Dependence Modelling of GDP components

Observation driven models applied to Growth-at-Risk

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Abstract

Gross domestic product (GDP) is one of the most important macroeconomic measurements of overall welfare in a country. Since it comprises of components such as private consumption, correct modelling of its growth is of paramount importance. Recently, macroprudential policymakers have constructed Growth-at-Risk (GaR) to mitigate downside tail risks, which are events that can lead to financial crises. To obtain GaR estimates, policymakers often rely on quantile regression (QR) models. Yet it has multiple shortcomings, such as relying on a set of downside risk predictors. In this paper, I propose an integral approach that analyses both the structure of individual GDP components, as well as modelling their dependency with bivariate copulas augmented with a general autoregressive score (GAS) framework. Coverage tests on obtained GaR forecasts of a panel of six OECD countries indicate that copula-GAS models perform as well as QR. However, comparative backtesting on a set of loss functions applied to GaR forecasts show that copula-GAS models consistently provide significantly smaller losses, especially at higher coverages, which suggests that QR produces GaR with higher opportunity costs.

Contents

1 Abbreviations

Table 1: A list of abbreviations and acronyms used in this paper.

Abbreviation	Explanation
AR	Autoregressive
BIC	Bayesian information criterion
CAViaR	Conditional autoregressive Value-at-Risk
CDF	Cumulative distribution function
CPI	Consumer price index
FLF	Firm's loss function
GaR	Growth-at-Risk
GARCH	Generalised autoregressive conditional heteroskedasticity
GAS	Generalised autoregressive score
GDP	Gross domestic product
GGAS	Gaussian copula-GAS
GPL	Generalised Piecewise Linear
IMF	International Monetary Fund
IP	Industrial production
LB	Ljung-Box
LR	Likelihood ratio
MAD	Mean absolute deviation
MEM	Multiplicative error model
MLE	Maximum likelihood estimation
MSE	Mean squared error
OECD	Organisation for Economic Co-operation and Development
OLS	Ordinary least squares
PCA	Principal component analysis
PDF	Probability density function
QR	Quantile regression
RLF	Regulator's loss function
SCGAS	Symmetrised Clayton copula-GAS
VaR	Value-at-Risk
VAR	Vector autoregressive

2 Introduction

Gross domestic product (GDP) is well known among policymakers and the general public alike. GDP quantifies the value that is created by the production of goods and services in a country during a certain period. Real GDP per capita is still widely regarded as the main indicator of economic prosperity. At the same time, however, this economic indicator is criticised for its supposed inability to capture intangible human welfare. Besides this normative assessment, macroeconomic models applied by policymakers mostly focussed on forecasting expected mean growth in the past, while failing to recognise higher-order moments of the GDP growth distribution.

This paper addresses the latter critique. After the Financial Crisis of 2008, there has been increased interest in mitigating downside tail risk. Aikman et al. (2019) and Gertler and Gilchrist (2018) show that the fragile financial system in combination with large asset price imbalances played a big role. Policymakers of national and central banks concluded that a new, 'macroprudential' framework was necessary to ensure a crisis of such magnitude would not happen again. A key indicator for macroprudential policymakers is growth-at-risk (GaR). Originally introduced by Wang and Yao (2001) and in detail assessed by Prasad et al. (2019), GaR (usually) is an $\alpha\%$ quantile of the distribution of GDP growth rates. It can be seen as estimated future GDP growth which in (1-α)% of all cases will be lower than actual realised growth. This metric bears resemblance to value-at-risk (VaR), widely used in portfolio management and asset pricing models. Since macroeconomic downturns and financial vulnerabilities primarily have an impact on the left tail of the distribution of GDP growth, it is an intuitive metric to assess extreme downside risk of large macroeconomic systems.

Policymakers often apply quantile regression (QR) analysis to model GaR. QR is able to model the left tail of GDP growth distribution, which implies that this approach can be directly linked to GaR. A wide variety of factors that impact GaR, such as systemic risk measures (Giglio et al., 2016) and productivity shocks (Loria et al., 2019) have been studied by means of QR. Although its popularity and proven usefulness, recent contributions (e.g. Adrian et al. (2016) and Carriero et al. (2020)) emphasise that downside risk varies more than upside risk over time, pointing at asymmetrical business cycle fluctuations. Brownlees and Souza (2021) conclude that QR should not be relied on too heavily, since it is dependent on macroeconomic and financial factors that might not contribute to the predictive ability of GaR.

One approach to correct for asymmetric tail behaviour is to consider copula models. Proposed by Sklar (1959), a d-dimensional copula $\mathcal{C} : [0,1]^d \rightarrow [0,1]$ is a cumulative distribution function (CDF) that couples uniform marginal distributions. A well-known copula is the Gaussian copula, which captures correlation between random variables. Other types, such as Clayton and Gumbel copulas, are capable of capturing asymmetrical tail dependence in time-series. Nowadays, this method is widely applied in a financial and risk management context to model dependence structures, but has not been considered as much in a macroeconomic setting. This paper aims to fill that gap by assessing quarterly GDP growth rates in the years 1970Q1 to 2019Q4 of six OECD countries. To obtain a dependence structure, I divide GDP growth into private final consumption, which represents the largest proportion of total GDP (often more than 50%), and an aggregate time-series of investments, net exports and government spending, called a 'residual' term. Both components are filtered by estimating an AR(p)-GARCH model. I consider a bivariate Gaussian copula to capture correlation, as well as a symmetrised bivariate Clayton copula, which can estimate lower and upper tail dependence simultaneously.

Next, I augment both copula models with a Generalised Autoregressive Score (GAS) framework. GAS models are observation-driven models which feature time-varying parameters dependent on past observations. In this way, copula parameters can be estimated in a score-driven manner using maximum likelihood, which implies the parameters can be modelled time-varying as such. An advantage of this specification is its computational convenience, since likelihood functions are relatively straightforward to define. Moreover, as GAS relies on the gradient of the log-likelihood, its recursion will be locally optimal. I define a grid search over all parameters, to ensure that a global optimum is attained. In addition to GDP growth rate and its components, macroeconomic indicators can also be included in the GAS recursion. Prasad et al. (2019) argue that such indicators might suffer from multicollinearity in GaR analysis, and proposes to use principal components instead. Finally, to derive a set of GaR estimates for a given country, I will employ the obtained time-varying dependence structure in a Monte Carlo simulation by using the time-varying mean and variance obtained from the marginal AR-GARCH models.

The relationship between macroeconomic variables have been studied extensively in recent history, see for example Kydland and Prescott (1982) and Long and Plosser (1983). In addition, literature as early as Keynes (1936) pointed at asymmetrical boom and bust cycles and nonlinearities in macroeconomic dependence. Whereas this previous empirical research focussed mainly on (asymmetrical) dependence between GDP components, relating macroeconomic relationships to a concrete assessment of GaR using augmented copula models is a novel approach.

Ultimately, I want to assess in this paper whether GaR obtained from copula-GAS models are more accurate than GaR obtained from QR. I evaluate GaR by using backtesting tools frequently applied in VaR literature. The accuracy of GaR can be measured in terms of average empirical coverage and can be assessed using coverage tests of Kupiec (1995) and Christoffersen (1998). Another approach is in terms of losses, which can be constructed using loss functions introduced by Lopez (1999). The advantage of loss functions is their ability to rank GaR models. The model that produces the lowest penalty score is preferred. Testing their significance can be achieved by conducting a Diebold-Mariano test. I use loss functions defined by Caporin (2008), which considers the magnitude between an estimated GaR and its realised GDP growth. Since these functions are not consistent, i.e. belonging to the Generalised Piecewise Linear (GPL) family of functions, I refer to these results as a secondary standard to the (un)conditional coverage tests. In addition, I will not only consider losses in case of an exception, but also when GaR is overestimated. This approach is especially useful when the observed data frequency is low, e.g. quarterly growth.

The results of this analysis show that in terms of coverage tests, copula-GAS models perform as well as QR for all six countries. Both conditional and unconditional coverage tests do not reject the null of accurate GaR forecasts in almost all cases (that is, for all countries and different coverage levels considered). When the significance is tested of scores of the models, it can be seen that at higher coverages, copula-GAS models consistently outperform the QR benchmark. Additionally, in almost no cases the QR GaR forecasts are preferred to copula-GAS forecasts. Since the loss functions I use determine losses at every time in terms of magnitude, it implies that QR overestimates downside risk, producing GaR estimates that are lower than required. Conversely, copula-GAS models produce GaR estimates with a lower opportunity cost. In conclusion, policymakers should also consider incorporating copula-GAS models in their macroprudential toolkit.

The following sections comprise the remainder of this report: Section 3 provides an extensive literature review. Section 4 describes the methodology of this paper, which consists of multiple parts, each representing a step to construct GaR forecasts. Section 5 describes the empirical application that is used in this paper. Section 6 presents the results of the empirical application, and Section 7 concludes. The Appendix presents supplementary figures and tables to this paper.

3 Literature Review

One of the aims of central banks is to control macrofinancial events through macroprudential policy. A vector autoregressive (VAR) model is one of the more popular econometric tools to assess the effects of monetary and fiscal policy. White et al. (2015), Prasad et al. (2019) and Duprey and Ueberfeldt (2020) identify exogenous fiscal shocks through models based on VAR. Mountford and Uhlig (2009) analyse policy shocks by imposing sign restrictions, and Lof and Malinen (2014) analyse the relationship between sovereign debt and economic growth, all through a VAR perspective. Quantile regression (QR), originally introduced by Koenker and Bassett (1978) as 'regression quantiles', is also widely considered in academic literature to analyse relationships of variables. In addition, financial and (macro-)economic data often exhibit excess kurtosis, a phenomenon QR can overcome. Mello and Perrelli (2003) analyse the effect of control variables on GDP growth with estimated quantiles. Central and regional banks commonly implement QR as well, see for example White et al. (2015). Koenker (2017) provides an extensive review of QR models.

Since the Financial Crisis of 2008, there has been increased attention to the impact of systemic risk. GaR, originally introduced by Wang and Yao (2001), is a measurement for economic downside risk, and has been developed further by the IMF in its Global Financial Stability Report of 2017, Adrian, Grinberg, Liang, and Malik (2018) and Prasad et al. (2019). GaR, akin to value-at-risk (VaR) is defined as a growth rate such that the probability of expected growth rate does not exceed a certain quantile. The advantage of this approach is that, in contrast to point forecasts of mean growth often communicated by policymakers, GaR focuses on empirically forecasting the entire probability distribution of future GDP growth. This allows policymakers to quantify the impact of both upper and lower tail risks in addition to point forecasts. Conversely, GaR can provide a probability level at which, for example, one-year ahead GDP growth will be negative. Adrian et al. (2019) demonstrate that lower quantiles of GDP growth fluctuate more than upper quantiles, in addition to their findings that financial and macroeconomic conditions have a greater influence on the lower quantiles. Although QR is the main approach to construct GaR, Brownlees and Souza (2021) suggest that this approach should be applied with caution when interest lies in forecasting, since a macroeconomic environment usually exhibits scarce information (i.e. low-frequency data) and requires a representative set of downside risk predictors.

GaR is often analysed with QR in combination with vector autoregressive (VAR) models.

Whereas QR is robust to outliers and VAR allows one to analyse dynamics of the endogenous variables, this approach refrains from making assumptions about the shape of the probability distribution. Conversely, these symmetric models do not account for asymmetric or non-normal tail behaviour, which are exactly the parts of empirical growth distributions policymakers are concerned with. Moreover, while VAR models could capture information from different marginals by incorporating cross-terms, multivariate data may quite possibly suffer from a certain dependence structure, which a VAR model might be unable to capture.

One method to overcome the shortcomings of VAR and QR is to consider copulas, an approach proposed by Sklar (1959). Copula models are an intuitive methodology to characterise the dependence structure between marginals. The key benefit of this methodology is that the dependence structure and univariate distribution functions of variables can be separated. Sklar (1959) formulates this in a theorem which states that for any combination of distribution functions, there exists a unique copula function. Patton (2006) (among others) shows that financial assets are often documented to exhibit asymmetric tail dependence, an issue which copulas may solve. This advantage has ensured that it is now widely applied in the financial sector (Cherubini et al. (2004) provide a review of copula functions in a mathematical finance setting).

Dynamic copula models of Patton (2006) are examples of so-called observation driven models, as categorised by Cox et al. (1981). Observation driven models contain parameters that can be written as a function of (lagged) endogenous and exogenous variables. This category also encompasses well known generalised autoregressive conditional heteroskedasticity (GARCH) models as introduced by Engle (1982) and Bollerslev (1986), but also the conditional autoregressive Value-at-Risk (CAViaR) model of Engle and Manganelli (2004), and vector multiplicative error models (MEM) of Cipollini et al. (2006). This paper builds upon another observation driven model developed by Creal et al. (2013), called GAS models. GAS considers an autoregressive updating equation for time-varying parameters, which has innovations equal to the score of the likelihood of the observations. My research relates to the research of Creal et al. (2013) by constructing both Gaussian and symmetrised Clayton copulas with GAS updating dynamics. The Gaussian copula combined with GAS results in time-varying correlation, whereas the symmetrised Clayton copula with GAS produces time-varying upper and lower tail dependence. As Patton (2006) showed that financial assets exhibit asymmetric tail dependence, the more myopic approach to construct GaR would be using a Gaussian GAS model. By contrast, a symmetrised Clayton copula is capable of capturing these dynamics.

The performance of GaR forecasts can be measured using different methods. One of these methods considers the coverage of GDP growth rates. When GaR is required at, for example, a confidence level of α, then an accurate model would cover about (1-α) of all growth rates. Kupiec (1995) shows that the difference between a nominal level α and its resulting empirical level $\hat{\alpha}$ can be used to construct a likelihood ratio test, by assuming a binomial relationship between VaR violations and VaR acceptances. When this unconditional test is not rejected, then the VaR estimates are deemed accurate. Christoffersen (1998) builds upon this methodology and pointed out that VaR violations can be clustered, therefore potentially yielding unreliable testing results. He proposes a conditional coverage test, which assumes a binary Markov chain as a transition mechanism between VaR violations. However, Lopez (1999) shows with a simulation exercise that the statistical power of an unconditional or conditional coverage test can be low. This implies that there exists a high probability of classifying inaccurate VaR forecasts as accurate.

Instead, Lopez (1999) proposes a second evaluation method that entails loss functions. This approach considers the magnitude of a failure in the event of an exception. In the case of observed (portfolio) returns, an exception means that the return at that time is lower than a forecasted VaR, i.e. resulting in a loss. Loss functions with different characteristics can be defined. For example, Caporin (2008) proposes regulator's loss functions that evaluate losses in case of exceptions. By contrast, Sarma et al. (2003) argue that a firm's risk manager is confronted with the challenge of being too safe (thereby requiring too much capital) or being too reckless, and proposes firm's loss functions. These loss functions evaluate the entire data sample, which implies that opportunity costs are also taken into account. Another advantage of this approach is that it has increased power over firm's loss functions. For example, a low number of exceptions can incorrectly favour a certain VaR model, as a small variation in the data can result in different outcomes. This benefit is especially useful in the case of low-frequency data, such as quarterly GDP growth rates. Finally, loss functions are capable of ranking VaR models, since the model with the smallest loss is preferred. Nolde and Ziegel (2017) formalise the ranking of scores by introducing a comparative backtest based on a Diebold-Mariano test statistic.

4 Methodology

In the following sections, GaR will be obtained through different methods. The copula-GAS methods can be summarised by four steps: marginal modelling, copula-GAS estimation, Monte-Carlo simulation and (comparative) backtesting. Backtesting comprises coverage tests and loss functions. All aspects of these steps are expounded in the following sections. In addition, the methodology to obtain GaR estimates with QR is given as well. I refer to these QR GaR estimates as a benchmark. Table 1 lists all abbreviations and acronyms used in this paper.

4.1 General autoregressive score models

In this section, I will construct a model with score driven time-varying parameters. First, I assume a conditional density for GDP growth rates y_t at time t, such that

$$p(y_t|\psi), \tag{1}$$

with parameter vector ψ. Furthermore, assume that only a subset of ψ is time-varying, which means that we can write the parameter vector ψ_t as $\psi_t = (f_t; \theta)$, where f_t is the time-varying parameter of interest, and where θ consists of the remaining fixed parameters of the model. Next, I consider the t-th log-likelihood

$$l_t = \log p(y_t|\mathcal{F}_{t-1}; \theta), \tag{2}$$

where $\mathcal{F}_{t-1}$ is the information set known at time $t-1$, consisting of realised time-varying parameters f_{t-1} and data. The main idea of a GAS model is the method of determining the time-varying parameter value f_{t+1} for the next period. A GAS model assumes an autoregressive updating function for f_t,

$$f_{t+1} = \omega + \sum_{i=0}^{p-1} a_i s_{t-i} + \sum_{j=0}^{q-1} b_j f_{t-j} + CX_t, \tag{3}$$

with ω a constant, a_i, b_j and c coefficients of the model, X_t exogenous regressors and s_t a driving mechanism or innovation term. Using the autoregressive order p and q, the updating algorithm defined in Equation 3 will be abbreviated to GAS(p, q). Following Creal et al. (2013), the innovation term s_t is equal to the scaled *score* of l_t defined in Equation 2 with respect to f_t and is defined as

$$s_t = \nabla(y_t, f_t)S(f_t) = \frac{\partial \log p(y_t|f_t; \theta)}{\partial f_t} S(f_t; \theta), \tag{4}$$

where $\nabla(y_t, f_t)$ is the score of the log-density, and $S(f_t)$ a scaling parameter. The intuition of considering the score in a time-varying setting, is to update the time-varying parameter f_t by taking the steepest ascent direction s_t. The scaling factor $S(f_t)$ is applied to improve the update by accounting for the curvature of the score. This scaling factor is defined as

$$S(f_t) = \mathcal{I}_t^{-1} = \mathrm{E}\Big(\nabla^2(y_t, f_t)\Big)^{-1} = -\mathrm{E}_t\left(\frac{\partial^2 \log p(y_t|f_t;\theta)}{\partial f_t^2}\right), \tag{5}$$

where $\mathcal{I}_t^{-1}$ is the inverse of the Fisher information matrix.

4.2 Copula models

Since the introduction of Sklar (1959), copulas have been analysed and applied extensively. One of the reasons copulas have become popular in literature is that the method allows one to separately specify the dependence structure and marginals. Additionally, copulas and their dependence structure can be estimated fully parametrically, or be estimated semiparametrically by specifying a parametric copula while marginals are specified by a nonparametric model, as shown by Genest et al. (1995) or, lastly, fully nonparametric as shown by Chen and Huang (2007). Because of these advantages, copula models have proved to be especially useful in a financial context, not only because financial data often exhibit complex dependence structures, but also because it suffers from heavy tails and skewed distributions.

Copulas can be applied to both bivariate and multivariate data. However, when estimating a multivariate copula, one implicitly assumes that the dependence structure between marginals is the same for all variables included. Moreover, especially in the case of analysing GDP growth rates of a variety of countries, this assumption may often ignore characteristics of the data. Therefore, in my analysis of GDP growth rates of different countries, I will focus on two components of GDP, namely private final consumption and a sum of investments, net exports and government spending. This methodology characterises the dependence structure between the most important macroeconomic indicators of a given country.

McNeil et al. (2015) classify copulas into different families. They showed that copulas are classified as elliptical or as non-elliptical functions. The Gaussian copula is classified as an elliptical copula and is defined as

$$\mathcal{C}_\Psi^G(u_1, \ldots, u_n) = F_\Psi\Big(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\Big), \tag{6}$$

where F_Ψ is a multivariate normal distribution $MN_n(0, \Psi)$, and with $\Phi(u_i)$ the standard univariate normal CDF over marginals u_i. Although Gaussian copulas are models that are easy to work with and to implement, the marginals are assumed to have elliptical innovations. Moreover, Gaussian copulas have zero tail dependence, which implies that it underestimates kurtosis, a phenomenon that is widely observed in macrofinancial time-series. Contemporary macroprudential policy is concerned with modelling the fat tails of Future GDP growth. Therefore Gaussian copulas are less capable to capture these characteristics. However, these models are able to capture time-varying dependence, characteristics of the data QR is unable to account for.

Non-elliptical (or Archimedean) copulas allow for non-elliptical distributions with tail dependencies. Archimedean copulas are defined as $\mathcal{C}(u_1, \ldots, u_n) = \phi^{-1}(\sum \phi(u_i))$, with ϕ^{-1} the inverse of ϕ. The function ϕ is called the generator of the copula. A specific Archimedean copula is the Clayton copula,

$$\mathcal{C}_\theta^{Cl}(u_1, \ldots, u_n) = \Big(\sum_{i=1}^{n} u_i^{-\theta} - (n+1)\Big)^{-\theta^{-1}} \qquad \theta > 0, \tag{7}$$

where $\mathcal{C}_\theta^{Cl}$ is the cumulative distribution function of the copula. The Clayton copula can be used to model tail dependency. To arrive at the density of the Clayton copula, note that for any copula $\mathcal{C}(u_1, \ldots, u_n)$ the PDF $c(u_1, \ldots, u_n)$ can be obtained as $c(u_1, \ldots, u_n) = \partial^n \mathcal{C}((u_1, \ldots, u_n)/\partial u_1, \ldots \partial u_n$. The density for the Clayton copula then is obtained as

$$c_\theta(u) = \prod_{k=0}^{d-1}(\theta k + 1)\Big(\prod_{i=1}^{n} u_i\Big)^{-(1+\theta)}\Big(\sum_{i=1}^{n} u_i^{-\theta} - n + 1\Big)^{-(n+1/\theta)}. \tag{8}$$

See Hofert et al. (2012) for comprehensive clarifications of multivariate Archimedean copulas.

4.3 Copula-GAS

Firstly, I will consider a Gaussian copula in an observation driven GAS framework. Patton (2006) proposes to write the updating equation of the bivariate Gaussian copula as

$$f_t = \omega + A \sum_{i=0}^{m-1} z_{1,t-i} z_{2,t-i} + B f_{t-1}, \tag{9}$$

where $z_{i,t}, i \in \{1, 2\}$ and f_t the time-varying parameter of interest are transformed univariate marginals using the inverse CDF, and m a lag order parameter. The link function

$$\rho_t = \frac{1 - \exp(-f_t)}{1 + \exp(-f_t)} \tag{10}$$

is a modified logistic transformation, which ensures that the correlation parameter ρ_t is defined between [-1,1]. Using the PDF of a bivariate Gaussian copula the score can be obtained by taking the gradient. The score $\nabla_t(f_t)$ of the bivariate copula can be found by taking the derivative of the Gaussian copula c^G with respect to the time-varying correlation parameter ρ_t,

$$\nabla(z_{1,t}, z_{2,t}, \rho_t) = \frac{\partial \log c^G(z_{1,t}, z_{2,t}|\rho_t)}{\partial \rho_t} = \frac{(1+\rho_t^2)(z_{1,t}-\rho_t) - \rho_t(z_{2,t}-2)}{\left(1-\rho_t^2\right)^2}. \tag{11}$$

Subsequently scaling the score with $S_t(\rho_t) = (1+\rho_t^2)/(1-\rho_t^2)^2$ (see Creal et al. (2013) for an overview of scores and scaling factors of different GAS models) yields a scaled score

$$s_t(z_{1,t}, z_{2,t}, \rho_t) = \frac{2\left(\omega_{1,t} - \rho_{t-1} - \rho_{t-1}\frac{\omega_{2,t}-2}{1+\rho_{t-1}^2}\right)}{1-\rho_{t-1}^2}, \tag{12}$$

where $\omega_{2,t} = z_{1,t}^2 + z_{2,t}^2$ and $\omega_{1,t} = z_{1,t}z_{2,t}$. The score in Equation 12 can finally be implemented in a GAS(1,1) recursion

$$f_t = \omega + as_{t-1} + bf_{t-1} + cX_{t-1}, \tag{13}$$

and the optimal time-invariant copula parameters ω, a, b and c can be estimated by maximising the scaled log likelihood.

A Gaussian copula is not capable of capturing either lower or upper tail dependence, which is frequently observed in financial time-series. Therefore, considering copulas that are able to capture either latent dynamics could be beneficial, especially in macroprudential policymaking which concerns extreme tail events. A mixture of Clayton copulas, proposed by Junker and May (2005), is such a model that can capture lower and upper tail dependence. This model mixes a normal Clayton copula (to capture lower tail dynamics) and a survival Clayton copula (to capture upper tail dynamics). The score of the Clayton copula required for the GAS framework can be obtained by taking the gradient of the logarithmic density as shown in Equation 8. The information matrix that is used for scaling the score on the other hand is harder to find analytically, therefore it is often estimated using numerical optimisation, or by simply setting the scaling matrix equal to an identity matrix.

When mixing two Clayton copulas, one is capable of modelling both lower and upper tail dependence. This results in a symmetrised Clayton copula, defined by the CDF

$$\mathcal{C}^{Cl}(u_1, u_2; \alpha) = p\mathcal{C}_1^{Cl}(u_1, u_2; \alpha_1) + (1-p)\mathcal{C}_2^{Cl}(u_1, u_2; \alpha_2), \tag{14}$$

where α_1 and α_2 are the respective dependence parameters of Clayton copulas $\mathcal{C}_1^{Cl}$ and $\mathcal{C}_2^{Cl}$. Tail dependencies are captured by taking a convex linear combination of bivariate Clayton $\mathcal{C}_1^{Cl}$ and rotated 'survival' Clayton $\mathcal{C}_2^{Cl}$, where $\mathcal{C}_1^{Cl} = (u_1^{-\alpha_1} + u_2^{-\alpha_1} - 1)^{-1/\alpha_1}$ with lower tail dependence parameter α_1, and $\mathcal{C}_2^{Cl} = u_1 + u_2 - 1 + \big((1-u_1)^{-\alpha_2} + (1-u_2)^{-\alpha_2} - 1\big)^{-1/\alpha_2}$ with upper tail dependence parameter α_2.

The GAS updating equation as given in Equation 3 has to be altered slightly to accommodate for a mixture of two copulas. In this case, the updating equation becomes two-dimensional,

$$\mathbf{f}_t = \boldsymbol{\omega} + \mathbf{A}\mathbf{s}_{t-1} + \mathbf{B}\mathbf{f}_{t-1}, \tag{15}$$

where $\mathbf{f}_t$, $\boldsymbol{\omega}$, $\mathbf{A}$ and $\mathbf{B}$ now are vectors. Similar to the Gaussian copula-GAS (GGAS) case, a link function needs to be imposed to ensure well-defined tail dependence parameters. Ayala and Blazsek (2018) use a link function $\alpha_t = \exp(f_t)$. However, I found that using the link function $\alpha_t = |f_t|$ tends to be more stable when computing maximum likelihood estimations.

Furthermore, following Creal et al. (2013), copulas c_1 and c_2 are defined without dependence parameters of the other copula. Therefore I can apply univariate Clayton copula results to construct the score of the symmetrised Clayton copula. Hofert et al. (2012) show that the score of an individual Clayton copula can be written as a function of its generator. A d-dimensional Archimedean copula is uniquely defined by its generator $\psi : [0,\infty] \to [0,1]$ by writing

$$\mathcal{C}(\mathbf{u}) = \psi\Big(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\Big), \mathbf{u} \in [0,1]^d. \tag{16}$$

By defining $Q_\alpha(\mathbf{u}) = \sum_{j=1}^{d} \psi(u_j)$ the score of the symmetrised Clayton copula defined in Equation 14 can be written as

$$\frac{\partial \log c}{\partial \alpha} = \mathbf{w} \begin{pmatrix} \frac{\partial \log c_1}{\partial \alpha_1} \\ \frac{\partial \log c_2}{\partial \alpha_2} \end{pmatrix} = \mathbf{w} \begin{pmatrix} \frac{1}{\alpha_1+1} - \log(u_1u_2) + \frac{\log\big(1+Q_{\alpha_1}(\mathbf{u})\big)}{\alpha_1^2} - (2+1/\alpha_1)\frac{Q_{\alpha_1}(\mathbf{u})}{1+Q_{\alpha_1}(\mathbf{u})} \\ \frac{1}{\alpha_2+1} - \log(u_1u_2) + \frac{\log\big(1+Q_{\alpha_2}(\mathbf{u})\big)}{\alpha_2^2} - (2+1/\alpha_2)\frac{Q_{\alpha_2}(\mathbf{u})}{1+Q_{\alpha_2}(\mathbf{u})} \end{pmatrix}, \tag{17}$$

where $c_i, \alpha_i, i \in \{1,2\}$ are the PDF and dependence parameter of the Clayton and survival Clayton copula, respectively, and $\mathbf{w}$ defined as $\mathbf{w} = \big(w_1, w_2\big)' = \big(p_1c_1/(p_1c_1 + p_2c_2), p_2c_2/(p_1c_1 + p_2c_2)\big)'$. Equation 17 shows the key difference between modelling a single copula GAS recursion (i.e. the Gaussian copula score given in Equation 12) and a mixture of copula GAS. It can be seen from Equation 17 that the scores from the Clayton and survival Clayton copula are updated independently from each other. By doing so, given the definition of w_i, observations either partly contribute to upper or lower tail dependence. This allows to model the upper and lower tail dependence of the data simultaneously. Therefore, compared to a myopic Gaussian GAS approach, a symmetrised Clayton GAS model yields a more complete view of macroeconomic dependencies in either tail of its distribution.

4.4 Maximum likelihood estimation

Since both GGAS and symmetrised Clayton copula-GAS (SCGAS) are observation driven models, maximum likelihood can be applied to obtain optimal parameter estimates. When applying maximum likelihood to a copula-GAS model, the maximisation problem can be formulated as

$$\max_{\theta} \sum_{t=1}^{n} \ell_t(y_t|f_t;\theta) = \max_{\theta} \sum_{t=1}^{n} \log\Big(c(u_1,u_2|f_{t-1};\theta)\Big), \tag{18}$$

where $\ell_t(y_t|f_{t-1};\theta)$ is the log-likelihood of observation at time t. In addition, θ and f_t are the static and time-varying parameters of the model, and c denotes the PDF of the copula of interest. Since the log-likelihood can be determined for every observed value y_t (or observed pair u_1, u_2 in a copula setting), the GAS model as given in Equation 3 defines a filter. In addition, recursions can be defined for the score and scaled score. These recursions are required to be able to update the time-varying parameter. Equation 12 shows the recursion for the scaled score of a Gaussian copula, whereas Equation 17 shows the vector of scores of the symmetrised Clayton copula.

To obtain well-behaved time-varying parameters through a GAS recursion, some restrictions need to be applied to ensure stability. First, a link function needs to be defined in order to acquire well-defined time-varying parameters (see section 4.3 for link functions applicable to GGAS or SCGAS). Secondly, by imposing the restriction $|B| < 1$ in Equation 9 and $|\mathbf{B}| < \iota_2$ in Equation 15, the updating dynamics are enforced to remain stable over time. These mild regularity conditions ensure that the maximum likelihood estimator will be consistent and exhibits asymptotic normal properties. Creal et al. (2013) analyse MLE in a copula-GAS setting. Furthermore, see Heij et al. (2004) and Blasques et al. (2014) for a comprehensive discussion regarding maximum likelihood estimation (MLE) and MLE in a GAS framework, respectively.

By maximising the log-likelihood as defined in Equation 18, one obtains maximised likelihood parameter estimates

$$\hat{\theta}_{ML} = \arg\max_{\theta} \ell_t(y_t|f_{t-1};\theta). \tag{19}$$

Using standard regularity conditions, it is possible to write

$$\sqrt{n}\big(\hat{\theta}_{ML} - \theta_0\big) \xrightarrow{d} \mathrm{N}\big(0, \mathcal{I}_0^{-1}\big), \tag{20}$$

where $\mathcal{I}_0^{-1}$ is the observed information matrix evaluated at the true parameter θ_0. The observed information can be calculated as the negative of the Hessian,

$$\mathcal{I}_n = E\left[\frac{\partial l}{\partial \theta}\frac{\partial l}{\partial \theta'}\right] = -E\left[\frac{\partial^2 l}{\partial \theta \partial \theta'}\right]. \tag{21}$$

Since the approximate distribution of the maximum likelihood estimator is known, I obtain the distribution of the estimated parameter $\hat{\theta}_{ML} \approx \mathrm{N}\Big(\theta_0, \mathcal{I}_n^{-1}(\hat{\theta}_{ML})\Big)$. This result can then be used to construct the variance of $\hat{\theta}_{ML}$,

$$\mathrm{var}\big(\hat{\theta}_{ML}\big) \approx \frac{1}{n}\mathcal{I}_0^{-1} \approx \mathcal{I}_n^{-1}\big(\theta_0\big) \approx \mathcal{I}_n^{-1}\big(\hat{\theta}_{ML}\big). \tag{22}$$

I will use Matlab R2020b built-in functions `fminunc` and `fminsearch` to maximise the likelihood.

4.5 Marginals

Financial and macroeconomic time series are generally considered to be non-normal in terms of skewness and excess kurtosis. Fang and Miller (2009) note, for example, that the evidence of kurtosis in GDP growth could point at volatility in the mean growth rate. Blanchard and Simon (2001) further contribute to this reasoning by showing that large shocks in GDP growth rates This means that, before the estimation procedure of the models presented above, the marginals of copula-GAS models should be assessed as well. A general approach to correct for non-normal behaviour, but also for a possible presence of autocorrelation and conditional heteroskedasticity, is to apply autoregressive (AR) and general autoregressive conditional heteroskedasticity (GARCH) models, further discussed by Grégoire et al. (2008). For example, Creal et al. (2013) apply AR(1)-GARCH(1,1) to filter out conditional means and variances in a GGAS framework, whereas Manner and Reznikova (2012) exercise AR(p)-GARCH(1,1) models to returns of exchange rates. I will follow a similar approach in the empirical illustration of section 5 by applying an AR(p)-GARCH(1,1) model to the marginal distributions of log GDP growth rates $y_{i,t}$,

$$\mu_{i,t} = \phi_{i,0} + \sum_{j=1}^{p} \phi_{i,j} y_{i,t-1} + \varepsilon_{i,t}; \tag{23}$$

$$\varepsilon_{i,t} = \sigma_{i,t} z_{i,t}; \tag{24}$$

$$\sigma_{i,t}^2 = \gamma_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \tag{25}$$

for country i at time t, $Z_{i,t}$ being student-t distributed with ν_i degrees of freedom, and $\mu_{i,t}$ being the conditional mean. The number of lags p in equation 23 will be selected by considering Bayesian information criteria (BIC) of the models.

This process yields standardised residuals $\varepsilon_{i,t} = (y_{i,t} - \mu_{i,t})/\sigma_{i,t}$. The final step in processing the marginals is transforming them into univariate marginals (refer back to equation 9 for the copula models). I will transform the marginals nonparametrically by applying empirical cumulative distribution functions

$$u_{i,t} = F_i(\varepsilon_{i,t}) = \frac{1}{T+1} \sum_{t=1}^{T} I_{[\varepsilon_{i,t} \leq \varepsilon]}, \tag{26}$$

where T is the number of observations and where F_i is rescaled by $T+1$ to ensure stability in the first-order condition of the log-likelihood. Finally, the transformed marginals $u_{i,t}$ can be used to estimate the GGAS and SCGAS models.

4.6 Quantile regression models

In standard regression analysis, the relationship between a dependent variable y_t and independent variables X_t is modelled through the conditional mean $E(y_t|X_t)$. A linear regression model $y_t = x_t'\beta + \varepsilon_t$, yields an estimate $\hat{\mu}(x)$ of this conditional mean given $X_t = x$ by minimising the mean squared error (MSE)

$$E(y_t|X_t) = \arg\min_z E\Big((y_t - z)^2 | X_t = x\Big), \tag{27}$$

where $z = \beta_0 + \beta_1 x_{1,t} + \cdots + b_k x_{k,t}$. However, this approach only covers a small part of the relationship between the response variable and regressors. One may be interested in, for example, a certain quantile in the estimation procedure. An evident method capable of finding a measure that reflects a level of confidence is QR. QR models the relationship using the conditional function $Q_\alpha(y|x) = F_y^{-1}(\alpha)$, where F is is the cumulative distribution function of y and α a quantile of the distribution. Contrasting minimising MSE in a standard linear regression, a QR model

$$y_t = x_t'\beta(\alpha) + \varepsilon_t(\alpha) \tag{28}$$

minimises the median absolute deviation (MAD)

$$Q(\beta_\alpha) = \arg\min_z E\Big(\rho_\alpha\big(y_t - z(\alpha)\big)|X_t = x\Big), \tag{29}$$

where $z(\alpha) = \beta_0(\alpha) + \beta_1(\alpha)x_{1,t} + \cdots + b_k(\alpha)x_{k,t}$, and ρ_α an asymmetric loss function defined as

$$\rho_\alpha(\varepsilon_t) = \varepsilon_t\big(\alpha - I(\varepsilon_t < 0)\big), \tag{30}$$

where ε_t is the error of a given observation at time t. By defining two additional variables $u_t = \varepsilon_t I(\varepsilon_t > 0)$ and $v_t = |\varepsilon_t|I(\varepsilon_t < 0)$ it is possible to rewrite equation 28 to $y_t = x_t'\beta(\alpha) + (u_t - v_t)$. Then minimising MAD as defined in 29 can be reformulated as

$$Q(\beta_\alpha) = \alpha \sum_{t:y_t \geq x_t'\beta(\alpha)} |y_t - z(\alpha)| + (1-\alpha) \sum_{t:y_t < x_t'\beta(\alpha)} |y_t - z(\alpha)|, \tag{31}$$

subject to the constraints $Y - X'\beta = u - v$ and $u_t > 0, v_t > 0$, where Y is a $T \times 1$ vector of endogenous outcomes and X a $T \times k$ matrix of exogenous variables. Equation 31 can be seen as a linear programming problem, and can further be rewritten to

$$\min_{(u,v,\beta)} \begin{pmatrix} \alpha\iota_T \\ (1-\alpha)\iota_T \\ 0\iota_k \end{pmatrix}' \begin{pmatrix} u \\ v \\ \beta \end{pmatrix} \tag{32}$$

$$\text{such that} \tag{33}$$

$$\begin{pmatrix} I_T & -I_T & X_{t,k} \\ -I_T & I_T & -X_{t,k} \\ -I_T & 0 & 0 \\ 0 & -I_T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \beta \end{pmatrix} \leq \begin{pmatrix} Y \\ Y \\ 0\iota_T \\ 0\iota_T \end{pmatrix}, \tag{34}$$

with ι_T a $T \times 1$ vector of ones and I_T a $T \times T$ identity matrix. This linear programming problem can be solved using algorithms such as the Frisch-Newton interior point method, and dual simplex approach, introduced by Koenker and D'Orey (1987). I will implement the latter more straightforward technique, since it ensures an optimal solution within a finite number of iterations.

QR has a number of appealing advantages over the more 'traditional' ordinary least squares (OLS). One evident benefit of QR is the ability to model a quantile, thereby having the liberty of analysing different levels of confidence. In addition, it can be shown that monotonic transformations do not influence the outcome of QR. Finally, no assumptions are made about the distributions of the parameters. QR implements a loss function as defined in 30. This implies that QR is more robust to outliers and non-normal errors, since the mean in OLS assigns equal $1/T$ weights to all observations. The above advantages have stimulated policymakers of regional and central banks to implement QR in their analysis (see for example Ghosh et al. (2012) who analyse the accumulation of reserves of emerging market economies; and White et al. (2015) who determine VaR of international financial institutions, all conducted in a QR framework).

4.7 Exogenous variables

As potentially many macroeconomic indicators could explain GDP growth, a subset of those variables could co-move over time, and therefore suffer from multicollinearity. Prasad et al. (2019) propose partitions of macroeconomic variables to overcome this issue, and distinguish financial conditions and macrofinancial vulnerabilities as potential partitions. Using principal component analysis (PCA), these partitions (or factors) can be extracted. I will apply PCA to the explanatory variables listed in the Data section, and add the components to the GAS model as shown in Equation 3.

4.8 Monte Carlo simulation

GaR will be analysed using a backtesting procedure. Backtesting is widely applied in financial risk management, because the procedure enables a risk manager to determine how well a certain model estimates VaR. Since VaR is an analogue of GaR, backtesting methods can also be applied to measure the accuracy of GaR predictions. For every country, I will analyse log growth rates of private final consumption and the residual term by applying an AR model to the conditional mean and a GARCH model to the conditional variance, which yields residuals

$$\varepsilon_{i,t} = \frac{J_{i,t} - \mu_{t,i}}{\sigma_{i,t}}, \tag{35}$$

where $J_{i,t}$ is either the log growth rate of private final consumption ($i = 1$) or the residual term ($i = 2$) in quarter t, and $\mu_{i,t}$ and $\sigma_{i,t}$ the conditional mean and variance in quarter t, respectively.

Assume that the time-varying dependence parameters of both Gaussian and mixed Clayton models, as specified in Section 4.3 are estimated at this stage. This dependency structure will then be used in a Monte Carlo simulation for GaR. The GaR backtesting procedure can be formulated as follows:

1. By specifying AR-GARCH models for the marginal distributions, assume that conditional means $\hat{\mu}_{i,t}$ and conditional variances $\hat{\sigma}^2_{i,t}, i = 1, 2$ are known at this stage.

2. Assume time-varying dependence parameter $\hat{f}_t$ is obtained by maximising the log-likelihood of either the Gaussian $\mathcal{C}^G$ or Clayton $\mathcal{C}^{Cl}$.

3. Perform $Q = 1000$ Monte Carlo simulations and simulate marginals using $\hat{f}_t$ obtained from the fitted copula. This yields paths of marginal samples $U = \left(u^{(1)}_{i,t}, \ldots, u^{(Q)}_{i,t}\right)', i = 1, 2.$

4. Transform the simulated marginal samples to GDP innovation levels by applying the inverse Student-t distribution to simulated marginal samples, $F_i^{-1}\left(u^{(q)}_{i,t}\right) = \epsilon^{(q)}_{i,t}, i = 1, 2$ and $q = 1, \ldots, 1000$. The degrees of freedom are obtained in step 1.

5. Construct simulated consumption or residual growth rates $i = \{1, 2\}$ and simulated path q by computing $\hat{J}^{(q)}_{i,t} = \hat{\mu}_{i,t} + \hat{\sigma}_{i,t}\epsilon^{(q)}_{i,t}$, and construct simulated GDP growth $\hat{y}^{(q)}_t$ by calculating a weighted average of simulated consumption and residual growth, $\hat{y}^{(q)}_t = w_{1,t}\hat{J}^{(q)}_{1,t} + w_{2,t}\hat{J}^{(q)}_{2,t}$. The weights $w_{i,t}$ are obtained by dividing the size of consumption or the residual term with the size of GDP in quarter t.

6. Sort all paths q for every quarter t in descending order to obtain a historical simulation estimate of GaR_t. If one wishes a GaR at, say, a confidence level of 5%, then the 50th lowest simulated estimate should be selected (assuming 1000 simulated GDP growth rate paths).

7. Repeat steps 1 to 6 for all countries.

4.9 Coverage tests

GDP growth rates will be compared to either marginal or joint GaR at time t, which results in empirical coverage

$$\hat{\pi}(p) = \frac{1}{T}\sum_{t=1}^{T} I_{\{y_t < GaR^p_t\}}, \tag{36}$$

which measures exceedances of GDP growth rates below GaR at a confidence level of p, and where T is the total amount of quarters considered in the sample. Note that the empirical coverage is implicitly dependent on the nominal coverage p. This means that accurate GaR obtained from the Monte Carlo simulation specified in Section 4.8 should therefore be similar.

The difference between empirical coverage and nominal coverage can be tested using likelihood ratio (LR) tests. Kupiec (1995) proposes the unconditional coverage test. He showed that relatively large samples are required to produce reliable VaR estimates with, at that time, standard statistical techniques. Instead, the paper proposed the unconditional coverage test, which tests the difference in the expected number of VaR violations and observed VaR violations. It is defined as an LR-test, with the null hypothesis of the expected number of VaR violations and observed violations. The likelihood of the series of violations $\{I_1, I_2, \ldots, I_T\}$ under the null hypothesis can be expressed as

$$L(p; I_1, I_2, \ldots, I_T) = (1-p)^{n_0} p^{n_1}, \tag{37}$$

with p denoting the 'true' nominal coverage, $n_1 = \sum_{t=1}^{T} I_{\{y_t > GaR_t\}}$ and $n_0 = T - n_1$. The likelihood under the alternative hypothesis of different number of expected and observed violations is

$$L(\pi; I_1, I_2, \ldots, I_T) = (1-\pi)^{n_0} \pi^{n_1}. \tag{38}$$

Using 37 and 38, the likelihood ratio test is given as

$$LR_u = -2\log\Big(L(p; I_1, I-2, \ldots, I_T)/L(\hat{\pi}; I_1, I_2, \ldots, I_T)\Big) \tag{39}$$

$$= -2\log\Big((1-p)^{n_0} p^{n_1}\Big) + 2\log\Big((1-\hat{\pi})^{n_0} \hat{\pi}^{n_1}\Big). \tag{40}$$

Under the null hypothesis, this test is asymptotical $\chi^2(1)$ distributed. The advantage of this approach is that one can test (and therefore potentially reject) a model based on VaR predictions that are both higher and lower than the expected number of violations. However, Kupiec's test is not able to distinguish the time-variation of the data, which means that it only can effectively be applied to data with 'conditional' VaR violations only.

Instead, Christoffersen (1998) proposes the conditional coverage test. He notes that the majority of forecasting literature focuses on the evaluation of point forecasts and, more importantly, that researchers often make the assumption of homoskedastic error terms. Therefore this test is an improvement over the unconditional coverage test, since it also addresses predicted VaR violations that are clustered in the data. The conditional coverage test is a combination of the unconditional coverage test and an independence test. To derive the test of independence, consider $\{I_1, I_2, \ldots, I_T\}$ as a binary Markov chain. y defining the probability $\pi_{ij} = P(I_t = j|I_{t-1} - i)$, a first-order transition matrix can be written as

$$\Pi_1 = \begin{bmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{bmatrix}. \tag{41}$$

Denote n_{ij} as the number of observations with consecutive observations of VaR violations, with $I_{t-1} = i$ and $I_t = j$. This process yields an approximate likelihood of

$$L(\Pi_1; I_1, I_2, \ldots, I_T) = (1 - \pi_{01})^{n_{00}} \pi_{01}^{n_{01}} (1 - \pi_{11})^{n_{10}} \pi_{11}^{n_{11}}. \tag{42}$$

The independence test that follows tests whether there is clustering of VaR violations in the data, and could potentially be conducted separately from the unconditional and conditional tests. The last step is to combine the unconditional coverage test and an independence test. Christoffersen (1998) shows that the LR-test of conditional coverage is

$$\begin{aligned} LR_c &= -2 \log \Big(L(p; I_1, I - 2, \ldots, I_T) / L(\hat{\Pi}_1; I_1, I_2, \ldots, I_T) \Big) && (43) \\ &= -2 \log \Big((1-p)^{n_0} p^{n_1} \Big) + 2 \log \Big((1 - \hat{\pi}_{01})^{n_{00}} \hat{\pi}_{01}^{n_{01}} (1 - \hat{\pi}_{11})^{n_{10}} \hat{\pi}_{11}^{n_{11}} \Big) && (44) \\ &= LR_u + LR_{ind}, && (45) \end{aligned}$$

where p denotes the true coverage, and $\hat{\pi}_{ij}, i, j \in \{0, 1\}$ denotes the estimated transition probability. Although the literature expounded above considers VaR, the techniques can of course also be applied in a GaR framework. Moreover, the unconditional and conditional coverage tests can be applied to the benchmark QR model, as well as to both copula-GAS models assessed in this paper. Therefore I will provide both LR_u and LR_c test statistics in the empirical illustration, to show the predictive accuracy of the models.

4.10 Loss functions and comparative backtesting

Lopez (1999) argues that evaluating VaR estimates based on specific information, i.e. (un)conditional coverage tests, does not provide enough information about its accuracy, and fail to distinguish VaR estimates from different models. Instead, Lopez (1999) proposes regulatory loss functions to discern VaR estimates. Using loss functions, one is able to evaluate the magnitude of experienced losses. A general loss function can be defined as

$$L_{m,t} = \begin{cases} f\Big(y_t, GaR_{m,t}(\alpha)\Big) & \text{if } y_t < GaR_{m,t}; \\ g\Big(y_t, GaR_{m,t}(\alpha)\Big) & \text{if } y_t \geq GaR_{m,t}, \end{cases} \tag{46}$$

where $L_{m,t}$ is the loss when using GaR from model m at time t, y_t a GDP growth rate at time t and α the usual coverage rate. Furthermore, a distinction can be made between loss functions of firms and policymakers, since their objective of capital requirement usually differs. Abad et al. (2015) categorises firm's loss functions (FLF) and regulator's loss functions (RLF). FLF incorporate opportunity costs, whereas RLF take into account the magnitude of (noncovered) losses. For example, Lopez (1999) uses a quadratic specification of f in equation 46 to penalise large exceedances. In addition, the restriction $f\big(y_t, GaR_{m,t}(\alpha)\big) \geq g\big(y_t, GaR_{m,t}(\alpha)\big)$ is imposed as well. Historically, the function g is set equal to 0, which implies $L_{m,t}$ only accounts for exceedances. Next, a score from model m can be constructed by adding all losses over the sample period,

$$S_m = \sum_{t=1}^{T} L_{m,t} \tag{47}$$

Caporin (2008) proposes multiple definitions for the function f that take into account failure of VaR estimations, but also the magnitude of the loss, namely

$$\begin{aligned} f_1\Big(y_t, GaR_{m,t}(\alpha)\Big) &= \left|1 - \left|\frac{y_t}{GaR_{m,t}(\alpha)}\right|\right|, \\ f_2\Big(y_t, GaR_{m,t}(\alpha)\Big) &= \frac{\Big(|y_t| - |GaR_{m,t}(\alpha)|\Big)^2}{|GaR_{m,t}(\alpha)|}. \end{aligned} \tag{48}$$

It should be noted that the definitions of the loss functions given above are not strictly consistent. Consistent loss functions should be GPL (see Thomson (1979) and Saerens (2000)). However, I am especially interested in the relative size (or magnitude) of both covered and uncovered losses. This attribute can be modelled by dividing standard loss functions by GaR. GPL loss functions are not able to incorporate this factor. Whereas Nieto and Ruiz (2016) points out that RLF often fail to be GPL, they refrain from providing a consistent loss function that incorporates relative magnitude of losses. To the best of my knowledge, no other literature discussed such loss functions. In addition, there exists a strand of literature (e.g. Abad and Benito (2013) and Abad et al. (2015)) that does not consider the consistency of their evaluated loss functions. Therefore I consider loss functions defined in equations 48 only as a secondary standard to the conditional and unconditional tests, since these tests might not provide a definitive answer to the main objective of this paper, i.e. determine the best GaR predicting model.

Caporin (2008) argues that the entire sample should be taken into account, instead of only considering exceedances, in order to obtain reliable results. This can be achieved by setting $f_j = g_j$, where $j = 1, 2$, which implies that such loss functions can be both seen as FLF and RLF. It incorporates the relative size of the uncovered losses by dividing by GaR, as well as quantifying opportunity costs by additionally penalising covered losses. I will construct scores using loss functions defined in equations 48.

Scores of different models should be tested if one wishes to determine which model performs best. GaR obtained with QR I consider as benchmark predictions. GaR obtained with copula-GAS methods challenge the predictive performance of the benchmark model. In this backtesting procedure, I am interested in two null hypotheses,

$$\begin{aligned} H_0^- &: \text{ The competing model performs at least as well as the benchmark model,} \\ H_0^+ &: \text{ The competing model performs at most as well as the benchmark model.} \end{aligned}$$

Nolde and Ziegel (2017) define the test statistic

$$\bar{S} = \frac{1}{T} \sum_{t=1}^{T} \Big(S_k - S^* \Big), \tag{49}$$

where S_k is the total score obtained from model k, and where S^* is the total score obtained from the benchmark model. Under H_0^-, the test statistic $\bar{S}$ should provide a value lower than 0. Conversely, $\bar{S}$ should be non-negative under H_0^+. Next, $\bar{S}$ can be standardised by calculating

$$\frac{\bar{S} - \mathbf{E}(\bar{S})}{\hat{\sigma}_n / \sqrt{n}}, \tag{50}$$

which is asymptotically standard normal and where $\hat{\sigma}_n$ is the HAC estimator of the asymptotic variance. Using the test statistic $\hat{\theta} = \bar{S}/(\hat{\sigma}_n/\sqrt{n})$, the null hypotheses H_0^- and H_0^+ can be tested. This approach is colloquially called a Diebold-Mariano test (see Diebold and Mariano (1995)). The null hypothesis H_0^- is rejected when $1 - \Phi(\hat{\theta}) \leq \alpha$, and the null hypothesis H_0^+ is rejected when $\Phi(\hat{\theta}) \leq \alpha$, for a certain level α. Three regions are defined in Fissler et al. (2015) to jointly assess the significance of the test statistics. If H_0^- is rejected, then the competing model is situated in the red region. If H_0^+ is rejected, then the competing model is situated in the green region. Finally, it may be possible that both null hypotheses are not rejected. Then further investigation of the competing model is required. This aspect, however, falls beyond the scope of this paper.

5 Data and macroeconomic indicators

In the next sections, I will turn to an empirical application using the methodology defined in the previous part. Six OECD countries are selected for this purpose: Australia, Germany, Great Britain, Japan, The Netherlands and The United States. The GaR of these countries will be analysed by modelling their quarterly GDP growth from 1970-Q1 to 2019-Q4, for a total of 200 observations. In addition to the above variable of interest, several exogenous variables are selected for predictive purposes. These exogenous variables include 10-year government bond yields, consumer price index (CPI) growth compared to a year ago, housing price growth, credit to the private non-financial sector growth, industrial production (IP) growth and credit gap. All growths are compared to a year ago. Some indicators are not available from 1970-Q1 (for example, 10-year bond yields of Japan start from 1989-Q1. Missing values are therefore replaced appropriately, by standardising the global equivalent of the missing indicator series, and then rescaling that series with the standard deviation of the incomplete series.

Figure 1 presents GDP growth rates of Australia, Germany, Japan, The Netherlands, United Kingdom and the United States. In addition, Table 2 presents descriptive statistics for all countries. It can be seen that on average GDP growth has been positive. Moreover, Germany, Japan and The Netherlands show the worst GDP decline of all considered countries of around -5% (occurring in the Financial Crisis of 2008). Volatility clustering can be observed until 1990. Finally, a Jarque-Bera test suggests that the GDP growth distributions of all countries deviate from normality, showing excess kurtosis for all countries and negative skewness (with the exception of Australia and the United Kingdom).

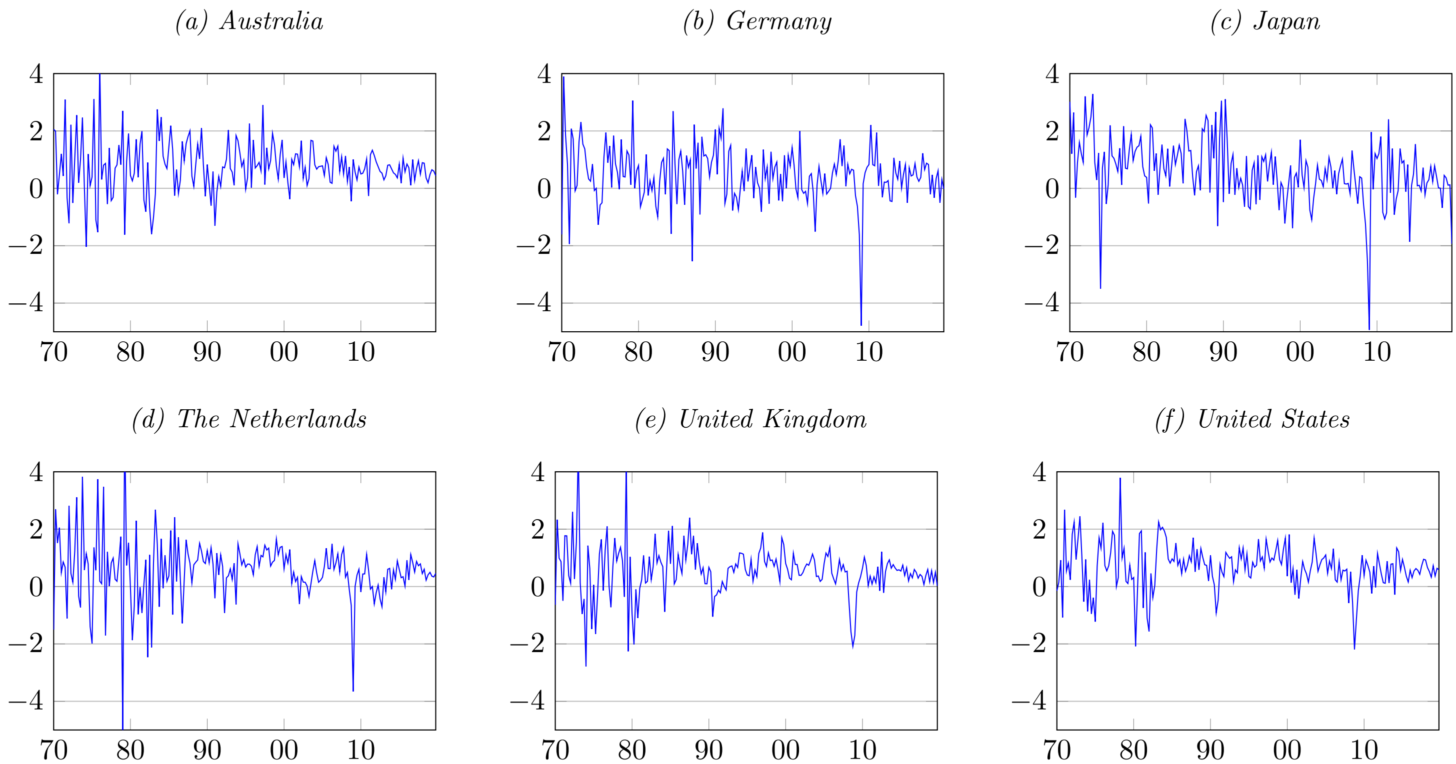

Figure 1: GDP growth rates of six OECD countries, from 1970Q1 to 2019Q4, in percentages.

Potentially many macroeconomic and macrofinancial indicators can explain GDP growth rates. At the same time, however, those indicators could also be dependent on each other, as they might reflect the same trends observed in macroeconomic data. Prasad et al. (2019) show that it is therefore useful to partition macroeconomic and macrofinancial variables into broad sets of similar variables. In the case of the empirical illustration of this paper, partitioning is achieved by applying PCA to the correlation matrix of the six exogenous variables mentioned above. I add the three most informative principal components (z_1, z_2, z_3) to the GAS recursion. This ensures that about 80% or more of the variance explained is included in the model. Table 3 shows the variance explained per country.

Country	Mean	Std. dev.	Min.	Max.	Skewness	Kurtosis	JB*
Australia	0.759	0.891	-2.038	4.328	0.192	4.753	0.001
Germany	0.484	0.982	-4.787	3.899	-0.616	7.291	0.000
Japan	0.579	1.108	-4.935	3.290	-0.708	6.336	0.000
Netherlands	0.561	1.113	-5.030	5.661	-0.337	9.072	0.000
United Kingdom	0.548	0.903	-2.786	4.841	0.151	7.790	0.000
United States	0.680	0.787	-2.188	3.791	-0.288	5.354	0.000

Table 2: Descriptive statistics of GDP growth rates of 6 OECD countries. (∗); p-values of Jarque-Bera test statistic. Shown values of the mean, standard deviation minimum and maximum are in percentages.

Table 3: Variance explained (in %) of the first three principal components obtained with PCA.

PC	AUS	GER	JPN	NLD	GBR	USA
z_1	39.588	39.603	42.810	40.740	40.808	37.142
z_2	24.632	23.156	20.657	21.587	28.268	28.994
z_3	15.043	19.491	18.124	16.305	16.600	14.496
Total explained	79.263	82.249	81.591	78.631	85.676	80.633

6 Results

6.1 Quantile regression

In this section, I present GaR estimates obtained from QR. Since QR is the main tool for macro-prudential policymakers to assess downside risk, I will consider it a benchmark for the copula-GAS models. Figure 2 shows GaR estimates at 90% coverage (in brown), at 95% coverage (in blue), at 99% coverage (in red) and GDP growth rates in black of the United States. GaR estimates obtained from the remaining countries are shown in the Appendix. For the United States GaR estimates it can be seen that the GaR at different coverage levels is somewhat alike. Around 1987 and 2005 the biggest differences in GaR can be observed. Empirical exceedances are given in Table 4 for all countries. The results suggest that on average a QR approach underestimates all considered nominal coverages. For example, Germany and Japan exhibit empirical coverages that fall below their respective nominal values. Section 6.5 will provide statistical coverage tests of Kupiec (1995) and Christoffersen (1998) to determine the significance of the observed empirical coverage.

Table 4: Empirical exceedances of GaR estimates obtained with QR, conducted at a nominal coverage of 90%, 95% and 99%.

	Nominal coverage		
Country	90%	95%	99%
Australia	0.910	0.950	0.995
Germany	0.870	0.920	0.980
Japan	0.860	0.915	0.960
Netherlands	0.915	0.955	0.995
United Kingdom	0.905	0.955	0.980
United States	0.900	0.950	0.985
Average	0.893	0.941	0.983

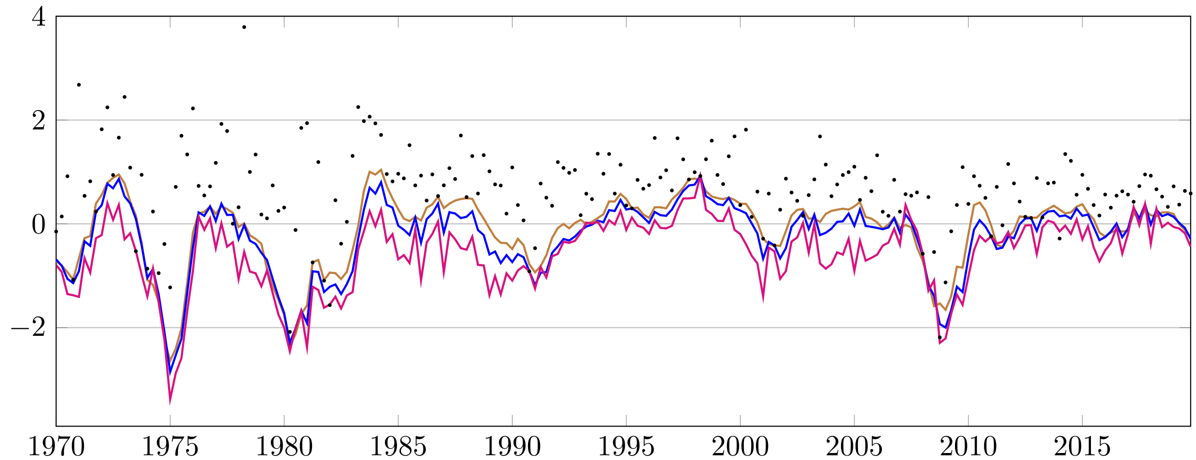

Figure 2: Growth-at-risk (brown 10% coverage, blue 5% coverage, red 1% coverage) obtained with QR, using long term bond yields, CPI growth, house pricing growth, credit growth, IP and credit gap data as indicators for GDP growth.

6.2 Marginals

The first stage of estimating copula-GAS models is modelling the marginal distributions of GDP growth rates. An AR(p)-GARCH(1,1) model is used for this purpose, where lag p is selected based on the lowest BIC. The results are shown in Table 5. AR(p)-GARCH(1,1) results estimated on PC growth rates and the residual term growth rates are given in the Appendix. It can be seen that the optimal AR lag for Australia and Germany is 1, whereas GDP growth rates of Japan, Netherlands, United Kingdom and the United States exhibit an optimum of 2 lags.

Table 6 shows descriptive statistics for the standardised residuals of the AR(p)-GARCH(1,1) model. It contains p-values for Ljung-Box Q tests, at 1 and 4 lags, conducted on the standardised

Table 5: AR(p)-GARCH(1,1) estimation results for quarterly GDP growth rates (1970-Q1 to 2019-Q4) of Australia, Germany, Japan, Netherlands, United Kingdom and United States. Standard errors are given in parenthesis.

	AR(p)			GARCH(1,1)				
Country	ϕ_0	ϕ_1	ϕ_2	γ	α	β	d.o.f. ν	BIC
Australia	0.689	0.017	-	0.000	0.055	0.934	18.453	480.98
	(0.064)	(0.068)		(0.004)	(0.031)	(0.028)	(25.304)	
Germany	0.442	0.052	-	0.217	0.204	0.564	6.240	560.56
	(0.069)	(0.082)		(0.136)	(0.098)	(0.200)	(2.768)	
Japan	0.352	0.181	0.179	0.151	0.213	0.663	7.183	599.78
	(0.090)	(0.077)	(0.081)	(0.101)	(0.100)	(0.137)	(4.028)	
Netherlands	0.357	0.142	0.243	0.009	0.133	0.862	3.870	523.13
	(0.069)	(0.070)	(0.068)	(0.011)	(0.058)	(0.047)	(1.005)	
United Kingdom	0.282	0.295	0.192	0.022	0.292	0.708	4.132	426.50
	(0.059)	(0.074)	(0.070)	(0.016)	(0.123)	(0.078)	(1.272)	
United States	0.391	0.232	0.209	0.016	0.177	0.804	5.121	426.44
	(0.071)	(0.073)	(0.068)	(0.016)	(0.075)	(0.072)	(2.011)	

residuals (LB Q) and squared standardised residuals (LB Q^2) of the marginal models. The null hypothesis for the Ljung-Box Q test on standardised residuals claims there exists no autocorrelation for a given lag. The null hypothesis for the Ljung-Box Q test on squared standardised residuals claims there exists no residual time-varying volatility for a given lag. Based on the figures in Table 6, there is not enough evidence to reject the null hypothesis of homoskedastic standardised residuals for all countries at 5% significance, at 1 and 4 lags (except Australia at 1 lag). In addition, for all countries except the Netherlands, the null hypothesis of no autocorrelation cannot be rejected as well at 5% significance, both at 1 and 4 lags. These results resemble findings of e.g Bhar and Hamori (2003) and Brownlees and Souza (2021). This suggests that the marginal AR(p)-GARCH(1,1) models entertained in this methodology adequately capture GARCH dynamics in the GDP growth rates.

6.3 Gaussian copula-GAS

This section presents Gaussian copula-GAS (GGAS) estimation results. These results are obtained by estimating the GGAS model on the uniformly transformed residuals from the AR(p)-GARCH(1,1) model. The second step of constructing quarterly GaR estimates is obtaining the maximum likelihood estimates of the GGAS method. I include the three most important principal components to the GAS recursion. Table 7 presents GGAS estimation results for all countries. It

Table 6: Descriptive statistics for the standardised residuals obtained with the AR(p)-GARCH(1,1) model. Provided are p-values for the Ljung-Box test conducted on residuals and squared residuals, at 1 and 4 lags.

Country	LB Q(1)	LB Q(4)	LB $Q^2(1)$	LB $Q^2(4)$
Australia	0.422	0.328	0.039	0.067
Germany	0.785	0.080	0.910	0.631
Japan	0.746	0.369	0.486	0.680
Netherlands	0.032	0.044	0.118	0.634
United Kingdom	0.624	0.911	0.572	0.757
United States	0.845	0.633	0.766	0.399

can be seen that the GAS specification yields a highly persistent time-varying correlation process (based on the estimates of the parameter β). These results are similar to the findings of Creal et al. (2013), which apply a GGAS model to exchange rates. In addition, the estimated parameters of the three principal components, as shown by the estimates of c_1, c_2 and c_3, are sometimes insignificant. Since the estimated parameters of α and β are significant and consistent across all countries, the parameters of principal components may indicate that external macroeconomic factors play a less prominent role in capturing time-varying dependence dynamics.

Table 7: Gaussian copula-GAS estimation results for quarterly GDP growth rates (1970-Q1 to 2019-Q4) of Australia, Germany, Japan, Netherlands, United Kingdom and United States. Standard errors are given in parenthesis.

Country	ω	α	β	c_1	c_2	c_3
Australia	-0.866	0.039	0.955	0.047	0.012	0.003
	(0.227)	(0.019)	(0.020)	(0.015)	(0.028)	(0.032)
Germany	1.996	0.110	0.982	-0.020	0.055	0.044
	(0.165)	(0.012)	(0.009)	(0.017)	(0.027)	(0.023)
Japan	2.974	0.116	0.982	-0.002	-0.007	0.008
	(0.254)	(0.022)	(0.006)	(0.020)	(0.029)	(0.033)
Netherlands	-0.434	0.062	0.745	0.025	-0.090	0.357
	(0.176)	(0.029)	(0.064)	(0.042)	(0.044)	(0.093)
United Kingdom	1.999	0.120	0.982	0.077	0.096	0.100
	(0.621)	(0.020)	(0.005)	(0.019)	(0.026)	(0.050)
United States	4.331	0.075	0.982	0.245	0.249	0.174
	(0.190)	(0.004)	(0.006)	(0.020)	(0.023)	(0.016)

The time-varying correlation process for US private consumption and residual term growth rates is shown in Figure 3. It can be observed that the correlation between private consumption and the residual term varies between -1 and 1 in the sample period. From 1970 to around 1985 the correlation process showed a relatively high amount of disturbance, which can be attributed to the macroeconomic uncertainty at that time (see also Figure 1f which shows highly varying GDP growth rates in the US). Overall, it can be observed that, whenever the US economy is in distress, the GDP components are highly uncorrelated. Conversely, periods accompanied with economic growth exhibit highly correlated GDP components. The Appendix contains time-varying correlation processes for the remaining countries.

The third and final step is constructing time-varying GaR estimates. I use a Monte Carlo simulation approach, explained in Section 4.8 to obtain GaR estimates. I employ the time-varying correlation process in order to construct simulated paths of private consumption and the residual term. A historical estimate subsequently yields the GaR estimate at a required coverage level. Figure 4 presents quarterly GaR estimates for the US between 1970-Q1 and 2019-Q4, with a nominal coverage of 10% (in brown), 5% (in blue), and 1% (in red). I refer the reader to the Appendix for GaR figures associated with the remaining countries.

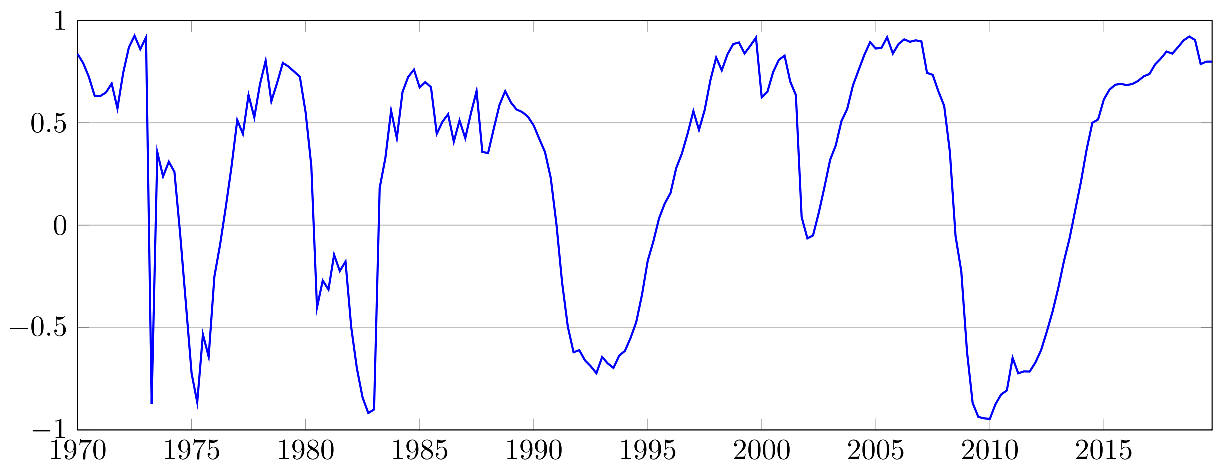

Figure 3: Time-varying correlation of private final consumption and the residual term of the United States. Shown between 19070-Q1 and 2019-Q4. Obtained with the Gaussian copula-GAS method.

The GaR estimates provide empirical coverage figures, as presented in Table 8. The table additionally shows weighted average empirical coverages. These figures suggest that a GGAS model provides higher empirical coverages than their respective nominal values (see for example the results of Japan), which is also reflected in the average empirical coverages. This result suggests that GGAS overestimates GaR, as the model does not capture enough exceedances. The results of Table

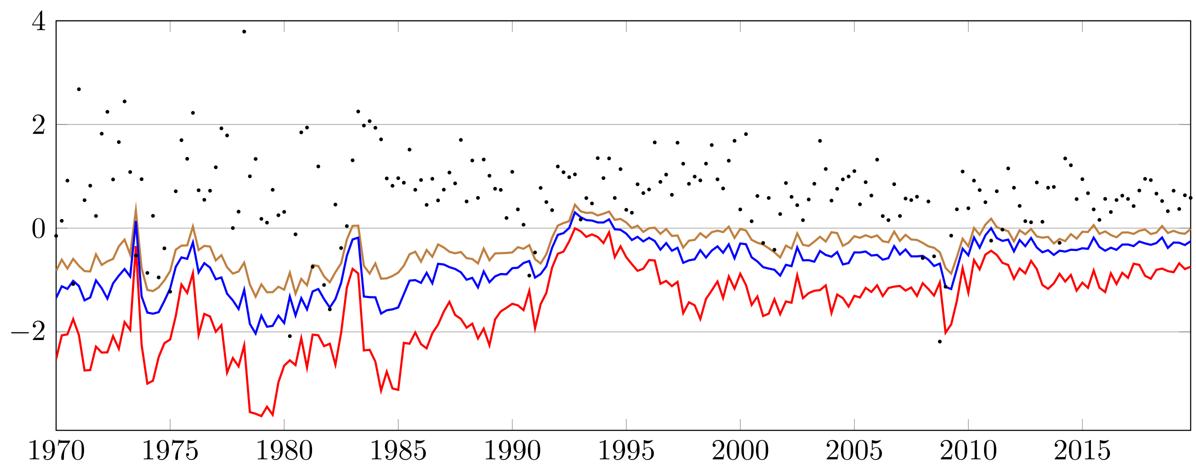

Figure 4: Growth-at-risk (brown 10% coverage, blue 5% coverage, red 1% coverage) for US GDP growth rates from 1970-Q1 to 2019-Q4, shown in black points. Obtained with the Gaussian copula-GAS approach.

8 contrasts the results of QR (shown in Table 4), which show an underestimation of GaR.

Table 8: Empirical exceedances of GaR estimates obtained with a Gaussian copula-GAS approach. Conducted at a nominal coverage of 90%, 95% and 99%.

	Nominal coverage		
Country	90%	95%	99%
Australia	0.895	0.935	0.975
Germany	0.945	0.970	0.990
Japan	0.950	0.975	0.995
Netherlands	0.925	0.965	0.985
United Kingdom	0.930	0.945	0.995
United States	0.920	0.950	0.990
Average	0.928	0.957	0.988

6.4 Symmetrised Clayton copula-GAS

This section presents estimation results of the SCGAS model. The second step in the estimation procedure (that is, after obtaining AR(p)-GARCH(1,1) estimates) is to obtain maximum likelihood estimates of the parameters of the model. Analogous to the first step of estimating the GGAS model, I will be using the bivariate GDP components (namely its residual term and private final consumption) for this purpose. These components have been demeaned and devolatised using an AR(p)-GARCH(1,1) model, as specified in Section 4.5. Table 9 shows estimation results of the SCGAS model applied to Australia, Germany, Japan, Netherlands, United Kingdom and the United

States, with standard errors given in parenthesis. It shows both maximum likelihood estimators of the Clayton copula $\mathcal{C}_1^{Cl}$ (capturing lower tail dependence) and rotated Clayton copula $\mathcal{C}_1^{Cl}$ (capturing upper tail dependence). The principal components influence the GAS recursion directly, which means that the maximum likelihood estimators of c_1, c_2, c_3 are estimated once. The last column shows the maximum likelihood estimation of p, the fraction in the mixed Clayton copula

$$\mathcal{C}^{Cl} = p\mathcal{C}_1^{Cl} + (1-p)\mathcal{C}_2^{Cl}. \tag{1}$$

Table 9: SCGAS estimation results for quarterly GDP growth rates (1970-Q1 to 2019-Q4) of Australia, Germany, Japan, Netherlands, United Kingdom and United States. $\mathcal{C}_1^{Cl}$ is the Clayton copula capturing lower tail dependence, $\mathcal{C}_2^{Cl}$ is the rotated Clayton copula capturing upper tail dependence. Standard errors are given in parenthesis.

Country	Copula	ω	α	β	c_1	c_2	c_3	p
Australia	$\mathcal{C}_1^{Cl}$	0.339	0.614	0.412				
		(0.064)	(0.082)	(0.031)	0.249	0.080	0.295	0.762
	$\mathcal{C}_2^{Cl}$	0.210	1.492	0.743	(0.039)	(0.020)	(0.020)	(0.033)
		(0.039)	(0.425)	(0.021)				
Germany	$\mathcal{C}_1^{Cl}$	1.754	1.313	0.569				
		(0.114)	(0.248)	(0.060)	0.020	0.312	0.171	0.462
	$\mathcal{C}_2^{Cl}$	1.074	0.144	0.608	(0.007)	(0.001)	(0.010)	(0.063)
		(0.012)	(0.002)	(0.006)				
Japan	$\mathcal{C}_1^{Cl}$	0.207	0.283	0.713				
		(0.048)	(0.053)	(0.008)	0.336	0.146	0.159	0.792
	$\mathcal{C}_2^{Cl}$	0.598	1.801	0.388	(0.012)	(0.011)	(0.027)	(0.039)
		(0.088)	(0.349)	(0.037)				
Netherlands	$\mathcal{C}_1^{Cl}$	0.918	2.332	0.863				
		(0.287)	(0.488)	(0.043)	0.056	0.233	0.381	0.143
	$\mathcal{C}_2^{Cl}$	1.030	0.101	0.531	(0.021)	(0.008)	(0.006)	(0.022)
		(0.030)	(0.013)	(0.015)				
United Kingdom	$\mathcal{C}_1^{Cl}$	0.367	1.234	0.484				
		(0.067)	(0.070)	(0.023)	0.064	0.195	0.035	0.517
	$\mathcal{C}_2^{Cl}$	0.244	1.227	0.605	(0.019)	(0.010)	(0.009)	(0.042)
		(0.044)	(0.186)	(0.035)				
United States	$\mathcal{C}_1^{Cl}$	0.210	1.170	0.535				
		(0.052)	(0.096)	(0.011)	0.264	0.121	0.242	0.421
	$\mathcal{C}_2^{Cl}$	1.177	1.016	0.377	(0.004)	(0.009)	(0.031)	(0.027)
		(0.024)	(0.094)	(0.019)				

Compare the results of Table 9 with the results of Table 7. Note the persistent correlation process across all countries, as shown by the estimated parameters of β in Table 7 obtained with the GGAS method. These results contrast the estimation results of β as given in Table 9, which are lower for all countries. Both lower and upper tail dependence are thus less persistent than the correlation process captured by the GGAS method. In case of the United States, the maximum likelihood estimates of the SCGAS approach yield time-varying lower and upper tail dependence between private consumption and the residual term, as presented in Figure 5. It can be seen that the dependence measures capture significant US macroeconomic events, and reveal economic stability starting from around 1985, colloquially known as the Great Moderation. In addition, a very sharp increase in both upper and lower tail dependence can be seen around the Financial Crisis of 2008. For all other countries, I refer to the Appendix for figures containing their respective dependencies.

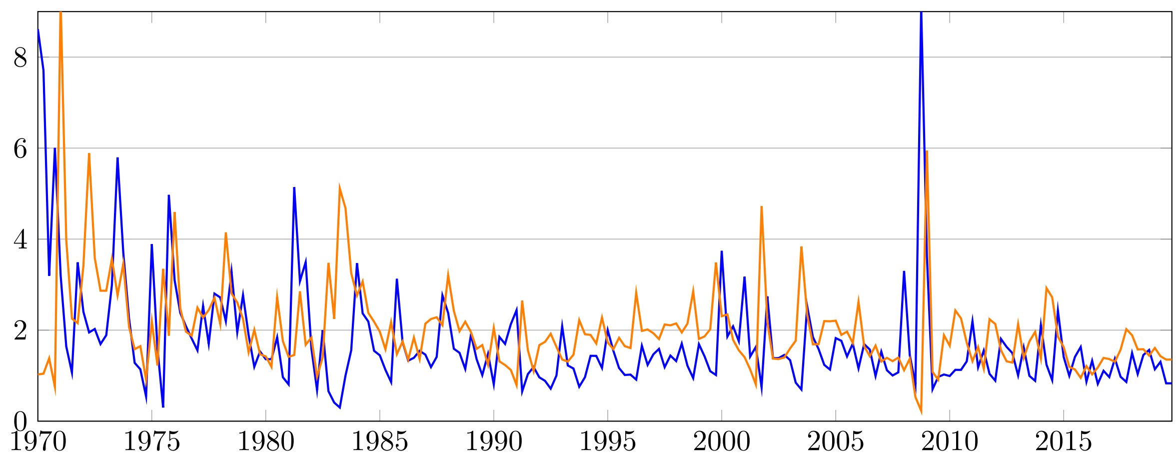

Figure 5: Time-varying lower tail dependence (blue) and time-varying upper tail dependence (orange) between private final consumption and the residual term of the United States. Obtained with the symmetrised SCGAS method.

Similar to the estimation process of the GGAS approach, the third step of obtaining GaR is to construct Monte Carlo simulations. The time-varying tail dependencies, which were estimated in step two, are employed in this process. Multiple paths of simulations can be constructed by using the definition of the mixed Clayton copula as defined by Equation 14. By using the obtained time-varying dependencies, one instance of a Monte Carlo simulation can be constructed. Refer back to Section 4.8 for the complete methodology of obtaining GaR with Monte Carlo simulations. Entertaining the third step ultimately provides GaR estimates, which are shown for the US in Figure 6, with a nominal coverage of 10% (in brown), 5% (in blue), and 1% (in red). The Appendix contains GaR constructed with the symmetrised copula-GAS method for the remaining countries.

Finally, Table 10 provides accompanying empirical coverage levels. The results demonstrate that

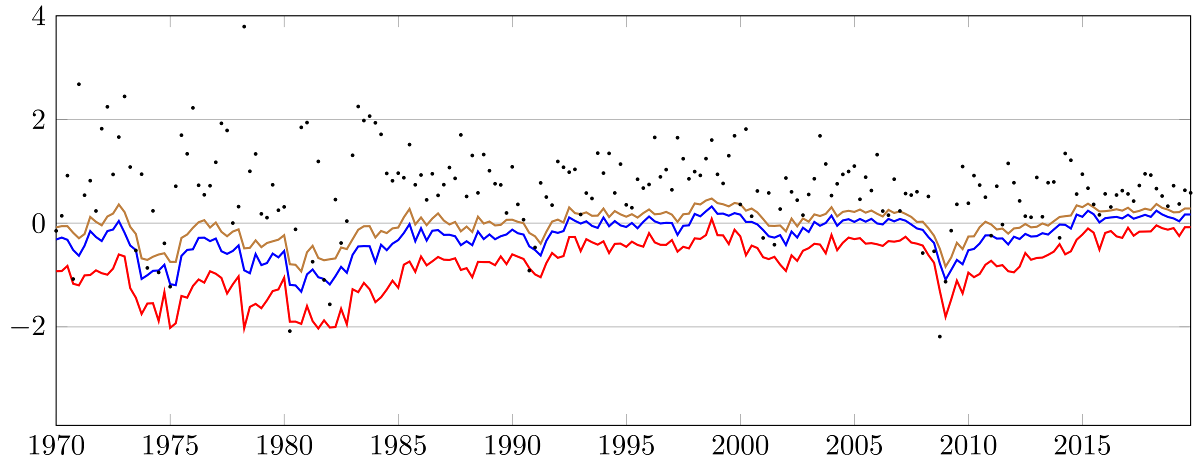

Figure 6: Growth-at-risk (brown 10% coverage, blue 5% coverage, red 1% coverage) for US GDP growth rates from 1970-Q1 to 2019-Q4, shown in black points. Obtained with the SCGAS approach.

a SCGAS model neither excessively overestimates nor underestimates nominal coverages, relative to the empirical coverages obtained with a GGAS model (shown in Table 8). A summarised value (denoted with an average empirical coverage level), suggest that a SCGAS model fairly accurately captures GDP growth rate exceedances. These results are crucial to a policymaker and (central) banks, since their aim is to quantify the likelihood of risk scenarios and develop appropriate risk measures. If GaR is underestimated (overestimated), then preemptive action in terms of macro-prudential instruments (e.g. minimal capital requirement for banks, setting a target inflation rate), would be too weak (strong). Policymakers that only consider these figures would therefore arrive at premature conclusions. The next section will determine the significance of all empirical coverage results, thereby aiding a policymaker in determining the model that best captures GaR.

Table 10: Empirical exceedances of GaR estimates obtained with SCGAS, conducted at a nominal coverage of 90%, 95% and 99%.

	Nominal coverage		
Country	90%	95%	99%
Australia	0.900	0.940	0.985
Germany	0.885	0.945	0.995
Japan	0.930	0.975	0.995
Netherlands	0.910	0.960	0.995
United Kingdom	0.925	0.950	0.995
United States	0.875	0.920	0.980
Average	0.904	0.948	0.991

6.5 Performance tests

In this section, GaR obtained with all three methods (QR, GGAS and SCGAS) is backtested using the unconditional and conditional coverage tests as introduced by Kupiec (1995) and Christoffersen (1998), respectively. These tests will ascertain whether the empirical coverages as presented in Tables 4, 8 and 10 differ significantly from their nominal counterparts. Table 11 shows the test statistics of the unconditional coverage test conducted on all countries in combination with the three models used in this paper. In addition, Table 12 shows test statistics obtained with the conditional coverage test. Both tests are conducted with nominal coverages of 90%, 95% and 99%. The tables show that all models have about equal performance in terms of correct coverage. In almost all cases both an unconditional test and a conditional test failed to be rejected, which implies that all three models are well suited to predict GaR. This result is in stark contrast with the empirical coverages that were obtained with the GGAS method, since Table 8 points out that GaR is overestimated in almost all cases when this model is used (prematurely indicating low predictive performance). In conclusion, by only considering the results of Tables 11 and 12, a macroprudential policymaker would be indifferent in choosing a specific model if his or her aim is to correctly predict GaR.

Table 11: P-values for the unconditional coverage test, conducted at multiple nominal coverages in combination with different models (QR: quantile regression, GGAS: Gaussian copula-GAS and SCGAS: symmetrised Clayton copula-GAS).

Nominal Coverage	90%			95%			99%		
Model	QR	GGAS	SCGAS	QR	GGAS	SCGAS	QR	GGAS	SCGAS
Australia	0.632	0.815	0.999	0.999	0.351	0.529	0.432	0.073	0.508
Germany	0.174	0.021	0.489	0.072	0.162	0.749	0.211	0.999	0.432
Japan	0.073	0.010	0.137	0.038	0.074	0.074	0.001	0.432	0.432
Netherlands	0.469	0.220	0.632	0.742	0.305	0.502	0.432	0.508	0.432
United Kingdom	0.812	0.137	0.220	0.742	0.749	0.999	0.211	0.432	0.432
United States	0.999	0.330	0.255	0.999	0.999	0.072	0.508	0.999	0.211

6.6 Comparing GaR with loss functions

Empirical coverage tests, conducted in the last section, do not disclose the full picture of GaR performance. Consider Figure 7. It shows the distribution of US GDP growth rates from 1970Q1 to 2019Q4 (in gray), as well as 5% GaR estimates. GaR obtained with QR is shown in red and GaR obtained with SCGAS is shown in blue. The figure suggests that QR GaR estimates have a

Table 12: P-values for the conditional coverage test, conducted at multiple nominal coverages in combination with different models (QR: quantile regression, GGAS: Gaussian copula-GAS and SCGAS: symmetrised Clayton copula-GAS).

Nominal Coverage	90%			95%			99%		
Model	QR	GGAS	SCGAS	QR	GGAS	SCGAS	QR	GGAS	SCGAS
Australia	0.758	0.825	0.759	0.591	0.638	0.381	0.730	0.177	0.768
Germany	0.057	0.003	0.095	0.005	0.136	0.044	0.421	0.980	0.730
Japan	0.092	0.027	0.184	0.109	0.182	0.182	0.004	0.730	0.734
Netherlands	0.065	0.024	0.043	0.029	0.008	0.473	0.730	0.768	0.730
United Kingdom	0.006	0.184	0.008	0.139	0.244	0.217	0.421	0.730	0.730
United States	0.759	0.217	0.060	0.591	0.591	0.069	0.768	0.980	0.421

higher variance compared to SCGAS GaR estimates, since it covers a much larger region of growth rates. Additionally, QR GaR estimates have a larger mean than SCGAS GaR estimates. Since the last sections provided evidence that all approaches produced reliable GaR estimates (in terms of empirical coverage and coverage tests), this observation is striking.

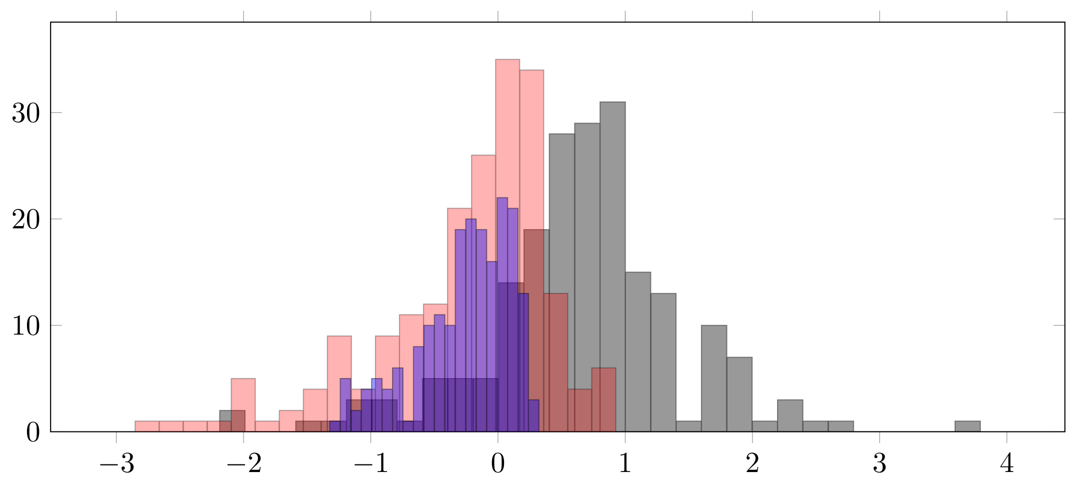

Figure 7: Distributions of 5% GaR of United States GDP growth rates (gray histogram), from 1970Q1 to 2019Q4. Estimated with QR (red histogram) and SCGAS (blue histogram).

Lopez (1999) suggests regulatory loss functions to distinguish VaR estimates from competing models. I employ the loss functions given in Caporin (2008) as a secondary standard to the unconditional and conditional tests. The results of these loss functions are given in the Appendix (Tables 16, 17 and 18 for coverage levels of 10%, 5% and 1%, respectively). Nolde and Ziegel (2017) provide a Diebold-Mariano testing procedure in order to determine the significance of competing GaR predictions. Tables 13a, 13b and 13c presents traffic light matrices for coverage levels of 10%, 5% and

1%, respectively, an approach suggested by Fissler et al. (2015). Green cells indicate a rejection of H_0^+ (the null hypothesis claiming the competing model performs *at most* as well as the benchmark model). For example, when GaR obtained from GGAS is comparatively backtested against GaR obtained from QR, Table 13b suggests that H_0^+ is rejected using loss function f_1. In addition I also performed comparative backtesting on GaR estimates obtained with GGAS and SCGAS, and included these results in the last columns of the tables.

A few remarks can be made from the tables combined. First, copula-GAS methods produce significantly better GaR estimates when considering coverage levels of 5% and 1%. Second, the tables suggest that when coverage levels increase, it becomes more clear that copula-GAS methods significantly outperform QR. Conversely, at a coverage of 10%, the Diebold-Mariano test fails to distinguish GaR performance of QR, GGAS and SCGAS models in case of Australian, Dutch and United States GDP growth rates. However, it cannot be concluded from the tables that QR produces better performing GaR estimates, since only one case rejects H_0^-. Third, the last column presents comparative backtests of models GGAS and SCGAS. The tables suggest that, as the coverage level increases, SCGAS GaR forecasts are significantly better than their GGAS counterparts. This phenomenon can be due to the fact that Clayton copula models capture tail dependence of the marginals, and are therefore more suitable to be used as models for high coverage GaR forecasts.

Table 13: Traffic light matrices for GaR forecasts based on a Diebold-Mariano test. Green cells indicate a rejection of H_0^+, red cells indicate a rejection of H_0^-. Yellow cells indicate that H_0^+ nor H_0^- can be rejected.

(a) 10% coverage.

Compared models	GGAS-QR		SCGAS-QR		SCGAS-GGAS	
Loss function	f_1	f_2	f_1	f_2	f_1	f_2
Australia						
Germany						
Japan						
Netherlands						
United Kingdom						
United States						

(b) 5% coverage.

Compared models	GGAS-QR		SCGAS-QR		SCGAS-GGAS	
Loss function	f_1	f_2	f_1	f_2	f_1	f_2
Australia						
Germany						
Japan						
Netherlands						
United Kingdom						
United States						

(c) 1% coverage.

Compared models	GGAS-QR		SCGAS-QR		SCGAS-GGAS	
Loss function	f_1	f_2	f_1	f_2	f_1	f_2
Australia						
Germany						
Japan						
Netherlands						
United Kingdom						
United States						

6.7 Discussion

In the methodology of this paper, several steps were taken in order to determine the performance of GaR forecasts of different models. The first exploratory measure to discern GaR performance of QR, GGAS and SCGAS models was to determine their respective empirical coverage levels. The results suggested that QR slightly underestimated GaR on average and both GGAS and SCGAS slightly overestimated GaR on average. Based on these results alone, one is unable to select a model that produces the best GaR estimates. The second step considered testing of GaR coverage, based on the (un)conditional coverage tests of Kupiec (1995) and Christoffersen (1998). The results of both tests were insignificant in almost all cases, pointing out that all three models performed equally well, which implied that this step also does not provide certainty in selecting a model. The last step encompassed the calculation of penalty scores, based on loss functions proposed by Lopez (1999) and extended by Caporin (2008). I used loss functions as a secondary measure to the aforementioned coverage tests. This approach allows a macroprudential policymaker to rank any model that produces GaR estimates, and in particular the models covered in this paper. Nolde and Ziegel (2017) provide a comparative backtesting tool to formally test the significance of the scores. Using this test, I distillised the outcomes by assigning traffic light colours. The traffic light matrices suggested that copula-GAS models are preferred in almost all cases to the more conventional QR model, since QR GaR estimates yield higher opportunity costs. However, at lower coverage levels it became less clear which model produces the best GaR forecasts in terms of losses. Although more types of loss functions exist (such as loss functions that only penalise exceedances, discussed in e.g. Gneiting (2011) and Abad et al. (2014)), those that cover the entire set of GDP growth rates and their respective GaR are preferred in the case when the frequency is low. In the case of this paper, a 5% coverage rate in a sample of 200 observations would produce around 10 exceedances. Loss functions that address that limited amount of observations would show low predictive performance, thus less preferred when analysing quarterly data. Combining all results, in conclusion, demonstrates that a copula-GAS method for obtaining GaR estimations is superior to the conventional QR approach. Therefore copula-GAS models should be incorporated, or considered at the least, in a macroprudential surveillance framework.

7 Conclusion

GDP is a well-known macroeconomic indicator that can act as a measurement of a country's overall prosperity. It comprises of elements such as private consumption, government expenses and net export. This implies that whenever one of these factors shows a decline (or increase) in output, the shock also will be reflected in GDP. Therefore it is of paramount importance to model and forecast GDP growth rates, as this may lead to an overall more resilient macroeconomic system. However, until the Financial Crisis of 2008, policymakers of (central) banks mainly focussed on forecasting an expected mean GDP growth, which has led to a reevaluation of their methods. Policymakers and risk managers concluded that a more prudent approach was required to mitigate tail risks of GDP growth. One element of this macroprudential framework is GaR analysis, first developed by Wang and Yao (2001), and is recently assessed in more detail by Adrian et al. (2018) and Prasad et al. (2019). GaR is defined as a certain quantile of GDP growth. Hence this approach is able to sketch and model the entire probability distribution of GDP growth rates, a major improvement to point forecasts which policymakers originally evaluated. To construct GaR forecasts, policymakers predominantly implement a method called quantile regression (QR) in combination with macroeconomic indicators, see for example Mello and Perrelli (2003) and White et al. (2015). Since QRs can assess the left tail of GDP growth distribution, it is a natural candidate to predict GaR. Although this approach has benefits such as resilience to outliers, it does not make any distributional assumptions, which can be beneficial when the frequency of the data is relatively low. Moreover, QR requires a representative set of macroeconomic downside predictors as demonstrated by Brownlees and Souza (2021), whereas a GAS approach only relies on growth rate data.

In this paper, I construct GaR forecasts by applying copulas (proposed by Sklar (1959)) and GAS models (proposed by Creal et al. (2013)) to quarterly GDP growth rates of six OECD countries between 1970Q1 and 2019Q4. The main question is whether this integral approach (which should capture non-normal behaviour and tail dynamics) can produce more accurate GaR forecasts than QR. Since GDP consists of multiple elements, a copula can be applied to analyse the dependence between those marginals. I assess the dependence structure between private final consumption, which often accounts for more than 50% of GDP, and a residual term. The methodology comprises of two separate copula models that are used to construct GaR estimates, namely a Gaussian copula and a symmetrised Clayton copula. These models are augmented with a GAS structure, to obtain time-varying dependence parameters of the copulas. I constructed a 4-step methodology (marginal modelling, model estimation, Monte Carlo simulation and backtesting) to determine the quality and accuracy of the quarterly GaR forecasts obtained with the copula-GAS models. Additionally, the performance of GaR forecasts obtained with QR is used as a benchmark. The analysis of GaR performance comprises of coverage tests (to assess the accuracy) and implementation of loss functions as a secondary measure (to rank the models). The results show that both copula-GAS models cover GDP growth rates as accurately as the QR benchmark, for all six countries considered. However, significance tests on the obtained scores show that the copula-GAS models mainly produce the best GaR, with SCGAS forecasts consistently outperforming QR forecasts at high coverage levels. In this sense, my results underline the findings of Brownlees and Souza (2021), which concluded that macroprudential policymakers should not rely on QR too heavily when their objective is producing reliable risk estimates. Therefore, my proposed methodology could provide policymakers and financial institutions with GaR forecasts that are well suitable in managing downside risk of GDP growth, and can be implemented to cover extreme events more appropriately in terms of opportunity costs.

Some aspects of my approach can be investigated for further research. For example, the bivariate copula I used can be extended to a multivariate (hierarchical) copula, which allows modelling a more granular dependence structure between GDP components. It can also be used to model the dependence between GDP growth rates of different countries, thereby establishing a more resilient region of countries. Secondly, my methodology can be applied to expected shortfall, a coherent risk measure recently introduced by the Basel Committee on Banking Supervision to develop a more resilient financial sector. A macroeconomic framework could benefit from this approach. In conclusion, the modest scope of my methodology sets the stage for the development of a more sophisticated macroprudential toolkit for policymakers.

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8 Appendix

The Appendix contains additional Tables and Figures for the remaining countries analysed in this paper, as well as estimation results of applying an AR(p)-GARCH(1,1) model to private consumption growth component and the residual component of GDP. The United States is covered in the main text, whereas supplementary material of Australia, Germany, Japan, The Netherlands and United Kingdom are given here. Table 14 and Table 15 contain AR(p)-GARCH(1,1) estimation results applied to the private final consumption component and residual component, respectively. Figure 8 shows GaR results obtained with QR. Figures 9 and 10 present time-varying correlation and GaR, respectively, obtained with GGAS. Figures 11 and 12 present time-varying tail dependence figures and GaR estimates, respectively, obtained with SCGAS.

Table 14: AR(p)-GARCH(1,1) estimation results for the quarterly consumption growth rate component (1970-Q1 to 2019-Q4) of Australia, Germany, Japan, Netherlands, United Kingdom and United States.

	AR(p)						GARCH(1,1)				
Country	ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	γ	α	β	d.o.f. ν	BIC
Australia	0.577	0.191	-	-	-	-	0.004	0.161	0.836	199.973	440.22
	(0.070)	(0.073)					(0.008)	(0.069)	(0.058)	(2935.66)	
Germany	0.168	-0.116	0.015	0.219	0.230	0.268	0.259	0.700	0.149	4.219	495.98
	(0.070)	(0.076)	(0.061)	(0.051)	(0.049)	(0.057)	(0.105)	(0.307)	(0.131)	(1.758)	
Japan	0.296	0.033	0.195	0.303	-	-	0.477	0.394	0.296	3.746	585.25
	(0.081)	(0.071)	(0.062)	(0.061)			(0.244)	(0.200)	(0.242)	(1.052)	
Netherlands	0.208	0.003	0.209	0.338	-	-	0.000	0.039	0.954	5.913	525.41
	(0.064)	(0.058)	(0.066)	(0.066)			(0.008)	(0.024)	(0.028)	(2.815)	
United Kingdom	0.299	0.143	0.192	0.200	-	-	0.040	0.248	0.723	7.416	526.49
	(0.086)	(0.072)	(0.077)	(0.075)			(0.036)	(0.109)	(0.095)	(3.480)	
United States	0.228	0.186	0.190	0.294	-	-	0.002	0.084	0.907	6.702	333.98
	(0.056)	(0.067)	(0.068)	(0.069)			(0.005)	(0.043)	(0.042)	(3.143)	

Table 15: AR(p)-GARCH(1,1) estimation results for the quarterly residual growth rate component (1970-Q1 to 2019-Q4) of Australia, Germany, Japan, Netherlands, United Kingdom and United States.

	AR(p)			GARCH(1,1)				
Country	ϕ_0	ϕ_1	ϕ_2	γ	α	β	d.o.f. ν	BIC
Australia	0.777	-0.098	-	0.000	0.085	0.907	199.995	784.06
	(0.105)	(0.068)		(0.025)	(0.043)	(0.043)	(3510.50)	
Germany	0.547	0.036	-	0.566	0.168	0.686	4.420	825.43
	(0.114)	(0.076)		(0.402)	(0.087)	(0.156)	(1.527)	
Japan	0.530	0.020	0.171	1.626	0.487	0.000	7.568	781.94
	(0.118)	(0.082)	(0.069)	(0.500)	(0.171)	(0.162)	(5.843)	
Netherlands	0.678	-0.070	-	0.022	0.181	0.819	5.427	767.80
	(0.085)	(0.076)		(0.032)	(0.070)	(0.062)	(1.934)	
United Kingdom	0.417	-0.012	-	0.118	0.086	0.864	13.128	774.73
	(0.105)	(0.082)		(0.144)	(0.057)	(0.095)	(13.349)	
United States	0.609	-0.046	-	0.088	0.130	0.840	6.507	771.56
	(0.102)	(0.069)		(0.085)	(0.063)	(0.067)	(2.770)	

Table 16: Penalty scores obtained with loss functions f_1 and f_2, using 10% GaR estimations in the period 1970Q1 to 2019Q4. The lowest penalty scores for every country and loss function are emphasised in bold.

Loss function	f_1			f_2		
Model	QR	GGAS	SCGAS	QR	GGAS	SCGAS
Australia	1353.14	**1204.52**	1353.32	1357.21	**1331.11**	1458.57
Germany	2155.12	**137.22**	437.43	2502.48	**143.66**	523.73
Japan	983.76	**241.67**	726.32	1034.07	**383.66**	1032.38
Netherlands	**1239.23**	1817.12	3323.62	1666.04	**1418.90**	3053.39
United Kingdom	2138.44	**504.53**	4165.21	1644.59	**440.59**	3796.59
United States	1820.37	2208.59	**1656.28**	2803.20	2098.59	**1731.50**

Table 17: Penalty scores obtained with loss functions f_1 and f_2, using 5% GaR estimations in the period 1970Q1 to 2019Q4. The lowest penalty scores for every country and loss function are emphasised in bold.

Loss function	f_1			f_2		
Model	QR	GGAS	SCGAS	QR	GGAS	SCGAS
Australia	2287.31	**857.96**	871.19	3159.95	955.28	**783.03**
Germany	824.35	**118.94**	130.66	703.36	145.97	**105.48**
Japan	1120.04	**125.98**	173.75	1441.68	165.79	**163.25**
Netherlands	823.06	397.57	**254.68**	728.45	414.17	**204.70**
United Kingdom	1146.35	1111.22	**419.83**	1122.48	997.76	**358.68**
United States	**1115.95**	1154.60	2295.05	1377.56	**691.63**	2312.22

Table 18: Penalty scores obtained with loss functions f_1 and f_2, using 1% GaR estimations in the period 1970Q1 to 2019Q4. The lowest penalty scores for every country and loss function are emphasised in bold.

Loss function	f_1			f_2		
Model	QR	GGAS	SCGAS	QR	GGAS	SCGAS
Australia	827.32	**372.20**	398.64	795.37	335.44	**281.48**
Germany	210.71	143.96	**111.63**	282.64	356.24	**137.92**
Japan	242.07	135.00	**105.30**	442.68	327.05	**127.96**
Netherlands	214.40	196.68	**106.47**	355.25	305.01	**160.82**
United Kingdom	667.85	550.01	**100.66**	619.76	418.44	**94.87**
United States	1201.02	499.91	**248.92**	1261.18	527.36	**197.37**

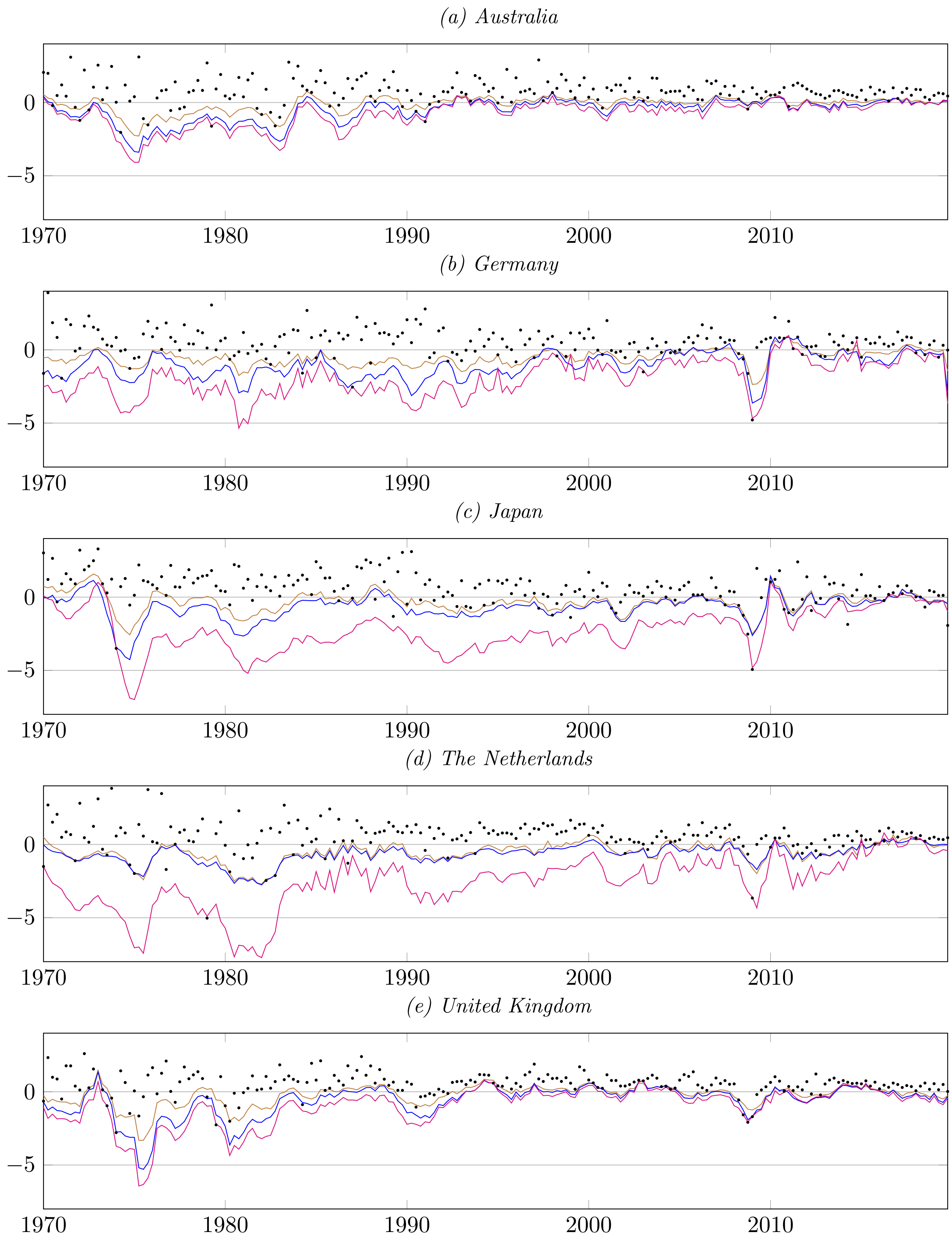


Figure 8: Growth-at-risk (brown 10% coverage, blue 5% coverage, red 1% coverage) for the remaining countries obtained with quantile regression, using long term bond yields, CPI growth, house pricing growth, credit growth, IP and credit gap data as indicators for GDP growth.

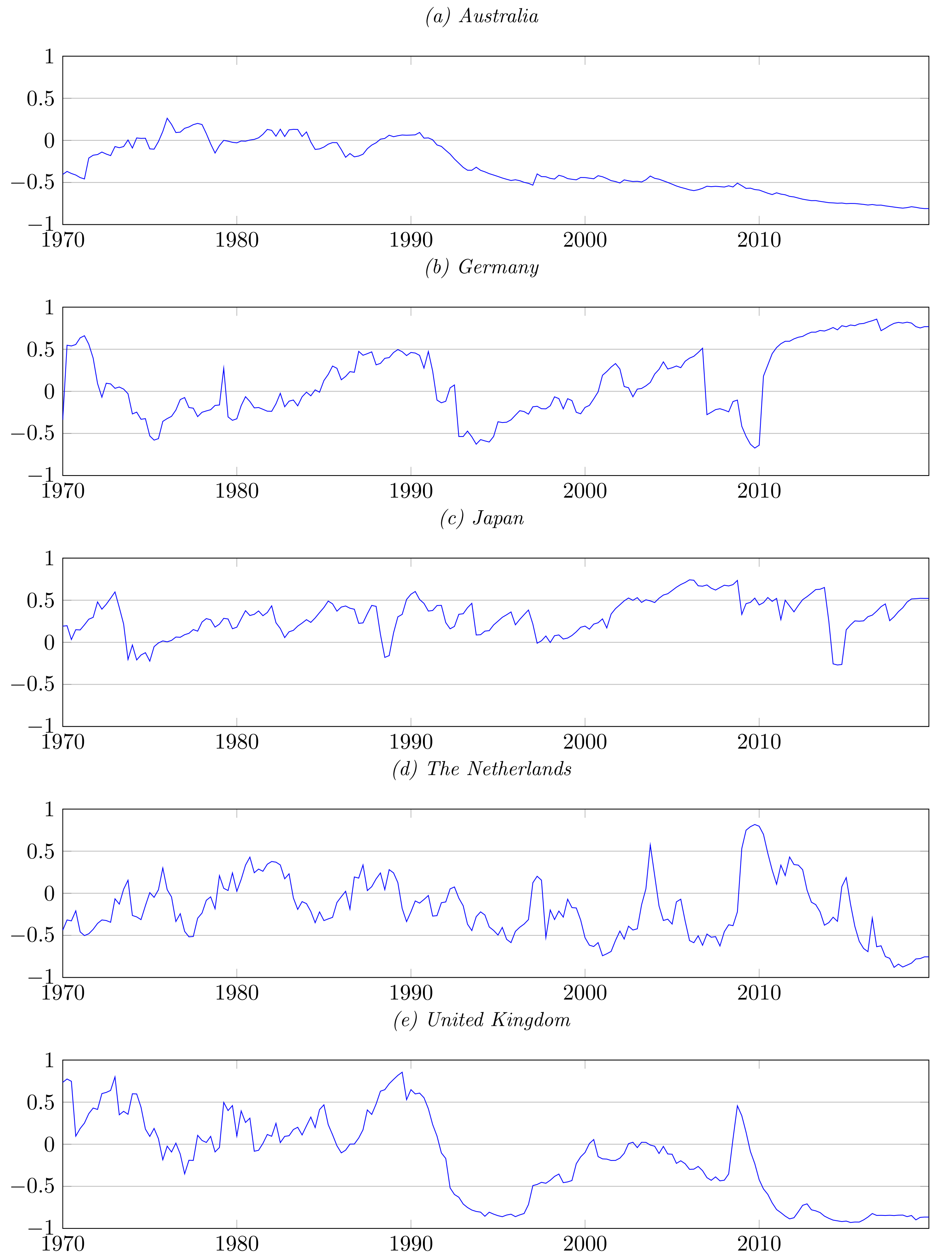


Figure 9: Time-varying correlations between private consumption and the residual components for the remaining countries. Obtained with Gaussian copula-GAS.

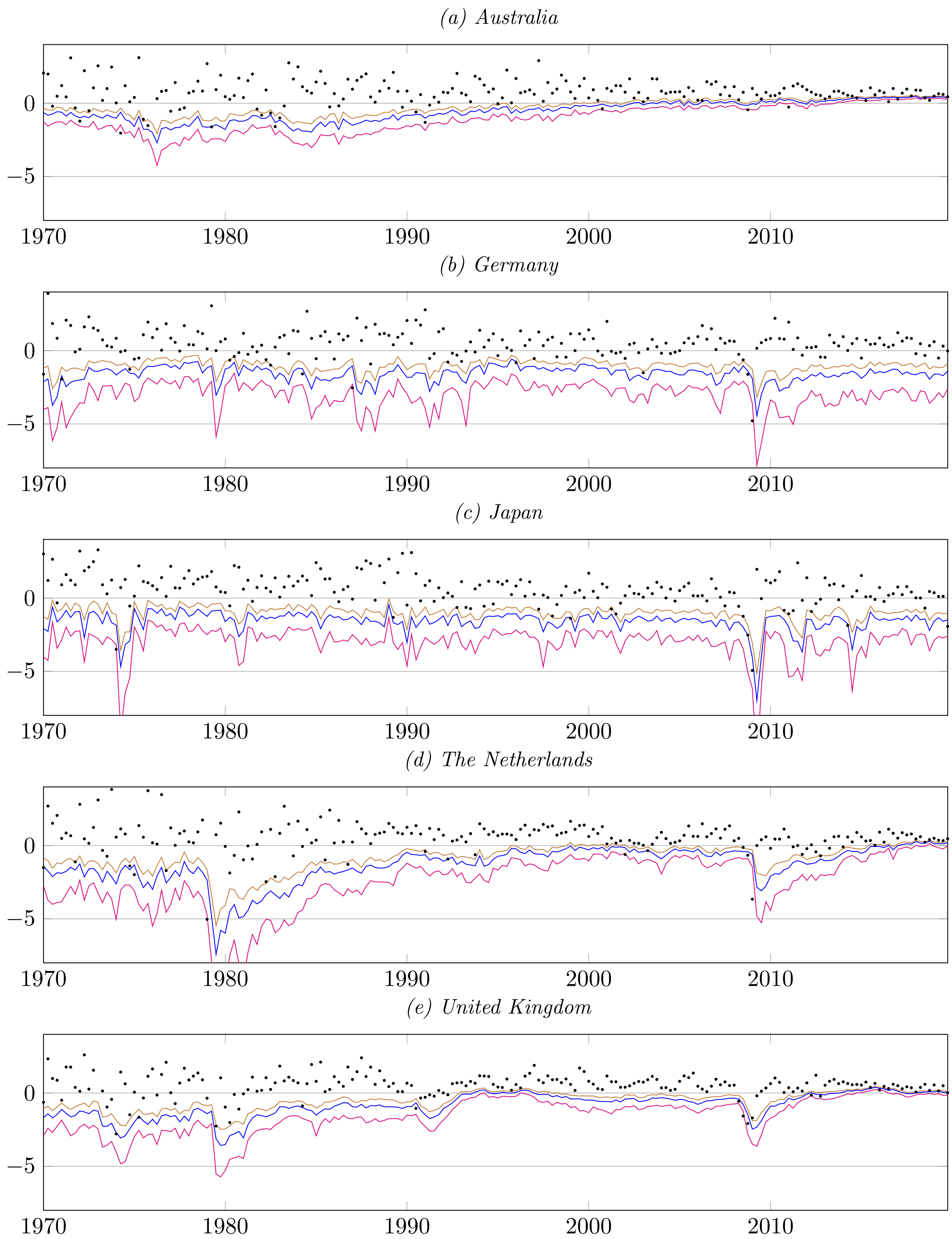
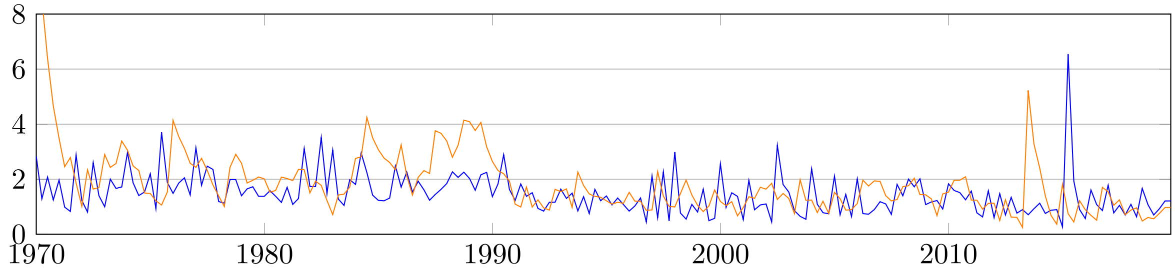
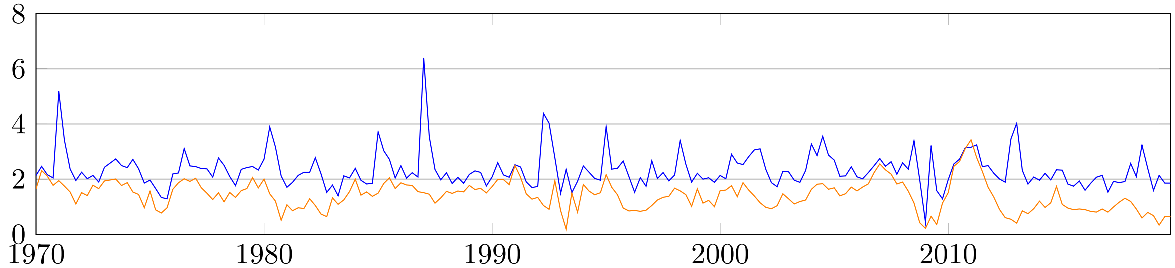


Figure 10: Growth-at-risk (brown 10% coverage, blue 5% coverage, red 1% coverage) for the remaining countries obtained with Gaussian copula-GAS.

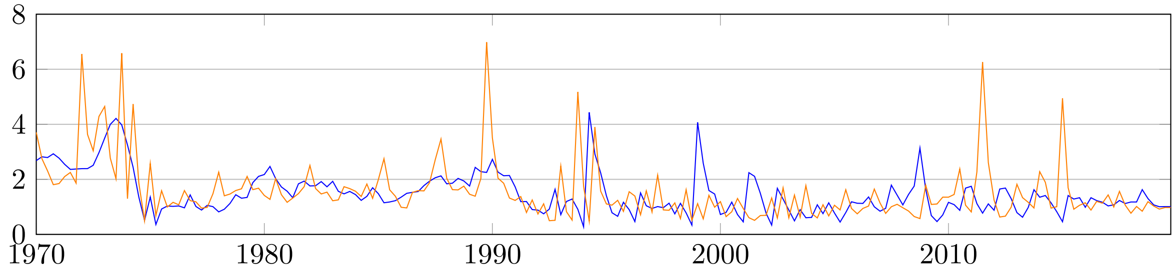
(a) Australia



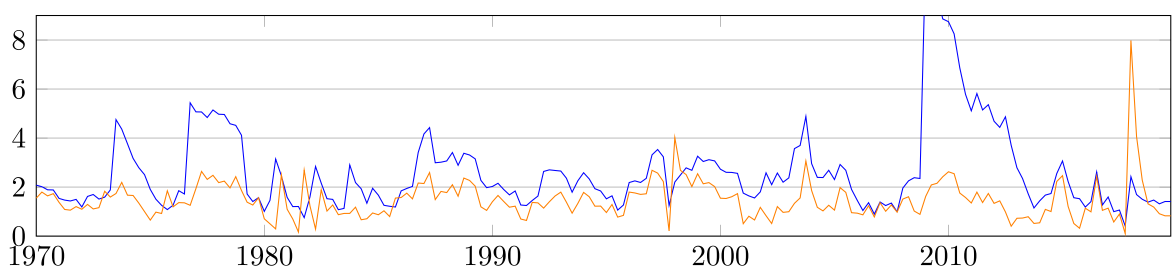
(b) Germany



(c) Japan



(d) The Netherlands



(e) United Kingdom

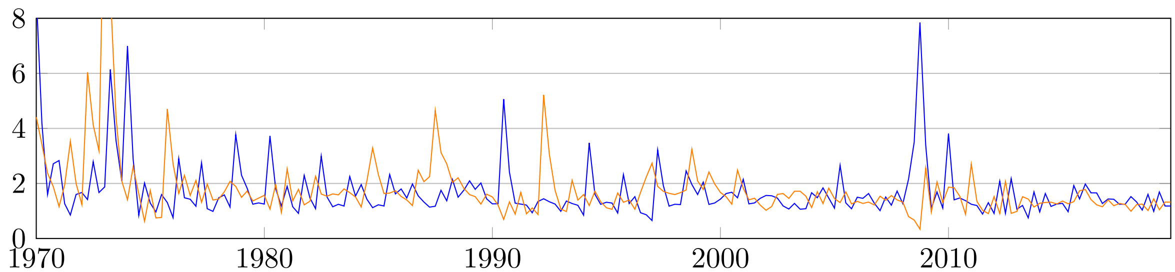


Figure 11: Time-varying lower tail dependence (blue) and time-varying upper tail dependence (orange) between private final consumption and the residual term. Obtained with the symmetrised Clayton copula-GAS method.

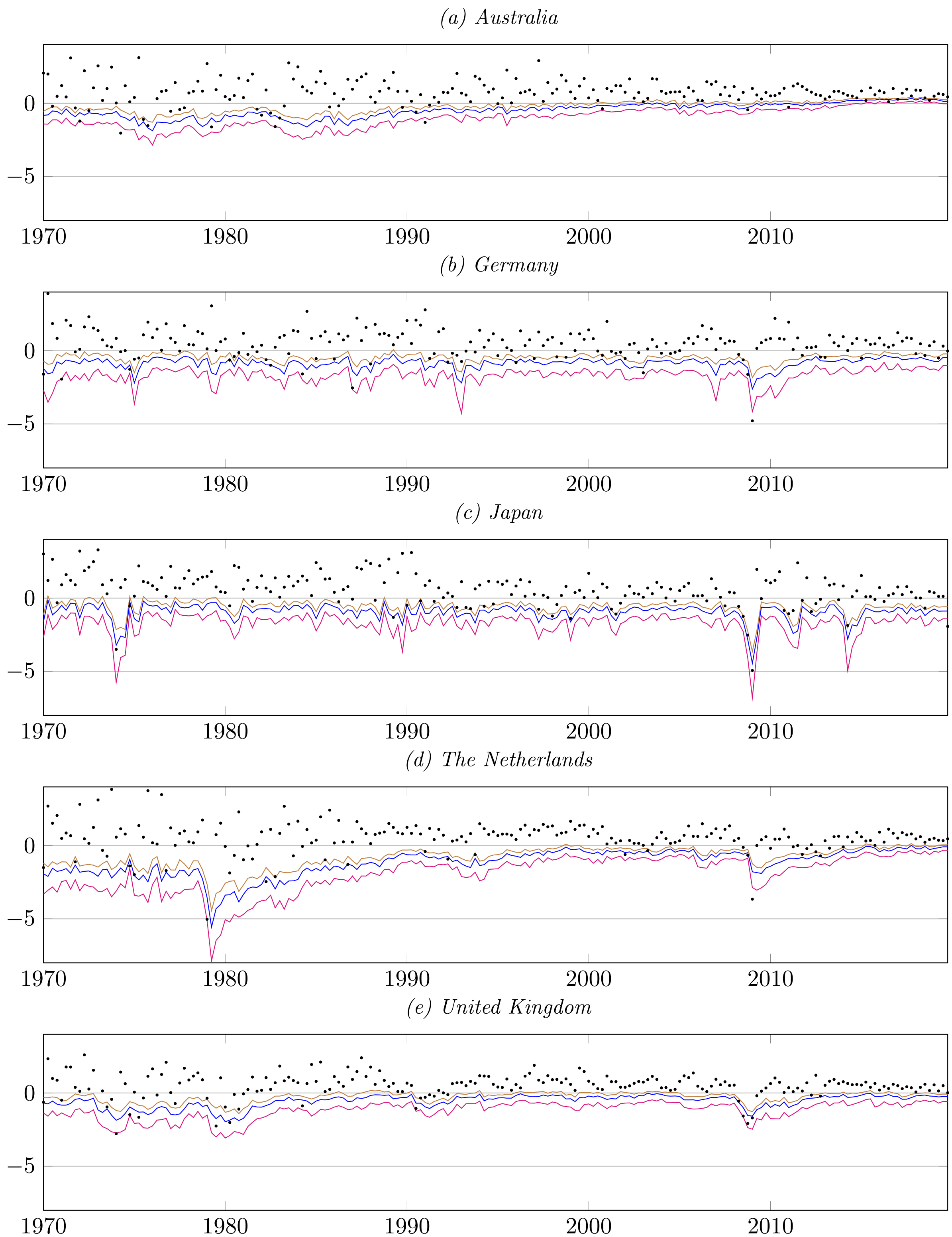


Figure 12: Growth-at-risk (brown 10% coverage, blue 5% coverage, red 1% coverage) for the remaining countries obtained with symmetrised Clayton copula-GAS.