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A comparative analysis of tail index based safety-first portfolios

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Abstract

It is a fact that asset returns experience heavy tails. Numerous studies exploit this fact by identifying the behaviour in the tails, be it with a first- or second order convolution. The difference between the two approaches is what interests us. It has been shown that including second order terms will result in more balanced optimal safety-first portfolios. We focus solely on a two asset portfolio, consisting of a relatively risky and a relatively safe asset. The portfolios of our interest contain a major US stock index, bond index and stock. First, an empirical analysis is performed on concepts found in available literature, after which we add a new perspective. We extent to Hyung and De Vries (2007) by considering a longer investment horizon, and implementing second order tail index terms into Expected Shortfall estimation. We find that when increasing the time horizon, a safety-first investor's portfolio will consist of roughly the same asset allocation, a result also found by Jansen et al. (2000). Furthermore, this paper shows how the second order tail index estimation can be included within Expected Shortfall estimation. This is first done analytically, after which findings are shown empirically. We see that properties for ES hold empirically.

Keywords: Safety-first, Heavy tails, Value-at-Risk, Expected Shortfall, Investment horizon

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Returns:

$r_{i,t} = \ln \frac{P_{i,t}}{P_{i,t-1}}$	log-returns of asset i at time t
$X_{i,t} = -r_{i,t}$	loss returns of asset i at time t
$X_{(m)}$	the m-th largest (descending) order statistic
$Y_{i,k} = \sum_{j=1}^{k} X_{i,j}$	k-period (loss-)return of asset i

Safety-first portfolio:

$V_{i,t}$	value of asset i at time t
ω_i	weight of asset i in portfolio
$R = \frac{\sum \omega_i V_{i,t+1}}{\sum \omega_i V_{i,t}}$,	mixed portfolio (simple) returns
W_t	wealth level of the investor at time t
b	investor's amount of lending ($b < 0$ is borrowing)
r	risk-free rate
α	(first order) tail index
β	second order tail index
$VaR(\omega, \delta)$	Value-at-Risk with asset weights ω and loss probability δ
$ES_{(\omega,\delta)}$	Expected Shortfall corresponding with $VaR(\omega, \delta)$

1 Introduction

Countless studies emphasise the importance of taking downside risk into consideration in portfolio management. As the downside risk displays the probability of losing money, this is often considered of more importance than its upside counterpart to investors. For this reason, generally, investors opt to limit the downside risk, as is shown by Tversky and Kahneman (1991). This paper incorporates asset and portfolio risk through the safety-first principle by Roy (1952) and Arzac and Bawa (1977). From literature we know that the distribution of asset returns is typically fat-tailed, where we see that riskier assets generally experience fatter tails. This characteristic is utilised by Jansen et al. (2000) and Hyung and De Vries (2007), who study a semi-parametric method to describe tail behaviour using extreme value theory. The second order expansion used in Hyung and De Vries (2007) finds optimal safety-first portfolios switched away from the corner solutions found by Jansen et al. (2000).

This paper focuses on a safety first investor's portfolio management, considering multiple risk measures and investment horizons, and follows a similar structure to Hyung and De Vries (2007) and Jansen et al. (2000). Previously found results are replicated with a different data set. We consider daily observations from a major US stock- and bond index, and the Apple stock (AAPL). It is shown that the findings in Hyung and De Vries (2007) hold for this data set, yet we also find an exception. Furthermore, a different method is executed to find tail index estimates.

We continue our research by investigating several extensions to Hyung and De Vries (2007). First, the extension from Jansen et al. (2000) is applied to the second order expansion from Hyung and De Vries (2007), to research the effect of the second order terms on a varying investment horizon. Jaggia and Thosar (2000) and Lenoir and Tuchschmid (2001) conducted research on expanding investing horizons and yield interesting results, making it an interesting perspective to highlight in this paper. Their most prominent finding is the conversion to more risky assets as the time horizon increases. We find results similar to Jansen et al. (2000) for the first order convolution, and see that with a relatively small horizon extension the second order safety-first portfolios' allocations also yield those results.

Secondly, we are interested in the prominent use of VaR, within the second order tail index frame work. To our knowledge research on a different risk measure is not or scarcely available. We turn to the use of VaR as risk measure in the semi-parametric tail index approach. Emmer et al. (2013) argue that Expected Shortfall is theoretically a superior risk measure to VaR. They find that ES is preferred due to its sub-additivity and sensitivity for tail risk. VaR fails to cover tail risk beyond the VaR level, whereas ES intuitively shows the tail risk with the expected loss past this level. Furthermore, VaR does not satisfy all axioms for a coherent risk measure, where ES does. We show how the second order terms can be included into the ES estimation process. This is first done for single assets, after which we focus on theoretically mixed portfolios. We find expressions for the second order single asset terms, with which we can calculate the ES. Finally, we show that the intuitive principle, $ES_{(\omega,\delta)} \geq -VaR(\omega,\delta)$, holds empirically for both single assets as mixed portfolios.

This paper adds to current literature in the sense that mostly VaR is used as risk measure in combination with a second order convolution approach. We have found little literature on the effect of the second order terms on an expanding time window, and believe to present some useful results. Finally, we suggest some aspects that could be interesting to further research following from our findings.

This paper proceeds as follows. Section 2 broadly covers the most important available literature on this topic. Section 3 discusses the data which is used to conduct the research. Section 4 provides the most prominent methods and techniques which are applied in the research, section 5 presents the results and section 6 concludes.

2 Literature

This section covers available literature on this research topic. First, some important psychological findings from previous literature will be presented to clarify the motivation for risk averse investing. Hereafter, important research on the safety first criterion will be presented in a chronological manner. Finally, available literature on the introduced extensions will be presented.

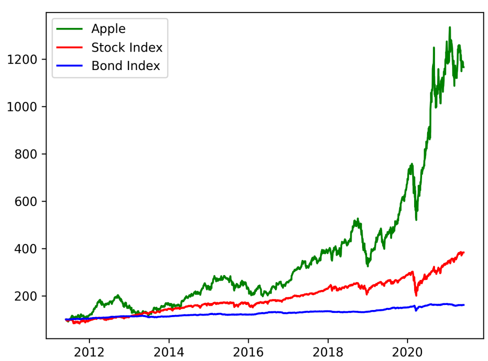
A lot has been written on behavioural decision research. It is generally stated that losses outweigh gains, resulting in loss aversive decision making. Kahneman and Tversky (1979) first introduced the prospect theory, analysing decision making under uncertainty. They find that risk aversion is commonly preferred, illustrated by a gambling example. Further, the endowment effect, introduced by Thaler (1980), states that opportunity costs are generally underweight, as individuals value goods included in their endowment more than goods that are not included. Kahneman et al. (1990) adds to this by showing empirically that individuals treat gains and losses asymmetrically in some cases, thus finding evidence for loss aversion. More recent research, such as Jarrow and Zhao (2006), shows that indeed loss averse preferences exist within portfolio management, and, although using a different approach than in this research, find significant differences to Markowitz (1952) mean-variance efficient portfolio. This wide range of papers emphasises the motives for research on safety-first investors.

To form a good understanding of the evolution of this problem field, a selection of the most important papers covering safety-first principles are summarised in chronological order. Roy (1952) first introduced the safety-first criterion, a portfolio selection criterion incorporating limited downside risk. Arzac and Bawa (1977) showed that, with a lexicographic approach, the safety-first criterion can be separated into two problems. First, a safety first investor finds the optimal risky asset proportions under a certain loss probability constraint, after which the investor optimises the choice between the asset allocation and borrowing or lending. Roy (1952) initiated the use of the Chebyshev inequality as risk measure, yet de Haan et al. (1994) showed clear indications that this bound does not suffice, proposing extreme value theory as an alternative. Their train of thought was that exploiting the fat tail property of asset returns would tighten the bounds, which they proved empirically. This reasoning proved to be of importance to further research as the use Value-at-Risk (VaR) became the most prominent risk measure in the problem field of portfolio selection by safety-first investors. Gourieroux et al. (2000) show how to check for the convexity of VaR and show that considering VaR in this context is equivalent to portfolio selection with safety-first criterion. The difference between Gourieroux et al. (2000) and Jansen et al. (2000) is that the latter exploits the asset returns' fat tailed property, while the former opts for a kernel based approach. Jansen et al. (2000) exploits this by focussing on the behaviour of the only the tails of the returns, rather than the complete return distribution. This makes sense as safety-first investors focus primarily on downside risk, i.e. risk in the lower tail of the returns distribution. Jansen et al. (2000) also researches what happens to portfolio allocations when extending the investment window. An interesting study as it adds to the robustness of the approach and gives a more empirical rather than theoretical perspective. Hyung and De Vries (2007) follows the Jansen et al. (2000) approach, adding a second order expansion resulting in less corner solutions of portfolio asset allocation. The reasoning behind this is the inclusion of both tail indices in the compilation of the optimal asset selection. The importance of investigating varying investment windows is stressed by Milevsky (1999), who shows the effect of the investment time horizon on the choice of risky assets. This paper focuses on a 10 day time horizon, as is typically done in financial application. We therefore need to scale the found VaR values to this new investment window. Dicks (2013) states three different methods to do this, the most commonly known square root of time rule, the alpha root rule and empirically established scaling laws. We follow the same rule as used in Jansen et al. (2000), i.e. the alpha root rule, which follows from Danielsson and De Vries (2000). It is preferred over the more commonly known square root of time rule as it solves the normality issue which occurs with asset returns, which is discussed in Dacorogna et al. (1995).

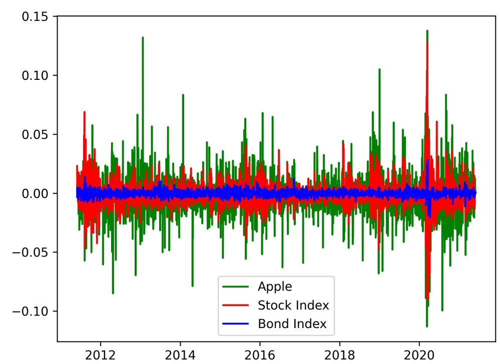
This paper builds on available literature by considering a second order convolution for the extension in Jansen et al. (2000), considering varying investment windows. Furthermore, Expected Shortfall (ES) is used as risk measure. Both Emmer et al. (2013) and Mak and Meng (2014) compare risk measures and conclude ES to be best combination of robustness and intuition. Acerbi and Tasche (2002) proof the four axioms that need to hold for a coherent risk measure. A function is called a coherent risk measure if it is monotonous, sub-additive, positively homogeneous and translation invariant. The second axiom of sub-additivity does not hold for VaR in all cases, meaning it lacks coherency. The mathematical notation and proofs for these axioms can be found in Acerbi and Tasche (2002), though they do not add to the understanding of this paper.

3 Data

This paper considers two data pairs. The data consists of daily closing prices from 31-05-2011 till 01-06-2021, with a total of 2557 observations, for the S&P500 (stock) index, S&P500 bond index and a stock, Apple Inc. (AAPL). The two pairs that are considered are the stock index - bond index pair, and the stock index - single stock pair. Note that a different index pair than in Hyung and De Vries (2007) and Jansen et al. (2000) is used, thus the results cannot be compared one-on-one. It will become apparent why we choose for an index-stock pair, which is not considered in previous literature, later. The index data is collected from S&P Global and the AAPL data is collected from Yahoo Finance. Figure 1 shows the daily closing prices and the daily returns for both indices and the stock, respectively. The daily returns are calculated as $r_{i,t} = \ln \frac{P_{i,t}}{P_{i,t-1}}$, with $P_{i,t}$ the price of asset i, $i = 1, 2$, at time t, $t = 2, 3, ..., T$. Note that all computations will be done using the loss returns, i.e. $X_{i,t} = -r_{i,t}$, as we focus on downside risk and are thus interesting in the lower tail behaviour. Missing data points are accounted for by attributing the previous available data point, i.e. zero-returns. The closing prices in figure 1a have been normalised with a base price of $100 to give a better comparison. These figures give an indication that the single stock experiences higher volatility than the stock index, which experiences higher volatility than the bond index, just as would be expected from literature. Figure 1b illustrates more "noisy" returns for the stock than the indices, as can be seen by "the amount of ink used". A similar indication follows from Figure 1a where it seems the prices are correlated as they follow the same pattern, but the effects seem larger for the single stock where we see larger spikes than for indices. These indications are confirmed in Table 1. Table 1 presents the summary statistics for the daily returns of the S&P 500 index and S&P 500 bond index and the AAPL stock, respectively. The mean return of the stock index, 0.0005 per day, is three times larger than the mean return of the bond index, 0.0002 per day, but only half of the single stock, 0.0010 per day. On the other hand, the standard deviation of stock index returns, 0.0110, is more than four times larger than that of bond returns, 0.0028, and 2/3rd of the stock's standard deviation, 0.0180. We see that all assets have negatively skewed returns, with a skewness of -0.9284, -1.2056 and -0.3349, respectively. Moreover, we see all series have a high kurtosis, corresponding with fat tails. This confirms what is stated in literature regarding asset returns. These fat tails allow for a semi-parametric method of modelling tail events through a tail expansion. The Jarque-Bera test rejects normality for all asset returns, with p-values of 0.0000, although the AAPL stock is relatively seen closest to normality. This can be also be seen by noting that the skewness and kurtosis are closer to 0 and 3, respectively, which is expected for normally distributed data.



(a) Closing prices



(b) Daily returns

Figure 1: S&P 500 equity- and bond index and AAPL stock (normalised) price and returns

Table 1: Summary Statistics

	Stock Index	Bond Index	Apple Stock
Maximum	0.0898	0.0206	0.1132
Minimum	-0.1276	-0.0284	-0.1377
Mean	0.0005	0.0002	0.0010
Std. Dev	0.0110	0.0028	0.0180
Skewness	-0.9284	-1.2056	-0.3349
Kurtosis	17.9513	15.4730	6.5912
JB test	24174.1	17188.0	1421.3
p-val	0.0000	0.0000	0.0000

Note: The table shows the statistics for daily returns from 01-06-2011 till 01-06-2021 (2556 observation).

4 Methodology

This section introduces the methods and techniques used for this research. First, the extreme value theory (EVT) is discussed, also paying attention to the regular variation property, and specifically focussing on Theorem 1 from Hyung and De Vries (2007). After this the safety-first principle will be described, following the two stage optimisation approach from Arzac and Bawa (1977). Clarification on the tail index estimator will be given, along with multiple estimation procedures. Finally, the different risk measures will be discussed and the effect of time horizon diversification.

4.1 Extreme value theory

As previously stated, Hyung and De Vries (2007) and Jansen et al. (2000) exploit the fat tailed property seen in asset returns. The tails of these heavy-tailed distributions decline by a power, rather than with exponential decline as seen in the normal distribution, which is eventually always faster. This can be understood by the regular variation property, formulated by Jansen et al. (2000) as

$$\lim_{s\to\infty} \frac{F(-ts)}{F(-t)} = s^{-\alpha} \tag{1}$$

where distribution F(s) is said to be varying regularly at negative infinity, and $s, \alpha > 0$. Here, α is known as the tail index, which will be discussed in more extent later. The ratio in Eqs. 1 goes to 1 as $s \to \infty$. Asymptotically, it holds that, as $s \to \infty$ and $A > 0$

$$F(-s) \approx As^{-\alpha} \tag{2}$$

More concrete, if we assume returns to be i.i.d. and to have tails that vary regularly at infinity, it corresponds to a first order

$$P\{X > s\} = As^{-\alpha} + o(s^{-\alpha}), \text{ as } s \to \infty, \tag{3}$$

where $o(f(x))$ is known as the "little oh notation", described in Appendix D, and $A, \alpha > 0$.

De Haan and Stadtmüller (1996) note the two non-trivial second order expansions at infinity, of which only one is important to this research, using notation as in Hyung and De Vries (2007),

$$P\{X > s\} = As^{-\alpha}[1 + Bs^{-\beta} + o(s^{-\beta})], \text{ as } s \to \infty, \tag{4}$$

with $A, \alpha, \beta > 0$ and $B \in \mathbb{R}$.

We assume two assets to have tails that are different, yet symmetric, and to vary regularly at infinity, resulting in second order expansions as $s \to \infty$, as in Hyung and De Vries (2007)

$$P\{X_i > s\} = P\{X_i < -s\} = A_i s^{-\alpha_i}\left[1 + B_i s^{-\beta_i} + o(s^{-\beta_i})\right] \tag{5}$$

for i = 1,2.

With this notation we consider convolution $X_1 + X_2$, such that we can investigate its tail properties. Hyung and De Vries (2007) (page 385) formulate Theorem 1 as follows, where Eqs. (1) and (2) are equivalent to Eqs. (5) in this paper:

*"**Theorem 1:** Suppose that the tails of the distributions of X_1 and X_2 satisfy Eqs. (1) and (2). Moreover, assume $2 < \alpha_1 < \alpha_2$ so that $E[X]$ and $E[X^2]$ are bounded. When X_1 and X_2 are independent, the asymptotic 2-convolution up to the second order terms is*

(I) if $\alpha_2 - \alpha_1 < \min(\beta_1, 1)$, *then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + A_2 s^{-\alpha_2} + o(s^{-\alpha_2})$

(II) if $1 < \alpha_2 - \alpha_1$ *and* $1 < \beta_1$,*then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + A_1 \alpha_1 E[X_2] s^{-\alpha_1 - 1} + o(s^{-\alpha_2})$

(III) if $\beta_1 < \alpha_2 - \alpha_1$ *and* $\beta_1 < 1$, *then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + A_1 B_1 s^{-\alpha_1 - \beta_1} + o(s^{-\alpha_1 - \beta_1})$

(IV) if $\alpha_2 - \alpha_1 = 1 < \beta_1$, *then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + \{A_2 + A_1 \alpha_1 E[X_2]\} s^{-\alpha_2} + o(s^{-\alpha_2})$

(V) if $\alpha_2 - \alpha_1 = \beta_1 < 1$, *then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + \{A_2 + A_1 B_1\} s^{-\alpha_1} + o(s^{-\alpha_1})$

(VI) if $\alpha_2 - \alpha_1 = \beta_1 = 1$, *then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + \{A_2 + A_1 \alpha_1 E[X_2] + A_1 B_1\} s^{-\alpha_2} + o(s^{-\alpha_2})$."

Finally, we have case *(VII)* where the tail indices are equal, such that $\alpha_1 = \alpha_2$. As $s \to \infty$, this results in $P\{X_1 + X_2 > s\} = (A_1 + A_2) s^{\alpha_1} + o(s^{-\alpha_1})$.

4.2 Safety-first principle

As stated before the safety-first criterion is a portfolio selection criterion incorporating downside risk. More specifically, the allocation of assets is based on the trade-off between expected return and its corresponding risk. This paper follows the lexicographic design as in Arzac and Bawa (1977), expanded by Hyung and De Vries (2007). Lexicographically, the safety-first principle can be defined as

$$\max_{\omega_i, b} (\pi, \mu), \tag{6}$$

$$s.t. \sum_i \omega_i V_{i,t} + b = W_t, \tag{7}$$

with

$$\pi = \begin{cases} 1 & \text{if } p = P\left\{\sum \omega_i V_{i,t+1} + br \le s\right\} \le \delta \\ 1 - p & \text{otherwise,} \end{cases} \tag{8}$$

where μ is equal to the expected value of wealth at time t, $E[\sum \omega_i V_{i,t+1}] + br$. Further, $V_{i,t}$ denotes the value of asset i at time t, with corresponding weight in portfolio ω_i. W_t represents the wealth level at time t, b takes the value of the investor's lent amount, so $b < 0$ represents borrowing, and r represents the risk-free rate. Further, s corresponds with a certain downside threshold for wealth, with δ being the maximum probability to reach this threshold. The mixed portfolio returns are defined as

$$R = \frac{\sum \omega_i V_{i,t+1}}{\sum \omega_i V_{i,t}}, \tag{9}$$

with $\bar{R} = E(R)$. Note that the definition for the portfolio returns differ from the single asset returns defined earlier. The portfolio returns are defined as simple returns, where as the single asset returns are defined as log-returns.

We follow the two problem set up of the safety-first criterion as in Arzac and Bawa (1977), where first the probability is rewritten using returns R, such that

$$P\left\{R \leq r + \frac{s - W_t r}{W_t - b}\right\} \leq \delta \tag{10}$$

We define $q_\delta(R)$, known as VaR, described in more detail later, so there is a probability of $\delta\%$ that negative returns exceed this value, such that $P\{R \leq q_\delta(R)\} = \delta$. The safety-first investor now first maximises the ratio, formulated as

$$\max_{\omega_i} \frac{(\bar{R} - r)}{(r - q_\delta(R))} \tag{11}$$

which reads the ratio of the risk premium to the return opportunity loss that he is willing to incur with probability δ. The second stage of the Arzac and Bawa (1977) approach sees the investor determine the scale of the risky part of his portfolio from the budget constraint, such that

$$W_t - b = \frac{s - W_t r}{q_\delta(R) - r}, \tag{12}$$

This implies that the safety-first investor, as expected, displays decreasing absolute risk aversion. As the initial wealth increases, as will the invested amount in the riskier assets. For more information on this part, we would like to refer to Arzac and Bawa (1977).

4.3 Tail index estimator

This section focuses on the estimation of the tail index α. For the first order tail index, α, Jansen et al. (2000) and Hyung and De Vries (2007) opt for the Hill (1975) estimator, which is a semi-parametric approach as the assumption of Pareto distribution in tails only hold asymptotically. The Hill estimator can be seen as a maximum likelihood estimator (MLE) for the conditional Pareto distribution. Assume descending order statistics, $X_{(1)}, ..., X_{(n)}$, such that $X_{(1)} \geq X_{(2)} \geq ... \geq X_{(n)}$, so the Hill estimator approximates the m-th largest observation to estimate $\frac{1}{\hat{\alpha}}$, such that

$$\frac{1}{\hat{\alpha}} = \frac{1}{m} \sum_{i=1}^{m} \left[\ln X_{(i)} - \log X_{(m+1)}\right] \tag{13}$$

where $X_{(i)}$ are the highest, positive order statistics as we consider the left tail through loss returns. Here the choice of m is important for the tail index estimate's properties. The method used in Jansen et al. (2000) and Hyung and De Vries (2007) for the choice of m is the Hall (1990) bootstrap, where an initial estimation of m has to be given. This paper uses tea Package (R) for the computation of the Hall bootstrap.

Huisman et al. (2001) find that for small samples it is better to use a different method. They propose to compute κ Hill estimates, where the estimation is robust to the choice of κ, with the modified Hill estimator being a weighted average of κ Hill estimators. The matrix notation for this weighted least squares (WLS) estimator follows from

$$\frac{\mathbf{1}}{\hat{\boldsymbol{\alpha}}}^{*} = Z\beta + \epsilon \tag{14}$$

where $\frac{\mathbf{1}}{\hat{\boldsymbol{\alpha}}}^{*}$ is a vector of Hill estimates $\frac{1}{\hat{\alpha}}(m)$, $m = 1, 2, .., \kappa$, Z is a matrix of size $(\kappa * 2)$ with a column of ones and a column with $\{1, 2, ..., \kappa\}'$. Due to the heteroskedasticity in ϵ Huisman et al. (2001) propose to use WLS with weight matrix $(\kappa * \kappa)$ W, with diagonal elements $\{\sqrt{1}, \sqrt{2}, ..., \sqrt{\kappa}\}$ and 0 elsewhere. The point estimated of tail index, $\hat{\alpha}$, follows from $\frac{1}{\hat{\alpha}}$ which is the intercept, i.e. the first element of the estimated vector $\hat{\beta}_{WLS} = (Z'W'WZ)^{-1}Z'W'W\frac{\mathbf{1}}{\hat{\boldsymbol{\alpha}}}^{*}$. Huisman et al. (2001) show that the estimation is robust to the choice of κ thus we choose $\kappa = 51 \approx \sqrt{n}$, with $n = 2556$, the starting value for m in Jansen et al. (2000) and Hyung and De Vries (2007). We calculate tail estimates both through the Huisman et al. (2001) as the Hall (1990) procedures, but continue the research with the estimated $\hat{\alpha}_{Hall}$, as the sample is sufficiently large to do so.

As already introduced in the section on EVT this paper focuses not solely on the first order tail estimates α. The second order tail index β is estimated using a consistent estimator for the ratio of β and α, as formulated in Hyung and De Vries (2007) as

$$\frac{\widehat{\beta}}{\alpha} = \frac{\ln \hat{m}}{2\ln n - 2\ln \hat{m}} \tag{15}$$

where m corresponds with Hall bootstrap's choice of m of the estimation of the tail index.

4.4 Risk measures

This paper extends to available literature by evaluating the effect of the inclusion of a different risk measure in the safety-first principle. Though a one-on-one comparison is not possible because safety-first portfolio selection is based on VaR, we do implement second order terms within another risk measure. Emmer et al. (2013) compare VaR, ES and Expectiles, and conclude ES to be the best risk measure in practice, making it an interesting alternative to the VaR approach by Hyung and De Vries (2007). This section introduces these two risk measures, compares them, and shows how the second order terms are are implemented in the risk measures. First, VaR is discussed after which ES is explained in more detail. This paper does not consider Expectiles.

Value-at-Risk

Value-at-Risk (VaR) is a measure of (downside) risk, that represents the expected minimum one day return, for a probability δ and given portfolio. In a two-asset portfolio, $VaR(\omega, \delta)$ thus corresponds with the VaR of a portfolio with weights ω and loss probability δ, such that

$$P\{\omega X_1 + (1-\omega)X_2 > VaR(\omega, \delta)\} = \delta \tag{16}$$

where time index t is omitted omitted from loss returns $X_{i,t}$.

Note that, if the portfolio return R is used as notation, it is easily seen that $q_\delta(R) = -VaR(\omega, \delta)$ and that the safety-first criterion is violated whenever $q_\delta(R) < r + \frac{s - W_t r}{W_t - b}$. The unit-investment VaR is estimated as

$$\hat{q}_p = X_{(m)} \left(\frac{m}{np}\right)^{1/\hat{\alpha}} \tag{17}$$

In the context of a second order convolution and a two-asset portfolio, $q_\delta(R)$ is estimated for given ω and δ by solving

$$\omega^{\alpha_1} A_1 q_\delta^{-\alpha_1} + (1-\omega)^{\alpha_2} A_2 q_\delta^{-\alpha_2} \approx \delta \tag{18}$$

after plugging $A_i = \frac{m_i}{n} X_{(m_i)}^{\alpha_i}$ into Proposition 3 (Appendix C) from Hyung and De Vries (2007). This follows from the β and α estimates, which will be presented in the results section, and Theorem 1 from Hyung and De Vries (2007). The q_δ's are simply found by finding the intersects with the x-axis after subtracting δ from both sides in Eqs. 18.

Expected Shortfall

Expected Shortfall (ES), also known as Conditional Value-at-Risk, is a risk measure that, just like VaR, can be used intuitively. Where VaR is defined as the loss level that will not be exceeded with a certain probability, ES is the expected loss conditional on exceedence of the loss level. Formally, ES is defined as the expected value of loss return, X, conditional on the fact that X exceeds $VaR(\omega, \delta)$, so

$$ES_{(\omega,\delta)} = \mathbb{E}\{X|X > VaR(\omega, \delta)\} \tag{19}$$

$$= \frac{1}{1-\delta}\int_{\delta}^{1} VaR(\omega, u)du \tag{20}$$

with $VaR(\omega, \delta)$ as defined before. One significant, and intuitive, property of ES is that it is greater or equal to the VaR with same δ for portfolio with given ω. Another is that the ES estimate increases as the quantile decreases. We have to reformulate Eqs. 19 in the form of a survival function, $S(t) = 1 - F(t) = P\{T_i \geq t\}$, to incorporate the first and second order convolutions we are interested in. In this context the threshold t corresponds with a loss level. We thus reformulate Eqs. 19 as

$$ES_{(\omega,\delta)} = \mathbb{E}[X|X \geq Var(\omega, \delta)] \tag{21}$$

$$= VaR(\omega, \delta) + \frac{1}{\delta}\int_{VaR(\omega,\delta)}^{\infty} P\{X > s\}ds \tag{22}$$

for which the derivation can be found in Appendix B2, where the indicator function is defined as $\mathbb{I}_{\{X>s\}} = 0$ if $x \leq s$ and 1 if $x > s$, such that $\int_0^\infty \mathbb{I}_{\{X>s\}}ds = X$. From this expression the second order approximation follows. For a single asset we use second order expansion $P\{X > s\} \approx As^{-\alpha} + ABs^{-\alpha-\beta}$ and for a two asset portfolio we can use the notation as in case *(I)* of Theorem 1 in Hyung and De Vries (2007), $P\{\omega X_1 + (1-\omega)X_2 > s\} \approx \omega^{\alpha_1}A_1 s^{-\alpha_1} + (1-\omega)^{\alpha_2}A_2 s^{-\alpha_2}$, to calculate the integrals. The estimates of A_i for the mixed portfolio are the same as in Eqs.18. A and B for the single asset are estimated simultaneously, for which the derivation can be found in Appendix B3.

4.5 Longer investment horizons

Jaggia and Thosar (2000) describe the effect of time horizon on risk aversion. Though several academics find that utility function optimisation, with incorporated constant relative risk aversion, is independent of time horizon, investment managers claim that a larger part of the portfolio should be allocated to risky assets as the time horizon shifts. This is due to the fact that a longer time horizon allows for more time for a portfolio to recover from stock market dips. Riskier assets usually have higher mean returns, implying a favourable position in the long run. Jaggia and Thosar (2000) incorporate this risk tolerance into the utility function. Furthermore, Lenoir and Tuchschmid (2001) confirm the investment manager's perspective by finding that optimal portfolio allocation shifts from relatively safer to riskier assets as the time horizon increases. This makes it an interesting path to explore in terms of safety-first investors.

When considering longer investment windows, i.e. k-periods, we get $Y_{i,k}$, the k-period loss return for asset i, with $X_{i,j}$ assumed to be i.i.d. for each asset i and

$$Y_{i,k} = \sum_{j=1}^{k} X_{i,j} \tag{23}$$

This paper will focus solely on one day and 10 day time horizons, as this is typically done in financial applications. Though quite a strong assumption, we assume returns to be i.i.d. in order to apply EVT and compute probabilities. Consider Eqs. 2, then it follows from Feller (1957), who deals with sums of heavy-tailed random variables, that, as $s \to \infty$,

$$P\{Y_{i,k} \geq s\} = P\{X_{i,1} + X_{i,2} + ... + X_{i,k} \geq s\} \tag{24}$$

$$\approx kA_i s^{-\alpha_i} \tag{25}$$

For normally distributions the *"square-root-of-time rule"* can be used so that the multi-period VaR differs with factor $\sqrt{k}$, yet this does not hold for heavy tailed distributions. In a k-period analysis for heavy tailed distributions we use the so called *"α-root-of-time rule"*, which implies a scaling factor of $k^{1/\alpha}$. It thus follows follows that k-period unit VaR equals $k^{1/\hat{\alpha}}\hat{q}_\delta$, using the α root of time rule. The same principle holds for the second order convolution such that in case *(I)* of Theorem 1 in Hyung and De Vries (2007), if $\alpha_2 - \alpha_1 < \min(\beta_1, 1)$, we get

$$P\{Y_{1,k} + Y_{2,k} > s\} \approx k(A_1 s^{-\alpha_1} + A_2 s^{-\alpha_2}) \tag{26}$$

for which a derivation can be found in Appendix B1. We thus use the α-root-of-time rule for the second order convolution, as for the first order convolution, and get for k-period unit VaR's, $k^{1/\hat{\alpha}}\hat{q}_\delta$. A good thing to notice that using the statistically preferred α-root-of-time rule for heavy-tailed distributions yields lower k-period VaR estimates as $k^{1/2} > k^{1/\alpha}$ when $\alpha > 2$.

One final remark we would like to stress is a finding by Dacorogna et al. (1995). They find that for lower frequency data the tail indices, α, need not be re-estimated, such that the highest available frequency can, and should, be used for tail index estimation. This is because $\hat{\alpha}$ generally has higher efficiency as data frequency increases.

5 Results

This section covers and elucidates the results found during the research. First the results in Jansen et al. (2000) and Hyung and De Vries (2007) are replicated with different data. The tail index estimates, the estimated VaR levels and the safety first portfolios are presented consecutively. All first order convolution results can be found in the Appendix A. In some tables certain results from the Jansen et al. (2000) approach are summarised, information regarding this can be found in the text and table notes. Finally, results for the stated Hyung and De Vries (2007) extensions will be presented, where first a longer investment horizon is considered after which second order ES estimation is presented.

5.1 Tail index estimates and exceedence values

The top part of Table 2 presents the tail index point estimates for the lower tails. Both the Huisman approach as the Hall bootstrap procedure estimates are presented. The values for m, and $X_{(m)}$ correspond with the Hall bootstrap procedure. Here m is the number of order statistics included in the Hall bootstrap and $X_{(m)}$ is the corresponding loss return. The remainder of this paper uses the $\hat{\alpha}_{Hall}$ estimates for reasons stated earlier. The bottom part of the table shows the unit-investment VaR estimates, computed as in Eqs. 17. These values can be interpreted as follows, for example if we take $p = 1/n$, there is a 1 in 2556 chance that daily stock index returns exceed -10.68%. Especially the probabilities with lower frequencies than $1/n$ are interesting since these cannot be estimated using the empirical distribution.

Table 2: Tail index estimates and exceedence values

	$\hat{\alpha}_{Huisman}$	$\hat{\alpha}_{Hall}$	m	$X_{(m)}$
Stock Index	2.7316	2.7130	49	0.0254
Bond Index	1.9044	2.9199	87	0.0045
Apple Stock	3.5455	3.4273	10	0.0680
	X_p: Exceedences for given p			
	$p = 0.05$	$p = 1/n$	$p = 1/1.5n$	$p = 1/2n$
Stock Index	-0.0179	-0.1068	-0.1240	-0.1379
Bond Index	-0.0040	-0.0209	-0.0241	-0.0265
Apple Stock	-0.0323	-0.1331	-0.1498	-0.1629

Note: the top part of the table presents the tail estimate with a Huisman and Hall approach. The bottom part presents the single asset VaR levels for different probabilities.

5.2 Estimated VaR levels

Here the results for the estimated VaR levels are presented. First the results of of the portfolio mix with the two indices are presented, after which the stock index and Apple stock portfolio's are shown. For the second order convolution note that $\beta_{SI} = 1.3350$ and $\alpha_{BI} - \alpha_{SI} = 0.2069$, $\alpha_{AAPL} - \alpha_{SI} = 0.7143$. So for both mixed portfolios we get case *(I)* from Theorem 1 Hyung and De Vries (2007): "*(I) if* $\alpha_2 - \alpha_1 < \min(\beta_1, 1)$*, then* $P\{X_1 + X_2 > s\} = A_1 s^{-\alpha_1} + A_2 s^{-\alpha_2} + o(s^{-\alpha_2})$". We consider 11 different hypothetical portfolio mixes, with weights for asset 1 decreasing from 100% to 0% in steps of 10%. The same assumptions and steps as in Jansen et al. (2000) are taken into account for the first order convolution, from which the results are found in Appendix A. The most significant part follows from Geluk and deHaan (1987), who state that in a mixed portfolio the tail index of the asset with the fattest tail dominates. In practice this means the stock index tail index, $\alpha_{SI} = 2.7130$, is used for all portfolio mixes except for the 100% alternative asset portfolios, be it bond index, $\alpha_{BI} = 2.9199$, or AAPL stock, $\alpha_{AAPL} = 3.4273$. The estimated first order VaR levels are computed as in Eqs. 17. In Table 3 the second order VaR estimates for the hypothetical portfolios are presented.

Table 3: Second order VaR estimates

Port. weights	Probabilities: expected number of occurrences			
	p=0.05	p=1/n	p=1/1.5n	p=1/2n
Portfolios of S&P 500 (stock) index and S&P 500 bond index				
100% S.I.	-0.0179	-0.1068	-0.1240	-0.1379
90% S.I.	-0.0161	-0.0961	-0.1116	-0.1241
80% S.I.	-0.0143	-0.0855	-0.0992	-0.1103
70% S.I.	-0.0125	-0.0748	-0.0868	-0.0966
60% S.I.	-0.0107	-0.0642	-0.0745	-0.0828
50% S.I.	-0.0090	-0.0536	-0.0622	-0.0692
40% S.I.	-0.0073	-0.0432	-0.0501	-0.0557
30% S.I.	-0.0056	-0.0332	-0.0385	-0.0428
20% S.I.	-0.0043	-0.0247	-0.0285	-0.0317
10% S.I.	-0.0038	-0.0202	-0.0232	-0.0256
0% S.I.	-0.0040	-0.0209	-0.0241	-0.0265
Portfolios of S&P 500 (stock) index and AAPL stock				
100% S.I.	-0.0179	-0.1068	-0.1240	-0.1379
90% S.I.	-0.0161	-0.0962	-0.1117	-0.1242
80% S.I.	-0.0146	-0.0860	-0.0998	-0.1109
70% S.I.	-0.0141	-0.0778	-0.0900	-0.0998
60% S.I.	-0.0150	-0.0740	-0.0850	-0.0937
50% S.I.	-0.0171	-0.0764	-0.0869	-0.0953
40% S.I.	-0.0198	-0.0840	-0.0949	-0.1036
30% S.I.	-0.0228	-0.0946	-0.1067	-0.1161
20% S.I.	-0.0259	-0.1068	-0.1203	-0.1309
10% S.I.	-0.0291	-0.1198	-0.1349	-0.1467
0% S.I.	-0.0323	-0.1131	-0.1498	-0.1629

Note: VaR level estimates of mixed portfolios consisting of the stated percentage of stock index, with the remainder appointed to the other asset option.

5.3 Safety-first portfolio selection

We next turn to the computation of the safety-first portfolios, for which the estimated VaR values are used. The investor first maximises $(R-r)/(r-q_\delta(R))$, where we set $r=1$, as for daily observations other values would make little sense. The results to this step for the first order can be found in Appendix A, and the results to the second order are displayed in Table 4. Recall that the second stage of the portfolio selection consists of determining the scale of the risky part of his portfolio from the budget constraint. In other words, the safety-first investor determines the amount borrowed against the risk free rate, $r=1$ in our case. The borrowed amount is dependent on the critical wealth level, s, set by the investor. As an example we will set $s=0.70W$ and take $\delta=0.05$. For the stock index - AAPL stock pair portfolio we then get an optimal second order safety-first portfolio consisting of 60% stock index and the amount borrowed equals $b=(0.70-q_\delta(R))W/(q_\delta(R)-r)=(0.70-(1-0.0150))W/((1-0.0150)-1)=19W$. This implies that the investor borrows an additional 19 times its initial wealth to invest in a portfolio consisting of 60% stock index and 40% AAPL stock, with a mean return of 1.000589. This results in a mean return of $(1.000589*20)-19=1.0118$, and an downside risk of $((1 - 0.0150) * 20) - 19 = 0.70$, as is demanded by the safety-first criterion.

The safety-first criterion thus determines the amount that is borrowed. One might think that investing only in the riskier asset yields a higher mean return, i.e. 100% AAPL stock with a mean return of 1.0011, but this is not the case. The amount borrowed will be significantly less, $(0.70 - (1 - 0.0323))W/((1 - 0.0323) - 1) = 8.2879W$, with a mean return of $(1.0011 * 9.2879) - 8.2879 = 1.0104 < 1.0118$. Of course the optimal portfolios depend on the investor's preferences. For this example we followed the proposed levels of δ and s by Jansen et al. (2000), but clearly different risk preferences will yield different returns. Taking a closer look at the top part of Table 4 gives one remarkable insight. The first order optimal portfolio, indicated by "J", consists of a larger part of the riskier asset than the second order portfolio, something that is not according to our expectations, as we would expect more balanced portfolios for the latter. Hyung and De Vries (2007) showed that the second order expansion should result in a more balanced portfolio mix, moving away from corner solutions.

To conclude, we find slightly different results to Hyung and De Vries (2007). Though we see a more balanced portfolio for the stock index - AAPL stock pair, with a relatively high value for δ, we do not find this in general for all optimal portfolios.

Table 4: Second order safety-first portfolio selection

Port. weights	$q_\delta(R)$, $\delta = 0.05$	$\frac{(\bar{R}-r)}{(r-q_\delta(R))}$	$q_\delta(R)$, $\delta = 1/n$	$\frac{(\bar{R}-r)}{(r-q_\delta(R))}$
Portfolios of S&P 500 (stock) index and S&P 500 bond index				
100% S.I.	1-0.0179	0.0327	1-0.1068	0.0055
90% S.I.	1-0.0161	0.0361	1-0.0961	0.0060
80% S.I.	1-0.0143	0.0402	1-0.0855	0.0067
70% S.I.	1-0.0125	0.0454	1-0.0748	0.0076
60% S.I.	1-0.0107	0.0521	1-0.0642	0.0087
50% S.I.	1-0.0090	0.0606	1-0.0536	0.0102
40% S.I.	1-0.0073	0.0723	1-0.0432	0.0122
30% S.I..	1-0.0056	0.0898	1-0.0332	0.0151
20% S.I.	1-0.0043	0.1076^{*J}	1-0.0247	0.0187^{J}
10% S.I.	1-0.0038	0.1020	1-0.0202	0.0192*
0% S.I.	1-0.0040	0.0476	1-0.0209	0.0091
Portfolios of S&P 500 (stock) index and AAPL stock				
100% S.I.	1-0.0179	0.0327	1-0.1068	0.0055
90% S.I.	1-0.0161	0.0364	1-0.0962	0.0061
80% S.I.	1-0.0146	0.0402	1-0.0860	0.0068
70% S.I.	1-0.0141	0.0417	1-0.0778	0.0076
60% S.I.	1-0.0150	0.0392*	1-0.0740	0.0080
50% S.I.	1-0.0171	0.0345	1-0.0764	0.0077
40% S.I.	1-0.0198	0.0299	1-0.0840	0.0070
30% S.I..	1-0.0228	0.0261	1-0.0946	0.0063
20% S.I.	1-0.0259	0.0232	1-0.1068	0.0056
10% S.I.	1-0.0291	0.0213	1-0.1198	0.0052
0% S.I.	1-0.0323	0.0348^{J}	1-0.1131	0.0099^{*J}

Note: $r = 1$, the optimal second order portfolio among available choices is indicated by an asterisk (*), while the optimal first order counterpart is indicated by a "J". Note that this value does not correspond with the first order portfolio, and is only indicated as reference.

5.4 Longer investment horizon

Next we turn to the investigation of the effect of longer investment horizons on the safety-first principle. We conduct the same steps as in previous parts which result in VaR estimates and safety-first portfolios for 11 hypothetical portfolios, as illustrated in Table 5. As argued in the methodology section this paper only focuses on a 10-period investment horizon. The left side of the table corresponds with the first order convolution, while the right side of the table corresponds with the second order convolution. All safety-first portfolio are computed with $r = 1$, for which the reasoning remains the same as stated earlier as this time-frame is too short to make a difference.

For the first order we see similar results to the 1-period investment window, which is in line with findings from Jansen et al. (2000). Both asset pairs have the same resulting portfolio allocations as under the daily horizon. For the $r = 1$ and $s = 0.70W$ safety-first case, we find an average return on the optimal portfolio, consisting of 0% stock index and 100% AAPL stock, of 1.0255. As for the index-pair we find 1.0528.

We next turn to the second order results. As with the first order mixed-portfolio VaR estimates, the tail index used for the α-root-of-time rule is the one of the dominant tail. Thus only for the hypothetical portfolios with 0% stock index the tail index estimate of the other asset is used. Here we see the same as for the first order convolution. The optimal portfolio allocations are almost the same. This makes sense for the stock index - AAPL stock pair as the riskier asset already contributed to 100% of the portfolio. For the index-pair we see a slight shift towards the riskier stock index, though only marginal. We find average returns of optimal portfolios of 1.0557 and 1.0255, respectively. With the latter corresponding with the 100% AAPL stock portfolio. The reason for the small effect of the second order convolution could be the fact that the α-root-of-time rule only implements the first order term, α. It would be interesting to see what results a "second order" α-root-of-time rule would yield, but this is left for further research. Finally, it would be interesting to see how the optimisation would react to an even further investment horizon extension, say one month or a year.

Table 5: 10-day investment horizon, VaR estimates and SF portfolios

	First order		Second order	
Port. weights	$q_\delta(R)$, $\delta = 10/n$	$\frac{(\bar{R}-r)}{(r-q_\delta(R))}$	$q_\delta(R)$, $\delta = 10/n$	$\frac{(\bar{R}-r)}{(r-q_\delta(R))}$
Portfolios of S&P 500 (stock) index and S&P 500 bond index				
100% S.I.	1-0.1068	0.0550	1-0.1068	0.0550
90% S.I.	1-0.0959	0.0607	1-0.0960	0.0607
80% S.I.	1-0.0843	0.0684	1-0.0855	0.0674
70% S.I.	1-0.0732	0.0777	1-0.0748	0.0761
60% S.I.	1-0.0623	0.0900	1-0.0643	0.0871
50% S.I.	1-0.0507	0.1081	1-0.0535	0.1025
40% S.I.	1-0.0403	0.1318	1-0.0432	0.1231
30% S.I.	1-0.0296	0.1712	1-0.0334	0.1519
20% S.I.	1-0.0226	0.2075*[1]	1-0.0252	0.1858*
10% S.I.	1-0.0202	0.1950	1-0.0213	0.1851[1]
0% S.I.	1-0.0209	0.0920	1-0.0209	0.0921
Portfolios of S&P 500 (stock) index and AAPL stock				
100% S.I.	1-0.1068	0.0550	1-0.1068	0.0550
90& S.I.	1-0.1064	0.0552	1-0.0963	0.0610
80% S.I.	1-0.1092	0.0539	1-0.0865	0.0680
70% S.I.	1-0.1144	0.0515	1-0.0801	0.0735
60% S.I.	1-0.1245	0.0474	1-0.0801	0.0736
50% S.I.	1-0.1320	0.0448	1-0.0869	0.0680
40% S.I.	1-0.1348	0.0440	1-0.0984	0.0602
30% S.I.	1-0.1398	0.0426	1-0.1122	0.0531
20% S.I.	1-0.1470	0.0410	1-0.1273	0.0473
10% S.I.	1-0.1586	0.0391	1-0.1430	0.0433
0% S.I.	1-0.1331	0.0851*[1]	1-0.1331	0.0851*[1]

Note: $r = 1$, the optimal k-period portfolio among available choices is indicated by an asterisk (*), while the optimal 1-period counterpart is indicated by a "1". Note that this value does not correspond with the 1-period portfolio, and is only indicated as reference.

5.5 Expected Shortfall as risk measure

This section presents the ES estimates using second order tail index terms. First the single asset estimates are shown, after which the same hypothetical portfolios are constructed as for the safety-first investor.

We first turn to the single asset ES estimation. We take the hall bootstrap values for $\hat{\alpha}_{Hall}$, m and $X_{(m)}$ from Table 2 and compute $\hat{\beta}$ from Eqs. 15. Note that the first order statistic is just the minimum value in the data set, as we consider loss returns. Table 6 presents the found ES estimates. We see that for all probabilities the ES estimate is at least as high as the corresponding VaR estimate and that the ES estimate increases as the probability decreases. The estimates thus satisfy the properties stated in the methodology section. Note that the ratios between the ES- and VaR estimates remain nearly constant per asset, and increase slightly as the quantile (or expectation) decreases. The interpretation of the ES estimate is as follows, if we take the stock index as example, with a probability level of $p = 1/n$, the stock index is expected to have a loss of $20,93\%$, given that negative returns exceed the $1/n$'th quantile.

Table 6: Single asset ES estimates

	$\hat{\beta}$		$X_{(1)}$	
Stock Index	1.3350		0.1276	
Bond Index	1.9288		0.0284	
Apple Stock	0.7118		0.1377	
	X_p:			
	$p = 0.05$	$p = 1/n$	$p = 1/1.5n$	$p = 1/2n$
Stock Index	0.0290	0.2093	0.2438	0.2716
Bond Index	0.0072	0.0476	0.0547	0.0605
Apple Stock	0.0444	0.1968	0.2223	0.2423

Note: the top part of the table presents the second order tail estimates and the additionally needed order statistic. The bottom part of the table presents the ES estimates for different probabilities. Note that the missing information for the computation can be found in Table 2.

We next compute ES estimates for the hypothetical portfolios used to examine the safety-first investor. Here we see the same principle as with the single assets, where the ratio between VaR and ES remains almost constant. For all portfolios and probabilities the ES estimate is at least as high as the VaR estimate and increase as the quantile decreases. Once again we see that the estimates satisfy the properties stated earlier.

Table 7: Mixed portfolio ES estimates

Port. weights	Probabilities: p=0.05	p=1/n	p=1/1.5n	p=1/2n
Portfolios of S&P 500 (stock) index and S&P 500 bond index				
100% S.I.	0.0283	0.1689	0.1961	0.2180
90% S.I.	0.0254	0.1520	0.1765	0.1962
80% S.I.	0.0226	0.1351	0.1569	0.1744
70% S.I.	0.0198	0.1182	0.1372	0.1526
60% S.I.	0.0170	0.1014	0.1177	0.1309
50% S.I.	0.0142	0.0846	0.0983	0.1093
40% S.I.	0.0114	0.0682	0.0791	0.0879
30% S.I.	0.0089	0.0523	0.0606	0.0674
20% S.I.	0.0067	0.0384	0.0445	0.0493
10% S.I.	0.0057	0.0307	0.0353	0.0390
0% S.I.	0.0060	0.0316	0.0363	0.0401
Portfolios of S&P 500 (stock) index and AAPL stock				
100% S.I.	0.0283	0.1689	0.1961	0.2180
90% S.I.	0.0255	0.1520	0.1765	0.1963
80% S.I.	0.0230	0.1357	0.1575	0.1741
70% S.I.	0.0216	0.1216	0.1409	0.1563
60% S.I.	0.0222	0.1130	0.1300	0.1437
50% S.I.	0.0246	0.1128	0.1287	0.1413
40% S.I.	0.0281	0.1209	0.1369	0.1495
30% S.I.	0.0322	0.1345	0.1517	0.1652
20% S.I.	0.0366	0.1512	0.1702	0.1852
10% S.I.	0.0411	0.1693	0.1906	0.2072
0% S.I.	0.0457	0.1946	0.2116	0.2301

Note: ES estimates of mixed portfolios consisting of the stated percentage of stock index, with the remainder appointed to the other asset option.

6 Conclusion

Research on safety-first investors is available in abundance, though there remain aspects that have yet to be researched. The aim of this paper is to address part of this unexplored region, and to replicate findings by Jansen et al. (2000) and Hyung and De Vries (2007) with a different data set.

First we computed safety-first portfolios as in Jansen et al. (2000) and Hyung and De Vries (2007) with different data pairs. The researched differed from them as we also considered a comparison between an index and a stock. We found similar results, though one remarkable outcome was the less balanced optimal safety-first portfolio in one of the second order computations. Although not completely clear, a possible reasoning behind this is the closely valued tail index estimates of the the available assets.

We next implemented multiple extensions to Hyung and De Vries (2007). First of all, we considered a longer investment horizon for the second order convolution. Something that was researched for the the first order by Jansen et al. (2000). Next we were interested how a second order terms could be implemented within other risk measures. The risk measure we opted for in this paper was Expected Shortfall as literature finds several properties of this risk measure to be preferred over VaR.

The extension on longer investment horizon found similar results for the second order convolution as Jansen et al. (2000) found for the first order safety-first portfolios. The optimal portfolios differed very little from their one-day counterparts. We do think it is interesting for further research to investigate the possibilities of a "second order" α-root-of-time rule and the consideration of even longer time horizons, to see what the effect would be on the optimal portfolios.

Finally, we implemented a conceptually superior risk measure to VaR in the second order expansion framework. We showed how ES estimates can be computed with the second order terms, and computed estimates for single assets and mixed portfolios. The results illustrated the formulated properties from the methodology section.

It could be interesting to further research to expand this paper's research to multi-asset portfolios. This paper shows the conceptual power of the second order expansion, yet is practically far from robust as an investor's portfolio rarely consists of two assets. The basis for such an approach can be found in Hyung and De Vries (2007). Expanding that research with the proposed adjustments from this paper is something that would add to literature. Furthermore, we suggest more researched should be done on a "second order" α-root-of-time rule. Finally, we showed how the second order expansion can be implemented in ES, but further research could be done on its applications.

Appendices

Appendix A: First order results

Table 8: First order VaR estimates

Port. weights	Probabilities: expected number of occurrences p=0.05	p=1/n	p=1/1.5n	p=1/2n
Portfolios of S&P 500 (stock) index and S&P 500 bond index				
100% S.I.	-0.0179	-0.1068	-0.1240	-0.1379
90% S.I.	-0.0160	-0.0959	-0.1114	-0.1238
80% S.I.	-0.0141	-0.0843	-0.0978	-0.1088
70% S.I.	-0.0123	-0.0732	-0.0850	-0.0945
60% S.I.	-0.0104	-0.0623	-0.0723	-0.0804
50% S.I.	-0.0085	-0.0507	-0.0589	-0.0655
40% S.I.	-0.0067	-0.0403	-0.0468	-0.0521
30% S.I.	-0.0050	-0.0296	-0.0344	-0.0383
20% S.I.	-0.0038	-0.0226	-0.0262	-0.0291
10% S.I.	-0.0034	-0.0202	-0.0235	-0.0261
0% S.I.	-0.0040	-0.0209	-0.0241	-0.0265
Portfolios of S&P 500 (stock) index and AAPL stock				
100% S.I.	-0.0179	-0.1068	-0.1240	-0.1379
90% S.I.	-0.0178	-0.1064	-0.1236	-0.1374
80% S.I.	-0.0183	-0.1092	-0.1268	-0.1410
70% S.I.	-0.0191	-0.1144	-0.1328	-0.1477
60% S.I.	-0.0208	-0.1245	-0.1445	-0.1607
50% S.I.	-0.0221	-0.1320	-0.1533	-0.1704
40% S.I.	-0.0226	-0.1348	-0.1566	-0.1741
30% S.I.	-0.0234	-0.1398	-0.1624	-0.1806
20% S.I.	-0.0246	-0.1470	-0.1707	-0.1898
10% S.I.	-0.0265	-0.1586	-0.1842	-0.2048
0% S.I.	-0.0323	-0.1331	-0.1498	-0.1629

Note: VaR level estimates of mixed portfolios consisting of the stated percentage of stock index, with the remainder appointed to the other asset option.

Table 9: First order safety-first portfolio selection

Port. weights	$q_\delta(R)$, $\delta = 0.05$	$\frac{(\bar{R}-r)}{(r-q_\delta(R))}$	$q_\delta(R)$, $\delta = 1/n$	$\frac{(\bar{R}-r)}{(r-q_\delta(R))}$
Portfolios of S&P 500 (stock) index and S&P 500 bond index				
100% S.I.	1-0.0179	0.0328	1-0.1068	0.0055
90% S.I.	1-0.0160	0.0362	1-0.0959	0.0061
80% S.I.	1-0.0141	0.0408	1-0.0843	0.0068
70% S.I.	1-0.0123	0.0463	1-0.0732	0.0077
60% S.I.	1-0.0104	0.0535	1-0.0623	0.0090
50% S.I.	1-0.0085	0.0643	1-0.0507	0.0108
40% S.I.	1-0.0067	0.0783	1-0.0403	0.0131
30% S.I.	1-0.0050	0.1014	1-0.0296	0.0170
20% S.I.	1-0.0038	0.1225*	1-0.0226	0.0205*
10% S.I.	1-0.0034	0.1146	1-0.0202	0.0192
0% S.I.	1-0.0040	0.0479	1-0.0209	0.0091
Portfolios of S&P 500 (stock) index and AAPL stock				
100% S.I.	1-0.0179	0.0328	1-0.1068	0.0055
90% S.I.	1-0.0178	0.0329	1-0.1064	0.0055
80% S.I.	1-0.0183	0.0321	1-0.1092	0.0055
70% S.I.	1-0.0191	0.0307	1-0.1144	0.0054
60% S.I.	1-0.0208	0.0283	1-0.1245	0.0051
50% S.I.	1-0.0221	0.0267	1-0.1320	0.0047
40% S.I.	1-0.0226	0.0262	1-0.1348	0.0045
30% S.I.	1-0.0234	0.0254	1-0.1398	0.0044
20% S.I.	1-0.0246	0.0244	1-0.1470	0.0043
10% S.I.	1-0.0265	0.0233	1-0.1586	0.0039
0% S.I.	1-0.0323	0.0348*	1-0.1331	0.0084*

Note: $r = 1$, the optimal first order portfolio among available choices is indicated by an asterisk (*). Note that in the bottom part of the rightmost column the 100% and 90% stock index portfolios only differ by a margin not visible in this table.

Appendix B: Derivations

B1. Derivation second order longer investment horizon VaR

From Feller (1957) it follows that, if $\alpha_2 - \alpha_1 < \min(\beta, 1)$ and as $s \to \infty$ with X_i i.i.d. then

$$\begin{aligned} P\{\omega Y_{1,k} + (1-\omega) Y_{2,k} > s\} &= P\{\sum_{j=1}^{k} \omega X_{1,j} + (1-\omega) X_{2,j} > s\} \\ &\approx \sum_{j=1}^{k} P\{\omega X_{1,j} + (1-\omega) X_{2,j} > s\} \\ &= k\left[\omega^{\alpha_1} A_1 s^{-\alpha_1} + (1-\omega)^{\alpha_2} A_2 s^{-\alpha_2}\right] \end{aligned}$$

B2. Derivation Expected Shortfall to survival function

$$\begin{aligned} ES_{(\omega,\delta)} &= \mathbb{E}[X | X \geq Var(\omega,\delta)] \\ &= \frac{1}{P\{X \geq VaR(\omega,\delta)\}} \mathbb{E}[L \mathbb{I}_{\{X > VaR(\omega,\delta)\}}] \\ &= \frac{1}{\delta} \mathbb{E}\left[\int_0^\infty \mathbb{I}_{\{X>t\}} \mathbb{I}_{\{X>VaR(\omega,\delta)\}} dt\right] \\ &= \frac{1}{\delta} \int_0^\infty \mathbb{E}\left[\mathbb{I}_{\{X>t\}} \mathbb{I}_{\{X>VaR(\omega,\delta)\}}\right] dt \\ &= \frac{1}{\delta} \int_0^\infty P\{X > t, X > VaR(\omega,\delta)\} dt \\ &= \frac{1}{\delta} \int_0^{VaR(\omega,\delta)} P\{X > VaR(\omega,\delta)\} dt + \frac{1}{\delta} \int_{VaR(\omega,\delta)}^\infty P\{X > t\} dt \\ &= VaR(\omega,\delta) + \frac{1}{\delta} \int_{VaR(\omega,\delta)}^\infty P\{X > t\} dt \end{aligned}$$

B3. Derivation A and B for single asset Expected Shortfall

We have a set of two equations:

$$\begin{aligned} A X_{(m)}^{-\alpha} \left(1 + B X_{(m)}^{-\beta}\right) &= \frac{m}{n} \\ A X_{(1)}^{-\alpha} \left(1 + B X_{(1)}^{-\beta}\right) &= \frac{1}{n} \end{aligned}$$

We solve the first equation for A, which yields:

$$A = \frac{m}{n} X_{(m)}^{\alpha} \left(1 + B X_{(m)}^{-\beta}\right)^{-1}$$

We plug A into the second equation:

$$\frac{m}{n}\left[\frac{X_{(m)}^{\alpha} X_{(1)}^{-\alpha}}{1 + B X_{(m)}^{-\beta}} + \frac{B X_{(m)}^{\alpha} X_{(1)}^{-\alpha-\beta}}{1 + B X_{(m)}^{-\beta}}\right] = \frac{1}{n}$$

We now multiply both sides with the n*denominator of the fraction:

$$m\left(X_{(m)}^{\alpha}X_{(1)}^{-\alpha}+BX_{(m)}^{\alpha}X_{(1)}^{-\alpha-\beta}\right)=1+BX_{(m)}^{-\beta}$$

We now solve for B:

$$B\left(mX_{(m)}^{\alpha}X_{(1)}^{-\alpha-\beta}-X_{(m)}^{-\beta}\right)=1-mX_{(m)}^{\alpha}X_{(1)}^{-\alpha}$$
$$B=\frac{1-mX_{(m)}^{\alpha}X_{(1)}^{-\alpha}}{mX_{(m)}^{\alpha}X_{(1)}^{-\alpha-\beta}-X_{(m)}^{-\beta}}$$

Plug this expression into A to get A and B estimates, for which the integral can be calculated.

Appendix C: Proposition 3 Hyung and De Vries (2007)

"**Proposition 3.** *Under assumptions of* Theorem 1 *and if* $\alpha_2-\alpha_I<\min(\beta_l,1)$*, consider the downside risk level*

$$P\{\omega X_1+(1-\omega)X_2>s\}=\omega^{\alpha_1}A_1s^{-\alpha_1}\left[1+\frac{(1-\omega)^{\alpha_2}A_2}{\omega^{\alpha_1}A_1}s^{-\alpha_2+\alpha_1}+o\left(s^{-\alpha_2+\alpha_1}\right)\right]$$

and define the VaR implicitly as follows $P\{\omega X_1+(1-\omega)X_2>\text{VaR}(\omega,p)\}=p$*. By De Bruijn's theory on asymptotic inversion*

$$\text{VaR}(\omega,p)=\omega A_1^{\frac{1}{\alpha_1}}p^{-\frac{1}{\alpha_1}}\left[1+\frac{(1-\omega)^{\alpha_2}}{\omega^{\alpha_2}}\frac{A_2}{\alpha_1A_1^{\alpha_2/\alpha_1}}p^{\frac{\alpha_2-\alpha_1}{\alpha_1}}+o(1)\right]$$

for any $0<\omega<1$."

Appendix D: Little oh notation

$o(f(x))$ is known as the "little oh notation", which is the set of functions, $G(x)$, with a smaller or the same rate of growth as $f(x)$. Mathematically this can be illustrated as

$$o(f(x))=\{\forall g(x)\in G(x)|\forall c>0,\exists k>0,g(x)\leq cf(x),\forall x\geq k\} \tag{27}$$

which means that the remainder, $o(s^{-\alpha})$, will go to 0 faster than $s^{-\alpha}$. Equivalently, we could write Eqs. (3) as $As^{-\alpha}+f(s)$. The $o(s^{-\alpha})$ indicates that "the remainder" $f(s)$ goes to 0 faster than $s^{-\alpha}$.

Appendix E: Code

E1. Python script

```
import pandas as pd
import numpy as np
from numpy.linalg import inv
import matplotlib.pyplot as plt


### Get data from excel
df = pd.read_excel(r'/Users/wouternommensen/Desktop/thesis/Data.xlsx',
                   sheet_name='Data')
apple = pd.read_excel(r'/Users/wouternommensen/Desktop/thesis/AppleData.xlsx',
                      sheet_name='Apple')
compare = pd.read_excel(r'/Users/wouternommensen/Desktop/thesis/AppleData.xlsx',
                        sheet_name='Comparison')

### Compute returns
log_ret = pd.DataFrame(np.zeros((len(df) - 1, 2)))
app_ret = pd.DataFrame(np.zeros((len(df) - 1, 1)))
for j in range(2):
    for i in range(1, len(df)):
        i_log_ret = np.log(df.iat[i, j + 1]) - np.log(df.iat[i - 1, j + 1])
        log_ret.iat[i - 1, j] = i_log_ret
for i in range(1, len(apple)):
    i_log_ret = np.log(apple.iat[i, 2]) - np.log(apple.iat[i - 1, 2])
    app_ret.iat[i - 1, 0] = i_log_ret

returns = pd.DataFrame(np.zeros((len(df) - 1, 0)))
returns[[0, 1]] = log_ret
returns[[2]] = app_ret

### negatives to focus on lower tail
loss_ret = pd.DataFrame(np.zeros((returns.shape[0], returns.shape[1])))
for i in range(returns.shape[0]):
    for j in range(returns.shape[1]):
        loss_ret.loc[i, j] = -returns.loc[i, j]


def stats(input):
    stat = pd.DataFrame(np.zeros((6, input.shape[1])))
    for i in range(input.shape[1]):
        column = input[i]
        stat.iat[0, i] = column.max()
        stat.iat[1, i] = column.min()
        stat.iat[2, i] = column.mean()
```

```
        stat.iat[3, i] = column.std()
        stat.iat[4, i] = column.skew()
        stat.iat[5, i] = column.kurtosis()
    return stat


summary = pd.DataFrame(0, index=range(6), columns=range(1))
summary[0] = ["max", "min", "mean", "std", "skew", "kurt"]
summary[[1, 2]] = stats(log_ret)
summary[[3]] = stats(app_ret)


def only_pos(input):
    for i in range(len(input)):
        if input.loc[i] <= 0:
            input = input.drop([i])
    input = input.reset_index(drop=True)
    return input


### only positives for logret as input for hall bootstrap
pos_si = only_pos(loss_ret[0])
pos_bi = only_pos(loss_ret[1])
pos_app = only_pos(loss_ret[2])
positives = [pos_si, pos_bi, pos_app]
pos_ret = pd.concat(positives, axis=1)

#### write results, summary stats to .dat file
returns.to_csv(r'/Users/wouternommensen/Desktop/thesis1/01_results.dat',
    index=False, sep=' ', header=False)
returns.to_excel(r'/Users/wouternommensen/Desktop/thesis1/01_returns.xlsx')
summary.to_excel(r'/Users/wouternommensen/Desktop/thesis1/02_summary.xlsx')
pos_ret.to_excel(r'/Users/wouternommensen/Desktop/thesis1/03_pos_ret.xlsx')

#### Plots of index price and log returns
plt.plot(compare['Date'], compare['Apple'], 'g',
        compare['Date'], compare['StockIndex'], 'r',
        compare['Date'], compare['BondIndex'], 'b', )
plt.legend(['Apple', 'Stock Index', 'Bond Index'])
plt.savefig(r'/Users/wouternommensen/Desktop/thesis1/04_index_price1.png', dpi=300)
plt.clf()
plot_ret = plt.plot(df['Date'].iloc[1:, ], returns[2], 'g',
    df['Date'].iloc[1:, ], returns[0], 'r', df['Date'].iloc[1:, ], returns[1], 'b')
plt.legend(['Apple', 'Stock Index', 'Bond Index'])
plt.savefig(r'/Users/wouternommensen/Desktop/thesis1/05_index_returns1.png', dpi=300)
plt.clf()
```

```
### Hill estimator, Huisman approach
def hill(input):
    n = len(input)  # 2556
    ord_stat = input.sort_values(0)
    ord_stat = ord_stat.reset_index(drop=True)
    kappa = int(round(np.sqrt(n)))  # 51
    z_matrix = pd.DataFrame(np.ones((kappa, 2)))
    hill_vec = pd.DataFrame(np.zeros((kappa, 1)))
    hill_vec.index += 1  # index 1:51
    z_matrix.index += 1  # index 1:51
    ord_stat.index += 1

    ### vector with hill estimates 1,...,kappa
    for m in range(1, kappa + 1):  # 1:51
        sum_hill = 0
        for i in range(1, m + 1):  # 1:m
            elem_hill = np.log(ord_stat.loc[n + 1 - i] / ord_stat.loc[n - m])
            sum_hill += elem_hill
        hill_est = 1 / m * sum_hill
        z_matrix.loc[m, 1] = m
        hill_vec.loc[m, 0] = hill_est

    ### weight matrix
    w_matrix = pd.DataFrame(np.zeros((kappa, kappa)))
    for i in range(kappa):
        w_matrix.loc[i, i] = np.sqrt(i + 1)

    ### estimated beta
    z_matrix.index += -1  # get same indices as w
    hill_vec.index += -1
    z_trans = z_matrix.transpose()
    w_trans = w_matrix.transpose()
    beta = inv(z_trans.dot(w_trans).dot(w_matrix).dot(z_matrix)).dot(z_trans).
        dot(w_trans).dot(w_matrix).dot(hill_vec)

    alpha = 1 / beta.item((0, 0))
    return alpha


SI_huisman = hill(loss_ret[0])
BI_huisman = hill(loss_ret[1])
app_huisman = hill(loss_ret[2])
huisman = {'si_huis': [0, 0, SI_huisman], 'bi_huis': [0, 0, BI_huisman],
    'app_huis': [0, 0, app_huisman]}
```

```
huis_df = pd.DataFrame(data=huisman)
huis_df.to_excel(r'/Users/wouternommensen/Desktop/thesis1/06_huisman.xlsx')

#### from R, epsilon = 0.7
m_si = 49
m_bi = 87
m_app = 10

xm_si = 0.02544643
xm_bi = 0.004535064
xm_app = 0.06796481

alpha_si = 2.71298491
alpha_bi = 2.919877125
alpha_app = 3.42732702


def quant(m, x_m, hill, n):
    levels = {'p': [0.05, 1 / n, 1 / (1.5 * n), 1 / (2 * n)]}
    lev_df = pd.DataFrame(data=levels)
    for i in range(len(lev_df)):
        lev_df.loc[i, 1] = x_m * pow((m / (n * lev_df.loc[i, 'p'])), (1 / hill))
    return lev_df


si_p_quants = quant(m_si, xm_si, alpha_si, len(loss_ret[0]))
bi_p_quants = quant(m_bi, xm_bi, alpha_bi, len(loss_ret[1]))
app_p_quants = quant(m_app, xm_app, alpha_app, len(loss_ret[2]))
si_p_quants.to_excel(r'/Users/wouternommensen/Desktop/thesis1/07_si_p_quants_1.xlsx')
bi_p_quants.to_excel(r'/Users/wouternommensen/Desktop/thesis1/08_bi_p_quants_1.xlsx')
app_p_quants.to_excel(r'/Users/wouternommensen/Desktop/thesis1/09_app_p_quants_1.xlsx')

################################################################################
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################################################################################
#### Mixed portfolios
omega = pd.concat([pd.DataFrame([i / 10], columns=[0]) for i in range(11)],
    ignore_index=True)


def port_mix(asset_1, asset_2):
    if len(asset_2) != len(asset_1):
        print("Assets not same dimensions")
    else:
        mixed_ret = pd.DataFrame(np.zeros((len(asset_1), len(omega))))
        for i in range(len(omega)):
```

```
            for j in range(len(asset_1)):
                mixed_ret.iat[j, i] = (1 - omega.loc[i]) * asset_1.loc[j] +
                    omega.loc[i] * asset_2.loc[j]
        return mixed_ret


mix_returns_app = port_mix(loss_ret[0], loss_ret[2])
mix_returns_redo = port_mix(loss_ret[0], loss_ret[1])


def mix_quant(input, alpha, m):
    ord_stat = input.sort_values(0, ascending=False)
    ord_stat = ord_stat.reset_index(drop=True)
    n = len(ord_stat)
    x_m = ord_stat.loc[m - 1]
    levels = {'p': [0.05, 1 / n, 1 / (1.5 * n), 1 / (2 * n)]}
    lev_df = pd.DataFrame(data=levels)
    output = pd.DataFrame(np.zeros((1, len(lev_df))))
    for i in range(len(lev_df)):
        output.loc[0, i] = x_m * pow((m / (n * lev_df.loc[i, 'p'])), (1 / alpha))
    return output


mixed_quants_app = pd.DataFrame(np.zeros((0, 4)))
mixed_quants_redo = pd.DataFrame(np.zeros((0, 4)))
for i in range(len(omega) - 1):
    row = mix_quant(mix_returns_app[i], alpha_si, m_si)
    row1 = mix_quant(mix_returns_redo[i], alpha_si, m_si)
    mixed_quants_app = mixed_quants_app.append(row, ignore_index=True)
    mixed_quants_redo = mixed_quants_redo.append(row1, ignore_index=True)
row = mix_quant(mix_returns_app[10], alpha_app, m_app)
row1 = mix_quant(mix_returns_redo[10], alpha_bi, m_bi)
mixed_quants_app = mixed_quants_app.append(row, ignore_index=True)
mixed_quants_redo = mixed_quants_redo.append(row1, ignore_index=True)
mixed_quants_app.to_excel(r'/Users/wouternommensen/Desktop/thesis1/
    10_mixed_quants_apple.xlsx')
mixed_quants_redo.to_excel(r'/Users/wouternommensen/Desktop/thesis1/
    11_mixed_quants_redo.xlsx')
### these are the first order results, second order are found through plot

################################################################################
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################################################################################
#### Safety First
mixed_port = pd.DataFrame(np.zeros((len(df['SP500']) - 1, len(omega))))
mixed_port_apple = pd.DataFrame(np.zeros((len(df['SP500']) - 1, len(omega))))
```

```
R_hat = pd.DataFrame(np.zeros((len(omega), 1)))
R_hat_app = pd.DataFrame(np.zeros((len(omega), 1)))
for i in range(len(omega)):
    for j in range(len(df['SP500']) - 1):
        mixed_port.loc[j, i] = ((1 - omega.loc[i, 0]) * df.loc[j + 1, 'SP500'] +
                omega.loc[i, 0] * df.loc[j + 1, 'SP500_BI'])/
                ((1 - omega.loc[i, 0]) * df.loc[j, 'SP500'] +
                omega.loc[i, 0] * df.loc[j, 'SP500_BI'])
        mixed_port_apple.loc[j, i] = ((1 - omega.loc[i, 0]) * df.loc[j + 1, 'SP500'] +
                omega.loc[i, 0] * apple.loc[j + 1, 'Apple'])/
                ((1 - omega.loc[i, 0]) * df.loc[j, 'SP500'] +
                omega.loc[i, 0] * apple.loc[j, 'Apple'])
    column = mixed_port[i]
    column_app = mixed_port_apple[i]
    R_hat.loc[i] = column.mean()
    R_hat_app.loc[i] = column_app.mean()

R_hat.to_excel(r'/Users/wouternommensen/Desktop/thesis1/12_R_hat_SI.xlsx')
R_hat_app.to_excel(r'/Users/wouternommensen/Desktop/thesis1/13_R_hat_Apple.xlsx')


def opt_port(q, r, R):
    opts = pd.DataFrame(np.zeros((len(omega), 1)))
    for i in range(len(omega)):
        opts.loc[i] = (R.loc[i] - r) / (r - (1 - q.loc[i]))
    return opts


d_05_redo = opt_port(mixed_quants_redo[0], 1, R_hat)
d_0004_redo = opt_port(mixed_quants_redo[1], 1, R_hat)
d_05_app = opt_port(mixed_quants_app[0], 1, R_hat_app)
d_0004_app = opt_port(mixed_quants_app[1], 1, R_hat_app)
tab_sf = pd.DataFrame(np.zeros((0, 0)))
tab_sf[0] = mixed_quants_redo[0]
tab_sf[1] = d_05_redo
tab_sf[2] = mixed_quants_redo[1]
tab_sf[3] = d_0004_redo
tab_sf[4] = mixed_quants_app[0]
tab_sf[5] = d_05_app
tab_sf[6] = mixed_quants_app[1]
tab_sf[7] = d_0004_app
tab_sf.to_excel(r'/Users/wouternommensen/Desktop/thesis1/14_tab_sf_new.xlsx')

beta_index = alpha_si * ((np.log(m_si))
                        / (2 * np.log(len(returns[0])) - 2 * np.log(m_si)))
```

```
####################################################
####################################################
####################################################
#### Longer investment horizon


def long_quant(input, alpha, m, k):  # k-period
    ord_stat = input.sort_values(0, ascending=False)
    ord_stat = ord_stat.reset_index(drop=True)
    n = len(ord_stat)
    x_m = ord_stat.loc[m - 1]
    long_level = k / n
    output = pd.DataFrame(np.zeros((1, 1)))
    output.loc[0, 0] = (x_m * pow((m / (n * long_level)), (1 / alpha))) *
        pow(k, 1 / alpha)
    return output


long_quants_app = pd.DataFrame(np.zeros((0, 1)))
long_quants_redo = pd.DataFrame(np.zeros((0, 1)))
for i in range(len(omega) - 1):
    row_long = long_quant(mix_returns_app[i], alpha_si, m_si, 10)
    row_long1 = long_quant(mix_returns_redo[i], alpha_si, m_si, 10)
    long_quants_app = long_quants_app.append(row_long, ignore_index=True)
    long_quants_redo = long_quants_redo.append(row_long1, ignore_index=True)
row_long = long_quant(mix_returns_app[10], alpha_app, m_app, 10)
row_long1 = long_quant(mix_returns_redo[10], alpha_bi, m_bi, 10)
long_quants_app = long_quants_app.append(row_long, ignore_index=True)
long_quants_redo = long_quants_redo.append(row_long1, ignore_index=True)
long_quants_app.to_excel(r'/Users/wouternommensen/Desktop/thesis1/
    15_long_quants_apple.xlsx')
long_quants_redo.to_excel(r'/Users/wouternommensen/Desktop/thesis1/
    16_long_quants_redo.xlsx')


### 10-day R-hat
long_mixed_port = pd.DataFrame(np.zeros((len(df['SP500']) - 10, len(omega))))
long_mixed_port_apple = pd.DataFrame(np.zeros((len(df['SP500']) - 10, len(omega))))
long_R_hat = pd.DataFrame(np.zeros((len(omega), 1)))
long_R_hat_app = pd.DataFrame(np.zeros((len(omega), 1)))
for i in range(len(omega)):
    for j in range(len(df['SP500']) - 10):
        long_mixed_port.loc[j, i] = ((1 - omega.loc[i, 0]) * df.loc[j + 10, 'SP500'] +
                omega.loc[i, 0] * df.loc[j + 10, 'SP500_BI'])/
                ((1 - omega.loc[i, 0]) * df.loc[j, 'SP500'] +
```

```
                omega.loc[i, 0] * df.loc[j, 'SP500_BI'])
        long_mixed_port_apple.loc[j, i] = ((1 - omega.loc[i, 0]) * df.loc[j+10,'SP500']
                + omega.loc[i, 0] * apple.loc[j + 10, 'Apple'])/
                ((1 - omega.loc[i, 0]) * df.loc[j, 'SP500'] +
                omega.loc[i, 0] * apple.loc[j, 'Apple'])
    long_column = long_mixed_port[i]
    long_column_app = long_mixed_port_apple[i]
    long_R_hat.loc[i] = long_column.mean()
    long_R_hat_app.loc[i] = long_column_app.mean()


long_R_hat.to_excel(r'/Users/wouternommensen/Desktop/thesis1/17_long_R_hat_SI.xlsx')
long_R_hat_app.to_excel(r'/Users/wouternommensen/Desktop/thesis1/
    18_long_R_hat_Apple.xlsx')


d_long_si = opt_port(long_quants_redo, 1, long_R_hat)
d_long_app = opt_port(long_quants_app, 1, long_R_hat_app)
long_sf = pd.DataFrame(np.zeros((0, 0)))
long_sf[0] = long_quants_redo[0]
long_sf[1] = d_long_si
long_sf[2] = long_quants_app[0]
long_sf[3] = d_long_app
long_sf.to_excel(r'/Users/wouternommensen/Desktop/thesis1/19_long_sf_.xlsx')
```

E2. R script

```
library(evir)
library(tea)
library(openxlsx)
#https://github.com/cran/tea/blob/master/R/hall.R
pos_dat = read.xlsx("/Users/wouternommensen/Desktop/thesis/pos_ret.xlsx",
            colNames = TRUE, rowNames = TRUE)
pos_app = read.xlsx("/Users/wouternommensen/Desktop/thesis/pos_app.xlsx",
            colNames=TRUE, rowNames =TRUE)

si_pos = as.numeric(pos_dat[,1])
bi_pos = as.numeric(pos_dat[,2])
app_pos = as.numeric(pos_app[,1])

hall_adj <-
  function(data,B=1000,epsilon=0.955,kaux=2*sqrt(length(data))){
    n=length(data)
    n1=floor(n^epsilon)

    helphill=function (k) {
      xstat = sort(data, decreasing = TRUE)
      xihat = mean((log(xstat[1:k]) - log(xstat[k + 1])))
      xihat
    }

    help=helphill(kaux)

    mse=matrix(nrow=B,ncol=n1-1)
    for (l in 1:B){
      i=1:(n1-1)
      x1=sample(data,n1,replace=TRUE)
      x1=sort(x1,decreasing=TRUE)
      h=(cumsum(log(x1[i]))/i)-log(x1[i+1])
      mse[l,]=h-help
    }
    mse=mse^2
    msestar=colMeans(mse)
    k1star=which.min(msestar)

    k0star=floor(k1star*(n/n1)^(2/3))
    u=sort(data,decreasing=TRUE)[k0star]
    ti=1/helphill(k0star)
    results=c(k0=k0star,threshold=u,tail.index=ti)
    results
    return(results)
  }
```

```
app_pos_hall <- hall_adj(app_pos, B=10000, epsilon=0.8, kaux=sqrt(length(data)))
si_pos_hall <- hall_adj(si_pos, B=1000, epsilon = 0.955 ,kaux=sqrt(length(data)))
bi_pos_hall <- hall_adj(bi_pos, B=1000, epsilon = 0.955 ,kaux=sqrt(length(data)))


res_vec <- as.data.frame(list(c('k0','threshold','tail index')))
row_names <-c('k0','threshold','tail index')
df_pos <- data.frame(row_names, si_pos_hall, bi_pos_hall)
write.table(df_res, file="/Users/wouternommensen/Desktop/thesis/hall_est.dat",
            append = FALSE, sep = " ", dec = ".", row.names = TRUE, col.names = TRUE)
write.table(df_pos, file="/Users/wouternommensen/Desktop/thesis/pos_hall_est.dat",
            append = FALSE, sep = " ", dec = ".", row.names = TRUE, col.names = TRUE)
```

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