Bachelor Thesis Econometrie & Operationele Research

Selecting Optimal Sample Fraction for Tail Index Estimation in a Safety First Portfolio Problem

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Date final version:
4 July 2021

Abstract

This paper considers the problem a safety first investor faces when optimizing a mixed portfolio of a stock index mutual fund and a bond index mutual fund. It focuses on constructing a Value at Risk (VaR) based on the tail index estimates modelled by a second order Hall expansion, and particularly on finding the optimal number of extreme values to compute these estimates. The tail parameters are estimated for several methods of selecting the optimal the amount of extreme values and each of these estimates are used to construct a VaR and an optimal portfolio mix. These are used to compare each of the selection methods such that the solution for the portfolio problem can be improved. It is shown that improvement can be made when consciously choosing a selection method. The best performing method depends on the size of the estimation sample.

Contents

1 Introduction

An important aspect of investing is finding a measure of risk for an investment. This also seems to be a difficult aspect, as the distribution of stock and bond returns is uncertain. Especially for a safety first investor this is a relevant issue, as downside risk is his most important motivation when constructing a portfolio. The safety first investor wishes to maximize the expected return of a portfolio under the condition that it has a certain small probability of falling below a threshold. This problem was introduced by Roy (1952), where he proposes the Value at Risk (VaR) as a measure for this downside risk. In order to make the right decisions, a good estimation of the VaR is of great importance. Therefore, constructing the VaR may be considered as a salient "subproblem" of the safety first portfolio problem. This subproblem can be challenging, especially when the investor allows for small probabilities of VaR violations, i.e. the thresholds for minimum returns are far in the tail of the distribution. The challenge lies in the fact that asset return distributions experience heavy tails.

Jansen et al. (2000) consider the above described portfolio problem while accounting for the heavy tails of asset return distributions. They examine the decision problem that a safety first investor faces when allocating a portfolio that consists of a U.S. stock index mutual fund and/or a U.S. bond index mutual fund. They propose to estimate the VaR by means of extreme value theory and address the problem of modelling the heavy tails in the asset distributions with a semi-parametric approach. The tails are modelled by a distribution that to the first order is equal to the Pareto distribution, with an unknown first order tail parameter. They estimate this parameter by the commonly used tail index estimator from Hill (1975), which relies on the m most extreme values of a data sample. This tail index estimate is used to construct a VaR such that an optimal portfolio allocation can be selected. The procedure by Jansen et al. (2000) tends to select corner solutions, assigning the full weight of the portfolio to the asset with the thinnest tails. Hyung and de Vries (2007) revise the portfolio allocation problem from Jansen et al. (2000) and model the tails from the asset return distributions by the second order expansion introduced by Hall (1990). The second order expansion additionally contains a second order tail index, which is estimated by an estimator proposed by Danielsson et al. (2000). This estimator depends on the first order tail index estimate and on m as well. Hyung and de Vries (2007) further replicates the estimates and values from Jansen et al. (2000) and find that when accounting for the second order tail index, optimal interior solutions can be found where Jansen et al. (2000) selected optimal solutions in the corner.

Another problem arises in selecting the optimal value for the sample fraction size m in both tail index estimators. Accurate estimates crucially depend on this value and in turn, the VaR computation and portfolio allocation essentially depend on the tail index estimates. More specifically, increasing the sample fraction size causes an increase in the bias of the Hill estimator, while it simultaneously reduces its variance. The optimal value for m should balance the bias and variance in order to obtain reliable estimates. Selecting the optimal value is a widely encountered problem in the literature and several methods have been proposed. However, not one has been found to be overall superior. Which method performs best may depend on, among other things, the sample size and the underlying tail distribution. Jansen et al. (2000) use the method introduced by Hall (1990) to select m. In order to present their improvement, Hyung and de Vries (2007) adopt the exact values from Jansen et al. (2000) even though the method by Hall (1990) imposes undesirable restrictions on the second order tail index. Consequently, further improvement may be achieved when applying a different method for the selection of m. Therefore, the goal of this paper is to improve the allocation of a mixed stock- and bond index portfolio for a safety first investor, by solving the problem with an alternative method of sample fraction selection. The results from solving this problem for multiple methods of selecting m will be compared in order to find an improvement of the method from Hall (1990).

I will consider four methods, which I will compare with the Hall method and with one another. The first method is introduced by Danielsson et al. (2000), and focuses on selecting the sample fraction size that minimizes the asymptotic mean squared error(AMSE) of the Hill estimator. This method requires a double bootstrap procedure to estimate the AMSE and provides an estimator for a control variate to replace the theoretical true value of the tail index. The second method is introduced by Drees and Kaufmann (1998). This is a sequential method that considers the maximum random fluctuation of the AMSE. If this maximum is exceeded, the fluctuation can not be random and therefore must be dominated by a bias. Both these methods rely on asymptotic arguments and favour large estimation samples. The remaining two methods I discuss in this paper are of a more heuristic nature. They initially estimate the tail index using a small amount of observations. This amount is then increased until the fluctuations of the Hill estimates are sufficiently low. The two methods to measure the fluctuations that will be discussed in this paper are proposed by Schouten (2017) and by Danielsson et al. (2019). Furthermore, I will discuss the differences between the estimation results obtained in this paper and those obtained by Hyung and de Vries (2007). I find that the optimal portfolio selected by the safety first criterion is not affected by the selection methods in their particular problem. However, I show that for some other examples, different selection methods may result in different and improved optimal portfolios. In addition, this paper concludes that the estimate of the VaR can be significantly improved when using the method by Drees and Kaufmann (1998) or the one by Schouten (2017) to estimate the tail indices, depending on the size of the estimation sample. Improvement is measured by regarding violations of the VaR. As tail events occur rarely by definition, I will construct a cross-sectional data set by pooling similar data from other countries in order obtain enough informative observations.

Danielsson et al. (2019) have done a similar study but their focus is on which method produces the best estimate of the first order tail index only. They perform several simulations of known distributions and examine which sample fraction selection method results in the tail index estimates closest to the true value. They conclude that their own introduced distance metric method performs best. This method measures the distance between the quantile function of a distribution that satisfies a first order expansion and the empirical quantile. As this paper only focuses on distributions that satisfy the second order expansion I will not consider this distance metrics method.

The remainder of this paper is outlined as follows: in the next section I provide a mathematical description of the safety first criterion and extreme value theory, in Section 3 I will elaborate on the mentioned sample fraction size selection methods, in Section 4 the data is described that is used for estimation and evaluation, Section 5 provides an overview of the estimation and evaluation results and finally, I will draw my conclusions in Section 6.

2 Theoretical Framework

2.1 Safety First Criterion

The portfolio problem of choosing the optimal mix between a U.S. stock index fund and a U.S. bond index fund is based on the safety first criterion. An investor decides a level of wealth, s, and a probability δ such that the invested wealth falls below s with a probability of at most δ. Now, as in Roy (1952), define

$$\pi = \begin{cases} 1 & \text{if } p = P[\sum_i w_i V_{i,t+1} + br \leq s] \leq \delta \\ 1 - p & \text{else,} \end{cases}$$

where w_i denotes the weight of invested amount in risky asset i, which has value $V_{i,t}$ at time t and b is the amount lent or borrowed at risk free rate r. They then define the safety first problem as

$$\max_{w_i,b}(\pi,\mu)$$

subject to

$$\sum_i w_i V_{it} + b = W_t,$$

where W_t is the initial wealth of the investor. As in Hyung and de Vries (2007), I approach the problem in a way proposed by Arzac and Bawa (1977). They find that it can be separated into two problems. For the first part, define the gross return $R = \sum_i w_i V_{i,t+1} / \sum_i w_i V_{i,t}$ and quantile $q_\delta(R)$ such that $P[R \leq q_\delta(R)] = \delta$. The first step is to solve

$$\max_{w_i} \frac{E[R] - r}{r - q_\delta(R)}. \tag{1}$$

In the second step the scale of the risky portfolio and the amount borrowed are obtained from the budget constraint

$$W_t - b = \frac{s - rW_t}{q_\delta(R) - r}.$$

This paper focuses on the first step and for more details on the second step I refer to Arzac and Bawa (1977). Now the problem remains in finding a creditable value for $q_\delta(R)$, which I will discuss in the next subsection.

2.2 Extreme Value Theory

The value $q_\delta(R)$ defined in Section 2.1 can be interpreted as a VaR level corresponding to probability δ. Finding a suitable value requires accurate modelling of the tails of the underlying distribution. To account for the heavy tails, consider asset return distributions that come from a class of regularly varying distributions, such that

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \tag{2}$$

with tail index $\alpha > 0$. Here, a larger(lower) index corresponds to thinner(fatter) tails. The heavy tails property from distributions that satisfy Equation (2) follows from the fact that their tails decline by a power, thus slower than tails of distributions that decline exponentially, e.g. the normal distribution. This slow decline causes moments larger than α to be unbounded. In the context of this paper, only the mean returns are required to compute the optimal portfolio allocation and therefore α should be greater than 1. Now, following Hyung and de Vries (2007), we consider a more specific class of distributions, those that satisfy the so called Hall expansion

$$1 - F(X) = Ax^{-\alpha}[1 + Bx^{-\beta} + o(x^{-\beta})], \tag{3}$$

for $s \to \infty$, with first order tail index $\alpha > 0$, second order tail index $\beta > 0$, $A > 0$, B a real constant and little-oh notation $o(\cdot)$[1]. One can verify that the Hall expansion satisfies the property of Equation (2), so that it belongs to the class of regularly varying distributions. Hyung and de Vries (2007) find that for two different, assets with returns X_1 and X_2, that have a symmetric distribution such that

$$1 - F(s) = P[X_i > s] = P[X_i \leq -s] = A_i s^{-\alpha_i}[1 + B_i s^{-\beta_i} + o(s^{-\beta_i})], \tag{4}$$

for $i = 1, 2$ and large loss threshold s, the following theorem holds for the convolution $X_1 + X_2$:

[1] See Appendix A for further details on little-oh

Theorem 1 *Suppose that the tails of the distributions of X_1 and X_2 satisfy Equation (4). Moreover, assume $1 < \alpha_1 \leq \alpha_2$ so that $E[X]$ is bounded. When X_1 and X_2 are independent, the asymptotic 2-convolution up tot the second order terms is*

(i). if $\alpha_2 - \alpha_1 < min(\beta_1, 1)$, then $P[X_1 + X_2 > s] = A_1 s^{-\alpha_1} + A_2 s^{-\alpha_2} + o(s^{-\alpha_2})$

(ii). if $1 < \alpha_2 - \alpha_1$ and $1 < \beta_1$, then $P[X_1 + X_2 > s] = A_1 s^{-\alpha_1} + A_1\alpha_1 E[X_2] s^{-\alpha_1 - 1} + o(s^{-\alpha_2})$

(iii). if $\beta_1 < \alpha_2 - \alpha_1$ and $\beta_1 < 1$, then $P[X_1 + X_2 > s] = A_1 s^{-\alpha_1} + A_1 B_1 s^{-\alpha_1 - \beta_1} + o(s^{-\alpha_1 - \beta_1})$

(iv). if $\alpha_2 - \alpha_1 = 1 < \beta_1$, then $P[X_1 + X_2 > s] = A_1 s^{-\alpha_1} + \{A_2 + A_1\alpha_1 E[X_2]\} s^{-\alpha_2} + o(s^{-\alpha_2})$

(v). if $\alpha_2 - \alpha_1 = \beta_1 < 1$, then $P[X_1 + X_2 > s] = A_1 s^{-\alpha_1} + \{A_2 + A_1 B_1\} s^{-\alpha_2} + o(s^{-\alpha_1})$

(vi). if $\alpha_2 - \alpha_1 = \beta_1 = 1$, then $P[X_1 + X_2 > s] = A_1 s^{-\alpha_1} + \{A_2 + A_1\alpha_1 E[X_2] + A_1 B_1\} s^{-\alpha_2} + o(s^{-\alpha_2})$

For proof of this theorem I refer to Hyung and de Vries (2007). Furthermore, they use Theorem 1 to prove that the VaR(w, p), for which

$$P[wX_1 + (1 - w)X_2 > \text{VaR}(w, p)] = p$$

with probability p, is a convex function of the portfolio allocation w, This convexity ensures, under certain conditions, interior solutions for the portfolio allocations with optimal VaR levels. The first order tail index will be estimated by means of the Hill (1975) estimator, as in the majority of the literature. This is defined as follows:

$$\hat{\gamma} = \frac{1}{\hat{\alpha}_i} = \frac{1}{m_i} \sum_{j=1}^{m_i} [\log(X_{i,(n+1-j)}) - \log(X_{i,(n-m_i)})], \tag{5}$$

for asset $i = 1, 2$, where n is the number of observations, m_i the tail fraction size for estimation of tail index α_i and $X_{i,(m)}$ the m-th ascending order statistic from the sample $\{X_{i,1}, ..., X_{i,n}\}$. Danielsson et al. (2000) find the following estimator for the ratio between the first and second order tail indices

$$\frac{\hat{\beta}_i}{\hat{\alpha}_i} = \frac{\log(\hat{m}_i)}{2\log(n) - 2\log(\hat{m}_i)}, \tag{6}$$

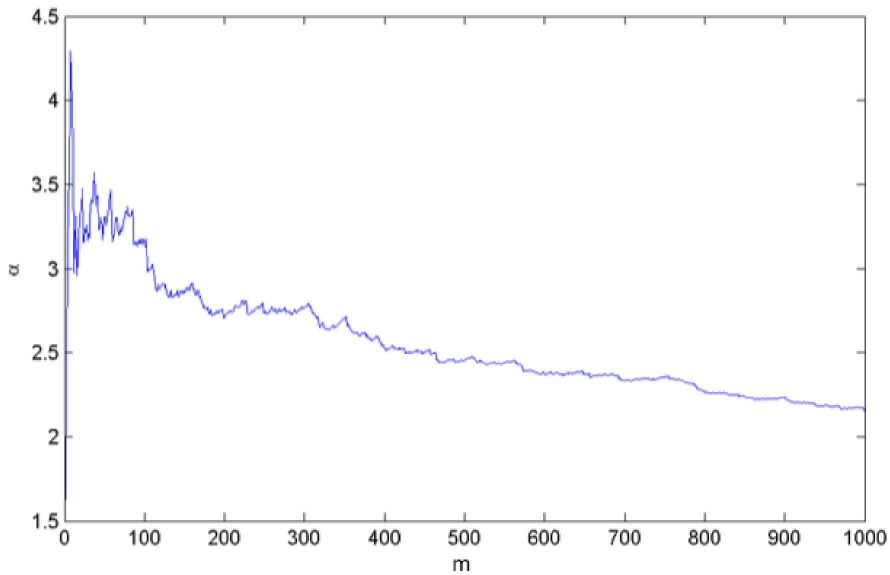
which is shown to be consistent if $\hat{m}$ is a consistent estimator of the asymptotically optimal value for m. The exact meaning of $\hat{m}$ being a consistent estimator will become clear in Section 3.1. Note that both tail index estimators are a function of the sample tail fraction size, thus it seems important to compute a suitable value for it. Several methods have been proposed in the literature to select the level of m, of which I will discuss four of them in the following section.

3 Methodology

3.1 Sample Fraction Selection

The optimal value of m, which will be denoted as m^*, should balance the variance and bias of the Hill estimator. A large m could cause the Hill estimator to select values which are too close to the center of the distribution to be considered a tail event. This would therefore lead to a bias in the tail index estimator, which grows as m becomes larger. However, when m is large, the Hill estimator will select observations that occur more often and closer to one another, which inherently decreases the variance. As an illustration of this phenomenon, Figure 1 shows the plotted Hill estimates from a sample drawn from the Student-t distribution with 4 degrees of freedom - ergo $\alpha = 4$ -, for different levels of m. The Student-t distribution is one of the theoretical distributions that satisfy the Hall expansion. Clearly, the Hill estimates fluctuate heavily for small values of m and stabilize as m gets larger, but then drift away from the true value of 4. A mathematical illustration of the bias-variance trade off is given in Appendix B.

Figure 1: Hill Plot



Hill estimates for first order tail index α for different levels of sample fraction size m. The estimates are based on 10.000 random draws from a Student-t distribution with 4 degrees of freedom.

Indeed, the term "balance" here is rather vague and there are several ideas of what value for m can be considered as optimal. I will discuss four methods of selecting optimal value m^*, which can be considered to be one of two approaches: an approach based on asymptotic arguments or a more heuristic approach. The asymptotic methods discussed in Section 3.1.1 and 3.1.2 are based on a theorem introduced by Mason (1982), which states that if sequence $m = m_n \to \infty$ and $m_n/n \to 0$ when $n \to \infty$, then $\hat{\gamma}(m_n) \to \gamma$. Sequences that satisfy these asymptotic properties are called intermediate sequences. The optimal value for m is the one that minimizes the mean squared error of $\hat{\gamma}$, but this value depends on the unknown value γ. Therefore, to obtain the estimator in Equation (6), the idea is to find a consistent estimator $\hat{m} = \hat{m}_n$ for the optimal sequence m_n, based on the available sample. The estimate computed by this estimator is then used as m^*.

As the above described estimation methods are based on asymptotic arguments, they may not perform very well in finite samples. It therefore seems logical to also examine an alternative approach. Section 3.1.3 and 3.1.4 describe methods that consider the variance-bias trade off for the Hill estimator $\hat{\gamma}$ by trying several values m and take the lowest such that the fluctuations of $\hat{\gamma}$ are sufficiently small in the region of m. These methods do not yield a consistent estimator $\hat{m}$ as described for the asymptotic methods, however, since Equation (6) can be written as

$$\hat{\beta} = \frac{\log(m)}{2\log(n) - 2\log(m)} \cdot \hat{\gamma}^{-1}, \tag{7}$$

regions with small fluctuations in $\hat{\gamma}$ should also have small fluctuations in $\hat{\beta}$. Therefore the optimal values for m found by these methods may be optimal for both tail index estimates.

3.1.1 Double Bootstrap Method

Danielsson et al. (2000) propose to find such an estimator $\hat{m}$ by applying two subsample bootstrap procedures. Their approach is to find a sample fraction size that minimizes the asymptotic mean squared error (AMSE) of $\hat{\gamma}$, which is defined as

$$\text{AMSE}(\hat{\gamma}) = AsyE[(\hat{\gamma}(m) - \gamma)^2], \tag{8}$$

with Hill estimate $\hat{\gamma}$ and true value γ. However, regular bootstrap does not necessarily provide an estimate for the AMSE that is asymptotically equivalent to it. The unsmooth nature of $\hat{\gamma}$ causes its bias to be a major contributor to the AMSE and regular bootstrap to seriously underestimate the bias. Hall (1990) solve this problem by drawing reboot samples of size n_1, such that it is smaller than the original sample size n. They use this difference between n_1 and n to smooth out $\hat{\gamma}$ and estimate and minimize the AMSE. This procedure is adopted by Danielsson et al. (2000) and in addition they propose to replace the unknown value γ from Equation (8) with the control variate

$$Z(m) = \frac{1}{m} \sum_{i=1}^{m} (\log(X_{n-i+1}) - \log(X_{n-m}))^2,$$

for which the AMSE has the same rate of convergence as the AMSE($\hat{\gamma}$). Then the following bootstrap estimate for the mean squared error is used

$$Q(n_1, m_1) := E\big[\big((Z_{n_1}(m_1) - 2(\hat{\gamma}_{n_1}(m_1)^2)^2\big], \tag{9}$$

where $Z_{n_1}(m_1)$, $\hat{\gamma}_{n_1}(m_1)$ and m_1 are the control variate, Hill estimator and sample fraction, respectively, corresponding to the bootstrap samples of size n_1. They show that the values for m that minimize AMSE($\hat{\gamma}$) and $Q(n, m)$ are of the same order. $Q(n_1, m_1)$ is then estimated for each m_1 and the value m_1^* such that $m_1^* = \underset{m_1}{\text{argmin}}\, Q(n_1, m_1)$ is stored. This process is repeated for a bootstrap sample of size $n_2 = \frac{n_1^2}{n}$ to obtain m_2^*. Finally, the optimal sample fraction based on n_1 is computed as

$$m_{db}^* = \frac{(m_1^*)^2}{m_2^*} \left(\frac{\log^2(m_1^*)}{(2\log(n_1) - \log(m_1^*))^2} \right)^{\frac{log(n_1) - log(m_1^*)}{log(n_1)}},$$

which is the value for m to be used in Equations (5) and (6) to estimate the tail indices. To determine n_1, they suggest repeating the whole procedure for a range of values of n_1, such that $n_1 = n^{1-\epsilon}$ for $0 < \epsilon < 1/2$, and use the value that minimizes the ratio

$$\frac{Q(n_1, m_1^*)^2}{Q(n_2, m_2^*)}.$$

The range of values for n_1 and the amount of bootstrap resamples are to be decided by personal preferences or available computing time. The implementation of this method is summarized in the following algorithm

Algorithm 1: Double bootstrap method

input : Amount of bootsrap resamples b, bootstrap resample size n_1, full sample size n
output: Optimal fraction of sample size

Draw b bootstrap resamples of size n_1
for $m_1 = 1, ..., n_1$ **do**
  $Q(n_1, m_1) := E\big[\big((Z_{n_1}(m_1) - 2(\hat{\gamma}_{n_1}(m_1)^2\big)^2\big]$
end
$m_1^* = \underset{m_1}{\text{argmin}}\{Q(n_1, m_1)\}$
Draw b bootstrap resamples of size $n_2 = n_1^2/n$
for $m_2 = 1, ..., n_2$ **do**
  $Q(n_2, m_2) := E\big[\big((Z_{n_2}(m_2) - 2(\hat{\gamma}_{n_2}(m_2)^2\big)^2\big]$
end
$m_2^* = \underset{m_2}{\text{argmin}}\{Q(n_2, m_2)\}$
return $m_{db}^* = \frac{(m_1^*)^2}{m_2^*}\left(\frac{\log^2(m_1^*)}{(2\log(n_1)-\log(m_1^*))^2}\right)^{\frac{log(n_1)-log(m_1^*)}{log(n_1)}}$

3.1.2 Sequential Method

Another approach that yields a consistent estimate of the tail index ratio is proposed by Drees and Kaufmann (1998). They use the following slightly different notation for the Hall expansion

$$1 - F(X) = Ax^{-\frac{1}{\gamma}}[1 + Bx^{-\frac{\rho}{\gamma}} + o(x^{-\frac{\rho}{\gamma}})],$$

which is equivalent to Equation (3) when $\alpha = \frac{1}{\gamma}$ and $\rho = \frac{\beta}{\alpha}$. They build on the result from Hall and Welsh (1985) that the AMSE of the Hill estimator is minimal for

$$m \sim \left(\frac{A^{2\rho}(\rho+1)^2}{2B^2\rho^3}\right)^{\frac{1}{2\rho+1}} n^{\frac{2\rho}{2\rho+1}}.$$

They state that the maximum random fluctuation of $i^{1/2}(\hat{\gamma}(i) - \gamma)$ for $2 \leq i \leq m_n$, is of order $\sqrt{\log(\log(n))}$ for all intermediate sequences m_n and introduce a sequential procedure to obtain an asymptotically consistent estimator to compute m^*. Furthermore, they define the stoppage time of a sequence of Hill estimators by

$$\bar{m}_n(r_n) = \min\left\{m \in 2, ..., n | \max_{2 \leq i \leq m_n} i^{1/2}|\hat{\gamma}^{-1}(i) - \hat{\gamma}^{-1}(m_n)| > r_n\right\},$$

where $r_n = 2.5\tilde{\gamma}n^{1/4}$ is a sequence such that $\sqrt{\log(\log(n))} < r_n < n^{1/2}$ and $\tilde{\gamma}$ is an initial estimator of γ. Under the conditions $r_n = o(n^{1/2})$ and $\sqrt{\log(\log(n)} = o(r_n)$, it is shown that $\left(\bar{m}_n(r_n^\epsilon)/\bar{m}_n(r_n)^\epsilon\right)^{1/(1-\epsilon)}$, with $\epsilon \in (0, 1)$, has the optimal order for $\hat{m}_n$. This leads to the optimal value

$$m_{DK}^* = \left[(2\hat{\rho}+1)^{-1/\hat{\rho}n}(2\hat{\gamma}_n^2\hat{\rho}_n)^{1/(2\hat{\rho}+1)}\left(\bar{m}_n(r_n^\epsilon)/\bar{m}_n(r_n)^\epsilon\right)^{1/(1-\epsilon)}\right]$$

with

$$\hat{\rho}(r_n, \lambda) = \log \frac{\max\limits_{2 \le i \le [\lambda \bar{m}_n(r_n)]} \left\{ i^{1/2} \left| \hat{\gamma}^{-1}(i) - \hat{\gamma}^{-1}\left([\lambda \bar{m}_n(r_n)]\right) \right| \right\}}{\max\limits_{2 \le i \le \lambda \bar{m}_n(r_n)} \left\{ i^{1/2} \left| \hat{\gamma}^{-1}(i) - \hat{\gamma}^{-1}\left(\lambda \bar{m}_n(r_n)\right) \right| \right\}} / \log(\lambda) - \frac{1}{2},$$

where $\lambda \in (0, 1)$. After extensive simulations they find the best results for $\tilde{\gamma} = \hat{\gamma}(2\sqrt{n^+})$, where n^+ is the amount of positive observations, $\epsilon = 0.7$ and $\lambda = 0.6$. It may occur that the threshold r_n is too large. In that case, Drees and Kaufmann (1998) suggest repeatedly replacing r_n by $0.9r_n$ until $\bar{m}_n(r_n)$ is defined. The implementation of the Sequential method is summarized in the following algorithm

Algorithm 2: Sequential method

input : Data sample, parameters ϵ and λ
output: Optimal fraction of sample size

$\tilde{\gamma} = \hat{\gamma}(2\sqrt{n^+})$

$r_n = 2.5\tilde{\gamma}n^{1/4}$

$\bar{m}_n(r_n) = \min\left\{ m \in 2, ..., n \middle| \max\limits_{2 \le i \le m} \{\sqrt{i}|\hat{\gamma}(i)^{-1} - \hat{\gamma}(m)^{-1}| > r_n\} \right\}$

$\hat{\rho}(r_n, \lambda) = \log \frac{\max\limits_{2 \le i \le [\lambda \bar{m}_n(r_n)]} \{i^{1/2}|\hat{\gamma}(i)^{-1} - \hat{\gamma}([\lambda \bar{m}_n(r_n)])^{-1}|\}}{\max\limits_{2 \le i \le \lambda \bar{m}_n(r_n)} \{i^{1/2}|\hat{\gamma}_{n,i}^{-1} - \hat{\gamma}(\lambda \bar{m}_n(r_n))^{-1}|\}} / \log(\lambda) - \frac{1}{2}$

return $m^*_{DK} := \left[(2\hat{\rho}+1)^{-1/\hat{\rho}n} (2\hat{\gamma}_n^2\hat{\rho})^{1/(2\hat{\rho}+1)} \left(\bar{m}_n(r_n^\epsilon)/\bar{m}_n(r_n)^\epsilon \right)^{1/(1-\epsilon)} \right]$

3.1.3 Stability Method

Schouten (2017) has found a way to quantify fluctuation -or lack of fluctuation as he measures stability- and developed an algorithm to find the optimal fraction size m^*. First, define a domain of sample fraction sizes that is allowed, i.e. $m \in \{c_1 n^p, ..., c_2 n^p\}$, where $c_1 n^p$ and $c_2 n^p$ are the minimum and maximum accepted value for m, respectively. c_1, c_2 and p are chosen such that $0 < c_1 < c_2$ and $0 < p < 1$. A value for stability in a region around m is expressed as

$$S(m) = \sum_{i=m-k}^{m+k-1} (|\hat{\gamma}(i+1) - \hat{\gamma}(i)|),$$

where $k \in N$ is the region around m and should be small enough such that the stability measure is representative, but not too small since that could compute a small value (indicating more stability) for a region with coincidental stability. Then the optimal value for m is found by

$$m^*_{stable} = \underset{m}{\operatorname{argmin}} \{S(m) \cdot m^q\},$$

where $q > 0$. I will stick to his method by using the same values for $c_1 = 0.2$, $c_2 = 3$, $p = 0.5$ and $q = 0.5$. For the latter, he finds that 0.5 yields the best results, among a sequence of values, for several distributions of the underlying CDF that satisfy the Hall expansion. A value for k will be assigned such that the moving window $\{m - k, ..., m + k\}$ is about 1% of the sample. The implementation of the Stability method is summarized in the following algorithm

Algorithm 3: Stability method

input : Minimum acceptable value m_{min} for m, maximum acceptable value m_{max} for m, size k of region around m and parameter q

output: Optimal fraction of sample size

for $m = m_{min}, ..., m_{max}$ **do**

  $S(m) = \sum_{i=m-k}^{m+k-1} (|\hat{\gamma}(i+1) - \hat{\gamma}(i)|)$

end

return $m^*_{stable} = \underset{m}{argmin}\{S(m) \cdot m^q\}$

3.1.4 Eye-Ball Method

The Eye-Ball method derives its name from "eyeballing" a Hill plot and choose m by simply observing where the fluctuations are relatively small and the bias not too evident. This method is quantified by Danielsson et al. (2019), who propose that m should be chosen such that a certain percentage of Hill estimates over a moving window should fall within a small region of the first estimate in that window. The optimal value m^* is then defined as

$$m^*_{eye} = \min \big\{k \in 2, ..., n-w | h < \frac{1}{w} \sum_{i=1}^{w} \mathrm{I}\{\hat{\gamma}(m+i) < \hat{\gamma}(m) \pm \epsilon\}\big\},$$

where w is the size of the moving window, ϵ the permitted bound around $\gamma(m)$ such that no less than $h\%$ of the estimates on window w should be within this bound for m to be a candidate. They state that typically $w = 1\%$ of the sample, $h = 90\%$ and $\epsilon = 0.3$. Therefore these values will also be used in this paper. The implementation of the Eye-Ball method is summarized in the following algorithm

Algorithm 4: Eye-ball method

input : Moving window size w, size ϵ of region around $\hat{\gamma}(m)$ and percentage h of estimates that should be within this region

output: Optimal fraction of sample size

for $m = 2, ..., n^+ - w$ **do**

  $z = \frac{1}{w} \sum_{i=1}^{w} \mathrm{I}[\hat{\gamma}(m+i) \in \{\hat{\gamma}(m) \pm \epsilon\}]$

  if $h < z$ **then**

    return $m^*_{eye} = m$

  end

end

3.2 Evaluation

Each of the methods mentioned in Section 3.1 will be used to calculate an optimal stock- and bond index portfolio for k countries over a training dataset of n periods. Then, for all k portfolios, the amount of violations over a test dataset are counted. For a good measure of the VaR, the fraction of observed violations should be close to the predefined probability δ. To measure "closeness", following van Os (2021), first define the indicator function

$$I_{i,n+j} = \begin{cases} 1 & \text{if } R_{i,n+j} \leq q_\delta(R) \\ 0 & \text{if } R_{i,n+j} > q_\delta(R) \end{cases},$$

for country $i = 1, 2, ..., k$ and $j = 1, 2, ..., T$ and where $R_{i,t}$ and $q_\delta(R)$ are respectively the gross portfolio return at time t and the VaR, as defined in Section 2.1. Furthermore, let $T_1 = \sum_{i=1}^{k} \sum_{j=1}^{T} I_{i,n+j}$, where T is the total sum of observations in each of the countries' test datasets, and $T = T_1 + T_0$. Then, assuming tail events occur independently, I will test for correct unconditional coverage by testing $H_0 : P[I_{n+j} = 1] = \delta$ against $H_1 : P[I_{t+1} = 1] = \pi \neq \delta$. The likelihood function under the null becomes

$$\mathcal{L}(\delta) = (1 - \delta)^{T_0} \delta^{T_1}$$

and under the alternative

$$\mathcal{L}(\pi) = (1 - \pi)^{T_0} \pi^{T_1}.$$

Here, π will be replaced by maximum likelihood estimator $\hat{\pi} = \frac{T_1}{T}$, so that we can compute the likelihood ratio test statistic

$$LR = -2 \log \left(\frac{\mathcal{L}(\delta)}{\mathcal{L}(\hat{\pi})} \right) \sim \chi^2(1). \tag{10}$$

This test statistic will be computed for the four discussed sample fraction selection methods and used for comparison.

4 Data

To compare the results obtained in this paper to the results from Hyung and de Vries (2007), the data in both papers should be comparable. As it is unclear which exact stock and bond indices they used, I will use the S&P 500 and the Dow Jones Equal Weight corporate bond index. The latter tracks the total returns of 100 large and liquid investment-grade bonds issued by companies in the U.S. The monthly simple returns for these indices are computed and the sample used to optimize the portfolio runs from January 1926 until December 1992. Data over the period from January 1993 until December 2020 is left for evaluating the discussed methods. As tail events occur rarely by definition, the evaluation period is probably too short to form meaningful conclusions. Therefore, I collected similar data of stock and bond indices from other countries to build a cross sectional data sample. This data was available over roughly the same period for the countries Italy, France, the Netherlands, Switzerland and the U.K. At least for the larger part of this period, these countries were relatively stable first world countries and should therefore make good comparisons for a U.S. stock and bond portfolio. Table 1 shows some summary statistics for the returns of the stock- and bond indices of these countries. N denotes the number of available observations from January 1926 - December 1992, which are used for estimation. Only the observations of the U.K. bond index begin from July 1932. T denotes the number of observations in the test data sample, which all begin in January 1993 but have varying end dates. Both numbers may vary slightly per country due to a lack of observations for various reasons. The large values for the Jarque-Bera statistics reject the hypothesis of a normal distribution. All values for kurtosis are substantially larger than 3, which confirms the presence of fat tails in all the index returns. Stock indices generally have fatter tails than bond indices but for Italy, the kurtosis for the bond index is higher than for the stock index. However, considering the large difference in standard deviations of both indices, outliers of the bond index returns may still be smaller in absolute value than those of the stock index returns.

Table 1: Summary statistics for simple monthly asset returns

		Mean	SD	Kurtosis	Jarque-Bera	N	T	Full Sample Period
US	Stocks	0.0066	0.0673	52	81578	804	335	Jan 1926 - Dec 2020
	Bonds	0.0054	0.0181	10	1514	804	335	Jan 1926 - Dec 2020
UK	Stocks	0.0008	0.0578	16	6285	804	339	Jan 1926 - Mar 2021
	Bonds	0.0041	0.0027	10	1659	714	339	Jul 1932 - Mar 2021
It	Stock	0.0053	0.0757	9	1388	803	340	Jan 1926 - Apr 2021
	Bonds	0.0013	0.0024	15	2878	804	340	Jan 1926 - Apr 2021
Swi	Stocks	0.0046	0.0444	9	1194	804	340	Jan 1926 - Apr 2021
	Bonds	0.0016	0.0109	8	26770	804	340	Jan 1926 - Apr 2021
Fra	Stocks	0.0070	0.0613	16	6652	794	341	Jan 1926 - May 2021
	Bonds	0.0006	0.0026	13	3449	804	341	Jan 1926 - May 2021
NL	Stock	0.0041	0.0543	19	9371	780	340	Jan 1926 - Apr 2021
	Bonds	0.0006	0.0158	18	7462	786	340	Jan 1926 - Apr 2021

A corporate bond index is not available for the sample period for all countries so I will use a government bond index for the Netherlands, France and Italy. Jansen et al. (2000) state that corporate bond returns and government bond returns are highly correlated and are therefore interchangeable in this particular problem. This correlation may not apply to all countries but as the goal of this paper is to compare different sample selection methods, the most important thing is that all methods are applied to the same data. As stock indices I selected the FTSE All-Share indices for the U.K and Italy, the CAC-40 index for France, the SBC index for Switzerland and an All-Shares index for the Netherlands. All index time series were obtained from the Global Financial Data(GFD) database, except for the S&P 500 index which comes from the CRSP database.

5 Results

5.1 Revisit to Hyung and de Vries (2007)

As in Hyung and de Vries (2007), I will first estimate the parameters using each of the sample fraction sizes and use those to calculate a value at risk with probability $\delta = 2/n$ for several hypothetical portfolio combinations. For the portfolio combinations the fraction of stocks is varied from 0% to 100% with steps of 10%. In Hyung and de Vries (2007) they copy the sample fraction sizes from Jansen et al. (2000), where they use the selection method introduced by Hall (1990). As the data in this paper is not identical to Hyung and de Vries (2007) and Jansen et al. (2000), I applied their method to this data and included the estimates in the results. The result differ slightly, where they select a sample fraction size of 13 and 16 for U.S. stocks and bonds, respectively, I will select 18 and 15. These sample fraction sizes and corresponding parameter estimates are displayed in Table 2, along with the sample fraction sizes and parameter estimates from the methods discussed in Section 3.1. All results were computed in R. Except for the Stability method, I used functions from Ossberger (2020), with some slight modifications, to estimate the tail indices and to compute m^*. Here, one might notice a few remarkable results. The first is the relatively large U.S. bond tail index estimate of 4.4442 when using the Sequential method, compared to the other estimates. Furthermore, the U.S. bond tail index estimate when using the Stability method is curious in two ways. Its value of 1.7378 is less than 2, suggesting the variance of U.S. bond returns is infinite. In addition, its value is lower than the estimate for the U.S. stock tail index (2.0754) when using the same method. This would indicate that the U.S. bond index suffers more extreme losses than the U.S. stock index, which contrasts with the other estimates. A possible explanation is that the input parameters c_2 and q in Algorithm 3 favour a large sample size, therefore causing a bias in the estimate.

Table 2: Tail indices estimates, optimal fraction sizes and order statistics for U.S. assets sample January 1926 - December 1992

	U.S. Stocks				U.S. Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	18	2.6468	1.0068	-0.1291	15	2.6923	0.9156	-0.0381
DB	28	2.2171	1.1002	-0.1003	14	2.7580	0.8985	-0.0382
Seq	45	2.2285	1.3791	-0.0812	3	4.4442	0.4366	-0.0680
Stable	49	2.0754	1.4434	-0.0758	52	1.7378	1.2537	-0.0164
Eye	16	2.6053	0.9221	-0.1365	16	2.8415	1.0057	-0.0369

Except for the estimation results from the sequential method, all results fall under case (i) from Theorem 1. Using this theorem and following Hyung and de Vries (2007), the VaR levels q_δ for net returns for each portfolio allocation can be estimated by solving

$$w^{\alpha_1} A_1 q_\delta^{-\alpha_1} + (1-w)^{\alpha_2} A_2 q_\delta^{-\alpha_2} \approx \delta \tag{11}$$

for corresponding portfolio allocation w and probability δ. Here α_i is the first order tail index estimate for asset i and A_i is the Weissman (1978) estimator $A_i = \frac{m_i}{n}(-X_{(m_i)})^{\alpha_i}$, where $X_{(m)}$ is the $m-th$ largest negative return and n the number of observations in the full estimation sample. The results from the Sequential method fall under case (ii) from Theorem 1 and the VaR levels can be estimated in a similar way by solving

$$w_i^{\alpha_i} A_i q_\delta^{-\alpha_i} + w_i^{\alpha_i} A_i \alpha_i E[(1-w_i)X_2] q_\delta^{-\alpha_i - 1} \approx \delta, \tag{12}$$

where i is the asset 1 or 2 with the lowest first order tail index estimate, and w_i the portfolio fraction of asset i. Since the fraction w_i is a factor in both terms, it is impossible to use these estimates to compute the VaR levels for a portfolio with 0% asset i. The VaR levels with $\delta = 1/402 = 0.0025$ for each of the portfolio allocations are displayed in Table 3. For example, for an 80% stock portfolio and tail indices constructed by using the Double Bootstrap method, solve the equation

$$0.8^{2.2171} * \frac{28}{804} * 0.1003^{2.2171} * q_\delta^{-2.2171} + 0.2^{2.7580} * \frac{14}{804} * 0.0382^{2.7580} * q_\delta^{-2.7580} \approx 0.0025$$

to obtain $q_\delta \approx 0.2962$. Since the order statistics in Equation (11) are mapped on the positive quadrant, the additive inverse of q_δ is taken such that losses can be considered in terms of negative returns again. The most optimal levels in Table 3 are marked with an asterisk. As in Hyung and de Vries (2007), these VaR levels for all methods favour a portfolio with 10% stock. This portfolio mix has the lowest expected loss for a risk level of 0.25%. Its VaR levels obtained from the Hall-, Double Bootstrap- and Eye-Ball method are quite similar but differ substantially from those obtained from the Sequential- en Stability method.

Table 3: VaRs for net portfolio returns with probability $\delta = 0.0025$

q_δ	Hall	DB	Seq	Stable	Eye
100% stock	-0.2956	-0.3291	-0.3276	-0.3532	-0.3026
90% stock	-0.2660	-0.2962	-0.2965	-0.3183	-0.2724
80% stock	-0.2365	-0.2633	-0.2632	-0.2841	-0.2422
70% stock	-0.2071	-0.2305	-0.2309	-0.2506	-0.2120
60% stock	-0.1780	-0.1980	-0.1987	-0.2182	-0.1820
50% stock	-0.1494	-0.1659	-0.1664	-0.1871	-0.1525
40% stock	-0.1221	-0.1349	-0.1342	-0.1580	-0.1239
30% stock	-0.0977	-0.1065	-0.1019	-0.1323	-0.0979
20% stock	-0.0801	-0.0844	-0.0694	-0.1124	-0.0784
10% stock	-0.0748*	-0.0742*	-0.0368*	-0.1026*	-0.0715*
0% stock	-0.0802	-0.0772	NA	-0.1066	-0.0764

The results in Table 3 are used to maximize $\frac{E[R]-r}{r-q_\delta(R)}$ from Equation (1). Here, I will use risk free rate $r = 1$ and $q_\delta(R)$ is the VaR of gross return R, therefore $q_\delta(R) = 1 + q_\delta$. The expected return $E[R]$ for a portfolio is computed by weighting the mean gross returns -which are obtained from the mean net returns in Table 1- of both assets according to the corresponding portfolio allocation. For example, the mean gross returns for the U.S. stock index and U.S. bond index are 1.0066 and 1.0054, respectively, such that a 60% stock portfolio with VaR constructed by the Eye-Ball method is valued $\frac{0.6*1.0066+0.4*1.0054-1}{1-(1-0.1820)} = 0.0336$. These values are displayed in Table 4 for each selection method and portfolio allocation. Maximum values for all methods are obtained from a portfolio with 10% stocks, where Hyung and de Vries (2007) favour a portfolio with 20% stocks under the same conditions. Although the sample fraction selection methods do not affect the portfolio allocation in this case, it can still be of interest to examine how well the VaR estimates perform.

Table 4: Safety first optimization values with $r = 1$ and $\delta = 0.0025$

$\frac{E[R]-r}{r-q_\delta(R)}$	Hall	DB	Seq	Stable	Eye
100% stock	0.0223	0.0201	0.0201	0.0186	0.0218
90% stock	0.0244	0.0219	0.0219	0.0204	0.0238
80% stock	0.0269	0.0242	0.0241	0.0224	0.0263
70% stock	0.0301	0.0271	0.0270	0.0249	0.0294
60% stock	0.0344	0.0309	0.0308	0.0280	0.0336
50% stock	0.0402	0.0361	0.0361	0.0321	0.0393
40% stock	0.0482	0.0436	0.0438	0.0372	0.0475
30% stock	0.0590	0.0541	0.0565	0.0435	0.0588
20% stock	0.0704	0.0668	0.0813	0.0502	0.0719
10% stock	0.0738*	0.0744*	0.1500*	0.0538*	0.0772*
0% stock	0.0673	0.0699	NA	0.0507	0.0707

5.2 Evaluation

For the countries U.K., Switzerland, Italy, France and the Netherlands, the same computations as in Section 5.1 were made. The estimation results are displayed in Tables 13-17 in Appendix C, along with the sample fraction sizes. These estimations are used to compute the VaR levels and select the optimal portfolios. Table 5 shows the stock fraction in the selected optimal portfolio in the parentheses and the corresponding VaR levels for each country and each of the sample size selection methods. Unlike the case with a U.S. mixed portfolio, different sample sizes may lead to different optimal allocations for portfolios from these countries. Smaller differences in the VaR levels and/or large difference between the mean stock return and the mean bond return cause Equation (1) to be maximized for different stock fractions. For the same reason it is also possible that this portfolio selection criterion selects corner solutions -as is the case for two sample selections methods for the U.K. and France- even though the optimal VaR levels occur for interior portfolio allocations.

Table 5: Optimal stock fractions and corresponding VaRs

q_δ (Optimal Stock Fraction)	UK	It	Swi	Fra	NL
Hall	-0.0946 (10%)	-0.1042 (30%)	-0.1265 (10%)	-0.0408 (10%)	-0.0956 (30%)
DB	-0.1000 (10%)	-0.0904 (30%)	-0.0934 (30%)	-0.0102 (0%)	-0.0933 (30%)
Seq	-0.1118 (0%)	-0.0923 (30%)	-0.0981 (30%)	-0.0197 (10%)	-0.1108 (40%)
Stab	-0.1217 (0%)	-0.1433 (50%)	-0.1092 (20%)	-0.013 (0%)	-0.1898 (60%)
Eye	-0.0977 (10%)	-0.0896 (30%)	-0.0947 (20%)	-0.0213 (10%)	-0.1234 (40%)

Table 6: Violations as fraction of test sample size and test statistics

Violations/T	Hall	DB	Seq	Stab	Eye
US	0/335	0/335	3/335	0/335	0/335
UK	0/339	0/339	0/339	0/339	0/339
It	0/340	1/340	1/340	0/340	1/340
Swi	0/340	2/340	2/340	0/340	2/340
Fra	0/341	0/341	1/341	0/341	0/341
NL	0/340	0/340	0/340	0/340	0/340
Total	0/2035	3/2035	6/2035	0/2035	3/2035
LR	10.1877	1.0081	0.1551	10.1877	1.0081
(p-value)	(0.0014)	(0.3154)	(0.6937)	(0.0014)	(0.3154)

To examine the influence of the sample size selection on the portfolio allocation for a safety first investor, the monthly returns of each the selected optimal portfolios are computed over the period January 1993 - December 2020 (may vary slightly per country depending on available data). Every time a return falls below the corresponding VaR level $q_\delta(R)$, a violation is counted for the fraction selection method it was computed with. For example, the Double Bootstrap method computed VaR level $q_\delta(R) = 1 - 0.0934 = 0.9066$ and selected 30% stock as the optimal portfolio mix of the Swiss stock index and the Swiss bond index. The monthly gross return of this particular portfolio fell below 0.9066 two times since January 1993. Table 6 shows that the Sequential method has the most violations, with a total of 6. The Double Bootstrap method and the Eye-Ball method both have a total of 3 violations and the Hall en Stability method have none. The absence of violations may indicate that these two methods computed VaR levels which are too safe for the corresponding probability level. For a good fit the amount of violations should be near $2035 * 0.0025 = 5$, where 2035 is the sum of the test sample sizes given in Table 1. In the last column of Table 6 the LR statistics from Equation (10) are given for each method, with T_1 the total amount of violations and $T = 2035$. In the parentheses are the corresponding p-values. The large numbers for the LR statistics for the Hall and the Stability method confirm that their computed VaR levels were indeed too safe. We may reject the null hypothesis of correct unconditional coverage at a probability of 0.014%. We fail to reject the null hypothesis for the remaining three methods and the highest p-value for the Sequential method suggests this sample fraction selection method results in the most "correct" VaR level. Both methods with an asymptotic approach perform quite well and from the heuristic methods only the Eye-Ball method performs reasonably well. In this case, the estimation sample sizes of 804 are quite large, which is favorable for the asymptotic approaches. In the next section I will examine if these methods perform worse relative to the heuristic methods in smaller estimation samples.

5.3 Small Estimation Sample

I re-estimated the tail indices for each of the methods using a sample of 300 observations from January 1950 - December 1974. The results for the U.S. stock and bond indices are displayed in Table 7. The estimates from the smaller sample are somewhat larger, which could be expected since smaller samples have less extreme values. The Stability method computed a first-order tail parameter for the U.S. stock index that seems out of line with the other estimates, as it is substantially larger. The estimates for the other countries can be found in Tables 18-22 in Appendix D. For both the Swiss stock index and bond index, the Double Bootstrap method and the Sequential method computed infinite first order tail indices. The small sample size and lack of extreme values caused them to select an optimal sample fraction smaller than two. The Swiss portfolios will not be included in the evaluation for these two methods.

The estimates are used to compute the VaR levels again with probability $2/n = 0.0067$. Table 8 shows that the losses that occur with this probability are lowest for the 10% portfolio allocation for all methods, as was the case for the large estimation sample. The mean returns of the indices over the new sample are needed to maximize Equation (1) and are shown in Table 9. The computations of Equation (1) for the eleven U.S. portfolio allocations are show in Table 10. This time, the different sample fraction selection methods may select different optimal portfolio allocations. The Sequential method and the Stability method pick a portfolio with 20% in the stock index, where the other three allocate only 10% to the stock index. These computations were made for the other countries and the optimal portfolios for each method are displayed in Table 11, along with the corresponding VaR levels.

Table 7: Tail indices estimates, optimal fraction sizes and order statistics for U.S. assets sample January 1950 - December 1974

	US Stocks				US Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	14	3.4642	1.4899	-0.0625	13	4.9953	2.0388	-0.0153
DB	2	3.7728	0.2573	-0.1184	3	4.0781	0.4861	-0.0195
Seq	13	3.3136	1.3539	-0.0629	17	4.3052	2.1221	-0.0139
Stable	7	5.1929	1.3433	-0.0818	5	4.7782	0.9384	-0.0189
Eye	12	3.1187	1.2026	-0.0646	12	5.1052	1.9685	-0.0154

Table 8: VaRs for net portfolio returns with probability $\delta = 0.0067$

q_δ	Hall	DB	Seq	Stable	Eye
100% stock	-0.1094	-0.1182	-0.1105	-0.1040	-0.1146
90% stock	-0.0985	-0.1064	-0.0994	-0.0936	-0.1034
80% stock	-0.0876	-0.0946	-0.0884	-0.0832	-0.0923
70% stock	-0.0766	-0.0828	-0.0773	-0.0728	-0.0811
60% stock	-0.0657	-0.0710	-0.0663	-0.0624	-0.0700
50% stock	-0.0547	-0.0591	-0.0553	-0.0520	-0.0588
40% stock	-0.0438	-0.0474	-0.0443	-0.0416	-0.0476
30% stock	-0.0331	-0.0358	-0.0336	-0.0314	-0.0363
20% stock	-0.0237	-0.0252	-0.0244	-0.0223	-0.0250
10% stock	-0.0205*	-0.0201*	-0.0211*	-0.0207*	-0.0198*
0% stock	NA	-0.0212	-0.0228	-0.0229	NA

Table 9: Mean simple monthly returns over January 1950 - December 1974 and test sample sizes

	US		UK		It	
	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds
Mean	0.0095	0.0032	0.0027	0.0003	0.0038	0.0016
T	551	551	555	555	547	547

	Swi		Fra		NL	
	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds
Mean	0.0048	0.0010	0.0055	0.0002	0.0052	0.0012
T	555	555	556	556	557	557

Table 10: Safety first optimization with $r = 1$ and $\delta = 0.0067$

$\frac{E[R]-r}{r-q_\delta(R)}$	Hall	DB	Seq	Stable	Eye
100% stock	0.0864	0.0799	0.0855	0.0909	0.0825
90% stock	0.0893	0.0830	0.0888	0.0943	0.0853
80% stock	0.0930	0.0867	0.0928	0.0986	0.0889
70% stock	0.0977	0.0916	0.0980	0.1041	0.0934
60% stock	0.1040	0.0980	0.1049	0.1114	0.0994
50% stock	0.1126	0.1071	0.1146	0.1217	0.1077
40% stock	0.1253	0.1205	0.1289	0.1370	0.1199
30% stock	0.1460	0.1422	0.1514	0.1616	0.1398
20% stock	0.1860	0.1770	0.1824*	0.1967*	0.1780
10% stock	0.1889*	0.1911*	0.1824	0.1852	0.2041*
0% stock	NA	0.1493	0.1407	0.1404	NA

Table 11: Optimal stock fractions and corresponding VaRs

q_δ (Optimal Stock Fraction)	UK	It	Swi	Fra	NL
Hall	-0.0098 (0%)	-0.0813 (40%)	-0.0236 (10%)	-0.0149 (10%)	-0.0531 (30%)
DB	-0.0099 (0%)	-0.0819 (50%)	-	-0.0168 (10%)	-0.0497 (30%)
Seq	-0.0045 (0%)	-0.0821 (50%)	-	0.0172 (10%)	-0.0451 (30%)
Stab	-0.0129 (0%)	-0.0888 (40%)	-0.0315 (10%)	-0.0147 (10%)	-0.0512 (30%)
Eye	-0.0096 (0%)	-0.0584 (30%)	-0.0257 (10%)	-0.0141 (10%)	-0.0497 (20%)

In the same way as in Section 5.2, the returns of the optimal portfolios are observed and the amount of violations of the corresponding VaR level are counted over the sample from January 1975 - December 2020. The amount of violations, the LR statistics and the corresponding p-values are shown in Table 12. Adding up the sizes of all the test samples (see Table 9) gives $T = 2766$ for the Double Bootstrap and Sequential method and $T = 3321$ for the remaining methods. For good VaR levels, the total amount of violations should be $2766 * 0.0067 = 18$ and $3321 * 0.0067 = 22$ for the respective methods. Note that these numbers are already exceeded by the U.S. portfolios only. A probable explanation for the high numbers of violations in the U.S. is that the estimations are based on a sample period with rather low volatility. Then 13 of the violations occur during one of three periods with high volatility: shortly after Silver Thursday in March 1980, shortly after Black Monday in 1987 or during the Financial Crisis from 2007-2009.

It is interesting to see that the Sequential method performs worst with a LR statistic of 103, even though this method performed best with the larger estimation sample. Especially when taking in mind that this procedure failed to produce tail index estimates for both Swiss indices, there is clear evidence that it performs poorly in smaller samples. The Stability method, which was one of the two worst performers in the large sample, is now the best performer with a LR statistic of 18. Yet, the null hypothesis of correct unconditional coverage is rejected for all methods, which is largely due to the high amount of violations of the U.S. portfolios. If they were to be excluded one would obtain LR statistics (p-values) of 13.6078(0.0002), 5.2755(0.0216), 66.1593(0.0000), 0.4016(0.5263) and 33.7940(0.0000) for the respective methods in the same order as Table 12. Now, we fail to reject correct unconditional coverage for the VaR levels constructed by the Stability method and the Double Bootstrap method with respective significance levels of 52.63% and 2.16%.

Table 12: Violations as fraction of test sample sizes and test statistics

Violations/T	Hall	DB	Seq	Stab	Eye
US	26/551	28/551	22/551	24/551	26/551
UK	5/555	5/555	30/555	0/555	5/555
It	4/547	6/547	6/547	1/547	15/547
Swi	13/555	-	-	4/555	11/555
Fra	5/556	3/556	3/556	6/556	7/556
NL	9/557	10/557	16/557	10/557	10/557
Total	62/3321	52/2766	77/2766	45/3321	74/3321
LR	48.0527	40.7741	103.6591	18.0447	75.1679
(p)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

6 Conclusion

This paper examines and compares different methods of selecting the optimal sample fraction size when estimating the tail indices of tails modelled by a second order Hall expansion. The different estimation results are then used to construct Value at Risk levels and ultimately to select an optimal portfolio allocation between a mutual fund that tracks a stock index and one that tracks a bond index. This selection is done conform the Safety First Criterion, which deals with the expected return of a portfolio allocation and the maximum risk the investor is willing to take. I have shown that the choice of fraction size is important to construct a good estimate of the VaR and that in some cases, it affects the optimal portfolio allocation. The goal of this paper was to improve the portfolio allocations constructed by Hyung and de Vries (2007), where they used the Hall method to select the fraction size. Although the allocation for this particular problem was not affected, the Hall method was outperformed by three of the discussed methods in terms of violations of the VaR. Therefore, replacing the Hall method will yet lead to an improvement in the sense that it gives the investor a more accurate threshold of the disaster return corresponding to a small probability. The Sequential method, introduced by Drees and Kaufmann (1998), came out best in the test for this problem for the larger estimation sample with 804 observations. This same method performed worst when the estimation sample is smaller, containing 300 observations, and even failing to compute tail index estimates in certain occasions. In the small estimation sample, the Stability method from Schouten (2017) resulted in the best performing estimate of the VaR. The Double Bootstrap method performed reasonably well for both sample sizes in terms of violations, although it also failed to produce tail index estimates in some cases. The Eye-Ball method performed well on the large estimation sample but quite poorly on the small sample. The Hall method performed poorly on both estimation samples.

As different estimates of the Hill estimator can affect the VaR performance and the portfolio allocation, it may as well be of interest to examine how different estimators could have an effect on these issues. Other estimators are proposed by de Haan and Resnick (1984), Hall and Welsh (1985), Pickands (1975) and Mason (1982). Furthermore, the second order tail index in this paper is estimated by an estimator proposed by Danielsson et al. (2000), even though there are other estimators proposed in the literature. A few examples are those by Gomes and Martins (2002), Ljunberg and Enqvist (2002) or Peng and Qi (2004).

Appendices

A Little-oh Notation

The little-oh notation $o(g(x))$ is an asymptotic notation and is used to describe a loose upper bound of a function $f(x)$. By mathematical definition, from Landau (1909), $f(x) = o(g(x))$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

From this definition follows that if $f(x) = h(x)o(g(x))$, which implies $\frac{f(x)}{h(x)} = o(g(x))$, then

$$\lim_{x\to\infty} \frac{\frac{f(x)}{h(x)}}{g(x)} = \lim_{x\to\infty} \frac{f(x)}{h(x)g(x)} = 0,$$

such that $f(x) = h(x)o(g(x))$ is equivalent to $f(x) = o(h(x)g(x))$. For the little-oh notation in the Hall expansion this means $Ax^{-\alpha}o(x^{-\beta}) = o(Ax^{-\alpha-\beta})$. Note that constant A does not affect the rate at which the function converges to 0, such that Equation (3) can be rewritten as

$$P[X > x] - Ax^{-\alpha} - ABx^{-\alpha-\beta} = o(x^{-\alpha-\beta}),$$

and therefore

$$\lim_{x\to\infty} = \frac{P[X > x] - Ax^{-\alpha} - ABx^{\alpha-\beta}}{x^{-\alpha-\beta}} = 0.$$

Intuitively, you can say that, as $x \to \infty$, the function in the numerator declines faster than the term in the denominator.

B Bias-Variance Trade Off

Danielsson et al. (2019) show that for random variable X, with a distribution that satisfies the Hall expansion, the asymptotic bias for the Hill estimator

$$\mathrm{E}\left[\frac{1}{\hat{\alpha}} - \frac{1}{\alpha}|X > s\right] = \frac{-\beta B s^{-\beta}}{\alpha(\alpha+\beta)} + o(s^{-\beta}) \tag{13}$$

is a function of threshold s. As the threshold becomes smaller, the sample fraction size m will become larger since it will include more observed values closer to the center of the distribution. From Equation (13) it is obvious that if s decreases, i.e. m increases, the bias increases. For the asymptotic variance of the Hill estimator they show that is also a function of s

$$\mathrm{var}\left(\frac{1}{\hat{\alpha}}\right) = \frac{s^{\alpha}}{nA\alpha^2} + o\left(\frac{s^{\alpha}}{n}\right).$$

This equation shows that if s decreases, the variance of the Hill estimator decreases.

C Tables Large Sample

Table 13: Tail indices estimates, optimal fraction sizes and order statistics for U.K. assets sample January 1926 - December 1992 for the stock index and sample January 1932 - December 1992 for the bond index

	U.K. Stocks				U.K. Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	50	2.4664	1.7369	-0.0765	21	3.4234	1.4778	-0.0504
DB	11	2.6582	0.7426	-0.1233	4	3.0218	0.4040	-0.0845
Seq	45	2.2285	1.4713	-0.0812	8	2.9207	0.5624	-0.0669
Stable	49	2.0754	1.4435	-0.0758	28	2.4295	1.2498	-0.0392
Eye	16	2.6053	0.9221	-0.1365	11	3.2290	0.9277	-0.0609

Table 14: Tail indices estimates, optimal fraction sizes and order statistics for Italian assets sample January 1926 - December 1992

	Italian Stocks				Italian Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	55	2.7962	2.0897	-0.0908	23	2.9633	1.3071	-0.0462
DB	11	3.5460	0.9909	-0.1583	5	4.2213	0.6687	-0.0857
Seq	2	3.7767	0.2183	-0.2591	15	3.0490	1.0369	-0.0536
Stable	22	3.4479	1.4813	-0.1286	48	1.8588	1.2766	-0.0275
Eye	14	3.9113	1.2746	-0.1516	14	3.0384	0.9898	-0.0541

Table 15: Tail indices estimates, optimal fraction sizes and order statistics for Dutch assets sample January 1926 - December 1992

	Dutch Stocks				Dutch Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	61	2.2297	1.7984	0.0611	23	3.1452	1.3963	0.0382
DB	33	2.3565	1.3026	0.0826	3	3.3660	0.3321	0.0784
Seq	22	2.5980	1.1253	0.1035	4	2.9647	0.3891	0.0720
Stable	43	2.1245	1.3786	0.0701	78	1.7886	1.6865	0.0169
Eye	10	2.2349	0.5906	0.1353	13	2.3620	0.7385	0.0454

Table 16: Tail indices estimates, optimal fraction sizes and order statistics for French assets sample January 1926 - December 1992

	French Stocks				French Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	30	3.4946	1.8141	-0.0928	52	2.4903	1.7966	-0.0023
DB	14	3.2381	1.0908	-0.1168	9	2.6412	0.6459	-0.0058
Seq	39	3.7506	2.2798	-0.0846	2	3.5779	0.2068	-0.0134
Stable	56	2.7821	2.1116	-0.0705	39	2.6365	1.5960	-0.0026
Eye	14	3.3381	1.0908	-0.1168	17	2.8742	1.0551	-0.0041

Table 17: Tail indices estimates, optimal fraction sizes and order statistics for Swiss assets sample January 1926 - December 1992

	Swiss Stocks				Swiss Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	52	1.7928	1.5852	-0.0644	51	2.1971	1.2363	-0.0304
DB	4	2.6772	0.3565	-0.1418	9	2.7273	0.6547	-0.0696
Seq	10	3.5280	0.9259	-0.1312	20	2.4962	1.0425	-0.0511
Stable	36	2.4325	1.4032	-0.0781	12	2.7454	0.8113	-0.0669
Eye	15	2.6773	1.1492	-0.1168	10	3.3793	0.7026	-0.0695

D Tables Small Sample

Table 18: Tail indices estimates, optimal fraction sizes and order statistics for U.K. assets sample January 1950 - December 1974

	UK Stocks				UK Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	16	2.8863	1.3635	-0.0906	21	1.9901	1.1378	-0.0030
DB	3	3.5964	0.4036	-0.1736	4	3.6542	0.5862	-0.0082
Seq	14	3.1772	1.3665	-0.0924	5	3.2312	0.6346	-0.0034
Stable	48	1.5951	1.6817	-0.0416	45	1.5326	1.5349	-0.0017
Eye	10	2.8641	0.9685	-0.1095	10	2.1921	0.7413	-0.0046

Table 19: Tail indices estimates, optimal fraction sizes and order statistics for Italian assets sample January 1950 - December 1974

	Italian Stocks				Italian Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	38	2.5034	2.2001	-0.0501	17	1.8914	0.9323	-0.0281
DB	12	3.2421	1.2501	-0.0855	7	3.9734	1.0278	-0.0864
Seq	4	4.5132	0.0724	-0.1270	20	3.5762	1.9756	-0.0658
Stable	33	2.5568	2.0220	-0.0537	32	1.4828	1.1464	-0.0159
Eye	5	4.3495	0.8542	-0.1196	5	2.8109	0.5520	-0.0532

Table 20: Tail indices estimates, optimal fraction sizes and order statistics for Dutch assets sample January 1950 - December 1974

	Dutch Stocks				Dutch Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	19	2.5848	1.3775	-0.0658	23	2.5253	1.5395	-0.0172
DB	2	3.0394	0.2102	-0.1403	6	2.5814	0.5907	-0.0325
Seq	4	4.0770	0.6540	-0.1167	3	3.5439	0.4224	-0.0407
Stable	15	2.7143	1.2254	-0.0711	24	2.5177	1.5819	-0.0171
Eye	9	2.7403	0.8577	-0.1087	8	2.3349	0.6692	-0.0263

Table 21: Tail indices estimates, optimal fraction sizes and order statistics for French assets sample January 1950 - December 1974

	French Stocks				French Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	22	2.5564	1.5103	-0.0566	24	2.2605	1.4203	-0.0018
DB	7	2.2798	0.5903	-0.0943	14	1.9026	0.8183	-0.0023
Seq	7	2.2798	0.5903	-0.0943	2	2.4067	0.1664	-0.0079
Stable	28	2.6951	1.8907	-0.0518	27	1.9094	1.3049	-0.0016
Eye	12	2.7606	1.0645	-0.0714	10	2.1607	0.7373	-0.0024

Table 22: Tail indices estimates, optimal fraction sizes and order statistics for Swiss assets sample January 1950 - December 1974

	Swiss Stocks				Swiss Bonds			
	m	α	β	$X_{(m)}$	m	α	β	$X_{(m)}$
Hall	19	2.4553	1.1947	-0.0786	27	1.5593	0.8798	-0.0017
DB	2	2.0752	0.1302	-0.1589	74	1.2861	1.6378	-0.0005
Seq	20	2.3545	1.1773	-0.0712	18	1.4782	1.6689	-0.0024
Stable	63	1.8421	1.8629	-0.0379	36	1.2856	0.9556	-0.0013
Eye	11	2.6524	0.8328	-0.0965	6	2.1028	0.4257	-0.0050

References

Arzac, E., & Bawa, V. (1977). Portfolio choice and equilibrium in capital markets with safety-first investors. *Journal of Financial Economics*, *4*, 277–288.

Danielsson, J., de Haan, L., Peng, L., & de Vries, C. (2000). Using a bootstrap method to choose the sample fraction in tail index estimation. *Journal of Multivariate Analysis*, *76*, 226–248.

Danielsson, J., Ergun, L., de Haan, L., & de Vries, C. (2019). Tail index estimation: Quantile-driven threshold selection. *Working Paper*.

de Haan, L., & Resnick, S. (1984). Tail estimates motivated by extreme value theory. *The Annals of Statistics*, *12*, 1467–1487.

Drees, H., & Kaufmann, E. (1998). Selecting the optimal sample fraction in univariate extreme value estimation. *Stochastic Processes and their Applications*, *75*, 149–172.

Gomes, M., & Martins, M. (2002).
asymptotically unbiased" estimators of the tail index based on external estimation of the second order parameter. *Extremes*, *1*, 5–31.

Hall, P. (1990). Using the bootstrap to estimate mean squared error and select smoothing parameter in nonparametric problems. *Journal of Multivariate Analysis*, *32*, 177–203.

Hall, P., & Welsh, A. (1985). Adaptive estimates of parameters of regular variation. *The Annals of Statistics*, *13*, 331–341.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *The Annuals of Statistics*, *3*, 1163–1174.

Hyung, N., & de Vries, C. G. (2007). Portfolio selection with heavy tails. *Journal of Empirical Finance*, *14*, 383–400.

Jansen, D., Koedijk, K., & de Vries, C. (2000). Portfolio selection with limited down-side risk. *Journal of Empirical Finance*, *7*, 247–269.

Landau, E. (1909). *Handbuch der lehre von der verteilung der primzahlen* [2 volumes. Reprinted by Chelsea, New York, 1953]. Teubner.

Ljunberg, F., & Enqvist, M. (2002).
asymptotically unbiased" estimators of the tail index based on external estimation of the second order parameter. *Extremes*, *1*, 5–31.

Mason, D. (1982). Laws of large numbers for sums of extreme values. *Institute of Mathematical Statistics*, *10*, 754–764.

Ossberger, J. (2020, April 19). *Threshold estimation approaches* (Version 1.1). https://rdrr.io/cran/tea/

Peng, L., & Qi, Y. (2004). Estimating the first- and second-order parameters of a heavy-tailed distribution. *Australian New Zealand Journal of Statistics*, *46*, 305–312. https://doi.org/10.1111/j.1467-842X.2004.00331.x

Pickands, J. (1975). Statistical inference using extreme order statistics. *The Annals of Statistics*, *3*, 119–131.

Roy, A. D. (1952). Safety first and the holding of assets. *Econometrica*, *30*, 431–449.

Schouten, T. (2017). *Modelling dependence in dynamic factor models with copulas and extreme value theory* (Master's thesis). Delft University of Technology.

van Os, B. (2021). *Quantitative methods for finance* [Lecture].

Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *Journal of the American Statistical Association*, *73*, 812–815.