbol{\theta}_t}{X_t \Delta t}\right|^{\alpha}\right), \tag{19}$$

where $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are the positive cost parameter vectors. $\boldsymbol{\theta}_t$ denotes the amount of stocks, such that a Δ operator denotes the change in that amount. We again denote total wealth by the scalar X_t, which we will define later. Products, absolute values and powers in this equation are element-wise, that is $\hat{S}_t^i = S_t^i \left(1 + \varepsilon_{1,i} \,\mathrm{sgn}(\Delta\theta_{t,i}) + \varepsilon_{2,i} \left|(S_t^i \Delta\theta_{t,i})/(X_t \Delta t)\right|^{\alpha}\right)$. Generally, it will be clear from context what kind of product is used. The first costs term denotes a linear costs term, which resembles the costs often associated with a bid-ask spread (Guasoni and Weber (2020)). The second costs term denotes the power costs, we associate with market impact. This formulation uses the investors wealth as a proxy for the total wealth in the financial system. Such that a trade of one million dollars will have less effect nowadays, when compared to 100 years ago, as would be logical. Let Δt be the time interval on which the trading takes place, when we buy a lot in a short time this will cause more costs than when Δt is bigger. This formulation is used in Guasoni and Weber (2020) as well. We move to a setting with infinitesimal trading by taking a limit and thus $\Delta\boldsymbol{\theta}_t / \Delta t \to \dot{\boldsymbol{\theta}}_t$. We also introduce $\boldsymbol{u}_t = \boldsymbol{S}_t \dot{\boldsymbol{\theta}}_t / X_t$, which denotes the wealth turnover. We now have

$$\hat{\boldsymbol{S}}_t = \boldsymbol{S}_t + \boldsymbol{\varepsilon}_1 |\boldsymbol{u}_t| + \boldsymbol{\varepsilon}_2 |\boldsymbol{u}_t|^{\alpha+1} \tag{20}$$

We denote the total costs function as $K(\boldsymbol{u}) = \boldsymbol{\varepsilon}_1'|\boldsymbol{u}_t| + \boldsymbol{\varepsilon}_2'|\boldsymbol{u}_t|^{\alpha+1}$. As the cash position is only affected by the interest rate and the trading of stocks. We model the cash position as

$$\mathrm{d}C_t = r_t C_t \,\mathrm{d}t - \hat{\boldsymbol{S}}_t' \dot{\boldsymbol{\theta}}_t \,\mathrm{d}t. \tag{21}$$

We define the total wealth as $X_t = \boldsymbol{\theta}_t' \boldsymbol{S}_t + C_t$ and proportion of wealth in a risky asset as $\boldsymbol{Y}_t = \boldsymbol{\theta}_t \boldsymbol{S}_t / X_t$. In a simpler setting Guasoni and Weber (2017) concludes that short selling or borrowing is never optimal. As their arguments still hold for our setting C_t, X_t and $\boldsymbol{Y}_t$ are all non-negative and $\sum_i \boldsymbol{Y}_t^i \leq 1$. Using Îto's lemma, we find

$$\frac{dX_t}{X_t} = r_t dt + \boldsymbol{Y}_t'(\boldsymbol{\sigma\lambda} dt + \boldsymbol{\sigma} d\boldsymbol{W}_t) - K(\boldsymbol{u_t})dt, \tag{22}$$

$$d\boldsymbol{Y}_t = (\boldsymbol{u}_t + \boldsymbol{P_y}(\boldsymbol{\sigma\lambda} - \boldsymbol{\Sigma Y}_t) + K(\boldsymbol{u}_t)\boldsymbol{Y}_t)\, dt + \boldsymbol{P_y \sigma} d\boldsymbol{W}_t, \tag{23}$$

where $\boldsymbol{P_y} = \mathrm{diag}(\boldsymbol{Y}_t) - \boldsymbol{Y}_t \boldsymbol{Y}_t'$ and $\boldsymbol{\Sigma} = \boldsymbol{\sigma\sigma'}$. Due to the non-linearity in the costs, we cannot use the $\boldsymbol{Y}_t$ as control variables as one would in a setting without transaction costs. However, we will use the previously defined wealth turnover $\boldsymbol{u}_t$ as a control variable. As we are looking at an infinite time period, we wish to optimize the control variable over the equivalent safe rate (ESR) as we did in the single variable problem as well. To solve this optimization problem, we use the principles of Bellman and set up the requirements a value function for this problem should satisfy. We know that the value function is dependent on both the current wealth and the proportion of wealth in the risky assets.

Ansatz 3.4 *Due to homogeneity and the limited influence time should have in such a setting we can also take an ansatz for the value function to be*

$$v(t, x, \boldsymbol{y}) = U(x) \exp(\beta(1-\gamma)(T-t))\varphi(\boldsymbol{y}). \tag{24}$$

In this ansatz $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ *denotes the isoelastic utility function,* β *denotes the ESR and the function* $\varphi : [0,1]^2 \to [0,1]$ *is unknown.*

We set up a Hamilton-Jacobi-Bellman equation for $\varphi(\boldsymbol{y})$ by using Îto's lemma on the value function (with respect to wealth and risky asset proportion) and setting the drift to zero, as the value function should be a martingale in the optimal scenario. Consequently, we find an optimal trading policy based on the value function and the equation the value function should solve.

Proposition 3.5 *Assuming* $\varphi \neq \boldsymbol{y}'\nabla\varphi, \quad \forall \boldsymbol{y}$ *in the domain the optimal trading policy is given by*

$$\boldsymbol{u}^*(\boldsymbol{y}) = \mathrm{sgn}(\nabla\varphi) \left(\frac{\left(\frac{|\nabla\varphi|}{\varphi - \boldsymbol{y}'\nabla\varphi} - \boldsymbol{\varepsilon}_1 \right)^+}{(\alpha+1)\boldsymbol{\varepsilon}_2} \right)^{1/\alpha}, \tag{25}$$

where $(\cdot)^+$ *equals* $\max(0,\cdot)$. *Here* $\varphi(\boldsymbol{y})$ *solves the following equation in the unit simplex.*

$$
\begin{aligned}
0 = \max_{\boldsymbol{u}} \Bigg(& \varphi(\boldsymbol{y})(-(1-\gamma)\beta + (1-\gamma)(r_t + \boldsymbol{y}'\boldsymbol{\sigma}\boldsymbol{\lambda} - K(\boldsymbol{u})) - \frac{1}{2}\gamma(1-\gamma)\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}) \\
& + (\nabla_{\boldsymbol{y}}\varphi)'((1-\gamma)(\boldsymbol{P_y}\boldsymbol{\Sigma}\boldsymbol{y}) + \boldsymbol{u} + \boldsymbol{P_y}(\boldsymbol{\sigma}\boldsymbol{\lambda} - \boldsymbol{\Sigma}\boldsymbol{y}) + K(\boldsymbol{u})\boldsymbol{y}) \\
& + \frac{1}{2}\operatorname{Tr}\left[\boldsymbol{P_y}\boldsymbol{\Sigma}\boldsymbol{P_y}D^2_{\boldsymbol{y}}\varphi(\boldsymbol{y})\right]\Bigg), \qquad (26)
\end{aligned}
$$

here we have omitted subscripts for time and lower-case characters are used to ease notation. $\nabla_{\boldsymbol{y}}$ *and* $D^2_{\boldsymbol{y}}$ *denote the gradient and Hessian operator respectively.*

The proof for this proposition can be found in the appendix and follows similar steps as the single-asset setting. Again, we assume it is obvious where element-wise operations are used. The interpretation behind this optimal value is also quite straightforward. The $\text{sgn}(\nabla\varphi)$ term makes sure the trading is in the right direction, that is, we buy when this has a positive effect on the value function and sell when an increase in risky asset position leads to a worse value function. The numerator of the term in brackets causes a no-trade zone when the the marginal gain in the value function is less than the marginal costs caused by the linear transaction costs. When the numerator is positive, the denominator influences the size of the trade. If $\boldsymbol{\varepsilon}_2$ is large, which would imply larger costs for larger trades, we would trade in smaller steps. However this solution for the optimal $\boldsymbol{u}$ is dependent on the unknown function φ, this makes the above equation not easily solvable. However, we are able to intuitively comment about the form of the function. We expect the function to have a maximum at the Merton proportion and to be decreasing moving away from this proportion. Due to the increasing effect of quadratic costs, the further away from the Merton proportion we are, we also expect the function to decrease faster further away from this proportion. This would mean we have a parabolic shaped function with a peak at the Merton proportion. When transaction costs are zero, that is the Merton problem, we know the solution of the value function is not dependent on a function $\varphi(\boldsymbol{y})$, which means this function equals the multiplicative identity 1, due to our ansatz. Due to our ansatz it is also not possible that our function is negative. Therefore, we conclude that our function has an upper bound of 1 and a lower bound of 0.

The differential equation as described above is found to be hard to solve. Both forms, that is the single-asset and multi-asset descriptions, have non-linear terms due to the optimal control. To find a solution to this problem we have implemented algorithms to iteratively find a solution. That is, we start by a given policy (for example, do not trade) and solve the equation numerically for the value function, then use this value function to find a new policy. We could repeat this process until convergence, however after a few steps of this algorithm the trading policy blows up to non-sensible values. We have tried this for the single-asset as well as the multi-asset model. For the single-asset have also implemented the Runge-Kutta method to solve the differential equation. This method solves initial value problems starting at one boundary and integrating numerically, however we notice some extreme behavior at the boundaries making this method also unfit to handle the problem. This extreme behaviour at the boundary is mainly caused by terms like $1/(y(1-y))$, which explode near $y = 0$ and $y = 1$. Therefore, we propose two methods to solve this equation. In both methods, we propose a certain Ansatz for the value function. In the first, we assume that costs are small and that are value function can be approximated around the Merton proportion. In the second method, we propose to take a Taylor polynomial around the Merton proportion and minimize the error in the differential equation.

3.3 Small Costs Asymptotic Approximation

A method to find the value function instead of solving the above Hamilton-Jacobi-Bellman equation, is to approximate it for small costs. Reasonably, we would expect the value function to be concave with a maximum value at the Merton proportion. This is also quite common in other literature and a main example is Guasoni and Weber (2018), where an approximation is used for small quadratic costs. In their work it is assumed that the costs function is of the form $\epsilon \boldsymbol{u}'\boldsymbol{\Lambda}\boldsymbol{u}$, translating this to our specification of the cost function this would mean that our costs parameter vector $\boldsymbol{\varepsilon}_2 = \binom{\epsilon\Lambda_{1,1}}{\epsilon\Lambda_{2,2}}$ and the off-diagonal elements of $\boldsymbol{\Lambda}$ are zero. In this setting we would take $\alpha = 1$. They find that the value function is approximated by

$$\varphi(\boldsymbol{y}) \approx \exp\left(-\left(\frac{\epsilon\gamma}{2}\right)^{1/2}(\bar{\boldsymbol{Y}} - \boldsymbol{y})'\boldsymbol{C}(\bar{\boldsymbol{Y}} - \boldsymbol{y})\right), \tag{27}$$

where $\bar{\boldsymbol{Y}}$ is the Merton proportion and $\boldsymbol{C}$ is a known matrix dependent on Λ and the covariance matrix of the risky assets. We see that this approximation has the expected form, it is a parabolic shaped function with a maximum at the Merton proportion. Unfortunately, this approximation is only applicable in presence of quadratic costs. This means that α must equal 1 and that proportional costs are not taken into account. As the approximation is made when ϵ approaches zero, we could argue that the proportional costs are of a higher order of ϵ than the quadratic costs and that they disappear 'quicker'. As Equation (25) is given, we would still end up with the two properties the optimal solution must have, that is the no-trade zone close to the target and increased trading towards the target further away.

To illustrate our value function, we find below the value function in the unit simplex.

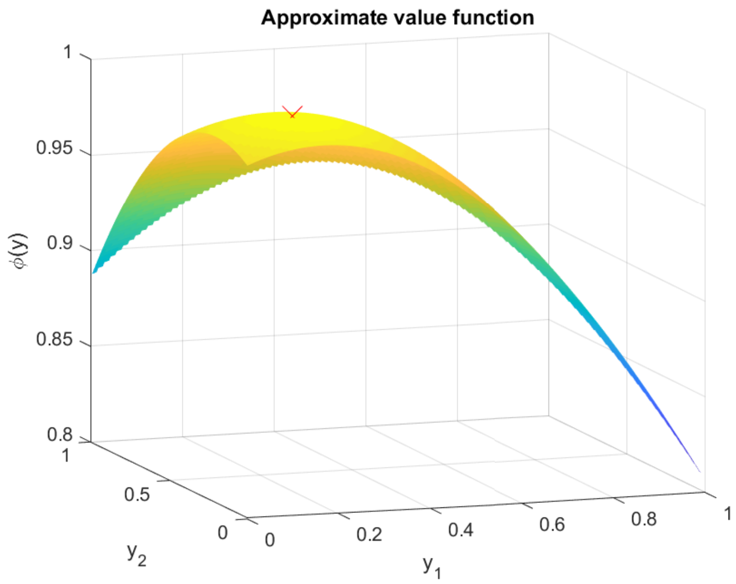
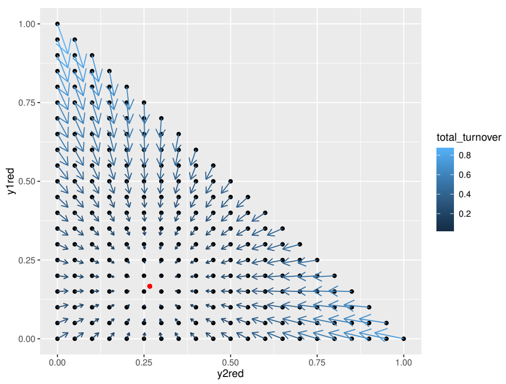
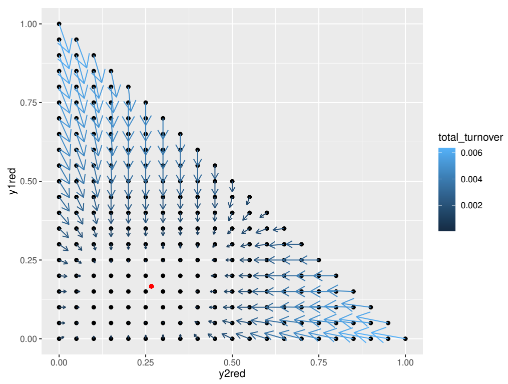


Figure 1: Asymptotic approximation for the value function in the unit simplex with a marker at the optimum which coincides with the Merton point

As the trading strategy depends directly on the value function and its derivatives, we find the following trading strategies.



(a) Optimal trading in the Merton scenario without transaction costs. The red dot denotes the Merton point which we would optimally like to be in.



(b) Optimal trading in a scenario with transaction costs. Notice the no trade zone around the Merton point and the lower trading volume compared to the other figure

Figure 2: The asymptotically optimal trading strategy according to the chosen value function and the Merton trading strategy

It is very obvious that in both trading strategies trading is always towards the Merton proportion. However, in the asymptotically optimal strategy we can see a no-trading zone around this Merton point. This is caused by numerator of the formula for optimal trading. In the no-trade zone the positive effect of trading (i.e. the increase in the value function) does not outweigh the negative effect (i.e. the proportional trading costs). A plot clearly indicating the no-trade zone can be found in the appendix. Furthermore, we see that the trading volume is much smaller at each point in the asymptotically optimal strategy. This is caused by the denominator of the optimal trading formula, where we correct for the trading costs such that volume shrinks as costs increase.

3.4 Polynomial Approximation

Whereas the previous approach was a linear expansion, we could also imagine that the value function has higher-order terms. In trading we often find ourselves close to but not immediately on the Merton proportion, therefore this point is of most interest. We propose to estimate the value function by a Taylor approximation of which the coefficients need to be found. This approach allows us to keep the minimization problem feasible, while still allowing flexibility in the Ansatz. In the single-asset case we assume that $q(y) = \xi_0 + \xi_1(y - \bar{y}) + \xi_2(y - \bar{y})^2 + \mathcal{O}\left((y - \bar{y})^3\right)$. In the following section we only mention the ξ_1 and ξ_2 values, as the other values are too low to have a significant impact on the value function.

As we have mentioned before, applying numerical methods to solve the differential equation directly does not lead to a solution. However, as we have now given the unknown equation a specific form the problem is more well behaved. Thus, we set up a discretized model for the differential equation (12) with the given boundary conditions using a central difference scheme. We define $y_i = i/N$, where N is the number of nodes on the domain and $i \in \{0, ..., N\}$. For every y_i, we fill in the Ansatz in the differential equation and for the y_i closest the the known and unknown boundaries we apply the boundary conditions. This approach gives us N equations, with unknowns $\beta, y^-, y^+, \xi_0, \xi_1$ and ξ_2. To make the problem a minimization problem, we take the sum of the squares of the N equations.

To further make sure that $\beta \in [0, \mu^2/\gamma\sigma^2]$, $y^- \in [0, \bar{y}]$ and $y^+ \in [\bar{y}, 1]$ we also add the squares of the deviation of the constraint. For example to constraint y^-, we add $\max(0, -y^-)^2$ and $\max(0, y^- - \bar{y})^2$ to the sum of squares.

To pose this approach in equations let differential equation (12) be denoted as $\mathcal{L}q = 0$, note that this is not a linear operator. We can make a discreet version of this operator $L(\boldsymbol{q}) = \mathbf{0}$, where we use brackets to emphasize that this is not a matrix vector multiplication, however $\boldsymbol{q}$ and $\mathbf{0}$ are vectors. Let the boundary conditions given by equations (13), (16), (14) and (15) be enforced as above and call this boundary penalty $N_b(\boldsymbol{q}, \beta, y^-, y^+)$. We minimize the total error in this problem as follows

$$(\boldsymbol{\xi}^*, \beta^*, y^{-*}, y^{+*}) = \arg \max_{\boldsymbol{\xi}, \beta, y^-, y^+} ||L(\tilde{q})||_2^2 + N_b(\tilde{q}(\boldsymbol{y}), \beta, y^-, y^+), \tag{28}$$

where $\tilde{q}(y) = \xi_0 + \xi_1(y - \bar{y}) + \xi_2(y - \bar{y})^2$ and $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2)$.

As this model is computationally simple and minimization is only done on a few variables, we find that optimization can be done for large N. We optimize the model for $N = 1000$, but see that for higher values of N the model still gives similar results. We see that for $\mu = 0.08$, $\sigma = 0.16$ and $\gamma = 5$ this gives the following functions $q(y)$ and the following optimal trading strategies $u(y)$

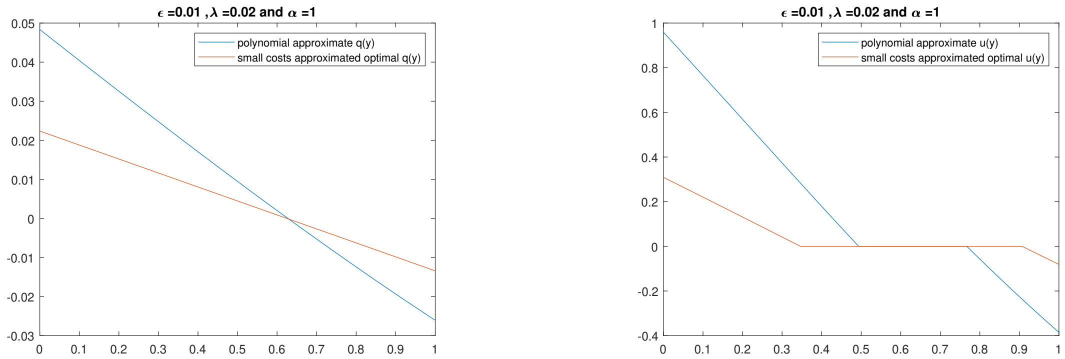


Figure 3: The optimal $q(y)$ on the left and trading policy $u(y)$ on the right for $\varepsilon = 0.01$, $\lambda = 0.02$ and $\alpha = 1$

We see that the optimal polynomial approximation has a gradient which is a bit steeper than the small costs approximation. We also notice that indeed the linear approximation seems to fit well. Due to the steeper gradient, we also see a higher optimal turnover and smaller no-trade zone in the right graph. For these costs parameters we get the following optimal values

Table 2

Parameter	β	y^-	y^+	ξ_1	ξ_2
Value	0.019	0.49	0.74	-0.077	-0.005

We see that the values for y^- and y^+ are exactly as we see them in the graph. Furthermore, we notice that indeed ξ_2 has quite a small absolute value. With these parameters, we would expect the return on the Merton portfolio (corrected for risk) to be 0.025. We thus see that for small costs β is five basispoints lower. In the small costs approximation the gradient is -0.36, which is about half as steep as the polynomial approximation.

As the small costs approximation is mainly concerned with the quadratic costs, we will also show what happens to the model if we change other parameters ε and α

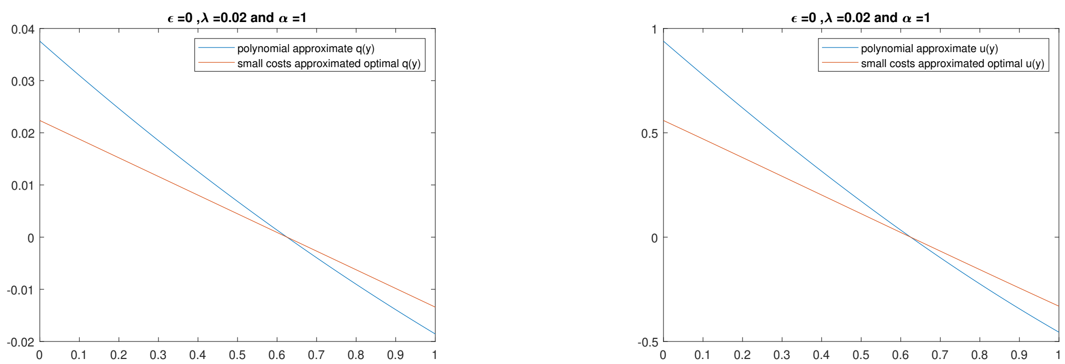


Figure 4: The optimal $q(y)$ on the left and trading policy $u(y)$ on the right for $\varepsilon = 0$, $\lambda = 0.02$ and $\alpha = 1$

In the above graphs we assume no proportional costs. Thus we would expect the small costs approximation and the polynomial approximation to be almost equal. However, we do still see a difference in gradient en subsequently in trading policy. We still see that the polynomial approximation is practically linear.

Table 3

Parameter	β	y^-	y^+	ξ_1	ξ_2
Value	0.019	0.63	0.63	-0.054	-0.011

When looking at the values in the above table, we notice that the β is almost the same as before. We also see that the no-trade zone has vanished as was expected. Furthermore, we do see that the gradient is less steep compared to the situation with proportional costs.

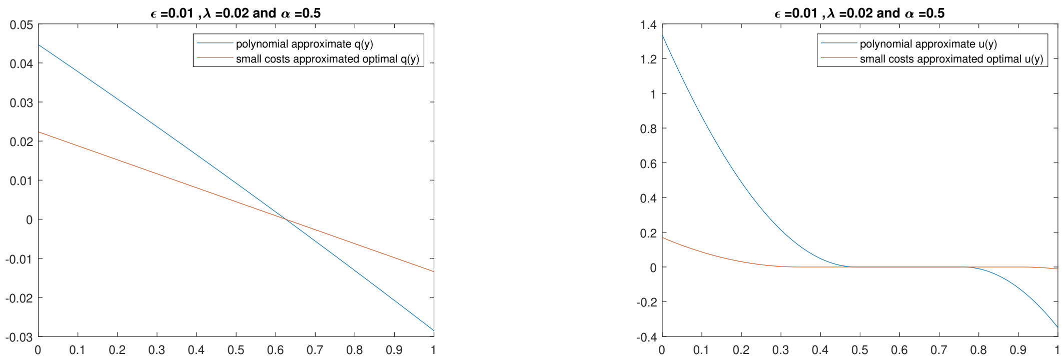


Figure 5: The optimal $q(y)$ on the left and trading policy $u(y)$ on the right for $\varepsilon = 0.01$, $\lambda = 0.02$ and $\alpha = 0.5$

In the above graphs we have changed the costs structure from quadratic to another power. We see that the trading is now not linear anymore. We also see that the polynomial is almost equivalent to the first one. Further away from the Merton proportion the trading explodes, which is not very realistic. However, this is not problematic as the strategy itself would probably keep us close to the optimum. This could also be explained as we approximate the function near the optimum, so naturally it will deviate as we move further.

Table 4

Parameter	β	y^-	y^+	ξ_1	ξ_2
Value	0.019	0.50	0.74	-0.074	-0.005

Looking at the above values, we see again a similar value for the ESR. We also see that the no-trade zone has become marginally smaller than in the first setting. The value function is similar as the first scenario.

4 Simulation

To see how a rebalancing strategy as described above would act in a real-world scenario we simulate multiple portfolios and calculate their performance. We will describe the simulation setting, the performance measures we use and see the results of the simulation.

4.1 Simulating Strategies

We simulate five different strategies. We simulate a buy-and-hold portfolio as a benchmark. The strategy in this portfolio holds the position we start with. This has as consequences that better performing stocks represent a larger part of the portfolio after time. However, this could also imply higher volatility and lead to more extreme positions. This strategy does also mean we do not incur transaction costs. It is widely accepted in literature that when transaction costs are zero, that the Merton proportion gives the optimal portfolio in terms of utility. Therefore, we simulate this portfolio as well, however we calculate both portfolio value while considering transaction costs and without considering transactions costs. In the former, bankruptcy is expected to happen quite fast and we do not mention the results any longer as this is indeed the case, the latter can be an (unbeatable) benchmark to check performance against. The last four portfolios we simulate are the asymptotically optimal trading strategies, two considering transactions costs and another two where we set transaction costs to zero in the simulation. The former will show how good the performance is in the optimal scenario, the latter will be used to see if the strategy is also robust in a misspecified trading scenario.

In the simulation, we need to choose certain parameters. We need to choose parameters for the (co-)variance, market price of risk and risk-free rate. We also need to choose the costs specification. As it is known from literature that the α-parameter for market impact costs is approximately one and because our asymptotic approximation also uses this in its derivation, we will choose this accordingly. We will furthermore test for a situation negative correlation, no correlation and positive correlation. We will keep the proportional costs constant, as these costs are associated with bid-ask spreads which we assume do not cause issues. We will also keep risk-free rate and the market price of risk constant in all situations as we do not expect these parameters to influence the results a lot. For all simulations we will use the following parameters. Due to the manner in which we vary the correlation, we also increase or decrease the total asset price variation. This is of interest as it affects the size of the no-trade zone and the speed of rebalancing outside this zone.

Table 5: The parameters used for all simulations. Note: the $\boldsymbol{\sigma}$ does differ among simulations to change the correlation of the stock prices. For extra clarity a description of the parameter is repeated in the results.

Parameter	Value	Description
r	0.04	Risk less return rate
$\boldsymbol{\lambda}$	$\binom{0.5}{0.5}$	Market price of risk
$\boldsymbol{\sigma}$	$\begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}$	
γ	5	Risk-aversion parameter
$\boldsymbol{\varepsilon_1}$	$\binom{0.05}{0.05}$	Proportional costs
$\boldsymbol{\varepsilon_2}$	$\binom{0.1}{0.05}$	Market-impact costs
α	1	Market-impact parameter

The parameters are chosen such that they are not too far from a real world situation, similar parameters were found in literature (see for example Liu et al. (2017)) and in conversation with practitioners these seemed suitable. We have made sure that the Merton portfolio according to these variables lies inside the unit simplex. We also make sure that the costs specification does imply a non-trivial no trade-zone. The risk-aversion parameter is chosen to be 5 as this is generally regarded as a good value for institutional investors.

4.2 Performance Measures

To measure performance of our different portfolios we consider two main routes to take. We can evaluate the means of different criteria, but we can also look at tail events to measure risk with certain portfolio. If the differences between a buy-and-hold strategy and a rebalancing strategy are for example very small when comparing means, we would still expect a difference when looking at tail events. Thus rebalancing could also be used as a sort of risk-management tool.

For overall statistics, we look at the mean or median of wealth and utility. Also we look at the total amount spend on trading costs and the total turnover. Using these last two measure to see what rebalancing method works best. We expect that rebalancing strategies outperform buy-and-hold strategies in utility. We look at the mean utility by mentioning the empirical ESR, which is calculated using the formula given in the definition of the ESR. We also report the increase in utility with the buy-hold portfolio as a base case. Furthermore, we report return, volatility and trading costs for the portfolios. Although trading costs is already considered within the other measures, we can still see why certain portfolio's might not work.

To measure tail events we look at the Value-at-Risk at the 95% for utility and wealth. Here, it is logical that buy-and-hold is outperformed in terms of utility. However, it will be interesting to see whether rebalancing methods will also work as a risk-management tool in terms of wealth. As rebalancing makes sure there will be less outliers compared to the buy and hold, we expect this will be the case.

4.3 Simulation findings

We will now present results from the simulations, for each scenario we report two tables. We first report the means of different performance measures, second we report the tail performance measures. We will start with the negative asset correlation results, secondly we will discuss independent assets, lastly we will discuss assets with positive correlation.

4.3.1 Results for assets with negative correlation

To consider results with negative correlation between stock prices we set $\sigma_{1,2} = -0.2$. The other parameters stay the same.

Table 6: Key portfolio performance measures when asset prices are generated with negative correlation. The portfolios are a Buy-and-hold (BH) portfolio, the asymptotically optimal small costs approximation (ASY - SC) portfolio, asymptotically optimal polynomial (ASY - PL) portfolio, a monthly rebalanced Merton (mMRT) portfolio, and the Merton and asymptotically optimal portfolios without taking costs into account (MRT NC, ASY - SC NC and ASY - PL NC, respectively). We report the empirical value for the ESR (β), the increase in utility compared to the BH portfolio, the mean return, volatility of return and the trading costs

Portfolio	**Emperical ESR (%)**	**Increase in Utility (%)**	**Mean return (%)**	**Volatility of return (%)**	**Total trading costs (% of initial wealth)**
BH	1.81	0	16.4	11.8	0
ASY - SC	3.16	41.75	14.59	6.89	1.008
ASY - PL	3.23	43.22	14.42	5.50	1.011
mMRT	1.49	-13.40	11.24	5.87	5.458
MRT NC	4.54	66.49	14.53	5.39	0
ASY - SC NC	3.34	45.88	14.32	7.77	0
ASY - PL NC	3.57	50.43	14.44	6.10	0

It is clear that the asymptotically optimal portfolios outperform all other portfolios when considering costs. And within these two asymptotic portfolios, the polynomial optimized version performs best. It is also apparent that a buy-and-hold portfolio is better than a monthly rebalance strategy. We see that the monthly rebalance strategy has a high number of costs, implying this could be problematic. The asymptotically optimal strategy (ASY - PL) outperfoms BH by approximately 43 percent. Furthermore, we note that when no costs are specified that the Merton portfolio, which should be optimal, has indeed very good performance. We also see that the performance of the asymptotic portfolios is about a percent off in terms of ESR. Logically, the asymptotic polynomial strategy is a bit closer to Merton as this has more aggressive rebalancing. We furthermore see that the BH strategy has the highest return, however also the highest volatility of return. Among other strategies this is more or less equal, expect the monthly strategy which underperforms in terms of return.

Table 7: Performance measures for tail events at the 95 percent level when assets have negative correlation. The portfolios are a Buy-and-hold (BH) portfolio, the asymptotically optimal (ASY) portfolio, a monthly rebalanced Merton (mMRT) portfolio, and the Merton and asymptotically optimal without taking costs into account (MRT NC, ASY - SC NC and ASY - PL NC, respectively)

Portfolio	**VaR95 ESR (%)**	**VaR95 return (%)**	**Total trading costs (% of initial wealth)**
BH	-2.43	3.11	0
ASY - SC	-0.68	4.19	0.24
ASY - PL	-0.43	4.54	0.55
mMRT	-1.88	3.22	43.62
MRT NC	1.25	6.54	0
ASY - SC NC	-0.52	4.33	0
ASY - PL NC	-0.23	5.33	0

We see that the BH strategy performs very badly in tail events as is expected. We furthermore see that the differences between the other strategies have become smaller. Especially for the simulation without costs the difference is much smaller. We also see that the asymptotically optimal with costs is quite close the the no costs specified simulations in terms of ESR. We furthermore notice that the Merton portfolio performances a lot better than the other portfolios. Therefore, in tail events the asymptotic portfolios are not necessarily close to the (unreachable theoretical) optimum, but they may be the better option among others practically viable options. We note that when compared to the previous table the costs for all strategies are much higher, this could imply that in tail events costs play an even more important role.

4.3.2 Results for assets without correlation

For simulation without correlation we use the base parameters as given in the parameter table.

Table 8: Key portfolio performance measures when asset prices are generated without correlation. The portfolios are a Buy-and-hold (BH) portfolio, the asymptotically optimal small costs approximation (ASY - SC) portfolio, asymptotically optimal polynomial (ASY - PL) portfolio, a monthly rebalanced Merton (mMRT) portfolio, and the Merton and asymptotically optimal without taking costs into account (MRT NC, ASY - SC NC and ASY - PL NC, respectively). We report the empirical value for the ESR (β), the increase in utility compared to the BH portfolio, the mean return, volatility of return and the trading costs

Portfolio	**Empirical ESR** (%)	**Increase in Utility** (%)	**Mean return** (%)	**Volatility of return** (%)	**Total trading costs (% of initial wealth)**
BH	2.54	0	17.33	12.20	0
ASY - SC	3.39	28.97	14.33	6.99	0.85
ASY - PL	3.44	30.39	13.89	6.10	0.93
mMRT	2.11	-18.62	11.22	5.78	4.54
MRT NC	4.54	55.17	14.34	5.56	0
ASY - SC NC	3.49	31.72	14.40	7.45	0
ASY - PL NC	3.62	35.12	14.36	0	0

We see similar results in this case as we saw in the negative correlation setting. The Empirical ESR has increased for all parameters. Although the ranking of the strategies remains similar, we see that the monthly rebalancing strategy performs even worse. We also see that the asymptotic strategies outperform the BH strategy a lot less by only around 30 percent. Theoretically, the effect of adding covariance will have a linear effect on the expected return and a quadratic effect on the variance. Thus the difference between adding covariance and simulating no covariance has a difference on the increase in utility, possibly implying that the performance of rebalancing strategies increases as covariance increases. However, it could also imply rebalancing strategies perform better with higher total variance in the model, which is trivial.

Table 9: Performance measures for tail events at the 95 percent level when assets have no correlation. The portfolios are a Buy-and-hold (BH) portfolio, the asymptotically optimal (ASY) portfolio, a monthly rebalanced Merton (mMRT) portfolio, and the Merton and asymptotically optimal without taking costs into account (MRT NC, ASY - SC NC and ASY - PL NC, respectively)

Portfolio	**VaR95 ESR** (%)	**VaR95 return** (%)	**Total trading costs (% of initial wealth)**
BH	-1.77	2.34	0
ASY - SC	-0.62	4.14	0.12
ASY - PL	-0.49	4.40	0.25
mMRT	-1.25	3.20	27.22
MRT NC	1.24	6.11	0
ASY - SC NC	-0.54	3.96	0
ASY - PL NC	-0.10	4.86	0

We also notice here that in the tail events the differences have also become smaller. Their is almost no difference in the values for utility for the no-cost strategies and the asymptotically optimal with costs. Although, this could be said already for the previous setting. We see that the asymptotic strategies do improve the tail ESR and return, however when looking at the no-costs specification, we see that the asymptotic strategies do not perform near the optimum. Thus when a risk of misspecification is present, these asymptotic portfolios might not be a good method to decrease portfolio risk.

4.3.3 Results for assets with positive correlation

To simulate stock prices with positive correlation we consider $\sigma_{1,2} = 0.2$. The other parameters remain as in the above table.

Table 10: Key portfolio performance measures when asset prices are generated with positive correlation. The portfolios are a Buy-and-hold (BH) portfolio, the asymptotically optimal small costs approximation (ASY - SC) portfolio, asymptotically optimal polynomial (ASY - PL) portfolio, a monthly rebalanced Merton (mMRT) portfolio, and the Merton and asymptotically optimal without taking costs into account (MRT NC, ASY - SC NC and ASY - PL NC, respectively). We report the empirical value for the ESR (β), the increase in utility compared to the BH portfolio, the mean return, volatility of return and the trading costs

Portfolio	**Empirical ESR (%)**	**Increase in Utility (%)**	**Mean return (%)**	**Volatility of return (%)**	**Total trading costs (% of initial wealth)**
BH	2.77	0	20.21	15.34	0
ASY - SC	3.57	27.27	14.98	8.03	0.97
ASY - PL	3.67	29.88	15.21	7.80	1.02
mMRT	2.50	-11.36	11.91	5.43	3.91
MRT NC	4.50	50.00	14.34	4.93	0
ASY - SC NC	3.67	30.30	14.72	7.78	0
ASY - PL NC	3.84	34.12	14.53	7.21	0

We see that again the ranking of strategies on ESR is equal. The differences have again become smaller with ASY - SC, ASY - PL and BH, where ASY - SC outperforms BH by only 27 percent. Also the difference in standard deviation between the two is very small. However, the volatility of return of BH is almost double that of ASY - PL. We see again that total costs are even smaller than the independent assets scenario. What is noticeable is that even with higher total variance the difference between models has again decreased. This could imply that rebalancing works best against certain stock movements. In the 2D unit simplex, we could thus imagine that movement towards the hypotenuse are harder to rebalance. An explanation for this could be that to rebalance such a movement (to the hypotenuse) would require us to trade in both assets, whereas other movements would only require trading in one asset.

Table 11: Performance measures for tail events at the 95 percent level when assets have negative correlation. The portfolios are a Buy-and-hold (BH) portfolio, the asymptotically optimal (ASY) portfolio, a monthly rebalanced Merton (mMRT) portfolio, and the Merton and asymptotically optimal without taking costs into account (MRT NC, ASY - SC NC and ASY - PL NC, respectively)

Portfolio	**VaR95 ESR (%)**	**VaR95 return (%)**	**Total trading costs (% of initial wealth)**
BH	-1.68	2.99	0
ASY - SC	-0.43	4.04	0.13
ASY - PL	-0.41	4.09	0.15
mMRT	-0.89	3.98	23.06
MRT NC	1.23	6.43	0
ASY - SC NC	-0.37	4.23	0
ASY - PL NC	-0.33	4.66	0

We see that the tail event results are marginally better in the positive correlation scenario when compared to the other scenarios. However, we do not see a lot of change between strategies. It is also noticeable that the total trading costs is fairly low in this last scenario.

5 Discussion and conclusion

In this thesis, we have investigated a portfolio problem where proportional costs and market impact costs are considered. This problem arises when a trader, such as a pension fund, has trades of a large enough size to cause influence on the price. We start on the foundations of modern portfolio theory, which does not consider transaction costs. Cases with transaction costs proportional to the trade size have previously been considered. In more recent literature quadratic costs are considered, we continue this route. In literature approaches are taken for combinations of proportional costs and market-impact costs, as well as, single-asset and multi-asset portfolios. We look into the case of proportional and market-impact costs in a single-asset setting and also in a multi-asset setting. We define the state-variables on which the problem relies and by a dynamic programming approach we find a Hamilton-Jacobi-Bellman equation, which should be satisfied. This approach is used often in recent research. From this equation we derive an optimal trading strategy depending on a value function, which can be found by solving the equation. The equation we find is highly non-linear and we see that it is hard to solve, either analytically or numerically. To approximate the solution we continue on two roads. In literature a small costs approximation is found and we show that this can be used in our case as well. We also optimize a polynomial on the domain to approximate the value function. Both the small costs approximation and the polynomial approximation result in a linear (reduced) value function. The optimal strategy, we find also has an understandable form. We expect from the proportion costs setting that there is a no-trade zone and from the quadratic costs we expect a correction in trading size, which we both see. We see that due to the linear value function, that the trading strategy has a no-trade zone and is linear outside the no-trade zone for proportional and quadratic costs. If instead of quadratic costs, we take 1.5-power costs, we see trading change accordingly. We continue with these optimal strategies and make assumptions on the form of the value function. Simulating different scenarios tells us how the new trading strategy performs compared to other conventional strategies.

We find that the asymptotically optimal strategy outperforms both the buy-and-hold and a monthly rebalancing strategy in every scenario. We see that as correlation between assets moves form negative correlation to independence to positive correlation the differences become smaller. This shift in differences is notable as with the shift from negative to positive correlation we implicitly also increase the variation of the stock price, in turn this influences no-trade zones and trading speed. We see that We see that our method also outperforms all methods in tail events. Especially for positive correlation, but this was also seen in other scenarios, is that when costs are not taken into account the asymptotically optimal strategy still performs remarkably good compared to the optimal strategy, therefore we conclude that our proposed strategy is also quite robust for misspecifications. We see especially for monthly rebalancing that the associated costs really impact the performance.

Our proposed methods therefore provide a simple and intuitive trading strategy, which has good performance in a simulation setting. However, our method has not been proven for a non-stylized setting. In our research we have optimized a polynomial to minimize the sum of squares of the loss for the differential equation we have derived. However, we eventually look into how this performs in a simulation setting. We could also imagine optimizing the polynomial in the simulation setting. This approach is more suitable for future research for practitioners as it drifts away from the theoretical discussion in this paper. The field of transaction costs, particularly market-impact costs, still has lots of uncovered problems and the methods used in practice are heuristic at best. As we see that using an approximation to a theoretical optimum improves utility and returns drastically, there is still a lot to be gained in this field.

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A HJB equation derivation for the single-asset with constant risk-free rate setting

We use the process for wealth

$$\frac{dX_t}{X_t} = Y_t(\mu dt + \sigma dW_t) - \varepsilon |u_t| dt - \lambda |u_t|^{\alpha+1} dt \tag{29}$$

and the process for risky asset weight

$$\begin{aligned} dY_t = \big(Y_t(1-Y_t)(\mu - Y_t\sigma^2) + u_t + \varepsilon|u_t|Y_t + \lambda Y_t|u_t|^{\alpha+1}\big)dt \\ +\sigma Y_t(1-Y_t)dW_t \end{aligned} \tag{30}$$

to set up Îto's formula for the value function as follows

$$dV(t,x,y) = V_t dt + V_x dX_t + V_y dY_t + \frac{V_{xx}}{2} d\langle X_t, X_t\rangle + \frac{V_{yy}}{2} d\langle Y_t, Y_t\rangle + V_{xy} d\langle X_t, Y_t\rangle \tag{31}$$

$$\begin{aligned} &= V_t dt + V_x \left((\mu Y_t X_t - \varepsilon|u_t|X_t - \lambda|u_t|^{\alpha+1})dt + \sigma Y_t X_t dW_t\right) \\ &\quad + V_y \left((Y_t(1-Y_t)(\mu - \sigma^2 Y_t) + u_t + \varepsilon|u_t|Y_t + \lambda|u_t|^{\alpha+1}Y_t)dt + Y_t(1-Y_t)dW_t\right) \\ &\quad + \frac{V_{xx}}{2}\sigma^2 X_t^2 Y_t^2 dt + \frac{V_{yy}}{2}\sigma^2 Y_t^2(1-Y_t)^2 dt \\ &\quad + V_{xy}\sigma^2 X_t Y_t^2(1-Y_t)dt. \end{aligned} \tag{32}$$

As argued in Liu et al. (2017) the value function must be a supermartingale for every admissible strategy and a martingale for the optimal one by the martingale optimality principle of stochastic control. Thus focusing on the drift of the above function we find

$$\begin{aligned} 0 &= V_t + V_x \mu X_t Y_t + V_y Y_t(1-Y_t)(\mu - \sigma^2 Y_t) \\ &\quad + \sigma^2 \left(\frac{V_{xx}}{2} X_t^2 Y_t^2 + \frac{V_{yy}}{2} Y_t^2 (1-Y_t)^2 + V_{xy} X_t Y_t^2 (1-Y_t)\right) \\ &\quad + \max_{u_t} \left\{X_t V_x(-\varepsilon|u_t| - \lambda|u_t|^{\alpha+1}) + V_y(u_t + \varepsilon|u_t|Y_t + \lambda|u_t|^{\alpha+1}Y_t)\right\} \end{aligned} \tag{33}$$

By homotheticity of the power utility function and the assumption that in the long run utility grows at a constant exponential rate, we take the following ansatz for the value function. This ansatz is also used in the literature in Guasoni and Weber (2017), Guasoni and Weber (2018), Guasoni and Weber (2020) and Liu et al. (2017).

$$V(t,x,y) = \frac{x^{1-\gamma}}{1-\gamma} \exp\left\{(1-\gamma)\left(\beta(T-t) + \int_{y_0}^{y} q(z)dz\right)\right\} \tag{34}$$

combining this with (33) and dividing by $(1-\gamma)V(t,x,y)$ (we also replace X_t and Y_t by x and y, respectively). We find the following ordinary differential equation of $q(y)$

$$\begin{aligned}
0 = &-\beta + \mu y - \frac{\gamma\sigma^2}{2}y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q \\
&+ \frac{\sigma^2}{2}y^2(1-y)^2(q' + (1-\gamma)q^2) \\
&+ \max_u \left\{-\varepsilon|u| - \lambda|u|^{\alpha+1} + (u + \varepsilon|u|y + \lambda|u|^{\alpha+1}y)q\right\}.
\end{aligned} \tag{35}$$

If we split u into a positive and negative part, that is $u = u_+ - u_-$, we find

$$\begin{aligned}
0 = &-\beta + \mu y - \frac{\gamma\sigma^2}{2}y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q \\
&+ \frac{\sigma^2}{2}y^2(1-y)^2(q' + (1-\gamma)q^2) \\
&+ \max_{u_+\geq 0} \left\{-\varepsilon u_+ - \lambda u_+^{\alpha+1} + (u_+ + \varepsilon u_+ y + \lambda u_+^{\alpha+1}y)q\right\} \\
&+ \max_{u_-\geq 0} \left\{-\varepsilon u_- - \lambda u_-^{\alpha+1} + (-u_- + \varepsilon u_- y + \lambda u_-^{\alpha+1}y)q\right\}.
\end{aligned} \tag{36}$$

Solving this we find that the optimal u_+ and u_- are given as follows

$$u_\pm = \max\left\{0, \left[\frac{1}{\lambda(\alpha+1)}\left(\pm\frac{q(y)}{1-yq(y)} - \varepsilon\right)\right]^{1/\alpha}\right\}. \tag{37}$$

Based on previous literature and following the methods of Liu et al. (2017), we assume a that the no balancing region $\left\{y : -\varepsilon < \frac{q(y)}{1-yq(y)} < \varepsilon\right\}$ is given by an interval $[y_-, y_+]$. The filling in the optimal trading strategy and the no-trade region in the differential equation gives us

$$\begin{aligned}
0 = &-\beta + \mu y - \frac{\gamma\sigma^2}{2}y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q \\
&+ \frac{\sigma^2}{2}y^2(1-y)^2(q' + (1-\gamma)q^2) \\
&+ \begin{cases}
\alpha\lambda^{-1/\alpha}(\alpha+1)^{-\frac{\alpha+1}{\alpha}} \frac{(q-\varepsilon(1-yq))^{\frac{\alpha+1}{\alpha}}}{(1-yq)^{1/\alpha}}, & \text{if} \quad y \in [0, y_-) \\
0, & \text{if} \quad y \in [y_-, y_+] \\
\alpha\lambda^{-1/\alpha}(\alpha+1)^{-\frac{\alpha+1}{\alpha}} \frac{(-q-\varepsilon(1-yq))^{\frac{\alpha+1}{\alpha}}}{(1-yq)^{1/\alpha}}, & \text{if} \quad y \in (y_+, 1]
\end{cases}.
\end{aligned} \tag{38}$$

As there are four unknown elements in this equation, we will need an additional four conditions. We find these by taking limits at the boundaries and imposing continuity restrictions at the no-trade boundary. We find the following conditions

$$q(0^+) = \varepsilon + (\alpha+1)\left(\lambda\beta^\alpha\alpha^{-\alpha}\right)^{\frac{1}{\alpha+1}}, \tag{39}$$

$$q(y_-) = \frac{\varepsilon}{1-y_-\varepsilon}, \tag{40}$$

$$q(y_+) = \frac{-\varepsilon}{1-y_+\varepsilon}, \tag{41}$$

$$\frac{(-q(1^+)-\varepsilon(1-q(1^+)))^{\frac{\alpha+1}{\alpha}}}{(1-q(1^+))^{1/\alpha}} = \left(\beta-\mu+\frac{\gamma\sigma^2}{2}\right)\alpha\lambda^{1/\alpha}(\alpha+1)^{\frac{\alpha+1}{\alpha}}, \tag{42}$$

where $q(0^+)$ and $q(1^-)$ denote $\lim_{y\downarrow 0} q(y)$ and $\lim_{y\uparrow 1} q(y)$ respectively.

B HJB equation derivation for the multi-Asset with O-U risk-free rate setting

The derivation of the differential equation used in the multi-asset setting is primarily the same as the single-asset setting. We derive both the differential equation in a setting with wealth and a setting with funding ratio. In essence the single-asset and multi-asset setting are similar, although some terms from correlation between Brownian motions of assets and liabilities might arise in a setting with funding ratio. We will denote $K = K(\boldsymbol{\varepsilon}, \boldsymbol{u}) = \boldsymbol{\varepsilon}_1'|\boldsymbol{u}| + \boldsymbol{\varepsilon}_2'|\boldsymbol{u}|^{\alpha+1}$, which should be evaluated elementwise on some operations, as the transaction cost function. In the follow we will assume two assets and all matrix dimensions to be such that the products work out. We assume the risk-free rate follows an Ohrstein-Uhlenbeck process $dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r dW_t^r$. We assume the ith risky asset follows $\frac{dS_t^i}{S_t^i} = (r_t + \boldsymbol{\sigma_i'\lambda})dt + \boldsymbol{\sigma}_i' d\boldsymbol{W_t}$, where $\boldsymbol{W_t}$ is an independent two-dimensional Brownian motion and $\langle dW_t^r, d\boldsymbol{W}_t \rangle = \left(\begin{smallmatrix}\rho_1 \\ \rho_2\end{smallmatrix}\right) dt$. We also introduce the liability process L_t as given by a bond with maturity τ_L, this follows $\frac{dL_t}{L_t} = (r_t - D_L\sigma_r\lambda_r)dt - D_L\sigma_r dW_t^r$. For simplicity we denote $\boldsymbol{\sigma}$ as the matrix of individual $\boldsymbol{\sigma}_i'$ stacked and we denote $\boldsymbol{\Sigma} = \boldsymbol{\sigma\sigma'}$ The cash position follows

$$dC_t = r_t C_t dt - \boldsymbol{S}_t' d\boldsymbol{\theta}_t - K X_t dt, \tag{43}$$

as derived in the single-asset case. Following from the above and defining the wealth as $X_t = \boldsymbol{\theta}_t'\boldsymbol{S}_t + C_t$ and the risky asset weight $y_t^i = \frac{\theta_t^i S_t^i}{X_t}$, an application of Îto's lemma shows that

$$\frac{dX_t}{X_t} = (r_t + \boldsymbol{y}_t'\boldsymbol{\sigma\lambda} - K)dt + \boldsymbol{y}_t'\boldsymbol{\sigma} d\boldsymbol{W_t} \tag{44}$$

$$\begin{aligned} d\boldsymbol{y}_t = (\boldsymbol{u}_t + P(\boldsymbol{y}_t)(\boldsymbol{\sigma\lambda} - \boldsymbol{\Sigma}\boldsymbol{y}_t) + K(\boldsymbol{\varepsilon}, \boldsymbol{u}_t)\boldsymbol{y}_t)dt \\ + P(\boldsymbol{y}_t)\boldsymbol{\sigma} d\boldsymbol{W_t}. \end{aligned} \tag{45}$$

With these processes for the state-variables of the value-function. We find

$$dV_t = \partial_t v dt + \partial_x v dX_t + (\nabla_{\boldsymbol{y}} v)' d\boldsymbol{y}_t + \frac{1}{2}\partial_{xx} v d\langle X_t, X_t\rangle + \partial_x (\nabla_{\boldsymbol{y}} v)' d\langle X_t, \boldsymbol{y}_t\rangle + \frac{1}{2}\langle d\boldsymbol{y}_t, D_{\boldsymbol{y}}^2 v d\boldsymbol{y}_t\rangle \tag{46}$$

Using a similar ansatz as the single-asset case which is also seen in Guasoni and Weber (2018). $v(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} \exp\{(1-\gamma)(-\beta(T-t) + w(\boldsymbol{y}))\}$, setting $\boldsymbol{q} = \nabla_{\boldsymbol{y}} w$, and seeing that

$$\begin{aligned} d\langle X_t, X_t\rangle &= X_t^2 \boldsymbol{y}_t' \boldsymbol{\Sigma} \boldsymbol{y}_t dt \\ d\langle X_t, \boldsymbol{y}_t\rangle &= X_t P(\boldsymbol{y}_t) \boldsymbol{\Sigma} \boldsymbol{y}_t dt \\ \langle d\boldsymbol{y}_t, D_{\boldsymbol{y}}^2 v d\boldsymbol{y}_t\rangle &= \mathrm{Tr}\left[P(\boldsymbol{y}_t)\boldsymbol{\Sigma} P(\boldsymbol{y}_t) D_{\boldsymbol{y}}^2 v\right] dt. \end{aligned}$$

We also see that the ansatz implies

$$
\begin{aligned}
X_t \partial_x v &= (1-\gamma) v & X_t^2 \partial_{xx} v &= -\gamma(1-\gamma) v \\
\nabla_{\boldsymbol{y}} v &= (1-\gamma) v \boldsymbol{q} & D_{\boldsymbol{y}}^2 v &= (1-\gamma) v ((1-\gamma)\boldsymbol{q}\boldsymbol{q}' + \boldsymbol{J}\boldsymbol{q}) \\
X_t \partial_x (\nabla_{\boldsymbol{y}} v) &= (1-\gamma)^2 v \boldsymbol{q},
\end{aligned}
$$

where $\boldsymbol{J}\boldsymbol{q}$ denotes the Jacobian operator. As we are only interested in the drift part of the process, we can fill in the following:

$$
\begin{aligned}
dV_t = (...)d\boldsymbol{W}_t + \Big(& -\beta + \boldsymbol{y}_t' \boldsymbol{\sigma}\boldsymbol{\lambda} - K && (47)\\
& + \boldsymbol{q}'(\boldsymbol{u}_t + P(\boldsymbol{y}_t)(\boldsymbol{\sigma}\boldsymbol{\lambda} - \boldsymbol{\Sigma}\boldsymbol{y}_t) + K\boldsymbol{y}_t) && (48)\\
& + (1-\gamma)\boldsymbol{q}' P(\boldsymbol{y}_t)\boldsymbol{\Sigma}\boldsymbol{y}_t - \gamma \boldsymbol{y}_t \boldsymbol{\Sigma}\boldsymbol{y}_t \\
& + \frac{1}{2}\operatorname{Tr}[P(\boldsymbol{y}_t)\boldsymbol{\Sigma}P(\boldsymbol{y}_t)((1-\gamma)\boldsymbol{q}\boldsymbol{q}' + \boldsymbol{J}\boldsymbol{q})] \Big) dt.
\end{aligned}
$$

The process should be a martingale for the optimal value function and thus for an optimal $\boldsymbol{u}$ the drift will be equal to 0. The optimal value for $\boldsymbol{u}_t$ will be divided in the optimal buy u_t^+ and optimal sell policy u_t^-. The optimal values are

$$
u_{it}^{\pm} = \max \pm \left[0, \left(\frac{-\varepsilon_{1i}(1-\boldsymbol{q}'\boldsymbol{y}) \pm q_i}{(\alpha+1)\varepsilon_{2i}(1-\boldsymbol{q}'\boldsymbol{y})} \right)^{1/\alpha} \right] \tag{49}
$$

This divides the optimal policy for every asset into three areas, in case of two assets we thus have $3^2 = 9$ different areas. To incorporate this into the equation (47), we define

$$
g_i(\boldsymbol{q}_t, \boldsymbol{y}_t) = \begin{cases} q_{it}u_{it}^+ - (1-\boldsymbol{y}_t'\boldsymbol{q}_t)(\varepsilon_{1i}u_{it}^+ + \varepsilon_{2i}(u_{it}^+)^{\alpha+1}), & \text{if} \quad \varepsilon_{1i} < \frac{q_{it}}{1-\boldsymbol{y}_t'\boldsymbol{q}_t} \\ 0, & \text{if} \quad -\varepsilon_{1i} \le \frac{q_{it}}{1-\boldsymbol{y}_t'\boldsymbol{q}_t} \le \varepsilon_{1i} \ , \\ -q_{it}u_{it}^- - (1-\boldsymbol{y}_t'\boldsymbol{q}_t)(\varepsilon_{1i}u_{it}^- + \varepsilon_{2i}(u_{it}^-)^{\alpha+1}), & \text{if} \quad -\varepsilon_{1i} > \frac{q_{it}}{1-\boldsymbol{y}_t'\boldsymbol{q}_t} \end{cases} \tag{50}
$$

. Then the PDE we must solve will be equal to (26).

If we were to extend the same to a setting with funding ratio, which equals $F_t = \frac{X_t}{L_t}$. Where L_t follows

$$
\frac{dL_t}{L_t} = (r_t - D_L\sigma_r\lambda_r)dt + D_L\sigma_r dW_t^r. \tag{51}
$$

We should replace all processes of X_t with F_t which follows

$$
\frac{dF_t}{F_t} = (\boldsymbol{y}_t\boldsymbol{\sigma}\boldsymbol{\lambda} - K(\boldsymbol{\varepsilon}, \boldsymbol{u}_t) + D_L\sigma_r\lambda_r + \frac{1}{2}D_L^2\sigma_0^2)dt + \boldsymbol{y}_t'\boldsymbol{\sigma}d\boldsymbol{W}_t + D_L\sigma_r dW_t^r. \tag{52}
$$

Possible correlation between the Brownian motion of bonds and risky assets causes extra terms, if we let $\langle dW^r, d\boldsymbol{W}_t\rangle = \boldsymbol{rho}dt$, we see

$$d\langle F_t, F_t\rangle = F_t^2(\boldsymbol{y}_t\boldsymbol{\Sigma}\boldsymbol{y}_t + D_L^2\sigma_r^2 + 2D_L\sigma_r\boldsymbol{y}_t'\boldsymbol{\sigma}\boldsymbol{\rho})dt \tag{53}$$

$$d\langle F_t, \boldsymbol{y}_t\rangle = F_t(P(\boldsymbol{y}_t)\boldsymbol{\Sigma}\boldsymbol{y}_t + D_L\sigma_r P(\boldsymbol{y}_t)\boldsymbol{\sigma}\boldsymbol{\rho})dt. \tag{54}$$

Seeing that this does not affect the optimal $\boldsymbol{u}_t$, the previous optimal policy still holds. The equation we have to solve is

$$\begin{aligned} 0 = &-\beta + \boldsymbol{y}'\boldsymbol{\sigma}\boldsymbol{\lambda} + D_L\sigma_r(\lambda_r + \frac{1}{2}D_L\sigma_r) \\ &+ (\boldsymbol{\sigma}\boldsymbol{\lambda} - \gamma\boldsymbol{\Sigma}\boldsymbol{y} - D_L\sigma_r\boldsymbol{\sigma}\boldsymbol{\rho})'P(\boldsymbol{y})\boldsymbol{q} \\ &- \frac{1}{2}\gamma(\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y} + D_L\sigma_r(D_L\sigma_r + 2\boldsymbol{y}_t'\boldsymbol{\sigma}\boldsymbol{\rho})) \\ &+ \mathrm{Tr}[P(\boldsymbol{y})\boldsymbol{\Sigma}P(\boldsymbol{y})((1-\gamma)\boldsymbol{q}\boldsymbol{q}' + \boldsymbol{J}\boldsymbol{q})] \\ &+ g_1(\boldsymbol{q}, \boldsymbol{y}) + g_2(\boldsymbol{q}, \boldsymbol{y}), \end{aligned} \tag{55}$$

C No-trade zones

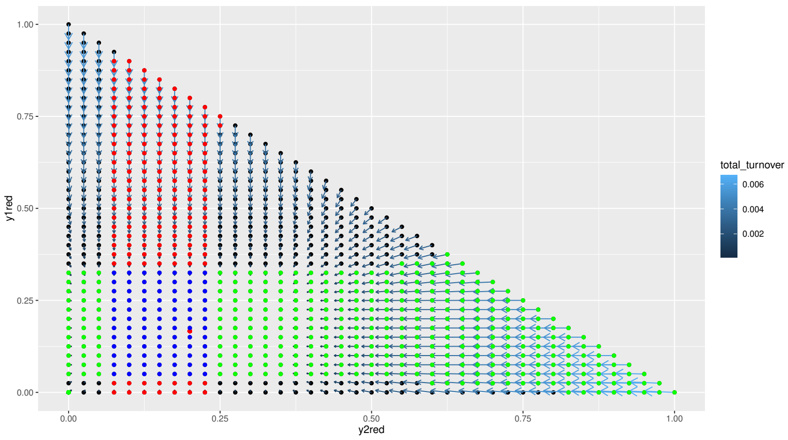

Figure 6: The optimal trading strategy in terms of turnover with marked no-trade zones.

In the above plot we can see the no-trade zone specified for each proportion of assets held. We see that the no-trade zone is approximately a square. This makes sense as we could look at the form of $q(y)/(1 - yq)$ in the single-asset case. This term can be found in the optimal trading strategy. Filling in the asymptotic approximation leads to a linear term in the numerator and a quadratic term in the denominator, due to the coefficients of these polynomial we get a function that is almost linear around 0 and thus the no-trade zone will be of this square form.