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An Analysis of the Joint Macroeconomic Downside Risk of Economies under the GaR Statistic

Master Thesis

Econometrics and Management Science-Quantitative Finance

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Abstract

This paper investigates the out-of-sample predictive performance of joint Growth-at-Risk (GaR) for 5 economies. We consider forecasts constructed from univariate and multivariate GARCH models. The univariate specifications are based on the standard and asymmetric GARCH(1,1) with skewed Student-t residuals, whereas their multivariate Student-t extensions on the Dynamic Conditional Correlation GARCH and the Copula-GARCH. The backtesting results show that asymmetric GARCH has to a certain extend better performance than the standard volatility model, while the multivariate GARCH cases that incorporate cross-sectional information generate more accurate predictions.

Keywords: joint GaR, GARCH, asymmetry, skewed Student-t, DCC, Copula

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1 Introduction

The global financial crisis of 2007-2009 had an unprecedented impact on the systemic risk in national financial systems which led policymakers to adopt macroprudential policies to address similar phenomena and sustain financial stability. The risk associated with the economic growth is essential since negative economic growth would imply financial instability. GDP can constitute an important indicator of contracting or expanding economy. In this context, the International Monetary Fund (IMF) has developed a novel risk measure for GDP growth rate, named Growth-at-Risk (GaR), which is analogous to Value at Risk (VaR) used in risk management. Particularly, the univariate $(1-p)$ Growth-at-Risk is defined as the lower one-sided prediction interval that contains future realizations of GDP growth rates of a given country at $(1-p)$ confidence level.

However, univariate GaR is too restrictive from a central monitoring perspective. Global financial systems such as the European Central Bank desire to track economies under a joint region and determine the impact of small peripheral economies on global central economies. That is why, the main goal of the thesis is to provide an out-of-sample empirical analysis of joint GaR. The $(1-p)$ joint GaR is defined as the prediction region that contains the GDP growth rates of all economies with $(1-p)$ coverage probability. The definition of joint GaR is built upon the work of Brownlees & Souza (2021) who introduced a generalization of univariate GaR prediction region. This region is based on the prediction of joint rectangular regions by a bootstrap method of Wolf & Wunderli (2015).

We consider joint GaR forecasts at different time horizons for a panel of dependent economies. The forecast horizons are $1, 2, 4$, and 8 quarters ahead, which allow policymakers to proceed in long-term informed decisions. The estimated GaR provides them with an indication of an expanding or loosening economy since it represents how much the GDP growth rate will fall given a particular time frame. The forecasts are based on Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) specifications. GARCH is the workhorse model for handling volatility clustering in the literature. We particularly utilize the standard GARCH of Bollerslev (1986) and the GJR GARCH of Glosten et al. (1993) with skewed Student-t residuals. The GJR belongs to the family of asymmetric GARCH models and weights negative shocks more than the positive ones.

This paper further contributes to the current literature of joint GaR proposed by Brownlees & Souza (2021) by incorporating cross-sectional information to the joint GaR forecasts. This is achieved by the multivariate DCC and Copula GARCH models. The former one allows for the cross-sectional correlation of the economies by the decomposition of the conditional covariance matrix into conditional standard deviations and dynamic correlations. The latter one captures the cross-sectional dependence of the panel of economies with a multivariate Student-t copula. This model links the marginal distributions of the economies with their joint distribution using a copula function introduced by Sklar (1959) characterized by a unique dependence parameter. Both models assume a multivariate Student-t distribution for the innovations.

To our best knowledge, our work is the first to incorporate cross-sectional information and address joint GaR in the multivariate GARCH framework, namely the DCC and the Copula GARCH models. The backtesting of joint GaR forecasts is based on the unconditional coverage test of Kupiec (1995) and two variants of the Dynamic Quantile test of R. F. Engle & Manganelli (2004) which can be interpreted as an overall goodness-of-fit test for the GaR predictions. To conclude, the main research objective can be summarized as follows:

"(How) does the cross-sectional information enhance the accuracy of joint GaR predictions?"

The backtesting results show that asymmetric modeling of volatility adequately improves the forecasts when the standard algorithm of joint GaR and the multivariate Copula-GARCH are considered. The multivariate GARCH processes, DCC and Copula, deliver more accurate predictions in comparison with their counterpart that does not consider cross-sectional information. However, forecasts of all models are characterized by poor long-term predictive performance. Between DCC and Copula, the former one produces fairly better forecasts with smaller average length and empirical coverage closer to the nominal in short-term horizons. On the other side, Copula specifications present a stable behavior in producing predictions across the different time horizons with empirical coverage closer to the nominal, at the cost of excessively high average length.

Several reasons substantiate these results. DCC and Copula incorporate cross-sectional information by considering dynamic correlation and allowing for non-Gaussian dependence, respectively across the economies. Thus, they outperform the algorithm for joint GaR predictions without cross-sectional information. As far as the comparison of the multivariate GARCH models is concerned, their main difference is detected in the simulation process and standardization of residuals. Particularly, in the DCC context, the residuals are simulated and subsequently standardized based on the shape parameter ν of the joint Student-t distribution. On the contrary, Copula utilizes both a joint shape parameter and an extra dependence parameter θ for the simulations, whereas the standardization step makes usage of the central moments of the skewed Student-t distribution.

The rest of the paper proceeds as follows. Section 2 gives an overview of the existing literature. Section 3 describes the data used for the study. Section 4 analyses the implemented methodology. Section 5 presents the empirical evidence and Section 6 concludes the thesis. Additional plots, empirical results, and proof of the skewed Student-t moments are provided in the Appendices.

2 Literature Review

As the GaR framework is heavily related with VaR literature, we firstly make a thorough review of the existing VaR methodologies. Secondly, we briefly comment on univariate GaR since it constitutes a building block of joint GaR, and thus provides insights about the general context of GaR. Lastly, as the literature background of joint GaR is limited, we remark on several attempts to capture cross-sectional information with multivariate GARCH models.

2.1 VaR

VaR background can be summarized into three major segments: distributional assumptions required to model VaR, methods to estimate VaR, and backtesting techniques to evaluate the accuracy of these predictions.

C. W. Chen et al. (2012) highlight the importance of distributional assumptions over the volatility modeling. Financial data present a variety of stylized facts such as skewness and leptokurtosis and usually deviate from normality. Therefore, the mixture of Normal distributions, Student-t as in the context of the multivariate GARCH models in this paper, or Generalized Error Distribution (GED) provide more precise VaR estimates than the Normal distribution as shown in Zhang & Cheng (2005), Nelson (1991), and Kuester et al. (2006).

Forecasting methods of VaR can be divided into three groups aparametric, semi-parametric and parametric. Abad et al. (2014) and McNeil et al. (2015) offer a solid background of the current established VaR techniques as well of their possible combinations. Historical Simulation (HS) is an indicative example of an aparametric technique to estimate VaR and utilizes the empirical distribution of financial returns. However, it is heavily based on the data. As far as parametric approaches are concerned, RiskMetrics, firstly launched by J.P. Morgan, is the most straightforward and assumes that the financial data follow a normal distribution over time which ignores many of the observed stylized facts of financial returns. On the other side, semi-parametric methods do not directly model the returns, but the errors. Thus, Filtered Historical Simulation (FHS) and Monte Carlo are conducted to the residuals resulted from the specified volatility dynamics. This paper follows the parametric path by modelling the errors resulted by univariate GARCH specifications with a skewed Student-t distribution.

The adequacy of the VaR measures plays an important in the comparison of different forecasting methods. The backtesting of VaR methods can be bisected as follows: standard tests about accuracy of VaR and tests which evaluate the magnitude of the losses. Kupiec (1995) proposes an unconditional coverage test which tests whether the observed failure rate is equal to the failure rate suggested by the confidence interval p. However, this test does not take into account the conditional coverage since violation can cluster over time. Escanciano & Pei (2012) overcome that limitation by implementing a data-driven weighted back-test to evaluate HS and FHS forecasts. Alternatively, Christoffersen (1998) proposes a conditional coverage test and considers a likelihood ratio test to examine whether the probability of observing an exception on a particular day depends on whether a violation occurred. The Dynamic Quantile test of R. F. Engle & Manganelli (2004) tests whether the current violations are uncorrelated with past violations, while it can be extended to include a variety of explanatory variables, among others the returns or squared returns.

The second view on the backtesting context is based on the magnitude of the losses experienced in the case of a violation. The interest in this approach lies on which VaR model offers the minimum loss function. Abad et al. (2014) and Nieto & Ruiz (2016) offer a wide spectrum of backtesting context and several loss functions such as the quadratic loss function. In our framework, we follow the work of Kupiec (1995) and R. F. Engle & Manganelli (2004) to evaluate the distinct joint GaR predictions. Backtesting joint GaR forecasts with a loss function is not feasible since joint GaR is a multi-dimensional region.

2.2 Univariate GaR

As earlier denoted, univariate GaR is not the primary focus of this paper. That is why, we briefly remark on factor-based approaches to it which incorporate (macro)economic information. Alter et al. (2019) provide a practical guide to conduct GaR analysis consisting of three stages: a) selection of macrofinancial variables and construction of partition of financial factors, b) estimation of the non-linear relationship between the exogenous explanatory financial variables and the future GDP growth by Quantile Regression (QR), and c) derivation of the conditional future growth distribution.

In the same direction, Adrian et al. (2019) assume a skewed-t distribution for the conditional distribution of GDP growth and take into account how vulnerable the predicted GDP growth is to extreme events. They find that downside risks to GDP growth are predicted by financial conditions while upside risks are stable over time. De Polis et al. (2020) further consider time-varying parameters of skewed-t distribution motivated by a number of tests (Portmanteau test, Ljung-Box extension, and Nyblom test). They argue that GDP growth experiences significant changes in the long-run mean, shifts in the volatility, skewness of the distribution, countercyclicality of the volatility of GDP growth, and sharp fall of the skewness of the cycle during recessions.

Brownlees & Souza (2021) utilize the quantile regression and introduce several predictors, with NFCI (a dynamic factor constructed from an unbalanced panel of 105 mixed-frequency indicators of U.S. financial activity) being the most prominent among them. As QR directly links downside risk to predictors of interest, policymakers should give great attention to its forecasting power since the chosen macroeconomic variables of the regression must fairly represent the financial conditions of the economy. Indicative examples are the lending standards that provide useful information about the state of financial system of an advanced bank-dominated economy, or house prices that play a significant role for economies with large mortgage-debt. They also consider a series of univariate GARCH models which perform similarly with QR.

Our work diverges from the framework of QR or any other endeavour to associate GDP with macroeconomic risk variables. However, we consider a skewed Student-t distribution since we model the volatility dynamics of the growth rates with univariate GARCH processes characterized by skewed Student-t innovations.

2.3 Joint GaR

In the context of joint GaR, univariate GARCH models do not take into account cross-dependence among the several economies. Furthermore, macroeconomic data such as the GDP are available only in a short panel of time series. Brownlees & Souza (2021) overcome these limitations by estimating GARCH parameters with composite likelihood and by assuming that the dynamic GARCH coefficients are common across the distinct countries they consider. Pakel et al. (2011) point out the pitfall of standard estimation methods such as the common quasi maximum likelihood to perform well in small samples and consider cross-sectional information by suggesting also composite likelihood to pool information across the panel of time series.

There have been many attempts to capture cross-sectional dependence among variables by generalizing GARCH models to multiple dimensions. R. F. Engle & Kroner (1995) firstly introduced the BEKK model which handles the conditional covariance matrix in a systemwise regression and captures time-varying covariances. In the same context, R. Engle (2002) proposes a Dynamic Conditional Correlation model (DCC), and does not directly estimate the covariance matrix, but rather models it through a dynamic correlation matrix. This paper adopts the aforementioned approach of dynamic correlation between the selected economies. R. F. Engle et al. (2019) robustify DCC against large dimensions by using two tools, Composite Likelihood and non-linear shrinkage method of Ledoit et al. (2012). However, according to Caporin & McAleer (2013) DCC is not actually a model, bur rather a useful filter or diagnostic check for dynamic correlation.

Cross-sectional dependence in GARCH methodology can be further enhanced by combining them with copulas which are used to describe the dependence between two random variables. Jondeau & Rockinger (2006) shed light on the specification of the joint distribution of multivariate return series by exploiting the skewed Student-t copula proposed by Hansen et al. (1994). In comparison with the standard Student-t distribution, it introduces an extra parameter for asymmetry. A key feature of copulas is that they depend on easily conditioned parameters such as the dependence parameter or the degrees of freedom in the case of Student-t copula which is implemented in this paper.

The superiority of t copula to the Gaussian copula is substantiated by its ability to better capture the phenomenon of dependent extreme values which is often observed in financial data. Demarta & McNeil (2005) derive several copulas based on the t copula such as the skewed-t copula, grouped-t copula, extreme value t copula, and lower tail t copula. As an ultimate attempt to capture asymmetric, leptokurtic, and heavy-tail characteristics, the time-varying volatility characteristics and the extreme-tail dependence characteristics of financial asset returns Q.-a. Chen et al. (n.d.) take advantage of the Generalized-Hyperbolic distribution (G-H), a general class of distributions, among others, the Student's t distribution. They additionally consider a standard GARCH model whose standardized innovations follow a G-H distribution and they further combine it with a multivariate time-varying copula in contrast to our work, in which the copula is static.

The estimation of copula parameters is done in various ways. Standard approaches are the method of moments in which the theoretical moment is equated with the estimated from the sample moment, the maximum likelihood or the pseudo maximum likelihood dependent on the data set selected for the estimation (original or pseudo sample), and lastly the two-step estimation procedure of Inference of Margins. Salleh et al. (2016) discuss details regarding estimation of copula parameters. This paper follows the pseudo maximum likelihood approach. Finally, the prediction of joint GaR according to Brownlees & Souza (2021) follows the procedure of joint rectangular prediction region of Wolf & Wunderli (2015); a region that contains all the GDP growth rates of all the participated economies at the desired probability level.

In conclusion, the novelty of this paper is entrenched in the ensuing constituents: a) assumption of skewed

Student-t innovations in the univariate GARCH models, and b) enrichment of current joint GaR framework by considering a multivariate Student-t distribution in the case of multivariate GARCH models.

3 Data and Stylized Facts

We retrieve time series for GDP growth rates from the Federal Bank of St. Louis (Organization for Economic Co-operation and Development, Main Economic Indicators (2016)). The data set analyzed in this paper comprises monthly observations on the following five economies: i) OECD countries expanded with the major six OECD non-member countries, namely Brazil, China, India, Indonesia, Russian Federation and South Africa, ii) only OECD countries, iii) countries in Eurozone, iv) countries participating in the North American Free Trade Agreement (NAFTA) which includes Mexico, the USA, and Canada, and v) the Big 4 European countries; France, Germany, Italy, and United Kingdom. After the Brexit, OECD statistics still regards the United Kingdom in the four big European. The sample covers the time period from March 1961 to November 2020. In total, we obtain 717 time series observations for each of the five included economies which represent the seasonally adjusted growth rates of GDP.

Table 1: Sample statistics of the five economies.

Statistics \ Economies	OECD+6MajorNME	OECD	Eurozone	NAFTA	Big4Eur
Mean	3.810	2.972	2.510	3.029	2.285
Std. Dev.	1.774	2.086	2.467	2.227	2.380
Minimum	-8.769	-11.513	-14.952	-10.329	-16.642
Maximum	7.345	7.323	7.949	7.947	7.291
Skewness	-1.773	-1.574	-1.713	-0.930	-2.377
Kurtosis	7.108	7.008	8.522	3.180	13.522
JB-Test	1898.641	1776.243	2538.009	409.203	6177.970
	{0}	{0}	{0}	{0}	{0}
LB-Test$^{\textbf{growthrates}}$	3567.459	4608.721	4737.396	3908.14	3794.929
(24)	{0}	{0}	{0}	{0}	{0}
LB-Test$^{\textbf{growthrates}^2}$	3651.563	3821.626	2272.316	3496.556	1277.773
(24)	{0}	{0}	{0}	{0}	{0}
DCC-Test			17537.410		
(24)			{0}		

Note. This table reports the descriptive statistics of the five economies. The JB test examines jointly the null hypothesis of skewness and excess kurtosis equal to 0 and the LB test whether data are independently distributed. The multivariate DCC test provides one statistic value and utilizes a static GARCH copula to estimate the Constant Conditional Correlation (CCC) matrix of the null hypothesis. The curly braces include the p-values and the parentheses the number of lags used for each of the tests. All reported numbers are rounded to 3 decimals.

Table 1 shows the negative skewness and excessively higher kurtosis than that of the Gaussian distribution implying asymmetry in the tail events of the distribution, concentrated in the left tail, and fat-tailed growth rates. The rejection of null hypothesis at all standard confidence levels of Jarque-Bera test confirms the deviation from normality. The analogous rejection of null hypothesis of Ljung Box test verifies the serial correlation of growth rates and squared growth rates.

The pandemic crisis of 2020 had a major adverse impact in all the economies leading to their worst historical growth rate in May of 2020, whereas their best historical performance is recorded in different dates throughout the examined time period as Figure 2 depicts. This fact implies that economies possibly share concurrently their downside risk. Figure 1 illustrates that all economies demonstrate high positive dependence according to Kendall rank correlation coefficient. The proposed by Engle & Sheppard (2001) multivariate DCC test rejects the null hypothesis of constant correlation. Therefore, it is necessary to consider non-constant correlation across the economies. Sample correlations in Figure 3 also support that fact. The aforementioned stylized facts for growth rates suggest the need for using a skewed fat-tailed distribution, as well as allowing for asymmetric modelling of volatility and cross-sectional dependence of economies to gauge the leptokurtosis, asymmetry and cross-correlation of the data.

Kendall's Tau

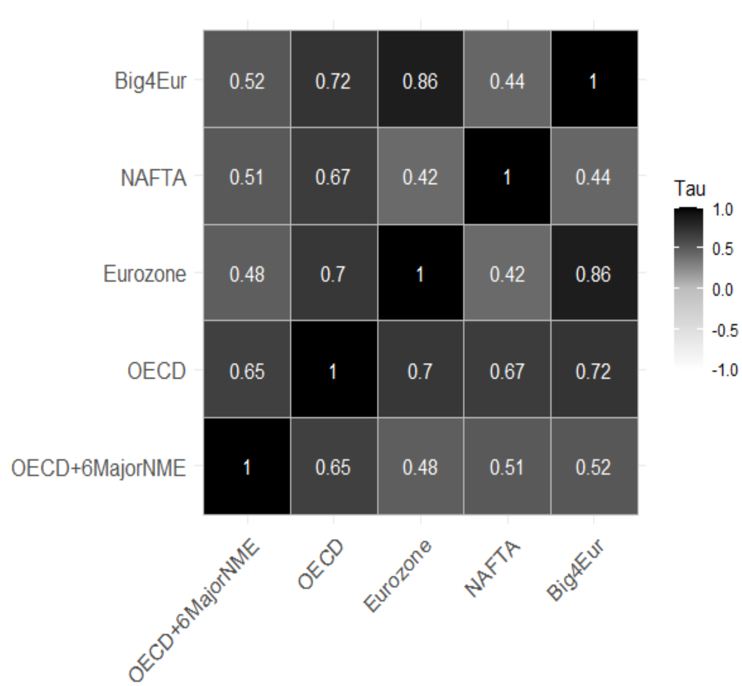


Figure 1: Dependence of the five economies based on the Kendall's Tau. The color scale fluctuates from white to grey and finally to black indicating negative, zero, and positive association respectively. Kendall rank correlation, in comparison with standard correlation measures as the Pearson's, does not proceed to any distributional or linearity assumptions but measures the dependence between two variables.

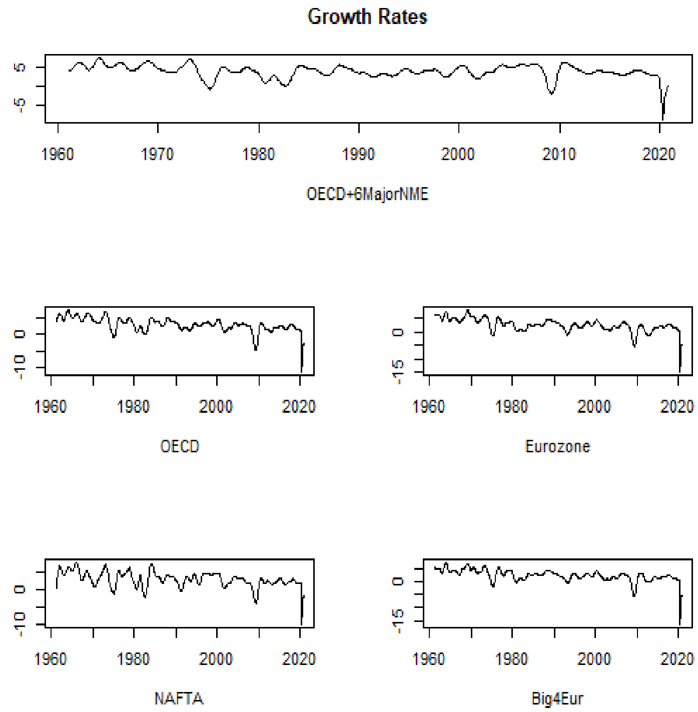


Figure 2: The growth rates of the five economies from 3/1960 to 11/2020

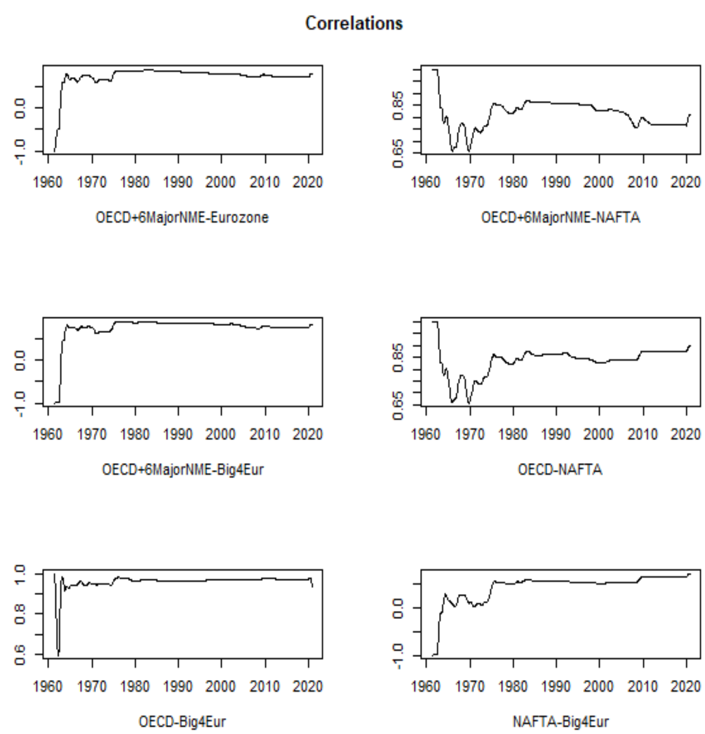


Figure 3: The sample correlations between a selection of economies.

4 Methodology

4.1 Volatility Modelling

In the ensuing GARCH models, we consider a standard GARCH(1,1), an asymmetric GARCH(1,1), a multivariate Dynamic Conditional Correlation GARCH(1,1), and a multivariate Copula-GARCH(1,1) model. For notational convenience, they will be addressed as S-GARCH, GJR-GARCH, DCC-GARCH, and C-GARCH respectively. As the S-GARCH is symmetric in the sense that positive and negative shocks have the same impact on volatility, the GJR-GARCH introduces a leverage term for modeling asymmetric volatility clustering; large negative changes are more likely to be clustered than positive changes. The multivariate DCC-GARCH of R. Engle (2002) models the individual volatilities by univariate GARCH specifications while it assumes a dynamic correlation between the economies. The C-GARCH model utilizes the univariate GARCH specifications and accounts for the cross-sectional dependence of the economies using a copula function.

Analytically, let Y_t denote the GDP growth rate of an economy at time t, then a GARCH model can be described by two equations. Firstly, the observation equation is defined as follows

$$Y_t = \mu + \sqrt{\sigma_t^2} z_t, \tag{1}$$

where μ denotes the conditional mean, σ_t^2 represents the conditional variance, and z_t are the innovations terms. The second equation describes the dynamics of volatility. The difference of the proposed distinct GARCH specifications lies on the mechanism with which they model σ_t^2.

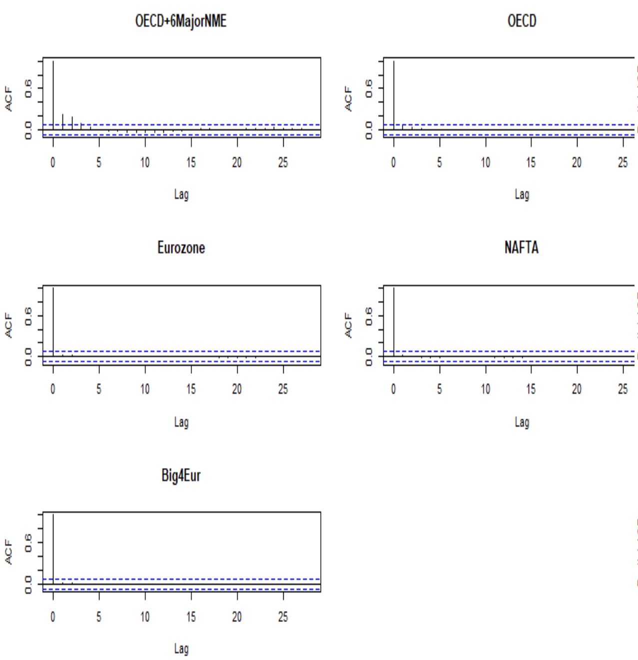
As the rejection of the null hypothesis of Jarque-Bera indicated, the data deviates from the normal distribution, presenting large skewness and kurtosis. Therefore, the distribution of the innovations z_t must assume heavy tails in order to allow for a higher likelihood of extreme events to happen and asymmetry. That is why, we choose them to follow skewed Student-t distribution. The QQ-plot in Figure 5 verifies that skewed Student-t distribution is better able to capture the tail events. QQ-plots of the rest economies are presented in the Appendix A. We specifically apply the skewed Student-t distribution introduced by Fernández & Steel (1998) and characterized by two extra parameters: the degrees of freedom (d.o.f) ν, which control the fat tails and the skewness parameter ξ, which controls the asymmetry of the extreme events.

Standard GARCH(1,1)

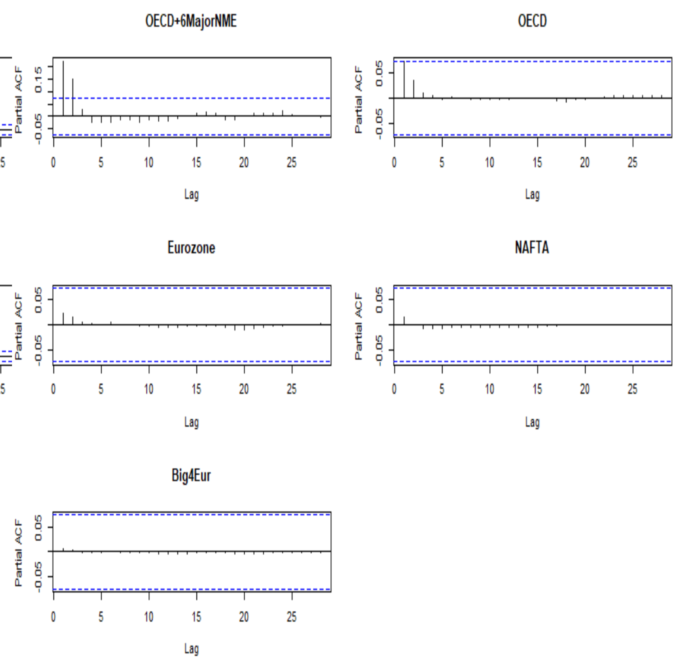
The S-GARCH models the conditional volatility as

$$\sigma_t^2 = \omega + \alpha(Y_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2, \tag{2}$$

in which ω is an offset term, the lowest value the variance can achieve in any time period, α represents how volatility reacts to new information, and β the persistence of volatility. Higher values of β indicate that large changes in the volatility will affect future volatilities for a long period. To guarantee that σ_t^2 is positive for all t the following constraints must be satisfied: $\omega > 0, \alpha \geq 0$, and $\beta \geq 0$. Assuming that the volatility is stationary, the unconditional variance in the long run is given by $\sigma^2 = \frac{\omega}{1-\alpha-\beta}$ with $\alpha + \beta < 1$. The unknown parameters $\omega, \alpha, \beta, \mu$, the d.o.f ν and the skewness parameter ξ are estimated by the maximum-likelihood. To validate the model selection, equivalently check for the adequacy of ARCH and GARCH orders, we construct the ACF and PACF plots of the resulted standardized squared residuals. The Partial autocorrelation plot shows solely the correlation of two lagged observations of a time series allegedly, X_t and X_{t-k}, after adjusting for the presence of all the in between terms $X_{t-1}, \ldots, X_{t-k-1}$. If the model is appropriate, then the plots should present no significant autocorrelations. As depicted in the Figure 4, there is serial correlation presented in the first lags of only the OECD+6MajorNME economy, indicating that the model is not adequate to capture the conditional heteroscedasticity of this economy and possibly higher orders of ARCH and GARCH could be considered.



(a) ACF Plots of the selected economies



(b) PACF Plots of the selected economies

Figure 4: The (P)ACF plots show the (partial) autocorrelation of the squared standardized residuals. The standardized residuals are estimated from fitting the entire GDP growth rates series of each economy to a standard GARCH(1,1) model.

GJR GARCH(1,1)

The GJR-GARCH accounts for a leverage effect by adding an extra term γ in volatility equation as follows

$$\sigma_t^2 = \omega + (\alpha + \gamma I_{t-1}) (Y_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2, \tag{3}$$

in which $I_{t-1} = \begin{cases} 0 & \text{if } Y_{t-1} \geq \mu \\ 1 & \text{if } Y_{t-1} < \mu \end{cases}$ and the additional term γ controls the degree of asymmetry in the conditional volatility response to the past shock. γ equal to 0 implies no asymmetric volatility, negative values of γ imply that negative shocks will increase volatility more than positive shocks, and positive values are interpreted analogously. To ensure the positivity of variance, we require $\omega > 0, \alpha > 0$, and $\beta \geq 0$. Stationarity assumption of volatility imposes for the long run that $\sigma^2 = \frac{\omega}{1-\alpha-\gamma\mathbb{P}(Y_{t-1}<\mu)-\beta}$ with $\alpha + \gamma\mathbb{P}\left(Y_{t-1} < \mu\right) + \beta < 1$. The $\mathbb{P}\left(Y_{t-1} < \mu\right)$ is directly related to the distributional assumption of innovations. Particularly, the probability is equal to $\frac{1}{1+\xi^2}$ and different from $\frac{1}{2}$ as the conditional distribution of growth rates, the skewed Student-t distribution, is asymmetric around μ. The proof is given in Appendix C along with other derivations concerning the skewed Student-t distribution. The unknown parameters ω, α, β, γ, μ, the d.o.f ν, and the skewness parameter ξ are estimated by the maximum-likelihood. Following the same reasoning with the standard GARCH, we check for the fit of GJR-GARCH with the (P)ACF plots of the squared standardized residuals. As presented in Figure 15, there is significant autocorrelation in OECD+6MajorNME and NAFTA economies. This indicates that the proposed GJR-GARCH specification has not sufficiently captured the heteroscedasticity of the respective economies.

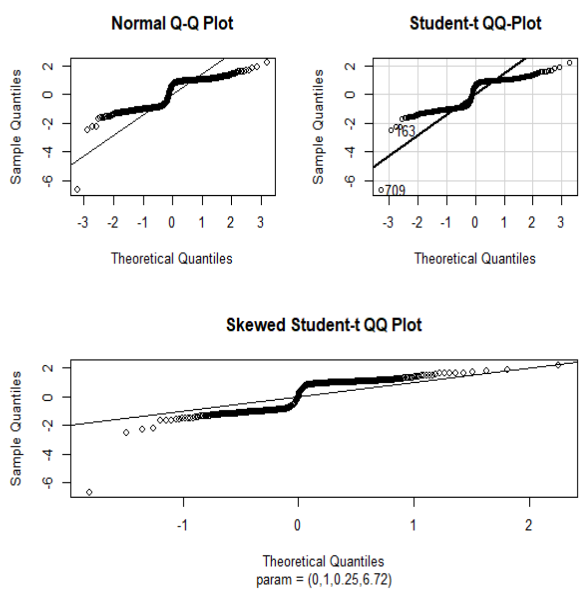


Figure 5: The QQ plots show the standardized residuals against the theoretical quantiles from Normal, Student-t and skewed Student-t distributions. The parameters in skewed Student-t are location, scale, skewness and degrees of freedom and are estimated from fitting the entire Eurozone GDP growth rates series to a standard GARCH(1,1) model by MLE. The standardized residuals are estimated from fitting the entire Eurozone GDP growth rates series to a standard GARCH(1,1) model by QMLE.

4.2 Multivariate GARCH

The suggested multivariate GARCH models, DCC and Copula, built upon the univariate GARCH specifications assume a multivariate Student-t distribution for the economies and take into account a particular dependence structure: correlation and copula, respectively. To substantiate the multivariate distribution of the residuals, we resort to a test of multivariate Student-t distribution with Monte Carlo simulations introduced by Santoso et al. (2021). The main idea of the test is to randomly generate simulation datasets which have the same dimension with our research data (in our case the standardized residuals $z_t = \frac{Y_t-\mu}{\sqrt{\sigma_t^2}}$ estimated by QMLE) and follow multivariate Student-t distribution with three different numbers of degrees of freedom. Subsequently, we compute the multivariate skewness and kurtosis of research data. Then, we compute the $p-value$ for both multivariate skewness and kurtosis by computing the percentage of skewness and kurtosis value of simulation data that is bigger than skewness and kurtosis value of research data. We reject the null hypothesis that innovations follow a multivariate Student-t distribution if $p-value < \alpha$ with α the specified significance level. Multivariate skewness and kurtosis according to Mardia (1970) are defined respectively as:

$$b_{1,p} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(\mathbf{x}_i-\overline{\mathbf{x}})'\,\mathbf{S}^{-1}\left(\mathbf{x}_j-\overline{\mathbf{x}}\right)\right]^3, \tag{4}$$

$$b_{2,p} = \frac{1}{n}\sum_{i=1}^{n}\left[(\mathbf{x}_i-\overline{\mathbf{x}})'\,\mathbf{S}^{-1}\left(\mathbf{x}_i-\overline{\mathbf{x}}\right)\right]^2, \tag{5}$$

where n is the sample size, x is a $p \times 1$ vector of random variables, and S is the biased sample covariance matrix of x equal to $\mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}\left[(\mathbf{x}_i-\overline{\mathbf{x}})\left(\mathbf{x}_i-\overline{\mathbf{x}}\right)'\right]$, and $\overline{\mathbf{x}}$ the sample mean. Table 2 shows the results. We notice that as the degrees of freedom increase, the p-values decrease. The hypothesis that the aparametric standardized residuals follow a multivariate Student-t distribution with d.o.f$= 3$ and 7 is accepted at 2% and 5% significance levels.

Table 2: Multivariate Student-t Test

Number of Simulations=10000								
			p-values				p-values	
Economy	Skewness	df=3	df=7	df=11	Kurtosis	df=3	df=7	df=11
OECD+6MajorNME	-0.119				-1.175			
OECD	-0.441				-0.713			
Eurozone	-0.558				0.804			
NAFTA	-0.0996				-0.442			
Big4Eur	-0.223				0.294			
Multi	5.680	0.949	0.026	0.000	54.307	1	0.415	0.009

This table reports the output results for Multivariate-t data test. The standardized residuals are estimated from fitting the entire GDP growth rates series of each economy to a standard GARCH(1,1) model by QMLE. All reported numbers are rounded to 3 decimals.

Dynamic Conditional Correlation GARCH

The DCC GARCH of R. Engle (2002) allows for time varying dynamics of the correlation and instead of directly estimating the covariance matrix Σ_t of the five distinct economies $Y_t^1, \ldots, Y_t^5$ at time t, it models Σ_t through a conditional correlation matrix R_t as

$$\Sigma_t = D_t R_t D_t \quad where \quad D_t = diag\{\sigma_{ii,t}\} \quad with \quad i = 1, \ldots, 5, \tag{6}$$

and the diagonal entries $\sigma_{ii,t}$ are modeled as univariate GARCH processes following equations 2 and 3. The i represents each one of the five economies. To render the conditional correlation matrix R_t positive definite, we firstly define Q_t as follows

$$Q_t = (1 - \tilde{\alpha} - \tilde{\beta})\tilde{Q} + \tilde{\alpha} z_{t-1} z'_{t-1} + \tilde{\beta} Q_{t-1} \quad with \quad z_{t-1} = D_{t-1}^{-1}(Y_{t-1} - \mu), \tag{7}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ have non-negative values and their sum must be smaller than 1 to ensure that Q_t is stationary and positive definite. $\tilde{Q}$ is the unconditional covariance matrix of the standardized residuals z_t. The conditional correlation matrix R_t can then be constructed as

$$R_t = Q_t^{*-1/2} Q_t Q_t^{*-1/2}, \tag{8}$$

where Q_t^* is the diagonal matrix of Q_t.

In order to comprehend the estimation process of the parameters in DCC models, we will firstly explain the estimation under the multivariate Gaussianity assumption which is done in the following two-step manner by decomposing the normal log-likelihood in two parts; the volatility and the correlation. In the first one, the parameters of the univariate GARCH models are estimated, and consequently the conditional variances σ_{ii}^2. Thus, the standardized disturbances are computed. In the second step, given the resulted parameters of the maximization of volatility component, the remaining parameters $\tilde{\alpha}$ and $\tilde{\beta}$ scalars, as well the $\tilde{Q}$ matrix, are estimated. Specifically, the unconditional covariance matrix $\tilde{Q}$ is estimated with the approach of correlation targeting; $\tilde{Q}$ is replaced by the unconditional correlation matrix of the standardized shocks. Nevertheless, in cases such as the multivariate Student-t, due to the presence of the shape parameter ν, the estimation must be done in one step to guarantee that the shape parameter is jointly estimated for all the GARCH-type models. Hence, to stay in the context of the two-step estimation, we follow the work of Ghalanos (2020), and estimate in the first step the five univariate GARCH models by quasi maximum likelihood, and subsequently estimate the correlation part along with the parameter ν of the joint Student-t density.

Copula-GARCH

Copula-GARCH models overcome the standard linear correlation coefficient and capture the dependence across the innovation terms z_t by specifying a function C. Algorithm 2 describes analytically the procedure of utilizing univariate GARCH specifications in the multivariate Copula-GARCH framework. That is why we focus now on modeling the joint distribution of the innovations terms across the five economies. According to

Sklar (1959) any joint distribution function may be decomposed into its marginal distributions and a copula function that completely describes the dependence between the n variables. Specifically, let $U_1, \ldots, U_n$ be random variables with CDFs $F_1, \ldots, F_n$ respectively, then:

$$F(u_1, \ldots, u_n) = C(F_1(u_1), \ldots, F_n(u_n)), \tag{9}$$

where F is the joint distribution function of $U_1, \ldots, U_n$. The copula function C is uniquely determined on $[0,1]^n$ when the margins $F_1, \ldots, F_n$ are continuous. To estimate the joint distribution in our framework, we deploy a multivariate meta-Student-t model to capture the fat tail property. A meta-Student-t model is defined according to Demarta & McNeil (2005) when arbitrary marginal distributions are coupled with a Student-t copula and these marginals are called meta-Student-t distributions. Specifically, the Student-t copula belongs to the class of elliptical copulas and is defined as:

$$C(u_1, \ldots, u_n) = t_{\theta,\nu}(t_{\nu_1}^{-1}(u_1), \ldots, t_{\nu_n}^{-1}(u_n)), \tag{10}$$

where $t_{\theta,\nu}$ is the multivariate Student-t with dependence parameter θ and degrees of freedom ν, and t^{-1} is the inverse function of the univariate Student-t. Here n is equal to 5; the total number of the economies. The copula parameters will be estimated using the residuals resulted from S-GARCH(1,1) and GJR-GARCH(1,1) processes with the skewed Student-t innovations.

As a result, the issue about the estimation of the parameters θ and ν is raised. In this paper, the proposed method to estimate them is the method of pseudo maximum likelihood. This method instead of directly utilizing the standardized residuals z_t resulted from the univariate GARCH models, it converts the variates to pseudo-observations (normalized ranked data). Consequently, we practically consider that there is no information regarding the marginals of the innovations and use their empirical CDF instead of their assumed parametric distribution. Each joint pseudo-observation $(u_{1t}, \ldots, u_{5t})$ is defined as $(\frac{Rank(z_{1t})}{n+1}, ..., \frac{Rank(z_{5t})}{n+1})$ for $t = 1, \ldots,$ end of training sample. Division by $n+1$ ensures that the maximum of $\frac{Rank(z_{kt})}{n+1}$ with $k = 1, \ldots, 5$ is $\frac{n}{n+1} < 1$. This is substantiated by the Sklar's Theorem that states that a copula is a function of uniform marginals. Hence, we need to transform the marginals of the standardized residuals to the standard uniform marginals to obtain the pseudo observations. Subsequently, we fit the pseudo sample to a five-variate Student-t distribution using the maximum likelihood, in which the total number of the unknown parameters is two: the dependence parameter θ and the degrees of freedom ν of the joint Student-t distribution. To summarize, the parametric assumptions for the marginals of the residuals are not taken into account and the copula parameters are estimated from the empirical residual observations using the pseudo-log-likelihood method.

4.3 Forecasting Joint GaR

Firstly, we repeat the joint GaR definition using mathematical notation. Analytically, the d-step ahead $(1-q)(1-p)$ joint GaR is defined as:

$$\mathbb{P}\Big(\text{ at least } \lceil qn \rceil \text{ growth rates } Y_{i,t+d} \text{ are not in } GaR_{t+d}^{joint}\Big) = p, \tag{11}$$

where $\lceil qn \rceil$ denotes the smallest integer greater than or equal to qn, n represents the total number of the economies, q the percentage of the system for which coverage is desired, $Y_{i,t+d}$ the d-month GDP growth rate of economy i, GaR_{t+d}^{joint} the d month n-dimensional GaR region and p the confidence level. Therefore, the $(1-q)(1-p)$ joint GaR contains the GDP growth rates of $(1-q)$ of the countries with $(1-p)$ probability.

No cross-sectional Information

Based on the volatility modelling, we can predict the joint GaR. As the d-month GaR does not have a closed form and the setting of all the considered GARCH processes is fully parametric, its estimation is based exclusively on Monte Carlo and Historical simulations. Algorithm 1 describes the procedure for estimating joint GaR in which no cross-sectional information is considered. We highlight step 1. Let z be a skewed Student-t distributed random variable. The absolute moments, required for deriving the central moments, are

$$M_1 = \frac{\sqrt{\frac{\nu}{\pi}}\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})} \quad \text{and} \quad M_2 = \frac{\nu}{\nu-2} \tag{12}$$

The mean and the variance are consequently defined as

$$E(z) = M_1(\xi - \xi^{-1}) \quad \text{and} \quad Var(z) = (M_2 - M_1^2)(\xi^2 + \xi^{-2}) + 2M_1^2 - M_2, \tag{13}$$

and expressed in terms of both the shape and skewness parameters. Thus, the standardization of a skewed Student-t random variable z to zero mean and unit variance is made by using the moments given above in the following manner $\frac{z-E(z)}{\sqrt{Var(z)}}$. Appendix C includes analytical derivation of absolute and central moments based on the work of Ghalanos (2020) and Kirkby et al. (2019).

All the estimated parameters result from the distinct GARCH specifications. The conducted number of simulations m is 3000, the quantiles of interest are 98% and 95%, and the coverage probabilities of the panel of the $n = 5$ economies are 60% and 20% and therefore according to definition from Equation 11, $p = 0.02, 0.05$, and $q = 0.4, 0.8$ respectively. We implement a rolling estimation window, where the end of the first training sample is the 607^{th} observation. The first five steps conclude the Monte Carlo simulation for each economy i which results in a 3000×5 simulated matrix. The last four steps describe the construction of an empirical quantile $d_p^{\lceil qn \rceil}$ that controls all the panel of the economies dependent on the desired probabilities and resulted from the standardized simulated residual matrix of step 6.

With cross-sectional Information

4.3.1 DCC-GARCH

For the prediction of GaR with DCC-GARCH, we will provide no analytical algorithm as it follows straightforward from the joint GaR with no cross-sectional information and we work similarly with the resulted residuals. Nevertheless, it will be given attention to the multi-step ahead forecast of the correlation matrix R which renders DCC computationally cumbersome. The d-step ahead evolution of Equations (7) and (8)

Algorithm 1 Joint d-month GaR($d = 3, 6, 12, 24$)

1: **Simulate** at the end of the sample d standardized innovations $z_{t+1}, \dots, z_{t+d}$ as i.i.d. skewed standardized Student-t distributed random variables with corresponding degrees of freedom $\hat{\nu}$ and skewness $\hat{\xi}$

2: **Predict** next month volatility $\hat{\sigma}_{t+1}$ utilizing the last period information Y_t and $\hat{\sigma}_t$ and the estimated parameters from GARCH ▷ Utilize equations 4 and 5

3: **Predict** Y_{t+1} as $\hat{Y}_{t+1} = \hat{\mu} + \hat{\sigma}_{t+1} z_{t+1}$

4: **Repeat** steps 2-3 d times to obtain one simulated path $\hat{Y}_{t+1}, \dots, \hat{Y}_{t+d}$ ▷ $\hat{Y}_{t+d}$ constitutes the first simulated d-month GDP growth rate symbolized as $\hat{Y}^1_{t+d}$

5: **Repeat** steps 1-4 m times for each country i and obtain a simulated sample of GDP growth rates $\hat{Y}^1_{i,t+d}, ..., \hat{Y}^m_{i,t+d}$ with $i = 1, \dots, 5$ and $m = 1, \dots, 3000$

6: Construct a **standardized simulated residual** $m \times n$ matrix as follows $\tilde{z}_{mi} = \left(\hat{Y}^m_{i,t+d} - \hat{\mu}_{i,t+d}\right) / \sqrt{\hat{\sigma}^2_{i,t+d}}$ where $\hat{\mu}_{i,t+d}$ and $\hat{\sigma}^2_{i,t+d}$ denote the mean and variance of the simulated sample $\hat{Y}^1_{i,t+d}, \dots, \hat{Y}^m_{i,t+d}$

7: Define $U_m{}^{\lceil qn \rceil}$ as the qn-smallest element of $\{\tilde{z}_{mi}\}^n_{i=1}$

8: Compute the multiplier $d_p^{\lceil qn \rceil}$ as the p^{th} empirical quantile of the statistics $U_1{}^{\lceil qn \rceil}, \dots, U_m{}^{\lceil qn \rceil}$

9: Construct the $(1-q)(1-p)$ joint GaR region as $GaR^{joint}_{t+d} = \left(GaR^{joint}_{i,t+d}, \infty\right) \times \cdots \times \left(GaR^{joint}_{n,t+d}, \infty\right)$ where $GaR^{joint}_{i,t+d} = \hat{\mu}_{i,t+d} + d_p^{\lceil qn \rceil} \sqrt{\sigma^2_{i,t+d}}$

are as follows:

$$Q_{t+d} = (1 - \tilde{\alpha} - \tilde{\beta})\tilde{Q} + \tilde{\alpha} E_t(z_{t+d-1} z'_{t+d-1}) + \tilde{\beta} Q_{t+d-1}, \tag{14}$$

$$R_{t+d} = Q^{*-1/2}_{t+d} Q_{t+d} Q^{*-1/2}_{t+d}, \tag{15}$$

with $E(z_{t+d-1} z'_{t+d-1} | I_{t+d-2}) = R_{t+d-1}$, and we set the unconditional matrix of the standardized residuals $\tilde{Q}$ approximately equal to the correlation matrix R ($\tilde{Q} \approx R$) and $E_t(Q_{d+1}) = E_t(R_{d+1})$ following the findings of Engle & Sheppard (2001) that this approximation provides the least bias.

The main difference from Algorithm 1 is that we employ for the simulations the standardized residuals resulted from the correlation matrix R with degrees of freedom ν of the multivariate Student-t. Hence, the skewed Student-t marginals of the innovations are not directly utilized for the standardization as in Algorithm 1. The individual degrees of freedom and skewness parameters are estimated though in the volatility component of the likelihood and participate in the estimation of the correlation matrix R and the shape parameter ν of the joint distribution. The standardization of residuals is done by rescaling with the factor $\sqrt{\frac{\nu}{\nu-2}}$ to obtain variance equal to 1.

4.3.2 Copula-GARCH

As already stated, copulas take into account a different dependence structure than the standard linear correlation. Algorithm 2 presents the procedure analytically. In comparison with DCC, besides the degrees of the multivariate Student-t distribution, copula introduces the extra parameter θ for the simulations and utilizes the central moments of skewed Student-t to transform the marginals of the simulated five-variate observations to standardized skewed Student-t innovations. The main difference from Algorithm 1 lies in steps 2-4, where we simulate the residuals from a multivariate Student-t distribution with dependence parameter θ and degrees of freedom ν. Steps 2-4 from Algorithm 2 are "analogous" to the first step of Algorithm 1, while the rest methodology is similar. In step 4, the copula remains invariant under the transformation of the marginals as Demarta & McNeil (2005) remark. The same property of invariance holds for the parameter θ since the dependence in the joint Student-t distribution is similar with the dependence in the copula of the multivariate Student-t distribution after the transformation. Equivalently, copula and marginals contain information that are mutually exclusive. This substantiates the independence copula when θ reaches its lower bound 0. Comonotonicity, the perfect positive dependence between the economies, is achieved at the upper bound of $\theta = 1$.

Algorithm 2 Copula-GARCH Joint d-month GaR($d = 3, 6, 12, 24$)

1: Estimate at the end of the sample the GARCH-skewed Student t model for each economy
2: **Fit** a Student-t copula to the standardized residuals
3: **Simulate** at the end of the sample d five-variate observations from the multivariate Student-t copula
4: **Transform** their marginals to standardized skewed Student-t innovations using their individual shape and skewness parameters as estimated by the univariate GARCH models
5: Repeat steps 2-9 from Algorithm 1

In summary, the joint GaR in all models is based on the simulated residual matrix resulted by the two univariate GARCH models and their multivariate extensions. The empirical quantile $d_p^{\lceil qn \rceil}$ constructed by taking the qn-smallest element of each row of the simulated residual matrix is the main factor that renders the predicted GaR region joint and distinct from the standard established approaches to forecast univariate GaR.

4.4 Backtesting

For the backtesting of joint forecasts, we will examine two approaches that examine the accuracy of the GaR models, namely the violation test of Kupiec (1995) and the Dynamic Quantile test of R. F. Engle & Manganelli (2004). We also report the empirical average coverage and average length of joint GaR forecasts. As we used a rolling estimation window, we recursively forecast and evaluate the joint GaR forecasts starting from the 610^{th} observation. Each estimation window dependent on the forecast horizon d will produce $T = 717 - 610 + 1 - d$ predictions.

Kupiec Test

The violation test of Kupiec Kupiec (1995) falls in the class of the unconditional coverage tests and examines whether the observed failure rate is equal with the one suggested by the model. In general, a violation occurs when the loss is more than the predicted GaR. However, in the joint GaR context the indicator function is more complicated. If the joint region was built upon the assumption that at least $\lceil qn \rceil$ growth rates are not in the joint GaR, then the indicator violation function is defined as $I_{\{\text{at least } \lceil qn \rceil \text{ growth rates } Y_{i,t} \text{ are below the } G\hat{a}R_{i,t}^{joint}\}}$ with $i = 1, \ldots, 5$. Analytically, the test is based on the binomial distribution approach and it employs a likelihood ratio to test whether the probability of exceptions concedes with the probability p implied by the GaR confidence level. If the data suggests that the probability of exceptions is different than p, the GaR model is rejected. The test statistic is:

$$LR = -2\log\left(\frac{(1-p)^{N-x}p^x}{\left(1-\frac{x}{N}\right)^{N-x}\left(\frac{x}{N}\right)^x}\right) \xrightarrow{d} \chi^2(1), \tag{16}$$

where x is the number of violations, and N the number of observations. The likelihood in the numerator results from the null hypothesis, while the one in the denominator is implied by the data. The statistic is asymptotically distributed as a chi-square variable with 1 degree of freedom $\chi^2(1)$.

In this context, we also report the average length of joint GaR forecasts following the work of Brownlees & Souza (2021) which is defined as

$$\hat{L}^{joint} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\hat{Q}_{0.99}(Y_i) - G\hat{a}R_{i,t}^{joint}\right), \tag{17}$$

where T the number of predictions, n the number of economies, and $\hat{Q}_{0.99}(Y_i)$ denotes the unconditional 0.99 empirical quantile of the i-th series estimated on the entire sample. The average length is particularly useful to check whether the GaR predictions are unnecessarily low. Hence, if the other criteria are equal, GaR forecasts with a smaller length are typically preferred. However, this does not mean certainly that $G\hat{a}R_{i,t}^{joint}$ must be closer to the empirical quantile $\hat{Q}_{0.99}(Y_i)$. Lastly, we define the empirical average of the joint GaR based on a success indicator function I as

$$\hat{C}^{joint} = \frac{1}{T}\sum_{t=1}^{T} I_{\left\{\text{at most } \lceil qn \rceil \text{ growth rates } Y_{i,t} \text{ are above the } G\hat{a}R_{i,t}^{joint}\right\}}, \tag{18}$$

Accurate GaR forecasts have an empirical coverage close to the nominal coverage $(1-p)$. It is straightforward that the empirical coverage is subpart of the Kupiec's unconditional coverage test. Nevertheless, its report facilitates the comprehension of the risk's under or over estimation. If the empirical coverage is smaller than $1-p$, then the risk is underestimated, and conversely.

Dynamic Quantile Test

However, the unconditional coverage property of Kupiec test does not give any information regarding the independence of violations. The independence property is essential for risk measures since they have to adopt to new information implied by the evolution in the dynamics of returns. The Dynamic Quantile (DQ) test of R. F. Engle & Manganelli (2004) belongs to the family of the standard statistical accuracy tests and examines whether the hit sequence is optimal with respect to the information set generated by the hit sequence itself.

Firstly, the notion of a hit sequence must be explained. For joint GaR, a hit function is defined as follows:

$$Hit_t(p) = I_{\left\{\text{at least } \lceil qn \rceil \text{ growth rates } Y_{i,t} \text{ are below the } \hat{GaR}^{joint}_{i,t}\right\}} - p \quad with \quad i = 1, .., 5 \tag{19}$$

The general form of the Dynamic Quantile test in our framework is given by an AR(K) process as follows:

$$Hit_t(p) = \beta_0 + \sum_{k=1}^{K} \beta_k Hit_{t-k}(p) + \varepsilon_t \tag{20}$$

where ε_t is the error term and follows a discrete *i.i.d.* process The null hypothesis of the above linear regression model tests jointly whether GaR forecasts have correct unconditional coverage, and whether the present violations of the GaR are not correlated with the past violations. This is equivalent to testing the nullity of the coefficients $\beta_k, \forall k = 0, \ldots, K$:

$$H_0 : \beta_0 = \ldots = \beta_k = 0, \quad \forall k = 0, \ldots, K$$

The test is based on a Wald statistic defined as:

$$W = \frac{\hat{\beta}' \left(\sum_{t-1} \mathbf{x}_{t-1} \mathbf{x}'_{t-1}\right) \hat{\beta}}{p(1-p)} \xrightarrow{d} \chi^2(k+1), \tag{21}$$

where $\hat{\beta} = (\beta_0, \beta_1, \cdots, \beta_k)$, $x_{t-1} = (1, Hit_{t-1}, \cdots, Hit_{t-k})$, and $k+1$ the number of the total variables for which we test the joint nullity. The above test, if we consider no hit lags and test alone the nullity of the coefficient β_0, is equivalent to the Kupiec test. Consequently, the GaR forecasts are optimal under the null hypothesis, if the variable $Hit_t(p)$ is uncorrelated with its own lagged values, and its expected value must be equal to zero. Thus, the DQ test examines whether the GaR predictions satisfy some basic requirements that every good quantile estimator must satisfy such as unbiasedness and independent hits. In this paper, we firstly test only for the correct unconditional coverage, and subsequently for the joint hypothesis of unbiasedness and independence of hits considering 4 lags, symbolized respectively as DQ(0) and DQ(4).

5 Results

5.1 In-Sample GARCH analysis

Table 3 reports for each economy the estimated coefficients of all the proposed GARCH models under the distributional assumptions and estimated methods that have been explained previously. All $\beta's$, except the one corresponding to OECD+6MajorNME, are equal to 0 and statistically insignificant. Thus, the volatility is not persistent. Possibly, the standout of OECD+6MajorNME economy in the S-GARCH model lies in the fact that includes both OECD and non-OECD countries such as the China, Russian Federation, the USA, and Germany rendering it more susceptible to volatility shocks of past realizations. As usually, under the stationarity assumption, lower values of β are followed by higher values of α which represent the adjustment to the past shocks of squared growth rates. All estimated $\alpha's$ are statistically significant and point out the volatility clustering. This behavioral switching is a partial repercussion of the heavy tails affirmed by the estimated degrees of freedom of the residuals, except in the case of OECD+6MajorNME. To summarize, only the ARCH term has a considerable effect on the conditional variance of future signifying spiky economies.

All values of γ are positive and indicate that negative shocks will increase the volatility more than positive shocks. However, not all $\gamma's$ are significant in the 5% confidence level. Specifically, in the cases of OECD and Eurozone economies, the assumption of the presence of a leverage effect is not validated. In the multivariate extension of univariate GARCH models, the shape parameter is also significant and points towards fat tails. The significance of $\tilde{\alpha'}s$ and $\tilde{\beta'}s$ in DCC confirm the account of a non-constant correlation model. These scalars explain how the correlations are evolving over time. Hence, the relatively higher values of $\tilde{\alpha'}s$ compared with the $\tilde{\beta'}s$, in both standard and asymmetric GARCH variants, provide a robust contribution of the realized correlation matrix from the last period and a moderate contribution of the "long-run" correlation matrix, respectively. We lastly find relatively high dependence parameters θ, 0.768 and 0.614, from both multivariate Student-t Copula-GARCH models, as expected since all economies are positively associated. Therefore, from a regulatory point of view, a negative major event such as the mortgage crisis of 2008 or the pandemic crisis of 2020, will influence similarly all the economies.

5.2 Out-of-Sample analysis

Tables 4, 5, 6, 7 report the p-values of the Kupiec and the two variants of Dynamic Quantile tests, along with the average empirical joint coverage and average length for each forecast horizon and forecasting method for joint GaR. We consider two fractions of economies coverage q equal to 0.4 and 0.8 and two coverage probabilities p equal to 0.02 and 0.05. In the Appendix B, we present the joint-GaR forecasts for other confidence levels for the same coverage levels of economies. The analysis of the results has a threefold purpose: i) to point out whether the GJR GARCH provides superior forecasts to the standard GARCH, ii) to argue for the predictive power of the models with the increase of forecast horizon, and iii) to substantiate the main argument of this paper that cross-sectional information improves the joint GaR forecasts.

Before proceeding to the analytical results, we clarify that in the cases in which the unconditional test of Kupiec rejects the null hypothesis, while DQ(0) does not, is due to the low power of the former one in finite samples. Additionally, when the average empirical joint coverage is 100 or 0, there is no exception or only violations respectively, and subsequently the test statistics are not defined. Thus, no p-values are provided. The criteria of length and coverage are mainly reported to contend for empirical properties of GaR forecasts such as unnecessarily low predictions or over and under estimation, and they do not comprise individually adequate measures to assess the forecasts. However, if the test of Kupiec or the variants of the Dynamic Quantile test that examine statistical properties of the predictions provide equivalent conclusions, then the average length or empirical coverage are utilized to determine the best model.

Algorithms 1 and 2 suffer from extremely high values of average length. In both these Algorithms, we standardized the resulted residuals from the GARCH specifications using the central moments of the skewed Student-t distribution, whereas in DCC we standardized them with the usage of the estimated shape parameter of the multivariate Student-t distribution. Particularly, between DCC and Copula GARCH models, the latter one burdens the simulation procedure with the extra dependence parameter θ. The marginals of the residuals have not been utilized in the standardization process of DCC as in Algorithms 1 and 2, and the resulted average length of the forecasts is moderately low. In Algorithm 2, we report even higher average length of the predictions in comparison with the Algorithm 1 due to the additional parameter of dependence introduced in the simulation process.

For illustrative purposes, Figures 6 and 7 compare the GaR predictions (red line) resulted from the standard univariate and multivariate GARCH processes for each economy with the actual GDP growth rates (black line) at confidence level $p = 0.05$, coverage fraction of economies $q = 0.4$ and forecasting horizons $d = 3, 24$. We notice that average length rapidly grows in all three models with the concurrent increase of time horizon. Nevertheless, in the case of DCC the rise is more modest since it generates lower GaR predictions, but fairly closer to the real GDP growth rates. It is noteworthy that our work is based on monthly GDP growth rates, and we thus produce monthly forecasts, while most of the work done in the literature related with macroeconomic variables is built upon quarterly data. Therefore, despite the fact that 24-months represent the same time horizon as 8-quarters, the former case is more susceptible to errors due to the longer forecasting time horizon.

Standard vs GJR GARCH

The standard GARCH underperforms when compared against its asymmetric counterpart, except for the setting of multivariate DCC. Analytically, DCC-S-GARCH produces d-month forecasts with average empirical coverage closer to the nominal at all p and q levels, but with higher average lengths. However, the differences in lengths belong to the same order of magnitude. On the contrary, the univariate and multivariate Copula asymmetric GARCH specifications ameliorate the joint GaR predictions, since we report average empirical coverage closer to the nominal at all p and q levels with the cost of higher average lengths. The Kupiec and DQ(0) tests, when defined, can not support the supremacy of either GARCH variant. Both models reject the hypothesis of independent violations. To better comprehend the performance switching of standard and asymmetric GARCH processes, we refer to Figures 8, 9, and 10. The conditional volatility from GJR-GARCH presents a stable pattern, while the one from S-GARCH abruptly increases. This is substantiated by the standard GARCH coefficients that marginally sum to 1 resulting in an exploding behavior of the volatility. In the case of OECD and NAFTA economies, the plots are similar since the coefficients resulted from both GARCH specifications approach the same sum. The superiority of asymmetric variant of Copula-GARCH model is clearly visible in the subplot 10a in which C-S-GARCH generates slightly more extreme bivariate observations in the lower tail. The worsened behavior of DCC-GJR-GARCH in comparison with its standard counterpart is due to estimating correlations between economies as negative, when in fact they are mainly positively associated.

Long-term predictive ability

The predictions of joint GaR exacerbate with the rise of the forecast horizon. Hence, it is essential to highlight in our context, the magnitude of such deteriorating performance and the corresponding implications for policymakers. The average length distinguishably increases with the concurrent increase of forecast horizon, while the average empirical coverage diverges even further from the nominal rate. This is due to the distributional misspecification errors resulted from the simulated residuals. The null hypothesis of the unconditional test of Kupiec, along with its analogous version DQ(0), when defined, is never rejected in the short-term horizon of 3-months. Nevertheless, independent violations are never observable. From a monitoring perspective, policymakers can accurately predict the joint GaR in the short term and proceed to relevant macroprudential regulations in the economy. As a final remark for the predictability power of the algorithms in terms of time horizon, we highlight that the thesis is based on monthly GDP growth rates and corresponding monthly forecasts, in contrast to the established macroeconomic research methods that engage with quarterly data. In terms of forecast horizon, our monthly predictions are equivalent to d-quarters ahead, but they are more susceptible to compounding errors due to model misspecifications.

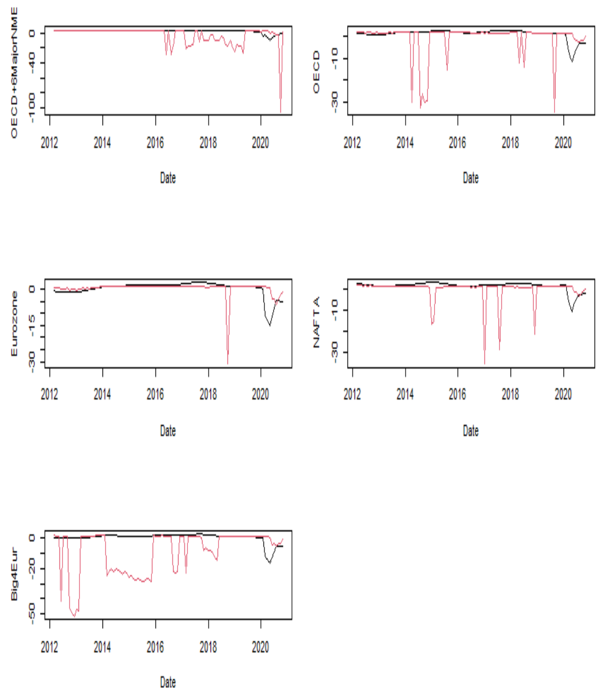
Without vs With Cross Sectional Information

Algorithm 1 does not incorporate cross-sectional information across the panel of selected economies, underperforming in comparison with the multivariate GARCH processes. This makes evident that in reality different economies affect each other, and consideration of no structure of dependence (dynamic correlation or copula function) worsens the predictions of macroeconomic variables. We distinguish two important aspects of the results that support the underperformance of Algorithm 1. Firstly, the unconditional test of Kupiec and its analogous version DQ(0) reject the null hypothesis in all the cases of d-month forecasts at all p and q levels. Secondly, as can be shown by Tables 4, 5, 6, and 7, DCC and Copula substantially outperform Algorithm 1 in all the respective backtesting criteria. Comparison of DCC and Copula is more challenging and no conclusive evidence can be obtained. Analytically, DCC consistently produces smaller average length in contrast to the excessively high length of Copula models. However, Copulas persistently generate forecasts with empirical coverage closer to the nominal in all d-month horizons forecasts, whereas in DCC the corresponding coverage substantially decreases. Particularly, C-S-GARCH succeeds to not reject the null hypothesis for the 24-month predictions at the higher fraction 0.8 at both probability levels. However, the reported average lengths are extremely high; a fact supported by subplot 7c. For this reason, the aforementioned success of independent hits is not ideal, since unnecessarily low joint GaR predictions are produced.

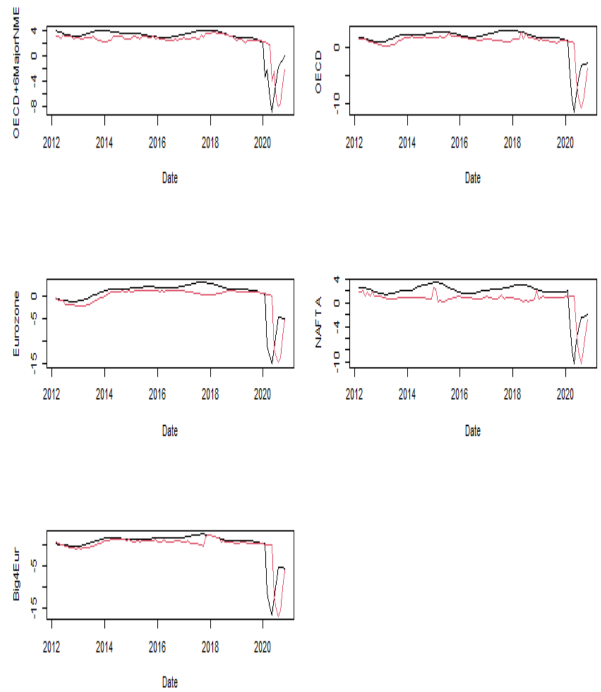
Table 3: In-sample GARCH analysis.

Model \ Economies	OECD+6MajorNME	OECD	Eurozone	NAFTA	Big4Eur
S-GARCH(1,1)					
μ	3.579	2.095	1.710	2.000	1.280
ω	0.028	0.115	0.092	0.219	0.099
α	0.825	0.999	0.999	0.999	0.999
	{0}	{0}	{0}	{0}	{0.007}
β	0.174	0.000	0.000	0.000	0.000
	{0}	{1}	{1}	{1}	{1}
ξ	0.399	0.269	0.255	0.320	0.290
	{0}	{0}	{0}	{0}	{0.015}
ν	58.143	6.200	6.727	6.781	6.850
	{0.580}	{0.035}	{0}	{0}	{0.010}
GJR–GARCH(1,1)					
μ	4.420	2.095	1.710	4.389	3.126
ω	0.056	0.115	0.092	0.228	0.090
α	0.822	1.000	1.000	0.669	0.810
	{0}	{0}	{0}	{0}	{0}
β	0.000	0.000	0.000	0.000	0.000
	{1}	{1}	{1}	{1}	{1}
γ	0.298	-0.006	-0.003	0.540	0.316
	{0.013}	{0.974}	{0.986}	{0.100}	{0.011}
ξ	2.874	0.269	0.255	3.310	3.327
	{0}	{0}	{0}	{0}	{0}
ν	6.946	6.185	6.709	4.774	5.994
	{0}	{0}	{0}	{0.004}	{0}
DCC-S-GARCH(1,1)					
ν	6.795	$\tilde{\alpha}$	0.782	$\tilde{\beta}$	0.188
	{0}		{0}		{0}
DCC-GJR-GARCH(1,1)					
ν	11.266	$\tilde{\alpha}$	0.718	$\tilde{\beta}$	0.255
	{0}		{0}		{0.002}
C-S-GARCH(1,1)					
ν	3.080	θ	0.768		
	{0.009}				
C-GJR-GARCH(1,1)					
ν	11.439	θ	0.614		
	{0.012}				

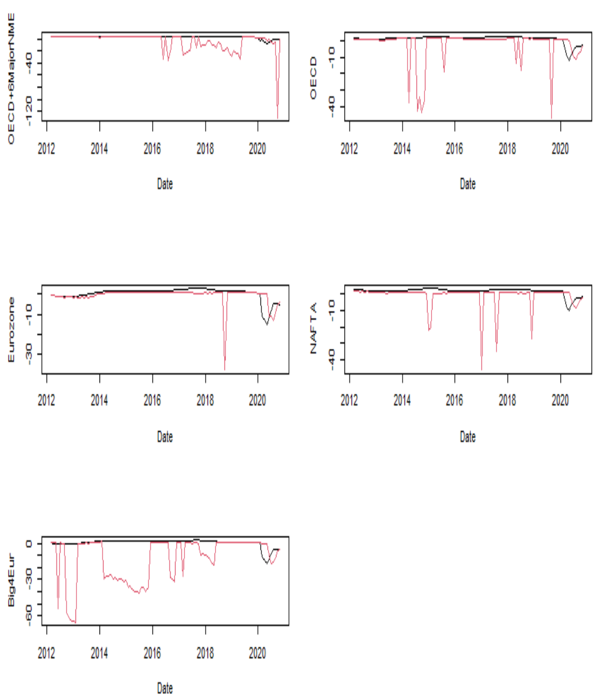
Note. This table reports the optimal parameters of GARCH processes as estimated from fitting the entire sample series to the S-GARCH and GJR-GARCH, and to their multivariate extensions, DCC and Copula . The curly braces include the p-values for the null hypothesis of significance of the estimated parameters. All reported numbers are rounded to 3 decimals.



(a) Standard GARCH joint GaR predictions vs real GDP growth rates

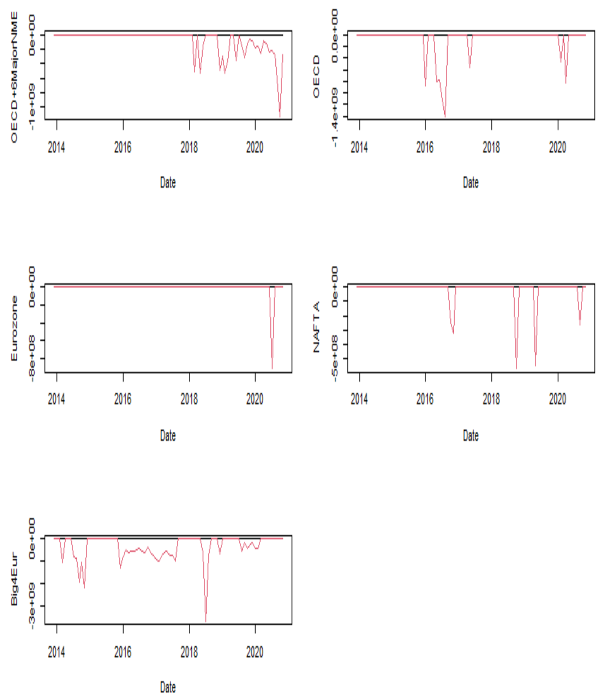


(b) Standard DCC joint GaR predictions vs real GDP growth rates

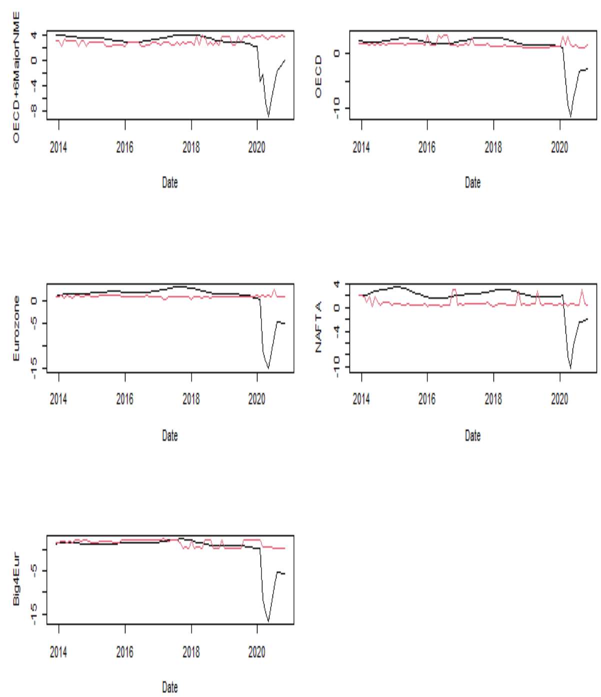


(c) Standard Copula-GARCH joint GaR predictions vs real GDP growth rates

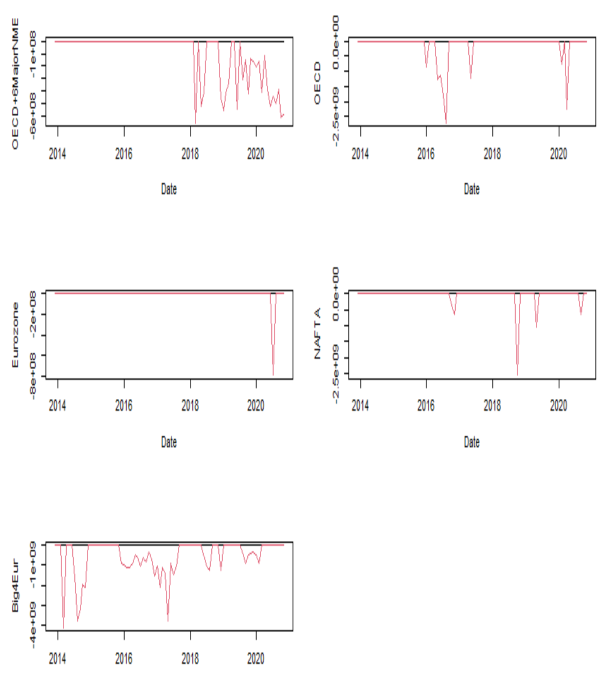
Figure 6: Plots of joint GaR predictions (red line) and real GDP growth rates (black line) for confidence level $p = 0.05$, economies coverage $q = 0.4$ and forecasting horizon $d = 3$



(a) Standard GARCH joint GaR predictions vs real GDP growth rates



(b) Standard DCC joint GaR predictions vs real GDP growth rates



(c) Standard Copula-GARCH joint GaR predictions vs Real GDP growth rates

Figure 7: Plots of joint GaR predictions (red line) and real GDP growth rates (black line) for confidence level $p = 0.05$, economies coverage $q = 0.4$ and forecasting horizon $d = 24$

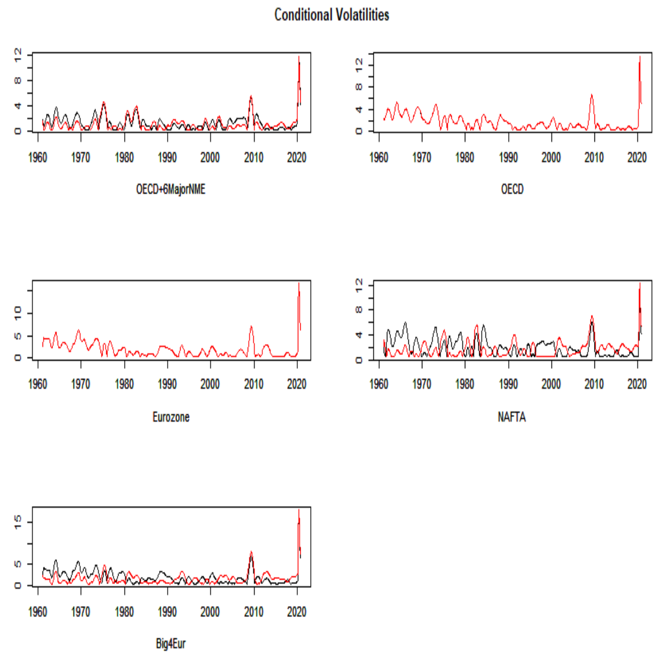


Figure 8: The plot shows the conditional volatilities as resulted from fitting the entire sample series to the considered standard (black line) and asymmetric (red line) GARCH(1,1).

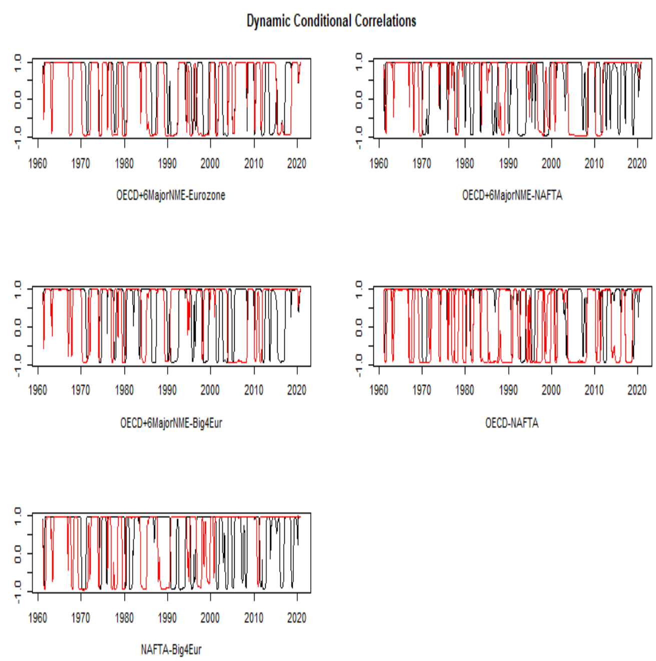
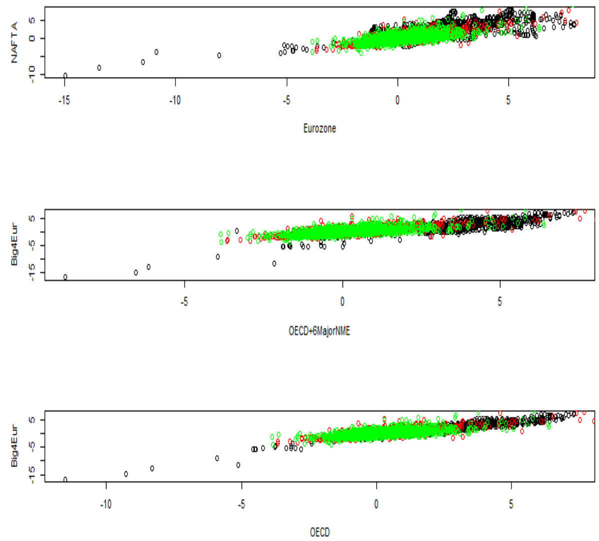
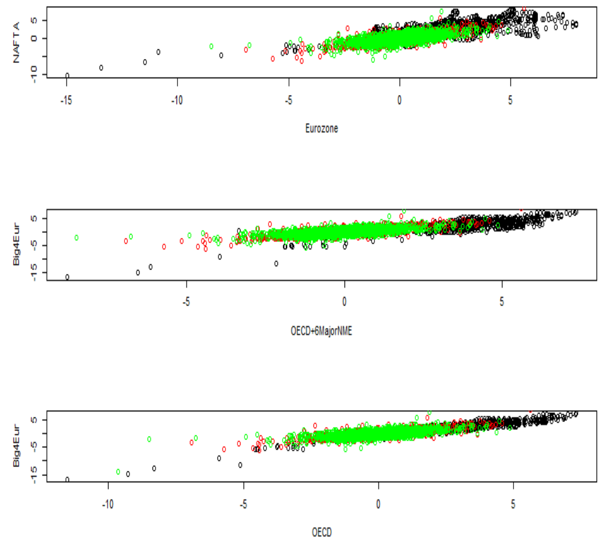


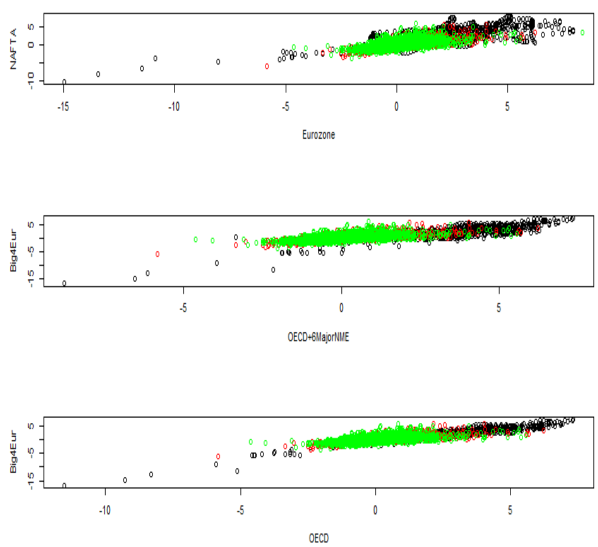
Figure 9: The plot shows the dynamic conditional correlations between a selection of economies. The black and red lines represent respectively the dynamic correlations as resulted from fitting the entire sample series to a multivariate DCC-S-GARCH and DCC-GJR-GARCH.



(a) Simulated bivariate observations from Student-t copula with skewed Student-t margins vs real GDP growth rates of the respective economies



(b) Simulated bivariate observations from Student-t copula with Student-t margins vs real GDP growth rates of the respective economies



(c) Simulated bivariate observations from Normal copula with skewed Student-t margins vs Real GDP growth rates of the respective economies

Figure 10: Plots of simulated bivariate observations from the multivariate C-S-GARCH(red dots), C-GJR-GARCH(green dots), and real GDP growth rates (black dots). The conducted number of simulations is 3000 with the estimated parameters from the in-sample GARCH analysis $(\theta, \nu) = (0.768, 3.080)$ and$(\theta, \nu) = (0.614, 11, 439)$ for the standard and asymmetric GARCH models respectively.

Table 4: Joint GaR forecast evaluation with $p = 0.02$ and $q = 0.4$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-0.918	75.238	0.000	0.000
	GJR-GARCH(1,1)	0.002	32.292	92.381	0.030	0.000
	DCC-S-GARCH(1,1)	0.555	-3.437	97.143	0.598	0.000
	DCC-GJR-GARCH(1,1)	0.555	-2.487	97.143	0.598	0.000
	C-S-GARCH(1,1)	0.555	2.142	97.143	0.598	0.000
	C-GJR-GARCH(1,1)	0.238	44.909	96.191	0.333	0.000
6	S-GARCH(1,1)	0.000	47.526	67.647	0.000	0.000
	GJR-GARCH(1,1)	0.000	444.379	88.235	0.002	0.000
	DCC-S-GARCH(1,1)	0.023	-4.354	94.118	0.096	0.000
	DCC-GJR-GARCH(1,1)	0.000	-4.986	77.451	0.000	0.000
	C-S-GARCH(1,1)	0.000	69.172	87.255	0.001	0.000
	C-GJR-GARCH(1,1)	0.000	638.623	91.177	0.015	0.000
12	S-GARCH(1,1)	0.000	7086.548	60.417	0.000	0.000
	GJR-GARCH(1,1)	0.000	51669.070	83.333	0.000	0.000
	DCC-S-GARCH(1,1)	0.000	-4.970	75	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.290	17.708	0.000	0.000
	C-S-GARCH(1,1)	0.000	12078.320	83.333	0.000	0.000
	C-GJR-GARCH(1,1)	0.000	80096.920	85.417	0.000	0.000
24	S-GARCH(1,1)	0.000	153882489	71.429	0.000	0.000
	GJR-GARCH(1,1)	0.000	641654209	82.143	0.000	0.000
	DCC-S-GARCH(1,1)	0.000	-5.083	71.429	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.481	14.286	0.000	0.000
	C-S-GARCH(1,1)	0.000	266645677	86.905	0.003	0.000
	C-GJR-GARCH(1,1)	0.000	1147168533	88.095	0.005	0.000

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

Table 5: Joint GaR forecast evaluation with $p = 0.02$ and $q = 0.8$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-3.752	59.048	0.000	0.000
	GJR-GARCH(1,1)	-	13.884	100	-	-
	DCC-S-GARCH(1,1)	0.026	-4.701	94.286	0.101	0.000
	DCC-GJR-GARCH(1,1)	0.002	-3.991	92.381	0.030	0.000
	C-S-GARCH(1,1)	0.002	-1.524	92.381	0.030	0.000
	C-GJR-GARCH(1,1)	-	22.591	100	-	-
6	S-GARCH(1,1)	0.000	20.919	59.804	0.000	0.000
	GJR-GARCH(1,1)	-	189.987	100	-	-
	DCC-S-GARCH(1,1)	0.000	-5.516	86.275	0.001	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.210	49.020	0.000	0.000
	C-S-GARCH(1,1)	0.000	36.009	82.353	0.000	0.000
	C-GJR-GARCH(1,1)	-	303.225	100	-	-
12	S-GARCH(1,1)	0.000	2860.446	70.833	0.000	0.000
	GJR-GARCH(1,1)	-	14813.230	100	-	-
	DCC-S-GARCH(1,1)	0.000	-5.802	78.125	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.959	13.542	0.000	0.000
	C-S-GARCH(1,1)	0.000	5867.204	85.417	0.001	0.000
	C-GJR-GARCH(1,1)	-	28587.600	100	-	-
24	S-GARCH(1,1)	0.000	31902429	71.429	0.000	0.000
	GJR-GARCH(1,1)	-	88055804	100	-	-
	DCC-S-GARCH(1,1)	0.000	-5.825	85.714	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.144	9.524	0.000	0.000
	C-S-GARCH(1,1)	0.009	99222677	92.857	0.067	0.000
	C-GJR-GARCH(1,1)	-	230153302	100	-	-

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

Table 6: Joint GaR forecast evaluation with $p = 0.05$ and $q = 0.4$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-1.882	55.238	0.000	0.000
	GJR-GARCH(1,1)	0.000	26.123	80.952	0.000	0.000
	DCC-S-GARCH(1,1)	0.126	-4.751	91.429	0.191	0.000
	DCC-GJR-GARCH(1,1)	0.251	-4.830	92.381	0.312	0.000
	C-S-GARCH(1,1)	0.251	-0.065	92.381	0.312	0.002
	C-GJR-GARCH(1,1)	0.251	34.454	92.381	0.312	0.000
6	S-GARCH(1,1)	0.000	38.615	52.941	0.000	0.000
	GJR-GARCH(1,1)	0.000	354.026	78.431	0.000	0.000
	DCC-S-GARCH(1,1)	0.000	-5.406	67.647	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.384	11.765	0.000	0.000
	C-S-GARCH(1,1)	0.000	51.866	77.451	0.000	0.000
	C-GJR-GARCH(1,1)	0.007	468.338	88.235	0.034	0.000
12	S-GARCH(1,1)	0.000	5681.884	45.833	0.000	0.000
	GJR-GARCH(1,1)	0.000	37639.130	78.125	0.000	0.000
	DCC-S-GARCH(1,1)	0.000	-5.666	53.125	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-6.903	0	-	-
	C-S-GARCH(1,1)	0.000	8817.945	77.083	0.000	0.000
	C-GJR-GARCH(1,1)	0.000	52545.610	82.292	0.001	0.000
24	S-GARCH(1,1)	0.000	113218951	50	0.000	0.000
	GJR-GARCH(1,1)	0.000	397525472	82.143	0.002	0.000
	DCC-S-GARCH(1,1)	0.402	-5.731	69.048	0.333	0.000
	DCC-GJR-GARCH(1,1)	-	-7.040	0	-	-
	C-S-GARCH(1,1)	0.000	181561159	83.333	0.004	0.000
	C-GJR-GARCH(1,1)	0.001	630391409	85.714	0.015	0.000

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

Table 7: Joint GaR forecast evaluation with $p = 0.05$ and $q = 0.8$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-4.210	52.381	0.000	0.000
	GJR-GARCH(1,1)	-	10.956	100	-	-
	DCC-S-GARCH(1,1)	0.000	-5.597	84.762	0.004	0.000
	DCC-GJR-GARCH(1,1)	0.000	-5.852	50.476	0.000	0.000
	C-S-GARCH(1,1)	0.000	-2.674	79.048	0.000	0.000
	C-GJR-GARCH(1,1)	-	10.956	100	-	-
6	S-GARCH(1,1)	0.000	16.347	51.961	0.000	0.000
	GJR-GARCH(1,1)	-	154.064	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.108	65.686	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.052	8.824	0.000	0.000
	C-S-GARCH(1,1)	0.000	27.906	79.412	0.000	0.000
	C-GJR-GARCH(1,1)	0.000	229.108	100	0.000	0.000
12	S-GARCH(1,1)	0.000	2158.608	56.250	0.000	0.000
	GJR-GARCH(1,1)	-	10696.200	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.203	66.667	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.362	6.250	0.000	0.001
	C-S-GARCH(1,1)	0.000	4296.103	84.375	0.000	0.000
	C-GJR-GARCH(1,1)	-	18993.280	100	-	-
24	S-GARCH(1,1)	0.000	15128055	58.333	0.000	0.000
	GJR-GARCH(1,1)	-	49191061	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.227	83.333	0.004	0.000
	DCC-GJR-GARCH(1,1)	-	-7.502	0	-	-
	C-S-GARCH(1,1)	0.036	62428704	89.286	0.090	0.000
	C-GJR-GARCH(1,1)	-	116177088	100	-	-

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

6 Conclusion

In this paper, we backtest the out-of-sample joint GaR predictions for a panel of 5 economies. We rely on the standard and asymmetric univariate GARCH models and their multivariate extensions, DCC and Copula. Algorithm 1 provided fairly poorer results compared to the multivariate GARCH specifications that considered cross-sectional information across the panel of the selected economies. Nevertheless, the more accurate forecasts were produced in the 3-month horizon. Between DCC and Copula GARCH processes, the former model built upon the standard univariate GARCH process, generated more accurate joint GaR predictions under the average length criterion and moderately accurate empirical coverage. The asymmetric counterpart of the multivariate Copula-GARCH produces better predictions when the empirical coverage is concerned. Before proceeding to the recommended further research, we briefly highlight the distributional assumptions and estimation methods of models to underline the reasons for their aforementioned predictive performance.

Algorithm 1 is built upon univariate GARCH specifications estimated by maximum likelihood and assuming skewed Student-t residuals. Hence, it is subjected to distributional misspecification errors, and additionally, it does not take into account dependence across the panel of the economies. On the other side, Copula-GARCH introduces a dependence parameter θ among the economies and a joint shape parameter ν, estimated aparametrically, and makes usage of the skewed Student-t marginals of the innovations. Thus, Copula-GARCH is also susceptible to distributional misspecification errors and burdens the simulation process with the parameter θ. In DCC however, we consider a dynamic correlation between the economies and proceed to a two-step estimation with quasi maximum likelihood and in the standardization process the marginals of the residuals are not utilized, but only the joint shape parameter ν.

Therefore, the superiority of DCC, in terms of reasonable average length values, reveals that the QML estimation method and cross-sectional information greatly improve the predictions of joint GaR, whereas the inferiority of Algorithm 1 to Algorithm 2 shows that dependence in variates in the joint GaR context is more crucial than the distributional assumptions. As a final remark, we highlight that copulas are used for modelling the joint behavior of random variables with a dependence structure. Combining a copula with the appropriate margins is quite challenging and may lead to severe problems in risk management. In Figure 10, we present several scatter plots of different pair of economies with simulated bivariate observations from Student-t and Normal copulas. We conduct three distinct combinations of copulas with margins to support the usage of our proposed model. As illustrated, all combinations can adequately capture extreme events especially in the lower tail. Nevertheless, bivariate Student-t copula with skewed Student-t margins is the only one to capture extreme events in the upper tail, rendering it more suitable.

Further Research

Our work engages with the prediction of joint GaR and in this context, we address issues such as the modeling of volatility, the distributional assumptions of residuals, and the incorporation of cross-sectional information in the multivariate GARCH framework. However, further research needs to shed light on multiple aspects of this paper. To begin with, the Student-t copula entails two main shortcomings according to Church (2012), i) it does not account for asymmetry in joint events, and ii) it introduces a common shape parameter ν for all economies. It would be more realistic, to utilize an asymmetric Student-t copula with individual degrees of freedom by attributing an individual skewness and shape parameter to each economy. Additionally, the parameters of the copula could be varying with time since a static copula does not consider dynamic dependence across the panel of the selected economies.

As the standard univariate GARCH specification is extended by introducing an asymmetric variant, this could also be applied to multivariate DCC. A multivariate asymmetric DCC GARCH model, proposed by Cappiello et al. (2006), gives more weight to negative standardized errors similarly with univariate GJR-GARCH by presenting the indicator function in Equation 7 that describes the dynamics of the matrix Q_t. Another equally important feature of this thesis that should be investigated is the modelling of mean, which could be more sophisticated and follow an ARMA process. Choosing the best order for ARMA-GARCH models has been extensively studied. A standard procedure to diagnose the orders of ARMA processes is by looking at the ACF and PACF (Partial Autocorrelation) plots of the time series as Box et al. (2015) propose. If both plots concurrently gradually decrease, then an ARMA process should be applied. On the other side, selection of the best ARMA-GARCH model can be conducted by the AIC (Akaike information criterion) as suggested by Brockwell et al. (2016).

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Appendices

A Plots

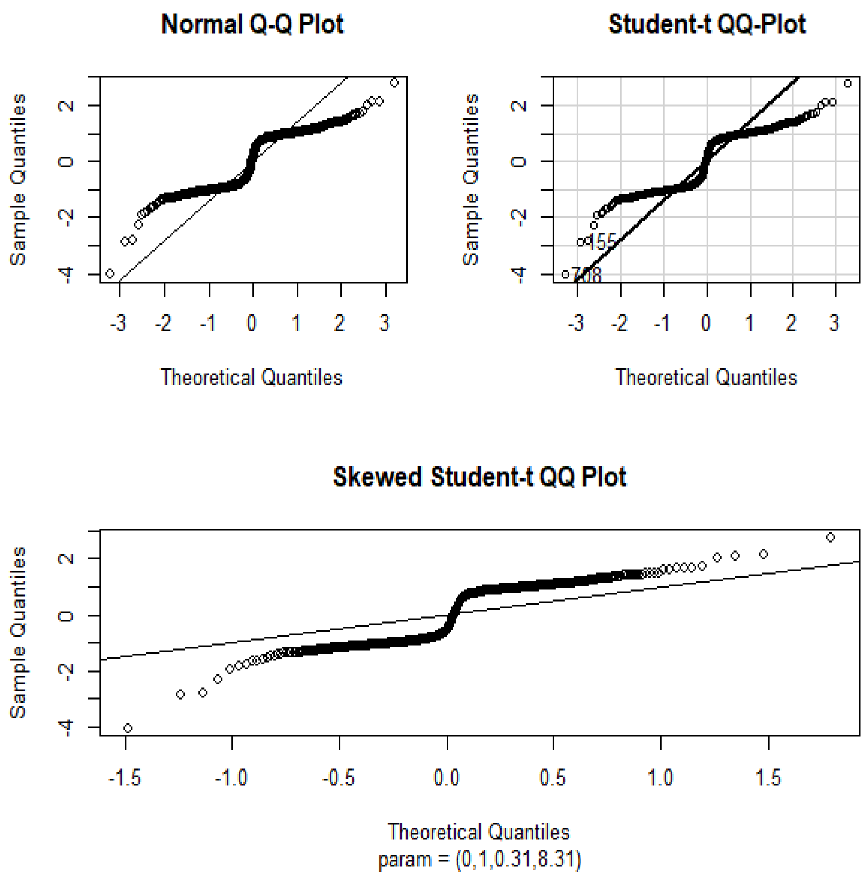


Figure 11: The QQ plots show the standardized residuals against the theoretical quantiles from Normal, Student-t and skew Student-t distributions. The parameters in skew Student-t are location, scale, skewness and degrees of freedom and are estimated from fitting the entire OECD+6MajorNME GDP growth rates series to a standard GARCH(1,1) model by MLE. The standardized residuals are estimated from fitting the entire OECD+6MajorNME GDP growth rates series to a standard GARCH(1,1) model by QMLE.

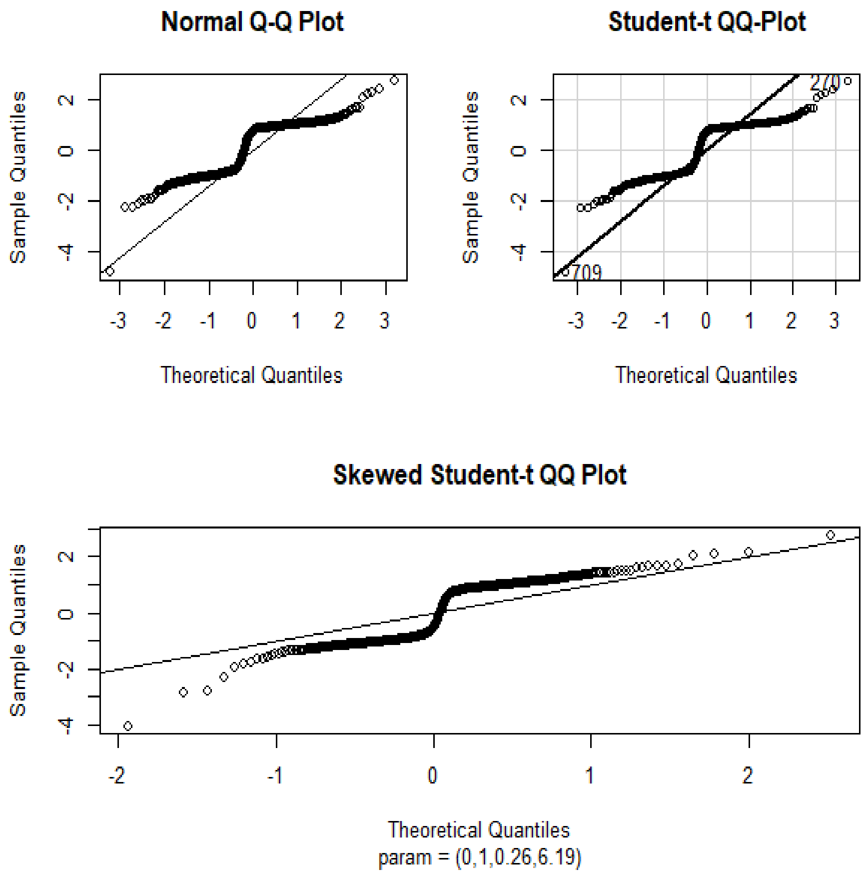


Figure 12: The QQ plots show the standardized residuals against the theoretical quantiles from Normal, Student-t and skew Student-t distributions. The parameters in skew Student-t are location, scale, skewness and degrees of freedom and are estimated from fitting the entire OECD GDP growth rates series to a standard GARCH(1,1) model by MLE. The standardized residuals are estimated from fitting the entire OECD GDP growth rates series to a standard GARCH(1,1) model by QMLE.

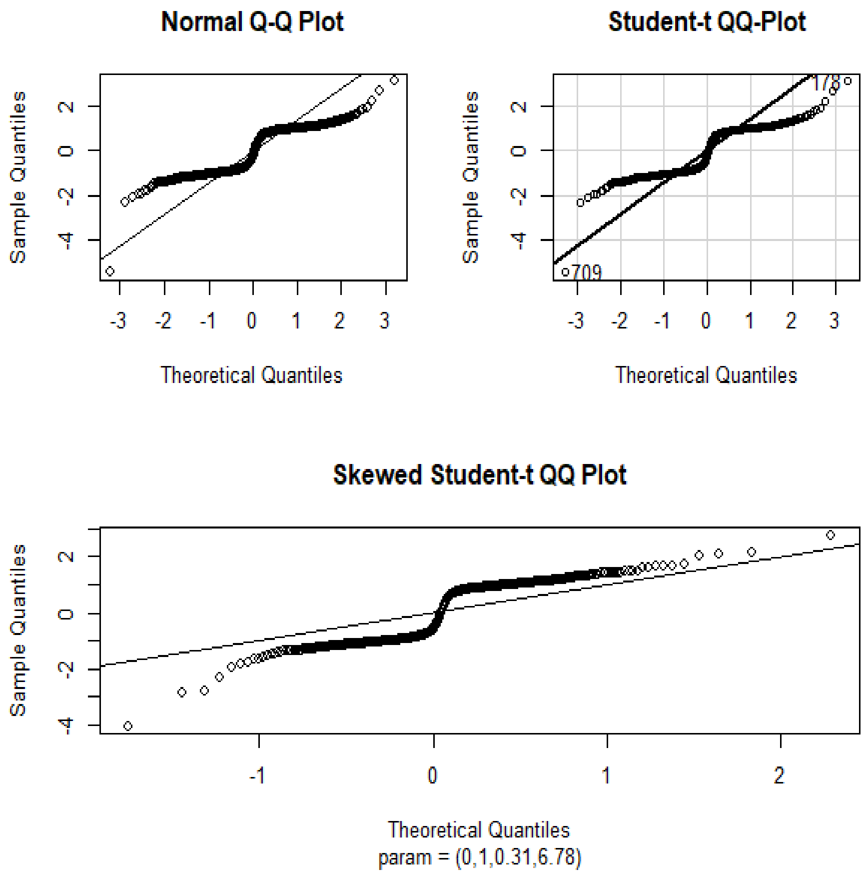


Figure 13: The QQ plots show the standardized residuals against the theoretical quantiles from Normal, Student-t and skew Student-t distributions. The parameters in skew Student-t are location, scale, skewness and degrees of freedom and are estimated from fitting the entire NAFTA GDP growth rates series to a standard GARCH(1,1) model by MLE. The standardized residuals are estimated from fitting the entire NAFTA GDP growth rates series to a standard GARCH(1,1) model by QMLE.

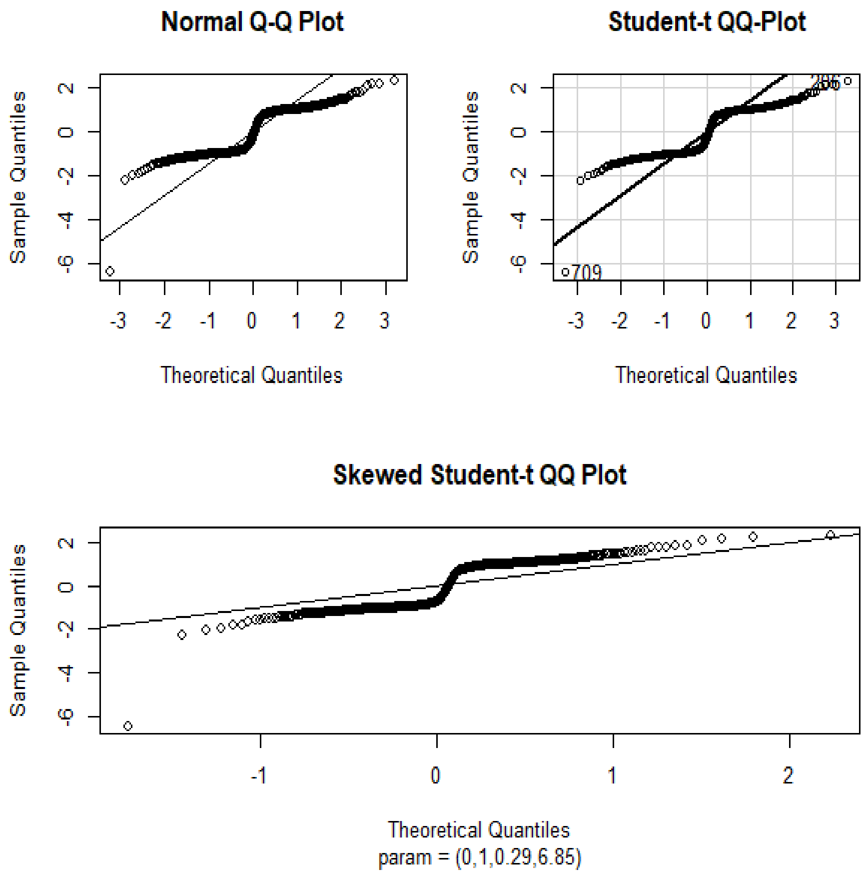
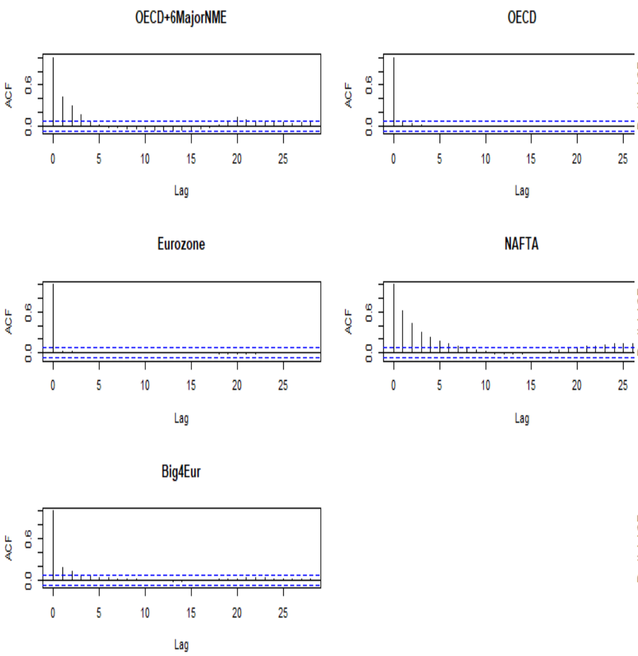
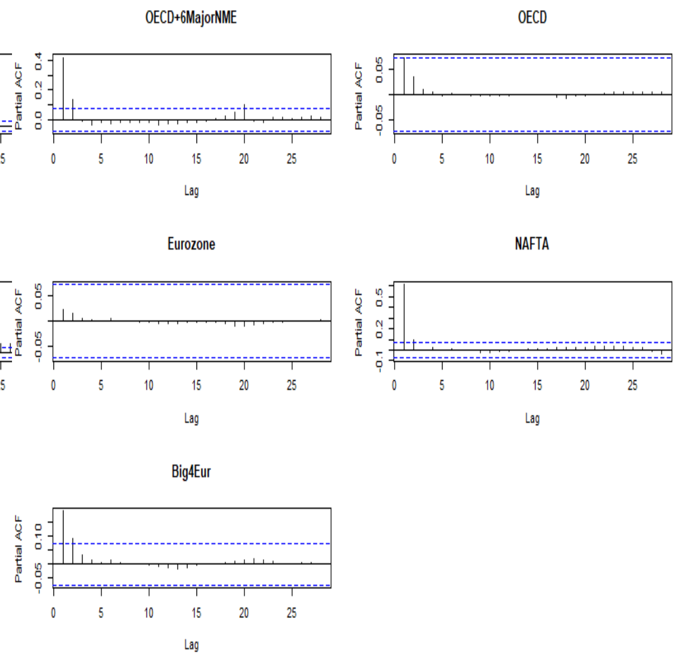


Figure 14: The QQ plots show the standardized residuals against the theoretical quantiles from Normal, Student-t and skew Student-t distributions. The parameters in skew Student-t are location, scale, skewness and degrees of freedom and are estimated from fitting the entire Big4Eur GDP growth rates series to a standard GARCH(1,1) model by MLE. The standardized residuals are estimated from fitting the entire Big4Eur GDP growth rates series to a standard GARCH(1,1) model by QMLE.



(a) ACF Plots of the selected economies



(b) PACF of the selected economies

Figure 15: The (P)ACF plots show the autocorrelation of the squared standardized residuals. The standardized residuals are estimated from fitting the entire GDP growth rates series of each economy to a GJR-GARCH(1,1) model.

B Additional Tables

Table 8: Joint GaR forecast evaluation with $p = 0.1$ and $q = 0.4$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-2.560	39.048	0.000	0.000
	GJR-GARCH(1,1)	0.000	21.507	76.191	0.001	0.000
	DCC-S-GARCH(1,1)	0.000	-5.490	53.333	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.117	3.810	0.000	0.000
	C-S-GARCH(1,1)	0.000	-1.447	69.524	0.000	0.000
	C-GJR-GARCH(1,1)	0.166	26.962	85.714	0.210	0.000
6	S-GARCH(1,1)	0.000	32.114	38.235	0.000	0.000
	GJR-GARCH(1,1)	0.000	289.717	76.471	0.001	0.000
	DCC-S-GARCH(1,1)	0.000	-5.899	38.235	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-6.991	0	-	-
	C-S-GARCH(1,1)	0.000	40.581	60.784	0.000	0.000
	C-GJR-GARCH(1,1)	0.000	359.250	77.451	0.002	0.000
12	S-GARCH(1,1)	0.000	4651.460	38.542	0.000	0.000
	GJR-GARCH(1,1)	0.001	28215.790	78.125	0.004	0.000
	DCC-S-GARCH(1,1)	0.000	-6.029	40.625	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.249	0	-	-
	C-S-GARCH(1,1)	0.000	6696.024	62.500	0.000	0.000
	C-GJR-GARCH(1,1)	0.001	36537.050	78.125	0.004	0.000
24	S-GARCH(1,1)	0.000	83160445	35.714	0.000	0.000
	GJR-GARCH(1,1)	0.029	258307014	82.143	0.060	0.000
	DCC-S-GARCH(1,1)	0.000	-6.083	58.333	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.367	0	-	-
	C-S-GARCH(1,1)	0.000	126108920	67.857	0.000	0.000
	C-GJR-GARCH(1,1)	0.029	364148022	82.143	0.060	0.000

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

Table 9: Joint GaR forecast evaluation with $p = 0.1$ and $q = 0.8$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-4.581	47.619	0.000	0.000
	GJR-GARCH(1,1)	-	8.695	100	-	-
	DCC-S-GARCH(1,1)	0.008	-6.098	59.048	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-6.784	8.571	0.000	0.000
	C-S-GARCH(1,1)	0.000	-3.428	65.714	0.000	0.000
	C-GJR-GARCH(1,1)	-	13.341	100	-	-
6	S-GARCH(1,1)	0.000	12.793	46.078	0.000	0.000
	GJR-GARCH(1,1)	-	127.535	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.382	51.961	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.428	8.824	0.000	0.000
	C-S-GARCH(1,1)	0.000	21.748	66.667	0.000	0.000
	C-GJR-GARCH(1,1)	-	178.449	100	-	-
12	S-GARCH(1,1)	0.000	1622.515	54.167	0.000	0.000
	GJR-GARCH(1,1)	-	8002.794	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.428	54.167	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.591	4.167	0.000	0.000
	C-S-GARCH(1,1)	0.004	3167.537	80.208	0.016	0.000
	C-GJR-GARCH(1,1)	-	12982.720	100	-	-
24	S-GARCH(1,1)	0.000	4606387	44.048	0.000	0.000
	GJR-GARCH(1,1)	-	17706366	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.459	58.333	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.728	0	-	-
	C-S-GARCH(1,1)	0.013	37798656	80.952	0.035	0.001
	C-GJR-GARCH(1,1)	-	55198241	100	-	-

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

Table 10: Joint GaR forecast evaluation with $p = 0.15$ and $q = 0.4$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-2.965	37.143	0.000	0.000
	GJR-GARCH(1,1)	0.018	18.894	76.191	0.034	0.000
	DCC-S-GARCH(1,1)	0.000	-5.855	23.810	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-6.683	0	-	-
	C-S-GARCH(1,1)	0.000	-2.195	46.667	0.000	0.000
	C-GJR-GARCH(1,1)	0.000	22.807	78.095	0.000	0.000
6	S-GARCH(1,1)	0.000	28.306	35.294	0.000	0.000
	GJR-GARCH(1,1)	0.024	253.818	76.471	0.042	0.000
	DCC-S-GARCH(1,1)	0.000	-6.121	19.608	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.248	0.000	-	-
	C-S-GARCH(1,1)	0.000	34.220	40.1960	0.000	0.000
	C-GJR-GARCH(1,1)	0.024	300.485	76.471	0.042	0.000
12	S-GARCH(1,1)	0.000	4044.049	35.417	0.000	0.000
	GJR-GARCH(1,1)	0.041	23365.100	77.083	0.065	0.000
	DCC-S-GARCH(1,1)	0.000	-6.206	27.083	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.417	0	-	-
	C-S-GARCH(1,1)	0.000	5519.040	44.792	0.000	0.000
	C-GJR-GARCH(1,1)	0.041	28520.150	77.083	0.065	0.000
24	S-GARCH(1,1)	0.000	66522731	32.143	0.000	0.000
	GJR-GARCH(1,1)	0.474	193911469	82.143	0.494	0.000
	DCC-S-GARCH(1,1)	0.000	-6.256	46.429	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.533	0	-	-
	C-S-GARCH(1,1)	0.000	95909366	46.429	0.000	0.000
	C-GJR-GARCH(1,1)	0.474	250133948	82.143	0.494	0.000

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

Table 11: Joint GaR forecast evaluation with $p = 0.15$ and $q = 0.8$.

d	Model	Kupiec	Length	Cov.	DQ(0)	DQ(4)
3	S-GARCH(1,1)	0.000	-4.815	45.714	0.000	0.000
	GJR-GARCH(1,1)	-	7.328	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.342	49.524	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.205	8.571	0.000	0.000
	C-S-GARCH(1,1)	0.000	-3.867	60.952	0.000	0.000
	C-GJR-GARCH(1,1)	-	11.078	100	-	-
6	S-GARCH(1,1)	0.000	10.625	44.118	0.000	0.000
	GJR-GARCH(1,1)	-	111.928	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.506	46.078	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.589	7.843	0.000	0.000
	C-S-GARCH(1,1)	0.000	18.003	62.745	0.000	0.000
	C-GJR-GARCH(1,1)	-	150.140	100	-	-
12	S-GARCH(1,1)	0.000	1299.287	53.125	0.000	0.000
	GJR-GARCH(1,1)	-	6488.822	100	-	-
	DCC-S-GARCH(1,1)	0.000	-6.546	52.083	0.000	0.000
	DCC-GJR-GARCH(1,1)	0.000	-7.709	3.125	0.000	0.000
	C-S-GARCH(1,1)	0.002	2498.753	72.917	0.008	0.000
	C-GJR-GARCH(1,1)	-	9979.517	100	-	-
24	S-GARCH(1,1)	0.000	-518012.800	33.333	0.000	0.000
	GJR-GARCH(1,1)	0.000	-90550.670	97.619	0.000	0.000
	DCC-S-GARCH(1,1)	0.000	-6.583	47.619	0.000	0.000
	DCC-GJR-GARCH(1,1)	-	-7.848	0	-	-
	C-S-GARCH(1,1)	0.002	24423102	71.429	0.006	0.000
	C-GJR-GARCH(1,1)	0.000	23526930	97.619	0.000	0.000

Note. This table reports the p-values of the GaR adequacy tests considered (Kupiec,DQ without and with Hits), the average empirical joint coverage and average length for each forecast horizon and forecasting method. All reported numbers are rounded to 3 decimals.

C Derivation of Skewed Student-t Moments

Statement: Let z be a random variable. Then, its density function given a shape parameter ν and a skewness parameter ξ according to Ghalanos (2020) can be represented as

$$f_{\nu,\xi}(z) = \frac{2}{\xi + \xi^{-1}} \left[f_\nu(\xi z) \mathbb{1}_{z<0} + f_\nu \left(\xi^{-1} z \right) \mathbb{1}_{z>0} \right], \tag{22}$$

where $\xi \in \mathbb{R}^+$. The absolute moments, required for deriving the central moments, are generated from the following function

$$M_r = 2 \int_0^\infty z^r f_\nu(z) dz \tag{23}$$

The mean and variance are then defined as

$$\begin{aligned} E(z) &= M_1 \left(\xi - \xi^{-1} \right) \\ \operatorname{Var}(z) &= \left(M_2 - M_1^2 \right) \left(\xi^2 + \xi^{-2} \right) + 2M_1^2 - M_2, \end{aligned} \tag{24}$$

where

$$M_1 = \frac{\sqrt{\frac{\nu}{\pi}} \Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})} \quad \text{and} \quad M_2 = \frac{\nu}{\nu - 2} \tag{25}$$

Proof: Based on equation 23, we have $M_1 = 2 \int_0^\infty z f_\nu(z) dz$ and $M_2 = 2 \int_0^\infty z^2 f_\nu(z) dz$. Hence,

$$E_{\nu,\xi}(z) = \tfrac{2}{\xi+\xi^{-1}} \left[\int_{-\infty}^0 z f_\nu(\xi z) dz + \int_0^\infty z f_\nu(z \mid \xi) dz \right] = \tfrac{2}{\xi+\xi^{-1}} \left[\tfrac{-1}{\xi^2} \int_{-\infty}^0 f_\nu(u) du + \xi^2 \int_0^\infty f_\nu(u) du \right] = \tfrac{M_1}{\xi+\xi^{-1}} (\xi - \xi^{-1})(\xi + \xi^{-1}) = M_1(\xi - \xi^{-1}),$$

and

$$E_{\nu,\xi}(z^2) = \tfrac{2}{\xi+\xi^{-1}} \left[\int_{-\infty}^0 z^2 f_\nu(\xi z) dz + \int_0^\infty z^2 f_\nu(z \mid \xi) dz \right] = \tfrac{2}{\xi+\xi^{-1}} \left[\tfrac{1}{\xi^3} \int_{-\infty}^0 u^2 f_\nu(u) du + \xi^3 \int_0^\infty u^2 f_\nu(u) du \right] = \tfrac{M_2}{\xi+\xi^{-1}} (\xi + \xi^{-1})(\xi^2 - 1 + \xi^{-2}),$$

and thus

$$Var_{\nu,\xi}(z) = M_2(\xi^2 - 1 + \tfrac{1}{\xi^2}) - M_1^2(\xi^2 - 2 + \tfrac{1}{\xi^2}) = (M_2 - M_1^2)(\xi^2 + \tfrac{1}{\xi^2}) - M_2 + 2M_1^2.$$

For $k \in \mathbb{N}_+$ with $0 < k < \nu$ and for $T \sim \mathrm{St}(t \mid 0, 1, \nu)$, the raw and absolute moments according to Kirkby et al. (2019) satisfy

$$\mathbb{E}\left(T^k\right) = \begin{cases} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}} \cdot \frac{\nu^{k/2}}{\prod_{i=1}^{k/2} \left(\frac{\nu}{2} - i\right)}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \tag{26}$$

and

$$\mathbb{E}\left(|T|^k\right) = \frac{\nu^{k/2} \Gamma((k+1)/2) \Gamma((\nu - k)/2)}{\sqrt{\pi} \Gamma(\nu/2)}, \tag{27}$$

where the moments M_2 and M_1 result respectively from Equations 26 and 27 by substituting $k = 2$ in the former equation and $k = 1$ in the latter one.

After the previous derivations, the proof of the probability $P(z < 0)$ results now directly as follows

$$P(z < 0) = \frac{2}{\xi + \xi^{-1}} \int_{-\infty}^0 f_\nu(\xi z) dz = \frac{2}{\xi + \xi^{-1}} \frac{1}{\xi} \int_{-\infty}^0 f_\nu(u) du = \frac{2}{\xi^2 + 1} \frac{1}{2} = \frac{1}{\xi^2 + 1} \tag{28}$$