ERASMUS UNIVERSITY ROTTERDAM

Erasmus School of Economics

Evaluation of VaR estimates based on second order expansion

Bachelor Thesis Financial Econometrics

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July 4, 2021

Abstract

This paper consists of two parts. First, the influence of applying the second order expansion on the tails of two assets is investigated. Similar to previous research, it is found that using this expansion results in a non-corner solution for a portfolio with US stocks and bonds and for a portfolio with the French stocks L'Oreal and Thomson CSF. In the second part of this study, the accuracy of VaR estimates, computed while using the second order expansion, is evaluated based on two tests. It turns out that VaR estimates with a very low probability level are quite accurate and VaR estimates corresponding to a higher probability level are not independent for most of the portfolios.

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1 Introduction

In this research, the problem of constructing a portfolio from stocks and bonds is analysed based on a downside risk constraint. This study replicates the techniques used by Hyung and De Vries (2007) and applies these on different data to evaluate whether the same conlusions hold. Moreover, the accuracy of the computation of Value at Risk (VaR) estimates based on the second order expansion is evaluated. For the first part, I analyse whether taking second order terms into account for the tail distribution, leads to not ending up in a corner solution with only bonds, which tends to happen when only considering a first order term. Hyung and De Vries (2007) investigate this problem and find that if second order terms are taken into account, this leads to an optimal portfolio with both stocks and bonds. This is due to the fact that the tail probabilities and the Value at Risk (VaR) are convex which is also proven in their research. I perform a similar research as Hyung and De Vries (2007), but with more recent data for the US stocks and bonds.

In addition, the research is extended by evaluating the accuracy of the VaR estimates obtained with the second order expansion. Jansen et al. (2000) compute VaR estimates for a portfolio of stocks and bonds when only the first order tail index parameter is taken into account and these estimates differ from the estimates computed by Hyung and De Vries (2007) where also the second order tail index is taken into account. Since Hyung and De Vries (2007) state that using the second order expansion results in an optimal portfolio with less downside risk and more potential on high returns, it is interesting to analyse whether the VaR estimates calculated with the second order expansion are accurate, which is the main goal of this research. To evaluate accuracy, independence and correct unconditional coverage of the VaR estimates are tested.

Similar to Hyung and De Vries (2007), I find that including second order terms for tail probability estimation, results in a non-corner solution for a portfolio with US stocks and bonds and for a portfolio with the French stocks L'Oreal and Thomson CSF. Also, I find that VaR estimates have correct unconditional coverage for three different probability levels. However, only for a very low probability level, the VaR estimates are also independent for all portfolios. The results are relevant for companies and institutions active in asset management, and especially for banks and pension funds since they have a strict policy concerning risk and this research constructs an optimal portfolio based on minimizing downside risk. Also for wealthy individual investors, the research might be useful since they can use it to construct an optimal portfolio on their own. For researchers in extreme value theory and portfolio optimization this research might be interesting because of the use of more recent data with which old findings are confirmed and because of the evaluation of the accuracy of VaR estimates obtained with the second order expansion.

The rest of this paper is structured as follows: in Section 2, the literature review is presented and Section 3 describes the theoretical framework. Section 4 covers the data, and Section 5 contains the methodology. In Section 6 the results are described and Section 7 is the conclusion.

2 Literature review

According to Rodriguez (2007), multivariate EVT has its limitations since an asymptotic dependency structure is assumed. This implies that observations in the tails are dependent on another, which is not always the case. As a result, the risk might be overestimated. Another downside is finding extreme observations, according to Rodriguez (2007) and Bensalah (2000). There are multiple ways to define extreme observations, but there is not a clear best approach. In this paper EVT is used as as well. However, in spite of dealing with multiple assets, I do not use multivariate EVT since the problem is a marginal one. EVT is applied in this study in a similar way as in Hyung and De Vries (2007) and Jansen et al. (2000). To determine the downside risk of a portfolio, one can apply the first order convolution approach or the second order expansion. Jansen et al. (2000) make use of this first order approach when constructing a portfolio of stocks and bonds and another portfolio with two stocks. When evaluating the first order convolution results, one tends to end up in a corner solution with only the asset with the highest tail coefficient (lower downside risk), whereas applying a second order asymptotic expansion yields a more balanced solution according to Hyung and De Vries (2007).

Gourieroux et al. (2000) analyse the sensitivity of VaR with respect to portfolio allocation and provide an empirical analysis of two French stocks. Gourieroux et al. (2000), Jansen et al. (2000), Hyung and De Vries (2007) all compute VaR estimates to evaluate downside risk and compare different portfolios. With these VaR esimates, the safety first criterion is computed. The safety first criterion is an accurate criterion to evaluate downside risk and is also used by Arzac and Bawa (1997).

Wong (2010) discusses a major shortcoming of VaR analysis, namely that losses beyond the threshold are not taken into account. However, VaR estimates are still a widely used tool to evaluate downside risk and also used in this research. Various methods have been proposed to evaluate the accuracy of VaR estimates. Kupiec (1995) defined the test for unconditional coverage, which evaluates whether the number of observations that fall below the VaR estimate are in line with the VaR probability level. This is done by means of a likelihood ratio test. Berkowitz and O'Brien (2002) investigated the performance of VaR estimates for six large US banks and found that the VaR estimates were quite conservative, so they easily passed the unconditional coverage test. However, they did find clustering in VaR violations due to difficulty in forecasting volatility. Since bankruptcy risk for companies active in asset management is much higher when violations occur in a short period of time, Christoffersen (2003) proposed an independence test with the goal of evaluating VaR estimates based on the clustering of violations, by means of a likelihood ratio test statistic. Iorgulescu (2012) compared eight different VaR models based on the unconditional coverage test and the independence test. Daily returns from the Romanian capital market are analysed and the first four years from the sample are used to estimate different (GARCH) VaR models, whereas the last year is used for backtesting. In this study, I analyse the accuracy of VaR estimates that are constructed with a model based on the second order expansion. In this way, the current literature is extended.

3 Theoretical framework

Extreme Value Theory (EVT) can be used to estimate the tail structure of asset returns because they usually have fatter tails than a normal distribution. The tails of the distributions are modeled by a power law instead of decaying at an exponential rate. Hyung and De Vries (2007) assume that the tails of two assets are different but symmetric, that returns are independent identically distributed (i.i.d) and that the tails vary regularly at infinity. This implies the first order equation

$$P(X > s) = As^{-\alpha} + o(s^{-\alpha}) \tag{1}$$

as s $\rightarrow \infty$. Where $A \geq 0$, $\alpha \geq 0$ and X the return of an asset. The little o notation is elaborated on in the appendix. Hyung and De Vries (2007) also apply the second order expansion which includes a second tail index parameter and, for asset i, is given by:

$$P(X_i > s) = P(X_i < s) = A_i s^{-\alpha_i}(1 + B_i s^{-\beta_i} + o(s^{-\beta_i})) \tag{2}$$

as $s \rightarrow \infty$. When the condition α_2 - α_1 $<$ $\min(\beta_1, 1)$ is satisfied, applying the second order expansion as given in equation (2) leads to an optimal portfolio with a non-corner solution, with α_1 and α_2 the tail index parameters for assets 1 and 2, and β_1 the second order tail index parameter of asset 1. This can be proven by proving the convexity of the tail probability and the VaR. For the convexity of the tail probability, the following statement holds (with X_1 and X_2 representing assets):
There exists a $w^* \in (0, 1)$ *for given large* s *such that:*

$$P(w^*X_1 + (1 - w^*)X_2) > s) \leq P(wX_1 + (1 - w)X_2) > s) \tag{3}$$

any $0 \leq w \leq 1$. *The equality holds only when* $w^* = w$.

For the VaR convexity the following statement holds:
There exists a $w^* \in (0, 1)$ *for given probability level* $\bar{p}$ *such that:*

$$VaR(w^*, \bar{p}) \leq VaR(w, \bar{p}) \tag{4}$$

for any $0 \leq w \leq 1$. *The equality holds only when* $w^* = w$.

Statements (3) and (4) are from Hyung and De Vries (2007) and the proof is also provided in their paper. The proof is given for the upper tail case and the proof for the lower tail case is similar with only small modifications. Both proofs follow from computing the first and second order derivatives, which imply that there is an interior minimum, meaning that the optimal portfolio with the lowest downside risk is a diversified one consisting of both asset 1 and 2. The portfolio does not end up in a corner-solution with only the less riskier asset as might have been the case if a first order approach was used.

4 Data

I evaluate monthly simple returns of a US stock index and US government bonds with 10 year maturity over the period January 1947 until December 2020, so in total 888 observations. The data from the US stock index is obtained from CRSP and US bond data is obtained from Swinkels (2019) via the data repository of the Erasmus University Rotterdam. In addition, I analyse daily simple returns of the French stocks Thomson CSF (currently known as Thales Group) and L'Oreal over the period 01-04-1997 until 04-05-1999, so in total 546 observations. The data on these stocks is obtained from Datastream. In table 1, the descriptive statistics are shown for the returns on all four assets. Stocks are considered more risky in general, which is also confirmed by the descriptive statistics with a higher standard deviation and geometric mean for stocks. L'Oreal has a much higher geometric mean return than Thomson CSF. The standard deviation however, does not differ much, indicating similar volatility for both assets. All assets have nonzero skewness and a kurtosis higher than 3. The excess kurtosis implies that the data is heavy-tailed compared to a normal distribution.

Table 1: Descriptive statistics for returns on US stocks and bonds and French stocks

	US Bonds	US stocks	L'Oreal	Thomson CSF
Geometric mean	0.00450	0.00881	0.00128	0.00011
Standard deviation	0.02071	0.04263	0.02610	0.02897
Skewness	0.562	-0.524	0.192	-0.107
Kurtosis	5.940	4.979	4.341	4.026

5 Methodology

In this research, I first investigate whether applying the second order expansion leads to a non-corner solution for a portfolio with two assets. The problem arises of choosing an appropriate threshold for the lower tail. For a threshold too low, the number of observations below the threshold is very small, leading to high variance and an unreliable result for the tail index estimation. For a threshold too high, the tail does not satisfy the convergence criterion which results in large bias. So the problem can be considered a bias variance trade-off (Pfaff, 2016). Similar to Jansen et al. (2000), I estimate the tail fraction with the bootstrap method from Hall (1990). When the number of tail observations is known, the Hill estimator is used to compute the tail index estimator. After this, the tail index estimator, threshold value and number of observations in the tail of both assets are used to compute VaR estimates for different portfolios with two assets. This is done for a portfolio with US stocks and bonds over the period 1947-2020 (monthly returns) and for a portfolio with the two French stocks L'Oreal and Thomson CSF over the period 01-04-1997 until 04-05-1999 (daily returns). Then, these VaR estimates are used to compute the safety first criterion for the different portfolios. This criterion takes both downside risk and expected returns into account. Based on the VaR and the safety first criterion, the different portfolios are compared. Subsequently, the accuracy of the VaR estimates, obtained with the second order expansion, is evaluated for a portfolio with US stocks and bonds. This is done by means of the unconditional coverage test and the independence test. Monthly simple returns from January 1947 until December 2003 are used to estimate the model and create VaR estimates and monthly observations in the period ranging from January 2004 until December 2020 are employed for backtesting.

5.1 Extreme Value Theory

As explained in the theoretical framework, EVT can be applied to estimate the tail structure of asset returns. This is also done in this research in a similar fashion as in Hyung and De Vries (2007). I estimate the tail index parameter α in equation (1) with the Hill moment estimator, as is done by Jansen et al. (2000). This is done for both assets individually and the estimator is given by $\hat{\alpha} = \hat{\gamma}^{-1}$ with $\hat{\gamma}$ as follows:

$$\hat{\gamma} = \frac{1}{m}\sum_{i=1}^{m} logX_{(n-i+1)} - logX_{(n-m)} \tag{5}$$

With the sample ordered as $X_n \geq X_{n-1} \geq ... \geq X_1$. I use the lowest m observations to estimate the tail index parameter. In practice, this means that all returns are multiplied by -1 and then the formula above is used to compute the tail index estimator for the lower tail. In order to use this estimator, a decision for the value of m needs to be made. Similar to Jansen et al. (2000), I use the bootstrap method from Hall (1990) to estimate an appropriate value for m. To apply this bootstrap method and estimate the tail index estimator α, I use the package *tea* in R. Jansen et al. (2000) use the first order theorem and end up in a corner solution with only bonds which have a higher tail index parameter indicating a thinner tail and lower downside risk. Hyung and De Vries (2007) apply the second order expansion which is given by equation (2) in the theoretical framework. I also apply this second order expansion to analyse whether this leads to a non-corner solution. In order to estimate the second order tail index parameter β, I use the following relation from Hyung and De Vries (2007):

$$\beta/\alpha = \frac{ln(k)}{2ln(n) - 2ln(k)} \tag{6}$$

5.2 VaR and safety first criterion

To evaluate downside risk I make use of the VaR. The higher the VaR value for a certain probability, the lower the downside risk for a portfolio. Since the goal is to create a portfolio which minimizes downside risk, the VaR is an appropriate metric. With the VaR estimates for different portfolios, it can be practically evaluated whether a corner solution with only bonds has the lowest downside risk or that the second order expansion provides a non-corner solution with the lowest downside risk. Similar to Hyung and De Vries (2007) I first verify whether the condition for an interior solution is satisfied. The condition, as mentioned earlier, is as follows:

$$\alpha_2 - \alpha_1 < min(\beta_1, 1) \tag{7}$$

Where β_1 can be computed with the ratio from equation (6). If the condition above is satisfied, the VaR estimates for a portfolio with two assets can be obtained by solving the following formula for q_δ

$$w^{\alpha_1} A_1 q_\delta^{-\alpha_1} + (1-w)^{\alpha_2} A_2 q_\delta^{-\alpha_2} = \delta \tag{8}$$

With q_δ the VaR estimate with corresponding probability δ and with $A_i = \frac{m_i}{n} X_{(m_i)}^{\alpha_i}$. To solve equation (8), I use the package *Rootsolve* in R.
With the VaR estimates, the corresponding safety first criterion value can be computed. This criterion takes both upside potential and downside risk into account and is as follows:

$$\max_{w_i} \quad \frac{E(R) - r}{r - q_\delta(R)}$$

Where w_i is the weight invested in asset i. R $= \sum w_i V_{it+1} / \sum w_i V_{it}$ with V_{it} the value of asset i at time t. $E(R)$ represents the expected value of R. For this expected return, the geometric mean return is used and for a portfolio with two assets, the expected return is based on the weights of the two assets. The geometric mean return is used because it takes into account the effect of compounding. $q_\delta(R)$ is the VaR for probability value δ and r represents the risk-free rate of return.

5.3 Extension methodology

VaR estimates for a portfolio consisting of US stocks and bonds are computed based on the second order expansion. The same steps are applied as the ones mentioned in section 5.1 and 5.2. However,this time the estimates are computed while only using part of the data, namely January 1947 until December 2003. In this way, 684 observations are used to estimate the VaR values. This specific sample period is chosen because it takes monthly data into account for 57 years, which is long enough to create a proper model with which VaR estimates can be computed, and because it contains the deep low in stock markets in the early 2000's as a result of the dot-com bubble and 9-11. When the VaR estimates are computed, the monthly portfolio returns over the period ranging from January 2004 until December 2020 are used to evaluate the accuracy of the estimates. This out-of-sample backtest period contains 204 estimates which does not differ much from the 244 observations in the backtest period used by Iorgulescu (2012). Especially interesting is that the out-of-sample period used in this research contains two crises, namely the financial crisis and the COVID-19 crisis with high volatility and negative returns on stock markets on a global scale.
There are other ways to evaluate the accuracy of the VaR estimates. One option is to compute VaR estimates over the period 1947 until 2020 and then evaluate them based on the returns over this period. However, it is more useful to analyse how the VaR estimates perform in the future, which is done with the method employed in this study, than how they perform in the past. Another possibility to evaluate accuracy is by using a moving window where the tail index parameters for both assets are estimated again each month and new VaR estimates are computed each month. However, since many companies active in asset management have a long-term and risk averse investment strategy, they hold their positions for a long time. They do not make considerable changes to their portfolio on a monthly basis so in this way it is an unrealistic scenario to estimate the VaR values again each month and make major adjustments to the portfolio based on these estimates. Of course, holding the same portfolio for 17 years is also not always the case, but for investors with a long-term investment strategy this is more realistic than making substantial changes very often. Similar to Iorgulescu (2012), I evaluate the accuracy of the VaR estimates

based on two criteria: correct unconditional coverage and independence. Another option is to combine both tests, but in this study is chosen to perform the tests separately to obtain a more detailed view on the specific areas in which the estimates are adequate or less adequate. Both tests are applied for the portfolios consisting of US stocks and/or bonds. VaR estimates with probabilities 0.05, 0.01 and 0.0049(1/204) are evaluated. The probability level of 0.05 is chosen because more violations (returns below the VaR estimate) are expected and this makes the test more reliable. Probability levels 0.01 and 0.0049 are chosen because these values are in line with the low probability levels, corresponding to very few violations, used for constructing VaR estimates by Hyung and De Vries (2007).

5.3.1 Correct unconditional coverage

Correct unconditional coverage means that the fraction of observations below the VaR estimate should be equal to the nominal coverage probability q, which is the probability for which the VaR estimate is computed. The indicator function I_{t+i} is defined as

$$I_{t+i} = \begin{cases} 1, \text{ if } r_{t+i} < VaR_t(1-q,1) \\ 0, \text{ if } r_{t+i} > VaR_t(1-q,1) \end{cases}$$

Here I_{t+i} indicates a "violation" of the VaR. As stated above, proper VaR estimates should have a fraction of violations equal to the nominal coverage probability. So the nul hypothesis can be stated as follows:

$$H0 : P(I_{t+i} = 1) = E(I_{t+i}) = q$$

Assuming independence of I_{t+1}, I_t, etc., the hypothesis can be tested with a likelihood ratio test with the following test statistic:

$$LR_{uc} = -2ln\frac{(1-q)^{T_0}q^{T_1}}{(1-\pi)^{T_0}\pi^{T_1}} \sim \chi^2(1)$$

Here T_0 is the number of observations that do not violate the VaR estimate and T_1 is the number of violations. q is the value of p under the null hypothesis of correct unconditional coverage. Under the alternative hypothesis $p = \pi$ for some $\pi \neq$ q with π the empirical coverage value represented by $\pi = \frac{T_1}{T_0+T_1}$. The higher the difference between q and π, the higher the value of the test statistic. So to perform this test, first the number of violations T_1 is computed, then π is calculated with the value for T_1 and then the Likelihood ratio test statistic is computed, based on which conclusions on correct unconditional coverage of the VaR estimates are made.

5.3.2 Independence

The estimates are also tested on independence, meaning that occurrences of observations that are below the threshold value should be spread over the sample and not occur in clusters. The null hypothesis is given by:

$$H0 : P(I_{t+1} = 1|I_t) = P(I_{t+1}), \quad \text{for all t}$$

The likelihood ratio test can be used to evaluate the null hypothesis and the test statistic is given by:

$$LR_{ind} = -2ln \frac{(1-\pi_2)^{(T_{00}+T_{10})}\pi_2^{(T_{01}+T_{11})}}{(1-\pi_{01})^{T_{00}}\pi_{01}^{T_{01}}(1-\pi_{11})^{T_{10}}\pi_{11}^{T_{11}}} \sim \chi^2(1)$$

Where T_{00} is the number of times that two consecutive observations do not violate the VaR estimate and T_{01} is the number of times that one observation does not violate the VaR estimate and the following observations does. T_{10} counts the number of times that one observation does violate the VaR estimate and the following observation does not. T_{11} represents the number of times that two consecutive observations violate the VaR estimate. Further, $\pi_{01} = \frac{T_{01}}{T_{01}+T_{11}}$, $\pi_{11} = \frac{T_{11}}{T_{10}+T_{11}}$ and $\pi_2 = \frac{T_{01}+T_{11}}{T_{00}+T_{01}+T_{10}+T_{11}}$. Under the null hypothesis, $\pi_{01} = \pi_{11} = \pi_2$, indicating an equal probability of violation of the VaR estimate regardless of whether the previous observation has been a violation or not. So to perform this test, first the values for T_{00}, T_{01}, T_{10} and T_{11} are computed based on the violations, then the corresponding ratios π_{01}, π_{11} and π_2 are calculated and then the test statistic is computed based on which conclusions on the independence of VaR violations are made.

6 Results

6.1 Tail estimation and VaR estimates

In table 2, results for the tail index estimation are shown for the US assets and French stocks. The first order tail index parameter is lower for US stocks implying higher risk. Also, the m-th lowest return, represented by X_m is more than twice as low for US stocks as for US bonds and the VaR for probability 1/2n is much lower for US stocks, also indicating higher risk for stocks. When looking at the two French stocks the first order tail index parameters do not differ much, implying similar downside risk. The m-th lowest return is higher for L'Oreal, but this might be due to a higher number of observations that are considered to be in the tail (38 versus 26). The VaR for probability 1/2n is slightly lower for Thomson CSF implying more downside risk, but the difference is small.

Table 2: Tail estimation statistics for simple returns on US stocks and bonds and French stocks

	US Bonds	US stocks	L'Oreal	Thomson CSF
No. Observations	888	888	546	546
m	19	25	38	26
X_m	-0.0387	-0.0793	-0.0351	-0.0496
α	4.185	3.513	3.239	3.487
$q_{1/2n}$	-0.092	-0.241	-0.134	-0.154

Note: m is the number of observations that form the tail, X_m shows the m-th lowest observation and α is the tail estimator. $q_{1/2n}$ represents the VaR at a probability level of 1/2n.

Table 3: VaR estimates of simple returns

Portfolio of two assets	US bonds and stocks		French stocks
probabilities	0.0025	0.000625	0.0018
100 % asset 2	-0.1579	-0.2343	-0.01084
90 % asset 2	-0.1421	-0.2109	-0.0975
80 % asset 2	-0.1263	-0.1874	-0.0870
70 % asset 2	-0.1106	-0.1640	-0.0779
60 % asset 2	-0.0949	-0.1407	-0.0723*
50 % asset 2	-0.0795	-0.1177	-0.0728
40 % asset 2	-0.0654	-0.0962	-0.0794
30 % asset 2	-0.0555	-0.0801	-0.0897
20 % asset 2	-0.0538*	-0.0757*	-0.1016
10 % asset 2	-0.0584	-0.0813	-0.1141
0 % asset 2	-0.0647	-0.0901	-0.1268

Note: Asset 2 is US stocks for the US assets and L'Oreal for the French assets. *denotes the minimum VaR level for all the portfolios.

The relation in equation (6) is used to compute the second order tail index parameters. I set US stocks and L'Oreal as X_1. For the US assets, $\alpha_2 - \alpha_1 = 0.6720$ and $\beta_1 = 1.580$ and for the French stocks $\alpha_2 - \alpha_1 = 0.2480$ and $\beta_1 = 2.210$. So in both cases the condition for an interior solution is satisfied. The VaR estimates for portfolios with US assets and a portfolio with French stocks are shown in table 3. It can be observed that for the US assets, the portfolio with 20% stocks and 80% bonds has the lowest downside risk. For the case of the two French stocks, the portfolio with the lowest downside risk contains 60% of L'Oreal and 40% of Thomson CSF. So, as expected and similar to Hyung and De Vries (2007), applying the second order expansion leads to a non-corner solution for a portofolio with two assets that have different (first order) tail index parameters. When only the first order terms are taken into account, as is done by Jansen et al. (2000), the portfolio with the lowest VaR consists for 100% of bonds (and 0% of stocks).

6.2 Safety first criterion

In table 4, the values for the safety first criterion for a portfolio consisting of US stocks and bonds are shown. The VaR estimates used correspond to a probability of 0.0025 and r = 1.00322 corresponds to the average monthly return on US Treasury bills over the period 1947-2020 (the average return on an annual basis is 4.0%). Table 5 is set up the same, only here VaR estimates with probability 0.000625 are used to compute the safety first criterion values. In both tables it can be observed that the optimal portfolio consists of 30% stocks and 70% bonds. This holds for the different values of r. The results are in line with the results obtained by Hyung and De Vries (2007) since using the second order expansion results in an optimal portfolio with both stocks and bonds in this study as well.

Table 4: Safety first criterion values US stocks and bonds for the period 1947-2020

	$q_\delta(R)$	(R-r)/(r-$q_\delta(R)$)	(R-r)/(r-$q_\delta(R)$)
portfolio with $\delta = 0.0025$		r =1	r=1.00322
100 % asset 2	1 - 0.1579	0.0558	0.0347
90 % asset 2	1 - 0.1421	0.0589	0.0355
80 % asset 2	1 - 0.1263	0.0629	0.0365
70 % asset 2	1 - 0.1106	0.0680	0.0378
60 % asset 2	1 - 0.0949	0.0747	0.0394
50 % asset 2	1 - 0.0795	0.0837	0.0415
40 % asset 2	1 - 0.0654	0.0953	0.0438
30 % asset 2	1 - 0.0555	0.1044*	0.0439*
20 % asset 2	1 - 0.0538	0.0996	0.0376
10 % asset 2	1 - 0.0584	0.0845	0.0278
0 % asset 2	1 - 0.0647	0.0696	0.0189

Note: * denotes the optimal portfolio based on the safety first criterion.

Table 5: Safety first criterion values US stocks and bonds for the period 1947-2020

	$q_\delta(R)$	(R-r)/(r-$q_\delta(R)$)	(R-r)/(r-$q_\delta(R)$)
portfolio with $\delta = 0.000625$		r =1	r=1.00322
100 % asset 2	1- 0.2343	0.0376	0.0235
90 % asset 2	1 - 0.2109	0.0397	0.0241
80 % asset 2	1 - 0.1874	0.0424	0.0248
70 % asset 2	1 - 0.1640	0.0458	0.0257
60 % asset 2	1 - 0.1407	0.0504	0.0269
50 % asset 2	1 - 0.1177	0.0565	0.0284
40 % asset 2	1 - 0.0926	0.0647	0.0302
30 % asset 2	1 - 0.0801	0.0724*	0.0309*
20 % asset 2	1 - 0.0757	0.0709	0.0272
10 % asset 2	1 - 0.0813	0.0607	0.0203
0 % asset 2	1 - 0.0901	0.0500	0.0138

Note: * denotes the optimal portfolio based on the safety first criterion

In table 6, the safety first criterion values for a portfolio consisting of the two French stocks are shown. The VaR estimates used for the computation correspond to a probability of 0.0018. It can be seen that the optimal portfolio consists of 80% L'Oreal and 20% Thomson CSF, although Thomson CSF has a higher first order tail index parameter implying lower downside risk. The optimal allocation can be explained by the relatively low VaR value for this portfolio. Besides this, L'Oreal has a much higher mean return leading to a higher value in the numerator of the safety first criterion when more weight is assigned to L'Oreal.

Table 6: Safety first criterion values French Stocks over the period 01-04-1997 until 04-05-1999

	$q_\delta(R)$	(R-r)/(r-$q_\delta(R)$)
portfolio with $\delta = 0.0018$		r =1
100 % asset 2	1 - 0.1084	0.0118
90 % asset 2	1 - 0.0975	0.0119
80 % asset 2	1 - 0.0870	0.0120*
70 % asset 2	1 - 0.0779	0.0119
60 % asset 2	1 - 0.0723	0.0112
50 % asset 2	1 - 0.0728	0.0095
40 % asset 2	1 - 0.0794	0.0073
30 % asset 2	1 - 0.0897	0.0051
20 % asset 2	1 - 0.1016	0.0034
10 % asset 2	1 - 0.1141	0.0020
0 % asset 2	1 - 0.1267	0.0009

Note: *denotes the optimal portfolio based on the safety first criterion

6.3 Evaluation of VaR estimates

The descriptive statistics of US bonds and stocks for the estimation period are shown in table 7. The statistics do not differ much from the descriptive statistics for the period ranging from 1947-2020. The geometric mean return for stocks is slightly higher for the shorter sample period. Furthermore, the number of observations that are considered to be in the lower tail based on the Hall bootstrap method is higher for bonds as well as stocks for the shorter sample period and the tail index parameters are lower for the shorter sample period, indicating higher (downside) risk.

Table 7: Descriptive statistics for monthly simple returns on US stocks and bonds over the period 1947-2003

	US Bonds	US stocks
Geometric mean	0.00468	0.00980
Standard deviation	0.02079	0.04239
Skewness	0.60058	-0.46549
Kurtosis	6.23507	4.84450
No. Observations	684	684
m	24	30
X_m	-0.0331	-0.0643
α	3.927	3.033
$q_{1/2n}$	-0.089	-0.248

Note: m is the number of observations that form the tail, X_m shows the m-th lowest observation and α is the tail estimator. $q_{1/2n}$ represents the VaR at a probability level of 1/2n.

Table 8: Results of unconditional coverage test for VaR probability 0.05

Portfolio of two assets	T_1	π	LR_{uc}	p-value
100 % asset 2	15	0.074	2.0898	0.148
90 % asset 2	13	0.064	0.7473	0.387
80 % asset 2	11	0.054	0.0645	0.800
70 % asset 2	11	0.054	0.0645	0.800
60 % asset 2	9	0.044	0.1545	0.694
50 % asset 2	8	0.039	0.5377	0.463
40 % asset 2	7	0.034	1.1819	0.177
30 % asset 2	8	0.039	0.5377	0.463
20 % asset 2	8	0.039	0.5377	0.463
10 % asset 2	7	0.034	1.1819	0.177
0 % asset 2	7	0.034	1.1819	0.177

Note: Asset 2 is US stocks and asset 1 is US bonds. T_1 represents the number of violations and π the ratio of violations to total observations.

In table 8, the results for the unconditional coverage test are shown for VaR estimates corresponding to a probability level of 0.05 (the VaR estimates are shown in table 14 in the appendix). It can be observed that values for π are increasing when the percentage of stocks in the portfolio increases, which is in line with the higher uncertainty in stock returns compared to bond returns and more violations of VaR estimates as a result. P-values are lower for portfolios that consist mainly of stocks or bonds, but the null hypothesis of correct unconditional coverage is not rejected for all portfolios.

In table 9, results for the independence test of VaR esimates corresponding to a probability of 0.05 are shown. It can be seen that for a portfolio with only bonds, there are no consecutive violations of the VaR estimate and the higher the percentage of stocks in the portfolio, the more consecutive violations of VaR estimates. The null hypthesis of independence is rejected for all portfolios at a 5% significance level, except for the portfolio with only bonds and the portfolio with 90% bonds and 10% stocks.

Table 9: Results of independence test for VaR probability 0.05

Portfolio of two assets	T_{00}	T_{01}	T_{10}	T_{11}	π_{01}	π_{11}	π_2	LR_{ind}	p-value
100 % asset 2	177	11	11	4	0.058	0.267	0.074	5.8290	0.016
90 % asset 2	181	9	9	4	0.047	0.308	0.064	8.0923	0.004
80 % asset 2	184	8	8	3	0.042	0.273	0.054	6.1280	0.013
70 % asset 2	184	8	8	3	0.042	0.273	0.054	6.1280	0.013
60 % asset 2	187	7	7	2	0.036	0.222	0.044	3.8963	0.048
50 % asset 2	189	6	6	2	0.031	0.250	0.039	4.8350	0.028
40 % asset 2	191	5	5	2	0.026	0.286	0.034	5.9640	0.015
30 % asset 2	189	6	6	2	0.031	0.250	0.039	4.8350	0.028
20 % asset 2	189	6	6	2	0.031	0.250	0.039	4.8350	0.028
10 % asset 2	190	6	6	1	0.031	0.143	0.034	1.5056	0.220
0 % asset 2	189	7	7	0	0.036	0.000	0.034	0.5001	0.479

Note: Asset 2 is US stocks and asset 1 is US bonds. T_{xx} represents consecutive observations and π_{xx} represents the accompanying ratio. Detailed computation is provided in section 5.3.2.

In table 10, results for the unconditional coverage test for VaR estimates corresponding to a probability of 0.01 are shown. It can be observed that the empirical coverage probability differs the most from the nominal level for a portfolio with 60% stocks. However, for all portfolios, the null hypothesis is not rejected.

Table 10: Results of unconditional coverage test for VaR probability 0.01

Portfolio of two assets	T_1	π	LR_{uc}	p-value
100 % asset 2	2	0.010	0.0008	0.977
90 % asset 2	2	0.010	0.0008	0.977
80 % asset 2	2	0.010	0.0008	0.977
70 % asset 2	2	0.010	0.0008	0.977
60 % asset 2	4	0.020	1.4858	0.223
50 % asset 2	3	0.015	0.3986	0.528
40 % asset 2	3	0.015	0.3986	0.528
30 % asset 2	3	0.015	0.3986	0.528
20 % asset 2	3	0.015	0.3986	0.528
10 % asset 2	3	0.015	0.3986	0.528
0 % asset 2	3	0.015	0.3986	0.528

Note: Asset 2 is US stocks and asset 1 is US bonds.

In table 11, results for the independence test of VaR estimates corresponding to a probability of 0.01 can be observed. Only for portfolios with 40%, 50% and 60% stocks, there is one occurence of two consecutive observations that violate the VaR estimate. Since there are less single violations (T_{01}, T_{10}) for the portfolios with 40% and 50% stocks, the likelood ratio test statistic is higher here and the p-value lower. At a 5% significane level, the null hypothesis is rejected for the portfolios with 40% and 50% stocks.

Table 11: Results of independence test for VaR probability 0.01

Portfolio of two assets	T_{00}	T_{01}	T_{10}	T_{11}	π_{01}	π_{11}	π_2	LR_{ind}	p-value
100 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.842
90 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.842
80 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.842
70 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.842
60 % asset 2	196	3	3	1	0.015	0.250	0.020	3.7146	0.054
50 % asset 2	198	2	2	1	0.010	0.333	0.015	5.0233	0.025
40 % asset 2	198	2	2	1	0.010	0.333	0.015	5.0233	0.025
30 % asset 2	197	3	3	0	0.015	0.000	0.015	0.0900	0.764
20 % asset 2	197	3	3	0	0.015	0.000	0.015	0.0900	0.764
10 % asset 2	197	3	3	0	0.015	0.000	0.015	0.0900	0.764
0 % asset 2	197	3	3	0	0.015	0.000	0.015	0.0900	0.764

Note: Asset 2 is US stocks and asset 1 is US bonds.

In table 12, results for the unconditional coverage test for VaR estimates corresponding to a probability of 0.0049 (1/204) are shown. It can be seen that for some portfolios there is exactly one violation which is in line with what would be expected if the VaR estimate is accurate. For the portfolio with only bonds there are zero violations and for the portfolios with a higher percentage of stocks, there are 2 violations, again indicating that bonds have less downside risk. For all portfolios, the null hypothesis of correct unconditional coverage is not rejected.

Table 12: Results of unconditional coverage test for VaR probability 0.0049 (1/204)

Portfolio of two assets	T_1	π	LR_{uc}	p-value
100 % asset 2	2	0.010	0.7783	0.378
90 % asset 2	2	0.010	0.7783	0.378
80 % asset 2	1	0.005	0.0000	1
70 % asset 2	1	0.005	0.0000	1
60 % asset 2	1	0.005	0.0000	1
50 % asset 2	1	0.005	0.7783	0.378
40 % asset 2	2	0.010	0.7783	0.378
30 % asset 2	2	0.010	0.0000	1
20 % asset 2	1	0.005	0.0000	1
10 % asset 2	1	0.005	0.0000	1
0 % asset 2	0	0.000	2.0041	0.157

Note: Asset 2 is US stocks and asset 1 is US bonds.

Table 13: Results of independence test for VaR probability 0.0049 (1/204)

Portfolio of two assets	T_{00}	T_{01}	T_{10}	T_{11}	π_{01}	π_{11}	π_2	LR_{ind}	p-value
100 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.8419
90 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.8419
80 % asset 2	201	1	1	0	0.005	0.000	0.005	0.0099	0.9207
70 % asset 2	201	1	1	0	0.005	0.000	0.005	0.0099	0.9207
60 % asset 2	201	1	1	0	0.005	0.000	0.005	0.0099	0.9207
50 % asset 2	201	1	1	0	0.005	0.000	0.005	0.0099	0.9207
40 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.8419
30 % asset 2	199	2	2	0	0.010	0.000	0.010	0.0398	0.8419
20 % asset 2	201	1	1	0	0.005	0.000	0.005	0.0099	0.9207
10 % asset 2	201	1	1	0	0.005	0.000	0.005	0.0099	0.9207
0 % asset 2	203	0	0	0	0.000	NaN (0)	0.000	0.000	1

Note: Asset 2 is US stocks and asset 1 is US bonds.

In table 13, results of the independence test of VaR estimates corresponding to a probability of 0.0049 are shown. It can be seen that for none of the portfolios there is an occurrence of two consecutive VaR estimate violations, which is also logical since only four portfolios have two violations and the others one or zero. As a result, p-values are very high and the null hypothesis of independence of violations is not rejected for all portfolios.

Now that the VaR estimates are evaluated, the performance of the second order expansion with regards to VaR estimation can be discussed. In terms of coverage, the null hypothesis of correct unconditional coverage is not rejected for all three scenarios with VaR estimates corresponding to different probabilities. When looking at independence however, it can be observed that for VaR estimates with higher probabilities, independence is rejected for more portfolios consisisting of stocks and bonds than for VaR estimates corresponding to lower probabilities. Especially in the case of the small probability 0.0049, independence is not rejected for any portfolio. This is also in line with what could be expected since for this very low probability there are hardly any violations. If these violations are not consecutive, which is the case, then the violations are considered independent. For the higher probabilities however, there are more violations and volatility clustering is a problem for the portfolios. The independence test evaluates whether the probability of a violation is higher if there was a violation on the previous day. Since the estimation period 1947-2003 is used to construct a model with which VaR estimates are obtained, these estimates are not conditional on returns in the backtesting period 2004-2020 and thus not conditional on the return on the previous day. Consequently, based on the model choice, it could be expected that for higher probabilities the VaR estimates might be dependent as a result of volatility clustering.

Since for the comparison of different portfolios in section 6.1 and section 6.2 very low probabilities are used to compute the VaR estimates, it can be concluded that these estimates are accurate. This is an interesting and useful result because implementing the second order expansion results in a portfolio with less downside risk and more potential on high returns at the same time.

7 Conclusion

In this research, it is investigated whether using a second order expansion with two tail index parameters leads to a non-corner solution for a portfolio with US stocks and bonds and a portfolio with the French stocks L'Oreal and Thomson CSF. First, the tail index parameters are computed for all assets and it is found that US bonds and Thomson CSF have the higher tail index parameters, indicating less downside risk. The different portfolios that are analysed, assign different weights to the two assets and the portfolios are compared based on their downside risk. The metrics used for this are VaR and the safety first criterion. It turns out that when monthly returns of US stocks and US bonds are analysed over the period January 1947 until December 2020, the optimal portfolio consists for 20% of stocks and 80% of bonds based on the VaR. According to the safety first criterion, the optimal portfolio consists for 30% of stocks. For the French stocks, the optimal portfolio consists for 60% and 80% out of L'Oreal when the VaR and safety first criterion are evaluated, respectively.

Since the use of the second order expansion results in a non-corner solution when analysing downside risk, it is also evaluated whether the VaR estimates computed based on the second order expansion are accurate for US stocks and bonds. This is done by means of two tests: the unconditional coverage test and the independence test. I find that for VaR estimates corresponding to a significance level of 0.05, 0.01 and 0.0049 correct unconditional coverage is not rejected for all portfolios. However, VaR estimates at a 0.05 probability level are not independent for almost all portfolios, and VaR estimates at a 0.01 probability level are not independent for two portfolios. VaR estimates corresponding to a 0.0049 probability level are independent for all portfolios. So, it can be concluded that the second order expansion can be a valuable tool to construct a portfolio with two assets where one asset has more downside risk than the other. However, the probability level corresponding to the VaR estimates should be kept low, preferably below 0.01, in order for the VaR estimates to be accurate and trustworthy.

For further research, it could be interesting to evaluate the accuracy of VaR estimates based on a model with daily returns. In this case, more observations can be used for the estimation period and for the backtest period. Especially for the backtest period this would be interesting because the results would be more reliable. In addition, it might be interesting to apply and evaluate other techniques to construct VaR estimates since for a very low probability, all the estimates for the different portfolios passed both tests for accuracy, so the estimates could possibly be less conservative for this low probability.

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8 Appendix

8.1 Tables

Table 14: VaR estimates for a portfolio of US stocks and bonds over the period 1947-2003

Probabilities	0.05	0.01	0.0049
100 % asset 2	-0.0616	-0.1048	-0.1326
90 % asset 2	-0.0555	-0.0943	-0.1193
80 % asset 2	-0.0493	-0.0838	-0.1061
70 % asset 2	-0.0432	-0.0734	-0.0928
60 % asset 2	-0.0371	-0.0630	-0.0797
50 % asset 2	-0.0314	-0.0530	-0.0669
40 % asset 2	-0.0267	-0.0442	-0.0554
30 % asset 2	-0.0244	-0.0387	-0.0476
20 % asset 2	-0.0250	-0.0381	-0.0461
10 % asset 2	-0.0273	-0.0411	-0.0494
0 % asset 2	-0.0302	-0.0456	-0.0546

Note: Asset 2 is US stocks.

8.2 Little o notation

Little o notation is used to express that the growth or decay rate of a certain function is faster than the growth or decay rate of another function. Let f(n) and g(n) be functions with real values, then the statement $f(n) = o(g(n))$ implies that for every positive constant ϵ, there exists a value for N such that:

$$|f(n)| \leq \epsilon g(n) \quad \text{for all} \quad n \geq N$$

This means that $g(n)$ grows much faster than $f(n)$ as n approaches infinity, so $f(n)$ becomes relatively insignificant. Mathematically this can be expressed as follows :

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

8.3 R code

8.3.1 Code for replication part

```
#Import simple returns of French Stocks L'oreal and Thomson CSF
StocksFrench <- read_excel("C:/Users/thijs/OneDrive - Erasmus University
    Rotterdam/Documents/Thesis/Loreal thomson simple returns.xlsx")
stocksFrench1 <- xts(StocksFrench[,-1], order.by=as.Date(StocksFrench$Date))
returnsFr <- as.data.frame(na.omit(stocksFrench1))

#Import simple returns on US stock index 1947-2020
US_Stocks47 <- read_excel("C:/Users/thijs/OneDrive - Erasmus University
    Rotterdam/Documents/Thesis/US stock index_returns_1947-2020.xlsx")
stocksUS47 <- xts(US_Stocks47[,-1], order.by=as.Date(US_Stocks47$Date))
returnsUS47 <- as.data.frame(na.omit(stocksUS47))

#Import simple returns on US bond index 1947-2020
bond_returns <- read_excel("C:/Users/thijs/OneDrive - Erasmus University
    Rotterdam/Documents/Thesis/10 year bond returns from 1947 onwards.xlsx")
bonds1 <- xts(bond_returns[,-1], order.by=as.Date(bond_returns$Date))
bonds2 <- as.data.frame(na.omit(bonds1))
```

```
#Hall bootstrap method for US stock index. Obtain number of observations in tail, threshold
    value and tail index estimator.
samplefrac <- hall(-1*returnsUS47[,1], B = 1000, epsilon = 0.955, kaux =
    sqrt(length(returnsUS47[,1])))
m_S  <- unlist(samplefrac[1])
X_S <- unlist(samplefrac[2])
a_S <- unlist(samplefrac[3])

#Hall bootstrap method for US bonds
samplefrac <- hall(-1* bonds2[,1], B = 1000, epsilon = 0.955, kaux = sqrt(length(bonds2[,1])))
m_B  = unlist(samplefrac[1])
X_B = unlist(samplefrac[2])
a_B = unlist(samplefrac[3])

#Computing parameters for VaR and safety first calculation
n = 888
A_S = (m_S/n) * -X_S ^a_S
A_B = (m_B/n) * -X_B^a_B
ratio = log(m_S)/ (2*log(n) - 2*log(m_S))
B_S = ratio * a_S
meanstocks = 0.008808
meanbonds = 0.004503
r = 1


#Computing VaR estimates for portfolio of stocks and bonds. Unisolutions represents the VaR
    estimates.
unisol<- vector()
safetyfirst <- vector()
for (k in 0:10) {
   w =(k)/10
  if (k<1) {
    fun <- function (x) (A_B * x^-a_B + (0.1))
  } else if (k>0 && k<10){
    fun <- function (x) ( (w^a_S) * A_S* x^-a_S + (1-w)^a_B * A_B * x^-a_B + 0.1)
  }else {
    fun <- function (x) ( A_S* x^-a_S + (0.1))
  }
   uni <- uniroot(fun, lower = 0, upper = 1, extendInt = "yes")$root
   unisol[k+1] <- uni
   safetyfirst[k+1] = ((1 + w * meanstocks + (1-w) * meanbonds) - r)/ (r - (1-uni))
}

unisolutions = -1 * unisol


#Hall bootstrap method for Loreal
samplefrac <- hall( -1* returnsFr[,1], B = 1000, epsilon = 0.955, kaux =
    sqrt(length(returnsFr[,1])))
m_L  <- unlist(samplefrac[1])
X_L <- unlist(samplefrac[2])
a_L <- unlist(samplefrac[3])

#Hall bootstrap method for Thomson CSF
samplefrac <- hall(-1 * returnsFr[,2], B = 1000, epsilon = 0.955, kaux =
    sqrt(length(returnsFr[,2])))
m_T  = unlist(samplefrac[1])
X_T = unlist(samplefrac[2])
a_T = unlist(samplefrac[3])

#Computing parameters for VaR and safety first calculation
n1 = 546
A_L = (m_L/n1) * -X_L ^a_L
```

```
A_T = (m_T/n1) * -X_T^a_T
ratio1 = log(m_L)/ (2*log(n1) - 2*log(m_L))
B1 = ratio1 * a_L
meanLoreal = 0.001282
meanThomson = 0.000109
r = 1

#VaR estimates for portfolio with L'oreal and Thomson CSF
unisol1<- vector()
safetyfirst1 <- vector()
for (k in 0:10) {
  w =(k)/10
  if (k<1) {
    fun <- function (x) (A_T * x^-a_T + (0.0018))
  } else if (k>0 && k<10){
    fun <- function (x) ( (w^a_L) * A_L* x^-a_L + (1-w)^a_T * A_T * x^-a_T + 0.0018)
  }else {
    fun <- function (x) ( (w^a_L) * A_L* x^-a_L + (0.0018))
  }
  uni1 <- uniroot(fun, lower = 0, upper = 1, extendInt = "yes")$root
  unisol1[k+1] <- uni1
  safetyfirst1[k+1] = ((1 + w * meanLoreal + (1-w) * meanThomson) - r)/ (r - (1-uni1))
}

unisolutions1 = -1 * unisol1
```

8.3.2 Code for extension

```
#Import US stock index simple returns
US_Stocks47 <- read_excel("C:/Users/thijs/OneDrive - Erasmus University
    Rotterdam/Documents/Thesis/US stock index_returns_1947-2020.xlsx")
stocksUS47 <- xts(US_Stocks47[,-1], order.by=as.Date(US_Stocks47$Date))
returnsUS47 <- as.data.frame(na.omit(stocksUS47))

#Import US bond index simple returns
bond_returns <- read_excel("C:/Users/thijs/OneDrive - Erasmus University
    Rotterdam/Documents/Thesis/10 year bond returns from 1947 onwards.xlsx")
bonds1 <- xts(bond_returns[,-1], order.by=as.Date(bond_returns$Date))
bonds2 <- as.data.frame(na.omit(bonds1))

#Take the observations for the period 1947-2003 (684 observations)
returnsUS_est<- returnsUS47[1:684,1]
bonds_est <- bonds2[1:684,1]

#Hall bootstrap method for US stock index
samplefrac <- hall((-1 * returnsUS_est), B = 1000, epsilon = 0.955, kaux =
    sqrt(length(returnsUS_est)))
m_S  <- unlist(samplefrac[1])
X_S <- unlist(samplefrac[2])
a_S <- unlist(samplefrac[3])
print (a_S)
print(X_S)

#Hall bootstrap method for US bonds
samplefrac <- hall((-1* bonds_est), B = 1000, epsilon = 0.955, kaux = sqrt(length(bonds_est)))
m_B  = unlist(samplefrac[1])
X_B = unlist(samplefrac[2])
a_B = unlist(samplefrac[3])
print (a_B)

#Computing parameters for VaR
n_est = 684
```

```
A_S = (m_S/n_est) * -X_S ^a_S
print(A_S)
A_B = (m_B/n_est) * -X_B^a_B
ratio = log(m_S)/ (2*log(n_est) - 2*log(m_S))
B_S = ratio * a_S
meanstocks = 0.009794
meanbonds = 0.004678
r = 1


#VaR estimates for portfolio of stocks and bonds over period 1947-2003
unisol_est<- vector()
#safetyfirst <- vector()
for (k in 0:10) {
  w =(k)/10
  if (k<1) {
    fun <- function (x) (A_B * x^-a_B + 0.0049)
  } else if (k>0 && k<10){
    fun <- function (x) ( (w^a_S) * A_S* x^-a_S + (1-w)^a_B * A_B * x^-a_B + 0.0049)
  }else {
    fun <- function (x) ( A_S* x^-a_S + 0.0049)
  }
  uni_est <- uniroot(fun, lower = 0, upper = 1, extendInt = "yes")$root
  unisol_est[k+1] <- uni_est
}

unisolutions_est = -1 * unisol_est


#Compute number of violations T1 (called T1_w here)
T1_w <- rep(0,11)

for (j in 0:10){
  weight = (j)/10
  if (j<1){ for(i in 1:204) {

    if (bonds2[(i+684),1]<unisolutions_est[1]){
      T1_w[1] = T1_w[1] +1
    }}}
  else if (j>0 && j<10) { for(i in 1:204) {

    if (weight * returnsUS47[i+684,1] + (1-weight) * bonds2[(i+684),1]<unisolutions_est[j+1]){
      T1_w[j+1] = T1_w[j+1] +1
    }}}
  else if ( j >9) { for(i in 1:204) {

    if (returnsUS47[(i+684),1]<unisolutions_est[11]){
      T1_w[11] = T1_w[11] +1
    }}}
  }

 print(T1_w)


#Compute statistics for correct unconditional coverage test
T = 204
q = 0.05
pi <- rep(0,11)
Lq <- rep (0,11)
Lpi <- rep(0,11)
LR_COC <- rep(0,11)
for (p in 1:11) {
pi[p] = T1_w[p] / T
```

```
Lq[p] <- (1-q)^ (T - T1_w[p]) * q^T1_w[p]
Lpi[p] = (1-pi[p])^(T - T1_w[p]) * (pi[p])^T1_w[p]
LR_COC[p] = -2 * log(Lq[p] / Lpi[p])
}



#Compute values for T00, T01, T10, T11
T01_w <- rep(0,11)
T10_w <- rep(0,11)
T11_w <- rep(0,11)
T00_w <- rep(0,11)

for (j in 0:10){
  weight = (j)/10

if (j<1){ for(i in 1:203) {
    if ((bonds2[(i+684),1]>unisolutions_est[1]) && (bonds2[(i+685),1]<unisolutions_est[1])) {
      T01_w[1] = T01_w[1] +1

    }else if ((bonds2[(i+684),1]<unisolutions_est[1]) &&
         (bonds2[(i+685),1]>unisolutions_est[1])) {
      T10_w[1] = T10_w[1] +1

   } else if ((bonds2[(i+684),1]<unisolutions_est[1]) &&
        (bonds2[(i+685),1]<unisolutions_est[1])) {
      T11_w[1] = T11_w[1] +1

   } else if ((bonds2[(i+684),1]>unisolutions_est[1]) &&
        (bonds2[(i+685),1]>unisolutions_est[1])) {
      T00_w[1] = T00_w[1] +1
    }
    }
}else if (j>0 && j<10) { for(i in 1:203) {

   if (((weight * returnsUS47[i+684,1] + (1-weight) * bonds2[(i+684),1])>unisolutions_est[j+1])
        && (weight * returnsUS47[i+685,1] + (1-weight) *
        bonds2[(i+685),1])<unisolutions_est[j+1]){
      T01_w[j+1] = T01_w[j+1] + 1
      }else if (((weight * returnsUS47[i+684,1] + (1-weight) *
          bonds2[(i+684),1])<unisolutions_est[j+1]) && (weight * returnsUS47[i+685,1] +
          (1-weight) * bonds2[(i+685),1])>unisolutions_est[j+1]){
      T10_w[j+1] = T10_w[j+1] +1

    }else if (((weight * returnsUS47[i+684,1] + (1-weight) *
        bonds2[(i+684),1])<unisolutions_est[j+1]) && (weight * returnsUS47[i+685,1] +
        (1-weight) * bonds2[(i+685),1])<unisolutions_est[j+1]){
      T11_w[j+1] = T11_w[j+1] +1

    }else if (((weight * returnsUS47[i+684,1] + (1-weight) *
        bonds2[(i+684),1])>unisolutions_est[j+1]) && (weight * returnsUS47[i+685,1] +
        (1-weight) * bonds2[(i+685),1])>unisolutions_est[j+1]){
      T00_w[j+1] = T00_w[j+1] +1
    }
    }
} else if ( j >9) { for(i in 1:203) {
   if ((returnsUS47[(i+684),1]>unisolutions_est[11]) &&
        (returnsUS47[(i+685),1]<unisolutions_est[11])) {
   T01_w[11] = T01_w[11] +1

  }else if ((returnsUS47[(i+684),1]<unisolutions_est[11]) &&
       (returnsUS47[(i+685),1]>unisolutions_est[11])) {
  T10_w[11] = T10_w[11] +1
```

```
  } else if ((returnsUS47[(i+684),1]<unisolutions_est[11]) &&
      (returnsUS47[(i+685),1]<unisolutions_est[11])) {
  T11_w[11] = T11_w[11] +1

  } else if ((returnsUS47[(i+684),1]>unisolutions_est[11]) &&
      (returnsUS47[(i+685),1]>unisolutions_est[11])) {
  T00_w[11] = T00_w[11] +1
   }
   }
}
}


#Compute statistics for independence test
pi01 <- rep(0,11)
pi11 <- rep(0,11)
pi2 <- rep(0,11)
Lpi2<- rep(0,11)
Lpi1 <- rep(0,11)
LR_IND <- rep(0,11)

for (z in 1:11) {
  pi01[z] = T01_w[z] / (T00_w[z] + T01_w[z])
  pi11[z] <- T11_w[z] / (T10_w[z] + T11_w[z])
  pi2[z] <- (T01_w[z] + T11_w[z]) / (T00_w[z] + T10_w[z] + T01_w[z] + T11_w[z])
  Lpi2[z] <- (1-pi2[z])^(T00_w[z] + T01_w[z]) * pi2[z] ^ (T01_w[z]+T11_w[z])
  Lpi1[z] <- ((1-pi01[z])^(T00_w[z]) * (pi01[z])^(T01_w[z]) * ((1-pi11[z])^(T10_w[z])) *
      (pi11[z] ^ T11_w[z]))
  LR_IND[z] = -2 * log(Lpi2[z]/Lpi1[z])
}
```