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Master Thesis [Quantitative Finance]

Robustness of the KNW model

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Abstract

This paper tests the robustness of the KNW model, as it is denoted by (Draper, 2014). The model is applied to the Netherlands, UK, and US, while model assumptions are alternated to improve the goodness-of-fit. This study confirms that the KNW model with the model assumptions as in (Draper, 2014) is mainly a model for the Northern European financial market with estimation results for the Netherlands that are in line with long-term trends and expectations. It is better to use monthly instead of quarterly data and to observe the three-month and three-year yields instead of the two-year and five-year yields. For the UK specifically, assuming all yields to be observed with a measurement error leads to improved model characteristics.

The model is estimated over a fifty year period, which leads to structural breaks in the parameter estimates. Therefore, it is advisable to estimate the KNW model using either time-varying parameters or a much shorter estimation period.

Keywords: KNW model, financial market model, structural break, parameter restrictions

Contents

1 Introduction

The KNW model by (R. S. Koijen, Nijman, & Werker, 2010) is a capital market model for the stock market, the bond market, and inflation. The model is used to generate scenarios for the overall equity index, the term structure of interest rates, and the development of price inflation over time. Although (R. S. Koijen et al., 2010) developed the model primarily for the US financial market, it is also used by leading Dutch institutions like the central bank of the Netherlands (Dutch: de Nederlandsche Bank; DNB), the Bureau for Economic Policy Analysis (Dutch: het Centraal Planbureau; CPB), and Netspar to generate future economic scenarios for policy evaluation. A scenario set can be used to, for example, perform comparable feasibility tests of pension funds ((Van den Berg, 2016)) or to simulate different investment strategies of a mutual fund for a life-cycle investor ((R. S. Koijen et al., 2010)).

The goodness-of-fit of the KNW model is derived from the value of the log-likelihood, significance of the parameter estimates, and the similarity between the model implied characteristics and their data counterparts. For the KNW model to give reliable results, the estimated parameters have to be robust to small changes to the estimation data. (Draper, 2012), (Draper, 2014), and (Van den Berg, 2016) all applied the KNW model to Dutch data, but even though the more recent papers only added one or two years to the dataset, some of the estimated parameters were quite different and led to different simulation results for the yields. Since 2014, the last estimation year of (Van den Berg, 2016), interest rates have further decreased, while inflation has risen. These changes can lead to different parameter estimates and implied characteristics for the yields and other variables. As this can put into question the reliability of the KNW model for generating economic scenarios, it is necessary to re-estimate the KNW model with a data sample that also includes data from 2015 until 2021.

To optimize the investment strategies of the pension funds and mutual funds, it is important that the simulated economic scenarios on which policy decisions are based are close to reality. Every five years a commission reconsiders the values of the parameters of the KNW model to make them fit for the financial market of the Netherlands. The most recent commission is the commission (Dijsselbloem et al., 2019), in which expectations for financial variables, like price inflation and stock return, that are important for evaluating pension funds, are presented. The expected values of these variables are generated by the model after calibration of some parameters. The expectations are based on the work of previous commissions and national and international research. For example, the expectation for price inflation is based on the historical average and is in line with the target inflation of the ECB. In general, the advice of the commission (Dijsselbloem et al., 2019) should lead to simulated economic scenarios based on the KNW model that are close to the future state of the economy.

To use the KNW model to make future economic scenario's, it is important to check the parameter stability over the sample period. As the KNW model is estimated over a period of almost fifty years, the input variables of inflation and yields have very different values across the sample period. For example, in the 1970s, inflation and bond yields were much higher than they are today. These shifts can lead to structural breaks of the parameters of the KNW model. Looking for structural breaks can give insight into the changing behaviour of the financial market. This can lead to improvements of the estimation procedure like adjusting the estimation period or making specific parameter time-varying.

Thus far, the KNW model has only been applied to data from the US ((R. S. Koijen et al., 2010)) and the Netherlands ((Draper, 2012), (Draper, 2014), and (Van den Berg, 2016)). The KNW model should also be applied to different countries, to see if the estimation and simulation results are significantly different to those from the US and the Netherlands. The parameter estimates of the Netherlands from (Draper, 2012) are less significant than those from (R. S. Koijen et al., 2010), meaning that they have a larger standard deviation. Also the persistence of the state variables is also larger in (Draper, 2012). The KNW model's accuracy in generating economic scenarios and the way in which inflation, stock return, bond yields and the state variables relate to each can differ across different countries. Therefore the KNW model will be applied to the UK in addition to the US and the Netherlands.

In the KNW model as described above, two yields are assumed to be observed without measurement error. This way the state variables follow directly from the value of these yields. Because it is difficult to predetermine which two yields should be chosen, it is important to look at different pairs of yields and see how the estimation results differ. Another possibility is to assume that all yields are observed with a measurement error. In this case the state variables have to be determined using a Kalman filter. There hasn't been research done on the difference on the estimation results between these two model assumptions. Contrary to the method used in (Draper, 2012), (Draper, 2014) and (Van den Berg, 2016), (Dijsselbloem et al., 2019) suggests that the KNW model should be estimated assuming all yields to be observed with error.

The goal of this paper is to test the robustness of the KNW model with respect to changes in the sample data, observation frequency, and model assumptions. This might lead to adjustments of the KNW model that will improve the overall fit of the model. The contribution of this paper can be split into five parts.

Firstly, the KNW model is applied to the Netherlands choosing either the three-month and three-year, or the two-year and five-year yields to be observed without error. I use quarterly data, so that the results can be more easily compared to the results from (Draper, 2014) and (Van den Berg, 2016). By comparing the results, the effect of adding the years 2015 until 2021, and the difference between the yield-pairs should become clear.

Secondly, the KNW model is applied to monthly datasets from the Netherlands, UK, and US to compare the parameter estimates and model characteristics. This way, it becomes clear for each country whether the KNW model is suitable as a financial market model. I use monthly instead of quarterly data, because a larger number of observations leads to a more informative dataset and improved asymptotic properties.

Thirdly, both the regular likelihood ratio test and the supremum likelihood ratio tests are applied to test for both known and unknown break points in the parameter estimates. For both these tests, the likelihood of the KNW model with parameter values that change at a certain break point is compared to the likelihood of the KNW model with constant parameter values over time.

Fourthly, the restrictions suggested by (Dijsselbloem et al., 2019) are applied to the unrestricted model. According to this paper, the process of the state variables should be stationary, and the term-structure of interest rates should be without oscillations. Also under these constraints, the unconditional geometric expected stock return, unconditional geometric expected inflation, and the UFR are equal to specific values. The last restriction is that not more than 2.5% of the ten-year yields from the unconditional distribution can be negative.

Lastly, the KNW model is estimated with the assumption that all yields are estimated with a measurement error for each country that is considered. The state variables are estimated using a Kalman filter, and an adjusted log-likelihood is maximised. The estimation results can be compared to the KNW model with two yields observed without error in terms of the parameter estimates and implied characteristics.

Section 7 contains a description of each dataset and summary statistics that are relevant when analyzing the model implied characteristics of the KNW model. The results of each subtopic are discussed in Section 8.

This paper gives some new insights with respect to improving the KNW model as a financial market model. One conclusion is that estimating the KNW model with monthly data instead of quarterly data leads to an improvement in the precision with which the parameters are estimated. Also, the estimation results greatly depend on the yields that are chosen to be observed without error. This paper shows that choosing the three-month and three-year yields is preferable. Lastly, the KNW model estimated over an estimation period comparable to the one in Draper (2014) is very susceptible to structural change of the parameters.

2 Literature review

The financial market model in (R. S. Koijen et al., 2010) is closely related to (Brennan & Xia, 2002), (Campbell & Viceira, 2001), and (Sangvinatsos & Wachter, 2005). These papers focus on the optimal allocation to long-term bonds and show that it is ideal to hedge time variation in real interest rates. An important characteristic of the KNW model is that both nominal bond yields and equity returns are related to the inflation process. In addition, the model accommodates time-varying interest rates, inflation rates, and bond risk premia.

Neither of the scientific papers that I considered applied the KNW model to the UK financial market. (Draper, 2012) used the KNW model on Dutch data in addition to US data and observed that the results for the Netherlands deviated in several aspects from those for the United States. In particular, the coefficient estimates were less significant. A possible explanation given by (Draper, 2012) is that exchange rate fluctuations are more relevant for Europe than for the US. The link between the bond yields of different durations is not as large in Northern Europe as in the US, while the duration of the inverse term structures in Europe is longer than in the US. Due to the small open economy character and the data construction this model can be considered a model for the Northern European capital market.

(Draper, 2012), (Draper, 2014), and (Van den Berg, 2016) all applied the KNW model to Dutch data with datasets starting in 1972, but ending in 2011, 2013, and 2014, respectively. The values from (Van den Berg, 2016) differ significantly to those of (Draper, 2012) with some parameters switching sign, but are very close to those of (Draper, 2014) for most parameters. However, the implied bond risk premia and Sharpe ratios for (Van den Berg, 2016) are significantly lower (more than 1% for the ten-year bond premium). Also the significance levels of the updated estimates are lower in comparison to those from (Draper, 2012) and (Draper, 2014). (Van den Berg, 2016) compared the likelihood of the estimated parameters from all three papers for the samples from both (Draper, 2014) and (Van den Berg, 2016). For both samples the parameters from (Van den Berg, 2016) lead to the highest likelihood, indicating that the estimation procedure from (Draper, 2014) did not lead to the maximum likelihood estimator for the parameter set. The CPB has also indicated that the estimation in (Draper, 2014) did not lead to the maximum likelihood. Most likely, the estimated set of parameters is a local optimum. This is a known problem for models with many dimensions. Likewise the estimation results from (Van den Berg, 2016) could also be a local optimum.

The fact that some of the parameter estimates and implied characteristics for (Draper, 2012), (Draper, 2014), and (Van den Berg, 2016) are significantly different, even though the datasets are almost identical and of a similar length, is an indication that the KNW model could be susceptible for structural change at some point in the estimation period. There has been a lot of scientific research on methods to finds structural breaks. Most results concern models that are too simple for economic applications. (Hanson, 1992) and (Zivot & Andrews, 1992) only focus on linear regression models. (Andrews, 1993) has provided tests that are applicable for a large number of nonlinear models. These test can be used to find both known and unknown change points and are applicable for maximum-likelihood estimation of the KNW model.

One way I will determine the goodness-of-fit of the KNW model is by making simulations based on the parameter estimates for the bond yields, inflation, and stock return. Previous research has mostly focused on the behaviour of the yield curve, and no research has been done on the model-implied simulations of inflation and stock return. (Van den Berg, 2016) made simulations of the yield curve for the parameter estimates from (Draper, 2012), (Draper, 2014), and (Van den Berg, 2016). The parameter estimates from each of these papers lead to simulated yield averages and volatilities that are close to the averages and volatilities of the real yields with the real yields all within the simulated 95% confidence intervals. Because of how close the averages and volatilities are to the real data, it can be concluded that the KNW model estimated with Dutch data is able to match the cross-sectional moments of bond yields. The smaller confidence interval of the volatility shows that the volatilities are estimated more precisely than the averages. The model from (Van den Berg, 2016) generates long-term (40 years or more) bond yields with implied averages that fit the data very well, but for the short-term bonds (20 years or less) the implied means for the model parameters in (Draper, 2012) and (Draper, 2014) are closer to the real data. This observation can be explained by the data sample on which the model parameters have been estimated. Bond rates have been decreasing over time since 1980. Also, the ECB's monetary policy in response to the European credit crisis has led to extremely low interest rates in the most recent years from the (Van den Berg, 2016) dataset, which aren't included in the (Draper, 2012) dataset. As interest rates have continued to fall since 2014, which is the last year of the sample from (Van den Berg, 2016), it can be expected that adding the most recent years to the data sample will further decrease the simulated averages and thereby increase the gap between the real and simulated averages for short term bonds.

3 KNW Model

The description of the KNW model in Sections 3.1-4.3 is largely taken from (R. S. Koijen et al., 2010), (Draper, 2014), and (Van den Berg, 2016).

3.1 Model assumptions

The KNW model is a financial market model that represents the behaviour of stock return, inflation, and bond yields over time. The model written in continuous time, but for estimation it is discretized and written as a VAR(1)-model. In addition to the economic factors two state variables are used that follow a mean-reverting process:

$$\begin{aligned} dX_t &= -KX_t dt + \Sigma'_X dZ_t \\ K &\text{ is } 2\times 2 \text{ and } \Sigma'_X = [I_{2\times 2} 0_{2\times 2}]. \end{aligned} \tag{1}$$

The matrix K controls the autocorrelation of the two state variables. Z_t is a vector containing four independent Brownian motions, each of which representing a distinct source of uncertainty in the financial market:

- uncertainty about the real interest rate
- uncertainty about the instantaneous expected inflation
- uncertainty about unexpected inflation
- uncertainty about the stock return

The state variables $X_t = (X_{1t}, X_{2t})$' are used to capture the dynamics of the real interest rate and expected inflation. Due to the composition of Σ_X, only the Brownian motions that drive uncertainty for real interest rates and expected inflation have impact on the state variables. For the instantaneous real interest rate r_t holds

$$r_t = \delta_{0r} + \delta'_{1r} X_t, \tag{2}$$

and for the instantaneous expected inflation π_t holds

$$\pi_t = \delta_{0\pi} + \delta'_{1\pi} X_t. \tag{3}$$

The correlation between the interest rate and inflation is governed by δ'_{1r} and $\delta'_{1\pi}$. The expected inflation is a factor in the price index process

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma'_\Pi dZ_t, \quad \text{with} \quad \sigma_\Pi \in R^4 \quad \text{and} \quad \Pi_0 = 1. \tag{4}$$

σ_Π governs the effect of the previously mentioned sources of uncertainty on the price index process.

The stock index process is governed by the nominal instantaneous interest rate R_t, which will be described in the next section, combined with shocks of uncertainty governed by σ_S:

$$\frac{dS_t}{S_t} = (R_t + \eta_s)dt + \sigma_S' dZt, \quad \text{with} \quad \sigma_S \in R^4 \quad \text{and} \quad S_0 = 1. \tag{5}$$

Here η_s is the constant equity risk premium. The equity returns have no constant expected value, as they are based on the nominal interest rates and the equity risk premium.

The risk aversion of the investors is governed by the price of risk

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t, \quad \text{with} \quad \Lambda_t, \Lambda_0 \in R^4 \quad \text{and} \quad \Lambda_1 : 4 \times 2. \tag{6}$$

The third element of Λ_0 and third row of Λ_1 are all zeros, as there is no risk premium for unexpected inflation. This restriction is imposed because the price of unexpected inflation risk cannot be identified on the basis of data on the nominal side of the economy alone. Therefore, I impose that the part of the price of unexpected inflation risk that cannot be identified using nominal bond data equals zero. As inflation-linked bonds were launched for the first time in the UK in 1981, the data available are insufficient to estimate this price of risk accurately. In fact, in the Netherlands inflation-linked bonds haven't been issued at all ((Westerhout & Ciocyte, 2017)).

The price of risk partially determines the process of the nominal stochastic discount factor

$$\frac{d\phi_t^N}{\phi_t^N} = -R_t dt - \Lambda_t' dZ_t \tag{7}$$

by governing the influence of the different sources of uncertainty. The stochastic discount factor ϕ_t^N represents the marginal utility ratio between future and present consumption and is used to discount future cash flows. In the case of a complete market, the marginal utility ratio is the same for everyone.

3.2 Fundamental valuation theory

In the KNW model the expected value of a discounted stock price is constant over time. In the model framework set up in Section 3.1, this restriction is given by the Fundamental valuation equation:

$$E[d\phi_t^N S_t] = 0. \tag{8}$$

Applying Itô Doeblin theorem gives

$$\begin{aligned}\frac{d\phi_t^N S_t}{\phi_t^N S_t} &= \frac{d\phi_t^N}{\phi_t^N} + \frac{dS_t}{S_t} + \frac{d\phi_t^N}{\phi_t^N} \cdot \frac{dS_t}{S_t} \\ &= (\eta_S - \Lambda_t' \sigma_S')dt - (\Lambda_t' - \sigma_S')dZ_t,\end{aligned} \tag{9}$$

because in the limit dt goes to zero, dt^2 and $dtdZ$ disappear, and dZ^2 approaches dt. Combining this expression with equation 8 gives $\eta_S = \Lambda_t' \sigma_S$, which implies the parameter constraints $\sigma_S' \Lambda_0 = \eta_S$ and $\sigma_S' \Lambda_1 = 0$.

3.3 Fundamental pricing theory

The fundamental pricing equation for a nominal zero coupon is given by

$$E[d\phi_t^N P^N] = 0. \tag{10}$$

This equation implies that the expected discounted value of the price of a nominal bond is constant over time. The bond prices are assumed to depend on the state X and a time trend t: $P^N = P^N(X,t)$. Likewise, the discounted value of the inflation corrected price of real bonds doesn't change over time either:

$$E[d\phi_t^N P^R \Pi_t] = 0. \tag{11}$$

Applying the Itô Doeblin theorem to the real stochastic discount factor $\phi_t^R = \phi_t^N \Pi_t$ gives

$$\begin{aligned}\frac{d\phi_t^R}{\phi_t^R} &= \frac{d\phi_t^N}{\phi_t^N} + \frac{d\Pi_t}{\Pi_t} + \frac{d\phi_t^N}{\phi_t^N}\frac{d\Pi_t}{\Pi_t} \\ &= -(R_t - \pi_t + \sigma_\Pi' \Lambda_t)dt - (\Lambda_t' - \sigma_\Pi')dZ_t \\ &= -r_t dt - (\Lambda_t' - \sigma_\Pi')dZ_t.\end{aligned} \tag{12}$$

This leads to the following expressing of the instantaneous nominal interest rate:

$$\begin{aligned}R_t &= r_t + \pi_t - \sigma_\Pi' \Lambda_t \\ &= (\delta_{0r} + \delta_{0\pi} - \sigma_\Pi' \Lambda_0) + (\delta_{1r} + \delta_{1\pi} - \sigma_\Pi' \Lambda_1) X_t \\ &= R_0 + R_1' X_t.\end{aligned} \tag{13}$$

3.4 Nominal term structure

Nominal bond prices for different maturities can be written as

$$P^N(X_t, t, t+\tau) = \exp(A^N(\tau) + B^N(\tau)' X_t). \tag{14}$$

with

$$A^N(\tau) = \int_0^\tau \dot{A}^N(s)ds \tag{15}$$

$$B^N(\tau) = (K' + \Lambda_1'\Sigma_X)^{-1}[\exp(-(K' + \Lambda_1'\Sigma_X)\tau) - I_{2\times2}]R_1 \tag{16}$$

with $I_{2\times2}$ the identity matrix of size two and

$$\dot{A}^N(\tau) = -R_0 - (\Lambda_0'\Sigma_X)B^N(\tau) + \frac{1}{2}B'^N(\tau)\Sigma_X'\Sigma_X B^N(\tau) \tag{17}$$

$$\dot{B}^N(\tau) = -R_1 - (K' + \Lambda_1'\Sigma_X)B^N(\tau) \tag{18}$$

the differentials of $A(\tau)$ and $B(\tau)$. The exact derivation can be found in Appendix B.

4 Estimation Procedure

4.1 Exact discretization

To estimate the model, the continuous KNW model is discretized. Exact discretization is done by first writing the model as a multivariate Ornstein-Uhlenbeck process

$$dY_t = (\Theta_0 + \Theta_1 Y_t)dt + \Sigma_Y dZ_t \tag{19}$$

with

$$Y_t' = [X \quad \ln(\Pi) \quad \ln(S)]. \tag{20}$$

Itô Doeblin theorem is applied for both log price index

$$\begin{aligned} d\ln(\Pi_t) &= \frac{\partial \ln(\Pi_t)}{\partial \Pi_t}d\Pi_t + \frac{1}{2}(\frac{\partial^2 \ln(\Pi_t)}{\partial \Pi_t^2})(d\Pi_t)^2 \\ &= (\pi_t dt + \sigma_\Pi' dZ_t) - \frac{1}{2}[\pi_t dt + \sigma_\Pi' dZ_t]^2 \\ &= (\pi_t - \frac{1}{2}\sigma_\Pi'\sigma_\Pi)dt + \sigma_\Pi' dZ_t \end{aligned} \tag{21}$$

and log stock index

$$\begin{aligned} d\ln(S_t) &= \frac{\partial \ln(S_t)}{\partial S_t}dS_t + \frac{1}{2}(\frac{\partial^2 \ln(S_t)}{\partial S_t^2})(dS_t)^2 \\ &= (R_t + \eta_S)dt + \sigma_S' dZ_t - \frac{1}{2}[(r_t + \eta_S)dt + \sigma_S' dZ_t]^2 \\ &= (R_0 + R_1' X_t + \eta_S - \frac{1}{2}\sigma_S'\sigma_S)dt + \sigma_S' dZ_t. \end{aligned} \tag{22}$$

Using Equation 3 for π_t and Equation 13 for R_t, this implies for the multivariate Ornstein-Uhlenbeck process

$$
\begin{aligned}
d\begin{bmatrix} X_t \\ \ln(\Pi_t) \\ \ln(S_t) \end{bmatrix} = &\left(\begin{bmatrix} 0_{2\times 1} \\ \delta_{0\pi} - \frac{1}{2}\sigma'_\Pi \sigma_\Pi \\ R_0 + \eta_S - \frac{1}{2}\sigma'_S \sigma_S \end{bmatrix} + \begin{bmatrix} -K & 0_{2\times 4} \\ \delta'_{1\pi} & 0_{1\times 4} \\ R'_1 & 0_{1\times 4} \end{bmatrix} \begin{bmatrix} X_t \\ \ln(\Pi_t) \\ \ln(S_t) \end{bmatrix} \right) dt \\
&+ \begin{bmatrix} \Sigma'_X \\ \sigma'_\Pi \\ \sigma'_S \end{bmatrix} dZ_t.
\end{aligned} \tag{23}
$$

By using the eigenvalue decomposition

$$\Theta_1 = UDU^{-1} \tag{24}$$

the exact discretization reads as

$$Y_{t+h} = \mu^{(h)} + \Gamma^{(h)} Y_t + \epsilon_{t+h} \quad \text{and} \quad \epsilon_{t+h} \sim \mathrm{N}(0, \Sigma^{(h)}) \tag{25}$$

in which:

(i) $\Gamma^{(h)}$ is defined as

$$\Gamma^{(h)} = \exp(\Theta_1 h) = U \exp(Dh) U^{-1} \tag{26}$$

whereby the matrix exponential is defined as $\exp(A) = I + \Sigma_{r=1}^{\infty} \frac{1}{r!} A^r$.

(ii) $\mu^{(h)}$ is defined as

$$\mu^{(h)} = UFU^{-1}\Theta_0 \tag{27}$$

where F is a diagonal matrix with elements $F_{ii} = h\alpha(D_{ii}h)$ with $\alpha(x) = \frac{exp(x)-1}{x}$ and $\alpha(0) = 1$.

(iii) $\Sigma^{(h)}$ is defined as

$$\Sigma^{(h)} = UVU' \tag{28}$$

with

$$V_{ij} = [U^{-1}\Sigma_Y \Sigma'_Y (U^{-1})']_{ij} h\alpha([D_{ii} + D_{jj}]h). \tag{29}$$

These relations are taken from (Bergstrom, 1984) and (R. Koijen, Nijman, & Werker, 2005). Formula 25 is used for $h = 1$ as the model should describe the one-period-ahead relation of the data used. This gives

$$\tilde{Y}_{t+1} = \mu + \Gamma \tilde{Y}_t + \epsilon_{t+1} \quad \text{and} \quad \epsilon_t \sim \mathrm{N}(0, \Sigma) \tag{30}$$

with $\tilde{Y}_t' = [X_t, \Delta\ln(\Pi_t), \Delta\ln(S_t)]$, while the parameters Γ, μ, and Σ are calculated using equations 26, 27, and 28 for $h = 1$. Because the log-difference is taken, the third and fourth element of $\tilde{Y}_t$ now represent inflation rates and stock returns, respectively. The columns of Γ linked to inflation and the equity returns are zero. Also the parameter that links X_2 to X_1 is zero, as this relationship is expressed by the parameter that links X_1 to X_2. The eigenvalues of Γ are all inside the unit circle, which means the unconditional expected value of $\tilde{Y}_t$ is given by

$$E(\tilde{Y}_t) = [I - \Gamma]^{-1}\mu \tag{31}$$

while the unconditional variance covariance matrix is

$$V(\tilde{Y}_t) = (I - \Gamma)^{-1}\Sigma(I - \Gamma')^{-1}. \tag{32}$$

4.2 Restriction on expected values and volatilities

After convergence of the state to zero, the expected long-run inflation and equity-returns are

$$\begin{bmatrix} E(\frac{d\Pi_t}{\Pi_t}) \\ E(\frac{dS_t}{S_t}) \end{bmatrix} = \begin{bmatrix} \delta_{0\pi} \\ R_0 + \eta_S \end{bmatrix}, \tag{33}$$

and the long-run volatilities are

$$\begin{bmatrix} E(\frac{d\Pi_t}{\Pi_t} - E(\frac{d\Pi_t}{\Pi_t}))^2 \\ E(\frac{dS_t}{S_t} - E(\frac{dS_t}{S_t}))^2 \end{bmatrix} = \begin{bmatrix} \sigma_\Pi'\sigma_\Pi \\ \sigma_S'\sigma_S \end{bmatrix}. \tag{34}$$

The values of these equations implied by parameter estimates can be used to measure the goodness-of-fit of the model, and to impose restrictions. The short-term values can deviate from these long-term expectations through the dynamics in the state variables.

4.3 Likelihood

This section contains the description of the bond yields. For the KNW model with the model assumptions in (Draper, 2014) and (Van den Berg, 2016), two yields are observed without measurement error. For those yields it holds that

$$y_t^\tau = (-A(\tau) - B(\tau)'X_t)/\tau, \quad \tau = \tau_5, \tau_6. \tag{35}$$

The observations can be used to determine the state vector X, given a set parameters which determine $A(\tau)$ and $B(\tau)$. The other four yields are observed with a measurement error

$$y_t^\tau = (-A(\tau) - B(\tau)'X_t)/\tau + v_t^\tau \quad \tau = \tau_1, \tau_2, \tau_3, \tau_4, \quad \text{and} \quad v_t' \sim \text{N}(0, \Sigma^\tau) \tag{36}$$

with $v_t' = [v_t^{\tau_1}, v_t^{\tau_2}, v_t^{\tau_3}, v_t^{\tau_4}]$. The matrix Σ^τ is a diagonal matrix as the measurement errors of different yields are assumed to be independent. The yields have volatility σ_{τ_1}, σ_{τ_2}, σ_{τ_3}, or σ_{τ_4}, depending on their maturity. These volatilities are estimated as free parameters.

The log-likelihood

$$\text{LL} = -0.5\left(T\ln(|\Sigma^\tau|) + \sum_{t=1}^{T} v_t(\Sigma^\tau)^{-1}v_t'\right) - 0.5\left(T\ln(|\Sigma|) + \sum_{t=1}^{T} \tilde{\epsilon}_t(\Sigma)^{-1}\tilde{\epsilon}_t'\right) - 0.5T\ln(|B|) \tag{37}$$

is a combination of the likelihood of the measurement errors of the yields, the likelihood of the error terms of the state variables, inflation, and stock return, and the likelihood of the yields that are observed without measurement error. The log-likelihood is maximized with respect to all parameters to find the global optimum. For B holds $B' = [B(\tau_5), B(\tau_6)]$ and $|\Sigma^\tau|$ is the determinant of Σ^τ. Details on the construction of the log-likelihood can be found in the Appendix C.1. The log-likelihood is maximized by minimizing the negative log-likelihood using the method of (Nelder & Mead, 1965). The standard errors of the estimated parameters are computed using the hessian:

$$\text{SE}_i = \sqrt{|((H^{-1})_{ii})|} \tag{38}$$

Here SE is the vector with the standard errors of the estimated parameters, H is the hessian, and $|x|$ is the absolute value of x.

The yields that are observed without error can be of any two maturities out of the six maturities that are used. I will select either the three-month and three-year, or the two-year and five-year yields as being observed without error. Other papers like for example (Shafiq, 2015) choose the two-year and five-year yields. I will also use the three-month and three-year yields to see the effect of fixing the shorter maturities of the yield curve, instead of the longer maturities. Also the three-month yield of the Netherlands is composed differently than the other yields, which will be discussed in Section 7. Choosing the three-year yield in combination with the three-month yield means that yields with both long and short maturities are fixed by the data. The KNW model will be estimated on quarterly data for both pairs of yields to see which combination leads to a higher likelihood and parameter estimates that economically are more feasible. Because (Draper, 2014) and (Van den Berg, 2016) also use quarterly data, it is easier to compare the results for quarterly data instead of monthly data. The yield-pair that performs the best is used for further estimations of the model with monthly data.

The KNW model can also be estimated with all yields being observed without measurement error. In this case Equation 36 holds for all six yields. Without two yields that are observed without measurement error, the values of the two state variables in X_t do not directly follow from Equation 35. Therefore the state vector X_t is estimated using the Kalman filter.
The log-likelihood is equal to

$$\text{LL} = -\frac{Tk}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}(\ln(|F_t|) + v_t F_t^{-1} v_t'), \tag{39}$$

Here k is the number of predictor variables, v_t are the error terms, and F_t the variance-covariance matrix of v_t. The fit of the adjusted model will be compared to the previous model based on the coefficient estimates and implied characteristics, which have to represent the data characteristics well. Therefore the maximum likelihood of the two models are not compared, as these can be susceptible to model misspecification. Therefore the log-likelihood can be reduced to

$$\text{LL} = -\frac{1}{2}\sum_{t=1}^{T}(\ln(|F_t|) + v_t F_t^{-1} v_t'), \tag{40}$$

after removing the constant, as it is irrelevant to the optimization process. The exact procedure of the Kalman filter including the chosen initial values can be found in Appendix C.2.

4.4 Simulations

Following (Van den Berg, 2016), simulations are made of the bond yields based on the estimated parameters. In addition, simulations are made of the inflation and stock return. For the state variables in X and the stock return and inflation, simulations are made using Equation 30. For the different bond yields either Equation 35 or Equation 36 is used, depending on the maturity. The simulations can be used to calculate the mean and volatility of these variables that are implied by the estimated model parameters. For each country the implied mean and volatility with the corresponding 95% confidence interval can be compared to the sample mean and standard deviation to see how well the model fits each dataset. First, 5000 sample paths with the same length as my data are simulated. Then, the average and standard deviation are computed for each sample path. Then, I compute the averages and 95% confidence interval of the means and standard deviations of all sample paths. The bounds of the 95% confidence interval are given by the 2.5% and 97.5% quantiles. With the simulations of the six different yields, the average and volatility of the yield curve can be constructed. It is important to see if the data are within their simulated confidence intervals, so that I know whether the KNW model is able to match the cross-sectional moments of the stock return, inflation rates and bond yields.

5 Parameter restrictions

The Dutch commission (Dijsselbloem et al., 2019) has advised on restricting certain parameters to make the model more in line with their expectations. To measure the impact of imposing these restrictions, the restricted KNW model will be estimated.

The first restriction is that the process is stationary, the term-structure of interest rates converges, and that oscillations in the term-structure of interest rates are avoided. These demands implicate the following three restrictions on the matrix $M = K + \Lambda_1' \Sigma_X$ ((Pelsser, 2019)):

$$(m_{11} - m_{22})^2 + 4m_{12}m_{21} > 0, \tag{41}$$

$$m_{11} + m_{22} > 0, \tag{42}$$

$$m_{11}m_{22} - m_{12}m_{21} > 0. \tag{43}$$

The commission also gives advice on imposing values for the ultimate forward rate (UFR), the unconditional geometric expected stock return over a one-year period, and the unconditional geometric expected inflation over a one-year period. The unconditional geometric expected return of Π_t over a one-year period is given by

$$\ln(1 + r_{\Pi}^g) = \lim_{t\to\infty} \mathrm{E}[\ln(\frac{\Pi_{t+1}}{\Pi_t})] = \delta_{0\pi} - \frac{1}{2}\sigma_\pi' \sigma_\pi \tag{44}$$

The unconditional geometric expected return of S_t over a one-year period is given by

$$\ln(1 + r_S^g) = \lim_{t\to\infty} \mathrm{E}[\ln(\frac{S_{t+1}}{S_t})] = R_0 + \eta_S - \frac{1}{2}\sigma_S' \sigma_S \tag{45}$$

The UFR is defined as the limit for $\tau \to \infty$ of the zero-coupon rates and is given by

$$\ln(1 + UFR) = \lim_{\tau\to\infty} - \frac{A(\tau)}{\tau} = R_0 + \Lambda_0' \Sigma_X B_\infty - \frac{1}{2} B_\infty' \Sigma_X' \Sigma_X B_\infty, \tag{46}$$

with $B_\infty = -(K + \Lambda_1' \Sigma_X)^{-1} R_1$. Following the commission (Dijsselbloem et al., 2019), I impose the values $r_S^g = 5.6\%$, $r_\Pi^g = 1.9\%$, and UFR= 2.1%.

The last restriction is that not more than 2.5% of the ten-year yields from the unconditional distribution can be negative. To this end I calculate the 2.5% quantile of the distribution of the ten-year yield at a horizon of five-years; i.e. $p(y_T^\tau|I_0)$ for $T = 60$ and $\tau = 10$. If this number is negative, it means that the probability of the ten-year yield at a ten-year horizon is larger than 2.5% and the restriction is not upheld. From Equation 30 follows the autoregressive Equation of X_{60}:

$$\begin{aligned} X_{t+1} &= \Gamma X_t + \epsilon_{t+1}, \quad \text{as} \quad \mu_X = 0 \\ X_T &= \Gamma^T X_0 + \sum_{t=1}^{T} \Gamma^{T-t}\epsilon_t. \end{aligned} \tag{47}$$

This leads to the following expectation and variance:

$$E(X_T|I_0) = 0 \tag{48}$$

$$V(X_T|I_0) = \sum_{t=1}^{T} \Gamma^{T-t}\Sigma(\Gamma^{T-t})'. \tag{49}$$

From Equation 36 then follow the mean and variance of $y_T^\tau|I_0$:

$$\begin{aligned} E(y_T^\tau|I_0) &= -(A(\tau) + B(\tau)'E(X_T|I_0))/\tau \\ &= -A(\tau)/\tau, \end{aligned} \tag{50}$$

$$V(y_T^\tau|I_0) = B(\tau)'V(X_T|I_0)B(\tau)/\tau^2 + (\sigma^\tau)^2, \tag{51}$$

and the corresponding distribution

$$y_T^\tau|I_0 \sim \mathrm{N}(E(y_T^\tau|I_0), V(y_T^\tau|I_0)). \tag{52}$$

An optimum for the model should imply a 2.5% quantile of the normal distribution in Equation 52 that is above zero. For optimizing the model with this restriction a penalty-function is invoked, which means the penalty on the likelihood increases as the 2.5% quantile becomes more negative.

6 Structural break

Since 1972, the beginning of the sample period of (Draper, 2014), the behaviour of the predictor variables and underlying state variables has varied over time. For example, the average stock return, inflation and bond yields are all lower in the last fifteen years compared to the period between 1972 and 1990. The exact numbers are shown in the data section. Inflation and bond yields even show a clear decreasing pattern over time. Because the KNW model implies a mean-reverting process for these variables, it can be difficult to give an economic interpretation to the parameter estimates. Given these changes over time, it is possible that there is a structural break in the parameter estimates of different time periods. This means the parameter estimates and implied model characteristics are significantly different between different time periods. (Andrews, 1993) has introduced methods of detecting structural change even when the point of change is unknown. Compared to many countries, the time period from 1972 until 2021 has been a period of relative political and economic stability for the Netherlands. Therefore, it is difficult to know which point in time is a moment of structural change. The method described in (Andrews, 1993) takes an entire grid of points into consideration and determines whether it contains a moment of structural break. Based on the test statistic the null hypothesis of no structural breaks is either accepted or rejected. $\pi \in \Pi$ is a potential point of change and Π the set of potential moments of change. π is a fraction indicating the proportion of the observations before the break point. If T is the total number of observations, then the null hypothesis is given by

$$H_0 : \beta_{1,[\pi T]} = \beta_{[\pi T]+1,T} = \beta_0, \quad \text{for some} \quad \beta_0 \subset R^p \tag{53}$$

Here p is the number of free parameters in the KNW model, $[x]$ is the integer closest to x, and $\beta_{i,j}$ is the vector containing the parameter estimates of the KNW model using observations i until j. The alternative hypothesis, implying a one-time structural break at point π is given by

$$H_1(\pi) : \beta_t = \begin{cases} \beta_1(\pi), & \text{for} \quad t = 1, ..., \pi T \\ \beta_2(\pi) & \text{for} \quad t = \pi T, ..., T \end{cases} \tag{54}$$

To check for a structural break at point π a likelihood ratio test can be applied. This test compares the maximum likelihood for the entire sample period to the sum of the maximum likelihood of the observations before πT and the maximum likelihood of the observations after πT. The test statistic is given by

$$LR_T(\pi) = -2(LL_{1,T} - (LL_{1,k} + LL_{k+1,T})), \tag{55}$$

where $LL_{i,j}$ is the likelihood in Equation 35 for the parameter estimates based on observations i until j. Because the model using a single estimation period is a restricted version of the model using two estimation periods, $LL_{1,k} + LL_{k+1,T}$ will always be larger than $LL_{1,T}$. The likelihood ratio test determines whether the difference between these two figures is significant. As the exact point of change is unknown, it is necessary to test for a wide range of points. If $\Pi \subset [0,1]$ is the set of possible change points, then the statistic that tests for an unknown change point is defined as

$$\sup_{\pi\in\Pi} LR_T(\pi). \tag{56}$$

As a large value for $LR_T(\pi)$ corresponds to a structural break at point π, a large value for $\sup_{\pi\in\Pi} LR_T(\pi)$ corresponds to a structural break somewhere in Π. For any set Π whose closure lies in $(0,1)$, and $\pi \in \Pi$ holds ((Andrews, 1993))

$$\sup_{\pi\in\Pi} LR_T(\pi) \rightarrow_d \sup_{\pi\in\Pi} Q_p(\pi) \tag{57}$$

where $Q_p(\pi) = (B_p(\pi) - \pi B_p(1))'(B_p(\pi) - \pi B_p(1))/(\pi(1-\pi)$ is the square of a standardize tied-down Bessel process of order p. $B_p(\pi)$ is a p-vector of independent Brownian motions on $[0,1]$ restricted to Π. $Q_p(\pi)$ has a chi-square distribution with p degrees of freedom. The distribution of the supremum is different, and its critical values were computed by simulation in (Andrews, 1993). The critical values depend on the parameter p and the boundaries of the interval Π. p is the difference in dimensionality between the parameter space under the null hypothesis and under the alternative hypothesis. Because under the alternative hypothesis the model is estimated for two separate time periods, p is equal to 27, the number of free parameters in the KNW model. Given the fact that the change point is unknown, it would make sense to choose $\Pi = (0,1)$, to choose a domain that is as wide as possible. However, in that case the supremum diverges to infinity in probability ((Andrews, 1993)). Also, given the large number of observations, estimating the model using every possible break point would be very time-consuming. If Π is bounded away from zero and one, the test statistic converges in distribution. I will use a lower bound of 0.15 and an upper bound of 0.85 for Π, as suggested by (Andrews, 1993). To limit the computational time of estimating the model for different time period, the minimal difference between two distinct values of π will be 0.05, implying $\Pi = [0.15, 0.2, ..., 0.85]$. The critical values of the supremum test can be found in (Andrews, 2003), which contains more accurate estimates of the critical values compared to (Andrews, 1993), as they are based on a larger number of simulations. As these papers only contain the critical values for $p <= 20$, the Figure for $p = 27$ is determined through linear extrapolation based on the critical values for

$p = 19$ and $p = 20$. This method will give estimates that are very close to the real values, because the critical values for $p = 14, ..., 20$ roughly follow a linear trend with a constant increase. It is a one-sided test, as the null hypothesis is only rejected for large values of $\sup_{\pi \in \Pi} LR_T(\pi)$.

7 Data

I will use quarterly and monthly data for inflation, yields and stock return. The quarterly dataset runs from the fourth quarter of 1972 until the fourth quarter of 2013, while the monthly dataset runs from September 1972 until January 2021. Six yields are used in estimation: three-month, one-year, two-year, three-year, five-year, and ten-year maturities, respectively. For the Netherlands, United States, and the United Kingdom data is available to estimate the KNW model. The yields are all quoted on a yearly basis. For both the inflation and stock return yearly log-returns are taken for each month in the estimation period. For the quarterly dataset every third observation is taken of these monthly observations. As a measure for stock returns for all countries, the returns on the MSCI World Index are taken. The MSCI World Index is taken from www.msci.com. Appendix A contains the data sources of inflation and the yields for each country. For some yields, different measures are used for different time periods, as not all rates are available for the entire sample period. Sometimes with the yield data, there are one-off missing observations. The values of these missing observations are estimated through interpolation. Some figures in the results section contain vertical shading for periods of recession. In these cases a recession is defined as a period of at least two quarters with negative growth. For each country quarterly GDP growth is retrieved from data.oecd.org.

This section contains figures with the movements of the data over time, and tables that contain summary statistics. It is important to verify whether these numbers are in line with the characteristics of stock returns, inflation rates, and bond yields. Also we can see whether there are major differences between countries that should also emerge when estimating the KNW model for the datasets of these different countries.

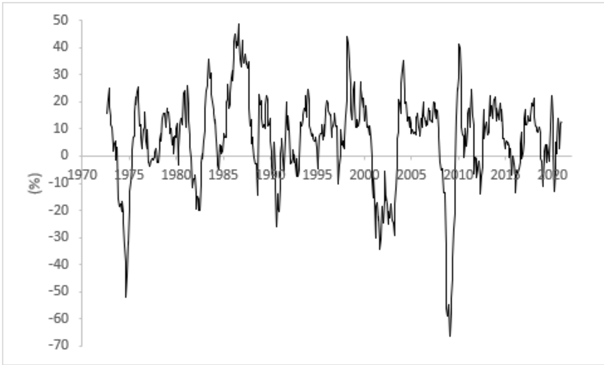

Figure 1: The yearly returns on the MSCI World Index.

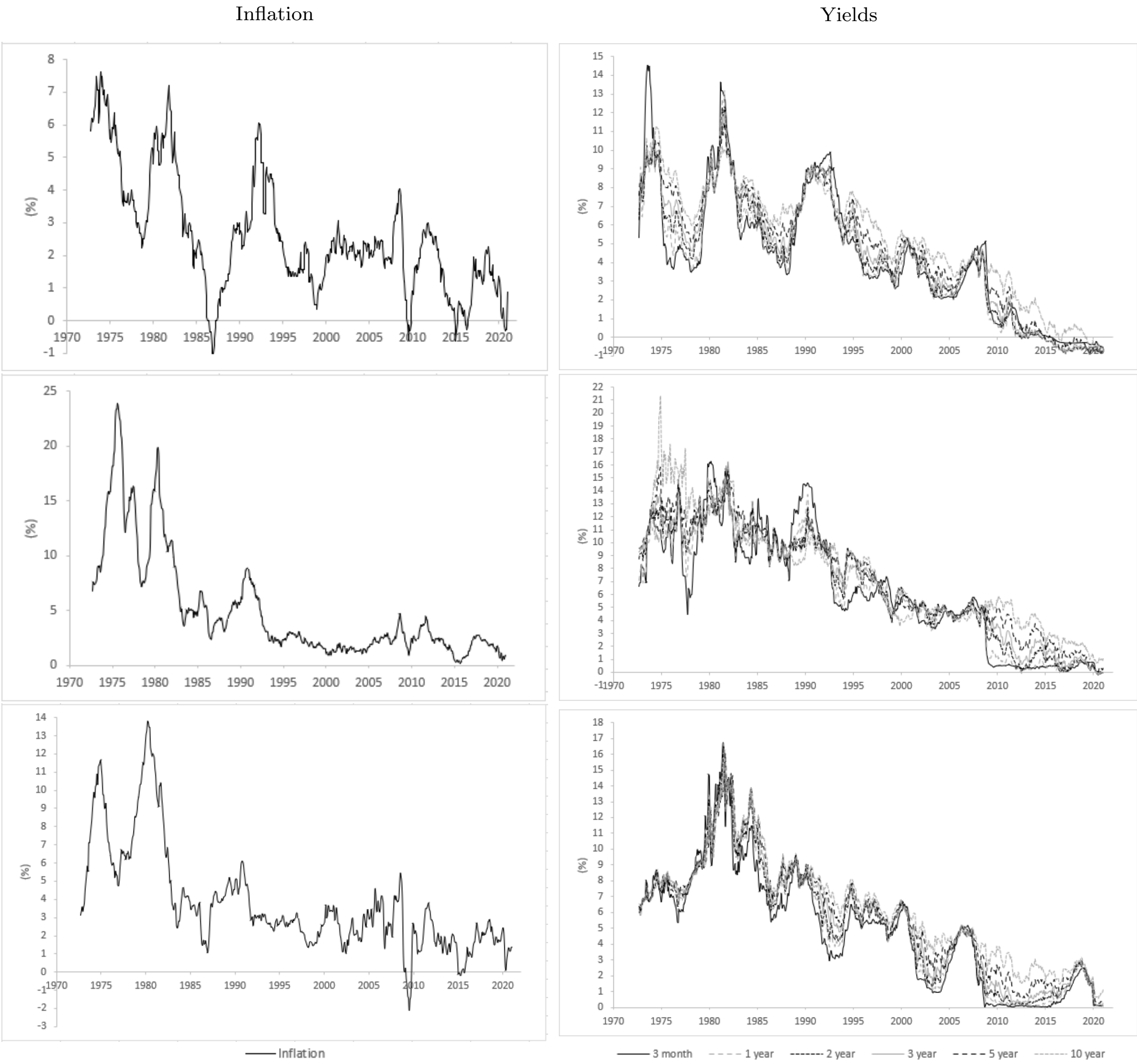

Figure 2: The inflation and the bond yields for six different maturities are noted for the Netherlands (top), UK (middle), and US (bottom).

Figure 1 contains the returns on the MSCI World Index over time, which is the measure for stock return for each country that is considered, while Figure 2 contains for each country the inflation and bond yields. Tables 1 and 2 contain the summary statistics of the quarterly and monthly datasets for the Netherlands, respectively. The summary statistics of the stock returns, that are the same for each country, are also noted in Table 2. Inflation and bond yields are high in the 1970s and 1980s but follow a decreasing trend since 1981. These observations are confirmed by the numbers in Table 3, which contains the average inflation and yields for three consecutive time periods and the estimation period from (Draper, 2014). The average inflation and yields are indeed lower for the more recent time periods.

Two important characteristics of the yield curve are that it has a positive slope and that a high positive correlation exists between yields of different maturities. Table 3 shows that the average yield is increasing with maturity with the exception of the three-month yield average, which is slightly higher than the one-year yield average. This could be caused by the fact that the German Interbank rates are slightly higher than the money market rates from the Bundesbank.

			Yields					
	Stock return	Inflation	3-month	1-year	2-year	3-year	5-year	10-year
Average	6.22%	2.85%	5.01%	4.87%	5.11%	5.35%	5.71%	6.21%
Std.dev.	17.50%	1.76%	3.06%	2.79%	2.71%	2.64%	2.52%	2.24%
Maximum	45.10%	7.19%	14.49%	12.69%	11.92%	11.64%	11.22%	11.3%
Minimum	-56.39%	-0.78%	0.20%	-0.10%	-0.08%	0.03%	0.35%	1.26%

Table 1: Summary Statistics of the quarterly dataset of the Netherlands

		Yields						
	Stock return	Inflation	3-month	1-year	2-year	3-year	5-year	10-year
Geometric average	4.65%	2.56%	2.96%	2.53%	2.71%	2.93%	3.40%	4.25%
Average	6.22%	2.58%	4.27%	4.12%	4.32%	4.53%	4.86%	5.37%
Std.dev.	16.71%	1.81%	3.38%	3.24%	3.21%	3.21%	3.16%	2.97%
Maximum	48.50%	7.62%	14.57%	13.17%	12.33%	12.00%	11.49%	11.30%
Minimum	-66.19%	-1.00%	-0.55%	-0.92%	-0.92%	-0.95%	-0.93%	-0.71%

Table 2: Summary Statistics of the monthly dataset of the Netherlands

			Yields					
	Stock return	Inflation	3-month	1-year	2-year	3-year	5-year	10-year
1972.09-1990.12	7.95%	3.64%	6.79%	6.75%	7.06%	7.33%	7.67%	8.01%
1991.01-2005.12	5.90%	2.42%	4.51%	4.42%	4.60%	4.81%	5.18%	5.73%
2006.01-2021.01	4.42%	1.44%	0.97%	0.63%	0.72%	0.84%	1.13%	1.79%
1972.09-2013.12	6.21%	2.86%	5.03%	4.90%	5.13%	5.36%	5.72%	6.22%
1972.09-1990.12		9.51%	10.93%	10.80%	11.15%	11.33%	11.73%	12.00%
1991.01-2005.12		2.44%	5.83%	6.06%	6.40%	6.53%	6.56%	6.37%
2006.01-2021.01		2.10%	1.28%	1.31%	1.71%	2.11%	2.71%	3.53%
1972.09-1990.12		6.27%	8.60%	9.03%	9.14%	9.31%	9.35%	9.41%
1991.01-2005.12		2.68%	4.03%	4.52%	4.79%	5.11%	5.46%	6.00%
2006.01-2021.01		1.88%	1.07%	1.24%	1.40%	1.61%	2.04%	2.71%

Table 3: Averages over time of the variables of the monthly dataset of the Netherlands (top), UK (middle), and US (bottom). The part of the table for the Netherlands also contains the averages over time of the stock returns.

Tables 4 and 5 contain the summary statistics of the monthly datasets of the UK and US, respectively. The main difference between the Netherlands, UK, and US are the average inflation and yields and the stability of the factors in terms of their standard deviation. The Netherlands on average has the lowest and most stable inflation rates and yields, while the UK has the highest. The average inflation in the Netherlands is 2.58% with a standard deviation of 1.81%, while the average inflation in the UK is 5.01% with a standard deviation of 4.92% over the entire sample period. Therefore, when estimating the KNW model, significant differences between the estimates of parameters related to inflation and bond yields can be expected.

		Yields					
	Inflation	3-month	1-year	2-year	3-year	5-year	10-year
Geometric average	4.90%	4.51%	4.63%	5.13%	5.52%	6.05%	6.57%
Average	5.01%	6.34%	6.38%	6.74%	6.97%	7.32%	7.62%
Std.dev.	4.92%	4.52%	4.26%	4.25%	4.18%	4.14%	4.17%
Maximum	23.80%	16.28%	15.05%	15.87%	16.26%	15.83%	21.26%
Minimum	0.20%	-0.08%	-0.14%	-0.21%	-0.20%	-0.01%	0.76%

Table 4: Summary Statistics of the monthly dataset of the UK

		Yields					
	Inflation	3-month	1-year	2-year	3-year	5-year	10-year
Geometric average	3.75%	3.42%	3.81%	4.09%	4.41%	4.87%	5.50%
Average	3.79%	4.84%	5.21%	5.38%	5.61%	5.87%	6.27%
Std.dev.	2.85%	3.71%	3.77%	3.71%	3.66%	3.48%	3.23%
Maximum	13.76%	16.73%	16.69%	16.45%	16.27%	16.05%	15.78%
Minimum	-2.12%	0.00%	0.07%	0.11%	0.11%	0.21%	0.55%

Table 5: Summary Statistics of the monthly dataset of the US

Given that the yields used for the Netherlands, UK, and US are composed of different statistics from several sources, a high correlation isn't self-evident. Table 6 contains the correlation matrix for the predictor variables of each country. The table shows that indeed a high correlation exists between the yields of all different maturities. There is also a strong correlation between inflation and bond yields, while for the stock returns the correlation is close to zero.

Because the dynamics of the stock returns, inflation rates and bond yields are modelled using two state variables, using highly correlated data can lead to a better fit of the model with more significant parameter estimates. Also, if two yields are measured without error, then for a given parameter set the observation error for the other yields will be smaller in case of a high correlation between all yields. The US has the highest correlation between the different yields, while the UK has the lowest.

	Inflation	Stocks	Yields 3-month	1-year	2-year	3-year	5-year	10-year
Inflation	1	-0.245	0.777	0.752	0.741	0.730	0.715	0.693
Stocks		1	-0.137	-0.086	-0.071	-0.059	-0.049	-0.052
3-month			1	0.976	0.960	0.944	0.919	0.882
1-year				1	0.994	0.985	0.967	0.934
2-year					1	0.997	0.986	0.960
3-year						1	0.995	0.976
5-year							1	0.991
10-year								1
Inflation	1	-0.108	0.664	0.673	0.707	0.724	0.751	0.846
Stocks		1	-0.012	-0.015	-0.023	-0.026	-0.025	-0.011
3-month			1	0.973	0.948	0.933	0.907	0.818
1-year				1	0.990	0.979	0.957	0.876
2-year					1	0.997	0.981	0.913
3-year						1	0.991	0.932
5-year							1	0.962
10-year								1
Inflation	1	-0.105	0.749	0.711	0.700	0.691	0.666	0.643
Stocks		1	0.115	0.120	0.115	0.096	0.081	0.086
3-month			1	0.992	0.986	0.971	0.951	0.931
1-year				1	0.998	0.989	0.974	0.958
2-year					1	0.993	0.981	0.968
3-year						1	0.992	0.982
5-year							1	0.995
10-year								1

Table 6: The correlation matrix for each explanatory variable for the Netherlands (top), UK (middle), and US (bottom).

Table 7 contains the autocorrelation of the variables. According to the efficient market hypothesis, stock returns from different time periods should be uncorrelated, because new information comes to the market in a random, independent way, and security prices quickly adjust to this new information. Table 7 shows that the serial correlation of stock returns is close to zero for longer lags. The autocorrelation for shorter lags is very high for shorter lags, because the data consists of monthly observations of yearly returns. This means there is overlap between observations that are within a year of each other, which leads to significantly large autocorrelation. In part, this also explains the large serial correlation that exists for both inflation rates and three-month yields.

For inflation a larger and longer persisting serial correlation exists compared to stock return. This is because the price indices for all countries consistently increase over time and therefore produce inflation rates that are mostly positive and of a similar magnitude. The Netherlands has the lowest serial correlation for all lags, while the UK has the highest.

The 3 month yields are very persistent even for longer time periods. The same pattern can be found for the one-year to ten-year yields, which can be seen in the autocorrelation tables in Appendix A.

	1 month	2 months	3 months	6 months	1 year	2 years
Stock return	0.934	0.851	0.775	0.520	-0.004	-0.103
Inflation						
Netherlands	0.983	0.962	0.939	0.869	0.700	0.447
US	0.990	0.973	0.954	0.894	0.750	0.535
UK	0.995	0.985	0.973	0.926	0.817	0.664
3-month yield						
Netherlands	0.993	0.982	0.968	0.916	0.792	0.548
US	0.991	0.980	0.969	0.943	0.880	0.754
UK	0.991	0.981	0.969	0.935	0.867	0.759

Table 7: Autocorrelation of each variable for lags between 1 month and 2 years

8 Results

8.1 Quarterly data

Table 8 contains the estimation results for the KNW model with two yields that are observed without measurement error using quarterly data. The goal is to determine which pair of yields should be observed without error when estimating the KNW model. In addition we can see the influence of adding the years 2014 to 2021 to the estimation period in Draper (2014) and Van den Berg (2016). To this end, inference is drawn from comparing the estimation results with those from Draper (2014) and Van den Berg (2016), and from comparing model implied characteristics with their data counterparts.

In the first column, the three-month and three-year yields (denoted as (3,36)) are observed without error, while in the second column the two-year and five-year yields (denoted as (24,60)) are observed without error. The third column are the estimates from (Van den Berg, 2016), and the fourth column are the estimates from (Draper, 2014). The main difference between my estimation and the estimation of (Van den Berg, 2016) and (Draper, 2014) is that the composition of the variables is slightly different.

The results in column (i) are comparable to those from (Van den Berg, 2016) and (Draper, 2014), except for the parameters related to the stock return process and the prices of risk. Also there are some differences between the estimates in column (i) and (ii), especially for the stock return process and the prices of risk.

Figure 3 shows the filtered values for X_1 and X_2 over the sample period. Because these state variables are determined using Equation 35, they are negatively correlated with the values of the yields that are observed without error. X_1 and X_2 have steadily increased over time, when the yields have decreased over time. The grey bars are the periods of recession for the Netherlands. Given that interest rates peak during recessions, it can be expected that the state variables reach their low points during recessions. This is indeed the case for X_1; while X_1 dips and then quickly increases during recessions, X_2 is more stable over the entire estimation period.

The state variables have a very high persistence with increased first-order autocorrelations for X_1 and X_2 of 0.97 and 0.89 in the left figure and 0.98 and 0.96 in the right figure compared to 0.88 and 0.91 for (Draper, 2014). The larger persistence of the state variables comes with a larger significance of the parameters κ_{11} and κ_{22}, as can be seen in Table 8. According to (Draper, 2012), a larger persistence in the state variables leads to less significant estimates. This is not in line with the results in Table 8, as many parameter estimates are more significant than the estimates from (Draper, 2014). This can partially be explained by the larger number of observations. The correlation between X_1 and X_2 is 0.73 for the $(3, 36)$-pair and 0.95 for the $(24, 60)$-pair, which comes with a larger significance of the parameter κ_{21} for estimation (ii). The state variables for the $(24, 60)$-pair are less stable over time compared to the state variables for the $(3, 36)$-pair.

Figure 3: The filtered state variables X_1 and X_2 with recession shading for the (3,36) and (24,60) pairs of yields that are observed without error.

Now that I've looked at the behaviour of the state variables over time, it is important to consider the parameter estimates and the behaviour of the filtered variables. The values of these variables are implied by the values of the state variables.

$\delta_{0\pi}$, representing the long-term average inflation expectations, is equal to 2.26% for (i) and 1.76% for (ii). This means the parameter estimates for (i) are closer to the average inflation of 2.85%, while the estimate in (ii) is closer to the estimates in column (iii) and (iv). The left part of Figure 4 contains the filtered expected inflation (π_t) for both estimations and the sample inflation. The expected inflation over time for both estimations is very similar. The correlation between the filtered expected inflation and the sample inflation of 0.73 for (i) is slightly better compared to the correlation of 0.70 for (ii).

η_S represents the historical risk premium on equities and is estimated to be 2.98% for (i) and 3.52% for (ii). The historical risk premium of the quarterly dataset is the difference between the average stock return, which is equal to 6.22%, and the average three-month yield, which is equal to 5.01%. This means the historical risk premium of the dataset is equal to 1.21%, which is significantly lower than the estimated value for η_S for both (i) and (ii), although the estimate for (i) is closer.

The value of R_0 in column (i) is different to the estimates from (Van den Berg, 2016) and (Draper, 2014), and about 1% lower than the average three-month yield. The estimate for (ii) is closer to the average three-month yield. The right part of Figure 4 contains the nominal interest rate R_t over time for both (i) and (ii), and the sample three-month yield. The movements of R_t over time for (i) and (ii) are very similar. The correlation between R_t and the sample three-month yield is 0.994 for estimation (i) and 0.965 for estimation (ii). The high correlation is also apparent when looking at Figure 4, as the filtered nominal interest rate follows almost exactly the same pattern as the sample yield.

In theory, the sum of the long-term money market rate (R_0) and the equity risk premium (η_S) should be equal to the expected equity return. Indeed, adding the parameters η_S and R_0 gives a value of 6.61%, which is close to the average stock return from the dataset (6.22%). In column (ii) the sum is equal to 7.46%, which means the estimates in column (i) better represent the sample data.

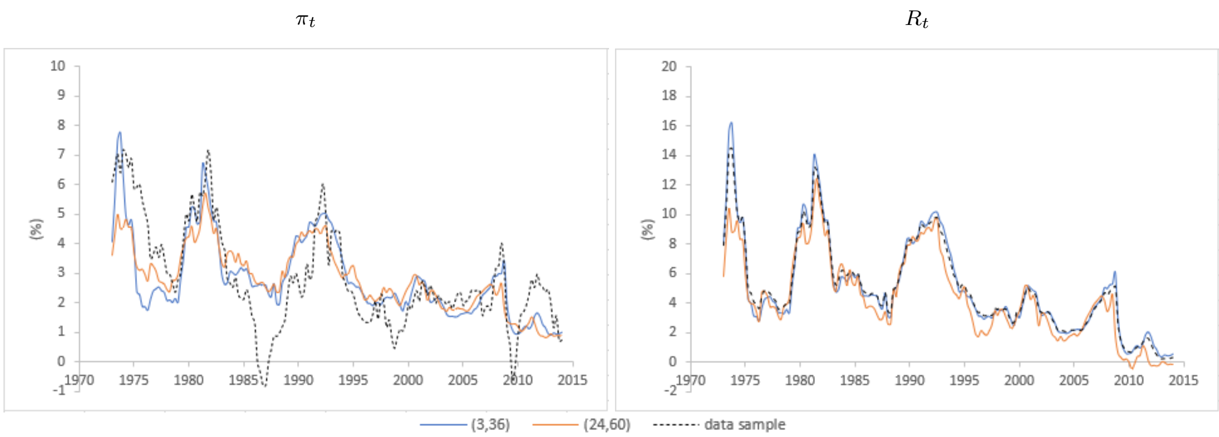

Figure 4: The filtered variables π_t and R_t with the sample inflation rates and three-month yields.

$\sigma_{S(4)}$, which represents the volatility of stock returns, is equal to 17.61% for (i) and 16.64% for (ii) which are both very close to the unconditional standard deviation of the sample stock return in Table 2 and quite close to the estimates in (iii) and (iv). The estimate for (ii) is closer to both the unconditional standard deviation of the sample stock return (16.71%) and the estimates of (Van den Berg, 2016) and (Draper, 2014). My estimates of elements one, two, and three of the vector σ_S are a lot higher in absolute terms compared to (Van den Berg, 2016) and (Draper, 2014), meaning that the impact of uncertainty about the real interest rate, instantaneous expected inflation, and unexpected inflation is larger. The estimates for $\sigma_{S(3)}$ are positive, contrary to (Van den Berg, 2016) and (Draper, 2014), which means that the uncertainty about unexpected inflation has a positive effect on the stock return process. For all three parameters the estimate in column (i) is closer to the estimates in columns (iii) and (iv) than the estimate in column (ii).

The unconditional price of real interest rate risk is lower than the unconditional price of expected inflation risk, as $\Lambda_{0(1)}$ is smaller than $\Lambda_{0(2)}$. This is contrary to (Van den Berg, 2016) and (Draper, 2014) where the unconditional price of real interest rate risk is larger.

	(i) 1972.4 - 2013.4 (3,36)		(ii) 1972.4 - 2013.4 (24,60)		(iii) 1972.4 - 2014.4		(iv) 1972.4 - 2013.4	
Parameter	Estimate	Std. error	Estimate	Std. error	Estimate	Std. error	Estimate	Std. error
Expected inflation $\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t$								
$\delta_{0\pi}$	2.26%	(1.21%)	1.76%	(0.55%)	1.98%	(4.05%)	1.81%	(2.79%)
$\delta_{1\pi(1)}$	-0.84%	(0.06%)	-0.74%	(0.19%)	-0.60%	(0.20%)	-0.63%	(0.10%)
$\delta_{1\pi(2)}$	1.25%	(0.24%)	1.40%	(0.57%)	0.27%	(0.41%)	0.14%	(0.24%)
Nominal interest rate $R_t = R_0 + R_1' X_t$								
R_0	3.63%	(2.95%)	3.94%	(3.42%)	1.98%	(10.40%)	2.40%	(6.06%)
$R_{1(1)}$	-1.95%	(0.08%)	-2.20%	(0.07%)	-1.44%	(0.38%)	-1.48%	(0.22%)
$R_{1(2)}$	2.75%	(0.41%)	4.64%	(0.65%)	0.56%	(0.97%)	0.53%	(0.56%)
Process real interest rate and expected inflation $dX_t = -KX_t dt + \Sigma_X' dZ_t$								
κ_{11}	0.05	(0.04)	0.01	(0.01)	0.06	(0.17)	0.08	(0.11)
κ_{21}	-0.36	(0.08)	-0.22	(0.02)	-0.22	(0.20)	-0.19	(0.08)
κ_{22}	1.13	(0.23)	0.49	(0.10)	0.32	(0.22)	0.35	(0.18)
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\pi' dZ_t$								
$\sigma_{\pi(1)}$	-0.01%	(0.21%)	-0.11%	(0.29%)	0.02%	(0.08%)	0.02%	(0.07%)
$\sigma_{\pi(2)}$	-0.14%	(0.25%)	0.40%	(0.63%)	-0.02%	(0.06%)	-0.01%	(0.06%)
$\sigma_{\pi(3)}$	-1.11%	(0.07%)	-1.10%	(0.13%)	0.61%	(0.04%)	0.61%	(0.04%)
Stock return process $\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma_S' dZ_t$								
η_S	2.98%	(1.55%)	3.52%	(1.55%)	4.20%	(3.77%)	4.52%	(3.73%)
$\sigma_{S(1)}$	1.69%	(2.65%)	2.55%	(2.40%)	-0.54%	(1.44%)	-0.53%	(1.44%)
$\sigma_{S(2)}$	-6.54%	(2.08%)	-6.93%	(4.39%)	-0.78%	(1.54%)	-0.76%	(1.54%)
$\sigma_{S(3)}$	4.87%	(1.59%)	7.11%	(1.54%)	-2.23%	(1.46%)	-2.11%	(1.51%)
$\sigma_{S(4)}$	17.61%	(1.03%)	16.64%	(0.97%)	16.39%	(0.93%)	16.59%	(0.96%)
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$								
$\Lambda_{0(1)}$	0.289	(0.027)	0.629	(0.177)	0.187	(0.513)	0.403	(0.333)
$\Lambda_{0(2)}$	0.583	(0.424)	1.067	(0.097)	0.137	(0.624)	0.039	(0.270)
$\Lambda_{1(1,1)}$	0.160	(0.045)	0.172	(0.003)	0.142	(0.184)	0.149	(0.156)
$\Lambda_{1(1,2)}$	-0.576	(0.036)	-0.345	(0.008)	-0.355	(0.037)	-0.381	(0.039)
$\Lambda_{1(2,1)}$	-0.047	(0.075)	0.167	(0.007)	0.144	(0.192)	0.089	(0.075)
$\Lambda_{1(2,2)}$	0.506	(0.237)	-0.194	(0.103)	-0.100	(0.211)	-0.083	(0.129)
LL	5103.8		5152.0		6720.4		6525.6	

Table 8: The parameters and standard errors of the maximum likelihood estimate of the KNW model with 2 out of 6 yields observed without a measurement error for different quarterly datasets. Column (i) and (ii) are the maximum likelihood estimates using quarterly observations of the data described in Section 7 over the sample period 1972.4 - 2013.4. For column (i) the 3 month and 3 year yields are observed without measurement error, while in column (ii) the 2 and 5 year yields are observed without measurement error. Column (iii) contains the estimates for data from (Van den Berg, 2016) over the sample period 1972.4 - 2014.4 and column (iv) contains the estimates for data from (Draper, 2014) over the sample period 1972.4 - 2013.4.

Most parameter estimates that I have considered lead to the conclusion that the model where the three-month and three-year yields are fixed, is more in accordance with the sample data. Next, I will consider a number of model implied characteristics to see if similar inference can be drawn from those statistics. Appendix D contains a description of how to calculate these statistics.

Table 9 contains the bond risk premia for the six different yields used for the parameter estimates in Table 8. The risk premia in column (i) are close to the risk premia in columns (iii) and (iv). However, the risk premia in the second column are all negative and significantly different to the risk premia in columns (iii) and (iv). Also the volatilities are much larger. This can be explained by the fact that both prices of risk in the vector Λ_0 in column (i) are closer to the estimates of (Van den Berg, 2016) and (Draper, 2014). Because the gap between the long-term and short-term yields for my estimates is larger compared to (Van den Berg, 2016) and (Draper, 2014), the Sharpe ratios are also more volatile across different maturities.

	1972.4-2013.4 (3,36)			1972.4-2013.4 (24,60)			(Van den Berg, 2016)			(Draper, 2014)		
	$y_\tau - y_0$	σ	S	$y_\tau - y_0$	σ	S	$y_\tau - y_0$	σ	S	$y_\tau - y_0$	σ	S
3-month	-0.18%	0.70%	-0.26	-0.84%	1.22%	-0.68	0.05%	0.37%	0.13	0.14%	0.38%	0.37
1-year	-0.20%	1.83%	-0.11	-2.75%	4.17%	-0.66	0.20%	1.32%	0.15	0.52%	1.33%	0.39
2-year	0.20%	2.75%	0.07	-4.26%	6.86%	-0.62	0.42%	2.35%	0.18	0.95%	2.35%	0.40
3-year	0.67%	3.63%	0.18	-4.95%	8.56%	-0.58	0.64%	3.23%	0.20	1.31%	3.24%	0.40
5-year	1.56%	5.37%	0.29	-4.95%	10.25%	-0.48	1.08%	4.90%	0.22	1.92%	4.94%	0.39
10-year	3.38%	9.07%	0.37	-2.43%	10.99%	-0.22	2.09%	9.02%	0.23	3.07%	8.97%	0.34

Table 9: This table contains the bond risk premia ($y_\tau - y_0$), their respective volatilities (σ) and the Sharpe ratios (S) implied by the estimates in Table 8.

Table 10 contains the implied correlation between stocks, bonds and bond risk premia for the parameter estimates in columns (i), (ii), and (iii) of Table 8. Because the parameter estimates of σ_S in column (i) and (ii) are significantly different to the estimates in (iii), the implied correlations between stocks and the different yields are also different. Instead of a correlation close to zero, the correlations are positive and decreasing with maturity. For column (i) the correlation coefficients are low for longer maturities which is in line with the data and with the implied correlations for (Van den Berg, 2016).

For column (ii), the implied correlations between the three-month yield and the other yields are close to one, as is indicated by the data. For column (i) however the implied correlation between the three-month bond and longer-term bonds is significantly smaller, although this is also the case for (Van den Berg, 2016).

	Stocks	3-month bond	1-year bond	2-year bond	3-year bond	5-year bond	10-year bond
Stocks	1	0.31	0.27	0.20	0.14	0.09	0.04
3-month bond		1	0.97	0.86	0.76	0.64	0.52
5-year risk premium	0.35	0.91	0.78	0.58	0.43	0.26	0.12
10-year risk premium	0.35	0.92	0.80	0.60	0.45	0.29	0.15
Stocks	1	0.38	0.37	0.37	0.37	0.36	0.33
3-month bond		1	1.00	1.00	0.99	0.98	0.92
5-year risk premium	0.23	0.52	0.50	0.47	0.44	0.36	0.14
10-year risk premium	0.37	0.96	0.95	0.94	0.92	0.89	0.76
Stocks	1	-0.02	-0.02	-0.03	-0.04	-0.05	-0.05
3-month bond		1	0.99	0.96	0.91	0.79	0.55
5-year risk premium	0.03	0.71	0.62	0.49	0.36	0.13	-0.20
10-year risk premium	0.02	0.76	0.68	0.56	0.43	0.20	-0.13

Table 10: This table contains the correlation between stocks, bonds, and risk premia implied by the estimates where the three-month and three-year yields are observed without error (upper part), the two-year and five-year yields are observed without error (middle part), and the estimates from (Van den Berg, 2016) (bottom part).

Lastly, I will consider the averages and volatility of the simulations of the yield curve implied by the parameter estimates in Table 8. These can be found in Figure 5. The real average yield curve is between 1.5% and 2% higher than the simulated average yield curve for the estimates in column (i), but is still comfortably within the bounds of the 95% confidence interval. The corresponding standard deviation of the simulations is very close to the standard deviation of the yields from the dataset. For the estimates in column (ii) of Table 8 the simulations aren't very accurate. The actual average yield curve is outside the bounds of the 95% confidence interval for larger maturities. Also the simulated average yield curve is inverted, which is not in line with the sample average yield curve. The volatility of the yields is close to the simulated average for larger maturities.

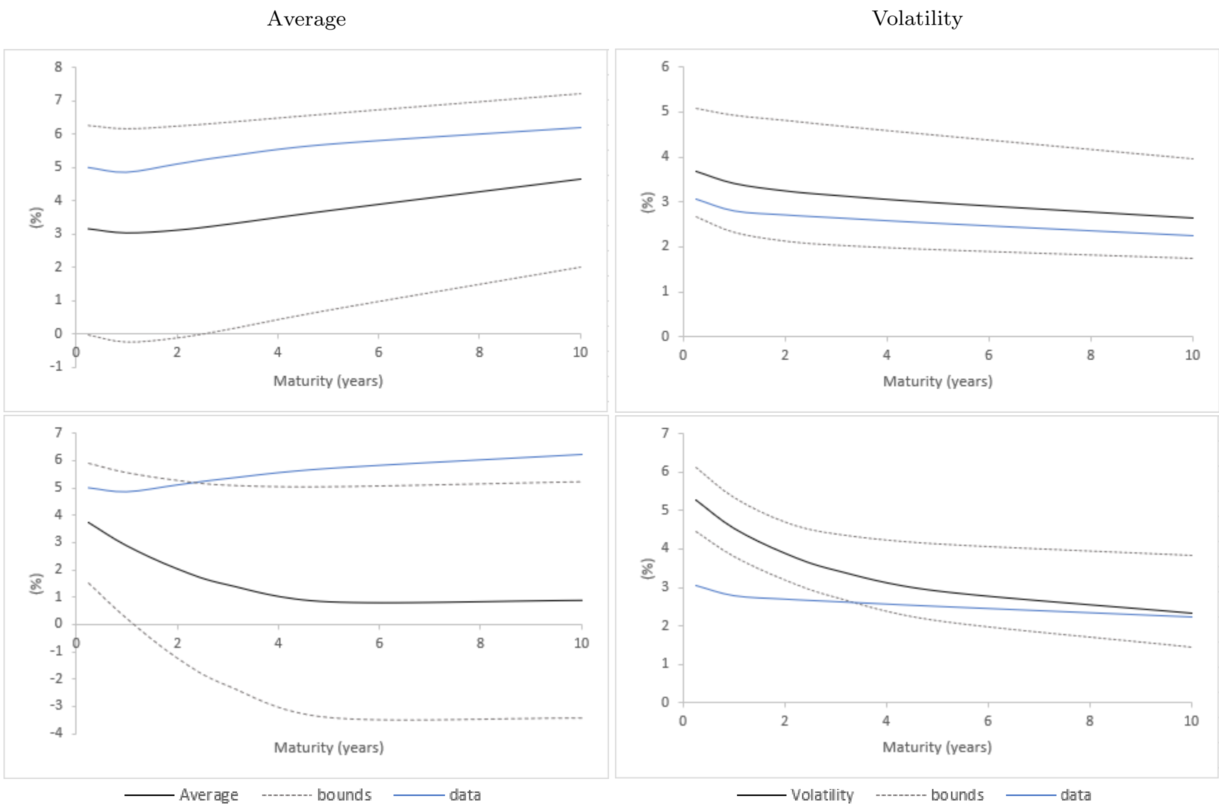

Figure 5: The simulated and real average and volatility of the yield curve with their respective confidence intervals for the estimates in column (i) (top) and (ii) (bottom) of table 8.

Overall, both the estimates in column (i) and (ii) are quite different from the estimates in column (iii) and (iv). The additional observations between 2014 and 2021 did not cause a large change in the average of the predictor variables, but did lead to significantly different parameter estimates. Given that the composition of the variables is different compared to Draper (2014) and Van den Berg (2016), it is tricky to attribute the increase in significance of the estimates to the increased length of the estimation period.

The estimates with the two-year and five-year yields being observed without error are closer to the estimates from (Van den Berg, 2016) and (Draper, 2014). However the estimates where the three-month and three-year yields are fixed are more in line with the sample data. For example, the state process X_t is more stable and mean-reverts around zero and the implied inflation, stock return and interest rate processes more closely resemble the actual data. The estimates in column (ii) lead to state variables that are exclusively negative, and negative bond risk premia. Lastly, the simulations implied by the parameter estimates are much more accurate for the estimates in column (i) than for the estimates in column (ii). This means the three-month and three-year yields are more indicative of the overall yield curve and the broader economy.

8.2 Monthly data

Table 11 contains the parameters estimates, standard errors and corresponding likelihood of the KNW model applied to monthly data from the Netherlands, UK and US. Following the results in Section 8.1, the three-month and three-year yields are observed without error. The goal is to compare the goodness-of-fit between quarterly and monthly datasets, and between the aforementioned countries. The sample period runs until January 2021 as opposed to the quarterly sample that runs until the fourth quarter of 2013. Compared to the results for the quarterly dataset in Table 8, most parameters estimates for the monthly dataset of the Netherlands are very different. All parameters with the exception of the unconditional price of real interest rate risk ($\Lambda_{0(1)}$) and of expected inflation risk ($\Lambda_{0(2)}$) are more significant than the estimates for the quarterly dataset. The standard errors of most parameter estimates are more than a factor $\sqrt{3.52}$ smaller than their quarterly counterparts. This number is an approximation of the square root of the factor difference between the number of observations of the monthly and quarterly datasets. This means that for most parameters, having a higher-frequency datasets leads to parameter estimates that are estimated with more precision.

The parameter $\delta_{0\pi}$ represents the long-term average inflation expectations. The parameter value for the Netherlands is relatively close to the average inflation (1.92% compared to 2.58%), while the values for the UK and US are very far off their respective averages. For both the UK and US, $\delta_{0\pi}$ is significantly lower than for the Netherlands despite the fact that both the UK and US have a higher average inflation than the Netherlands (5.01 and 3.79, respectively). This is in part because the inflation of both the UK and US have a higher volatility and a larger difference between their respective maximum and minimum value. The larger exposure of inflation to uncertainty in the market explains the larger values of the parameter vector σ_π. The lower value for $\delta_{0\pi}$ for the UK and US leads to a lower value for R_0, the expected nominal long-term money market rate.

Because the bond yields are different across different countries, the parameter η_S, representing the historical risk premium on equity, is also different. This is in spite of the fact that for all countries the same measure of stock return is used. Adding the parameters R_0 and η_S gives the expected yearly stock return. The values for the UK (3.14%) and US (1.93%) are significantly different to the average stock return. The values of the constants $\delta_{0\pi}$, R_0, η_S for the Netherlands are much closer to the average inflation, three-month yield, and stock risk premium than for the UK and US. One reason these constants can deviate from the sample averages is because the KNW model expects these variables to be mean-reverting around a certain constant, while especially the inflation and bond yields show a clear decreasing pattern over time.

$\sigma_{S(4)}$, representing the implied volatility of stock return, is equal to 17.00% for the Netherlands. This is closer to the unconditional sample standard deviation of 16.71% than is the case for the value of $\sigma_{S(4)}$ for the UK and US.

The unconditional price of real interest rate risk is higher than the unconditional price of expected inflation risk, as $\Lambda_{0(1)} > \Lambda_{0(2)}$ for all datasets. The parameters related to the price of risk for the Netherlands are very different than those for the UK and US. Previous estimations of the KNW model in (Draper, 2014) and (Van den Berg, 2016) have shown that even for very similar datasets the estimates for the price of risk can significantly differ. Also the estimates of the unconditional prices of risk with respect to real interest rates and with respect to expected inflation risk are not very significant, meaning that different estimates can give values for the log-likelihood that are relatively close.

The volatility of the measurement errors of most yields is highest for the UK. This can be explained by the high volatility of the bond yields of the UK compared to the US and the Netherlands. Also the correlation between the three-month yield and the other yields, and the correlation between the three-year yield and the other yields is lowest for the UK. This means that the three-month and three-year yields, that are assumed to be observed without measurement error, are least representative of the other yields, which will then lead to a larger error of the other yields.

It is clear that the parameter estimates of the Netherlands are more in line with their counterparts from the data. Next, I will look at the model implied characteristics, starting with statistics related to the parameter restrictions in Section 5. For each country all eigenvalues of the matrices K and M, defined in Section 3.1 and 5, are real and positive. Each eigenvalue of K being positive means the X_t-process is stationary. Therefore the mean and variance of the interest rates, inflation rate, and stock returns converge for long horizons. This means that for each country the unconditional expected returns of Π_t and S_t exist. The eigenvalues of M being real means there are no oscillations in the term-structure of interest rates.

The bottom part of table 11 contains the unconditional geometric expected inflation, unconditional geometric expected stock return and UFR, which are calculated using Equations 44, 45, and 46. For the Netherlands, these expected values are much closer to their data counterparts than for the UK and US. Also, for the Netherlands, these figures are almost exactly the same as their expected values from (Dijsselbloem et al., 2019) (1.9% and 5.6%). The implied UFR is equal to 3.49%, while the suggested value is 2.1%. The unconditional geometric expected inflation and the unconditional geometric expected stock return for the UK and US aren't at all close to the geometric returns from the data. These numbers strongly depend on the constants $\delta_{0\pi}$, R_0, and η_S, which are poorly estimated for the UK and US, as I explained before.

Parameter	(i) NL 1972.9 - 2021.1		(ii) UK 1972.9 - 2021.1		(iii) US 1972.9 - 2021.1	
	Estimate	Std. error	Estimate	Std. error	Estimate	Std. error
Expected inflation $\pi_t = \delta_{0\pi} + \delta'_{1\pi} X_t$						
$\delta_{0\pi}$	1.92%	(0.56%)*	-0.12%	(0.44%)	0.89%	(1.31%)
$\delta_{1\pi(1)}$	0.14%	(0.03%)*	-1.43%	(0.53%)	-0.80%	(0.08%)
$\delta_{1\pi(2)}$	0.20%	(0.00%)*	2.33%	(1.65%)	1.96%	(0.37%)
Nominal interest rate $R_t = R_0 + R'_1 X_t$						
R_0	2.91%	(1.30%)*	1.08%	(0.50%)	-1.16%	(2.73%)
$R_{1(1)}$	0.49%	(0.04%)*	-2.34%	(0.06%)	-1.15%	(0.04%)
$R_{1(2)}$	0.47%	(0.01%)*	5.28%	(0.30%)	1.98%	(0.28%)
Process real interest rate and expected inflation $dX_t = -KX_t dt + \Sigma'_X dZ_t$						
κ_{11}	0.13	(0.03)	0.02	(0.00)	0.01	(0.01)
κ_{21}	-0.00	(0.02)*	-0.43	(0.02)	-0.19	(0.03)
κ_{22}	0.01	(0.01)*	1.38	(0.08)	0.85	(0.20)
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma'_\pi dZ_t$						
$\sigma_{\pi(1)}$	0.10%	(0.08%)*	1.18%	(0.53%)	0.35%	(0.27%)
$\sigma_{\pi(2)}$	0.11%	(0.09%)*	-1.59%	(1.14%)	-0.17%	(0.31%)
$\sigma_{\pi(3)}$	1.13%	(0.03%)*	3.38%	(0.10%)	1.72%	(0.06%)
Stock return process $\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma'_S dZ_t$						
η_S	4.28%	(0.77%)*	2.06%	(0.78%)	3.09%	(0.71%)
$\sigma_{S(1)}$	-0.82%	(1.24%)*	2.71%	(1.32%)	-1.90%	(1.26%)
$\sigma_{S(2)}$	6.53%	(1.80%)	-0.00%	(1.33%)	0.92%	(1.09%)
$\sigma_{S(3)}$	-3.48%	(0.75%)*	-1.28%	(0.77%)	-5.23%	(0.73%)
$\sigma_{S(4)}$	17.00%	(0.52%)*	17.36%	(0.55%)	15.85%	(0.49%)
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$						
$\Lambda_{0(1)}$	-0.223	(0.679)	0.622	(0.036)	0.679	(0.166)
$\Lambda_{0(2)}$	-1.058	(0.618)	0.491	(0.094)	0.552	(0.538)
$\Lambda_{1(1,1)}$	1.350	(0.040)	0.160	(0.005)	0.084	(0.005)
$\Lambda_{1(1,2)}$	0.019	(0.015)*	-0.527	(0.003)	-0.212	(0.008)
$\Lambda_{1(2,1)}$	1.377	(0.022)*	-0.030	(0.021)	0.057	(0.045)
$\Lambda_{1(2,2)}$	0.039	(0.015)*	0.260	(0.084)	0.223	(0.212)
Volatility measurement error yields $v'_t \sim N(0, \Sigma^\tau)$						
$\sigma_{\tau_{12}}$	0.37%	(0.01%)*	0.55%	(0.01%)	0.26%	(0.00%)
$\sigma_{\tau_{24}}$	0.16%	(0.00%)	0.25%	(0.00%)	0.33%	(0.01%)
$\sigma_{\tau_{60}}$	0.23%	(0.00%)	0.49%	(0.01%)	0.40%	(0.01%)
$\sigma_{\tau_{120}}$	0.59%	(0.02%)	1.51%	(0.05%)	0.54%	(0.01%)
LL_{NL}	18354.4		16933.5		17095.6	
LL_{UK}	10783.5		15809.2		12477.0	
LL_{US}	15637.7		16523.2		17387.4	
min.ev(K)	0.0081		0.0211		0.0069	
min.ev(M)	0.0318		0.1622		0.1148	
r^g_Π	1.94%		-0.20%		0.88%	
r^g_S	5.62%		1.60%		0.52%	
UFR	3.49%		-3.46%		0.05%	

Table 11: This table contains for different monthly datasets the parameters and standard errors of the maximum likelihood estimate of the KNW model with the three-month and three-year yields observed without a measurement error. Columns (i), (ii), and (iii) are the maximum likelihood estimates using monthly observations of the Netherlands, UK, and US, respectively, over the sample period 1972.9 - 2021.1. The first column contains a *-symbol for each standard deviation that is more than a factor $\sqrt{3.52}$ smaller than its quarterly counterpart in column (i) of Table 8. The table also contains the lowest eigenvalues of K and M, respectively, and the unconditional geometric yearly returns of the price and stock index, and the UFR.

Table 12 contain the bond risk premia, volatilities, and Sharpe ratios implied by the parameter estimates in Table 11. Compared to the quarterly dataset, bond risk premia for the Netherlands have increased. The risk premia and volatilities for the shorter maturities for both the UK and US are lower than for the Netherlands. This is in contrast to the dataset where the UK and US have higher bond yields for all maturities.

	NL			UK			US		
	$y_\tau - y_0$	σ	S	$y_\tau - y_0$	σ	S	$y_\tau - y_0$	σ	S
Three-month	0.14%	0.14%	0.99	-0.20%	1.17%	-0.17	-0.05%	0.51%	-0.09
One-year	0.50%	0.46%	1.08	-0.22%	2.76%	-0.08	0.06%	1.52%	0.04
Two-year	0.89%	0.93%	0.96	0.21%	3.43%	0.06	0.48%	2.32%	0.21
Three-year	1.26%	1.47%	0.86	0.74%	3.86%	0.19	1.00%	2.95%	0.34
Five-year	1.95%	2.56%	0.76	1.79%	4.81%	0.37	2.03%	4.12%	0.49
Ten-year	3.52%	5.09%	0.69	3.96%	7.25%	0.55	4.18%	6.68%	0.62

Table 12: This table contains the bond risk premia ($y_\tau - y_0$), their respective volatilities (σ) and the Sharpe ratios (S) implied by the estimates in Table 11.

Table 13 contains the implied correlations between stock return, inflation, and bond yields. The implied correlation between stock returns and bond yields is close to zero for the UK and US, which is in line with the data. The implied correlations for the Netherlands, however, are significantly lower than zero for all maturities and thus not in line with the unconditional sample correlation. Also the implied correlation between the three-month bond yields and the other yields is significantly lower than the correlation from the data.

	Stocks	3-month bond	1-year bond	2-year bond	3-year bond	5-year bond	10-year bond
Stocks	1	-0.26	-0.34	-0.35	-0.34	-0.32	-0.30
3-month bond		1	0.87	0.62	0.47	0.35	0.27
5-year risk premium	0.03	0.63	0.16	-0.22	-0.38	-0.50	-0.58
10-year risk premium	0.02	0.65	0.19	-0.20	-0.36	-0.48	-0.56
Stocks	1	0.07	0.08	0.10	0.12	0.14	0.15
3-month bond		1	0.99	0.95	0.90	0.76	0.56
5-year risk premium	0.04	0.98	0.95	0.89	0.80	0.63	0.40
10-year risk premium	0.04	0.99	0.96	0.90	0.81	0.64	0.41
Stocks	1	-0.11	-0.12	-0.12	-0.13	-0.12	-0.12
3-month bond		1	0.99	0.94	0.88	0.76	0.62
5-year risk premium	-0.08	0.94	0.88	0.77	0.66	0.50	0.32
10-year risk premium	-0.09	0.97	0.92	0.83	0.73	0.58	0.41

Table 13: This table contains the correlation between stocks, bonds, and risk premia implied by the estimates in Table 11 for the Netherlands (upper part), UK (middle part), and US bottom part).

Table 14 contains the correlation between the sample inflation and the implied expected inflation, and the correlation between the sample three-month yields and the implied nominal interest rate. The time-series of these implied variables can all be seen in Figure 6. For each country the second column corresponds to the estimation results in Table 11. The first column contains the results corresponding to Table 16, which contains the estimation results for the KNW model where all yields are observed with a measurement error. These will be discussed later. The correlation is highest for the Netherlands for both variables compared to the UK and US, meaning that the model implied variables better represents the sample data.

	NL		UK		US	
	all with error	(3,36) without error	all with error	(3,36) without error	all with error	(3,36) without error
inflation	0.699	0.708	0.587	0.331	0.430	0.587
three-month yield	0.935	0.998	0.791	0.993	0.980	0.995

Table 14: This table contains for the Netherlands, UK, and US the correlation between the sample inflation and the implied expected inflation (first row) and the correlation between the sample three-month yield and the implied nominal interest rate (second row). The expected inflation and nominal interest rate are implied by the KNW model where either all yields are observed with error (Table 16), or the three-month and three-year yields are observed without error (Table 11).

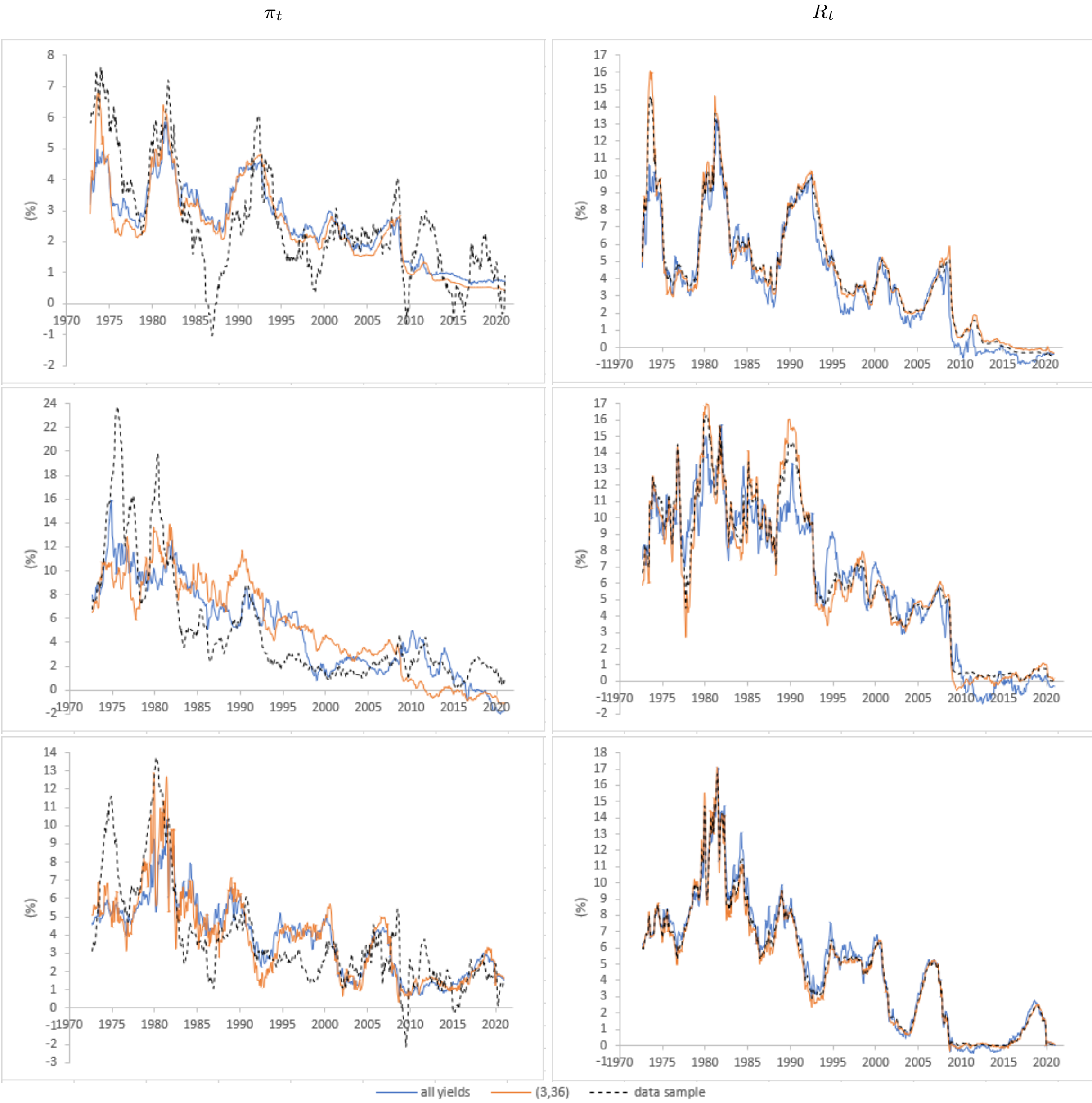

Figure 6: The implied expected inflation and nominal interest rate for the parameter estimates in Table 11 and the parameter estimates in Table 16. The upper two figures are for the Netherlands, the middle two figures are for the UK, and the bottom two figures are for the US.

Table 15 contains the average, volatility and their respective 95% confidence intervals of the simulated inflation and stock return, and the sample average and standard deviation. The average simulated inflation and stock return are closest to the data for the Netherlands. For the

UK the sample inflation and stock return are outside the 95% confidence interval of the average simulation. The standard deviation of inflation is close to the simulated standard deviation and within the confidence interval for each country. For stock return volatility only the US stays within the confidence interval.

	NL		UK		US	
	Inflation	Stock return	Inflation	Stock return	Inflation	Stock return
average	1.55%	4.57%	-0.28%	1.52%	0.99%	0.74%
2.5% quantile	-0.06%	0.43%	-2.82%	-1.27%	-2.35%	-6.04%
97.5% quantile	3.22%	8.86%	2.21%	4.26%	4.28%	7.37%
data	2.58%	6.22%	5.01%	6.22%	3.79%	6.22%
volatility	1.73%	18.91%	4.80%	17.99%	3.11%	17.56%
2.5% quantile	1.38%	17.78%	4.18%	16.92%	2.50%	16.44%
97.5% quantile	2.36%	20.04%	5.70%	19.04%	4.23%	18.81%
data	1.81%	16.73%	4.92%	16.73%	2.85%	16.73%

Table 15: Simulations statistics

Figure 7 contains the average and volatility of the yield curve with their respective confidence intervals. The yield curve is based on simulations of all six yields that are used. Both the simulated average and volatility of the yield curve are closest to the real data for the Netherlands. The UK real average is outside the 95% confidence interval.

For all variables the confidence interval of the volatility is smaller than that of the average. This shows that the volatilities are estimated more precisely than the averages.

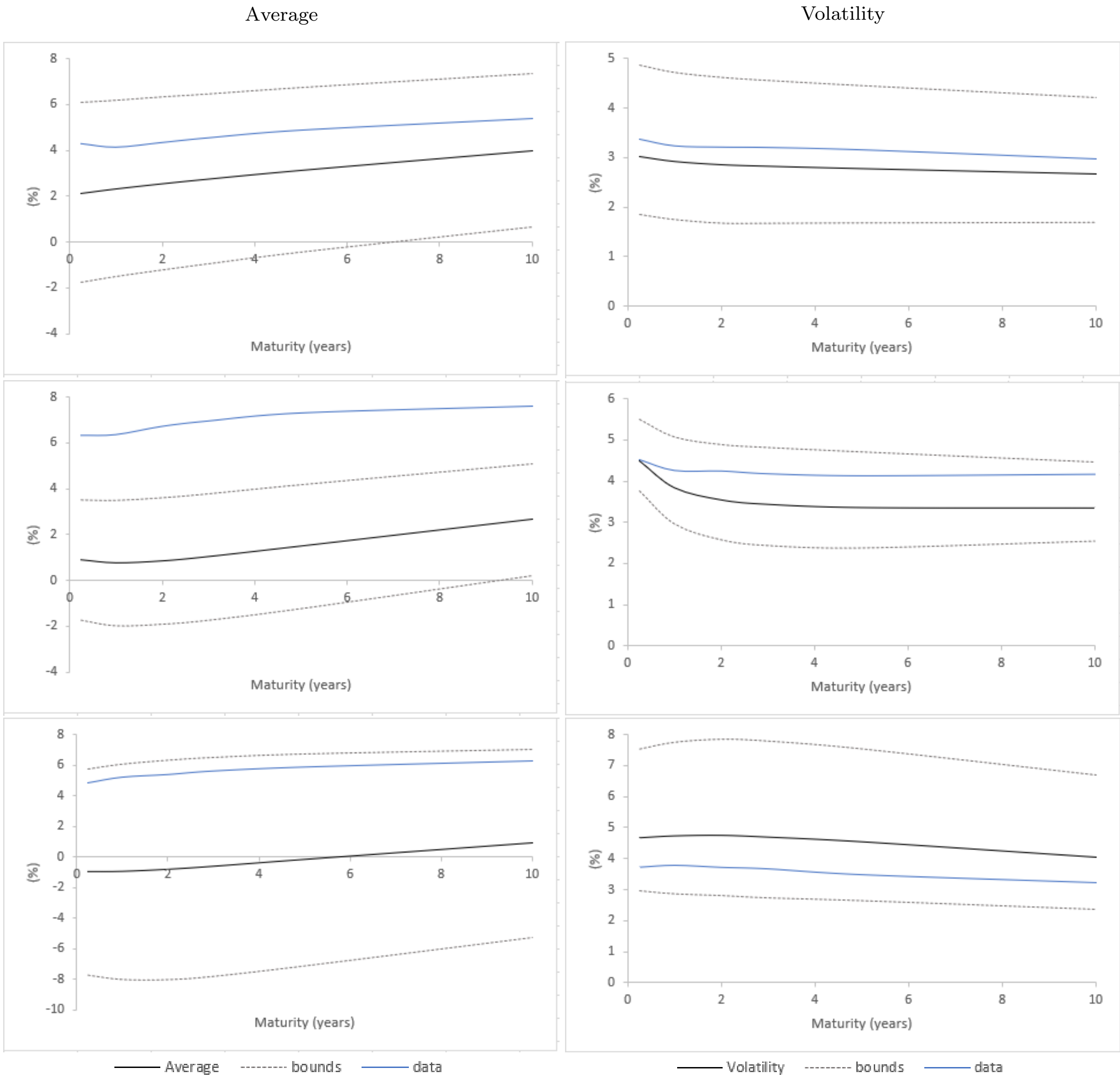

Figure 7: The simulated average and volatility with their respective 95% confidence intervals and the sample average or standard deviation for the Netherlands (upper part), UK (middle part), and US (bottom part). The simulations are based on the parameter estimates in Table 11.

Summarizing, the parameters estimates of each country lead to a stationary process of X_t, while oscillations in the term-structure of interest rates are avoided. Comparing the results to the quarterly estimates, the model with monthly data for the Netherlands has a larger precision for most parameter estimates, even when taking into account the natural increase in precision by using a larger number of observations. The increase in the number of observations is caused by using monthly instead of quarterly observations and by using a larger estimation period.

The parameter values for the monthly dataset of the Netherlands are more representative of the historical data than those of the quarterly dataset and the monthly datasets for the UK and US. Compared to the UK and US data, the parameters for the Dutch long-term average inflation expectations ($\delta_{0\pi}$) and the expected stock return ($R_0+\eta_S$) are much closer to the historical data.

Because the KNW model assumes inflation and bond yields to revert around a certain constant, these parameters are more difficult to interpret for our dataset where both these variables show a clear decreasing pattern. The correlations between the implied and sample variables are highest for the Netherlands, and the simulated average and volatility have a better fit for most variables. The KNW model for the Netherlands only performs worse in terms of the implied correlations between stock returns and bond yields and between the three-month yield and other yields. Also the simulated volatility of the stock returns is significantly different than the sample volatility, although this is also the case for the UK. One reason the KNW model doesn't replicate the characteristics of the UK dataset well might be the weaker correlation between the different yields. Especially the shortest and longest yields have a much smaller correlation compared to the Netherlands.

The superior results for the Netherlands seem to confirm the hypothesis from Draper (2012) that the KNW model is mainly a model for the Northern European markets.

8.3 All yields with measurement error

Table 16 contains the estimation results for the KNW model where all yields are observed with a measurement error. As observing two yields without error is a restriction on the KNW model, in this section I will refer to this model as the 'restricted model' and to the model where all yields are observed with error as the 'unrestricted model'. Naturally, estimating the unrestricted model leads to a higher likelihood for each country compared to the restricted model. Compared to the estimates for the KNW model with two yields that are observed without error most parameter estimates are significantly different. Both $\delta_{0\pi}$ and R_0 are much larger for the UK and US. The implied expected stock return ($R_0 + \eta_S$) for the UK is now much more in line with the sample average stock return. While this figure was underestimated for the US for the restricted model, this number is overestimated for the unrestricted model. For the Dutch data $R_0 + \eta_S$ is closer to the sample stock return for the restricted model. Table 14 shows that for the UK the correlation between implied expected inflation for the unrestricted model and sample inflation is substantially higher than for the restricted model, while it is substantially lower for the US. For the three-month yield this correlation is lower for all countries compared to the restricted model, which makes sense, as for the restricted model the three-month yield is observed without error. The volatility of the stock returns for the unrestricted model is close to the estimate of the restricted model for both the Netherlands and the UK. For the US, however, this parameter is equal to 11.72% which is significantly different than the sample standard deviation of the stock return (16.71%). This means that for the US the model is not able to capture the characteristics of the stock return process, as both the parameters related to the average and standard deviation are significantly different to their data counterparts.

As expected, the volatilities of the error terms of the yields that were observed with error in the restricted model are all lower for the unrestricted model for all countries. In the restricted model the fact that two yields were observed without error lead to a larger error for the other yields.

	(i) Netherlands		(ii) UK		(iii) US	
Parameter	Estimate	Std. error	Estimate	Std. error	Estimate	Std. error
Expected inflation $\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t$						
$\delta_{0\pi}$	2.67%	(0.58%)	1.77%	(1.52%)	5.41%	(1.07%)
$\delta_{1\pi(1)}$	0.06%	(0.02%)	-0.15%	(0.02%)	-0.29%	(0.02%)
$\delta_{1\pi(2)}$	0.11%	(0.01%)	-0.39%	(0.04%)	0.26%	(0.04%)
Nominal interest rate $R_0 + R_1' X_t$						
R_0	4.33%	(2.04%)	3.95%	(0.98%)	7.96%	(2.17%)
$R_{1(1)}$	0.27%	(0.01%)	-0.55%	(0.01%)	-0.45%	(0.01%)
$R_{1(2)}$	0.31%	(0.00%)	0.23%	(0.04%)	0.27%	(0.03%)
Process real interest rate and expected inflation $dX_t = -KX_t dt + \Sigma_X' dZ_t$						
κ_{11}	0.02	(0.01)	0.00	(0.00)	-0.00	(0.00)
κ_{21}	0.00	(0.01)	-0.02	(0.01)	-0.05	(0.01)
κ_{22}	0.00	(0.00)	0.03	(0.01)	0.10	(0.02)
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\pi' dZ_t$						
$\sigma_{\pi(1)}$	-0.84%	(0.16%)	0.79%	(1.04%)	1.93%	(0.50%)
$\sigma_{\pi(2)}$	-0.96%	(0.14%)	0.91%	(1.14%)	-0.65%	(0.36%)
$\sigma_{\pi(3)}$	0.06%	(0.57%)	2.83%	(0.35%)	0.60%	(0.62%)
Stock return process $\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma_S' dZ_t$						
η_S	4.51%	(0.75%)	1.65%	(0.71%)	2.73%	(0.70%)
$\sigma_{S(1)}$	5.50%	(2.86%)	-1.18%	(5.71%)	-5.62%	(5.06%)
$\sigma_{S(2)}$	2.28%	(2.79%)	3.55%	(6.88%)	7.18%	(1.78%)
$\sigma_{S(3)}$	1.68%	(4.43%)	-2.19%	(0.95%)	7.58%	(3.08%)
$\sigma_{S(4)}$	16.55%	(1.01%)	16.48%	(1.39%)	11.72%	(3.24%)
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$						
$\Lambda_{0(1)}$	0.248	(1.512)	0.352	(0.968)	0.562	(0.104)
$\Lambda_{0(2)}$	-0.495	(1.401)	-0.120	(0.079)	-0.042	(0.165)
$\Lambda_{1(1,1)}$	0.312	(0.012)	0.218	(0.006)	0.381	(0.001)
$\Lambda_{1(1,2)}$	0.031	(0.003)	-0.393	(0.013)	-0.678	(0.021)
$\Lambda_{1(2,1)}$	0.446	(0.013)	0.019	(0.005)	0.073	(0.012)
$\Lambda_{1(2,2)}$	0.106	(0.004)	0.007	(0.010)	-0.097	(0.021)
Volatility measurement error yields $v_t' \sim N(0, \Sigma^\tau)$						
σ_{τ_3}	0.86%	(0.02%)	1.29%	(0.04%)	0.46%	(0.01%)
$\sigma_{\tau_{12}}$	0.24%	(0.00%)	0.48%	(0.01%)	0.00%	(0.01%)
$\sigma_{\tau_{24}}$	0.00%	(0.01%)	0.01%	(0.01%)	-0.17%	(0.01%)
$\sigma_{\tau_{36}}$	0.05%	(0.01%)	0.14%	(0.01%)	0.30%	(0.00%)
$\sigma_{\tau_{60}}$	-0.00%	(0.01%)	0.25%	(0.01%)	0.28%	(0.00%)
$\sigma_{\tau_{120}}$	0.21%	(0.00%)	0.84%	(0.02%)	0.09%	(0.10%)
$LL_{Netherlands}$	22883.6		20731.5		21133.9	
LL_{UK}	6792.0		19340.1		14024.2	
LL_{US}	7886.6		18933.1		21181.1	

Table 16: This table contains, for the monthly datasets of the Netherlands, UK, and US, the parameters and standard errors of the maximum likelihood estimate of the KNW model where all yields are observed with a measurement error. The estimation period runs from September 1972 until January 2021. LL_x in column y is the log-likelihood value of the data-set of country x using the parameters in column y.

Table 17 contains the correlation between stocks, bonds and bond risk premia. The correlations between stock return and the different yields now strongly vary with maturity for each country, while this is not the case for the restricted model and also not for the data sample. The correlations between shorter and longer maturities are much lower and even below zero in some instances.

	Stocks	3-month bond	1-year bond	2-year bond	3-year bond	5-year bond	10-year bond
Stocks	1	0.29	0.27	0.22	0.17	0.07	-0.06
3-month bond		1	0.99	0.94	0.86	0.66	0.33
5-year risk premium	0.33	0.80	0.70	0.55	0.38	0.07	-0.31
10-year risk premium	0.34	0.84	0.76	0.61	0.45	0.15	-0.23
Stocks	1	0.14	0.12	0.09	0.05	-0.02	-0.13
3-month bond		1	0.99	0.96	0.90	0.70	0.24
5-year risk premium	0.22	0.78	0.70	0.58	0.43	0.10	-0.42
10-year risk premium	0.21	0.80	0.73	0.61	0.46	0.14	-0.39
Stocks	1	0.50	0.47	0.38	0.26	-0.02	-0.30
3-month bond		1	0.99	0.92	0.77	0.36	-0.18
5-year risk premium	0.54	0.86	0.77	0.59	0.34	-0.17	-0.65
10-year risk premium	0.54	0.87	0.79	0.61	0.36	-0.15	-0.63

Table 17: This table contains the correlation between stocks, bonds, and risk premia implied by the estimates in Table 16 for the Netherlands (upper part), UK (middle part), and US bottom part).

Table 18 contains the simulated average and volatility of inflation and stock return with their 95% confidence interval, and the data average and standard deviation. Compared to the restricted model, some averages and volatilities are closer to the data, and some are not. For the Netherlands the averages are further from the data averages compared to the restricted model, while for the UK the averages are closer to the data and within the confidence interval. For each country and for both variables the confidence interval of the average is larger compared to the restricted model. This means that the averages are estimated more precisely for the restricted model.

Figure 8 contains the simulated average and volatility of the yield curve. For the Netherlands, the average is not as close as for the restricted model. For the UK the average simulation is better compared to the restricted model, while for the US there isn't much difference. For both the Netherlands and the UK, the volatility of the yields curve are not as close as for the restricted model, while for the US the distance between the simulated volatility and the sample standard deviation is about the same for both models. Because of the lower estimates for the volatilities of the error terms of the yields, the simulated volatilities are lower for the unrestricted model. For all figures the confidence intervals are wider, meaning that the average and volatility are estimated with less precision for the unrestricted model.

	NL		UK		US	
	Inflation	Stock return	Inflation	Stock return	Inflation	Stock return
average	0.96%	2.57%	0.12%	2.80%	1.01%	0.61%
2.5% quantile	-1.42%	-4.40%	-4.93%	-2.64%	-3.44%	-8.25%
97.5% quantile	3.47%	9.69%	5.06%	8.01%	5.41%	9.44%
data	2.58%	6.22%	5.01%	6.22%	3.79%	6.22%
volatility	1.58%	17.99%	4.24%	17.41%	2.58%	17.09%
2.5% quantile	1.29%	16.92%	3.37%	16.35%	2.15%	16.07%
97.5% quantile	2.22%	19.13%	5.92%	18.48%	3.57%	18.22%
data	1.81%	16.73%	4.92%	16.73%	2.85%	16.73%

Table 18: Simulations statistics

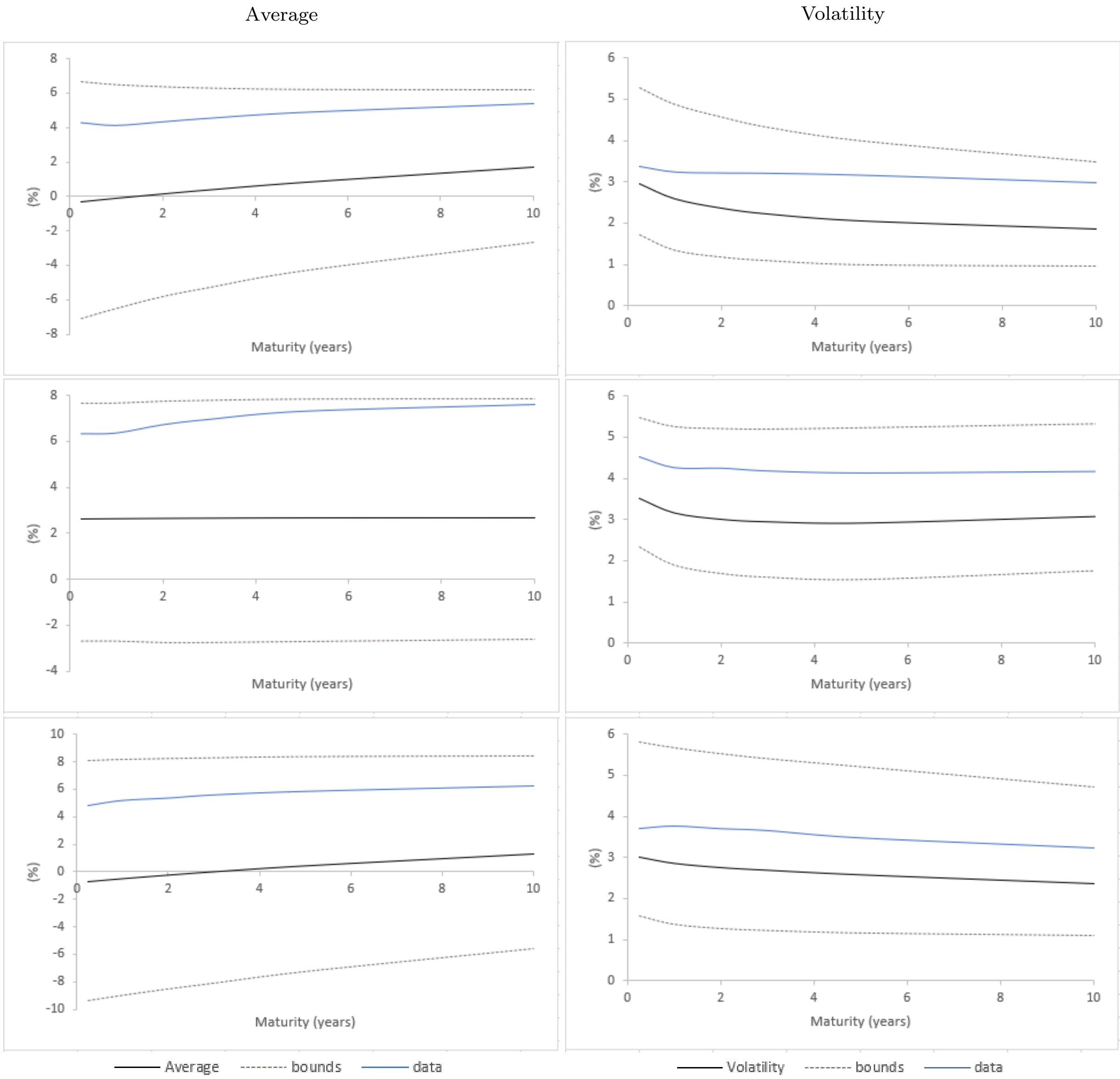

Figure 8: The simulated average (left) and volatility (right) with their respective 95% confidence intervals and the sample average or standard deviation for the Netherlands (upper part), UK (middle part), and US (bottom part). The simulations are based on the parameter estimates in Table 16. The black line is the simulated average or volatility, the grey dotted lines are the bounds of the confidence interval and the blue line is the sample average or volatility.

In summary, compared to the restricted model, the unrestricted model performs worse for most indicators for the Netherlands and the US. Of the three countries considered, the UK is the only one for which the fit of the data clearly improves; the parameter estimates are more in line with the data, the correlation between implied expected inflation and sample inflation is substantially higher, and the simulated averages and volatilities are closer to the data and within the 95% confidence interval. This might be caused by the low correlation between the yields compared to the other countries. If the correlation between yields is low, the assumption that all yields are observed with error might lead to a better fit compared to the assumption that two yields are observed without error.

For all countries, observing the three-month and three-year yields with error leads to some general tendencies. The correlation between the sample three-month yield and the implied nominal interest rate is lower compared to the restricted model. Also, the implied correlations between stocks, bond yields and bond risk premia strongly vary with maturity. The implied correlation between the three-month yields and long-maturity yields is substantially lower than the data would suggest. Finally, the estimates for σ_τ are lower, which leads to lower simulated volatilities of the yields. The simulated averages also have larger 95% confidence intervals, meaning the averages are estimated with less precision.

8.4 Supremum likelihood ratio test

This section contains the results of the supremum likelihood ratio test, as it was explained in Section 6. This test checks if a structural break exists at an unknown point in the dataset. Including structural breaks is an extension to the KNW model results in Section 8.2. The test is applied to both the Netherlands and the UK, as I would like to compare the results of a country that fits the KNW model well with those of a country with a poor goodness-of-fit. Figure 9 contains for each value $\pi \in \Pi$ with $\Pi = [0.15, 0.2, ..., 0.85]$ the log-likelihood with π as a break point, and the corresponding likelihood ratio. To calculate the likelihood ratio, each log-likelihood is compared to the log-likelihood of the monthly data under the null hypothesis. For both the Netherlands and the UK the test statistic $\sup_{\pi \in \Pi} LR_T(\pi)$ is much larger than the 5% critical value of 53.5. This means the null hypothesis of no structural break on Π is clearly rejected. The point where the maximum of $LR_T(\pi)$ is reached is $\pi = 0.5$ for the Netherlands and $\pi = 0.45$ for the UK.

The value of $LR_T(\pi)$ for a given π in Figure 9 represents the test statistic of a normal likelihood ratio test, which tests for a structural break at a known point π. This test statistic, which can be found in Figure 9 of each π, is chi-squared distributed with $p = 27$ degrees of freedom under the null hypothesis, as was shown in Section 6. In this case, the 5% critical value is equal to 40.1.

The critical value of this test is lower than that of the supremum test, as the expected value of $LR_T(\pi)$ for a randomly selected value for π is lower than the expected value of $\sup_{\pi \in \Pi} LR_T(\pi)$. For each of the regular likelihood ratio tests, the null hypothesis is rejected. This can largely be explained by the changing behaviour of inflation, stock return and bond yields over time. Both inflation rates and bond yields show a clear decreasing pattern over time. In addition, the large fluctuations make it difficult to estimate the expected nominal long-term money market rate (R_0), historical risk premium (η_S), and long-term average inflation ($\delta_{0\pi}$).

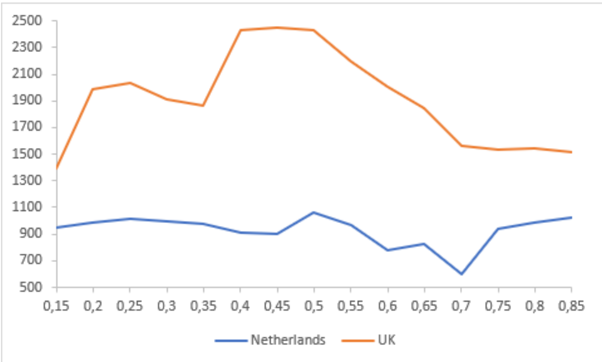

Figure 9: The value of the likelihood ratio test statistic as defined in Equation 55 for $\pi \in \Pi = [0.15, 0.2, ..., 0.85]$. The maxima for the Netherlands and the UK are $LR_T(0.5) = 1058.6$ and $LR_T(0.45) = 2451.6$, respectively.

Table 19 contains the parameter estimates of the most likely break point based on the supremum likelihood test for both the Netherlands and the UK. For the Netherlands, most parameters estimates for the split estimation samples are not too far off the parameter values of the one-period estimation in Table 11. Both the parameters representing the long-term average inflation expectations ($\delta_{0\pi}$) and the expected nominal long-term money market rate (R_0) are higher for the second half, even though the average inflation and three-month bond yields are lower. The implied expected long-term stock return ($R_0 + \delta_{0\pi}$) for the second period is double the number of the first period, even though the average stock return has decreased over time. This again shows that the parameters that represent long-term expectations are not well estimated for variables that decrease over time. For the second period, these three constants are closer to their counterparts from the sample data than for the first period. This means that a smaller part of the expected inflation and nominal interest rate processes are explained by the state variables, which leads to lower parameter estimates for $\delta_{1\pi}$ and R_1.

The parameters that differ the most are those related to the stock return process (σ_S). Given that the increase of the implied expected long-term stock return is not related to an increase in the actual long-term average stock return, a large part of the stock return process is related to the uncertainty in the financial market.

For the UK, there is a larger difference between most parameter values of the one-period and two-period estimations, which is in line with the large value of the supremum tests statistic compared to the Netherlands. The estimates of the expected long-term average inflation, nominal money market rate, and stock return are much more in line with their data counterparts compared to the one-period estimates in Table 11. These parameters now show a clear difference between the first and second period, as is the case for the sample data. The expected long-term average inflation and nominal money market rate are more accurate for the second estimation period than for the first.

	(i) Netherlands ($\pi = 0.5$)		(ii) UK ($\pi = 0.45$)	
Parameter	$\beta_{1,[\pi T]}$	$\beta_{[\pi T]+1,T}$	$\beta_{1,[\pi T]}$	$\beta_{[\pi T]+1,T}$
Expected inflation $\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t$				
$\delta_{0\pi}$	1.30%	1.77%	5.90%	1.96%
$\delta_{1\pi(1)}$	-0.16%	0.13%	-0.40%	-0.05%
$\delta_{1\pi(2)}$	0.38%	0.09%	0.45%	-0.00%
Nominal interest rate $R_t = R_0 + R_1' X_t$				
R_0	1.63%	2.45%	5.34%	3.60%
$R_{1(1)}$	0.51%	0.17%	1.55%	0.12%
$R_{1(2)}$	0.81%	0.22%	0.81%	0.29%
Process real interest rate and expected inflation $dX_t = -KX_t dt + \Sigma_X' dZ_t$				
κ_{11}	0.19	0.07	0.19	0.06
κ_{21}	-0.04	-0.01	0.04	0.04
κ_{22}	0.02	-0.02	-0.00	0.01
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\pi' dZ_t$				
$\sigma_{\pi(1)}$	0.05%	0.09%	0.63%	0.05%
$\sigma_{\pi(2)}$	-0.29%	0.26%	0.18%	-0.12%
$\sigma_{\pi(3)}$	1.34%	0.73%	4.90%	0.79%
Stock return process $\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma_S' dZ_t$				
η_S	2.19%	5.13%	-1.63%	5.22%
$\sigma_{S(1)}$	-2.65%	1.99%	-0.76%	1.88%
$\sigma_{S(2)}$	7.20%	15.90%	1.74%	10.42%
$\sigma_{S(3)}$	-6.87%	0.34%	-4.78%	1.10%
$\sigma_{S(4)}$	15.51%	15.22%	16.34%	15.54%
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$				
$\Lambda_{0(1)}$	0.653	0.571	0.845	0.575
$\Lambda_{0(2)}$	-1.026	-0.935	-0.537	0.040
$\Lambda_{1(1,1)}$	1.875	1.824	2.199	0.382
$\Lambda_{1(1,2)}$	-0.104	0.078	-0.019	0.002
$\Lambda_{1(2,1)}$	1.977	2.768	0.620	1.903
$\Lambda_{1(2,2)}$	0.056	0.144	0.028	0.394
Volatility measurement error yields $v_t' \sim N(0, \Sigma^\tau)$				
$\sigma_{\tau_{12}}$	0.42%	0.23%	0.66%	0.35%
$\sigma_{\tau_{24}}$	0.17%	0.11%	0.25%	0.20%
$\sigma_{\tau_{60}}$	0.23%	0.20%	0.60%	0.33%
$\sigma_{\tau_{120}}$	0.55%	0.52%	1.83%	0.81%

Table 19: The parameter estimates for both the Netherlands and UK for both the estimation periods before and after π.

For the Netherlands, I also test for a second break point in addition to the one previously mentioned at break point $\pi = 0.5$. For this test, I apply the grid $\Pi = [0.15, 0.2, ..., 0.85]$ to the observations before and after the first break point. Both the supremum and regular likelihood have the same critical values as for the test that was used to find the first break point. Similarly to the first break point, the conclusion for each point is that the null hypothesis is rejected. This means that each of these points can be considered to be a break point. The points where the maxima of $LR_T(\pi)$ are reached is $\pi = 0.15$ in the first half and $\pi = 0.65$ in the second half of the estimation period.

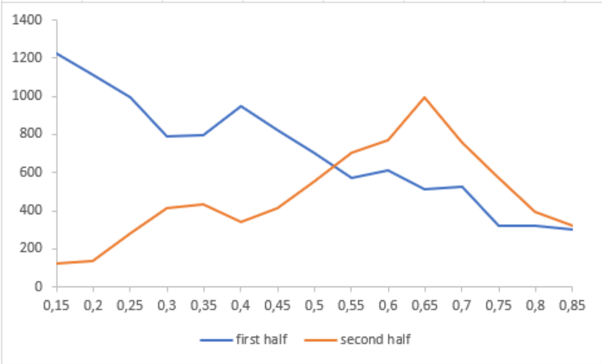

Figure 10: The value of the likelihood ratio test statistic as defined in Equation 55 for $\pi \in \Pi = [0.15, 0.2, ..., 0.85]$ for both the first and second half of the Dutch dataset. The maxima for the first and second half are $LR_T(0.15) = 1224.3$ and $LR_T(0.65) = 995.2$, respectively.

Overall, a break point is found for each point that is considered for both the Netherlands and the UK, including a second break point for the Netherlands. The UK, which showed the poorest fit using constant parameters over time, saw the largest improvement using time-varying parameters with break point π. This means that introducing time-varying parameters can be a solution to the poor fit of the KNW model. The parameter estimates for the Netherlands are less satisfactory than for the model with constant parameters over time. This means that even with a break point the KNW model can give parameter estimates that are not in line with long-term averages of model variables.

The fact that the null hypothesis of no structural break is rejected for each point, even for more than one structural break, indicates that the estimation period used is too long. A solution for this would be to estimate the KNW model using either a much shorter estimation period or parameters that change more frequently over time.

8.5 Restricted KNW model

Table 20 contains the parameter estimates, log-likelihood and relevant statistics for two combinations of the restrictions introduced in Section 5. These restrictions are imposed on the unrestricted model whose parameter estimates can be seen in column (i) of Table 11. For these estimates the process X_t is stationary, while oscillations in the term-structure of interest rates are avoided. In this section, the restrictions on the matrices K and M that imply such behaviour are imposed in combination with the other restrictions. In column (i) the restrictions related to the unconditional expected inflation, expected stock return and the UFR are imposed, while in column (ii) the restriction on negative ten-year yields is added to the previous constraints.

The table also includes the smallest eigenvalues of K and M to confirm that they are indeed positive. The last row is the 2.5% quantile of the distribution of the ten-year bond yield at a five-year ($T = 60$) horizon.

Column (i) shows that imposing the constraints in Equations 44-46 makes little difference for most parameters, but a large difference for some parameters. Compared to the unrestricted model, only the UFR decreases significantly (from 3.49% to 2.1%), as the unconditional geometric expected returns of Π_t and S_t for the unrestricted model were almost exactly the same as their restricted values. The reason the unconditional geometric expected returns of the price and stock indices are very close to their restricted values is that the restricted values are based on their historic average. The restricted UFR, however, is an infinite maturity zero coupon rate, which is above the long-term bond yields. The lower value for the UFR causes a drop in the value of the unconditional prices of risk with respect to real interest rate and expected inflation ($\Lambda_{0(1)}$ and $\Lambda_{0(2)}$). The drop in UFR leads to a large drop in the log-likelihood value of $18354.4 - 18316.7 = 37.7$, which means that the significantly different value of the UFR is not supported by the data.

The 2.5% percentile of the distribution of $y_{60}(120)$ is negative, meaning that the expected number of negative ten-year yields at a five-year horizon is larger than 2.5%. Because this number is already close to zero, the estimates in column (ii) are almost exactly the same as the estimates in column (i). The difference in the log-likelihood of $18316.7 - 18315.5 = 1.2$ is insignificant for most levels of significance. Imposing the non-negativity constraint leads to an increase in the lowest eigenvalue of K. This results in a reduced variance of X_{60}, which in turn reduces the variance of $y_{60}(120)$. The effects of adding different sets of constraint are in line with the results from (Pelsser, 2019).

Parameter	(i) parameter restrictions		(ii) positive ten-year yields	
	Estimate	Std. error	Estimate	Std. error
Expected inflation $\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t$				
$\delta_{0\pi}$	1.89%		1.89%	
$\delta_{1\pi(1)}$	0.25%	(0.03%)	0.24%	(0.03%)
$\delta_{1\pi(2)}$	0.16%	(0.01%)	0.16%	(0.00%)
Nominal interest rate $R_t = R_0 + R_1' X_t$				
R_0	2.51%		2.49%	
$R_{1(1)}$	0.85%	(0.01%)	0.86%	(0.00%)
$R_{1(2)}$	0.41%	(0.00%)	0.40%	(0.00%)
Process real interest rate and expected inflation $dX_t = -KX_t dt + \Sigma_X' dZ_t$				
κ_{11}	0.15	(0.04)	0.14	(0.00)
κ_{21}	0.03	(0.01)	0.03	(0.00)
κ_{22}	0.01	(0.01)	0.01	(0.00)
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\pi' dZ_t$				
$\sigma_{\pi(1)}$	0.03%	(0.10%)	0.09%	(0.10%)
$\sigma_{\pi(2)}$	0.04%	(0.08%)	0.05%	(0.08%)
$\sigma_{\pi(3)}$	1.13%	(0.03%)	1.13%	(0.02%)
Stock return process $\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma_S' dZ_t$				
η_S	4.53%		4.53%	
$\sigma_{S(1)}$	-0.16%	(1.87%)	-1.07%	(1.35%)
$\sigma_{S(2)}$	2.65%	(1.40%)	2.13%	(1.14%)
$\sigma_{S(3)}$	-3.95%	(0.80%)	-4.15%	(0.78%)
$\sigma_{S(4)}$	17.19%	(0.52%)	17.07%	(0.51%)
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$				
$\Lambda_{0(1)}$	-0.523	(0.428)	-0.563	(0.002)
$\Lambda_{0(2)}$	-1.196	(0.218)	-1.234	(0.001)
$\Lambda_{1(1,1)}$	1.525	(0.039)	1.521	(0.000)
$\Lambda_{1(1,2)}$	-0.017	(0.011)	-0.017	(0.001)
$\Lambda_{1(2,1)}$	0.806	(0.001)	0.801	(0.000)
$\Lambda_{1(2,2)}$	0.013	(0.009)	0.012	(0.000)
Volatility measurement error yields $v_t' \sim N(0, \Sigma^\tau)$				
$\sigma_{\tau_{12}}$	0.37%	(0.01%)	0.37%	(0.01%)
$\sigma_{\tau_{24}}$	0.16%	(0.00%)	0.16%	(0.00%)
$\sigma_{\tau_{60}}$	0.24%	(0.00%)	0.24%	(0.00%)
$\sigma_{\tau_{120}}$	0.60%	(0.02%)	0.60%	(0.01%)
LL	18316.7		18315.5	
min.ev(K)	0.0061		0.0090	
min.ev(M)	0.0123		0.0136	
$P_{2.5\%}[y_{60}(120)]$	-0.35%		0.03%	

Table 20: This table contains for the Dutch monthly dataset for different sets of constraints the parameters and standard errors of the maximum likelihood estimate of the KNW model with the three-month and three-year yields observed without a measurement error. For column (i) the matrices K and M and the values of r_Π^g, r_S^g, and UFR are restricted. These same restrictions hold for column (ii) with the added constraint of $P_{2.5\%}[y_{60}(120)] > 0$.

Overall, most restrictions do not lead to a large drop in the log-likelihood, because the unrestricted model estimates were in line with these restrictions. The only big change is the significant decrease of the UFR under the restricted model, which is caused by the extrapolating nature of the UFR. This means the current model assumptions and dataset lead to parameter estimates that are largely in line with the expectations from (Dijsselbloem et al., 2019).

9 Conclusion

The KNW model is a capital market model that can be applied to a country's financial market. The aim of this paper was to analyze the robustness of the KNW model, as defined in Draper (2014), to changing parameter restrictions and model assumptions. Also I tested the effect on the estimation results of changing the dataset by using additional observations, using monthly instead of quarterly observations or by applying the model to other countries.

In the first part, the KNW model was estimated using quarterly data to compare the estimation results with those from (Draper, 2014) and (Van den Berg, 2016) and to determine which pair of yields should be observed without error. This has lead to parameter estimates and implied characteristics that were significantly different to those from (Draper, 2014) and (Van den Berg, 2016). The estimates and implied characteristics where the three-month and three-year yields are observed without error are more in line with the sample data, and are therefore preferable when estimating the KNW model.

In the second part, the KNW model estimates for the monthly datasets of the Netherlands, UK, and US are compared. The parameter estimates for the monthly dataset are more precise and are more in line with the historical data compared to the quarterly dataset.

The parameter values give a better fit for the Netherlands than for the UK and the US. These findings are a confirmation of the hypothesis from Draper (2012) that the KNW model is mainly a model for the Northern European market.

In the third part, both the Dutch and UK datasets were tested for structural breaks. For all points in the datasets that were considered the null hypothesis of no structural break was rejected. This indicates that the estimation period of nearly fifty years is too long. Given the fluctuations of inflation, bond yields, and stock return over time, it is difficult to find parameter estimates that fit the data well for the entire sample period. Therefore, solutions could be to either use a much shorter estimation period, use multiple break point for certain parameters, or use a different model.

In the fourth part, restrictions suggested by (Dijsselbloem et al., 2019) are imposed on the model for the Netherlands. Most restrictions are already met by the unrestricted KNW model, which means the parameter estimates and implied model behaviour are in line with the expectations of the commission. The only major difference is the value of the UFR, which should be lower according to (Dijsselbloem et al., 2019). This is because the UFR is not influenced as much by the sample data and is an extrapolation to the future.

In the last part, the KNW model was estimated with all yields observed with a measurement error. Compared to the KNW model where the three-month and three-year yields are observed without error, the fit of the model for the UK clearly improves, while for the Netherlands and the US the results are worse. This might be caused the UK's weaker correlation between the bond yields, which means this model assumption should be chosen based on the link between a country's long- and short-maturity yields.

Applying the KNW model using the improvement found in this paper can lead to an improved economic scenario generator. For the Netherlands this can lead to better pension fund evaluation.

Next, I will discuss the reliability of the estimation results. Most conclusions that are drawn are based on a single estimation of the KNW model, which means they are strongly influenced by the specific estimation period. Using a long estimation period has the advantage that long-term trends and expectations in the data should become apparent in the model estimates. However, the conclusions with respect to the goodness-of-fit of different datasets and changing model assumptions could be sensitive to a change in the estimation period. The most reliable conclusion is that the parameter estimates using a long estimation period are very susceptible to structural changes. Namely, this result is based on many estimations of the KNW model which all clearly lead to the same conclusion.

Possible extensions to this research might be the application of the KNW model to countries whose financial market is characterized differently than the countries considered in this paper. Another idea is to consider more pairs of yields that are observed without error to better investigate the effect this choice has on the model estimates. In addition this idea could be extended to more countries to make it more easy to relate the conclusion of the optimal yield pair to the data characteristics.

One extension is related to the search for structural breaks. In this research, I only considered changes of all parameters at the same time. It is also possible to make certain parameters time-varying, while keeping others constant. With the estimation period I used it is probably better to make the parameters related to long-term expected averages of stock return, inflation and bond yields time-varying, as these variables have changed significantly over time.

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A Data

Variable	Time Period	Description	Source
inflation	September 1972 - December 1998	Returns on the (West) German CPI index	International Financial Statistics of the International Monetary Fund on https://data.imf.org/
	January 1999 - January 2021	Returns on the Harmonized Index of Consumer Prices of the Euro area	Federal Reserve Bank of St. Louis
Yields			
3-month	September 1972 - June 1990	Three-month money market rates	Bundesbank (www.bundesbank.de)
	July 1990 - January 2021	Three-month German Interbank rate	Federal reserve Bank of St.Louis. (https://fred.stlouisfed.org/)
1-year 2-year 3-year 5-year 10-year	September 1972 - January 2021	Term structure of interest rates on listed Federal securities	Bundesbank (www.bundesbank.de)

Table 21: This table contains for each time period the measures of Dutch inflation and the different yields and the source of the data.

Variable	Time Period	Description	Source
inflation	September 1972 - January 2021	Returns on the UK CPI index	International Financial Statistics of the International Monetary Fund on https://data.imf.org/
Yields			
3-month	September 1972 - June 2017	Yields on three-month Treasury securities	Federal Reserve bank of St. Louis
	July 2017 - January 2021	Three-month bond yields	www.investing.com
1-year 2-year 3-year 5-year 10-year	September 1972 - January 2021	UK instantaneous nominal forward curve	Bank of England

Table 22: This table contains for each time period the measures of UK inflation and the different yields and the source of the data.

Variable	Time Period	Description	Source
inflation	September 1972 - January 2021	Returns on the US CPI index	International Financial Statistics of the International Monetary Fund on https://data.imf.org/
Yields			
3-month 1-year 2-year 3-year	September 1972 - January 2021	Market yields on US Treasury securities	Federal Reserve
5-year	September 1972 - May 1976	Five-year Treasury rates	www.macrotrends.net
	June 1976 - January 2021	Market yield on US Treasury securities at five-year maturity	Federal Reserve
10-year	September 1972 - August 1981	Yields on ten-year bonds	www.investing.com
	September 1981 - January 2021	Market yield on US Treasury securities at ten-year maturity	Federal Reserve

Table 23: This table contains for each time period the measures of US inflation and the different yields and the source of the data.

	1 month	2 months	3 months	6 months	1 year	2 years
Netherlands	0.992	0.982	0.971	0.936	0.852	0.671
US	0.993	0.984	0.975	0.952	0.896	0.790
UK	0.991	0.981	0.973	0.950	0.912	0.854

Table 24: Autocorrelation 1 year yield

	1 month	2 months	3 months	6 months	1 year	2 years
Netherlands	0.993	0.984	0.974	0.943	0.872	0.726
US	0.993	0.985	0.977	0.954	0.901	0.806
UK	0.992	0.983	0.973	0.949	0.913	0.861

Table 25: Autocorrelation 2 year yield

	1 month	2 months	3 months	6 months	1 year	2 years
Netherlands	0.993	0.985	0.976	0.947	0.884	0.756
US	0.993	0.985	0.977	0.955	0.906	0.821
UK	0.992	0.983	0.974	0.950	0.914	0.857

Table 26: Autocorrelation 3 year yield

	1 month	2 months	3 months	6 months	1 year	2 years
Netherlands	0.994	0.986	0.977	0.951	0.894	0.781
US	0.993	0.984	0.976	0.953	0.905	0.829
UK	0.992	0.984	0.976	0.958	0.923	0.856

Table 27: Autocorrelation 5 year yield

	1 month	2 months	3 months	6 months	1 year	2 years
Netherlands	0.993	0.985	0.977	0.951	0.896	0.788
US	0.993	0.983	0.975	0.951	0.902	0.825
UK	0.989	0.980	0.971	0.956	0.912	0.821

Table 28: Autocorrelation 10 year yield

B Nominal term-structure

A second-order approximation of the fundamental pricing Equation of a nominal zero coupon bond in Equation 10 is

$$E[d\phi_t^N P^N + \phi_t^N dP^N + d\phi_t^N dP^N] = 0. \tag{58}$$

After applying Itô Doeblin theorem this results in

$$\begin{aligned} dP^N &= P_X^{N\prime} dX_t + P_t^N dt + \frac{1}{2} dX_t' P_{XX'}^N dX_t + dX_t' P_{Xt}^N dt + \frac{1}{2} dt P_{tt}^N dt \\ &= P_X^{N\prime}(-KX_t dt + \Sigma_X' dZ_t) + P_t^N dt + \frac{1}{2}(dZ_t)\Sigma_X P_{XX'}^N \Sigma_X' dZ_t. \end{aligned} \tag{59}$$

This Equation is substituted for the nominal stochastic discount factor in Equation 7 and the price change into the fundamental valuation Equation 10, which results in

$$0 = P_X^{N\prime}(-KX_t) + P_t^N + \frac{1}{2}\text{tr}(\Sigma_X P_{XX'}^N \Sigma_X') - P^N R_t - P_X^{N\prime} \Sigma_X' \Lambda_t. \tag{60}$$

where tr(a) represents the trace of a. The solution of this partial differential Equation is of the form

$$P^N(X_t, t, t+\tau) = \exp(A^N(\tau) + B^N(\tau)' X_t). \tag{61}$$

with $\tau = T - t$ is the bond maturity. The exact definition of $A^N(\tau)$ and $B^N(\tau)$ will be derived below.

The derivatives of the bond prices using Equation 61 are

$$\begin{aligned} \frac{1}{p^N} P_X^N &= B^N \\ \frac{1}{p^N} P_t^N &= -\frac{1}{P^N} P_\tau^N = -\dot{A}^N - \dot{B}'^N X_t \\ \frac{1}{p^N} P_{XX'}^N &= B^N B'^N. \end{aligned} \tag{62}$$

Substituting these relations into Equation 60 gives

$$0 = B^{N\prime}(-KX_t) + (-\dot{A}^N - \dot{B}^{N\prime} X_t) + \frac{1}{2}\mathrm{tr}(\Sigma_X B^N B^{N\prime} \Sigma_X') - R_0 - R_1' X_t - B^{N\prime} \Sigma_X' (\Lambda_0 + \Lambda_1 X_t) \tag{63}$$

and

$$\dot{A}^N(\tau) = -R_0 - (\Lambda_0' \Sigma_X) B^N(\tau) + \frac{1}{2} B'^N(\tau) \Sigma_X' \Sigma_X B^N(\tau) \tag{64}$$

$$\dot{B}^N(\tau) = -R_1 - (K' + \Lambda_1' \Sigma_X) B^N(\tau), \tag{65}$$

as both the stochastic term and the non-stochastic term have to be equal to zero. A nominal zero coupon bond with maturity $\tau = 0$ and payout 1 has a price of 1. Substituting these values into Equation 61 gives $A^N(0) = 0$ and $B^N(0) = 0$. The instantaneous nominal yield of a bond with duration τ is

$$-d\ln(P^N(X_t, t, t+\tau)) = -(\dot{A}^N(\tau) + \dot{B}^N(\tau)' X_t). \tag{66}$$

For $\tau = 0$ this expression is reduced to $-d\ln(P^N(X_t, t, t)) = -(\dot{A}^N(0) + \dot{B}^N(0)' X_t) = R_0 + R_1' X_t = R_t$. There is a closed form solution for these differential equations:

$$A^N(\tau) = \int_0^\tau \dot{A}^N(s) ds \tag{67}$$

$$B^N(\tau) = (K' + \Lambda_1' \Sigma_X)^{-1} [\exp(-(K' + \Lambda_1' \Sigma_X)\tau) - I_{2\times 2}] R_1 \tag{68}$$

C Estimation procedure

C.1 two yields without measurement error

The derivation of the log-likelihood is taken from (Shafiq, 2015). The log-likelihood consists of three parts. The first part of the likelihood is related to the four yields that are observed with a measurement error. Assuming that the measurement errors are Gaussian, the conditional distribution of the measurement error is given by:

$$\begin{aligned} f(v_t|0,\Sigma^\tau) &= (|2\pi\Sigma^\tau|)^{-\frac{1}{2}}\exp(-\frac{1}{2}v_t(\Sigma^\tau)^{-1}v_t') \\ &= ((2\pi)^4|\Sigma^\tau|)^{-\frac{1}{2}}\exp(-\frac{1}{2}v_t(\Sigma^\tau)^{-1}v_t'). \end{aligned} \tag{69}$$

This gives the likelihood

$$\mathrm{L}_1 = \prod_{t=1}^{T} f(v_t|0,\Sigma^\tau) = \prod_{t=1}^{T}((2\pi)^4|\Sigma^\tau|)^{-\frac{1}{2}}\exp(-\frac{1}{2}v_t(\Sigma^\tau)^{-1}v_t'), \tag{70}$$

which, after taking the natural logarithm, gives

$$\mathrm{LL}_1 = -\frac{T\cdot 4}{2}\ln(2\pi) - \frac{T}{2}\ln(|\Sigma^\tau|) - \frac{1}{2}\sum_{t=1}^{T} v_t(\Sigma^\tau)^{-1}v_t'. \tag{71}$$

The second part of the log-likelihood is related to the error term of the state variables, inflation and the stock returns. Let's define $\tilde{Y}' = [\hat{X}, \Delta\ln(\Pi), \Delta\ln(S)]'$. Here, $\hat{X}$ are the state variables that are determined based on the two yields that are measured without measurement error in Equation 35. The error term is denoted by

$$\tilde{\epsilon}_t = \tilde{Y}_{t+1} - \mu - \Gamma Y_t \tag{72}$$

Working out this Equation for each variable gives

$$\begin{aligned} \tilde{\epsilon}_{t+1,1} &= \tilde{X}_{1,t+1} - \mu_{X_1} - \Gamma_{X1,X1}\tilde{X}_{1,t}, \\ \tilde{\epsilon}_{t+1,2} &= \tilde{X}_{2,t+1} - \mu_{X_2} - \Gamma_{X2,X1}\tilde{X}_{1,t} - \Gamma_{X2,X2}\tilde{X}_{2,t}, \\ \tilde{\epsilon}_{t+1,3} &= \Delta\ln(\tilde{\Pi}_{t+1}) - \mu_{\ln(\Pi)} - \Gamma_{\ln(\Pi),X1}\tilde{X}_{1,t} - \Gamma_{\ln(\Pi),X2}\tilde{X}_{2,t}, \\ \tilde{\epsilon}_{t+1,4} &= \Delta\ln(\tilde{S}_{t+1}) - \mu_{\ln(S)} - \Gamma_{\ln(S),X1}\tilde{X}_{1,t} - \Gamma_{\ln(S),X2}\tilde{X}_{2,t}. \end{aligned} \tag{73}$$

Again, the assumption is made that the errors are Gaussian, which means that the conditional distribution of the error term $\tilde{\epsilon}_t$ at time t is given by

$$f(\tilde{\epsilon}_t|0,\Sigma) = ((2\pi)^4|\Sigma|)^{-\frac{1}{2}}\exp(-\frac{1}{2}\tilde{\epsilon}_t\Sigma^{-1}\tilde{\epsilon}_t'). \tag{74}$$

This gives the likelihood

$$\mathrm{L}_2 = \prod_{t=1}^{T} f(\tilde{\epsilon}_t|0,\Sigma) = \prod_{t=1}^{T}((2\pi)^4|\Sigma|)^{-\frac{1}{2}}\exp(-\frac{1}{2}\tilde{\epsilon}_t\Sigma^{-1}\tilde{\epsilon}_t'), \tag{75}$$

which, after taking the natural logarithm, gives

$$\mathrm{LL}_2 = -\frac{T \cdot 4}{2}\ln(2\pi) - \frac{T}{2}\ln(|\Sigma|) - \frac{1}{2}\sum_{t=1}^{T} \tilde{\epsilon}_t \Sigma^{-1} \tilde{\epsilon}_t', \tag{76}$$

The third and last part of the log-likelihood is a constant term related to the two yields that are observed without a measurement error:

$$\mathrm{LL}_3 = -\frac{T \cdot 2}{2}\ln(2\pi) - \frac{T}{2}\ln(|B|), \tag{77}$$

with $B' = [B(\tau_5), B(\tau_6)]$. Adding LL_1, LL_2, and LL_3 together, the log-likelihood function is:

$$\mathrm{LL} = -\frac{T \cdot 10}{2}\ln(2\pi) - \frac{1}{2}\left(T\ln(|\Sigma^\tau|) + \sum_{t=1}^{T} v_t (\Sigma^\tau)^{-1} v_t'\right) - \frac{1}{2}\left(T\ln(|\Sigma|) + \sum_{t=1}^{T} \tilde{\epsilon}_t (\Sigma)^{-1} \tilde{\epsilon}_t'\right) - \frac{1}{2}T\ln(|B|), \tag{78}$$

which is

$$\mathrm{LL} = -\frac{1}{2}\left(T\ln(|\Sigma^\tau|) + \sum_{t=1}^{T} v_t (\Sigma^\tau)^{-1} v_t'\right) - \frac{1}{2}\left(T\ln(|\Sigma|) + \sum_{t=1}^{T} \tilde{\epsilon}_t (\Sigma)^{-1} \tilde{\epsilon}_t'\right) - \frac{1}{2}T\ln(|B|), \tag{79}$$

after removing the part with $\ln(2\pi)$, as it is irrelevant to the overall optimization process.

C.2 All yields with measurement error

The measurement equation

$$y_t = c + ZX_t + G\epsilon_t, \tag{80}$$

and the transition equation

$$X_{t+1} = d + TX_t + H\eta_t, \tag{81}$$

Iteration

The filter iterations are implemented using the expected values

$$\begin{aligned} X_t &= \mathrm{E}[X_t | y_1, \ldots, y_{t-1}] \\ X_{t|t} &= \mathrm{E}[X_t | y_1, \ldots, y_t] \end{aligned} \tag{82}$$

and the variances

$$\begin{aligned} P_t &= \mathrm{Var}[X_t|y_1,\ldots,y_{t-1}] \\ P_{t|t} &= \mathrm{Var}[X_t|y_1,\ldots,y_t] \end{aligned} \tag{83}$$

of the state X_t in the following way:

Initialisation: Set $t = 1$ with $X_t = X_0$ and $P_t = P_0$

Updating equations:

$$\begin{aligned} v_t &= y_t - c_t - ZX_t \\ F_t &= ZP_tZ' + GG' \\ K_t &= P_tZ'F_t^{-1} \\ X_{t|t} &= X_t + K_tv_t \\ P_{t|t} &= P_t - P_tZ'K_t' \end{aligned} \tag{84}$$

Prediction equations:

$$\begin{aligned} X_{t+1} &= d + TX_{t|t} \\ P_{t+1} &= TP_{t|t}T' + HH' \end{aligned} \tag{85}$$

Next iteration: Set $t = t + 1$ and go back to “Updating equations”.

Input:
$X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as X is mean-reverting around zero.
$P_0 = \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix}$, my own choice
d $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as X is mean-reverting around zero.
c $= \begin{bmatrix} \mu_{\ln(\Pi)} \\ \mu_{\ln(S)} \\ -A(\tau_1)/\tau_1 \\ -A(\tau_2)/\tau_2 \\ -A(\tau_3)/\tau_3 \\ -A(\tau_4)/\tau_4 \\ -A(\tau_5)/\tau_5 \\ -A(\tau_6)/\tau_6 \end{bmatrix}$,
T $= \begin{bmatrix} \Gamma_{X_1,X_1} & 0 \\ \Gamma_{X_2,X_1} & \Gamma_{X_2,X_2} \end{bmatrix}$

$$Z = \begin{bmatrix} \Gamma_{\ln(\Pi),X_1} & \Gamma_{\ln(\Pi),X_2} \\ \Gamma_{\ln(S),X_1} & \Gamma_{\ln(S),X_2} \\ \Gamma_{y^{\tau_1},X_1} & \Gamma_{y^{\tau_1},X_2} \\ \Gamma_{y^{\tau_2},X_1} & \Gamma_{y^{\tau_2},X_2} \\ \Gamma_{y^{\tau_3},X_1} & \Gamma_{y^{\tau_3},X_2} \\ \Gamma_{y^{\tau_4},X_1} & \Gamma_{y^{\tau_4},X_2} \\ \Gamma_{y^{\tau_5},X_1} & \Gamma_{y^{\tau_5},X_2} \\ \Gamma_{y^{\tau_6},X_1} & \Gamma_{y^{\tau_6},X_2} \end{bmatrix}$$

$$H = \begin{bmatrix} \Sigma_{X_1,X_1} & \Sigma_{X_1,X_2} \\ \Sigma_{X_1,X_2} & \Sigma_{X_2,X_2} \end{bmatrix}$$

All measurement errors of the yields are assumed to be independent, both sequentially and cross-sectionally.

$$G = \begin{bmatrix} \Sigma_{\ln(\Pi),\ln(\Pi)} & \Sigma_{\ln(\Pi),\ln(S)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Sigma_{\ln(\Pi),\ln(S)} & \Sigma_{\ln(S),\ln(S)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{\tau_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{\tau_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\tau_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\tau_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\tau_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\tau_6} \end{bmatrix}$$

D Model implied statistics

Model implied risk premium of bond with maturity τ:

$$B(\tau)'\Sigma_X'\Lambda_0, \tag{86}$$

as Λ_t is set equal to its unconditional expectation Λ_0.

Model implied volatility of the risk premium of a bond with maturity τ:

$$\sigma_{B(\tau)'\Sigma_X'\Lambda_0} = \sqrt{B(\tau)'\Sigma_X'\Sigma_X B(\tau)} \tag{87}$$

Model implied correlation between stock return and bond yield with maturity τ:

$$\begin{aligned} \rho_{\ln(S),y_\tau} &= \frac{\mathrm{cov}(\ln(S), y_\tau)}{\sqrt{\mathrm{var}(\ln(S))\mathrm{var}(y_\tau)}} \\ &= \frac{B(\tau)'\Sigma_X'\sigma_S}{\sqrt{\sigma_S'\sigma_S}\sqrt{B(\tau)'\Sigma_X'\Sigma_X B(\tau)}} \end{aligned} \tag{88}$$

Model implied correlation between bond yield with maturity τ_1 and bond yield with maturity

τ_2:

$$\begin{aligned}\rho_{y_{\tau_1},y_{\tau_2}} &= \frac{\text{cov}(y_{\tau_1}, y_{\tau_2})}{\sqrt{\text{var}(y_{\tau_1})\text{var}(y_{\tau_2})}} \\ &= \frac{B(\tau_1)'\Sigma_X'\Sigma_X B(\tau_2)}{\sqrt{B(\tau_1)'\Sigma_X'\Sigma_X B(\tau_1)}\sqrt{B(\tau_2)'\Sigma_X'\Sigma_X B(\tau_2)}}\end{aligned} \tag{89}$$

Model implied correlation between risk premium of bond with maturity τ and stock return:

$$\begin{aligned}\rho_{y_\tau,\ln(S)} &= \frac{\text{cov}(y_\tau, \ln(S))}{\sqrt{\text{var}(y_\tau)\text{var}(\ln(S))}} \\ &= \frac{B(\tau)'\Sigma_X'\Lambda_1\Sigma_X'\sigma_S}{\sqrt{B(\tau)'\Sigma_X'\Lambda_1\Sigma_X'\Sigma_X\Lambda_1'\Sigma_X B(\tau)}\sqrt{\sigma_S'\sigma_S}}\end{aligned} \tag{90}$$

Model implied correlation between risk premium of bond with maturity τ_1 and yield of bond with maturity τ_2:

$$\begin{aligned}\rho_{y_{\tau_1},y_{\tau_2}} &= \frac{\text{cov}(y_{\tau_1}, y_{\tau_2})}{\sqrt{\text{var}(y_{\tau_1})\text{var}(y_{\tau_2})}} \\ &= \frac{B(\tau_1)'\Sigma_X'\Lambda_1\Sigma_X'\Sigma_X B(\tau_2)}{\sqrt{B(\tau_1)'\Sigma_X'\Lambda_1\Sigma_X'\Sigma_X\Lambda_1'\Sigma_X B(\tau_1)}\sqrt{B(\tau_2)'\Sigma_X'\Sigma_X B(\tau_2)}}\end{aligned} \tag{91}$$

E R programming code

For this paper a number of codes that are written in R are used. The following list contains the names of the codes and a description:

- 'installLoadPackages': This function installs and loads a given set of packages.
- 'data loading': This file loads the data used in this research from several excel files.
- 'Scriptie KNW model': This is the main file that is used to get the results described in this paper. With this file the KNW model can be estimated for all model assumption that are considered, the implied model characteristics and implied variables can be calculated, the structural break test can be performed, and simulations can be made. For estimating the KNW model, other functions are called.
- 'KNW_model_2': This function estimates the KNW model with two yields observed without measurement error, but without any further constraints.
- 'KNW_model': This function estimates the KNW model with all yields observed with a measurement error, and without any further constraints.
- 'KNW_model_2_restricted': This function estimates the KNW model with two yields without measurement error, with additional constraints.
- 'A': The result of this function is the value of $A(\tau)$.

- 'B': The result of this function is the value of $B(\tau)$.
- 'alpha': The result of this function is the value of $\alpha(x)$ for a given value of x.